# One Time-step Finite Element Discretization of the Equation of Motion of Two-fluid Flows 

V. Girault, ${ }^{1}$ H. López, ${ }^{2}$ B. Maury ${ }^{3}$<br>${ }^{1}$ Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 75252 Paris Cedex 05, France<br>${ }^{2}$ Centro de Cálculo Cientifico y Tecnológico, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela<br>${ }^{3}$ Laboratoire de Mathématiques, Université Paris-Sud, Campus d’Orsay, 91405 Orsay Cedex, France

Received 11 February 2005; accepted 18 July 2005
Published online 6 September 2005 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/num. 20117


#### Abstract

We discretize in space the equations obtained at each time step when discretizing in time a Navier-Stokes system modelling the two-dimensional flow in a horizontal pipe of two immiscible fluids with comparable densities, but very different viscosities. At each time step the system reduces to a generalized Stokes problem with nonstandard conditions at the boundary and at the interface between the two fluids. We discretize this system with the "mini-element" and establish error estimates. © 2005 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 22: 680-707, 2005


Keywords: two-fluid flow; Stokes system; free-surface boundary conditions; surface tension; curved boundary; "mini-element"

## INTRODUCTION

This work is devoted to the numerical solution of the equations obtained at each time step when discretizing the lubricated transportation of heavy crude oil in a horizontal pipeline. In the petroleum industry, an efficient way for transporting heavy crude oil in pipelines is by injecting water under pressure along the inner wall of the pipeline. The water acts as a lubricant by coating the wall of the pipeline, thus preventing the oil from adhering to the pipe. This behavior is made possible by

[^0]the facts that both fluids are immiscible and the oil is much more viscous than the water while both have comparable densities. For more details, the reader can refer to Joseph and Renardy [1].

The full problem is that of a three-dimensional flow in a cylindrical pipe of two immiscible fluids, water and oil, governed by the transient Navier-Stokes equations. On entering the pipe, the fluid with low viscosity (water) is adjacent to the pipe wall and it surrounds the fluid with high viscosity (heavy oil). It is assumed that the flow is sufficiently smooth so that this situation holds until a certain time $T$, and so that the interface between the two fluids, which is a free surface, can be suitably parametrized and is never adjacent to the pipe wall. The equation of the free surface is given by a transport equation and the transmission conditions on the interface are as follows:

1. the continuity of the velocity;
2. the balance of the normal stress with the surface tension.

On account of the above assumption, that rules out the phenomenon known as fingering, an Arbitrary Lagrangian-Eulerian method of approximation (ALE), which is based on a grid that moves with the fluid, appears to be well-adapted to the numerical solution of this problem. For discretizing the nonlinear convection term, we choose a method of characteristics. In addition to its good stability properties, it adapts easily to the ALE method. At each time step, the ALE and characteristics methods lead to a generalized Stokes problem with nonstandard boundary conditions. The object of this article is to apply and analyse the "mini-element" for solving numerically this problem, in the simplified situation of a horizontal pipeline in two dimensions. We shall not address here the specific properties of the ALE time discretization scheme. The important feature of this method, that plays a crucial role in our approach, is that the mesh follows the motion of the boundary. Thus the interface is approximated by a broken line, in contrast to approximations by Eulerian methods.

In the case of a two-dimensional horizontal pipeline, we can take advantage of symmetry and consider only one half of the domain, say the upper half, that we denote by $\Omega$. Then $\Omega$ is subdivided into two subdomains, $\Omega^{1}$ containing oil and $\Omega^{2}$ containing water (cf. Fig. 1), separated by the free surface $\Gamma$. The generalized Stokes equations have the form, in each $\Omega^{i}, i=1,2$ :

$$
\left\{\begin{align*}
\alpha \rho^{i} \mathbf{u}^{i}-\mu^{i} \Delta \mathbf{u}^{i}+\nabla p^{i} & =\rho^{i} \mathbf{g}+\alpha \rho^{i} \mathbf{w} & & \text { in } \Omega^{i},  \tag{0.1}\\
\nabla \cdot \mathbf{u}^{i} & =0 & & \text { in } \Omega^{i},
\end{align*}\right.
$$

with the interface conditions on $\Gamma$ :

$$
\begin{equation*}
\mathbf{u}^{1}=\mathbf{u}^{2},[\sigma]_{\Gamma} \cdot \mathbf{n}^{1}=-K \mathbf{n}^{1} \tag{0.2}
\end{equation*}
$$

and appropriate inflow and outflow conditions on the vertical boundaries of $\Omega$, a no-slip boundary condition on the top horizontal boundary of $\Omega$, and an artificial symmetry condition on the bottom horizontal boundary of $\Omega$. Here $\alpha$ stands for $1 / \delta t, \rho^{i}$, and $\mu^{i}$ are the given (positive) density and viscosity constants, $\mathbf{g}$ is the gravity, $\mathbf{w}$ is a given function that takes into account the convection of the previous velocity, $\mathbf{n}^{1}$ is the unit normal on $\Gamma$ to $\Omega^{1}$, pointing inside $\Omega^{2}, \sigma$ is the stress tensor, $[\cdot]_{\Gamma}$ denotes the jump across $\Gamma$ in the direction of $\mathbf{n}^{1}$, and $K$ is a given function that takes into account the surface tension.

The solvability of problems with a free surface submitted to surface tension has been addressed by many authors. For instance, as far as a single material is concerned, the reader will find in Saavedra and Scott [2] a variational formulation, theoretical analysis, numerical approximation, and error analysis for a steady elliptic equation with a free-surface boundary condition in the
presence of surface tension. Existence and uniqueness of the solution of a Stefan problem with surface tension is addressed by Friedman and Reitich [3] and by Solonnikov [4], but these are not flow problems.

Concerning the flow of a single fluid with a free surface or two immiscible fluids under surface tension, we refer for example to Solonnikov [4-6], Friedman and Velázquez [7, 8], or Socolowsky [9], but either the domain considered is unbounded in at least one direction, or its boundary is very smooth, a situation that does not cover lubricated transport of oil.

Apart from the above-mentioned reference [1], there are also many publications on the numerical solution of lubricated transport of oil such as for instance the work of Li and Renardy [10], but to our knowledge, the ALE method has not been applied before to solve this problem numerically.

This work is organized as follows. In Section 1, we recall the fully nonlinear equations, we describe their time discretization by the ALE method and the linear problem that must be solved at each time step. In Section 2, we recall some results of functional analysis for handling this linear problem, we write it in a variational form, and we establish that it is well-posed. In Section 3, this problem is discretized with the "mini-element," we analyze its error and we prove that this method is of order one-half, when taking into account the approximation error of the surface tension and free surface, and of order one without considering the error arising from the approximation of the surface tension and the free surface. Some more technical results are established in an Appendix at the end of this work.

We finish this introduction by recalling the notation that is used in the sequel. We shall use the standard Sobolev spaces, such as (cf. Adams [11] or Nečas [12]):

$$
H^{m}(\Omega)=\left\{v \in L^{2}(\Omega) ; \partial^{k} v \in L^{2}(\Omega) \forall|k| \leq m\right\}
$$

where $|k|=k_{1}+k_{2}$ with ( $k_{1}, k_{2}$ ) a pair of non-negative integers (in two dimensions) and

$$
\partial^{k} v=\frac{\partial^{|k|} v}{\partial x_{1}^{k_{1}} \partial x_{2}^{k_{2}}}
$$

This space is equipped with the seminorm

$$
|v|_{H^{m}(\Omega)}=\left[\sum_{|k|=m} \int_{\Omega}\left|\partial^{k} v\right|^{2} d \mathbf{x}\right]^{1 / 2}
$$

and is a Hilbert space for the norm

$$
\|v\|_{H^{m}(\Omega)}=\left[\sum_{0 \leq k \leq m}|v|_{H^{k}(\Omega)}^{2}\right]^{1 / 2}
$$

The scalar product of $L^{2}(\Omega)$ is denoted by $(\cdot, \cdot)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation.

Recall that $\mathcal{D}(\Omega)$ denotes the space of indefinitely differentiable functions with compact support in $\Omega, \mathcal{D}^{\prime}(\Omega)$ denotes the dual space of $\mathcal{D}(\Omega)$ and $\mathcal{D}(\bar{\Omega})$ coincides with $\mathcal{C}^{\infty}(\bar{\Omega})$. We refer to Lions and Magenes [13] for the definition of fractional-order spaces such as $H^{s}(\Omega)$, where $s$ is a real number. In particular, we denote by $H^{1 / 2}(\partial \Omega)$ the space of traces of functions of $H^{1}(\Omega)$ on the boundary $\partial \Omega$ and by $H^{-1 / 2}(\partial \Omega)$ its dual space. The trace $\gamma$ is a continuous mapping from $H^{1}(\Omega)$ onto $H^{1 / 2}(\partial \Omega)$ and there exists a constant $C$ such that

$$
\forall v \in H^{1}(\Omega), \quad\|\gamma v\|_{H^{1 / 2}(\partial \Omega)} \leq C\|v\|_{H^{1}(\Omega)}
$$

Finally, recall Poincaré's Inequality valid in the following subspace of $H^{1}(\Omega)$ : let $\Gamma$ be a part of $\partial \Omega$ with positive measure, $|\Gamma|>0$, and let

$$
H_{0, \Gamma}^{1}(\Omega)=\left\{v \in H^{1}(\Omega) ;\left.v\right|_{\Gamma}=0\right\} .
$$

Then there exists a constant $\mathcal{P}$, depending only on $\Omega$ and $\Gamma$ such that

$$
\begin{equation*}
\|v\|_{L^{2}(\Omega)} \leq \mathcal{P}\|\nabla v\|_{L^{2}(\Omega)} . \tag{0.3}
\end{equation*}
$$

Therefore, we equip $H_{0, \Gamma}^{1}(\Omega)$ with the seminorm $\|\nabla v\|_{L^{2}(\Omega)}=|v|_{H^{1}(\Omega)}$.

## 1. THE TWO-PHASE FLOW MODEL

Let us consider the $2-D$ flow illustrated by Fig. 1 that depicts the upper half $\Omega$ of the domain of interest.

For each time $t \in[0, T]$, the domain $\Omega$ is decomposed into two moving subdomains $\Omega^{1}(t)$ and $\Omega^{2}(t)$, with boundary

$$
\begin{equation*}
\partial \Omega^{i}(t)=\Gamma_{\mathrm{in}}^{i} \cup \Gamma_{0}^{i} \cup \Gamma_{\mathrm{out}}^{i}(t) \cup \Gamma(t), \quad i=1,2, \tag{1.1}
\end{equation*}
$$

where $\Gamma_{\text {in }}=\Gamma_{\text {in }}^{1} \cup \Gamma_{\text {in }}^{2}$ denotes the inlet boundary, $\Gamma_{\text {out }}=\Gamma_{\text {out }}^{1} \cup \Gamma_{\text {out }}^{2}$ the outlet boundary, $\Gamma_{0}^{2}$ the upper pipeline boundary, $\Gamma_{0}^{1}$ the artificial boundary in the middle of the pipeline, $\Omega^{1}(t)$ is the region occupied by the high-viscosity fluid (oil) and $\Omega^{2}(t)$ that occupied by the low-viscosity fluid (water). As stated in the introduction, it is assumed that the interface between the two fluids: $\Gamma(t)=\overline{\Omega^{1}}(t) \cap \overline{\Omega^{2}}(t)$, can be parametrized by a function $(x, t) \mapsto \Phi(x, t)$ such that the subdomains can be written

$$
\begin{array}{ccc}
\Omega^{1}(t)=\{(x, y) \in \Omega, & 0 \leq x \leq L, & 0 \leq y<\Phi(x, t)\}, \\
\Omega^{2}(t)=\{(x, y) \in \Omega, & 0 \leq x \leq L, & \Phi(x, t)<y \leq D\}, \tag{1.3}
\end{array}
$$

where $D$ denotes the radius of the pipeline and $L$ its length.


