Semi-classical methods in Ginzburg-Landau theory

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Dedicated to Laurent Schwartz (1915-2002).

Abstract

The theory of superconductivity has recently stimulated new researchs in semi-classical analysis concerning the ground state energy of the Neumann realization of a Schrödinger operator with magnetic field and the localization of its ground state.

As for the analysis of the tunneling for the Schrödinger operator, an important role is played by the Agmon estimates, permitting to determine a priori the decay of the ground state. This is rather clear already in the papers by Bernoff-Sternberg, Lu-Pan and Del Pino-Felmer-Sternberg but was more systematically developed in our works with A. Morame. The techniques which reappear here have strong intersection with former work by the author in collaboration with A. Morame (=A. Mohamed), which were inspired by a geometer Montgomery who was asking, by analogy with the celebrated question of M. Kac *"Can we hear the form of a drum"*, *"Can we hear the zero locus of a magnetic field"*.

But what is probably the most interesting is the unexpected specificity of the Neumann problem, in comparison with the Dirichlet problem. One can indeed see that the most interesting phenomena appear at the boundary. We shall describe in this survey after a short presentation of the physical origin of the problem the recent mathematical results which have been obtained in order to understand the mechanism of what is called boundary nucleation in superconductivity. In the case of 2- or 3-dimensional domains, we shall demonstrate how the geometry of the boundary (curvature, corners) is combined with the properties of the magnetic field.

1 Introduction: The question in superconductivity

Let us start by giving a physical presentation of the question and postpone the " mathematical translation of the question". This problem is described in

all the basic books in physics (see for example Saint-James-De Gennes [SdG], Saint-James-Sarma-Thomas [S-JSaTh], and Tinkham [Ti]).
One can for example find in Chapter 4 of [S-JSaTh] (Subsection 4.3) the following description:
" For type II superconductors, the above calculation shows that superconductivity is not entirely destroyed for $H_{c_2} < H < H_{c_3}$. A superconducting sheath remains close to the surface parallel to the applied field. Conversely, when the field is decreased below H_{c_3}, a superconducting sheath appears at the surface before superconductivity is restored in the bulk at $H = H_{c_2}$. If the sample is a long cylinder with the applied field parallel to the axis, the sheath will cover all the surface of the cylinder. If it is a sphere, the sheath will be restricted to a small zone near the equatorial plane when H is close to H_{c_3}. When the field is decreased towards H_{c_2} the sheath will progressively extend up to the poles."

More recently, a lot of articles appear which are devoted to this question. For mentioning some, let us cite the contributions by Chapman [Ch] and Bernoff-Sternberg [BeSt], which remain at a formal level, the nice paper by Bauman-Philips-Tang [BaPhTa] treating in detail the case of the disk and the papers by Giorgi-Phillips [GioPh], Lu-Pan [LuPa1, LuPa2, LuPa3, LuPa4, LuPa5] and Del Pino-Felmer-Sternberg [PiFeSt] for a mathematically rigorous analysis in general domains. But our presentation will be more concentrated around the quite recent contributions by Helffer-Morame [HeMo2, HeMo3, HeMo4], Lu-Pan [LuPa5], Helffer-Pan [HePa], Pan [Pan3] and Bonnaillie [Bon]. In particular, we would like to show how the geometry of the boundary of the sample can play an important role for the onset (our nucleation) of the superconductivity.

Let us now "translate" our main problem in a mathematical problem. It is naturally posed for domains in $\mathbb{R}^3$, but for cylindrical domains in $\mathbb{R}^3$, it is natural (but not completely justified mathematically) to consider a functional which is defined in a domain $\Omega \in \mathbb{R}^2$, where Ω is the section of the cylinder. This explains why we consider models in domains in $\mathbb{R}^3$ or in $\mathbb{R}^2$. The behavior of the sample can be read on the properties of the minimizers (ψ, A) in $H^1(\Omega;\mathbb{C}) \times H^1(\mathbb{R}^n;\mathbb{R}^n)$ of the Ginzburg-Landau functional $\hat{\mathcal{G}}$:

$$\hat{\mathcal{G}}(\psi\ ,\ A) = \int_\Omega \{|(\nabla - i\kappa A)\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\}\, dx + \kappa^2 \int_{\mathbb{R}^n} |\mathrm{curl}\, A - \mathcal{H}|^2\, dx\ . \tag{1.1}$$

Here Ω is a regular bounded set, ψ is called the order parameter and A is a magnetic potential defined on $\mathbb{R}^n$. $\mathcal{H}$ is a magnetic vector field when $n = 3$ and which is called the external magnetic field or the applied magnetic field. In the case $n = 2$, we identify this magnetic field to a function (thinking that it is the intensity of a magnetic field vector, which is parallel to the axis of the cylinder). It is initially defined on $\mathbb{R}^n$ but in the case when Ω is simply connected, one can reduce everything to Ω and consider the functional

$$\mathcal{G}(\psi\ ,\ A) = \int_\Omega \{|(\nabla - i\kappa A)\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\}\, dx + \kappa^2 \int_\Omega |\mathrm{curl}\, A - \mathcal{H}|^2\, dx\ . \tag{1.2}$$

We refer to [BeRu], [HHOO] for the analysis of specific problems with holes.

Here we will always asume that Ω is connected and simply connected.

The parameter κ is a characteristic of the sample. Traditionnally one makes the distinction between the type 1 materials corresponding to κ small and the type 2 materials corresponding to κ large. Mathematically, this leads to analyze various asymptotic regimes like $\kappa \to 0$ or $\kappa \to +\infty$. This is this last case which will be analyzed here.

As Ω is bounded, the existence of a minimizer is rather standard (see [DGP]). In order to kill the degeneracy of the problem due to the gauge invariance, one can consider instead the functional

$$\begin{aligned}&\tilde{\mathcal{G}}(\psi\ ,\ A)\\ &= \int_\Omega \{|(\nabla - i\kappa A)\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\}\, dx + \kappa^2 \int_\Omega |\mathrm{curl}\, A - \mathcal{H}|^2\, dx + \int_\Omega |\mathrm{div}\, \mathcal{A}|^2\, dx\ .\end{aligned} \tag{1.3}$$

One can also add the conditions that $\mathcal{A}$ should be divergence free (i.e. $\mathrm{div}\, \mathcal{A} = 0$) and tangent to the boundary.

