Chambers formulas and semiclassical analysis for generalized Harper's butterflies
(after Kerdelhué–Royo-Letelier and Helffer–Kerdelhué–Royo-Letelier)

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Abstract:
If the first mathematical results were obtained more than 30 years ago with the interpretation of the celebrated Hofstadter butterfly, more recent experiments in Bose-Einstein theory suggest new questions. I will start with a partial survey on old results of Helffer-Sjöstrand and Kerdelhué and then discuss more recent questions related to generalized butterflies (Dalibard and coauthors, Hou, Kerdelhué–Royo-Letelier). These new questions are strongly related to Harper on triangular or hexagonal lattices (in connection with the now very popular graphene). Our main focus will be on new semi-classical questions appearing in the analysis of the final model arising in the analysis of the Kagome model.

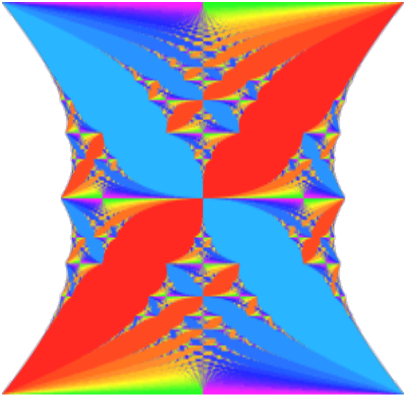
Introduction

The spectral properties of a charged particle in a two-dimensional system submitted to a periodic electric potential and a uniform magnetic field crucially depend on the arithmetic properties of the number γ representing the magnetic flux quanta through the elementary cell of periods, see e.g. Bellissard [Bel] for a description of various models.

The more popular model was proposed by Hofstadter. This is a model acting on $\ell^2(\mathbb{Z})$. For each (2π $\times$ rational) value of γ, Hofstadter writes the spectrum of the operator on an horizontal line. Since the works by Azbel [Az] and Hofstadter [Hof] it is generally believed that for irrational $\alpha = \gamma/2\pi$ the spectrum is a Cantor set, that is a nowhere dense (the interior of the closure is empty) and perfect set (closed $+$ no isolated point), and the graphical presentation of the dependence of the spectrum on γ shows a fractal behavior known as the Hofstadter butterfly.

The gaps in the spectrum.

This is the "colored" butterfly realized in 2003 by Y. Avron and his team.



After intensive efforts this was rigorously proved six years ago (Ten Martinis conjecture) for all irrational values of α for the discrete Hofstadter model, i.e. the discrete magnetic Laplacian admitting a reduction to the almost Mathieu equation, see Avila-Jitomirskaya [AvJi] and references therein and the talk of Svetlana Jitomirskaya in this conference.

Only few results are available for other models. Traditionally, various semiclassical methods play an important role in the analysis of the two-dimensional magnetic Schrödinger operators with periodic potentials, see e.g. Brüning-Dobrokhotov-Pankrashkin [BDP] for a review or Buslaev-Fedotov. In particular, the bottom part of the spectrum for strong magnetic fields can be described up to some extent using the tunnelling asymptotics, Wannier functions and this leads to simpler models like Harper.
Usually physicists have no problems to use these results without to come back to their proof and analyze directly the effective models.

A more detailed analysis (Bellissard, Helffer and Sjöstrand [HS1, HS2, HS3]) shows that the study of some parts of the spectrum for the Schrödinger operator with a magnetic field and a periodic electric potentials reduces to the spectral problem for a family of one-dimensional quasiperiodic pseudodifferential operators.

Under some symmetry conditions for the electric potentials, the analysis is reduced to the study of small perturbations of the continuous analog of the almost-Mathieu (=Harper) operator, which allowed one to carry out a rather detailed iterative analysis for special values of α.
In particular, in several asymptotic regimes a Cantor structure of the spectrum was proved.
This involved a pseudo-differential calculus, whose relevance in this context was predicted by the british physicist Wilkinson.

Pseudo-differential operators

In [HS1, HS2, HS3] (1988-1990) a machinery was developed for an iterative semiclassical analysis of a special class of pseudodifferential operators. One was concerned with the non-linear spectral problem (or, in other words, with the spectral problem for an operator pencil). Namely, for a family of self-adjoint operators $A(\mu)$ depending $\mu \in \mathbb{R}$ the μ-spectrum of $A(\mu)$ denotes the set of all μ such that $0 \in \operatorname{Spec} A(\mu)$.
The simplest case being the family $A - \mu$. This is what we do from now on.