FIG. 1. Geometry.

To describe the density and viscosity, we introduce the piecewise constant quantities $\rho$ and $\mu$ defined by:

$$
\begin{equation*}
\rho=\chi^{1} \rho^{1}+\chi^{2} \rho^{2}, \quad \mu=\chi^{1} \mu^{1}+\chi^{2} \mu^{2} \tag{1.4}
\end{equation*}
$$

where $\chi^{i}$ is the characteristic function of the domain $\Omega^{i}, \rho^{i}$ are the constant densities, and $\mu^{i}$ the constant viscosities, for $i=1,2$. To denote the velocity and pressure, we set

$$
\mathbf{u}=\mathbf{u}^{i}=\left(u_{x}^{i}, u_{y}^{i}\right), \quad p=p^{i} \text { in } \Omega^{i}, \quad i=1,2 .
$$

Then for each $t \in[0, T]$, the fluids must satisfy the following equations (to simplify, we suppress the dependence on $t$ ):

$$
\left\{\begin{align*}
\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \nabla \mathbf{u}\right)-\mu \Delta \mathbf{u}+\nabla p & =\rho \mathbf{g} & & \text { in each } \Omega^{i}, \quad i=1,2  \tag{1.5}\\
\nabla \cdot \mathbf{u} & =0 & & \text { in } \Omega,
\end{align*}\right.
$$

where $\mathbf{g}$ is the gravity and

$$
\mathbf{u} \cdot \nabla \mathbf{u}=\sum_{i=1}^{2} u_{i} \frac{\partial \mathbf{u}}{\partial x_{i}}
$$

The equation for the motion of the free surface $\Gamma$, stating the immiscibility of the fluids, is

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+u_{x} \frac{\partial \Phi}{\partial x}=u_{y} . \tag{1.6}
\end{equation*}
$$

The Equations (1.5) are complemented by an adequate initial condition, the following boundary conditions:

$$
\left\{\begin{align*}
\mathbf{u} & =\mathbf{U} & & \text { on } \Gamma_{\text {in }}  \tag{1.7}\\
\mathbf{u}^{2} & =\mathbf{0} & & \text { on } \Gamma_{0}^{2} \\
\mathbf{u}^{1} \cdot \mathbf{n} & =0 & & \text { on } \Gamma_{0}^{1} \\
\mathbf{t} \cdot \sigma^{1} \cdot \mathbf{n} & =0 & & \text { on } \Gamma_{0}^{1} \\
\sigma \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n} & & \text { on } \Gamma_{\text {out }},
\end{align*}\right.
$$

and interface conditions (continuity of the velocity and balance of the normal stress with the surface tension, across the interface)

$$
\begin{equation*}
[\mathbf{u}]_{\Gamma}=\mathbf{0}, \quad[\sigma]_{\Gamma} \cdot \mathbf{n}^{1}=-\frac{\kappa}{R} \mathbf{n}^{1} \tag{1.8}
\end{equation*}
$$

where $\mathbf{U}=\mathbf{U}^{i}$ on $\Gamma_{\text {in }}^{i}$ for $i=1,2$ denotes the given inlet velocity independent of time, $p_{\text {out }}$ a given exterior pressure on the outlet boundary, $\mathbf{n}$ is the unit exterior normal vector to the boundary of $\Omega, \mathbf{t}$ is the unit tangent vector to $\Gamma_{0}^{1}$, pointing in the direction of increasing $x$ (i.e., in the counterclockwise direction), $\mathbf{n}^{1}$ is the normal to $\Gamma$, exterior to $\Omega^{1},[\cdot]_{\Gamma}$ denotes the jump on $\Gamma$ in the direction of $\mathbf{n}^{1}$ :

$$
[f]_{\Gamma}=\left.f\right|_{\Omega^{1}}-\left.f\right|_{\Omega^{2}}
$$

$\kappa>0$ is a given constant, $R$ is the radius of curvature with the appropriate sign, i.e., with the convention that $R>0$ if the center of curvature of $\Gamma$ is located in $\Omega^{1}$, and the stress tensor $\sigma$ satisfies the constitutive equation of a Newtonian fluid:

$$
\sigma=\sigma(\mathbf{u}, p)=\mu A_{1}(\mathbf{u})-p I=\mu\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right)-p I .
$$

We assume that the inlet velocity $\mathbf{U}$ has the form:

$$
\begin{equation*}
\mathbf{U}=-U(y) \mathbf{n}=(U(y), 0)^{t}, U(y) \geq 0 \tag{1.9}
\end{equation*}
$$

i.e., the inlet velocity is parallel to the normal vector $\mathbf{n}$ and is directed inside $\Omega$. Moreover, we assume that $U(D)=0$; thus $\mathbf{U}$ satisfies the compatibility conditions:

$$
\begin{equation*}
\mathbf{U}^{2}\left(\Gamma_{0}^{2} \cap \Gamma_{\text {in }}^{2}\right)=\mathbf{0}, \quad \mathbf{U}^{1} \cdot \mathbf{t}^{1}\left(\Gamma_{\text {in }}^{1} \cap \Gamma_{0}^{1}\right)=0, \tag{1.10}
\end{equation*}
$$

where $\mathbf{t}^{1}$ is the unit tangent vector to $\Gamma_{\text {in }}^{1}$ (i.e., in the direction of the normal to $\Gamma_{0}^{1}$ ). Finally, (1.6) is complemented by the initial and boundary conditions,

$$
\begin{equation*}
\forall x \in[0, L], \quad \Phi(x, 0)=y_{0}, \forall t \in[0, T], \quad \Phi(0, t)=y_{0}, \tag{1.11}
\end{equation*}
$$

where $\left.y_{0} \in\right] 0, D[$ is a given constant.

## A. Semi-discretization in Time

The ALE method is based on the definition at all times $t$ of the domain's velocity of displacement. The trajectory of the fluid particles is not based on the velocity $\mathbf{u}$ : it is based on the relative velocity of the fluid with respect to that of the domain. This is explained briefly below; the reader can refer to Maury [14] for more details. As mentioned in the introduction, we choose to discretize the nonlinear convection term in the Navier-Stokes equations (1.5) by the method of characteristics. This is not the only possible choice, but it is motivated by the fact that it adapts easily to the ALE method. Indeed, the position $\mathbf{x}$ of a particle of fluid at time $t$ is a function of $t$, and the expression

$$
\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \nabla \mathbf{u}
$$

is in fact the total derivative $d \mathbf{u} / d t$. Now, if the domain is moving with velocity $\mathbf{c}$, the relative velocity of the particle is $\mathbf{u}-\mathbf{c}$. Therefore, we replace

$$
\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \nabla \mathbf{u} \quad \text { by } \quad \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}-\mathbf{c}) \cdot \nabla \mathbf{u}
$$

and we use the approximation

$$
\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}-\mathbf{c}) \cdot \nabla \mathbf{u} \simeq \frac{1}{\delta t}(\mathbf{u}(\mathbf{x}, t+\delta t)-\mathbf{u}(\mathbf{X}, t)),
$$

where $\mathbf{X}$ is the foot of the characteristic at time $t$, convected by $\mathbf{u}-\mathbf{c}$. Thus, in the combined ALE-characteristics method, each equation of (1.5) is discretized by

$$
\begin{equation*}
\rho \frac{1}{\delta t}\left(\mathbf{u}_{m}^{m+1}-\mathbf{u}_{m}\left(\mathbf{X}^{m}\right)\right)-\mu \Delta \mathbf{u}_{m}^{m+1}+\nabla p_{m}^{m+1}=\rho \mathbf{g}, \tag{1.12}
\end{equation*}
$$

where $\mathbf{u}_{m}^{m+1}$ is an approximation of the fluid's velocity at time $t^{m+1}$, defined on the approximate domain at time $t^{m}$.

At each time $t^{m}$, the momentum equations (1.12) are of the form (0.1) (to simplify, we suppress the dependence on $m$ ):

$$
\left\{\begin{array}{rlrl}
\alpha \rho \mathbf{u}-\mu \Delta \mathbf{u}+\nabla p & =\rho \mathbf{g}+\alpha \rho \mathbf{w} & & \text { in each } \Omega^{i}, \tag{1.13}
\end{array} \quad i=1,2\right.
$$

where $\alpha$ stands for $1 / \delta t$ and $\mathbf{w}$ stands for $\mathbf{u}_{m}\left(\mathbf{X}^{m}\right)$. The boundary conditions are given by (1.7):

$$
\left\{\begin{aligned}
\mathbf{u} & =\mathbf{U} & & \text { on } \Gamma_{\text {in }} \\
\mathbf{u}^{2} & =\mathbf{0} & & \text { on } \Gamma_{0}^{2} \\
\mathbf{u}^{1} \cdot \mathbf{n} & =0 & & \text { on } \Gamma_{0}^{1} \\
\mathbf{t} \cdot \sigma^{1} \cdot \mathbf{n} & =0 & & \text { on } \Gamma_{0}^{1} \\
\sigma \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n} & & \text { on } \Gamma_{\text {out }},
\end{aligned}\right.
$$

the first interface condition (1.8) is unchanged and since the position of the interface is now known, the second interface condition simplifies and (1.8) becomes:

$$
\begin{equation*}
[\mathbf{u}]_{\Gamma}=\mathbf{0}, \quad[\sigma]_{\Gamma} \cdot \mathbf{n}^{1}=-K \mathbf{n}^{1} \tag{1.14}
\end{equation*}
$$

where $K$, that stands for $\kappa / R$, is now a known function. This is a generalized Stokes problem with particular boundary conditions.

## 2. THE PROBLEM AT EACH TIME STEP

## A. Variational Formulation

Let us put problem (1.13), (1.7), (1.14) into an equivalent variational formulation. In this section, we do not take into account the interpretation of $K$ and therefore, it suffices to assume that the interface $\Gamma$ is Lipschitz-continuous. This is compatible with the fact that the interface is very smooth at initial time (in fact, its graph is a straight horizontal line); therefore we can reasonably assume that it remains a sufficiently smooth graph for some time $T$. Thus each subdomain $\Omega^{i}$ is also Lipschitz continuous. The given function $U$ belongs to $H^{1}(0, D)$, the outlet pressure $p_{\text {out }}$ belongs to $L^{2}\left(\Gamma_{\text {out }}\right)$ and the given function $K$ on the interface belongs to $L^{2}(\Gamma)$. Of course, $\mathbf{g}$ being the force of gravity is very smooth. First we consider the problem where the first equation in (1.7) is replaced by the homogeneous boundary condition with $\mathbf{U}=\mathbf{0}$ :

$$
\mathbf{u}=\mathbf{0} \quad \text { on } \Gamma_{\mathrm{in}}
$$

Afterward, an adequate lifting of $\mathbf{U}$ will enable us to solve (1.13), (1.7), (1.14). In view of the boundary conditions, we choose the following space for the velocity:

$$
\begin{equation*}
X=\left\{\mathbf{v} \in H^{1}(\Omega)^{2} ;\left.\mathbf{v}\right|_{\Gamma_{\text {in }}}=\mathbf{0},\left.\mathbf{v}\right|_{\Gamma_{0}^{2}}=\mathbf{0},\left.\mathbf{v} \cdot \mathbf{n}\right|_{\Gamma_{0}^{1}}=0\right\} \tag{2.1}
\end{equation*}
$$