The minimizer should necessarily satisfy the Euler-Lagrange equation, which is called in this context the Ginzburg-Landau system [GinLa, dG, S-JSaTh] and which reads in the case $n = 2$

$$\left\{\begin{array}{l} -(\nabla - i\kappa A)^2\psi = \kappa^2(1-|\psi|^2)\psi, \\ \mathrm{curl}\,^2 A = -\frac{i}{2\kappa}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2 A + \mathrm{curl}\,\mathcal{H}, \qquad \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} - i\kappa A\psi\cdot\nu = 0\ , \qquad \text{on } \partial\Omega \\ \mathrm{curl}\, A - \mathcal{H} = 0, \qquad \text{on } \partial\Omega. \end{array}\right. \tag{1.4}$$

Note here that ν denotes the external normal at a point of $\partial\Omega$ and that :

$$\mathrm{curl}\,^2 A = (\partial_2(\mathrm{curl}\, A), -\partial_1(\mathrm{curl}\, A))\ .$$

In this survey, we assume :

$$\mathcal{H}(x) \equiv \sigma H_e(x), \tag{1.5}$$

where $H_e(x)$ is a fixed, non identically zero, magnetic field in Ω and where σ is a parameter which will permit the variation of the intensity of the magnetic fields.

It is well known that there exists a unique vector field $\mathbf{F}$ on Ω such that

$$\mathrm{curl}\, \mathbf{F} = H_e \quad \text{and} \quad \mathrm{div}\, \mathbf{F} = 0 \quad \text{in } \Omega, \quad \mathbf{F}\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

We observe that $(0, \sigma\mathbf{F})$ is a trivial critical point of the functional $\mathcal{G}$. It is therefore natural to discuss in function of σ, if this pair is a local or a global minimizer. As σ is large, one can show [GioPh] that this is effectively the unique[1] global minimizer. One says that in this case the superconductivity is destroyed. In other words, the order parameter is identically zero in Ω. It is then natural to try to follow the property of the minimizers when decreasing σ starting from $+\infty$ and to determine when the trivial solution (also called the normal solution) is no more a global minimum or a local minimum.

[1] Up to gauge transformation. This will what we mean by unique below.

This leads (**assuming that H_e is constant and of intensity one**) to the definition

$$H_{C_3}(\kappa) = \inf\{\sigma > 0 : (0, \sigma \mathbf{F}) \text{ is the unique global minimizer of } \mathcal{G}\}. \tag{1.6}$$

So $H_{C_3}(\kappa)$ is the bottom of the set

$$\mathcal{N}(\kappa) := \{\sigma > 0 : (0, \sigma \mathbf{F}) \text{ is the unique global minimizer of } \mathcal{G}\}\ . \tag{1.7}$$

Note that it could be interesting to analyze $\tilde{H}_{C_3}(\kappa)$ which is the infimum of the unbounded connected component of $\mathcal{N}(\kappa)$ and to show that when κ is large we have $\tilde{H}_{C_3}(\kappa) = H_{C_3}(\kappa)$.
The first result that we would like to mention is essentially due to Lu-Pan (cf also Bauman-Phillips-Tang [BaPhTa] for the case of the disk).

Theorem 1.1 .
For any simply connected domain $\Omega \subset \mathbb{R}^2$ with regular boundary we have :

- $$\lim_{\kappa \to \infty} \frac{H_{C_3}(\kappa)}{\kappa} = \frac{1}{\Theta_0}\ , \tag{1.8}$$

  where Θ_0 is the bottom of the spectrum of an operator which will be introduced in (3.8).

- *If the external field satisfies :*

  $$\frac{\kappa}{\Theta_0} + o(1) < \sigma < H_{C_3}(\kappa)\ , \tag{1.9}$$

  then the superconductivity appears at the boundary of the sample. More precisely there exist C, $\alpha > 0$ and κ_0 such that the order parameter $\psi_{\kappa,\sigma}$ (corresponding to a minimizer $(\psi_{\kappa,\sigma}, A_{\kappa,\sigma})$ of $\mathcal{G}$) satisfies, for $x \in \Omega$ and $\kappa \geq \kappa_0$:

  $$|\psi_{\kappa,\sigma}(x)| \leq C\ \exp -\alpha\kappa d(x, \partial\Omega) ||\psi_{\kappa,\sigma}||_{L^\infty(\Omega)}\ . \tag{1.10}$$

The conclusion of the second part corresponds to a mathematical description of what is called boundary nucleation (or boundary onset of superconductivity).

But this inequality does not give the optimal localization for the boundary nucleation. Let us first discuss the case when the dimension is 2. If $c(x)$ denotes the curvature of $\partial\Omega$ (with the convention that the disk has positive curvature), the localization will appear near the points of maximal curvature under the condition that :

$$\delta(\kappa, \sigma)\ := H_{C_3}(\kappa) - \sigma\ ,$$

is sufficiently small.
This idea, which was arising from formal calculations of Bernoff-Sternberg [BeSt], has been made mathematically rigorous by the successive efforts of Lu-Pan

[LuPa1], Del Pino-Felmer-Sternberg [PiFeSt], and finally completed in the recent papers by Helffer-Morame [HeMo2] and Helffer-Pan [HePa].
Actually, this localization depends strongly on δ. It is shown in [LuPa1] (Theorem 1.5) that when $\delta\kappa^{-\frac{1}{3}}$ becomes large the localization becomes uniform in the boundary. Recent contributions [Pan2] are devoted to a more precise analysis of the phenomenon. Note also the recent contribution by Sandier-Serfaty [SaSe4] who analyze, for larger δ the onset of superconductivity inside the domain. For refined estimates, we will have to introduce a tubular neighborhood of the boundary

$$\Omega(\delta_1) := \{x \in \overline{\Omega} \mid d(x, \partial\Omega) < \delta_1\} .$$

In this case, and under the condition that δ_1 is small enough, we denote, for $x \in \Omega(\delta_1)$, by $s(x)$ the unique point of $\partial\Omega$ such that $d(x, \partial\Omega) = d(x, s(x))$.
The main results obtained in [HePa] are the following. The first one gives a fine asymptotics (compare with (1.8)), of $H_{C_3}(\kappa)$.

Theorem 1.2 .
There exists a universal constant $C_1 > 0$ such that, as $\kappa \to +\infty$,

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{C_1}{\Theta_0^{3/2}} c_{max} + \mathcal{O}(\kappa^{-\frac{1}{3}}) , \tag{1.11}$$

where $c_{max} = \max_{x\in\partial\Omega} c(x)$.

The second result gives a precise complementary information for the decay near the boundary.

