On nodal domains and spectral minimal partitions: a magnetic characterization

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Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition of Ω by k open sets ω_j, we can consider the quantity $\max_j \lambda(\omega_j)$ where $\lambda(\omega_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in ω_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(\omega_j)$, a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

In this talk, we consider the two-dimensional case and discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the rectangle and then give a "magnetic" characterization of these minimal partitions. This work has started in collaboration with T. Hoffmann-Ostenhof (with a preliminary work with M. and T. Hoffmann-Ostenhof and M. Owen) and has been continued with coauthors : V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Vial ... C. Lena

Section 1: Introduction

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

Let Ω be a regular bounded domain. Let $H(\Omega)$ be the Laplacian $-\Delta$ on $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition ($u_{\partial\Omega} = 0$). In the case of a Riemannian manifold we will consider the Laplace Beltrami operator.
We denote by $(\lambda_j(\Omega))_j$ the increasing sequence of its eigenvalues counted with multiplicity and by $(u_j)_j$ some associated orthonormal basis of eigenfunctions.

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For any $u \in C_0^0(\overline{\Omega})$, we introduce the nodal set of u by:

$$N(u) = \overline{\{x \in \Omega \,\big|\, u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u.

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and call the components of $\Omega \setminus N(u)$ the nodal domains of u.
The number of nodal domains of u is called $\mu(u)$. These $k = \mu(u)$ nodal domains define a partition of Ω.

The Courant nodal theorem says :

Theorem [Courant]

Let $k \geq 1$ and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated with λ_k.
Then, $\forall u \in E(\lambda_k) \setminus \{0\}\ ,\ \mu(u) \leq k\ .$

Except in dimension 1, the inequality is in general strict:

Theorem [Pleijel]

There exists k_0 such that if $k \geq k_0$, then

$$\mu(u) < k\,,\ \forall u \in E(\lambda_k) \setminus \{0\}$$

The main points in the proof of Pleijel's theorem are the Faber-Krahn inequality :

$$\lambda(\omega) \geq \frac{\pi j^2}{|\omega|} \; . \tag{2}$$

and the Weyl's law for the counting function.

Partitions

We first introduce the notion of partition.

Definition 1

Let $1 \leq k \in \mathbb{N}$. We call **partition** (or k-partition for indicating the cardinal of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint sets such that

$$\cup_{i=1}^{k} D_i \subset \Omega \, . \tag{3}$$

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We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected.

We denote by $\mathfrak{O}_k$ the set of open connected partitions.

Minimal partitions

We now introduce the notion of spectral minimal partition sequence.

Definition 2

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce the "energy" of $\mathcal{D}$:

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{4}$$

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Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{5}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

Remark A

If $k = 2$, it is rather well known (see [HH1] or [CTV3]) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition.

We discuss briefly the notion of regular and strong partition.

Definition 3: strong partition

A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{6}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4: Boundary set

$$N(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)}\,. \tag{7}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

Regular partitions

We now introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of its associated boundary set N, which should satisfy :

Definition 5: regular boundary set

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the nbhd of which N is the union of $\nu_i(x_i)$ smooth curves ($\nu_i \geq 2$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

Partitions and bicolorability.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

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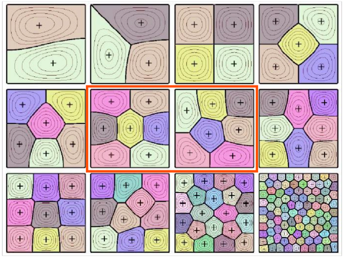
We recall that a collection of nodal domains of an eigenfunction is always bicolorable.
Here are two families of examples of regular partitions.

This family is supposed (unproved and not clearly stated) to correspond to minimal partitions of the square.

Figure 2: Examples of strong partitions non necessarily bicolorable.