Both the transmission condition on the interface and the outflow condition involve the stress tensor; thus the pressure has no indeterminate constant and hence the space for the pressure is

$$
\begin{equation*}
M=L^{2}(\Omega) \tag{2.2}
\end{equation*}
$$

and as usual, we define the space of the velocities with zero divergence:

$$
\begin{equation*}
V=\{\mathbf{v} \in X ; \nabla \cdot \mathbf{v}=0\} \tag{2.3}
\end{equation*}
$$

Now, for the variational formulation, since $\nabla \cdot \mathbf{v}=0$, we have the identity in each $\Omega^{i}$ :

$$
\Delta \mathbf{u}=\nabla \cdot A_{1}(\mathbf{u}) .
$$

Therefore, taking the scalar product of the first equation of (1.13) in $L^{2}\left(\Omega^{i}\right)^{2}$ with a test function $\mathbf{v} \in X$, applying formally Green's formula in each $\Omega^{i}$ (that will be justified afterward) and summing over $i$, we obtain

$$
\begin{array}{r}
\alpha \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} d \mathbf{x}+\sum_{i=1}^{2} \int_{\Omega^{i}}\left(\mu A_{1}\left(\mathbf{u}^{i}\right)-p^{i} I\right): \nabla \mathbf{v}^{i} d \mathbf{x}+\sum_{i=1}^{2} \int_{\partial \Omega^{i}}\left(-\mu A_{1}\left(\mathbf{u}^{i}\right) \mathbf{n}^{i}+p^{i} \mathbf{n}^{i}\right) \cdot \mathbf{v}^{i} d s \\
 \tag{2.4}\\
=\int_{\Omega} \rho \mathbf{g} \cdot \mathbf{v} d \mathbf{x}+\alpha \int_{\Omega} \rho \mathbf{w} \cdot \mathbf{v} d \mathbf{x} .
\end{array}
$$

The symmetry of the operator $A_{1}(\mathbf{u})$ gives $A_{1}(\mathbf{u}): \nabla \mathbf{v}=A_{1}(\mathbf{u}):(\nabla \mathbf{v})^{t}$ and therefore, as both $\mathbf{u}$ and $\mathbf{v}$ belong to $H^{1}(\Omega)^{2}$ we have

$$
\sum_{i=1}^{2} \int_{\Omega^{i}}\left(\mu A_{1}\left(\mathbf{u}^{i}\right)-p^{i} I\right): \nabla \mathbf{v}^{i} d \mathbf{x}=\frac{1}{2} \int_{\Omega} \mu A_{1}(\mathbf{u}): A_{1}(\mathbf{v}) d \mathbf{x}-\int_{\Omega} p \nabla \cdot \mathbf{v} d \mathbf{x}
$$

As far as the boundary terms are concerned observe that $\mathbf{v}=\mathbf{0}$ on $\Gamma_{\text {in }}$ and $\Gamma_{0}^{2}$ and

$$
\mathbf{v}=\left(v_{x}, 0\right)^{t}=v_{x} \mathbf{t} \quad \text { on } \quad \Gamma_{0}^{1} .
$$

Therefore the boundary term in (2.4) reduces to

$$
\begin{aligned}
\int_{\Gamma}\left(-\sigma\left(\mathbf{u}^{1}, p^{1}\right) \mathbf{n}^{1}, \mathbf{v}^{1}\right) d s+\int_{\Gamma}\left(\sigma\left(\mathbf{u}^{2}, p^{2}\right) \mathbf{n}^{1}, \mathbf{v}^{2}\right) d s+\int_{\Gamma_{0}^{1}}(-\sigma(\mathbf{u}, p) \mathbf{n}, \mathbf{t}) v_{x} d s & \\
& +\int_{\Gamma_{\text {out }}}(-\sigma(\mathbf{u}, p) \mathbf{n}, \mathbf{v}) d s .
\end{aligned}
$$

Substituting these equalities into (2.4) and using (1.14) and the last line of (1.7), we obtain the variational formulation of the homogeneous problem: Given the functions $\mathbf{w}, K, p_{\text {out }}, \mu, \rho$, and the constant $\alpha$, find $\mathbf{u} \in X$ and $p \in M$ solution of:

$$
\left\{\begin{array}{l}
\alpha \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} d \mathbf{x}+\frac{1}{2} \int_{\Omega} \mu\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right):\left(\nabla \mathbf{v}+(\nabla \mathbf{v})^{t}\right) d \mathbf{x}-\int_{\Omega} p \nabla \cdot \mathbf{v} d \mathbf{x}  \tag{2.5}\\
\quad=\int_{\Omega} \rho \mathbf{g} \cdot \mathbf{v} d \mathbf{x}+\alpha \int_{\Omega} \rho \mathbf{w} \cdot \mathbf{v} d \mathbf{x}-\int_{\Gamma} K \mathbf{v} \cdot \mathbf{n}^{1} d s-\int_{\Gamma_{\text {out }}} p_{\text {out }} \mathbf{v} \cdot \mathbf{n} d s, \quad \forall \mathbf{v} \in X \\
\quad \int_{\Omega} q \nabla \cdot \mathbf{u} d \mathbf{x}=0, \quad \forall q \in M
\end{array}\right.
$$



FIG. 2. Parametrization of $\omega_{\varepsilon}$.

## A Density Result

A rigorous proof of the equivalence between the boundary value problem (1.13), (1.7), (1.14) and its variational formulation (2.5) is based essentially on the density of smooth functions in the space below, that is appropriate for the Stokes operator on a Lipschitz domain $\omega$ in arbitrary dimension $n$ :

$$
W=\left\{(L, p) \in L^{2}(\omega)^{n \times n} \times L^{2}(\omega) ; \nabla \cdot(-\mu L+p I) \in L^{2}(\omega)^{n}, \nabla(\operatorname{tr} L) \in L^{2}(\omega)^{n}\right\},
$$

where $\operatorname{tr} L$ denotes the trace of the tensor $L$. This is a Hilbert space for the graph norm:

$$
\|(L, p)\|_{W}=\left(\|L\|_{L^{2}(\omega)^{n \times n}}^{2}+\|p\|_{L^{2}(\omega)}^{2}+\|\nabla \cdot(-\mu L+p I)\|_{L^{2}(\omega)^{n}}^{2}+\|\nabla(\operatorname{tr} L)\|_{L^{2}(\omega)^{n}}^{2}\right)^{1 / 2} .
$$

The Stokes operator is related to $W$ through the following identity:

$$
\forall \mathbf{u} \in \mathcal{D}^{\prime}(\omega)^{n}, \forall p \in \mathcal{D}^{\prime}(\omega), \quad \nabla \cdot\left(-\mu A_{1}(\mathbf{u})+p I\right)=-\mu \Delta \mathbf{u}-\mu \nabla(\nabla \cdot \mathbf{u})+\nabla p
$$

Hence, since $\nabla \cdot \mathbf{u}=\frac{1}{2} \operatorname{tr} A_{1}(\mathbf{u})$, if $\mathbf{u} \in H^{1}(\omega)^{n}$ and $p \in L^{2}(\omega)$ satisfy

$$
\begin{equation*}
-\mu \Delta \mathbf{u}+\nabla p=\mathbf{f}, \quad \nabla \cdot \mathbf{u}=0 \text { in } \omega, \tag{2.6}
\end{equation*}
$$

with $\mathbf{f}$ in $L^{2}(\omega)^{n}$, then the pair $\left(A_{1}(\mathbf{u}), p\right)$ belongs to the following subspace of $W$ :

$$
\begin{equation*}
W_{s}=\{(L, p) \in W ; L \text { is symmetric }\} . \tag{2.7}
\end{equation*}
$$

Theorem 2.1. Let $\omega$ be a bounded Lipschitz domain of $\mathbb{R}^{n}$, $n \geq 2$. Then $\mathcal{D}(\bar{\omega})^{n \times n} \times \mathcal{D}(\bar{\omega})$ is dense in $W$.

The proof is sketched in the Appendix.
Now we turn to Green's formula. Let $\mathcal{D}_{s}(\bar{\omega})^{n \times n}$ be the subspace of the symmetric tensors of $\mathcal{D}(\bar{\omega})^{n \times n}$. For each pair $(L, p) \in \mathcal{D}_{s}(\bar{\omega})^{n \times n}$ and for all $\mathbf{v} \in \mathcal{D}(\bar{\omega})^{n}$, we have

$$
\int_{\omega} \nabla \cdot(-\mu L+p I) \cdot \mathbf{v} d \mathbf{x}=\int_{\omega}(\mu L: \nabla \mathbf{v}-p \nabla \cdot \mathbf{v}) d \mathbf{x}+\int_{\partial \omega}(-\mu L+p I) \mathbf{n} \cdot \mathbf{v} d s .
$$

Therefore

$$
\begin{equation*}
\int_{\partial \omega}(-\mu L+p I) \mathbf{n} \cdot \mathbf{v} d s=\int_{\omega} \nabla \cdot(-\mu L+p I) \cdot \mathbf{v} d \mathbf{x}-\int_{\omega}(\mu L: \nabla \mathbf{v}-p \nabla \cdot \mathbf{v}) d \mathbf{x} . \tag{2.8}
\end{equation*}
$$

Then the next corollary follows from Theorem 2.1 by classical arguments.
Corollary. For all $(L, p) \in W$, the trace

$$
-\mu L \mathbf{n}+p \mathbf{n} \text { belongs to } H^{-1 / 2}(\partial \omega)^{n} ;
$$

it satisfies the bound, with a constant $C$ that depends only on $\omega$

$$
\|-\mu L \mathbf{n}+p \mathbf{n}\|_{H^{-1 / 2}(\partial \omega)^{n}} \leq C\|(L, p)\|_{W},
$$

and the Green's formula holds

$$
\begin{equation*}
\forall \mathbf{v} \in H^{1}(\omega)^{n}, \quad\langle(-\mu L+p I) \mathbf{n}, \mathbf{v}\rangle_{\partial \omega}=\int_{\omega} \nabla \cdot(-\mu L+p I) \cdot \mathbf{v} d \mathbf{x}-\int_{\omega}(\mu L: \nabla \mathbf{v}-p \nabla \cdot \mathbf{v}) d \mathbf{x}, \tag{2.9}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle_{\partial \omega}$ denotes the duality pairing between $H^{-1 / 2}(\partial \omega)^{n}$ and $H^{1 / 2}(\partial \omega)^{n}$.

## Equivalence

We have just seen that any solution $(\mathbf{u}, p)$ in $X \times M$ of (1.13), (1.7), (1.14) is a solution of the variational formulation (2.5). Conversely, if ( $\mathbf{u}, p$ ) is a solution of the variational problem (2.5), taking $\mathbf{v}=\mathbf{v}^{1} \in \mathcal{D}\left(\Omega^{1}\right)^{2}$ first, and then, $\mathbf{v}=\mathbf{v}^{2} \in \mathcal{D}\left(\Omega^{2}\right)^{2}$, we find the first equation of (1.13). The second equation is obtained readily by choosing adequate functions $q$. Of course, the fact that $\mathbf{u}$ belongs to $X$ implies the first equation in (1.14). To find the boundary conditions, we multiply each equation (1.13) in $\Omega^{i}$ by a function $\mathbf{v}^{i}$ verifying the boundary conditions of the $X$ space. Then, applying the Green's formula (2.9) and comparing with the variational formulation (2.5), we derive

$$
\begin{equation*}
\forall \mathbf{v} \in X, \quad \sum_{i=1}^{2}\left\langle\sigma\left(\mathbf{u}^{i}, p^{i}\right) \mathbf{n}^{i}, \mathbf{v}^{i}\right\rangle_{\mathrm{div}} \Omega^{i}=-\int_{\Gamma} K \mathbf{v} \cdot \mathbf{n}^{1} d s-\int_{\Gamma_{\text {out }}} p_{\text {out }} \mathbf{v} \cdot \mathbf{n} d s . \tag{2.10}
\end{equation*}
$$

Now, by choosing first $\mathbf{v} \in H_{0}^{1}(\Omega)^{2}$, we obtain $[\sigma]_{\Gamma} \cdot \mathbf{n}^{1}=-K \mathbf{n}^{1}$. Next, we choose $\mathbf{v} \in X$ with $\left.\mathbf{v}\right|_{\Gamma_{0}^{1}}=\mathbf{0}$, and we find $\sigma \cdot \mathbf{n}=-p_{\text {out }} \mathbf{n}$ on $\Gamma_{\text {out }}$. Finally, with this information, we recover from (2.10) the desired condition on $\Gamma_{0}^{1}$. Hence, we have established the following proposition.

