Subelliptic estimates for some systems of complex vector fields : quasihomogeneous case

B. Helffer (after Derridj-Helffer)
Département de Mathématiques, Univ Paris-Sud,
91 405 Orsay Cedex.

ESI, Vienna, November 2006

Abstract

For about twenty five years it was a kind of folk theorem that complex vector-fields defined on $\Omega \times \mathbb{R}_t$ (with Ω open set in $\mathbb{R}^n$) by

$$L_j = \frac{\partial}{\partial t_j} + i \frac{\partial \varphi}{\partial t_j}(\mathbf{t}) \frac{\partial}{\partial x}\ ,\ j = 1, \ldots, n\ ,\ \mathbf{t} \in \Omega, x \in \mathbb{R}\ ,$$

were subelliptic as soon as they were hypoelliptic when φ was analytic. This was the case when $n = 1$ but in the case $n > 1$, an inaccurate reading of the proof given by Maire (see also Trèves) of the hypoellipticity of such systems, under the condition that φ does not admit any local maximum or minimum (through a non standard subelliptic estimate), was supporting the belief for this folk theorem. Quite recently, J.L. Journé and J.M.Trépreau show by examples that there are very simple systems (with polynomial φ's) which were hypoelliptic but not subelliptic in the standard L^2-sense.

So it is natural to analyze this problem of subellipticity which is in some sense intermediate (at least when φ is C^∞) between the maximal hypoellipticity (which was analyzed by Helffer-Nourrigat and Nourrigat) and the simple local hypoellipticity (or local microhypoellipticity) and to start first with the easiest non trivial examples. The analysis presented here is a continuation of a previous work by M. Derridj and is devoted to the case of quasihomogeneous functions.

Introduction and Main result

Let Ω an open set in $\mathbb{R}^n$ with $0 \in \Omega$. We consider the regularity properties of the following system on $\Omega \times \mathbb{R}$

$$L_j = \frac{\partial}{\partial t_j} + i \frac{\partial \varphi}{\partial t_j}(\mathbf{t}) \frac{\partial}{\partial x} \ , \ j = 1, \dots, n \ , \ \mathbf{t} \in \Omega, x \in \mathbb{R} \ , \tag{1}$$

where $\varphi \in C^1(\Omega, \mathbb{R})$, with $\varphi(0) = 0$. We will concentrate our analysis near a point $(0,0)$.

Many authors have considered this type of system. They were in particular interested in the existence, for some pair (s, N) such that $s + N > 0$, of the following family of inequalities.

For any pair of open sets ω, I such that $\overline{\omega} \subset\subset \Omega$ and $I \subset\subset \mathbb{R}$, $\exists C_{s,N}(\omega, I)$ such that

$$
\begin{aligned}
&||u||_s^2 \leq C_N(\omega, I) \left(\sum ||L_j u||_0^2 + ||u||_{-N}^2\right) \ , \\
&\qquad \forall u \in C_0^\infty(\omega \times I) \ ,
\end{aligned}
\tag{2}
$$

where $|| \cdot ||_r$ denotes the Sobolev norm in $H^r(\Omega \times \mathbb{R})$.

If $s > 0$, we say that we have a subelliptic estimate. In [JoTre], there are also results where s can be arbitrarily negative. We will then speak about weak-subellipticity.

Note that in this case ($s \leq 0$) the existence of this inequality is not sufficient for proving hypoellipticity.

The system (1) being elliptic in the $\mathbf{t}$ variable, it is enough to analyze the subellipticity microlocally near $\tau = 0$, i.e. near $(0,(0,\xi))$ in $T^*(\omega \times I) \setminus \{0\}$ with $\{\xi > 0\}$ or $\{\xi < 0\}$.
This leads to the analysis of the existence of two constants C_s^+ and C_s^- such that the two following inequalities hold, for all $u \in C_0^\infty(\omega \times \mathbb{R})$:

$$\int_{\omega \times \mathbb{R}^+} \xi^{2s} |\widehat{u}(\mathbf{t},\xi)|^2 \, dtd\xi \leq C_s^+ \int_{\omega \times \mathbb{R}^+} |\widehat{Lu}(\mathbf{t},\xi)|^2 \, d\mathbf{t}d\xi \; , \tag{3}$$

where $\widehat{u}(\mathbf{t},\xi)$ is the partial Fourier transform of u with respect to the x variable, and

$$\int_{\omega \times \mathbb{R}^-} |\xi|^{2s} |\widehat{u}(\mathbf{t},\xi)|^2 \, d\mathbf{t}d\xi \leq C_s^- \int_{\omega \times \mathbb{R}^-} |\widehat{Lu}(\mathbf{t},\xi)|^2 \, d\mathbf{t}d\xi \; , \; . \tag{4}$$