Quantization

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a periodic smooth function, $L(x, \xi + 2\pi; h) = L(x + 2\pi, \xi; h) = L(x, \xi; h)$. Here h is a real parameter. By the Weyl quantization procedure one can assign to L an operator $\hat{L}_h$ in $L^2(\mathbb{R})$ by

$$\hat{L}_h f(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi(x-y)/h} L\left(\frac{x+y}{2}, \xi; h\right) f(y) d\xi \, dy. \qquad (1)$$

The operator $\hat{L}_h$ obtained is referred to as the Weyl h-quantization of L, and quantum Hamiltonians resulting from periodic symbols are often called Harper-like operators.

In particular, the symbol $L(x, \xi) := \cos x + \cos \xi$ produces the Harper operator on the real line,

$$f \mapsto \hat{L}_h f(x) = \frac{f(x+h) + f(x-h)}{2} + \cos x \, f(x). \tag{2}$$

In [HS3], in order to treat the Harper operator and perturbations of it occurring in a renormalization procedure, the following notion was introduced.

Definition: perturbation of Harper

A symbol $L(x, \xi; h)$ will be called an admissible perturbation of Harper's symbol if the following conditions are satisfied for some $h_0 > 0$:

(a) There exists $\varepsilon > 0$ such that, for all $h \in (0, h_0)$,

(a1) $L(x, \xi; h)$ is holomorphic in

$$D_\varepsilon = \left\{(x, \xi) \in \mathbb{C} \times \mathbb{C} : |\Im x| < \frac{1}{\varepsilon}, |\Im \xi| < \frac{1}{\varepsilon}, \right\},$$

(a2) for $(x, \xi) \in D_\varepsilon$, there holds

$$\left| L(x, \xi; h) - (\cos x + \cos \xi) \right| \leq \varepsilon.$$

Continuation of the definition

(b) The following symmetry conditions hold:

$$L(x,\xi;h) = L(\xi,x;h) = L(x,-\xi;h)$$
$$L(x,\xi;h) = L(x+2\pi,\xi;h) = L(x,\xi+2\pi;h).$$

By $\varepsilon(L)$ we will denote the minimal value of ε for which the above conditions hold.

In [HS1, HS2, HS3] a detailed analysis was performed for pseudodifferential operators associated with these symbols. One of the results was

Theorem 1

Let $L(h)$ be an admissible perturbation of Harper's symbol. There exist ϵ_0 , C s. t. if $\varepsilon(L) \leq \epsilon_0$ and if $(2\pi)^{-1}h$ is an irrational admitting a representation as a continuous fraction

$$\frac{h}{2\pi} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

with $n_j \geq C$, then the spectrum of the associated operators $\hat{L}_h$ is a zero measure Cantor set.

In particular, this applies to the spectrum of the Harper's model, for which one gets the Cantor structure, BUT with a very restrictive notion of irrationality.
But the theorem says also that this property is stable by perturbations respecting all the symmetries and gives a very accurate description of the spectrum.

Schrödinger operators with magnetic potentials

For operators

$$H = \sum_{j=1}^{2} (\hbar D_{x_j} - A_j)^2 + V$$

with periodic potentials V,

$$V(x_1 + 2\pi, x_2) \equiv V(x_1, x_2 + 2\pi) \equiv V(x_1, x_2)\ ,$$

(square lattice case) and constant (or periodic) magnetic fields

$$\mathrm{Curl}\, \vec{A} = B\ ,$$

it was shown in several asymptotic regimes that the study of some parts of the spectrum reduces to a non-linear spectral problem of the type above.
We will consider later other symmetries.

This is for example the case for

- B^{-1}-pseudodifferential operators with symbols close to $V(x, \xi)$ (see for example [HS4] which treats the strong magnetic case)
- B-pseudodifferential operators with symbols close to the first Floquet eigenvalue of the Schrödinger operator without magnetic field (Peierls substitution) (corresponding to the case of the weak magnetic field, see [HS1], [HS3] and [HS4] and earlier contributions by mathematicians and physicists (see the surveys by J. Bellissard in [Bel], G. Nenciu in [Ne2], J. Sjöstrand [Sj] and references therein).

Hence, in the first case perturbations of Harper appear for strong magnetic field when considering potentials V close to $\cos x_1 + \cos x_2$.