Theorem 1.3 .
Let

$$\rho := \frac{1}{C_1} \Theta_0^{\frac{3}{2}} (H_{C_3}(\kappa) - \sigma) .$$

1.
There exist L, C, κ_0, α_0 such that for ρ satisfying

$$0 \leq \rho \leq L\kappa^{-\frac{1}{3}} ,$$

the order parameter $\psi = \psi_{\kappa,\sigma}$ satisfies, for $\kappa \geq \kappa_0$,

$$\int_{\Omega(\delta_1)} \exp\left(\alpha_0\sqrt{\kappa}(c_{max} - c(s(x)))\right) |\psi|^2 \, dx \leq C \exp(C\kappa^{\frac{1}{6}}) \|\psi\|^2_{L^2(\Omega)} . \tag{1.12}$$

2.
There exist L, C, κ_0, α_0 such that, for ρ satisfying

$$L\kappa^{-\frac{1}{3}} \leq \rho \leq (c_{max} - c_{min}) ,$$

where $c_{min} = \min_{x\in\partial\Omega} c(x)$, then we have, for $\kappa \geq \kappa_0$:

$$\int_{\Omega(\delta_1)} \exp\left(\alpha_0\sqrt{\kappa}(c_{max} - \rho - c(s(x)))_+^{\frac{3}{2}}\right) |\psi|^2 \, dx \leq C\kappa^{\frac{1}{3}} \|\psi\|^2_{L^2(\Omega)} . \tag{1.13}$$

Note that this estimate of the decay is not optimal (one should probably speak of Agmon distance inside the boundary) but gives a much more precise information than in the initial papers by Lu-Pan [LuPa4] and Del Pino-Felmer-Sternberg [PiFeSt]. These theorems are related to the analysis of the Neuman realization of $-(\nabla - iA)^2$. It is useful to observe the strong connexions between the critical field $H_{C_3}(\kappa)$ and the smallest eigenvalue $\mu^{(1)}(A)$ of this realization. One first observe the following elementary lemma :

Lemma 1.4 .

- *If* $\mu^{(1)}(\kappa\sigma\mathbf{F}) < \kappa^2$, *then* $\mathcal{G}$ *has a non trivial minimizer.*
- *If* $\mathcal{G}$ *has a non trivial minimizer* $(\psi_{\kappa,\sigma}, A_{\kappa,\sigma})$ *then* $\mu^{(1)}(\kappa A_{\kappa,\sigma}) < \kappa^2$.

The second important remark is that $\psi_{\kappa,\sigma}$ is, using the first Ginzburg-Landau equation, a solution of :

$$-(h\nabla - i\frac{A_{\kappa,\sigma}}{\sigma})^2\psi_{\kappa,\sigma} + V_{\kappa,\sigma}\psi_{\kappa,\sigma} - \frac{1}{\sigma^2}\psi_{\kappa,\sigma} = 0 \ , \qquad (1.14)$$

where

$$h = 1/(\kappa\cdot\sigma) \ , \ V_{\kappa,\sigma} = \sigma^{-2}|\psi_{\kappa,\sigma}|^2 \ .$$

If one shows by a priori estimates that $\frac{A_{\kappa,\sigma}}{\sigma}$ is near $\mathbf{F}$ and that $\psi_{\kappa,\sigma}$ is small in L^∞ in the asymptotic regim considered here (properties established mainly in [LuPa4] and improved in [HePa]), it is not too surprising to think that the analysis which will be presented in another section of this survey on the decay of the ground state of $-(h\nabla - i\mathbf{F})^2$ as $h \to 0$ will still be valid.

2 Neumann realization of the magnetic Schrödinger operator

We would like to discuss the spectrum of selfadjoint realizations of the Schrödinger operator in an open set Ω in $\mathbb{R}^n$:

$$P_{h,A,\Omega} = \sum_{j=1}^{n}(h\,D_{x_j} - A_j)^2 \ ,$$

where $h > 0$ is a small parameter.
As we have seen in the previous section, one of the motivations comes from superconductivity but this is also one of the basic problems in the literature in mathematical physics in connexion with semi-classical problems (h being in this context the Planck constant). If one can naturally refer to Kato and, at the end of the seventie's, to Avron-Herbst-Simon [AHS] or Combes-Schrader-Seiler [CSS] for the mathematical analysis of the problem, the implementation of semi-classical techniques for the analysis of the ground state appears first in [HeSj4] and then in [HeMo2]. Very roughly, it is shown in [HeMo2] that, if $\Omega = I\!R^n$,

$h|\mathrm{curl}\, A(x)|$ plays the role of an effective electric potential. By this we mean that the analysis of the operator : $-h^2\Delta + h|B(x)|$ can give a good intuition for the localization of the ground state. The specialists in PDE will recognize that this problem is strongly related to the so-called Garding-Melin-Hörmander (cf [Mel], [HeRo]) inequalities for the operator :

$$P_{1,A,D_t} = \sum_j (\, D_{x_j} - A_j D_t)^2\, ,$$

in $\Omega \times I\!\!R$.
The boundary case was less analyzed. Of course the case of the Dirichlet realization does not lead to really new phenomena in comparison with the case $\Omega = I\!\!R^n$, at least if the condition

$$b < b'\, , \tag{2.1}$$

is satisfied, where we used the notations :

$$\inf_{x\in\bar\Omega} |B(x)| = b\; , \; \inf_{x\in\partial\Omega} |B(x)| = b'\; . \tag{2.2}$$

So it is not too difficult to prove

Theorem 2.1 .
Under assumption (2.1), the smallest eigenvalue $\lambda^{(1)}(h)$ of the Dirichlet realization $P^D_{h,A,\Omega}$ of $P_{h,A,\Omega}$ satisfies :

$$\frac{\lambda^{(1)}(h)}{h} = b + o(1)\; . \tag{2.3}$$

The points where the minima of $|B|$ are sometimes called magnetic wells for the energy b. The decay of the ground state outside the wells can be estimated (cf [Br], [HeNo2]) as a function of the Agmon distance associated to the so called Agmon metric $(|B| - b)dx^2$, where dx^2 denotes the euclidean metric. Note that this metric is degenerate.
We recall that this estimate is very easy to get in the special case when $n = 2$ and when the magnetic field has a constant sign. One can immediately write the inequality :

$$\pm h \int B(x)|u(x)|^2 \leq \langle P_{h,A}\, u \mid u\rangle\; ,\; \forall u \in C_0^\infty(\Omega)\; . \tag{2.4}$$

Here $\langle \cdot \mid \cdot\rangle$ denotes the scalar product in $L^2(\Omega)$ and $||\;\cdot\;||$ the corresponding norm.
In the general case, one can get a similar result but with a remainder in $\mathcal{O}(h^{\frac{5}{4}})||u||^2$ (cf [HeMo2], Theorem 3.1).