When (3) is satisfied, we will speak of microlocal subellipticity in $\{\xi > 0\}$ and similarly when (4) is satisfied, we will speak of microlocal subellipticity in $\{\xi < 0\}$. Of course, when $s > 0$, it is standard that these two inequalities imply (2).
We now observe that (3) for φ is equivalent to (4) for $-\varphi$, so it is enough to consider the first case.

The main result

In [De], Derrridj gave a sufficient condition on φ for getting (2) with $s > 0$. Here, we consider the case of quasihomogeneous functions φ on $\mathbb{R}^2$ (i.e. $n = 2$).

The conditions will be expressed for φ in C^1 but note that they become more simple in the analytic case.

More precisely, let ℓ and m in $\mathbb{R}$, such that

$$m \geq 2\ell \geq 2\ . \tag{5}$$

In the analytic case, we will assume $\ell \in \mathbb{Q}$.

We consider in $\mathbb{R}^2$ (t,s) as the variables (instead of $\mathbf{t}$) and

the functions $\varphi \in C^1(\mathbb{R}^2)$ will be quasihomogeneous in the following sense

$$\varphi(\lambda t, \lambda^\ell s) = \lambda^m \varphi(t,s)\ ,\ \forall (t,s,\lambda) \in \mathbb{R}^2 \times \mathbb{R}^+\ . \quad (6)$$

φ is determined by its restriction $\widetilde{\varphi}$ to the distorted circle $\mathcal{S}$

$$\widetilde{\varphi} := \varphi_{|\mathcal{S}}\ .$$

where $\mathcal{S}$ is defined by

$$\mathcal{S} = \{(t,s)\,;\, t^{2\ell} + s^2 = 1\}\ ,$$

Our main result is stated under the following assumption

Assumption. *(H2)*

(i) $\widetilde{\varphi}$ *is not strictly negative.*

(ii) $\widetilde{\varphi}$ *can not have a local maximum equal to* 0*.*

(iii) If $\mathcal{S}_j^+$ *is a component of* $\widetilde{\varphi}^{(-1)}(]0,+\infty[)$, *then one can write* $\mathcal{S}_j^+$ *as a finite union of arcs satisfying Property 2 below.*

(iv) If $\mathcal{S}_j^-$ *is a component of* $\widetilde{\varphi}^{(-1)}(]-\infty,0[)$, *then* $\tilde{\varphi}$ *has a unique minimum in* $\mathcal{S}_j^-$ *.*

(v) $\exists p \geq 1$, *s. t., if* θ_0 *is a zero of* $\widetilde{\varphi}$, *then* $\exists$ *an open arc* $\mathcal{V}_{\theta_0}$ *containing* θ_0 *and* $C_0 > 0$, *such that*

$$|\widetilde{\varphi}(\theta) - \widetilde{\varphi}(\theta')| \geq \frac{1}{C_0}|\theta - \theta'|^p \ , \ \forall \theta, \theta' \in \mathcal{V}_{\theta_0} \ , \quad (7)$$

with θ *and* θ' *in the same side.*

Here in the third item, we say that a closed arc $[\theta, \theta']$ has Property (P) if :

Property. [(P)]
There exists on this arc $\widehat{\theta}$ s. t.

(i) $\widetilde{\varphi}$ *is non decreasing on the arc* $\left[\theta, \widehat{\theta}\right]$ *and non increasing* $\left[\widehat{\theta}, \theta'\right]$.

(ii)

$$\langle \widehat{\theta} \,|\, \theta \rangle_\ell \geq 0 \textit{ and } \langle \widehat{\theta} \,|\, \theta' \rangle_\ell \geq 0 \ ,$$

where for $\theta = (\alpha, \beta)$ *and* $\widehat{\theta} = (\widehat{\alpha}, \widehat{\beta})$ *in* $\mathcal{S} \subset \mathbb{R}^2$,

$$\langle \widehat{\theta} \,|\, \theta \rangle_\ell := \widehat{\alpha}\alpha |\widehat{\alpha}|^{\ell-1} |\alpha|^{\ell-1} + \widehat{\beta}\beta \ .$$

We can now state our main theorem :

Theorem 1.
Let $\varphi \in C^1(\mathbb{R}^2, \mathbb{R})$ *satisfying* (6), *with* ℓ *and* m *satisfying* (5). *Then Assumption* $(H2)$ *implies that the system is microlocally* α*-subelliptic in the* $\{\xi > 0\}$ *direction with* $\alpha = \frac{1}{\max(m,p)}$.