Moreover in the semi-classical limit $\hbar \to 0$ or in the tight binding situation, it can be shown (case of a square lattice) that—up to the multiplication by an exponentially small term corresponding to the tunneling—– the lowest Floquet eigenvalue is close to $(\cos\theta_1 + \cos\theta_2)$.

Here it is important to assume the symmetry for V the additional symmetry

$$V(-x_2, x_1) = V(x_1, x_2)\,,$$

an assumption of non degenerate minima for V (one for each cell) and a geometric assumption on the geodesics for neighboring wells (the geometry is the Agmon metric $(V - \min V)dx^2$).

Symbols associated with some discrete operators

It has been established in [HS1] that the spectrum of the operator (2) as a set coincides with the spectrum of the discrete magnetic Laplacian acting on $\ell^2(\mathbb{Z}^2)$, see e.g. [HS1],

$$C_h f(m,n) = e^{ihn} f(m+1,n) + e^{-ihn} f(m-1,n) + f(m,n-1) + f(m,n+1).$$

More generaly consider a bounded linear operator C_h acting on $\ell^2(\mathbb{Z}^2)$ given by an infinite matrix $\big(C(p,q)\big)$, $p,q \in \mathbb{Z}^2$, satisfying

$$C(p+k, q+k) = e^{-ihk_2(p_1-q_1)} C(p,q), \quad p,q,k \in \mathbb{Z}^2, \qquad (3)$$

with some $h > 0$.

Proposition A

Let C_h be a bounded self-adjoint operator in $\ell^2(\mathbb{Z}^2)$ with the property (3) and satisfying $|C(p,q)| \leq ae^{-b|p-q|}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}^2$. Then the spectrum of C_h coincides with the spectrum of the Weyl h-quantization of the symbol T given by

$$T(x,\xi) = \sum_{m,n\in\mathbb{Z}} c(m,n)e^{-imnh/2}e^{i(mx+n\xi)}, \tag{4}$$

where $c(m,n) = C\big((0,0),(m,n)\big)$, $m, n \in \mathbb{Z}$.

Other examples of lattices

We first mention the triangular case

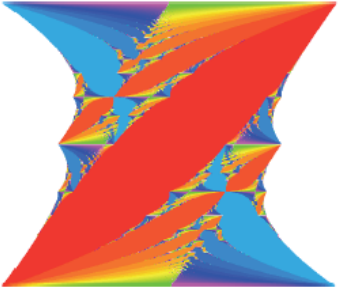


Figure 1: A phase diagram for the Hofstadter model on a triangular lattice where the flux through the down triangles $\Phi_d = \pi/2$. The vertical axis is the total flux Φ. The horizontal axis is the chemical potential. The colors represent the Chern numbers. The model is inversion symmetric, see sec. A.1.

Figure: Picture by J. E. Avron, O. Kenneth and G. Yeshoshua (2013).

One should add the graphene case (or hexagonal case).

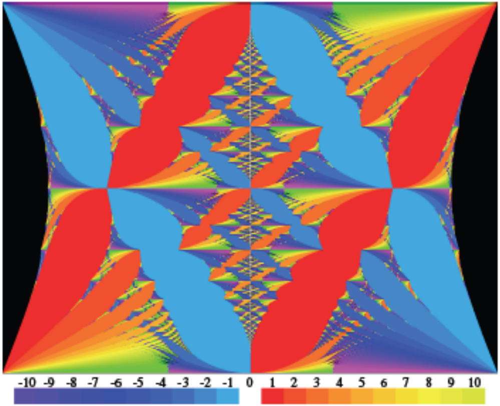


Figure 5: The colored Hofstadter butterfly for the honeycomb lattice, as obtained by the method of this paper. The vertical axis is the magnetic flux per unit cell Φ ranging from 0 to 1. The horizontal axis is the Fermi energy ranging from -3 to 3. The colors represent the Chern numbers. The resolution of this figure is 1920×1440 and the maximal value of q is $q_{\max} = 720$.

Figure: The colored Hofstadter butterfly for the Honeycomb lattice by A. Agazzi, J.-P. Eckmann, and G.M. Graf (2014) .

In her thesis J. Royo-Letelier has started (see [Hou]) to analyze rigorously the case of a Kagome lattice. This was extended in a paper in collaboration with P. Kerdelhué (Rev. Math. Physics 2014). Questions around the Chambers's formula have been analyzed by Helffer–Kerdelhué–Royo-Letelier. This involves new semi-classical problems related to "flat" bands.