As in the case when $A = 0$ but when an electric potential V is added, it is possible to discuss the various asymptotics in function of the properties of B near the minima (cf [HeMo2, Mon, Sh, Ue1, Ue2] or more recently [KwPa2]).

As we shall see later, this property is no more true in the case of the Neumann realization. The infimum b of $|B(x)|$ on $\bar{\Omega}$ is not necessarily the right quantity for analyzing the bottom of the spectrum as (2.1) is satisfied. Of course, by direct comparison of the variational spaces corresponding to Dirichlet and Neumann, one knows that the smallest eigenvalue $\mu^{(1)}(h)$ of the Neumann realization $P^N_{h,A,\Omega}$ of $P_{h,A,\Omega}$ is smaller than $\lambda^{(1)}(h)$ (but the lower bound (2.3) is not correct in general).

One important theorem that we would like to present is

Theorem 2.2 .
If the magnetic field is constant and not zero, then any ground state corresponding to the Neumann realization $P^N_{h,A,\Omega}$ is localized as $h \to 0$ near the boundary of Ω.

This theorem is general and does not depend on the dimension.

The second theorem (cf [LuPa5], [HeMo3]) is specific of the case of dimension 3. Let us introduce in $\partial\Omega$ the subset

$$\Gamma_B = \{x \in \partial\Omega \mid \langle B \mid N(x)\rangle = 0 \}\ , \tag{2.5}$$

where $N(x)$ is the normal at x to $\partial\Omega$, that is the set of points in $\partial\Omega$ where the magnetic field $B = (B_1, B_2, B_3)$ is tangent to $\partial\Omega$.

Theorem 2.3 .
Let us assume that $n = 3$. If the magnetic field is constant and if its intensity b is not zero, then any ground state corresponding to the Neumann realization $P^N_{h,A,\Omega}$, is localized as $h \to 0$ in Γ_B.

The typical case is the case of the ellipsoid : $\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 \leq 1$. In this case, the set Γ_B is the intersection of the boundary with the plane : $\alpha x_1 B_1 + \beta x_2 B_2 + \gamma x_3 B_3 = 0$.

These two theorems are not satisfactory in the sense that they are not necessarily optimal. In the case $n = 2$, we can state [HeMo2] :

Theorem 2.4 .
Let us assume that $n = 2$. If the magnetic field is constant and not zero, then any ground state corresponding to the Neumann realization is localized as $h \to 0$ near the boundary of Ω at the points of maximal curvature.

This gives the general answer for the case of dimension 2. The case of dimension 3 was more difficult and only solved quite recently. Although the methods of proof can also lead to localization results for the ground state (see [HeMo2], [HeMo3]) or more generally for minimizers of the Ginzburg-Landau functional (see [LuPa1]-[LuPa5], [HePa]), this will not be discussed here. This is actually explored in [Pan3].

The understanding of these results comes from the analysis of various reference models.

3 Models with constant magnetic fields in dimension 2

3.1 Preliminaries.

Let us consider in a regular domain Ω in $I\!R^2$ the Neumann realization (or the Dirichlet realization) of the operator $P_{h,bA_0,\Omega}$ with

$$A_0(x_1, x_2) = (\frac{1}{2}x_2, -\frac{1}{2}x_1) \ . \tag{3.1}$$

Note that by Neumann realization, we mean that the following boundary condition :

$$\nu \cdot (h\nabla - ibA_0)u = 0 \text{ on } \partial\Omega \ , \tag{3.2}$$

is satisfied and we recall that this was the natural condition considered in superconductivity (compare with the third line of (1.4)). We will assume $b > 0$ and we observe that the problem has a strong scaling invariance :

$$P_{h,bA_0} = h^2 P_{1,bA_0/h} \ . \tag{3.3}$$

As a consequence, the semi-classical analysis (b fixed) is equivalent to the analysis strong magnetic field (h being fixed). If the domain is invariant by dilation, one can reduce the analysis to $h = b = 1$. Let us denote by $\mu^{(1)}(h, b, \Omega)$ and by $\lambda^{(1)}(h, b, \Omega)$ the bottom of the spectrum of the Neumann and Dirichlet realizations of P_{h,bA_0} in Ω. Depending on Ω, this can correspond to an eigenvalue (if Ω is bounded) or to a point in the essential spectrum (for example if $\Omega = I\!R^2$ or if $\Omega = I\!R^2_+$). The analysis of basic examples will be crucial for the general study of the problem.

3.2 The case $\Omega = I\!R^2$.

We first recall that :

$$\mu^{(1)}(h, b, I\!R^2) = \lambda^{(1)}(h, b, I\!R^2) = hb \ . \tag{3.4}$$

The level hb is an eigenvalue of infinite multiplicity. It is called in the physical literature the first Landau level.

Note that by domain monotonicity, one can show that in the case of Dirichlet :

$$\lambda^{(1)}(h, b, \Omega) \geq \lambda^{(1)}(h, b, I\!R^2) = hb \ . \tag{3.5}$$

Such a lower bound is wrong in the Neumann case and this will make the problem more interesting.

3.3 The case $\Omega = I\!R^2_+$

The analysis in $I\!R^2_+ = \{(x_1, x_2) \in I\!R^2 | x_2 > 0\}$ is less known but will play an important role in the discussion.
Let us first observe that, by construction of quasimodes and by (3.5) :

$$\lambda^{(1)}(h, b, I\!R^2_+) = hb\lambda^{(1)}(1, 1, I\!R^2_+) = hb \; . \tag{3.6}$$