Remarks.

(i) [De] was considering the homogeneous case $\ell = 1$ *and* $m \geq 2$.

(ii) If φ *is analytic and* ℓ *is rational. The statement of the main theorem becomes simpler. (iii) and (v) are indeed automatically satisfied as soon that* $\widetilde{\varphi}$ *is not identically* 0. *Moreover, if we write* $\ell = \frac{\ell_2}{\ell_1}$ *(with* ℓ_1 *and* ℓ_2 *mutually prime integers), all the criteria on* $\widetilde{\varphi}$ *can be reinterpreted as criteria for the restriction* $\widehat{\varphi}$ *of* φ *on*

$$\mathcal{S}_{\ell_1,\ell_2} = \{(t,s)\ ;\ t^{2\ell_2} + s^{2\ell_1} = 1\}\ .$$

Derridj's subellipticity criterion.

Assumption. $(H_+(\alpha))$
$\exists \widetilde{\omega} \subset \omega$, *with full Lebesgue measure in* ω *and*

$$\widetilde{\omega} \times [0,1] \ni (\mathbf{t},\tau) \mapsto \gamma(\mathbf{t},\tau) \in \Omega\ ,$$

such that

(i) $\gamma(\mathbf{t},0) = \mathbf{t}\,;\ \gamma(\mathbf{t},1) \notin \omega\ ,\ \forall \mathbf{t} \in \widetilde{\omega}$.

(ii) γ *is* C^1 *outside a negligeable set* E *and* $\exists C_1 > 0$, $C_2 > 0$ *and* $C_3 > 0$ *s.t.*

(a)

$$|\partial_\tau \gamma(\mathbf{t},\tau)| \leq C_2\ ,\ \forall (\mathbf{t},\tau) \in \widetilde{\omega} \times [0,1] \setminus E\ .$$

(b)

$$|\det(D_{\mathbf{t}}\gamma)(\mathbf{t},\tau)| \geq \frac{1}{C_1}\ ,$$

where $\det D_{\mathbf{t}}\gamma$ *denotes the Jacobian of* γ *considered as a map from* $\widetilde{\omega}$ *into* $\mathbb{R}^2$.

(c)

$$\varphi(\gamma(\mathbf{t},\tau))-\varphi(\mathbf{t}) \geq \frac{1}{C_3}\tau^\alpha \, , \; \forall(\mathbf{t},\tau) \in \widetilde{\omega}\times[0,1] \, .$$

Let us recall the result of [De].

Theorem 2.
If φ satisfies $(H_+(\alpha))$, then the associated system $(1)_\varphi$ *is microlocally $\frac{1}{\alpha}$-subelliptic in $\{\xi > 0\}$.*

The proof is easy after taking the partial Fourier transform (with respect to x) and reexpressing u from Lu.

Quasihomogeneous structure

Distorted geometry

In the description of escaping curves, it appears useful to extend the usual terminology used in the Euclidean space $\mathbb{R}^2$. This is realized by introducing the *dressing* map :

$$(t,s) \mapsto d_\ell(t,s) = \left(t\,|t|^{\ell-1}, s\right) \ . \qquad (8)$$

The first example was the unit distorted circle $\mathcal{S}$ whose image by d_ℓ becomes the standard unit circle in $\mathbb{R}^2$ centered at $(0,0)$.
Similarly, we will speak of disto-sectors, disto-arcs, disto-rays.

The "disto" scalar product of two vectors in $\mathbb{R}^2$ (t,s) et (t',s') is then given by

$$\langle (t,s) \mid (t',s') \rangle_\ell = tt'|tt'|^{\ell-1} + ss' \ . \qquad (9)$$

(for $\ell = 1$, we recover the standard scalar product).