Kagome lattice

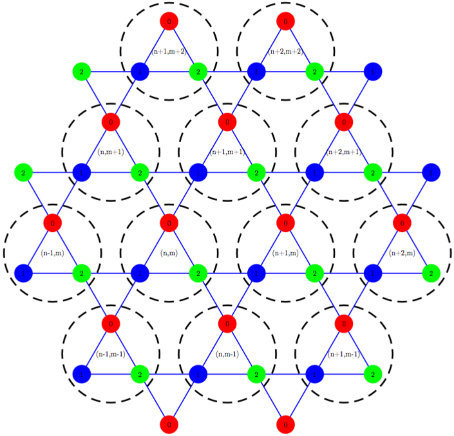
The kagome lattice is not a Bravais lattice, but is a discrete subset of $\mathbb{R}^2$ invariant under translations along a triangular lattice and containing three points per fundamental domain of this lattice. Each point of the lattice has four nearest neighbours for the Euclidean distance. The word *kagome* means a bamboo-basket (kago) woven pattern (me) and it seems that the lattice was named by the Japanese physicist K. Husimi .

Let $\Gamma_\triangle$ be the triangular lattice spanned by $\mathcal{B} = \{2\nu_1, 2\nu_2\}$, where

$$\nu_\ell = r^{\ell-1}(1,0)$$

and r is the rotation of angle $\frac{\pi}{3}$ and center the origin.
The kagome lattice can be seen as the union of three suitably translated copies of $\Gamma_\triangle$:

$$\Gamma = \left\{ m_{\alpha,\ell} = 2\alpha_1\nu_1 + 2\alpha_2\nu_2 + \nu_\ell \,;\, (\alpha_1,\alpha_2) \in \mathbb{Z}^2 \,,\, \ell = 1,3,5 \right\}.$$



(n+1,m+2)
(n+2,m+2)
(n,m+1)
(n+1,m+1)
(n+2,m+1)
(n-1,m)
(n,m)
(n+1,m)
(n+2,m)
(n-1,m-1)
(n,m-1)
(n+1,m-1)

Coming from a Schrödinger operator

As in the Harper model, it is possible to construct an electric potential whose minima are on a Kagome lattice. Moreover there are examples obtained with trigonometric polynomials. This means that they can be obtained by a combination of lasers.

Remembering the definitions of the vectors ν_j, we denote by $\nu^\perp$ the vector deduced from ν by a rotation of $\frac{\pi}{2}$ and for $j \in \{1, 3, 5\}$ we define

$$\mu_j = \sqrt{3}\, \nu_j^\perp \,.$$

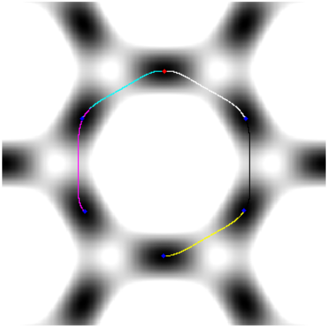
For $j = 1, 3, 5$ we set $\phi_j = 3\pi/2$ and define the potentials $V_j : \mathbb{R}^2 \to \mathbb{R}$ as

$$V_j(x) = \left[\cos\left(\pi x \cdot \mu_j + \phi_j\right) + 2\cos\left(\frac{\pi x \cdot \mu_j + \phi_j}{3}\right)\right]^2 ,$$

and

$$V = -V_1 - V_3 - V_5 \,.$$

V has local minima at the points of the kagome lattice.



The minima appear on the center of the black zone around an hexagon. The maximum at the center of the hexagon. Each minimum has four nearest neighbors (for the Agmon distance). These minima are living on a kagome lattice (subset of an hexagonal lattice). The figure is invariant by the double triangular lattice.

Analysis of the rational case and Chambers formula

Once a semi-classical (or tight-binding) approximation is done, involving a tunneling analysis and a construction of Wannier functions we arrive (modulo a controlled smaller error) in the case of a square lattice to the so-called Harper model, which is defined on $\ell^2(\mathbb{Z}^2, \mathbb{C})$ by

$$(Hu)_{m,n} := \frac{1}{2}(u_{m+1,n} + u_{m-1,n}) + \frac{1}{2} e^{i\gamma m} u_{m,n+1} + \frac{1}{2} e^{-i\gamma m} u_{m,n-1} \,,$$

where γ denotes the flux of the constant magnetic field through the fundamental cell of the lattice.
When $\frac{\gamma}{2\pi}$ is a rational, a Floquet theory permits to reduce the analysis to the analysis of the eigenvalues of a family of $q \times q$ matrices depending on a parameter $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$.

