Optimal elliptic estimates for solutions of Ginzburg-Landau systems revisited

Bernard Helffer
Mathématiques -
Univ Paris Sud- UMR CNRS 8628
(After S. Fournais and B. Helffer)

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Main goals

When analyzing the properties of the minimizers $(\psi, \mathbf{A})$ of a superconductor submitted to an external field, it is important to establish estimates on 'how far' the induced magnetic vector potential $\mathbf{A}$ is allowed to be from the external magnetic potential $\mathbf{F}$ corresponding to the external field.

It is in particular proved that when the intensity of the external field is between H_{C_2} and H_{C_3}, $\mathbf{A} - \mathbf{F} = o(1)$, in suitable norms.

Such estimates are very useful and can be found in various publications (c.f. Lu-Pan, Pan, Helffer-Pan, Almog, Fournais-Helffer, ...), we give here a simplified proof and present some improvments.

These estimates come in two types.

The first set of estimates is deduced from the ellipticity of the Ginzburg-Landau system. In this way one obtains the desired estimates in (Sobolev) norms, $W^{s,p}$, for $p < +\infty$ (by imbedding theorems also estimates in Hölder norms, $C^{s',\alpha}$, $\alpha < 1$, are obtained).

The challenge here is to get inequalities with the right dependence on the magnetic field strength (independently of the vector potential generating the field).

This part of the analysis is valid in a large parameter regime and is essentially functional analytical.

The second set of estimates corresponds to the cases $p = \infty$ above and uses the first set of estimates as input.

One proves that it is possible to go to these limiting cases essentially without loss in the parameter measuring the magnetic field strength—actually, in some cases one gets an *improved* behavior!

These inequalities are asymptotic in the sense that they depend on a certain parameter to be sufficiently large and are valid in a much smaller parameter regime ('above H_{C_2}').

The proof of these estimates uses the fact that a natural limiting equation has no non-trivial solutions and the proof is therefore much more intrinsically PDE in spirit. This technique is often called a 'blow-up argument' in the literature.

In this talk, we only discuss the 2-dimensional case.

This work was partially motivated by discussions with X-B. Pan and questions of S. Serfaty and we thank them for the encouragement.

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,\sigma}[\psi,\mathbf{A}] = \int_\Omega \Big\{ |\nabla_{\kappa\sigma\mathbf{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2\sigma^2|\operatorname{curl}\mathbf{A} - 1|^2 \Big\}\, dx \,,$$

with Ω simply connected in $\mathbb{R}^2$, $(\psi,\mathbf{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$ and $\nabla_{\mathbf{A}} = (\nabla + i\mathbf{A})$.

κ is the Ginzburg-Landau parameter and σ denotes the strength of the external magnetic field, which for simplicity is assumed to be constant.

We fix the choice of gauge by imposing that

$$\operatorname{div}\mathbf{A} = 0 \quad \text{in } \Omega\,,\ \mathbf{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Minimizers $(\psi, \mathbf{A})$ of the functional satisfy the Ginzburg-Landau equations,

$$\left.\begin{aligned} -\nabla^2_{\kappa\sigma\mathbf{A}}\psi = \kappa^2(1-|\psi|^2)\psi \\ \text{curl }^2\mathbf{A} = -\tfrac{i}{2\kappa\sigma}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\mathbf{A} \end{aligned}\right\} \quad \text{in} \quad \Omega\,; \tag{1a}$$

$$\left.\begin{aligned} (\nabla_{\kappa\sigma\mathbf{A}}\psi)\cdot\nu = 0 \\ \text{curl }\mathbf{A} - 1 = 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{1b}$$

Here $\text{curl }(A_1, A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$,

$$\text{curl }^2\vec{A} = (\partial_{x_2}(\text{ curl }\vec{A}), -\partial_{x_1}(\text{ curl }\vec{A}))\,.$$

Integration by parts

The starting point is a formula expressing the L^2-norm of mixed second derivatives of a function in terms of the L^2-norm of magnetic Laplacian on the function and lower order terms involving the magnetic field itself. For convenience, we will use the following notation for the magnetic derivatives

$$\mathbf{D} = (D_1, D_2) = (-i\nabla + B\mathbf{A}). \tag{2}$$

Our B will be later $\kappa\sigma$.