For $(t,s) \in \mathbb{R}^2$, we introduce also the quasihomogeneous positive function ϱ defined on $\mathbb{R}^2$ by :

$$\varrho(t,s)^{2\ell} = t^{2\ell} + s^2 \, . \tag{10}$$

With these notations, we observe that

$$(\widetilde{t}\,,\,\widetilde{s}) := \left(\frac{t}{\varrho(t,s)}, \frac{s}{\varrho(t,s)}\right) \in \mathcal{S} \, , \tag{11}$$

and

$$(t,s) \in \mathcal{R}_{(\widetilde{t},\widetilde{s})} \, .$$

The open disto-disk $D(R)$ is then defined by

$$D(R) = \{(x,y) \mid \varrho(x,y) < R\} \, .$$

Once an orientation is defined on $\mathcal{S}$, two points θ_1 and θ_2 (or (a_1,b_1) and (a_2,b_2)) on $\mathcal{S}$ will determine a unique “sector” $V \subset D(1)$.

Distorted dynamics

The parametrized curves γ permitting to satisfy Assumption will actually be "lines" (possibly broken) finally escaping from a neighborhood of the origin. In parametric coordinates, with

$$t(\tau) = t + \varrho\tau\,,\ \varrho = \pm c\,, \tag{12}$$

the curve γ starting from (t, s) and "parallel" to (c, d) is defined by writing that the vectors $(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}, s(\tau) - s)$ and $(c|c|^{\ell-1}, d)$ are collinear :

$$\left(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}\right) d = c|c|^{\ell-1}(s(\tau) - s)\ ,$$

and we find

$$s(\tau) = s + \frac{d}{c|c|^{\ell-1}} \left(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}\right)\ , \tag{13}$$

We consider the map $\sigma \mapsto f_\ell(\sigma)$ which is defined by

$$f_\ell(\sigma) = \sigma|\sigma|^{\ell-1} \ .$$

Note that

$$f'_\ell(\sigma) = \ell|\sigma|^{\ell-1} \geq 0 \ .$$

With this new function, (13) can be written as

$$df_\ell(t(\tau)) - s(\tau)f_\ell(c) = df_\ell(t) - sf_\ell(c) \ . \qquad (14)$$

This leads us to use the notion of distorted determinant of two vectors in $\mathbb{R}^2$.

$$\Delta_\ell(v;w) = f_\ell(v_1)w_2 - v_2 f_\ell(w_1) \ .$$

We will also write :

$$\Delta_\ell(v;w) = \Delta_\ell(v_1,v_2,w_1,w_2) \ .$$

With these notations, (14) can be written

$$\Delta_\ell(c,d,t(\tau),s(\tau)) = \Delta_\ell(c,d,t,s) \ ,$$

We now look at the variation of ψ which is defined (for a given initial point (t,s)) by

$$\tau \mapsto \psi(\tau) = \rho(\tau)^{2\ell} = t(\tau)^{2\ell} + s(\tau)^2 \,. \qquad (15)$$

Easy computations give also :

$$\psi'(\tau) = \frac{2\varrho}{f_\ell(c)} f'_\ell(t+\varrho\tau)\langle (c,d) \mid (t(\tau), s(\tau))\rangle_\ell \,.$$

We now analyze the variation of the "scalar product" $\langle (c,d) \mid (t(\tau), s(\tau))\rangle_\ell$ as a function of τ. We have the formula

$$\begin{aligned} &\langle (c,d) \mid (t(\tau), s(\tau))\rangle_\ell \\ &\quad = \langle (c,d) \mid (t,s)\rangle_\ell + \tfrac{1}{f_\ell(c)}(f_\ell(t(\tau)) - f_\ell(t)) \,. \end{aligned}$$

If we now assume that

$$c\varrho > 0 \;,\; \langle (c,d) \mid (a,b)\rangle_\ell \geq 0 \;, \qquad (16)$$

Then for (s,t) in the unit sector $\mathcal{V}_{abcd}$ associated to the arc $((a,b)\;,\;(c,d))$, we obtain :

$$\psi'(\tau) \geq \frac{1}{f_\ell(c)^2} \times (\, 2\varrho f'_\ell(t+\varrho\tau)\,(f_\ell(t(\tau)) - f_\ell(t))\,) \;.$$

We rewrite this inequality in the form

$$\psi'(\sigma) \geq \frac{1}{f_\ell(c)^2} \times \left((f_\ell(t(\sigma)) - f_\ell(t))^2\right)' \ , \ \forall \sigma \geq 0 \ .$$

Integrating over $[0, \tau]$, we get for $\tau \geq 0$:

$$\psi(\tau) \geq \frac{1}{f_\ell(c)^2} \times (f_\ell(t(\tau)) - f_\ell(t))^2 \ .$$