More precisely, when

$$\gamma = 2\pi p/q\,, \tag{5}$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ are relatively prime, the two following matrices play an important role:

$$J_{p,q} = \mathrm{diag}(e^{i(j-1)\gamma})\,, \tag{6}$$

and

$$(K_q)_{jk} = 1 \text{ if } k \equiv j+1[q]\,, \text{ 0 else.} \tag{7}$$

In the case of Harper, the family of matrices is

$$M_H(\theta_1, \theta_2) = e^{i\theta_1} J_{p,q} + e^{-i\theta_1} J^*_{p,q} + e^{i\theta_2} K_q + e^{-i\theta_2} K^*_q \,. \qquad (8)$$

The Hofstadter butterfly is then obtained as a picture in the rectangle $(-2, +2) \times [0, 1]$. A point $(\lambda, \gamma/2\pi)$ is in the picture if there exists θ_1, θ_2 such that
$\det(M_H(\theta_1, \theta_2) - \lambda) = 0$ for some $\frac{p}{q}$ with $p/q \in [0, 1]$ ($q \leq 50$).
The Chambers formula gives a very elegant formula for this determinant:

$$\det(M_H(\theta_1, \theta_2) - \lambda) = f^H_{p,q}(\lambda) + (-1)^q \left(\cos q\theta_1 + \cos q\theta_2\right) \,, \qquad (9)$$

where $f^H_{p,q}$ is a polynomial of degree q.

Many other models have been considered. In the case of a triangular lattice, the second model is, according to [Ke] (see also [Avetal]),

$$\begin{aligned} M_T(\theta_1,\theta_2,\phi) &= e^{i\theta_1} J_{p,q} + e^{-i\theta_1} J^*_{p,q} + e^{i\theta_2} K_q + e^{-i\theta_2} K^*_q \\ &\quad + e^{i\phi} e^{i(\theta_1-\theta_2)} J_{p,q} K^*_q + e^{-i\phi} e^{i(\theta_2-\theta_1)} K_q J^*_{p,q} \end{aligned} \tag{10}$$

with $\phi = -\gamma/2$.
The Chambers formula in this case takes the form

$$\begin{aligned} &\det(M_T(\theta_1,\theta_2,\phi) - \lambda) \\ &= f^T_{p,q}(\lambda) + (-1)^{q+1} \left(\cos q\theta_1 + \cos q\theta_2 + \cos q(\theta_2 - \theta_1 - \phi)\right) . \end{aligned} \tag{11}$$

In the case of the hexagonal lattice, which appears also in the analysis of the graphene, we have to analyze

$$M_G(\theta_1,\theta_2) := \begin{pmatrix} 0 & I_q + e^{i\theta_1} J_{p,q} + e^{i\theta_2} K_q \\ I_q + e^{-i\theta_1} J_{p,q}^* + e^{-i\theta_2} K_q^* & 0 \end{pmatrix} \tag{12}$$

We denote by P_G the characteristic polynomial of M_G. The resulting spectrum is given in Figure 3.

Finally, inspired by the physicist Hou, Kerdelhué and Royo-Letellier [KR] have shown that for the kagome lattice, the following approximating model is relevant. We consider the matrix:

$$M_K(\theta_1,\theta_2,\omega,\gamma) = \begin{pmatrix} 0 & A(\theta_1,\theta_2,\omega) & B(\theta_1,\theta_2,\omega) \\ A^*(\theta_1,\theta_2,\omega) & 0 & C(\theta_1,\theta_2,\omega) \\ B^*(\theta_1,\theta_2,\omega) & C^*(\theta_1,\theta_2,\omega) & 0 \end{pmatrix}, \tag{13}$$

with

$$\begin{aligned} A(\theta_1,\theta_2,\omega) &= e^{i(\omega+\frac{\gamma}{8})}(e^{-i\theta_1}J^*_{p,q} + e^{-i\frac{\gamma}{2}}e^{-i(\theta_1-\theta_2)}J^*_{p,q}K_q) \\ B(\theta_1,\theta_2,\omega) &= e^{-i(\omega+\frac{\gamma}{8})}(e^{-i\theta_1}J^*_{p,q} + e^{-i\theta_2}K^*_q) \\ C(\theta_1,\theta_2,\omega) &= e^{i(\omega+\frac{\gamma}{8})}(e^{-i\frac{\gamma}{2}}e^{i(\theta_1-\theta_2)}J_{p,q}K^*_q + e^{-i\theta_2}K^*_q)\,. \end{aligned} \tag{14}$$

Here ω is a parameter appearing in the model (most of the physicists consider without justification the case $\omega = 0$). We refer to [KR] for a discussion of this point.