Let us now consider the Neumann case (with also in mind in parallel the Dirichlet case). After elementary unitary transforms, it is enough to analyze :

$$\widetilde{S} = D^2_{x_2} + (D_{x_1} - x_2)^2 \; , \tag{3.7}$$

with Neumann or Dirichlet conditons. Note that we have now the standard Neumann condition on $x_2 = 0$.
For realizing the spectral analysis of this operator, one can then use a partial Fourier transform, and after a change of notations, we get :

$$\widehat{S}(t, \sigma, D_t) = D^2_t + (\sigma - t)^2 \tag{3.8}$$

considered as an operator defined on $I\!R^2_+ = \{(t, \sigma) \in I\!R^2 \mid t > 0\}$ with Neumann or Dirichlet conditions on $t = 0$.
The spectral analysis is then reduced to the analysis of a family (parametrized by σ) of differential operators $\widehat{S}(t, \sigma, D_t)$, which are now considered as defined on $I\!R^+$ with Neumann or Dirichlet conditions on $t = 0$, and denoted respectively by $H^{N,\sigma}$ and $H^{D,\sigma}$. The "Dirichlet" spectrum of P_{1,A_0} is then described as

$$\mathrm{Sp}\,(P^D_{1,A_0}) = \{\lambda \in I\!R^+ | \exists \sigma \in I\!R \text{ s. t. } \lambda \in \mathrm{Sp}\,(H^{D,\sigma})\} \; , \tag{3.9}$$

and a similar expression can be obtained for the "Neumann" spectrum. The eigenvalues of $H^{D,\sigma}$ and $H^{N,\sigma}$ are locally described by analytic non constant functions of σ. One consequently obtains a continuous spectrum (cf Reed-Simon, Vol. IV [ReSi]). The determination of the bottom of the spectrum is then obtained by analyzing the variation of the smallest eigenvalue (= ground state energy) of $H^{D,\sigma}$ or $H^{N,\sigma}$ as a function of σ.
Here the answer depends strongly on the boundary condition. For Dirichlet, the smallest eigenvalue $\lambda_+(\sigma)$ is monotonically decreasing from $+\infty$ to 1 as σ goes from $-\infty$ to $+\infty$.
For Neumann, the smallest eigenvalue $\mu_+(\sigma)$ is monotonically decreasing from $+\infty$ to 1 as σ goes from $-\infty$ to 0. One can then show [DaHe] that there exists a unique $\sigma_0 > 0$ such that $\mu_+(\sigma)$ is decreasing for $\sigma \in]-\infty, \sigma_0]$ till

$$\Theta_0 := \mu_+(\sigma_0) < 1 \tag{3.10}$$

and is then monotonically increasing to 1 as $\sigma \to +\infty$. Moreover, its minimum is non degenerate :

$$\delta_0 = \frac{1}{2}\mu''_+(\sigma_0) > 0 \; . \tag{3.11}$$

We then obtain :

Proposition 3.1 .
The Dirichlet spectrum in $I\!R^2_+$ *is given by :*

$$\mathrm{Sp}\,(P^D_{h,bA_0,I\!R^2_+}) = [bh, +\infty[\,. \tag{3.12}$$

The Neumann spectrum in $I\!R^2_+$ *is given by :*

$$\mathrm{Sp}\,(P^N_{h,bA_0,I\!R^2_+}) = [b\Theta_0 h, +\infty[\,, \tag{3.13}$$

with

$$0 < \Theta_0 < 1\,. \tag{3.14}$$

This property that the bottom of the Neumann spectrum is strictly below the bottom of the Dirichlet spectrum plays a fundamental role in all the discussion.

3.4 The case of the corner

After preliminary results devoted to the case $\Omega = I\!R_+ \times I\!R_+$ and obtained by [Ja] and [Pan1], a more systematic analysis have been performed by V. Bonnaillie in [Bon]. Let us mention her main results. We consider the Neumann realization of the Schrödinger operator with $h = 1$, $b = 1$ in a sector $\Omega_\alpha : \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leq \ \mathrm{tg}\,\frac{\alpha}{2} x_1\}$. One can first show, using Persson's Theorem (see for example [Ag]) that the bottom of the essential spectrum is equal to Θ_0. So the question is to know if there exists an eigenvalue below the essential spectrum. The first result obtained in [Bon] is that :

$$\lim_{\alpha \to 0} \frac{\mu^{corn}(\alpha)}{\alpha} = \frac{1}{\sqrt{3}}\,. \tag{3.15}$$

She actually obtains a complete expansion in powers of α. In addition, as byproduct of the proof, she has the universal estimate

$$\mu^{corn}(\alpha) \leq \frac{\alpha}{\sqrt{3}}\,. \tag{3.16}$$

This answers to the question about the existence of an eigenvalue below Θ_0 under the condition that

$$\frac{\alpha}{\sqrt{3}} < \Theta_0\,.$$

This does not permit to recover the case $\alpha = \frac{\pi}{2}$. Let us mention the following interesting conjecture :

Conjecture 3.2 .
The map $\alpha \mapsto \mu^{corn}(\alpha)$ *is monotonically increasing from* 0 *to* Θ_0.

3.5 The case of the disk.

The case of Dirichlet was considered by L. Erdös in connexion with an isoperimetric inequality [Er]. By using the techniques of [BoHe], one can then show [HeMo2] the following proposition which is a small improvment of his result

Proposition 3.3 .
As $R\sqrt{b}$ large, the following asymptotics holds :

$$\lambda^{(1)}(b, D(0,R)) - b \sim 2^{\frac{3}{2}}\pi^{-\frac{1}{2}}b^{\frac{3}{2}}R\exp(-\frac{bR^2}{2}) \ . \tag{3.17}$$

The Neumann case is treated in the paper by Baumann-Phillips-Tang [BaPhTa] (Theorem 6.1, p. 24) (see also [PiFeSt]) who prove the

Proposition 3.4

$$\mu^{(1)}(b, D(0,R)) = \Theta_0 b - 2M_3\,\frac{1}{R}\,b^{\frac{1}{2}} + \mathcal{O}(1) \ . \tag{3.18}$$

Here we recall that Θ_0 was introduced in (3.10) and that $M_3 > 0$ is a universal constant.

Another interesting case is the exterior of the disk. One first observes that the bottom of the essential spectrum is b and one can show that as b is large, there exists at least one eigenvalue below b. One shows also in [HeMo2] that the above formula for the smallest eigenvalue is still valid when replacing $\frac{1}{R}$ by $-\frac{1}{R}$. This permits to verify that it is indeed the algebraic value of the curvature which appears for all the models.