We now need the following

Lemma 1.
For any $\ell \geq 1$, $\tau \geq 0$, and $\gamma \in \mathbb{R}$, we have

$$f_\ell(\tau + \gamma) - f_\ell(\gamma) \geq f_\ell(\frac{\tau}{2}) \ . \tag{17}$$

But using Lemma 1, this leads to

Lemma 2.
Under Condition (16)*, we have, for any $\tau \geq 0$, for any $(t, s) \in \mathcal{V}_{abcd}$,*

$$\rho(\tau)^{2\ell} - \rho(0)^{2\ell} \geq (\frac{\varrho\tau}{2c})^{2\ell} \ . \tag{18}$$

If instead $\varrho c < 0$, we obtain :

$$\rho(\tau)^{2\ell} - \rho(0)^{2\ell} \leq -(\frac{\varrho\tau}{2c})^{2\ell} \,. \tag{19}$$

We continue by analyzing the variation of $s(\tau)$ and $t(\tau)$ and more precisely the variation on the disto-circle of :

$$\widetilde{t}(\tau) = \frac{t(\tau)}{\rho(\tau)} \,,\ \widetilde{s}(\tau) = \frac{s(\tau)}{\rho(\tau)^{\ell}} \,.$$

After some computations, we get, with

$$\varrho = \pm c \,,$$

$$\widetilde{t}'(\tau) = \pm|c|^{1-\ell}\frac{s(\tau)}{\rho(\tau)^{2\ell+1}}\Delta_{\ell}(c,d,t,s) \,,$$

which can also be written in the form

$$\widetilde{t}'(\tau) = \pm|c|^{1-\ell}\frac{\widetilde{s}(\tau)}{\rho(\tau)}\Delta_{\ell}(c,d,\widetilde{t}(\tau),\widetilde{s}(\tau)) \,.$$

Similarly, we get for $\widetilde{s}'$

$$\widetilde{s}'(\tau) = \mp\ell|c|^{1-\ell}\frac{t(\tau)^{2\ell-1}}{\rho(\tau)^{3\ell}}\Delta_\ell(c,d,t,s)\ ,$$

and

$$\widetilde{s}'(\tau) = \mp\ell|c|^{1-\ell}\frac{\widetilde{t}^{2\ell-1}(\tau)}{\rho(\tau)}\Delta_\ell(c,d,\widetilde{t}(\tau),\widetilde{s}(\tau))\ .$$

The analytic case and $\ell \in \mathbb{Q}$

We keep the previous assumptions but now assume that

$$\ell = \ell_2/\ell_1 \,,$$

with ℓ_1 and ℓ_2 mutually prime integers and that φ is analytic. In this case assumption (6) on φ implies that φ is actually a polynomial and we can write φ in the form

$$\varphi(t,s) = \sum_{\ell_1 j + \ell_2 k = \ell_1 m} a_{j,k} t^j s^k \,, \tag{20}$$

where (j,k) are integers and the $a_{j,k}$ are real.
We can of course apply the main theorem but it is nicer to have a criterion involving more directly the assumptions on φ instead those on $\widetilde{\varphi}$. It is indeed more natural to express the conditions on the restriction $\widehat{\varphi}$ of φ to the quasi-circle

$$\mathcal{S}_{\ell_1,\ell_2} := \{t^{2\ell_2} + s^{2\ell_1} = 1\} \,.$$

instead of the disto-circle $\mathcal{S}$. There are absolutely no problems if the critical points or zeroes of $\widetilde{\varphi}$ avoid

$\{t=0\}\cup\{s=0\}$ but one should be more careful in order to analyze Condition (7), if it is not satisfied.

Theorem 3.
Let φ be a real analytic non identically 0 quasihomogeneous function satisfying (6) *and* (5), *with $\ell=\ell_2/\ell_1$. Suppose that φ is not a negative function. Suppose in addition that :*
If $\mathcal{S}_k^- = (\theta_k,\theta_{k+1})$ is a maximal arc where $\widehat{\varphi}$ is negative, then $\widehat{\varphi}'$ has a unique zero on $]\theta_k,\theta_{k+1}[$.
Then φ satisfies (H_+) with $\alpha>0$. Hence the system (1) *is microlocally subelliptic in $\{\xi>0\}$.*

Example 4.
We recover some examples treated by H. Maire [Mai4]

$$\varphi(t,s)=t(s^2-t^{2\ell})\;,\;\ell\geq 1\;.$$

Here $m=2\ell+1$.