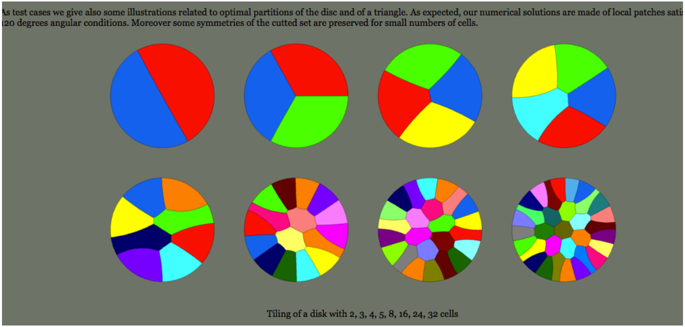
Multiple populations



“Minimization of the Renyi entropy production in the space-partitioning process”
Cybulski, Babin, and Holyst, Phys. Rev. E 71, 046130 (2005)

This family corresponds to computations of E. Oudet for the problem of dividing a bounded region of the plane into pieces of equal area such as to minimize the length of the boundary of the partition.

Figure 3



Section 2: Main results in the $2D$ case

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] that

Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

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Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

Other proofs of a somewhat weaker version of this statement have been given by Bucur-Buttazzo-Henrot [BBH], Caffarelli- F.H. Lin [CL].

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We have first the following converse theorem ([HH1], [HHOT1]):

Theorem 2

If the minimal partition is bicolorable this is a nodal partition.

A natural question is now to determine how general this previous situation is.

Surprisingly this only occurs in the so called Courant-sharp situation. We say that:

Definition 6: Courant-sharp

A pair (u, λ_k) is Courant-sharp if
$u \in E(\lambda_k) \setminus \{0\}$ and $\mu(u) = k$.

An eigenvalue is
called Courant-sharp if there exists an associated Courant-sharp pair.

For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction of $H(\Omega)$ with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.
In general, one can show, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega)\,. \tag{8}$$

The last result gives the full picture of the equality cases :

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k\,.$$

In addition, one can find a Courant-sharp pair (u, λ_k).

This answers a question in [BHIM] (Section 7).

Topological type of minimal partitions

The possible topological types of a minimal partition $\mathcal{D}$ rely essentially on

Euler's formula

Let Ω be an open set in $\mathbb{R}^2$ with piecewise-C^1 boundary and let N a closed set such that $\Omega \setminus N$ has k components and such that N is regular. Let b_0 be the number of components of $\partial\Omega$ and b_1 be the number of components of $N \cup \partial\Omega$. Denote by $\nu(x_i)$ and $\rho(z_i)$ the numbers of arcs associated with the $x_i \in X(N)$, respectively $z_i \in Y(N)$. Then

$$k = b_1 - b_0 + \sum_{x_i \in X(N)} \left(\frac{\nu(x_i)}{2} - 1\right) + \frac{1}{2} \sum_{z_i \in Y(N)} \rho(z_i) + 1\,. \qquad (9)$$

This allows us to analyze minimal partitions of a specific topological type. Assuming that the domain Ω is connected, simply connected ($b_0 = 1$, $k = 3$) and that the 3-partition is not nodal, we obtain that the only possible cases correspond to (forgetting some degenerate case)

- One critical point of type 3 and three boundary points;
- two critical points of type 3 and no boundary point;
- two critical points of type 3 and two boundary points.

In particular we get at most 2 (necessarily odd) critical points for $k = 3$. More generally there are at most $2k - 4$ odd critical points for minimal k-partitions.

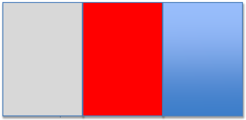
Section 3: Examples of k-minimal partitions for special domains

If in addition the domain has some symmetries and we assume that a minimal partition keeps some of these symmetries, then we find natural candidates for minimal partitions.