4 Models in the half space $\mathbb{R}^3_+$

The analysis of the case of dimension 3 is quite recent. The results are obtained by [LuPa5] with some improvments in [HeMo3]. So we consider :

$$P(h,b) := h^2D_{x_1}^2 + (hD_{x_2} - B_3x_1)^2 + (hD_{x_3} - B_2x_1 - B_1x_2)^2 \ ,$$

and it is well known that in $I\!R^3$, the spectrum is given by $[b, +\infty[$ where

$$b = \sqrt{\sum_i |B_i|^2} \ .$$

The only difference is that now the spectrum is continuous and that the discrete Landau levels have disappeared.
The case of the Neumann realization in $\{x_1 > 0\}$ is more interesting. One can again by a dilation argument assume that $b = 1$, but the estimate on the bottom of the spectrum will depend on the angle ϑ between the magnetic field and the

normal to the plane $x_1 = 0$. The problem is first reduced, after a rotation respecting $\{x_1 = 0\}$, to :

$$L(\vartheta, D_t) = D_{x_1}^2 + D_{x_2}^2 + (D_{x_3} - \cos\vartheta x_1 - \sin\vartheta x_2)^2 \; .$$

and a partial Fourier transform permits to get :

$$L(\vartheta, \tau) = D_{x_1}^2 + D_{x_2}^2 + (\tau - \cos\vartheta x_1 - \sin\vartheta x_2)^2 \; ,$$

in $\{x_1 > 0\}$ with the Neumann condition at $\{x_1 = 0\}$.
One observes that the bottom of the spectrum is given by :

$$\sigma(\vartheta) := \inf \mathrm{Sp}\; (L(\vartheta, D_t)) = \inf_\tau (\inf \mathrm{Sp}\; (L(\vartheta, \tau))) \; ,$$

and it can be shown that this function is continuous, decreasing on $[0, \frac{\pi}{2}]$ from 1 to Θ_0.

This is consequently when the magnetic field is parallel to the plane $\{x_1 = 0\}$ that the bottom of the spectrum is minimal. This fact is well known by the physicists already in the sixties but was not proved. This leads the physicists, in the case of a film, to the assumption that the magnetic field is parallel to the boundary of the film. This permits a reduction to a problem in dimension 1. See [GinLa], [Bo], [BoHe].
So we have explained how one can get the following

Theorem 4.1 .
The bottom of the spectrum is minimal when the magnetic field is parallel to the boundary.

This is the main point in the proof of Theorem 2.3.

5 Models with curvature in dimension 3

In order to understand what could be the curvature effect, we present here the following simplified model :

$$P_0 := (hD_r - \sin\theta\, t)^2 + (hD_s + \cos\theta\, t + \kappa\frac{r^2}{2})^2 + h^2 D_t^2 \tag{5.1}$$

on $I\!R^2 \times I\!R^+$. Here θ is assumed to be fixed and κ is a real parameter. In the application the two parameters are not independent but it is better to keep them as an independent parameter for the first part of the analysis. We have moreover assumed for simplification that $b = 1$. We are interested in the analysis of the bottom of the spectrum and we shall just sketch how to get a rather optimal lower bound.

Scaling .
We first introduce the following scaling : $t = h^{\frac{1}{2}}\tilde{t}$, $r = h^{\frac{1}{3}}\tilde{r}$ and the coefficients of the operator being independent of s we take a partial Fourier transform in s.

Dividing by h, we get that P_0/h is unitary equivalent to :

$$P_1 := (h^{\frac{1}{6}} D_{\tilde{r}} - \sin\theta\tilde{t})^2 + (h^{\frac{1}{2}}\sigma + \cos\theta\tilde{t} + \kappa h^{\frac{1}{6}}\frac{\tilde{r}^2}{2})^2 + D_{\tilde{t}}^2 \tag{5.2}$$

on $I\!R^2 \times I\!R^+$. We can rewrite the operator in the form :

$$\begin{aligned} P_1 \quad &= D_{\tilde{t}}^2 + \left(\tilde{t} - h^{\frac{1}{6}}\left[\sin\theta D_{\tilde{r}} + \cos\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)\right]\right)^2 \\ &+ h^{\frac{1}{6}}\left(\cos\theta D_{\tilde{r}} - \sin\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 . \end{aligned} \tag{5.3}$$

We note that we have a lower bound of the type :

$$\begin{aligned} P_1 \quad &\geq \mu_+(h^{\frac{1}{6}}\sin\theta D_{\tilde{r}} + \cos\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)) \\ &+ \left(h^{\frac{1}{6}}\cos\theta D_{\tilde{r}} - \sin\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 , \end{aligned} \tag{5.4}$$

where μ_+ appears in Subsection 3.3. This leads naturally to first consider the operator :

$$\begin{aligned} P_2 \quad &:= \mu_+(h^{\frac{1}{6}}\sin\theta D_{\tilde{r}} + \cos\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)) \\ &+ \left(h^{\frac{1}{6}}\cos\theta D_{\tilde{r}} - \sin\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 , \end{aligned} \tag{5.5}$$

now considered as an operator on $L^2(I\!R^2_{\tilde{r},\sigma})$. In order to find a suitable lower bound, it is better to first decompose it as an Hilbertian integral of operators $P_2(\sigma)$, this time defined on $L^2(I\!R)$, and to look for a minimization on σ. In this context, it is natural to replace μ_+ by its quadratic approximation at the bottom. We recover a differential model, which will give a good understanding of the problem, modulo an error term which has to be controlled.
We consequently analyze the family (depending on σ)

$$\begin{aligned} P_3(\sigma) \quad &:= \Theta_0 + \delta_0(h^{\frac{1}{6}}(\sin\theta D_{\tilde{r}} + \cos\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)) - \xi_0)^2 \\ &+ h^{\frac{1}{3}}\left(\cos\theta D_{\tilde{r}} - \sin\theta(h^{\frac{1}{3}}\sigma + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 . \end{aligned} \tag{5.6}$$

with δ_0 defined in (3.11). After easy transformations, we have to analyze the minimization over $\hat{\sigma}$ of the spectrum of the family

$$\begin{aligned} P_4(\hat{\sigma}) \quad &:= \Theta_0 + h^{\frac{1}{3}}\delta_0\left(\sin\theta D_{\tilde{r}} + \cos\theta(\hat{\sigma} + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 \\ &+ h^{\frac{1}{3}}\left(\cos\theta D_{\tilde{r}} - \sin\theta(\hat{\sigma} + \tfrac{\kappa}{2}\tilde{r}^2)\right)^2 . \end{aligned} \tag{5.7}$$

It is then natural to introduce :

$$P_5(\hat{\sigma}) := h^{-\frac{1}{3}}(P_4(\hat{\sigma}) - \Theta_0) , \tag{5.8}$$

which becomes independent of h :

$$\begin{aligned} P_5(\hat{\sigma}) \quad &= \delta_0\left(\cos\theta(\tfrac{\kappa}{2}r^2 + \hat{\sigma}) + \sin\theta D_r\right)^2 \\ &+ \left(\cos\theta D_r - \sin\theta(\tfrac{\kappa}{2}r^2 + \hat{\sigma})\right)^2 . \end{aligned} \tag{5.9}$$