Proposition 2.2. The problems (2.5) and (1.13), (1.7), (1.14) are equivalent.
Now, to handle the nonhomogeneous boundary condition on $\Gamma_{\text {in }}$, we must construct a lifting, say $\overline{\mathbf{U}}$, of the inlet velocity $\mathbf{U}$. Since for the moment, we do not require that $\overline{\mathbf{U}}$ be regular, we propose here a crude, but very simple lifting. Recall that owing to the geometry of $\Omega$ (see Fig. 1), the inlet velocity has the form (1.9)

$$
\mathbf{U}=(U(y), 0)^{t}
$$

where $U \in H^{1}(0, D)$ is a known function of $y$, that satisfies:

$$
U(D)=0
$$

Then $\overline{\mathbf{U}}$ is obtained by replicating these values for all $(x, y)$ in $\Omega$, i.e.,

$$
\begin{equation*}
\forall(x, y) \in \Omega, \quad \overline{\mathbf{U}}(x, y)=(U(y), 0)^{t}, \tag{2.11}
\end{equation*}
$$

which has clearly zero divergence, depends continuously on the function $U$, belongs to $H^{1}(\Omega)^{2}$ and satisfies the boundary conditions:

$$
\left.\overline{\mathbf{U}}\right|_{\Gamma_{0}^{2}}=\mathbf{0} \quad \text { and }\left.\quad \overline{\mathbf{U}} \cdot \mathbf{n}\right|_{\Gamma_{0}^{1}}=0
$$

Thus, we propose the following variational formulation for problem (1.13), (1.7), (1.14): Find $\mathbf{u} \in X+\overline{\mathbf{U}}$ and $p \in M$ solution of

$$
\left\{\begin{array}{l}
\alpha \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} d \mathbf{x}+\frac{1}{2} \int_{\Omega} \mu\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right):\left(\nabla \mathbf{v}+(\nabla \mathbf{v})^{t}\right) d \mathbf{x}-\int_{\Omega} p \nabla \cdot \mathbf{v} d \mathbf{x}  \tag{2.12}\\
\quad=\int_{\Omega} \rho \mathbf{g} \cdot \mathbf{v} d \mathbf{x}+\alpha \int_{\Omega} \rho \mathbf{w} \cdot \mathbf{v} d \mathbf{x}-\int_{\Gamma} K \mathbf{v} \cdot \mathbf{n}^{1} d s-\int_{\Gamma_{\mathrm{out}}} p_{\text {out }} \mathbf{v} \cdot \mathbf{n} d s, \quad \forall \mathbf{v} \in X \\
\quad \int_{\Omega} q \nabla \cdot \mathbf{u} d \mathbf{x}=0, \quad \forall q \in M .
\end{array}\right.
$$

## B. Well-Posedness

Problem (2.12) is a nonhomogeneous problem of mixed type. Indeed, setting

$$
\begin{gather*}
a(\mathbf{u}, \mathbf{v})=\alpha \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} d \mathbf{x}+\frac{1}{2} \int_{\Omega} \mu\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right):\left(\nabla \mathbf{v}+(\nabla \mathbf{v})^{t}\right) d \mathbf{x},  \tag{2.13}\\
b(\mathbf{v}, q)=\int_{\Omega} q \nabla \cdot \mathbf{v} d \mathbf{x},  \tag{2.14}\\
\mathbf{F}=\mathbf{g}+\alpha \mathbf{w},  \tag{2.15}\\
\ell(\mathbf{v})=\int_{\Omega} \rho \mathbf{F} \cdot \mathbf{v} d \mathbf{x}-\int_{\Gamma} K \mathbf{v} \cdot \mathbf{n}^{1} d s-\int_{\Gamma_{\text {out }}} p_{\text {out }} \mathbf{v} \cdot \mathbf{n} d s, \tag{2.16}
\end{gather*}
$$

it has the form: Find $\mathbf{u} \in X+\overline{\mathbf{U}}$ and $p \in M$ solution of

$$
\forall \mathbf{v} \in X, \quad a(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p)=\ell(\mathbf{v}), \forall q \in M, \quad b(\mathbf{u}, q)=0
$$

Then, since the extension $\overline{\mathbf{U}}$ has zero divergence, solving (2.12) is equivalent to solving: Find $\mathbf{u}_{0}=\left(\mathbf{u}_{0}^{1}, \mathbf{u}_{0}^{2}\right) \in V$ such that

$$
\begin{equation*}
\forall \mathbf{v} \in V, \quad a\left(\mathbf{u}_{0}, \mathbf{v}\right)=\ell(\mathbf{v})-a(\overline{\mathbf{U}}, \mathbf{v}) \tag{2.17}
\end{equation*}
$$

Thus the well-posedness of (2.12) (i.e., existence, uniqueness, and stability of its solution) is a consequence of two properties: the ellipticity on $V$ of the bilinear form $a(\cdot, \cdot)$, the inf-sup condition on $X \times M$ of the bilinear form $b(\cdot, \cdot)$, and the continuity of the right-hand side of (2.17).

The ellipticity of $a$ is an immediate consequence of Korn's inequality; it holds on $X$ (and not only on $V$ ) and is independent of the parameter $\alpha$ : there exists a constant $\lambda$, that depends only on $\Omega$, such that

$$
\forall \mathbf{u} \in X, \quad \frac{1}{2} \int_{\Omega}\left\|\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right\|^{2} d \mathbf{x} \geq \lambda\|\nabla \mathbf{u}\|_{L^{2}(\Omega)}^{2}
$$

Whence the ellipticity of $a$ :

$$
\begin{equation*}
\forall \mathbf{u} \in X, \quad \frac{1}{2} \int_{\Omega} \mu\left\|\nabla \mathbf{u}+(\nabla \mathbf{u})^{t}\right\|^{2} d \mathbf{x} \geq \lambda \operatorname{Min}\left(\mu^{1}, \mu^{2}\right)\|\nabla \mathbf{u}\|_{L^{2}(\Omega)}^{2} \tag{2.18}
\end{equation*}
$$

The following proposition establishes the inf-sup condition; its proof is written in the Appendix.
Proposition 2.3. There exists a constant $\beta>0$ such that

$$
\begin{equation*}
\forall q \in M, \quad \sup _{\mathbf{v} \in X} \frac{1}{\|\nabla \mathbf{v}\|_{L^{2}(\Omega)}} \int_{\Omega} q \nabla \cdot \mathbf{v} d \mathbf{x} \geq \beta\|q\|_{L^{2}(\Omega)} \tag{2.19}
\end{equation*}
$$

Finally, owing to the regularity assumptions on the data, the right-hand side of (2.12) clearly defines an element in the dual space of $H^{1}(\Omega)$. Then the next proposition follows from the Babuška-Brezzi's theory (cf. [15], [16]).

Proposition 2.4. The problem (2.12) is well-posed.

## 3. SPACE DISCRETIZATION

From now on, we must be aware that $K$ is related to the surface tension and hence we assume that, in addition to the hypotheses of Section 2, $\Gamma$ is a curve of class $\mathcal{C}^{2}$, with a horizontal tangent at the inlet point $\left(0, y_{0}\right)$, i.e., $\Phi^{\prime}(0)=0$. Then we triangulate separately each subdomain $\Omega^{i}$ with a regular triangulation $\mathcal{T}_{h}$, with maximum mesh-size $h$, and such that it is globally a conforming triangulation of $\Omega$. By regular, we mean (cf. Ciarlet [17]): there exists a constant $\zeta$ independent of $h$ such that

$$
\begin{equation*}
\forall T \in \mathcal{T}_{h}, \quad \frac{h_{T}}{\rho_{T}} \leq \zeta \tag{3.1}
\end{equation*}
$$

where $h_{T}$ denotes the diameter of $T$ and $\rho_{T}$ the diameter of the circle inscribed in $T$. We assume that $\mathcal{T}_{h}$ contains a polygonal line $\Gamma_{h}$ whose nodes belong to $\Gamma$; as pointed out in the introduction, this is a key feature of our discretization. We denote by $\Omega_{h}^{i}$ the domain approximating $\Omega^{i}$, i.e., the region bounded by $\Gamma_{h}, \Gamma_{0}^{i}, \Gamma_{\text {in }}^{i}$, and $\Gamma_{\text {out }}^{i}$.

Following Arnold [18], we approximate each component of the velocity in each triangle $T$ by a polynomial of degree one plus a "bubble" function $b_{T}$, i.e., a polynomial of degree three that vanishes on the three sides of $T$, i.e., in terms of the barycentric coordinates:

$$
b_{T}=27 \lambda_{1} \lambda_{2} \lambda_{3} .
$$

Similarly, we approximate the pressure in each triangle by a polynomial of degree one. Both approximations are continuous at inter-element edges except for the pressure on the edges of the
interface $\Gamma$, since the pressure is expected to be discontinuous across $\Gamma$. Thus we discretize $X$ and $M$ by:

$$
\begin{gather*}
X_{h}=\left\{\mathbf{v}_{h} \in \mathcal{C}^{0}(\bar{\Omega})^{2} ; \forall T \in \mathcal{T}_{h},\left.\mathbf{v}_{h}\right|_{T} \in\left(P_{1}+b_{T}\right)^{2}\right\} \cap X  \tag{3.2}\\
M_{h}=\left\{q_{h}=\left(q_{h}^{1}, q_{h}^{2}\right) \in \mathcal{C}^{0}\left(\bar{\Omega}_{h}^{1}\right) \times \mathcal{C}^{0}\left(\bar{\Omega}_{h}^{2}\right) ; \forall T \in \mathcal{T}_{h},\left.q_{h}\right|_{T} \in P_{1}\right\}  \tag{3.3}\\
V_{h}=\left\{\mathbf{v}_{h} \in X_{h} ; \forall q_{h} \in M_{h}, b\left(\mathbf{v}_{h}, q_{h}\right)=0\right\} \tag{3.4}
\end{gather*}
$$

Now, we must refine the lifting of the inlet velocity $\mathbf{U}$, so that the lifting can be appropriately approximated, considering that it is not reasonable to assume that $U$ is globally smooth. Indeed, it stems from physical considerations that the regularity of $U$ cannot be higher than that of the inflow velocity of the steady-state problem, namely $H^{3 / 2-\varepsilon}\left(\Gamma_{\text {in }}\right)$. Hence, whereas we can assume that $U^{i}$ belongs to $H^{2}\left(\Gamma_{\text {in }}^{i}\right)$, we cannot assume that $U$ belongs globally to $H^{2}\left(\Gamma_{\text {in }}\right)$; thus the simple function $\overline{\mathbf{U}}$ defined by (2.11) does not belong to $H^{2}\left(\Omega^{i}\right)^{2}$ and its approximation brings a nonoptimal error in the a priori error estimate. Of course, this lifting is not used numerically, but we must show that it exists. This is the object of the following proposition.