The magnetic Laplacian is now the operator

$$\mathcal{H} := \mathbf{D}^2 = D_1^2 + D_2^2\,.$$

Proposition 1 (Int. by Parts)
Let $\Omega \subset \mathbb{R}^2$ be a regular bounded domain. Suppose $\psi \in W^{2,2}(\Omega)$ satisfies magnetic Neumann bdry conditions

$$\nu \cdot D\psi\big|_{\partial\Omega} = 0. \tag{3}$$

Then

$$\begin{aligned}
&\textstyle\sum_{j,k} \|D_j D_k \psi\|^2_{L^2(\Omega)} \\
&= B^2 \textstyle\int_\Omega (\text{ curl } \mathbf{A})^2 |\psi|^2 \, dx + \int_\Omega |\mathcal{H}\psi|^2 \, dx \\
&\quad + 2B \textstyle\int_\Omega (\text{ curl } \mathbf{A}) \Im(D_1\psi \, \overline{D_2\psi}) \, dx.
\end{aligned} \tag{4}$$

Remark
This formula appears in [LuPa2] with an additional bndry term, which actually vanishes in the case of the magnetic Neumann-condition.

A similar formula exists in dimension 3 but the bndry term did not vanish anymore.

The proof consists of a tedious but elementary calculation.

Applying Hölder's inequality to the result of Proposition 1 yields an interesting elliptic inequality for $2D$ magnetic problems with Neumann bndry conditions.

Proposition 2

Let $\Omega \subset \mathbb{R}^2$ be a regular domain and $\beta \in L^\infty(\Omega)$. If $\psi \in C^\infty(\overline{\Omega})$ satisfies magnetic Neumann bndry conditions, then $\forall p_1, p_2 \in [1, +\infty]$ we have

$$\begin{aligned}\sum_{j,k} \|D_j D_k \psi\|^2_{L^2(\Omega)} \leq 3B^2 \|\beta\|^2_\infty \|\psi\|^2_2 + 2\|\mathcal{H}\psi\|^2_2 \\ + 2B^2 \|\,\text{curl }\mathbf{A} - \beta\|^2_{2p_1} \|\psi\|^2_{2q_1} \\ + 2B \|\,\text{curl }\mathbf{A} - \beta\|_{p_2} \|\mathbf{D}\psi\|^2_{2q_2},\end{aligned} \tag{5}$$

where $p_j^{-1} + q_j^{-1} = 1$.

The proof is direct using the identity in Proposition 1—replacing $\text{curl }\mathbf{A}$ by $(\,\text{curl }\mathbf{A} - \beta) + \beta$—and Hölder's inequality.

Regularity for the solutions of the Ginzburg-Landau system

We define $\mathbf{F}$ as the solution of

$$\text{curl } \mathbf{F} = 1 \,, \quad \text{div } \mathbf{F} = 0 \,, \; \mathbf{F} \cdot \nu = 0 \text{ on } \partial\Omega \,, \qquad (6)$$

Let us recall that a solution $(\psi, \mathbf{A})$ of the G-L system satisfies

$$\|\psi\|_\infty \leq 1 \,. \qquad (7)$$

Also recall that, as for $\mathbf{F}$ above, without loss of generality, we can, by a gauge transformation, assume that the vector potentials $\mathbf{A}$ belong to the space $H^1_{\text{div}}(\Omega)$, i.e.

$$A \in H^1(\Omega, \mathbb{R}^2) \,, \text{ div } \mathbf{A} = 0 \text{ in } \Omega \text{ and } \mathbf{A} \cdot \nu = 0 \quad \text{on } \partial\Omega \,. \qquad (8)$$