The case of a rectangle

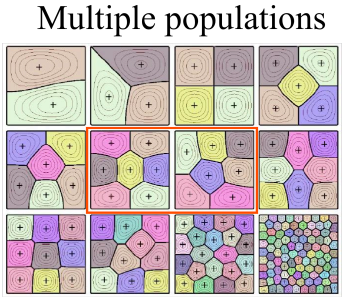
Using Theorem 3, it is now easier to analyze the situation for rectangles (at least in the irrational case), since we have just to look for Courant-sharp pairs.

In the long rectangle $]0,a[\times]0,1[$ the eigenfunction $\sin(k\pi x/a)\sin\pi y$ is Courant-sharp for $a \geq \sqrt{(k^2-1)/3}$. See the nodal domain for $k=3$.



The case of the square

We verify that $\mathfrak{L}_2 = \lambda_2$.
It is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$, and there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).



"Minimization of the Renyi entropy production in the space-partitioning process"

Assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet-Neumann problems.

Numerical computations performed by V. Bonnaillie-Noël and G. Vial lead to a natural candidate for a symmetric minimal partition.
See http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

Figure 3

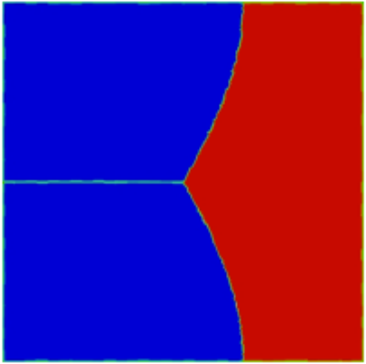


Figure: Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

The case of the square: $k = 3$ continued

In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

The case of the square: $k = 3$ continued

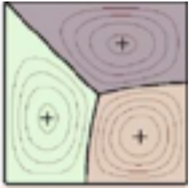
In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

But if we assume that the minimal 3-partition has one critical point and has the symmetry, then numerical computations lead to the Figure 3.
Numerics suggest more : the center of the square is the critical point of the partition.

This point of view is explored numerically by Bonnaillie-Helffer [BH] and theoretically by Noris-Terracini [NT].

Why this symmetry ?

The picture of Cybulski-Babin-Holst has another symmetry (with respect to the diagonal) and the same energy.



Actually there is a continuous family of candidates !!

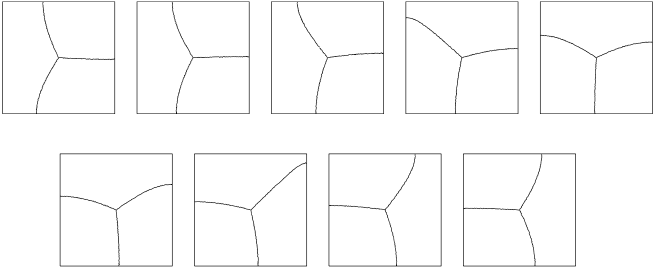


Figure: Continuous family of 3-partitions with the same energy.

We should give an explanation : Aharonov-Bohm spectrum !

Looking for minimal 5-partitions.

Using a variant of the Aharonov-Bohm approach (consider a double covering), we were able (BH+VB) to produce the following candidate for a minimal 5-partition of a specific topological type.

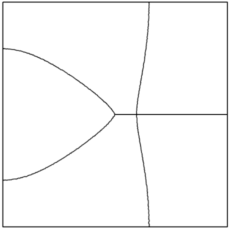


Figure: First candidate for the 5-partition of the square.

Inspired by numerical computations in [CyBaHo], one looks for a configuration which has the symmetries of the square and four critical points. We get two types of models that we can reduce to a Dirichlet-Neumann problem on a triangle corresponding to the eigth of the square. Moving the Neumann boundary on one side like in [BHV] leads to two candidates. One has a lower energy and coincides with the pictures in [CyBaHo].