Proposition 3.1. Assume that $\Gamma$ satisfies the above assumptions, $U$ belongs to $H^{1}\left(\Gamma_{\text {in }}\right)$ and its restriction to $\Gamma_{\mathrm{in}}^{i}$ belongs to $H^{2}\left(\Gamma_{\text {in }}^{i}\right)$ for $i=1$, Then there exists a function $\overline{\mathbf{U}} \in H^{1}(\Omega)^{2}$ such that $\operatorname{div} \overline{\mathbf{U}}=0$ in $\Omega, \overline{\mathbf{U}}=\mathbf{0}$ on $\Gamma_{0}^{2}, \overline{\mathbf{U}} \cdot \mathbf{n}=0$ on $\Gamma_{0}^{1}, \overline{\mathbf{U}}=(U, 0)^{t}$ on $\Gamma_{\mathrm{in}}$, its restriction $\overline{\mathbf{U}}^{i}$ to $\Omega^{i}$ belongs to $H^{2}\left(\Omega^{i}\right)^{2}$ for $i=1,2$ and

$$
\begin{equation*}
\|\overline{\mathbf{U}}\|_{2, \Omega^{i}} \leq C\|U\|_{2, \Gamma_{\text {in }}^{i}} \quad \text { for } i=1,2 \tag{3.5}
\end{equation*}
$$

with a constant $C$ that is independent of $U$.
Proof. Let us sketch the proof for $\Omega^{1}$; the proof is similar in $\Omega^{2}$. Our construction is based on the remark that if $\Phi(x)=y_{0}$ on $[0, L]$, then the function $\overline{\mathbf{U}}$ of (2.11) satisfies all the requirements of this proposition. Therefore, as a first step, we propose the intermediate lifting

$$
\begin{gather*}
\forall(x, y) \in \Omega^{1}, \quad \tilde{U}(x, y)=U\left(\frac{y_{0}}{\Phi(x)} y\right) a(x)+c \varrho(y),  \tag{3.6}\\
\tilde{\mathbf{U}}=(\tilde{U}, 0)^{t},
\end{gather*}
$$

where $a \in \mathcal{C}^{\infty}([0, L])$ is a truncating function satisfying:

$$
0 \leq a(x) \leq 1 \text { in }[0, L], \quad a(x)=1 \text { in }[0, L / 2], \quad a(x)=0 \text { in }[3 L / 4, L],
$$

$\varrho$ is the function defined in the proof of Proposition 2.3 given in the Appendix, and the constant $c$ is chosen so that $\int_{\partial \Omega^{1}} \tilde{\mathbf{U}} \cdot \mathbf{n} d s=0$. The above regularity assumptions imply that $\tilde{\mathbf{U}}$ satisfies all the desired conditions, except that its divergence is not necessarily zero. Furthermore, it can be checked that

$$
\frac{\tilde{U}}{\delta} \in L^{2}\left(\Omega^{1}\right)
$$

where $\delta(x, y)$ is the minimum distance of $(x, y)$ to the corners of $\partial \Omega^{1}$. Then, according to Kellogg and Osborn [19], there exists a function $\mathbf{v} \in\left[H^{2}\left(\Omega^{1}\right) \cap H_{0}^{1}\left(\Omega^{1}\right)\right]^{2}$ such that

$$
\operatorname{div} \mathbf{v}=\operatorname{div} \tilde{\mathbf{U}} \text { in } \Omega^{1}
$$

and

$$
\|\mathbf{v}\|_{H^{2}\left(\Omega^{1}\right)} \leq C\|\operatorname{div} \tilde{\mathbf{U}}\|_{H^{1}\left(\Omega^{1}\right)} .
$$

Note that this does not require the convexity of the domain $\Omega^{1}$; in our case, it only requires that the angles at the two corners of the inlet boundary $\Gamma_{\text {in }}^{1}$ be smaller than $\pi$, which is the case, since they are both right angles. Finally, we take

$$
\begin{equation*}
\overline{\mathbf{U}}=\tilde{\mathbf{U}}-\mathbf{v} \tag{3.7}
\end{equation*}
$$

and it can be readily checked that $\overline{\mathbf{U}}$ satisfies all the requirements of this proposition.
Since a function that belongs to $H^{1}\left(\Omega^{i}\right)$ for $i=1,2$ also belongs to $H^{1 / 2-\varepsilon}(\Omega)$ for any $\varepsilon>0$, Proposition 3.1 has the following corollary.

Corollary. Under the assumptions of Proposition 3.1, the function $\overline{\mathbf{U}}$ defined by (3.7) belongs to $H^{3 / 2-\varepsilon}(\Omega)^{2}$ for any $\varepsilon>0$ and there exists a constant $C(\varepsilon)$ independent of $\overline{\mathbf{U}}$ and $U$ such that

$$
\begin{equation*}
\|\overline{\mathbf{U}}\|_{H^{3 / 2-\varepsilon}(\Omega)} \leq C(\varepsilon)\|U\|_{H^{3 / 2-\varepsilon}\left(\Gamma_{\text {in }}\right)} \leq C(\varepsilon) \sum_{i=1}^{2}\|U\|_{2, \Gamma_{\text {in }}^{i}} . \tag{3.8}
\end{equation*}
$$

As $\overline{\mathbf{U}} \in \mathcal{C}^{0}(\bar{\Omega})^{2}$, the standard $P_{1}$ Lagrange interpolant on the nodes of $\mathcal{T}_{h}, I_{h}(\overline{\mathbf{U}})$, is well defined. In addition, since $I_{h}$ preserves the zero boundary condition, $I_{h}(\overline{\mathbf{U}})$ satisfies $I_{h}(\overline{\mathbf{U}})=\mathbf{0}$ on $\Gamma_{0}^{2}$ and $I_{h}(\overline{\mathbf{U}}) \cdot \mathbf{n}=0$ on $\Gamma_{0}^{1}$. Moreover, on $\Gamma_{\mathrm{in}}$,

$$
I_{h}(\overline{\mathbf{U}})=\left(I_{h}(U), 0\right)^{t}
$$

Finally, as the discrete divergence of $I_{h}(\overline{\mathbf{U}})$ does not necessarily vanish, we correct it and take

$$
\begin{equation*}
\overline{\mathbf{U}}_{h}=I_{h}(\overline{\mathbf{U}})+\mathbf{c}_{h}, \tag{3.9}
\end{equation*}
$$

where $\mathbf{c}_{h} \in X_{h}$ is chosen so that

$$
\begin{equation*}
\forall q_{h} \in M_{h}, \quad \int_{\Omega} q_{h} \operatorname{div} \overline{\mathbf{U}}_{h} d \mathbf{x}=0 . \tag{3.10}
\end{equation*}
$$

The existence and properties of $\mathbf{c}_{h}$ will be a consequence of Proposition 3.2.
Finally, we approximate carefully the term that takes into account the surface tension, namely $\int_{\Gamma} K \mathbf{v}_{h} \cdot \mathbf{n}^{1} d s$. Recall that

$$
K=\frac{\kappa}{R}
$$

and set

$$
\begin{equation*}
K(\mathbf{v})=\int_{\Gamma} K \mathbf{v} \cdot \mathbf{n}^{1} d s \tag{3.11}
\end{equation*}
$$

The approximation we propose is motivated by the fact that with the convention of sign used for $R$, we have

$$
\begin{equation*}
\frac{\mathbf{n}^{1}}{R}=-\frac{\mathbf{n}}{\bar{R}}=\frac{d \mathbf{t}}{d s} \tag{3.12}
\end{equation*}
$$

where $\mathbf{t}$ is the tangent to $\Gamma$ in the direction of increasing $s$, that is the same as that of increasing $x, \mathbf{n}$ is the principal normal to $\Gamma$, i.e., parallel to $\mathbf{n}^{1}$ and directed toward the center of curvature
of $\Gamma$, and $\bar{R}$ is the positive radius of curvature, i.e., $\bar{R}=R$ if the center of curvature is located inside $\Omega^{1}$ and $\bar{R}=-R$ otherwise. Hence in the case of the test function $\mathbf{v}_{h}$,

$$
\begin{equation*}
K\left(\mathbf{v}_{h}\right)=\int_{\Gamma} K \mathbf{v}_{h} \cdot \mathbf{n}^{1} d s=\kappa \int_{\Gamma} \mathbf{v}_{h} \cdot \frac{d \mathbf{t}}{d s} d s \tag{3.13}
\end{equation*}
$$

Now, let $\mathbf{x}_{i}=\left(x_{i}, y_{i}=\Phi\left(x_{i}\right)\right)$, for $0 \leq i \leq N$, be the mesh points of the triangulation $\mathcal{T}_{h}$ on $\Gamma$ with

$$
0=x_{0}<x_{1}<\cdots<x_{N-1}<x_{N}=L,
$$

and let $\tilde{S}_{i}$ denote the arc of $\Gamma$ with end points $\mathbf{x}_{i}$ and $\mathbf{x}_{i+1}$. Thus $\Gamma_{h}$ is the broken line joining the nodes $\mathbf{x}_{i}$ to $\mathbf{x}_{i+1}$, for $0 \leq i \leq N-1$, and we denote by $S_{i}$ the chord $\left[\mathbf{x}_{i}, \mathbf{x}_{i+1}\right]$. Then, we define the unit tangent $\mathbf{t}$ along the chord $S_{i}$ by

$$
\mathbf{t}_{i}=\frac{\mathbf{x}_{i+1}-\mathbf{x}_{i}}{\left|\mathbf{x}_{i+1}-\mathbf{x}_{i}\right|}, \quad 0 \leq i \leq N-1
$$

and approximate $K\left(\mathbf{v}_{h}\right)$ by

$$
\begin{equation*}
\forall \mathbf{v}_{h} \in X_{h}, \quad K_{h}\left(\mathbf{v}_{h}\right)=\kappa \sum_{i=1}^{N-1} \mathbf{v}_{h}\left(\mathbf{x}_{i}\right) \cdot\left(\mathbf{t}_{i}-\mathbf{t}_{i-1}\right) . \tag{3.14}
\end{equation*}
$$

Also, we define the approximate density and viscosity $\rho_{h}$ and $\mu_{h}$ by

$$
\begin{equation*}
\left.\rho_{h}\right|_{\Omega_{h}^{i}}=\rho^{i},\left.\quad \mu_{h}\right|_{\Omega_{h}^{i}}=\mu^{i}, \quad \text { for } i=1,2 \tag{3.15}
\end{equation*}
$$

Then, the discrete problem reads: Find $\mathbf{u}_{h} \in X_{h}+\overline{\mathbf{U}}_{h}$ and $p_{h} \in M_{h}$ solution of

$$
\left\{\begin{array}{l}
\alpha \int_{\Omega} \rho_{h} \mathbf{u}_{h} \cdot \mathbf{v}_{h} d \mathbf{x}+\frac{1}{2} \int_{\Omega} \mu_{h}\left(\nabla \mathbf{u}_{h}+\left(\nabla \mathbf{u}_{h}\right)^{t}\right):\left(\nabla \mathbf{v}_{h}+\left(\nabla \mathbf{v}_{h}\right)^{t}\right) d \mathbf{x}  \tag{3.16}\\
\quad-\int_{\Omega} p_{h} \nabla \cdot \mathbf{v}_{h} d \mathbf{x}=\int_{\Omega} \rho_{h} \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}-K_{h}\left(\mathbf{v}_{h}\right)-\int_{\Gamma_{\text {out }}} p_{\text {out }} \mathbf{v}_{h} \cdot \mathbf{n} d s, \quad \forall \mathbf{v}_{h} \in X_{h} \\
\quad \int_{\Omega} q_{h} \nabla \cdot \mathbf{u}_{h} d \mathbf{x}=0, \quad \forall q_{h} \in M_{h} .
\end{array}\right.
$$