Here we have omitted the tilde's. We observe that this operator has the structure :

$$P_5(\hat{\sigma}) = X_1^2 + X_2^2\ , \tag{5.10}$$

where X_1 and X_2 are selfadjoint vector fields (depending on the parameters θ, $\hat{\sigma}$ and κ) :

$$X_1 = \sin\theta D_r + \cos\theta(\frac{\kappa}{2}r^2 + \hat{\sigma})\ ,\ X_2 = \cos\theta D_r - \sin\theta(\frac{\kappa}{2}r^2 + \hat{\sigma})\ . \tag{5.11}$$

Here what is important is the rank 3 Lie structure of the two vector fields. We note that :

$$[X_1, X_2] = i\kappa r\ ,\ [X_1,[X_1,X_2]] = \kappa\sin\theta\ ,\ [X_2,[X_1,X_2]] = \kappa\cos\theta \tag{5.12}$$

and that higher brackets are equal to 0. Our aim is to minimize over $\hat{\sigma}$.
One can show via elementary transformations (scaling, gauge transforms) that all the analysis can be reduced to the analysis of a new simple model introduced by Montgomery [Mon] :

$$L^M(\rho) := \left(D_r^2 + (r^2-\rho)^2\right)\ .$$

The following lemma is true (see [Mon, HeMo1, KwPa1]) :

Lemma 5.1 *The infimum over $\rho \in \mathbb{R}$ of the bottom of the spectrum of $L^M(\rho)$ is obtained for an unique strictly positive $\rho = \rho_{min}$.*

We then define

$$\hat{\nu}_0 = \inf\sigma(L^M(\rho_{min}))\ . \tag{5.13}$$

This corresponds for the initial model to

$$\hat{\sigma}_{min} = \frac{\kappa}{2}\ ((\cos^2\theta + \delta_0\sin^2\theta)^2(\delta_0(\frac{\kappa}{2})^2)^{-1})^{\frac{1}{3}}\ \rho_{min}\ . \tag{5.14}$$

This leads to,

Proposition 5.2 .
The ground state energy of the model operator P_4 is given by :

$$\inf_{\hat{\sigma}} \inf \mathrm{Sp}\, P_4(\hat{\sigma}) = \Theta_0 + h^{\frac{1}{3}}\hat{\nu}_0(\frac{1}{2})^{\frac{2}{3}}\delta_0^{\frac{1}{3}}|\kappa|^{\frac{2}{3}}(\delta_0\sin^2\theta + \cos^2\theta)^{\frac{1}{3}}\ . \tag{5.15}$$

Moreover the minimum is obtained for $\hat{\sigma} = \hat{\sigma}_{min}$ defined in (5.14).

6 Main semi-classical results for general magnetic fields

In the case of a bounded regular domain, the first result is due to Lu-Pan [LuPa2] (for the main term), is improved in [HeMo2] and is concerned with general magnetic fields :

Theorem 6.1 .

$$\frac{\mu^{(1)}(h)}{h} = \min(\Theta_0 b' \ , \ b) + \mathcal{O}(h^{\frac{1}{2}}) \ . \tag{6.1}$$

This theorem exhibits two situations depending on the sign of $b - \Theta_0 b'$. When the sign is negative, the boundary condition is not important for the determination of the asymptotics of the ground state. If one adds the assumption (dimension 2 case) that $B(x)$ has a unique non zero minimum at a point z_o of Ω, which in addition is not degenerate, then, with :

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\, B(z_0)\right)^{1/2} \ , \tag{6.2}$$

it is shown in [HeMo2] the following theorem, which completes former results of [Mon] and [HeMo1].

Theorem 6.2 .
If A is C^∞, then there exists $C > 0$ such that

$$-C\, h^{19/8} \leq \inf\ \mathrm{Sp}\,(P_{h,A,\Omega}) \ - \ [b + \frac{a^2}{2b}h]h \leq C\, h^{5/2} \ . \tag{6.3}$$

The proof is based for the upper bound on a construction of quasimodes and for the lower bound on techniques inspired by the proof of Garding- Melin inequality [Mel]. One also uses the explicit computations concerning the spectrum of the harmonic magnetic oscillator whose spectrum is for example explicitely computed in [Mat]. Note that it is also established in [HeMo1] (cf also the previous works by [Br] and [HeNo2]) that any ground state decays exponentially outside the minima of B.

Let us now consider the second case. The first result, which was already observed in [PiFeSt] is the following.

Theorem 6.3 *Boundary Localization.*
Let us assume that the condition :

$$\Theta_0 b' < b \ , \tag{6.4}$$

is satisfied. Then there exist $C > 0$, $\alpha > 0$ and $\nu \in I\!R$, such that, if u_h is a ground state of $P^N_{A,h,\Omega}$, then :

$$\exp \alpha \frac{d(x,\partial\Omega)}{h^{\frac{1}{2}}} |u_h(x)| \leq C\ h^{-\nu} ||u_h||_{L^2} \ , \ \forall x \in \Omega \ . \tag{6.5}$$

Note that the condition (6.4) is always satisfied when the magnetic field is constant. We have indeed $b = b'$ and $\Theta_0 < 1$. This gives in particular Theorem 2.2.

The next result describes the localization inside the boundary

Theorem 6.4 .
Let

$$p(\partial\Omega) = \{\omega \in \partial\Omega \mid B(\omega) = b'\} \,. \tag{6.6}$$

Let us assume that : $\Theta_0 b' < b$. *Then, there exist* $\alpha_1 > 0$ *and, for any* $\epsilon > 0$, *constants* $C(\epsilon)$ *and* $h_0(\epsilon)$ *such that a normalized ground state satisfies :*

$$|| \exp \frac{\alpha_1 \tilde{d}(z, p(\partial\Omega))}{h^{\frac{1}{2}}} u_h || \leq C(\epsilon) \exp \frac{\epsilon}{h^{\frac{1}{2}}} \,, \tag{6.7}$$

for all $h \in]0, h_0(\epsilon)]$. *Here the "distance"* $\tilde{d}$ *is defined by :*

$$\tilde{d}(z, p(\partial\Omega)) = d(z, \partial\Omega) + \chi(d(z, \partial\Omega)) d_{\partial\Omega}(s(z)) \,, \tag{6.8}$$

where $d_{\partial\Omega}$ *is the Agmon distance associated inside the boundary to the metric* $(B(s,0) - b')ds^2$ *and* χ *is positive, equal to* 1 *on* $[0, \frac{\delta_1}{2}]$ *and with suppport in* $[0, \delta_1]$.

Before to go further for the regular case, let us present a recent result concerning the domain with corners [Bon].