$\lambda_{11}(\dot{\Omega}_X^{\mathcal{R}}) = 111.910$ $\quad\lambda_2^{DN} = 104.294$ $\quad\lambda_2^{DN} = 131.666$

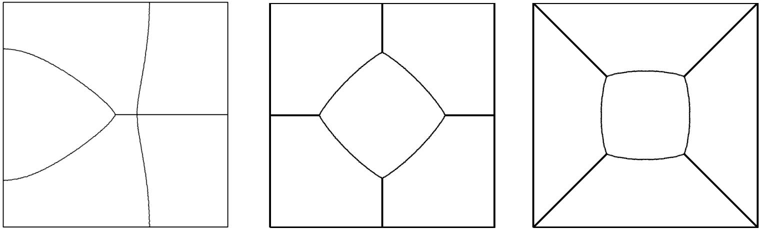


Figure: Three candidates for the 5-partition of the square.

Section 5: The Aharonov-Bohm Operator

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short **AB**X-Hamiltonian) with a singularity at X introduced in [BHHO, HHOO] and motivated by the work of Berger-Rubinstein.
We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with flux at X

$$\Phi = \pi$$

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right) . \quad (10)$$

We know that the magnetic field vanishes identically in $\dot{\Omega}_X$. The **AB**X-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \tag{11}$$

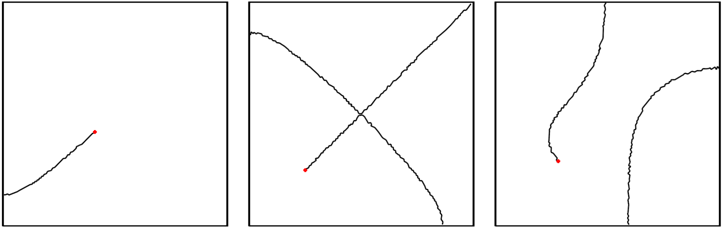
Let K_X be the antilinear operator

$$K_X = e^{i\theta_X}\,\Gamma\,,$$

with $(x - x_0) + i(y - y_0) = \sqrt{|x - x_0|^2 + |y - y_0|^2}\, e^{i\theta_X}$, and where Γ is the complex conjugation operator $\Gamma u = \bar{u}$.
A function u is called K_X-real, if $K_X u = u$.
The operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions.
Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X)\,,\; K_X\, u = u\}\,.$$

It was shown that the nodal set of such a K_X real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines meet at X.



For a "real" groundstate (one pole), one can prove that the nodal set consists of one line joining the pole and the boundary.

Extension to many poles

First we can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with ℓ distinct points $X_1, \ldots, X_\ell$ (putting a flux π at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j} \,,$$

where $\mathbf{X} = (X_1, \ldots, X_\ell)$.

We can also construct (see [HHOO]) the antilinear operator $K_{\mathbf{X}}$, where θ_X is replaced by a multivalued-function $\phi_{\mathbf{X}}$ such that $d\phi_X = 2\mathbf{A}^{\mathbf{X}}$ and $e^{i\phi_{\mathbf{X}}}$ is univalued and C^∞.
We can then consider the real subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$. It has been shown in [HHOO] (see in addition [1]) that the $K_{\mathbf{X}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j ($j = 1, \dots, \ell$) an odd number of half-lines should meet.
We denote by $L_k(\dot{\Omega}_{\mathbf{X}})$ the lowest eigenvalue (if any) such that there exists a $K_{\mathbf{X}}$-real eigenfunction with k nodal domains.

The explanation for the continuous family

The candidate for $\mathfrak{L}_3$ is the third eigenvalue of the Aharonov-Bohm operator with one pole at the center.

This eigenvalue has multiplicity 2.

The continuous family is associated to the nodal sets of a continuous family (in a real two-dimensional vector space) of K-invariant eigenfunctions.

Section 6: A magnetic characterization of a minimal partition

We now discuss the following theorem.