The trigonometric polynomial

$$(x,\xi) \mapsto p^{\triangle}(x,\xi) = \cos x + \cos \xi + \cos(x-\xi) \qquad (15)$$

which was playing an important role in the analysis of the triangular Harper model (see Claro-Wannier [CW] and Kerdelhué [Ke]) will also appear in our analysis.
We denote by $P_K(\theta_1,\theta_2,\lambda)$ the characteristic polynomial $\det(\lambda\, I_{3q} - M(\theta_1,\theta_2))$.
We prove that for the model considered by Hou [Hou], there exists a formula which is similar to the one obtained by Chambers [Ch] for the Harper model. (see also Helffer-Sjöstrand [HS1], [HS2], Bellissard-Simon [BelSim], C. Kreft [Kr], I. Avron (and coauthors) [Avetal]).

The first statement is probably well known in the physical literature.

Theorem [Graphene]

$$P_G(\theta_1, \theta_2, \lambda) = (-1)^q \det(M_T(\theta_1, \theta_2, 0) + 3 - \lambda^2)\,. \qquad (16)$$

The second statement was to our knowledge unobserved.

Theorem [Kagome]

There exists a polynomial Q_ω of degree $3q$, with real coefficients, depending on γ and ω, but not on (θ_1, θ_2), such that

$$P_K(\theta_1, \theta_2, \omega, \lambda) \\ = Q_\omega(\lambda) + 2p^{\triangle}(q(\theta_1 + p\pi), q(\theta_2 + p\pi)) \left(\lambda + 2\cos(3\omega - \tfrac{\gamma}{8})\right)^q . \tag{17}$$

Corollary

A flat band exists if and only if

$$Q_\omega(-2\cos(3\omega - \frac{\gamma}{8})) = 0 .$$

Let us illustrate by some examples mainly extracted of [KR].
In the case when $q = 1$ and $p = 0$, one finds, for the Hou's model:

$$P(\theta_1, \theta_2, \lambda) = -\lambda^3 + 6\lambda + 4\cos(3\omega) + 2\left(\lambda + 2\cos(3\omega)\right) p^{\triangle}(\theta_1, \theta_2)\,.$$

Hence, we have in this case:

$$Q_\omega(\lambda) = -\lambda^3 + 6\lambda + 4\cos(3\omega)\,.$$

The condition for a flat band reads:

$$Q_\omega(-2\cos(3\omega)\ ) = 0\,,$$

which takes the simple form: $(\cos 3\omega)^3 - \cos 3\omega = 0\,.$
Hence $\cos 3\omega = 0$ or $\cos 3\omega = \pm 1$. So the "flat bands" appear only for discrete value of ω, including the particular case $\omega = 0$, mostly considered in the physical literature. Note that in [KR], it is proved only that $\omega \to 0$ as a function of the initial semi-classical parameter.

We now consider other examples:

- For the triangular model, for $p/q = 1/6$, the spectrum is given by :

$$\lambda^6 - 18\lambda^4 - 12\sqrt{3}\lambda^3 + 45\lambda^2 + 36\sqrt{3}\lambda + 6 - 2p^{\triangle}(6\theta_1, 6\theta_2) = 0\,.$$

  $Q(\lambda)$ satisfies $Q(-\sqrt{3}) = Q'(-\sqrt{3}) = 0$. The second gap is closed.