Theorem 3
Let $\beta \in C^\infty(\overline{\Omega})$. Then there exists C, and, $\forall \alpha \in (0,1)$, $\forall p \in (1,+\infty)$, $\exists \widehat{C}_\alpha$ and $\exists \widetilde{C}_p$, such that, $\forall (\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\rm div}(\Omega)$ solution of the G-L system (1) with parameters $\kappa, \sigma > 0$,

$$\sum_{j,k} \|D_j D_k \psi\|_{L^2(\Omega)} \leq C(1 + \kappa\sigma + \kappa^2)\|\psi\|_2, \tag{9}$$

$$\| \text{curl } \mathbf{A} - 1\|_{C^{0,\alpha}(\overline{\Omega})} \leq \widehat{C}_\alpha \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty, \tag{10}$$

and

$$\| \text{curl } \mathbf{A} - 1\|_{W^{1,p}(\Omega)} \leq \widetilde{C}_p \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty \,. \tag{11}$$

Remarks

- Using the $W^{k,p}$-regularity of the Curl-Div system (see Agmon-Douglis-Nirenberg, and Temam for the case $p = 2$), we obtain from (11) the estimate

$$\|\mathbf{A} - \mathbf{F}\|_{W^{2,p}(\Omega)} \le \widetilde{D}_p \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty \, . \tag{12}$$

Hence, using the Sobolev injection Theorem,

$$\|\mathbf{A} - \mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \le \widehat{D}_\alpha \frac{1 + \kappa\sigma + \kappa^2}{\kappa\sigma} \|\psi\|_2 \|\psi\|_\infty , \tag{13}$$

for all $\alpha \in [0, 1)$.

- In the applications, σ is of the same order as κ, so (13) gives that $(\mathbf{A} - \mathbf{F})$ is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ in this regime, for any $\alpha < 1$.

- We have in particular obtained a complete proof of the basic Proposition 3.1 in [LuPa1] with actually an improvement of the right hand side and an extension of the regime of parameters (κ, σ) for which the estimate is true.

- When in addition, $\frac{\kappa}{\sigma} \geq 1 + b$ (with $b > 0$), V. Bonnaillie-Noël and S. Fournais [BonFo] have given recently a very simple proof (in comparison with Helf-Pan or Fou-Helf) showing that for a minimizer $(\psi, \mathbf{A})$ of the Ginzburg-Landau functional, one has for some constants $C_b, \kappa_b > 0$,

$$\|\psi\|_2 \leq C_b \, \kappa^{-\frac{1}{2}} \, \|\psi\|_\infty \, \leq C_b \, \kappa^{-\frac{1}{2}}, \tag{14}$$

  for all $\kappa \geq \kappa_b$.

  The proof in [BonFo] does not use the elliptic estimates that we discuss here.

Proof of Theorem 3

Using (7), we can get a number of *a priori* estimates on solutions to the Ginzburg-Landau equations (1).

Lemma 4

$\forall p \geq 2$, $\exists C = C(p) > 0$ s.t. $\forall (\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ satisfying (1), we have

$$\|p^2_{\kappa\sigma\mathbf{A}}\psi\|_p \leq \kappa^2 \, \|\psi\|_p, \tag{15}$$

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_2 \leq \kappa \, \|\psi\|_2, \tag{16}$$

$$\| \,\text{curl}\, \mathbf{A} - 1\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa\sigma} \, \|\psi\|_\infty \|p_{\kappa\sigma\mathbf{A}}\psi\|_p. \tag{17}$$

Combining (16) and (17) (with $p = 2$) yields

$$\| \,\text{curl}\, \mathbf{A} - 1\|_2 \leq \frac{C}{\sigma} \|\psi\|_\infty \|\psi\|_2. \tag{18}$$