Around Journé-Trépreau

For

$$\varphi(t,s) = -t^{2m} - t^2 s^{2p} + s^q \ ,$$

with

$$m \geq 1 \ , \ p \geq 2 \ , \ q \geq \frac{2mp}{m-1} \ ,$$

J.L. Journé and J.M. Trépreau show that one cannot obtain a better ρ-subellipticity than

$$\rho \leq -(1 - \frac{2p}{q} - \frac{1}{m})\frac{n-1}{4} + \frac{1}{2q} + \frac{m-1}{4mp} \ .$$

The right hand side can become strictly negative, but **Not** in the quasihomogeneous case !!

Inside this class ($m = 2, p = 2$), a particularly interesting example where the authors can obtain the optimal subellipticity is

$$\varphi(t, s) = -t^4 - t^2 s^4 + s^q \ ,$$

with $q \geq 8$.
The optimal subellipticity is $\rho_q = \frac{3}{2q} - \frac{1}{16}$. Here let us observe that the only quasihomogeneous case corresponds to $q = 8$ and that in this case their result is coherent with our result. This example show also that we loose the “positive” subellipticity for $q \geq 24$.

References

[De] M. Derridj. Subelliptic estimates for some systems of complex vector fields. Communication at a conference in Ferrara (March 2005) and to appear in the book "Hyperbolic problems and regularity questions". Series Trends in Mathematics. Edtrs : M. Padula and L. Zanghirati. Birkhäuser.

[HeNi] B. Helffer and F. Nier. Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Math. 1862, Springer Verlag, Berlin 2005.

[HelNo3] B. Helffer and J. Nourrigat. *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteur.* Progress in Mathematics, Birkhäuser, Vol. 58 (1985).

[Ho1] L. Hörmander. Hypoelliptic second order differential equations. Acta Mathematica 119 (1967), p. 147-171.

[Ho2] L. Hörmander. Subelliptic operators. Seminar on singularities of solutions of partial differential equations. Ann. Math. Studies 91 (1978), p. 127-208.

[JoTre] J.L. Journé and J.M. Trépreau. Hypoellipticité sans sous-ellipticité : le cas des systèmes de n champs de vecteurs complexes en $(n+1)$- variables. Séminaire EDP in Ecole Polytechnique, April 2006.

[Mai1] H.M. Maire. Hypoelliptic overdetermined systems of partial differential equations. Comm. Partial Differential Equations 5 (4), p. 331-380 (1980).

[Mai2] H.M. Maire. Résolubilité et hypoellipticité de systèmes surdéterminés. Séminaire Goulaouic-Schwartz 1979-1980, Exp. V, Ecole Polytechnique (1980).

[Mai3] H.M. Maire. Necessary and sufficient condition for maximal hypoellipticity of $\bar{\partial}_b$. Unpublished (1979).

[Mai4] H.M. Maire. Régularité optimale des solutions de systèmes différentiels et du

Laplacien associé : application au $\Box_b$. Math. Ann. 258, p. 55-63 (1981).

[No1] J. Nourrigat. *Subelliptic estimates for systems of pseudo-differential operators*. Course in Recife (1982). University of Recife.

[No2] J. Nourrigat. Inégalités L^2 et représentations de groupes nilpotents. J. Funct. Anal. 74 (2), p. 300-327 (1987).

[No3] J. Nourrigat. Réduction microlocale des systèmes d'opérateurs pseudo-différentiels. Ann. Inst. Fourier 36 (3), p. 83-108 (1986).

[No4] J. Nourrigat. Systèmes sous-elliptiques. Séminaire Equations aux Dérivées Partielles, 1986-1987, exposé V, Ecole Polytechnique (1987).

[No5] J. Nourrigat. Subelliptic systems II. Inv. Math. 104 (2) (1991), p. 377-400.

[RoSt] L.P. Rothschild and E.M. Stein. Hypoelliptic differential operators and nilpotent groups. Acta Mathematica 137, p. 248-315 (1977).

[Tr1] F. Trèves. A new method of proof of the subelliptic estimates. Comm. Pure Appl. Math. 24 (1971), p. 71-115.

[Tr2] F. Trèves. Study of a model in the theory of complexes of pseudo-differential operators. Ann. of Math. (2) 104, p. 269-324 (1976). See also erratum: Ann. of Math. (2) 113, p. 423 (1981).