Theorem 6

Let Ω be simply connected. Then

$$\mathfrak{L}_k(\Omega) = \inf_{\ell \in \mathbb{N}} \inf_{X_1,\dots,X_\ell} L_k(\dot{\Omega}_{\mathbf{X}}) \,.$$

Let us present a few examples illustrating the theorem. When $k = 2$, there is no need to consider punctured Ω's. The infimum is obtained for $\ell = 0$. When $k = 3$, it is possible to show that it is enough, to minimize over $\ell = 0$, $\ell = 1$ and $\ell = 2$. In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for the punctured domain at the center. For $k = 5$, it seems that the infimum is for $\ell = 4$ in the case of the square and for $\ell = 1$ in the case of the disk.

Let us give a sketch of the proof. Considering a minimal k-partition $\mathcal{D} = (D_1, \ldots, D_k)$, we know that it has a regular representative and we denote by $X^{odd}(\mathcal{D}) := (X_1, \ldots, X_\ell)$ the critical points of the partition corresponding to an odd number of meeting half-lines. Then the guess is that $\mathfrak{L}_k(\Omega) = \lambda_k(\dot{\Omega}_{\mathbf{X}})$ (Courant sharp situation). One point to observe is that we have proven in [HHOT1] the existence of a family u_i such that u_i is a groundstate of $H(D_i)$ and $u_i - u_j$ is a second eigenfunction of $H(D_{ij})$ when $D_i \sim D_j$.

Then we find a sequence $\epsilon_i(x)$ of $\mathbb{S}^1$-valued functions, where ϵ_i is a suitable[1] square root of $e^{i\phi_{\mathbf{X}}}$ in D_i, such that $\sum_i \epsilon_i(x)u_i(x)$ is an eigenfunction of the **ABX**-Hamiltonian associated with the eigenvalue $\mathfrak{L}_k$.
Conversely, any family of nodal domains of an Aharonov-Bohm operator on $\dot{\Omega}_{\mathbf{X}}$ corresponding to L_k gives a k-partition.

[1] Note that by construction the D_i's never contain any pole.

Numerical applications ??

For the square, we have (with VB) considered the minimization over the pole.

The analysis of the dependance on the pole (or on many poles) of the eigenvalues is an interesting problem.

M. Abramowitz and I. A. Stegun.
Handbook of mathematical functions,
Volume 55 of Applied Math Series. National Bureau of Standards, 1964.

B. Alziary, J. Fleckinger-Pellé, P. Takáč.
Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in $\mathbb{R}^2$.
Math. Methods Appl. Sci. **26**(13), 1093–1136 (2003).

G. Alessandrini.
Critical points of solutions of elliptic equations in two variables.
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 14(2):229–256 (1988).

G. Alessandrini.
Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains.
Comment. Math. Helv., 69(1):142–154, 1994.

A. Ancona, B. Helffer, and T. Hoffmann-Ostenhof.
Nodal domain theorems à la Courant.
Documenta Mathematica, Vol. 9, p. 283-299 (2004).

R. Band, G. Berkolaiko, H. Raz, and U. Smilansky.
On the connection between the number of nodal domains on quantum graphs and the stability of graph partitions.
ArXiv : 1103.1423v1, March 2011.

P. Bérard.
Transplantation et isospectralité. I.
Math. Ann. **292**(3) (1992) 547–559.

P. Bérard.
Transplantation et isospectralité. II.
J. London Math. Soc. (2) **48**(3) (1993) 565–576.

J. Berger, J. Rubinstein.
On the zero set of the wave function in superconductivity.
Comm. Math. Phys. **202**(3), 621–628 (1999).

V. Bonnaillie, and B. Helffer.

Numerical analysis of nodal sets for eigenvalues of Aharonov-Bohm Hamiltonians on the square and application to minimal partitions.
To appear in Journal of experimental mathematics.

V. Bonnaillie-Noël, B. Helffer and T. Hoffmann-Ostenhof.
spectral minimal partitions, Aharonov-Bohm hamiltonians and application the case of the rectangle.
Journal of Physics A : Math. Theor. 42 (18) (2009) 185203.