We use Proposition 2 with $p_1 = 1$, $p_2 = \infty$ and $B = \kappa\sigma$. After inserting (1a), (18), and the result of Lemma 4, this yields

$$\sum_{j,k} \|D_j D_k \psi\|_2^2$$
$$\leq C\Big\{\big(1+\kappa^4+(\kappa\sigma)^2\big)\|\psi\|_2^2+\kappa^3\sigma\|\psi\|_2^2\,\|\,\text{curl }\mathbf{A}-1\|_\infty\Big\}. \tag{19}$$

Using that $\|\psi\|_2 \leq |\Omega|$ and a Sobolev inequality for controlling $\|\,\text{curl }\mathbf{A} - 1\|_\infty$, this becomes, with a new C and for any $\epsilon > 0$,

$$\sum_{j,k} \|D_j D_k \psi\|_2^2$$
$$\leq C\Big\{(1+\epsilon^{-1})\big(1+\kappa^4+(\kappa\sigma)^2\big)\|\psi\|_2^2$$
$$+\,\epsilon(\kappa\sigma)^2\|\,\text{curl }\mathbf{A}-1\|_{W^{1,p}}^2\Big\}. \tag{20}$$

We now apply a Sobolev inequality and the (pointwise) diamagnetic inequality —for each function $(-i\partial_{x_k} + \kappa\sigma A_k)\psi$—to (17), in order to get, for a suitable C',

$$
\begin{aligned}
&\| \text{curl } \mathbf{A} - 1\|^2_{W^{1,p}(\Omega)} \\
&\leq \tfrac{C'}{(\kappa\sigma)^2}\|\psi\|^2_\infty \Big(\textstyle\sum_{j,k} \|D_jD_k\psi\|_2^2 + \|p_{\kappa\sigma A}\psi\|_2^2 \Big) \\
&\leq \tfrac{C'}{(\kappa\sigma)^2}\|\psi\|^2_\infty \textstyle\sum_{j,k} \|D_jD_k\psi\|_2^2 + \tfrac{C'}{\sigma^2}\|\psi\|^2_\infty \|\psi\|_2^2,
\end{aligned}
\tag{21}
$$

where the last inequality follows from (16).

Inserting (21) in (20) and choosing ϵ sufficiently small, yields (9).

Once (9) is established, we get (11) from (21). Finally, (10) follows from (11) and a Sobolev inequality. This finishes the proof of Theorem 3.

Asymptotic estimates

As explained in the introduction, we would like to treat the limiting case $p = +\infty$ and we start by proving the **Nonexistence of solutions to certain partial differential equations**

We will use the notation $\tilde{\mathbf{F}}$ for any vector potential on

- either $\mathbb{R}^2$

- or the half-space $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 \,\big|\, x_1 > 0\}$

satisfying curl $\tilde{\mathbf{F}} = 1$.

We know that $(-i\nabla + \tilde{\mathbf{F}})^2$ on $L^2(\mathbb{R}^2)$ has spectrum

$$\text{Spec } (-i\nabla + \tilde{\mathbf{F}})^2_{L^2(\mathbb{R}^2)} = \big\{2j + 1, j \in \mathbb{N} \cup \{0\}\big\}.$$

We also consider the Neumann-realization $\mathcal{H}$ of the same operator but restricted to the half-space $\mathbb{R}^2_+$. Here we define

$$\inf \ \text{Spec}\ \mathcal{H} = \Theta_0 \in]0,1[\ . \tag{22}$$

We will consider the following PDEs.

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda\psi \text{ on } \mathbb{R}^2, \text{ with } \lambda < 1, \tag{23a}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda(1 - S^2|\psi|^2)\psi \text{ on } \mathbb{R}^2, \text{ with } 0 \leq \lambda \leq 1, \tag{23b}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda\psi \text{ on } \mathbb{R}^2_+, \text{ with } \lambda < \Theta_0, \tag{23c}$$

$$(-i\nabla + \tilde{\mathbf{F}})^2\psi = \lambda(1 - S^2|\psi|^2)\psi \text{ on } \mathbb{R}^2_+, \text{ with } 0 \leq \lambda \leq \Theta_0. \tag{23d}$$

The last two equations are considered with Neumann bndry condition. So, in order for this bndry condition to be well-defined, we assume that $\psi \in H^2_{\text{loc}}(\mathbb{R}^2_+)$.