## A. Uniform Stability

The first step in the numerical analysis of problem (3.16) consists in proving a uniform discrete inf-sup condition for the pair of spaces $X_{h}, M_{h}$. This condition is well known for the "minielement" with globally continuous pressures, and its proof relies on the exact inf-sup condition (see Girault and Raviart [20] or Arnold [18]). But here the discrete pressure is discontinuous accross the interface $\Gamma_{h}$ and therefore the proof should rely on the exact inf-sup condition in each $\Omega_{h}^{i}$. This is somewhat delicate because $\Omega_{h}^{i}$ depends on $h$. It is very likely that the constant of the exact inf-sup condition depends "continuously" on the domain, and since $\Omega_{h}^{i}$ "tends" to $\Omega^{i}$ when $h$ tends to zero, the constant for $\Omega_{h}^{i}$ is bounded independently of $h$. However, we have not seen in the literature an explicit proof of this result and therefore, we propose to prove the discrete inf-sup condition without using the exact inf-sup condition in each $\Omega_{h}^{i}$. This is done at the expense of the following restriction on the triangulation: we assume that each $\Omega_{h}^{i}$ can be partitioned into
macro-elements $\mathcal{O}_{T}$ consisting of the union of a triangle $T$ and its three adjacent triangles. Let $\mathbf{a}_{1}$, $\mathbf{a}_{2}, \mathbf{a}_{3}$ be the three vertices of $T$, and denote by $F_{i}$ the two segments of $\partial \mathcal{O}_{T}$ that share the vertex $\mathbf{a}_{i}$. More precisely, we assume that there exists a set of $R$ triangles of $\mathcal{T}_{h},\left\{T_{i}\right\}_{i=1}^{R}$ such that

$$
\bar{\Omega}=\cup_{i=1}^{R} \mathcal{O}_{T_{i}}, \quad \forall 1 \leq i \leq R, \mathcal{O}_{T_{i}} \subset \Omega_{h}^{1} \text { or } \mathcal{O}_{T_{i}} \subset \Omega_{h}^{2}, \quad\left|\mathcal{O}_{T_{i}} \cap \mathcal{O}_{T_{j}}\right|=0 \quad \text { if } i \neq j,
$$

and if $F_{k}$ intersects $\partial \Omega$, then $F_{k}$ is entirely contained in $\Gamma_{\text {in }}$ or in $\Gamma_{\text {out }}$ or in $\Gamma_{0}^{i}$. Such a triangulation is easily obtained from an arbitrary triangulation by decomposing each triangle into four triangles, so that it stays conforming as in the $P_{2}$ - iso $P_{1}$ element.

Proposition 3.2. Let $\mathcal{T}_{h}$ satisfy (3.1) and the above assumptions. There exists a constant $\beta^{\star}>0$ such that the following inf-sup condition holds:

$$
\begin{equation*}
\forall q_{h} \in M_{h}, \quad \sup _{\mathbf{v}_{h} \in X_{h}} \frac{1}{\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}(\Omega)}} \int_{\Omega} q_{h} \nabla \cdot \mathbf{v}_{h} d \mathbf{x} \geq \beta^{\star}\left\|q_{h}\right\|_{L^{2}(\Omega)} . \tag{3.17}
\end{equation*}
$$

The proof is written in the Appendix. Note that in this proof, we construct an approximation operator $\Pi_{h} \in \mathcal{L}\left(X ; X_{h}\right)$ that preserves the discrete divergence and is stable in the $H^{1}$ norm. Moreover, it is easy to check that it has the same approximation error as the Scott and Zhang operator $R_{h}$ defined in [21]. However, in the sequel, we shall approximate functions whose components belong to $\hat{H}^{2}(\Omega)$, the space of all functions in $H^{1}(\Omega)$ whose restriction to $\Omega^{i}$ is in $H^{2}\left(\Omega^{i}\right)$, for $i=1,2$. In this case, it is simpler to correct the Lagrange interpolation operator $I_{h}$, in order to preserve the discrete divergence. This is the object of the following corollary.

Corollary. We retain the hypotheses of Proposition 3.2. For any $\mathbf{v} \in \hat{H}^{2}(\Omega)^{2}$ we define

$$
\tilde{I}_{h}(\mathbf{v})=I_{h}(\mathbf{v})+\mathbf{c}_{h},
$$

where $\mathbf{c}_{h} \in V_{h}^{\perp}$ is the solution of

$$
\begin{equation*}
\forall q_{h} \in M_{h}, \quad \int_{\Omega} q_{h} \operatorname{div} \mathbf{c}_{h} d \mathbf{x}=\int_{\Omega} q_{h} \operatorname{div}\left(\mathbf{v}-I_{h}(\mathbf{v})\right) d \mathbf{x} . \tag{3.18}
\end{equation*}
$$

Then there exists a constant $C$, independent of $h$ and $\mathbf{v}$, such that

$$
\begin{equation*}
\left\|\tilde{I}_{h}(\mathbf{v})-\mathbf{v}\right\|_{L^{2}(\Omega)}+h\left\|\nabla\left(\tilde{I}_{h}(\mathbf{v})-\mathbf{v}\right)\right\|_{L^{2}(\Omega)} \leq C h^{2}\|\mathbf{v}\|_{\hat{H}^{2}(\Omega)} \tag{3.19}
\end{equation*}
$$

Proof. By the Babuška-Brezzi's theory, the existence of $\mathbf{c}_{h} \in V_{h}^{\perp}$ follows from Proposition 3.2 and it satisfies

$$
\left|\mathbf{c}_{h}\right|_{H^{1}(\Omega)} \leq \frac{1}{\beta^{*}}\left\|\operatorname{div}\left(\mathbf{v}-I_{h}(\mathbf{v})\right)\right\|_{L^{2}(\Omega)} .
$$

Hence

$$
\left\|\mathbf{v}-\tilde{I}_{h}(\mathbf{v})\right\|_{H^{1}(\Omega)} \leq C_{1}\left\|\mathbf{v}-I_{h}(\mathbf{v})\right\|_{H^{1}(\Omega)}
$$

This reduces to an interpolation error estimate in each $\Omega_{h}^{i}$, a region where $\mathbf{v}^{i}=\left.\mathbf{v}\right|_{\Omega^{i}}$ is not necessarily in $H^{2}$. Hence, we extend $\mathbf{v}^{i}$ to $\Omega$ by the Calderón extension so that the extended
function, still denoted $\mathbf{v}^{i}$, belongs to $H^{2}(\Omega)^{2}$ and there exists a constant $P^{i}$, depending only on $\Omega^{i}$ and $\Omega$, such that

$$
\left\|\mathbf{v}^{i}\right\|_{H^{2}(\Omega)} \leq P^{i}\left\|\mathbf{v}^{i}\right\|_{H^{2}\left(\Omega^{i}\right)} .
$$

Therefore, the standard approximation properties of $I_{h}$ imply that

$$
\left\|\nabla\left(\mathbf{v}-\tilde{I}_{h}(\mathbf{v})\right)\right\|_{L^{2}(\Omega)} \leq C_{2} h \sum_{i=1}^{2}|\mathbf{v}|_{H^{2}\left(\Omega_{h}^{i}\right)} \leq C_{2} h \sum_{i=1}^{2} P^{i}\left\|\mathbf{v}^{i}\right\|_{H^{2}\left(\Omega^{i}\right)} \leq C_{3} h\|\mathbf{v}\|_{\hat{H}^{2}(\Omega)}
$$

with an analogous estimate for the $L^{2}$ norm.
In view of this corollary and (3.9), we have

$$
\begin{equation*}
\overline{\mathbf{U}}_{h}=\tilde{I}_{h}(\overline{\mathbf{U}}) \tag{3.20}
\end{equation*}
$$

and $\overline{\mathbf{U}}_{h}$ satisfies (3.10). Note that on $\Gamma_{\mathrm{in}}, \overline{\mathbf{U}}_{h}=\left(I_{h}(U), 0\right)^{t}$, a quantity that can be computed just by knowing the values of $U$ on $\Gamma_{\text {in }}$. In fact, from a practical point of view, the interior values of $\overline{\mathbf{U}}_{h}$ are not used.

Now, (3.16) is a square system of linear equations in finite dimension. Introducing the forms $a_{h}$ and $\ell_{h}$ :

$$
\begin{aligned}
a_{h}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)=\alpha \int_{\Omega} \rho_{h} \mathbf{u}_{h} \cdot \mathbf{v}_{h} d \mathbf{x}+\frac{1}{2} \int_{\Omega} \mu_{h}\left(\nabla \mathbf{u}_{h}+\left(\nabla \mathbf{u}_{h}\right)^{t}\right):\left(\nabla \mathbf{v}_{h}+\left(\nabla \mathbf{v}_{h}\right)^{t}\right) d \mathbf{x} \\
\ell_{h}\left(\mathbf{v}_{h}\right)=\int_{\Omega} \rho_{h} \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}-K_{h}\left(\mathbf{v}_{h}\right)-\int_{\Gamma_{\text {out }}} p_{\text {out }} \mathbf{v}_{h} \cdot \mathbf{n} d s
\end{aligned}
$$

it can be written: Find $\mathbf{u}_{h} \in X_{h}+\overline{\mathbf{U}}_{h}$ and $p_{h} \in M_{h}$ solution of:

$$
\begin{aligned}
\forall \mathbf{v}_{h} \in X_{h}, \quad a_{h}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)+b\left(\mathbf{v}_{h}, p_{h}\right) & =\ell_{h}\left(\mathbf{v}_{h}\right), \\
\forall q_{h} \in M_{h}, \quad b\left(\mathbf{u}_{h}, q_{h}\right) & =0 .
\end{aligned}
$$

Then we have the following result.
Proposition 3.3. Under the assumptions of Proposition 3.2, problem (3.16) has a unique solution $\left(\mathbf{u}_{h}, p_{h}\right)$. In addition, if the assumptions of Propositions 3.1 and 3.5 are satisfied, there exists $a$ constant $C$ that depends on $\rho, \mu, \alpha$, and $\lambda$ but not on $h$ such that

$$
\begin{equation*}
\left\|\mathbf{u}_{h}\right\|_{H^{1}(\Omega)}+\beta^{*}\left\|p_{h}\right\|_{L^{2}(\Omega)} \leq C\left(\|\mathbf{F}\|_{L^{2}(\Omega)}+\kappa+\left\|p_{\mathrm{out}}\right\|_{L^{2}\left(\Gamma_{\mathrm{out}}\right)}+\|U\|_{H^{5 / 4}\left(\Gamma_{\mathrm{in}}\right)}\right) \tag{3.21}
\end{equation*}
$$

Proof. Let us write

$$
\mathbf{u}_{h}=\mathbf{u}_{0, h}+\overline{\mathbf{U}}_{h}
$$

where, owing that $\overline{\mathbf{U}}_{h}$ satisfies (3.10), $\left(\mathbf{u}_{0, h}, p_{h}\right) \in X_{h} \times M_{h}$ is the solution of

$$
\begin{align*}
& \forall \mathbf{v}_{h} \in X_{h}, \quad a_{h}\left(\mathbf{u}_{0, h}, \mathbf{v}_{h}\right)+b\left(\mathbf{v}_{h}, p_{h}\right)=\ell_{h}\left(\mathbf{v}_{h}\right)-a_{h}\left(\overline{\mathbf{U}}_{h}, \mathbf{v}_{h}\right) \\
& \forall  \tag{3.22}\\
& \forall q_{h} \in M_{h}, \quad b\left(\mathbf{u}_{0, h}, q_{h}\right)=0 .
\end{align*}
$$