Theorem 6.5 .
Let Ω *a domain with* N *vertices* S_1, S_2,....,S_N, *with corresponding angle* α_1, ..., α_N. *Then the lowest eigenvalue of the Neumann realization in* Ω *satisfies*

$$\mu^{(1)}(h) = h \min \left(b, \Theta_0 b', \inf_{j=1,\cdots,N} \mu^{corn}(\alpha_j) B(S_j) \right) + \mathcal{O}(h^{\frac{5}{4}}) \,. \tag{6.9}$$

Modulo the proof of the conjecture 3.2, it says that the minimum of the main term in the right hand side of (6.9) is when B is constant obtained at the corner of smallest angle.

7 Curvature effects in the constant magnetic field case

7.1 The case in dimension 2

The next theorem was conjectured by Bernoff-Sternberg [BeSt] (see also [LuPa3] and [PiFeSt]) and its proof was achieved recently in a recent paper of the author with A. Morame [HeMo2]

Theorem 7.1 .
Let Ω *a regular bounded domain in* $\mathbb{R}^2$. *Let us assume that the magnetic field is constant and* > 0. *Then we have :*

$$\mu^{(1)}(h) \sim \Theta_0 h - 2 c_{max} M_3 h^{\frac{3}{2}} + \mathcal{O}(h^{\frac{5}{3}}) \,. \tag{7.1}$$

where M_3 *is the universal constant appearing in the case of the disk.*

The proof of the upper bound was given (at least formally) in Bernoff-Sternberg [BeSt] who construct approximate solutions. Lower bounds with remainders are proposed in [LuPa1] and [PiFeSt].The best remainder in the general case is obtained in [HeMo2] (who get a control in $\mathcal{O}(h^{\frac{7}{4}})$). The lower bound is based on the introduction of a fine partition of unity and after this localization on the results obtained for suitable models. Note that the proof of the lower bound in Theorem 7.1 gives simultaneously the localization the ground state.

7.2 The case in dimension 3

For this we need to introduce various notions and assumptions. It is natural to assume that the set Γ_B introduced in 2.5 :

$$\Gamma_B \text{ is a regular submanifold of } \partial\Omega\ . \tag{7.2}$$

At each point x of Γ_B, we define the normal curvature along the magnetic field B by :

$$\kappa_{n,B}(x) :=\ K_x(T(x)\wedge N(x)\ ,\ \frac{B}{|B|})\ , \tag{7.3}$$

where K denotes the second fundamental form on the surface $\partial\Omega$ and $T(x)$ is the unit oriented tangent vector to Γ_B at x. It is natural to assume that :

$$\kappa_{n,B} \neq 0\ ,\quad \text{on } \Gamma_B\ . \tag{7.4}$$

The last generic assumption is :

$$\text{The set of points where } B \text{ is tangent to } \Gamma_B \text{ is isolated.} \tag{7.5}$$

The following function on Γ_B will play an important role :

$$\tilde{\gamma}_0(x) = (\frac{1}{2})^{\frac{2}{3}}\hat{\nu}_0\delta_0^{\frac{1}{3}}\left(|\kappa_{n,B}(x)|^{\frac{2}{3}}\left(\delta_0+(1-\delta_0)|\langle T(x)\,|\,\frac{B}{|B|}\rangle|^2\right)^{\frac{1}{3}}\right)\ , \tag{7.6}$$

where $\hat{\nu}_0 > 0$ and $\delta_0 \in]0,1[$ are universal constants attached to spectral invariants related to two model Hamiltonians respectively defined on $\mathbb{R}$ and $\mathbb{R}^+$ and which will be defined in (5.13) and (3.11).

Associated to this function, which will play the role of effective curvature, we define :

$$\hat{\gamma}_0 = \inf_{x\in\Gamma_B}\tilde{\gamma}_0(x)\ . \tag{7.7}$$

Our main theorem, also obtained in collaboration with A. Morame, is :

Theorem 7.2 .
Let $P^N_{h,A,\Omega}$ be the Neumann realization on $L^2(\Omega)$ of the magnetic Laplacian $(hD-A)^2$, where $h\in]0,1[$ is a small parameter and A corresponds to constant magnetic field. Under assumptions (7.2), (7.4) and (7.5) , there exists $\eta>0$ and $\hat{\gamma}_0>0$ such that :

$$\inf\ \mathrm{Sp}\ (P^N_{h,A,\Omega}) = b\Theta_0 h + \hat{\gamma}_0 b^{\frac{2}{3}}h^{\frac{4}{3}} + \mathcal{O}(h^{\eta+\frac{4}{3}})\ , \tag{7.8}$$

where $\hat{\gamma}_0$ is defined in (7.7).

The proof used again a partition of unity and a fine comparison with models which were described above. When we are near a point x of Γ_B, we shall use the model described in Section 6 with $\kappa = \kappa_{n,B}(x)$ and $\theta = \theta(x)$, where $\theta(x)$ is the angle between B and $T(x)$. The expression appearing in (7.6) is then equal to (5.15).

8 Conclusion

The question of the effective localization when there is more than one point of maximal curvature remains open. One recovers here problems which are analogous to the problem of miniwells for the Schrödinger operator that was analyzed in collaboration with J. Sjöstrand [HeSj1]-[HeSj3].

As a result of the analysis which is presented here, we note that the exterior problem is also interesting. In the semi-classical limit, we obtain the existence of an interval $[\Theta_0\, b\, h + \mathcal{O}(h^{\frac{3}{2}})\ ,\ b\, h[$, in which the Neumann spectrum is discrete and corresponds to states which are localized near the boundary.

As suggested by J. Sjöstrand (and this seems to be used in the numerical computations by K. Hornberger and U. Smilansky [HoSm] for a different but connected problem-these authors look at highly excites states-), it would be interesting to develop a general theory of reduction to the boundary.

Note also that it would be interesting to explore a non homogeneous situation (at the origine for example of the pinning of the vortices see [AfSaSe]) and corresponding to the analysis of the functional

$$\mathcal{G}(\psi\ ,\ A) = \int_\Omega \{|(\nabla - i\kappa A)\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - a(x))^2\}\ dx + \kappa^2 \int_\Omega |\mathrm{curl}\, A - \mathcal{H}|^2\ . \quad (8.1)$$

Here $a(x)$ is a function describing the inhomogeneity of the sample which satisfies $0 < \alpha_0 \leq a(x) \leq 1$.

Finally, we have shown that the results presented here can be used in the non-linear case and give good informations on the nucleation. There is still a lot to do in continuation of the works by Helffer-Pan [HePa], Pan [Pan2], Sandier-Serfaty [SaSe4]....

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