Also, we assume that the parameter $S \geq 0$ in (23b) verifies $S \neq 0$ when $\lambda = 1$, and similarly, the parameter $S \geq 0$ in (23d) satisfies $S \neq 0$ when $\lambda = \Theta_0$.

The linear problems (23a), (23c), have no non zero solutions in L^2. That follows directly from the definition of the spectrum. One can prove that they do not have any non trivial bounded solutions either. In the case of Schrödinger operators without magnetic fields, there is a strong relation between the spectrum and generalized eigenfunction (Sch'nol's Theorem, see [CFKS]).

Proposition 5

Let (ψ, λ) be a solution to one of the equations (23a), (23b), (23c) or (23d) with λ in the indicated interval and ψ being globally bounded. Then $\psi = 0$.

The proof is reminiscent of Sch'nol's Theorem but we have now a magnetic potential and a non linear term.

We now explain a technique that we can initially find in the context of superconductivity in Lu-Pan and which was used in Helffer-Pan, Pan, Almog... The technique is based on a blowing up argument and on argument of subsequences.

First, we have tried to formalize what appears in many proofs as the **Extraction of convergent subsequences**
The non-existence result of Proposition 5 will then be combined with a compactness result which states that under certain circumstances we can construct bounded solutions to the equations (23).

Proposition 6

Let $\{(P_n,\kappa_n,\sigma_n)\}_{n\in\mathbb{N}} \subset \Omega\times\mathbb{R}_+\times\mathbb{R}_+$ be a sequence and let $(\psi_n,\mathbf{A}_n)_{\kappa_n,\sigma_n} \in H^1(\Omega)\times H^1_{\mathrm{div}}(\Omega)$ be an associated sequence of solutions to (1)
(with $(\kappa,\sigma)=(\kappa_n,\sigma_n)$ in the equation)
with $\psi_n \neq 0$.
Define $S_n := \|\psi_n\|_\infty$. Assume that $\kappa_n \to \infty$ and that $\kappa_n/\sigma_n \to \Lambda \in \mathbb{R}_+$.

Then there exist $P \in \overline{\Omega}, S \in [0,1], f \in \mathbb{C}$ and $\beta_0 \in \mathbb{R}$ such that—after possibly extracting a subsequence—we have

$$P_n \to P, \qquad S_n \to S, \tag{24}$$

$$\psi_n(P_n) \to f, \qquad \text{curl } \mathbf{A}_n(P_n) \to \beta_0, \tag{25}$$

as $n \to \infty$.

Furthermore:

Case 1.

If

$$\sqrt{\kappa_n \sigma_n}\ \text{dist}\ (P_n, \partial\Omega) \to \infty. \qquad (26)$$

Then there exists a function $\varphi \in C^{2,\alpha}(\mathbb{R}^2)$, for all $\alpha < 1$, satisfying $\|\varphi\|_\infty \leq 1$ and $|\varphi(0)| = \frac{|f|}{S}$ and a (linear) vector potential $\widetilde{\mathbf{F}} \in C^\alpha(\mathbb{R}^2)$ with $\text{curl}\ \widetilde{\mathbf{F}} = \beta_0$, and such that

$$(-i\nabla + \widetilde{\mathbf{F}})^2\varphi = \Lambda(1 - S^2|\varphi|^2)\varphi \qquad \text{in } \mathbb{R}^2. \qquad (27)$$

Case 2.
If there exists $C > 0$ such that

$$\text{dist}\,(P_n, \partial\Omega) \leq C/\sqrt{\kappa_n \sigma_n}. \qquad (28)$$