Since Korn's inequality holds on $X$ and hence on $X_{h}$, this problem has a unique solution $\mathbf{u}_{0, h} \in X_{h}$. Then the existence of a unique $p_{h} \in M_{h}$ follows from (3.17). To derive (3.21), choose $\mathbf{v}_{h}=\mathbf{u}_{0, h}$
in (3.22). The resulting left-hand side is bounded below by virtue of (2.18). As far as the right-hand side is concerned, to bound $K_{h}\left(\mathbf{u}_{0, h}\right)$, we use Proposition 3.5 stated further on:

$$
\left|K_{h}\left(\mathbf{u}_{0, h}\right)\right| \leq \kappa C_{1} h^{1 / 2}\left|\mathbf{u}_{0, h}\right|_{H^{1}(\Omega)}+\left|K\left(\mathbf{u}_{0, h}\right)\right|,
$$

and since $\Gamma$ is of class $\mathcal{C}^{2}$, we have

$$
\left|K\left(\mathbf{u}_{0, h}\right)\right| \leq \kappa C_{2}\left\|\mathbf{u}_{0, h}\right\|_{L^{2}(\Gamma)} \leq \kappa C_{3}\left|\mathbf{u}_{0, h}\right|_{H^{1}(\Omega)} .
$$

Therefore

$$
\left|K_{h}\left(\mathbf{u}_{0, h}\right)\right| \leq \kappa C_{4}\left|\mathbf{u}_{0, h}\right|_{H^{1}(\Omega)},
$$

and

$$
\left|\ell_{h}\left(\mathbf{u}_{0, h}\right)\right| \leq \max \left(\rho^{1}, \rho^{2}\right) \mathcal{P}\|\mathbf{F}\|_{L^{2}(\Omega)}+\kappa C_{4}+C_{5}\left\|p_{\text {out }}\right\|_{L^{2}\left(\Gamma_{\text {out }}\right.},
$$

where $\mathcal{P}$ is the constant of Poincare's inequality (0.3). Hence

$$
\begin{aligned}
\lambda \min \left(\mu^{1}, \mu^{2}\right)\left\|\nabla \mathbf{u}_{0, h}\right\|_{L^{2}(\Omega)} \leq \max \left(\rho^{1}, \rho^{2}\right) & \mathcal{P}\left(\|\mathbf{F}\|_{L^{2}(\Omega)}+\alpha\left\|\overline{\mathbf{U}}_{h}\right\|_{L^{2}(\Omega)}\right) \\
& +C_{6}\left(\kappa+\left\|p_{\text {out }}\right\|_{L^{2}\left(\Gamma_{\text {out }}\right)}\right)+2 \max \left(\mu^{1}, \mu^{2}\right)\left\|\nabla \overline{\mathbf{U}}_{h}\right\|_{L^{2}(\Omega)} .
\end{aligned}
$$

Then the estimate for $\mathbf{u}_{h}$ in (3.21) follows from the fact that by taking for instance $\varepsilon=1 / 4$ in (3.8), we obtain

$$
\left\|I_{h}(\overline{\mathbf{U}})\right\|_{H^{1}(\Omega)} \leq C_{1}\|\overline{\mathbf{U}}\|_{H^{5 / 4}(\Omega)} \leq C_{6}\|U\|_{H^{5 / 4}\left(\Gamma_{\text {in }}\right)}
$$

Thus,

$$
\left\|\overline{\mathbf{U}}_{h}\right\|_{H^{1}(\Omega)} \leq\left\|I_{h}(\overline{\mathbf{U}})\right\|_{H^{1}(\Omega)}+C_{7}\left|\mathbf{U}-I_{h}(\mathbf{U})\right|_{H^{1}(\Omega)} \leq C_{8}\|U\|_{H^{5 / 4}\left(\Gamma_{\text {in }}\right)} .
$$

The estimate for $p_{h}$ is an immediate consequence of the bound for $\mathbf{u}_{h}$ and the inf-sup condition (3.17).

## B. Error Analysis

Let $(\mathbf{u}, p)$ be the solution of (2.12) and $\left(\mathbf{u}_{h}, p_{h}\right)$ the solution of (3.16). Subtracting (2.12) with test function $\mathbf{v}_{h}$ from (3.16), we obtain

$$
\begin{aligned}
\forall \mathbf{v}_{h} \in X_{h}, \quad \alpha \int_{\Omega}(\rho \mathbf{u}- & \left.\rho_{h} \mathbf{u}_{h}\right) \cdot \mathbf{v}_{h} d \mathbf{x}+\frac{1}{2} \int_{\Omega}\left(\mu A_{1}(\mathbf{u})-\mu_{h} A_{1}\left(\mathbf{u}_{h}\right)\right): A_{1}\left(\mathbf{v}_{h}\right) d \mathbf{x} \\
& -\int_{\Omega}\left(p-p_{h}\right) \operatorname{div} \mathbf{v}_{h} d \mathbf{x}=\int_{\Omega}\left(\rho-\rho_{h}\right) \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}-\left(K\left(\mathbf{v}_{h}\right)-K_{h}\left(\mathbf{v}_{h}\right)\right) .
\end{aligned}
$$

The first term in the left-hand side can be split into

$$
\int_{\Omega}\left(\rho \mathbf{u}-\rho_{h} \mathbf{u}_{h}\right) \cdot \mathbf{v}_{h} d \mathbf{x}=\int_{\Omega}\left(\rho-\rho_{h}\right) \mathbf{u} \cdot \mathbf{v}_{h} d \mathbf{x}+\int_{\Omega} \rho_{h}\left(\mathbf{u}-\mathbf{u}_{h}\right) \cdot \mathbf{v}_{h} d \mathbf{x}
$$

and similarly for the second term. Thus reverting to the forms $a_{h}$ and $b$, we derive the error equation:

$$
\begin{aligned}
& \forall \mathbf{v}_{h} \in X_{h}, \quad a_{h}\left(\mathbf{u}-\mathbf{u}_{h}, \mathbf{v}_{h}\right)+b\left(\mathbf{v}_{h}, p-p_{h}\right)=\alpha \int_{\Omega}\left(\rho_{h}-\rho\right) \mathbf{u} \cdot \mathbf{v}_{h} d \mathbf{x} \\
& +\frac{1}{2} \int_{\Omega}\left(\mu_{h}-\mu\right) A_{1}(\mathbf{u}): A_{1}\left(\mathbf{v}_{h}\right) d \mathbf{x}+\int_{\Omega}\left(\rho-\rho_{h}\right) \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}-\left(K\left(\mathbf{v}_{h}\right)-K_{h}\left(\mathbf{v}_{h}\right)\right) \\
& \forall q_{h} \in M_{h}, b\left(\mathbf{u}_{h}-\mathbf{u}, q_{h}\right)=0 .
\end{aligned}
$$

Splitting $\mathbf{u}$ into $\mathbf{u}_{0}+\overline{\mathbf{U}}$ and $\mathbf{u}_{h}$ into $\mathbf{u}_{0, h}+\overline{\mathbf{U}}_{h}$, inserting $\tilde{I}_{h}\left(\mathbf{u}_{0}\right)$, and using the fact that $\operatorname{div} \overline{\mathbf{U}}=0$ and $\overline{\mathbf{U}}_{h}$ satisfies (3.10), this becomes, for any $q_{h} \in M_{h}$,

$$
\begin{gather*}
\forall \mathbf{v}_{h} \in V_{h}, \quad a_{h}\left(\tilde{I}_{h}\left(\mathbf{u}_{0}\right)-\mathbf{u}_{0, h}, \mathbf{v}_{h}\right)=a_{h}\left(\tilde{I}_{h}\left(\mathbf{u}_{0}\right)-\mathbf{u}_{0}, \mathbf{v}_{h}\right)-a_{h}\left(\overline{\mathbf{U}}-\overline{\mathbf{U}}_{h}, \mathbf{v}_{h}\right) \\
+\alpha \int_{\Omega}\left(\rho_{h}-\rho\right) \mathbf{u} \cdot \mathbf{v}_{h} d \mathbf{x}+\frac{1}{2} \int_{\Omega}\left(\mu_{h}-\mu\right) A_{1}(\mathbf{u}): A_{1}\left(\mathbf{v}_{h}\right) d \mathbf{x} \\
+b\left(\mathbf{v}_{h}, q_{h}-p\right)+\int_{\Omega}\left(\rho-\rho_{h}\right) \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}-\left(K\left(\mathbf{v}_{h}\right)-K_{h}\left(\mathbf{v}_{h}\right)\right),  \tag{3.23}\\
\forall q_{h} \in M_{h}, b\left(\tilde{I}_{h}\left(\mathbf{u}_{0}\right)-\mathbf{u}_{0, h}, q_{h}\right)=0 . \tag{3.24}
\end{gather*}
$$

This is the starting point for the main result of this section. To derive an error estimate from (3.23), we must find a bound for the terms involving $\rho_{h}-\rho, \mu_{h}-\mu$, and $K-K_{h}$. Consider for instance the term involving $\mu_{h}-\mu$; we can write

$$
\begin{equation*}
\left|\int_{\Omega}\left(\mu_{h}-\mu\right) A_{1}(\mathbf{u}): A_{1}\left(\mathbf{v}_{h}\right) d \mathbf{x}\right| \leq 4\left|\mu^{1}-\mu^{2}\right|\|\nabla \mathbf{u}\|_{L^{2}(\omega)}\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}(\omega)} \tag{3.25}
\end{equation*}
$$

where $\omega$ is the region where $\mu_{h}$ differs from $\mu$, i.e.,

$$
\omega=\left(\Omega^{1} \cap \Omega_{h}^{2}\right) \cup\left(\Omega^{2} \cap \Omega_{h}^{1}\right)
$$

Now, for $\varepsilon>0$, we consider

$$
\omega_{\varepsilon}=\{\mathbf{x} \in \Omega, \operatorname{dist}(\mathbf{x}, \Gamma)<\varepsilon\} .
$$

Then since $\omega \subset \omega_{h}$, we shall use the following lemma. Recall that $\hat{H}^{2}(\Omega)$ denotes the space of all functions in $H^{1}(\Omega)$ whose restriction to $\Omega^{i}$ is in $H^{2}\left(\Omega^{i}\right)$, for $i=1,2$.