V. Bonnaillie-Noël, B. Helffer and G. Vial.
Numerical simulations for nodal domains and spectral minimal partitions.
ESAIM Control Optim. Calc.Var. DOI:10.1051/cocv:2008074 (2009).

B. Bourdin, D. Bucur, and E. Oudet.
Optimal partitions for eigenvalues.
Preprint 2009.

D. Bucur, G. Buttazzo, and A. Henrot.
Existence results for some optimal partition problems.

Adv. Math. Sci. Appl. 8 (1998), 571-579.

K. Burdzy, R. Holyst, D. Ingerman, and P. March.
Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions.
J. Phys.A: Math. Gen. 29 (1996), 2633-2642.

L.A. Caffarelli and F.H. Lin.
An optimal partition problem for eigenvalues.
Journal of scientific Computing 31 (1/2) DOI: 10.1007/s10915-006-9114.8 (2007)

M. Conti, S. Terracini, and G. Verzini.
An optimal partition problem related to nonlinear eigenvalues.
Journal of Functional Analysis 198, p. 160-196 (2003).

M. Conti, S. Terracini, and G. Verzini.
A variational problem for the spatial segregation of reaction-diffusion systems.
Indiana Univ. Math. J. 54, p. 779-815 (2005).

M. Conti, S. Terracini, and G. Verzini.

On a class of optimal partition problems related to the Fučik spectrum and to the monotonicity formula.
Calc. Var. 22, p. 45-72 (2005).

O. Cybulski, V. Babin , and R. Holyst.
Minimization of the Renyi entropy production in the space-partitioning process.
Physical Review E71 046130 (2005).

B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, M. P. Owen.
Nodal sets for groundstates of Schrödinger operators with zero magnetic field in non-simply connected domains.
Comm. Math. Phys. **202**(3) (1999) 629–649.

B. Helffer, T. Hoffmann-Ostenhof.
Converse spectral problems for nodal domains.
Mosc. Math. J. **7**(1) (2007) 67–84.

B. Helffer, T. Hoffmann-Ostenhof.
On spectral minimal partitions : the case of the disk.
CRM proceedings **52**, 119–136 (2010).

B. Helffer, T. Hoffmann-Ostenhof.
On two notions of minimal spectral partitions.
Adv. Math. Sci. Appl. 20 (2010), no. 1, 249263.

B. Helffer, T. Hoffmann-Ostenhof.
On a magnetic characterization of spectral minimal partitions.
Submitted.

B. Helffer, T. Hoffmann-Ostenhof.
Spectral minimal partitions for a thin strip on a cylinder or a thin annulus like domain with Neumann condition
Submitted.

B. Helffer, T. Hoffmann-Ostenhof.
On spectral minimal partitions : the case of the torus.
In preparation.

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
Nodal domains and spectral minimal partitions.
Ann. Inst. H. Poincaré Anal. Non Linéaire (2009).

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.

On spectral minimal partitions : the case of the sphere.
Springer Volume in honor of V. Maz'ya (2009).

B. Helffer, T. Hoffmann-Ostenhof, S. Terracini.
On minimal spectral partition in $3D$.
To appear in a Volume in honor of L. Nirenberg.

D. Jakobson, M. Levitin, N. Nadirashvili, I. Polterovic.
Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond.
J. Comput. Appl. Math. 194, 141-155, 2006.

M. Levitin, L. Parnovski, I. Polterovich.
Isospectral domains with mixed boundary conditions
arXiv.math.SP/0510505b2 15 Mar2006.

B. Noris and S. Terracini.
Nodal sets of magnetic Schrödinger operators of Aharonov-Bohm type and energy minimizing partitions.
Indiana Univ. Math. J. **58**(2), 617–676 (2009).

A. Pleijel.

Remarks on Courant's nodal theorem.
Comm. Pure. Appl. Math., 9: 543–550, 1956.

O. Parzanchevski and R. Band.
Linear representations and isospectrality with boundary conditions.
arXiv:0806.1042v2 [math.SP] 7 June 2008.

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