Then there exists $\varphi \in \cap_{\alpha<1} C^{2,\alpha}(\mathbb{R}^2_+)$, satisfying
$\|\varphi\|_\infty \leq 1$ and $|\varphi(0)| = \frac{|f|}{S}$
and a (linear) vector potential $\widetilde{\mathbf{F}} \in C^\alpha(\mathbb{R}^2_+)$ with $\text{curl}\ \widetilde{\mathbf{F}} = \beta_0$, and such that

$$(-i\nabla + \widetilde{\mathbf{F}})^2 \varphi = \Lambda(1 - S^2|\varphi|^2)\varphi \qquad \text{in } \mathbb{R}^2_+. \quad (29)$$

Notice that, up to extraction of a subsequence, we can always assure that Case 1 or Case 2 occurs.The proof is based, in Case 1, on the change of variables :

$$x - P_n = y/\sqrt{\kappa_n \sigma_n}\ .$$

Asymptotic estimates

We now combine the non-existence result Proposition 5 with Proposition 6 to obtain strong estimates on solutions to the Ginzburg-Landau equations.

Our first result only uses the extraction of convergent subsequences as in Prop. 6. Proposition 7 is a slightly improved version of results of Helffer-Pan and Pan.

Proposition 7 Let $0 < \lambda_{\min} \leq \lambda_{\max}$. There exist constants C_0, C_1 s. t. if

$$\kappa \geq C_0, \qquad \lambda_{\min} \leq \kappa/\sigma \leq \lambda_{\max},$$

then any solution $(\psi, \mathbf{A})$ of (1) satisfies

$$\|p_{\kappa\sigma\mathbf{A}}\psi\|_{C(\overline{\Omega})} \leq C_1\sqrt{\kappa\sigma}\|\psi\|_\infty, \tag{30}$$

$$\|\operatorname{curl}\mathbf{A} - 1\|_{C^1(\overline{\Omega})} \leq \frac{C_1}{\sqrt{\kappa\sigma}}\|\psi\|_\infty^2, \tag{31}$$

$$\|\operatorname{curl}\mathbf{A} - 1\|_{C^2(\overline{\Omega})} \leq C_1\|\psi\|_\infty^2. \tag{32}$$

Remarks

What is weak in the assumption is the condition $0 < \lambda_{\min} \leq \frac{\kappa}{\sigma} \leq \lambda_{\max}$.

When $\lambda_{max} < 1$, one can improve the estimates by taking into account the localization (in the limit $\kappa \to +\infty$) at the boundary (surface superconductivity) of ψ and of curl $A - 1$ (Agmon estimates). This will not be discussed in this talk.

When $\lambda_{max} = 1$, one can under a weaker assumption that $\frac{\kappa}{\sigma} \leq 1 - \frac{C}{\kappa}$ give interesting improved estimates, for example for curl $A - 1$ in L^∞ (See also Almog or Almog-Helffer).

Proof of Proposition 7

Proof of (30)**.**

Suppose (30) is wrong. Then there exists a sequence $(\psi_n, \mathbf{A}_n)_{\kappa_n,\sigma_n}$ of solutions to (1), and a corresponding sequence of points $\{P_n\} \subset \Omega$ such that

$$\frac{|p_{\kappa_n\sigma_n\mathbf{A}_n}\psi_n(P_n)|}{\sqrt{\kappa_n\sigma_n}\|\psi_n\|_\infty} \to \infty.$$

After extracting subsequences as in the proof of Proposition 6 we find

$$\lim_{n\to\infty} \frac{|p_{\kappa_n\sigma_n\mathbf{A}_n}\psi_n(P_n)|}{\sqrt{\kappa_n\sigma_n}\|\psi_n\|_\infty} = |(-i\nabla - \tilde{\mathbf{F}})\varphi(z)| < \infty,$$

where

$z = 0$ in Case 1

and

$z = \lim_{n\to\infty} \sqrt{\kappa_n\sigma_n}\Phi_n^{-1}(P_n)$ in Case 2

(with Φ_n a suitable diffeomorphism).

This yields a contradiction; so (30) is correct.