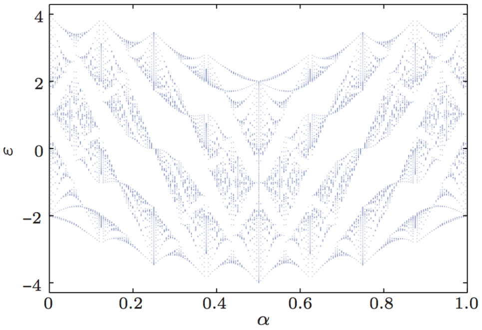
- For the graphene model, for $p/q = 1/2$, the spectrum is given by

$$\lambda^4 - 6\lambda^2 + 3 - 2(\cos(2\theta_1) + \cos(2\theta_2) - \cos(2(\theta_1 - \theta_2)))$$

  The bands are $[-\sqrt{6}, -\sqrt{3}]$, $[-\sqrt{3}, 0]$, $[0, \sqrt{3}]$ and $[\sqrt{3}, \sqrt{6}]$.

- For the Hou-model, as shown in [KR] for $\omega = \pi/8$ and $p/q = 3/2$, the bands are $\{-2\}$ (with multiplicity 2), $[1-\sqrt{6}, 1-\sqrt{3}]$, $[1-\sqrt{3}, 1]$, $[1, 1+\sqrt{3}]$ and $[1+\sqrt{3}, 1+\sqrt{6}]$.

The Kagome butterfly



Semi-classical analysis for Hou's butterfly near a flat band

The general study of Hou's butterfly near its flat bands seems difficult, but we can obtain an explicit reduction for the simplest one, corresponding to the flat band $\{0\}$ in the case when $\omega = 0$, $\gamma = 4\pi$. As shown in [KR], the spectrum of Hou's operator for $\omega = 0$, $\gamma = 4\pi + h$ is the spectrum of the Weyl h-quantization of the matrix symbol $M_K(x, \xi, \omega, \gamma)$

In our case, the principal symbol M_0 is given by

$$\begin{pmatrix} 0 & i(e^{-ix}+e^{-i(x-\xi)}) & -i(e^{-ix}+e^{-i\xi}) \\ -i(e^{ix}+e^{i(x-\xi)}) & 0 & i(e^{i(x-\xi)}+e^{-i\xi}) \\ i(e^{ix}+e^{i\xi}) & -i(e^{-i(x-\xi)}+e^{i\xi}) & 0 \end{pmatrix} \tag{18}$$

We first prove :

Proposition (at the level of the principal symbol)

There exists a familly $U_0(x,\xi)$ of unitary 3×3 matrices, depending smoothly on (x,ξ), 2π-periodic in each variable, and a familly $A(x,\xi)$ of selfadjoint 2×2 matrices such that

$$U_0^*(x,\xi)\,M_0(x,\xi)\,U_0(x,\xi) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$A(x,\xi)$}} \\ 0 & & \end{array}\right). \tag{19}$$

Moreover, for any $(x,\xi)\in\mathbb{R}^2$, the spectrum of $A(x,\xi)$ is contained in $[-2\sqrt{3},-\sqrt{3}]\cup[\sqrt{3},2\sqrt{3}]$.

Remark

The triviality of the fiber bundle whose fiber at (x,ξ) is the eigenspace of $M(x,\xi)$ associated with the two non vanishing eigenvalues can be proven directly. As observed by G. Panati [Pan], this is also the consequence of a general statement.

Using [HS2], we get:

Proposition (at the level of the operator)

There exist a unitary 3×3 pseudodifferential operator U with principal symbol $U_0(x, \xi)$, a selfadjoint scalar operator μ with principal symbol 0, and a selfadjoint 2×2 operator $\tilde{A}$ with principal symbol $A(x, \xi)$ such that

$$U^* \operatorname{Op}_h^W(M(x, \xi, h))\, U \sim \begin{pmatrix} \mu & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\tilde{A}} \end{pmatrix} \tag{20}$$

Moreover, the part of the spectrum of $\operatorname{Op}_h^W(M(x, \xi, h))$ in any compact subset of $]-\sqrt{3}, \sqrt{3}[$ is that one of μ for $|h|$ small enough.

The main result is then the computation of the subprincipal symbol of μ.

Proposition (Reduction near the flat band)

$$\sigma^W(\mu)(x,\xi,h) = -h\frac{3-p^{\triangle}(x,\xi)}{4(3+p^{\triangle}(x,\xi))} + \mathcal{O}(h^2)\,. \qquad (21)$$

Hence the analysis of the spectrum near the flat band 0 is given by a triangular like Harper model, suggesting a renormalization situation if we want to attack the possibly Cantor spectrum of the spectrum. But it would be probably extremely technical. This has not been done for the triangular or Hexagonal Harpers models (see however the work of Kerdelhué in the 90's.

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