Proof of (31)**.**
This inequality is a consequence of (30). Remember that

$$\text{curl}\,^2\mathbf{A} := (\partial_{x_2}\,\text{curl}\ \mathbf{A}, -\partial_{x_1}\,\text{curl}\ \mathbf{A}).$$

Thus, by the Ginzburg-Landau equation (1a) and (30)

$$\|\nabla(\,\text{curl}\ \mathbf{A}-1)\|_\infty = \|\,\text{curl}\ (\,\text{curl}\ \mathbf{A}-1)\|_\infty$$
$$= \frac{1}{\kappa\sigma}\|\Re\{\overline{\psi}\,p_{\kappa\sigma\mathbf{A}}\psi\}\|_\infty \le \frac{C}{\sqrt{\kappa\sigma}}\|\psi\|_\infty^2. \quad (33)$$

This is (31) for the derivatives.

Furthermore, since $\text{curl}\ \mathbf{A}-1=0$ on $\partial\Omega$ and Ω is bounded, we can integrate (33) 'from the boundary' and find $C>0$ such that

$$\|\,\text{curl}\ \mathbf{A}-1\|_\infty \le \frac{C}{\sqrt{\kappa\sigma}}\|\psi\|_\infty^2. \quad (34)$$

This finishes the proof of (31).

Proposition 8

Let $\epsilon_0, \epsilon_1 > 0$ be such that $0 < \Theta_0 - \epsilon_1 < 1 - \epsilon_0$. Then there exist $\kappa_0, C > 0$ such that if $(\psi, \mathbf{A})_{\kappa,\sigma}$ is a solution to (1) with $\psi \neq 0$,

$$\kappa > \kappa_0, \qquad \Theta_0 - \epsilon_1 \leq \kappa/\sigma \leq 1 - \epsilon_0,$$

and $P \in \overline{\Omega}$ is such that $|\psi(P)| = \|\psi\|_\infty$,

$$\text{then} \qquad \text{dist}\,(P, \partial\Omega) \leq \frac{C}{\sqrt{\kappa\sigma}}\,.$$

Proof

Suppose Proposition 8 is false. Then there exists a sequence $(P_n, \kappa_n, \sigma_n, \psi_n, \mathbf{A}_n)$ such that

$$\begin{aligned}
&\kappa_n \to \infty\,,\\
&\Theta_0 - \epsilon_1 \leq \kappa_n/\sigma_n \leq 1 - \epsilon_0\,,\\
&|\psi_n(P_n)| = \|\psi_n\|_\infty\,,\\
&\sqrt{\kappa_n\sigma_n}\;\text{dist}\,(P_n, \partial\Omega) \to \infty\,.
\end{aligned} \tag{35}$$

By Case 1 in Proposition 6, we find a continuous solution $\varphi \in L^\infty(\mathbb{R}^2)$ to (27), that is to

$$(-i\nabla + \widetilde{\mathbf{F}})^2\varphi = \Lambda(1 - S^2|\varphi|^2)\varphi \qquad \text{in } \mathbb{R}^2.$$

with $|\varphi(0)| = 1$, $\Lambda \in [\Theta_0 - \epsilon_1, 1 - \epsilon_0]$ and $S \le 1$.

By Proposition 5, we have $\varphi \equiv 0$.

This is in contradiction to $|\varphi(0)| = 1$.

Thus no such sequence can exist and Proposition 8 is true.

The next and last result is in the same spirit.

It gives a weak control of the $||\psi||_{L^\infty}$ when σ is closed to $H_{C3}(\kappa)$.

Proposition 9
Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy that $g(\kappa) \to 0$ as $\kappa \to \infty$. Then there exists a function $\tilde{g}$ with $\tilde{g}(\kappa) \to 0$ as $\kappa \to \infty$, such that if

$$\kappa(\Theta_0^{-1} - g(\kappa)) \leq \sigma \leq \kappa(\Theta_0^{-1} + g(\kappa)),$$

then any solution $(\psi, \mathbf{A})_{\kappa,\sigma}$ of (1) satisfies

$$\|\psi\|_\infty \leq \tilde{g}(\kappa).$$

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