Lemma 3.4. Let $\Gamma$ be of class $\mathcal{C}^{2}$ and set

$$
\begin{equation*}
\varepsilon_{0} \leq \frac{1}{2} \min |R| \tag{3.26}
\end{equation*}
$$

where $R$ is the signed radius of curvature of $\Gamma$. There exists $C>0$ such that, for all $\varepsilon \leq \varepsilon_{0}$,

$$
\begin{array}{cc}
\|u\|_{L^{2}\left(\omega_{\varepsilon}\right)} \leq C \sqrt{\varepsilon}\|u\|_{H^{1}(\Omega)} & \forall u \in H^{1}(\Omega), \\
\|\nabla u\|_{L^{2}\left(\omega_{\varepsilon}\right)} \leq C \sqrt{\varepsilon}\|u\|_{\hat{H}^{2}(\Omega)} & \forall u \in \hat{H}^{2}(\Omega) . \tag{3.28}
\end{array}
$$

Proof. For $\varepsilon$ sufficiently small, any $\mathbf{x} \in \omega_{\varepsilon}$ can be represented in a unique way by the pair $(s, \eta)$, where $s$ is the curvilinear coordinate along $\Gamma$ of the projection of $\mathbf{x}$ onto $\Gamma, 0 \leq s \leq \ell$, and $\eta$ is the signed distance from $\mathbf{x}$ to $\Gamma,-\varepsilon \leq \eta \leq \varepsilon$ (see Fig. 2). With the notation of (3.12), we have

$$
\mathbf{x}=\boldsymbol{\xi}(s)+\eta \mathbf{n}^{1}, \quad d \mathbf{x}=\mathbf{t} d s+\mathbf{n}^{1} d \eta+\eta \frac{\mathbf{t}}{R} d s,
$$

where $R$ is the radius of curvature of $\Gamma$ at the point $\boldsymbol{\xi}(s)$. The jacobian of the mapping $(s, \eta) \mapsto \mathbf{x}$ is

$$
J(s, \eta)=\left|\begin{array}{cc}
1+\frac{\eta}{R} & 0 \\
0 & 1
\end{array}\right| .
$$

From now on, we assume that $\varepsilon$ is smaller than $\varepsilon_{0}=\frac{1}{2} \min |R| ;$ since $\Gamma$ is of class $\mathcal{C}^{2}$, this minimum is strictly positive. Then

$$
\begin{equation*}
\frac{1}{2} \leq J(s, \eta) \leq \frac{3}{2} \tag{3.29}
\end{equation*}
$$

thus the mapping is invertible and

$$
\begin{equation*}
\frac{1}{J(s, \eta)} \leq 2 \tag{3.30}
\end{equation*}
$$

Let $v$ be a smooth function on $\Omega$. On has

$$
v(s, \eta)^{2}=\left(v(s, 0)+\int_{0}^{\eta} \nabla_{s, \eta} v \cdot \mathbf{n}(s) d \eta\right)^{2} \leq 2 v(s, 0)^{2}+2 \varepsilon \int_{-\varepsilon}^{\varepsilon}\left|\nabla_{s, \eta} v(s, \eta)\right|^{2} d \eta .
$$

Therefore,

$$
\begin{aligned}
& \int_{\omega_{\varepsilon}} v^{2}(\mathbf{x}) d \mathbf{x}=\int_{0}^{\ell} \int_{-\varepsilon}^{\varepsilon} v^{2}(s, \eta) J(s, \eta) d \eta d s \\
& \leq 2 \int_{0}^{\ell} \int_{-\varepsilon}^{\varepsilon} v(s, 0)^{2} J(s, \eta) d \eta d s \\
&+2 \varepsilon \int_{0}^{\ell} \int_{-\varepsilon}^{\varepsilon}\left(\int_{-\varepsilon}^{\varepsilon}\left|\nabla_{s, \eta} v(s, \eta)\right|^{2} d \eta\right) J(s, \gamma) d \gamma d s .
\end{aligned}
$$

Applying (3.29) and (3.30) in this inequality and finally applying the trace theorem, we derive

$$
\begin{equation*}
\int_{\omega_{\varepsilon}} v^{2}(\mathbf{x}) d \mathbf{x} \leq C \varepsilon\|v\|_{H^{1}(\Omega)}^{2} \tag{3.31}
\end{equation*}
$$

The estimate on $\|u\|_{L^{2}\left(\omega_{\varepsilon}\right)}$ follows from (3.31) and the density of smooth functions in $H^{1}(\Omega)$.
The second estimate is obtained in a similar way.

Now, applying (3.28) with $\varepsilon=h$ to $\nabla \mathbf{u}$ in (3.25), we obtain, if $\mathbf{u} \in X \cap \hat{H}^{2}(\Omega)^{2}$ :

$$
\begin{equation*}
\left|\int_{\Omega}\left(\mu_{h}-\mu\right) A_{1}(\mathbf{u}): A_{1}\left(\mathbf{v}_{h}\right) d \mathbf{x}\right| \leq 4 C\left|\mu^{1}-\mu^{2}\right| h^{1 / 2}\|\mathbf{u}\|_{\hat{H}^{2}(\Omega)}\left|\mathbf{v}_{h}\right|_{H^{1}(\Omega)} . \tag{3.32}
\end{equation*}
$$

Similarly, applying (3.27) with $\varepsilon=h$ to $\mathbf{u}$ and to $\mathbf{v}_{h}$, we derive

$$
\begin{equation*}
\left|\int_{\Omega}\left(\rho_{h}-\rho\right) \mathbf{u} \cdot \mathbf{v}_{h} d \mathbf{x}\right| \leq C\left|\rho^{1}-\rho^{2}\right| h\|\mathbf{u}\|_{H^{1}(\Omega)}\left|\mathbf{v}_{h}\right|_{H^{1}(\Omega)} . \tag{3.33}
\end{equation*}
$$

Finally, the term involving $\mathbf{F}$ is bounded by:

$$
\begin{equation*}
\left|\int_{\Omega}\left(\rho_{h}-\rho\right) \mathbf{F} \cdot \mathbf{v}_{h} d \mathbf{x}\right| \leq C\left|\rho^{1}-\rho^{2}\right| h\|\mathbf{F}\|_{H^{1}(\Omega)}\left|\mathbf{v}_{h}\right|_{H^{1}(\Omega)} \tag{3.34}
\end{equation*}
$$

Next, we derive a bound for the term involving $K-K_{h}$; this is the object of the next proposition.
Proposition 3.5. Let $\Gamma$ be of class $\mathcal{C}^{2}$ and suppose that $h \leq \varepsilon_{0}$, defined by (3.26). If $\mathcal{T}_{h}$ satisfies (3.1), there exists a constant $C$, independent of $h$, such that

$$
\begin{equation*}
\forall \mathbf{v}_{h} \in X_{h}, \quad\left|K\left(\mathbf{v}_{h}\right)-K_{h}\left(\mathbf{v}_{h}\right)\right| \leq C \kappa h^{1 / 2}\left|\mathbf{v}_{h}\right|_{1, \Omega} . \tag{3.35}
\end{equation*}
$$

Proof. Let us sketch the proof. Integrating by parts $K\left(\mathbf{v}_{h}\right)$ and using the fact that $\mathbf{v}_{h}\left(\mathbf{x}_{0}\right)=\mathbf{0}$ (since $\mathbf{x}_{0} \in \Gamma_{\text {in }}$ ), we obtain

$$
\begin{aligned}
K\left(\mathbf{v}_{h}\right)=-\kappa & \sum_{i=0}^{N-1} \int_{\tilde{S}_{i}} \mathbf{t} \cdot \frac{d \mathbf{v}_{h}}{d s} d s+\kappa \mathbf{t}\left(\mathbf{x}_{N}\right) \cdot \mathbf{v}_{h}\left(\mathbf{x}_{N}\right) \\
= & -\kappa \sum_{i=0}^{N-1} \int_{\tilde{S}_{i}}\left(\mathbf{t}-\mathbf{t}_{i}\right) \cdot \frac{d \mathbf{v}_{h}}{d s} d s-\kappa \sum_{i=0}^{N-1} \int_{\tilde{S}_{i}} \mathbf{t}_{i} \cdot \frac{d \mathbf{v}_{h}}{d s} d s+\kappa \mathbf{t}\left(\mathbf{x}_{N}\right) \cdot \mathbf{v}_{h}\left(\mathbf{x}_{N}\right) \\
& =-\kappa \sum_{i=0}^{N-1} \int_{\tilde{S}_{i}}\left(\mathbf{t}-\mathbf{t}_{i}\right) \cdot \frac{d \mathbf{v}_{h}}{d s} d s-\kappa \sum_{i=0}^{N-1} \mathbf{t}_{i} \cdot\left(\mathbf{v}_{h}\left(\mathbf{x}_{i+1}\right)-\mathbf{v}_{h}\left(\mathbf{x}_{i}\right)\right)+\kappa \mathbf{t}\left(\mathbf{x}_{N}\right) \cdot \mathbf{v}_{h}\left(\mathbf{x}_{N}\right) .
\end{aligned}
$$

By summation by parts, the second sum can be written

$$
-\kappa \sum_{i=0}^{N-1} \mathbf{t}_{i} \cdot\left(\mathbf{v}_{h}\left(\mathbf{x}_{i+1}\right)-\mathbf{v}_{h}\left(\mathbf{x}_{i}\right)\right)=\kappa \sum_{i=1}^{N-1} \mathbf{v}_{h}\left(\mathbf{x}_{i}\right) \cdot\left(\mathbf{t}_{i}-\mathbf{t}_{i-1}\right)-\kappa \mathbf{v}_{h}\left(\mathbf{x}_{N}\right) \cdot \mathbf{t}_{N-1} .
$$

Thus setting

$$
e_{h}=\kappa \sum_{i=0}^{N-1} \int_{\tilde{S}_{i}}\left(\mathbf{t}-\mathbf{t}_{i}\right) \cdot \frac{d \mathbf{v}_{h}}{d s} d s, \delta_{h}=\kappa\left(\mathbf{t}\left(\mathbf{x}_{N}\right)-\mathbf{t}_{N-1}\right) \cdot \mathbf{v}_{h}\left(\mathbf{x}_{N}\right),
$$

we find the error equation:

$$
K\left(\mathbf{v}_{h}\right)-K_{h}\left(\mathbf{v}_{h}\right)=-e_{h}+\delta_{h} .
$$

On one hand, the smoothness of $\Gamma$ implies that

$$
\left|\mathbf{t}\left(\mathbf{x}_{N}\right)-\mathbf{t}_{N-1}\right| \leq C h_{N},
$$

where $h_{N}$ denotes the maximum mesh length of the elements where $\mathbf{t}_{N-1}$ is defined. On the other hand, reverting to the reference element, where all norms are equivalent, we find

$$
h_{N}\left|\mathbf{v}_{h}\left(\mathbf{x}_{N}\right)\right| \leq C\left\|\mathbf{v}_{h}\right\|_{L^{2}\left(\omega_{h}\right)} .
$$

Hence, Lemma 3.4 gives

$$
\begin{equation*}
\left|\delta_{h}\right| \leq C \kappa h^{1 / 2}\left|\mathbf{v}_{h}\right|_{1, \Omega} \tag{3.36}
\end{equation*}
$$

The proof for estimating $e_{h}$ is similar, but somewhat more technical because it involves integrals along the $\operatorname{arcs} \tilde{S}_{i}$ and it makes use of the particular polynomial structure of $\mathbf{v}_{h}$. It gives

$$
\begin{equation*}
\left|e_{h}\right| \leq C \kappa h^{1 / 2}\left|\mathbf{v}_{h}\right|_{1, \Omega} \tag{3.37}
\end{equation*}
$$

In fact, by using a Sobolev imbedding, we can derive a sharper estimate for $\delta_{h}$, but improving (3.36) does not improve the final result since (3.37) is unchanged.

The next theorem follows easily by substituting (3.32)-(3.35) into (3.23), applying (2.18), (3.17), the properties of the lifting $\overline{\mathbf{U}}$ stated in Proposition 3.1, the properties of $\tilde{I}_{h}$ stated in the corollary of Proposition 3.2 and the standard approximation properties of $M_{h}$.

Theorem 3.6. Under the assumptions of Propositions 3.2 and 3.5, if $\left.\mathbf{u}\right|_{\Omega_{i}} \in H^{2}\left(\Omega_{i}\right)^{2}, U \in$ $H^{2}\left(\Gamma_{\mathrm{in}}^{i}\right)$, and $\left.p\right|_{\Omega_{i}} \in H^{1}\left(\Omega_{i}\right)$ for $i=1,2$, the scheme (3.16) is of order one half:

$$
\begin{equation*}
\left\|\mathbf{u}_{h}-\mathbf{u}\right\|_{H^{1}(\Omega)}+\beta^{*}\left\|p_{h}-p\right\|_{L^{2}(\Omega)} \leq C h^{1 / 2}\left(\sum_{i=1}^{2}\left(|\mathbf{u}|_{H^{2}\left(\Omega_{i}\right)}+|p|_{H^{1}\left(\Omega_{i}\right)}+\|U\|_{H^{2}\left(\Gamma_{\text {in }}^{i}\right)}\right)+\kappa\right) \tag{3.38}
\end{equation*}
$$

with a constant $C$ that depends on $\alpha, \rho, \mu, \lambda, \kappa$, and $\mathbf{F}$, but is independent of $h$.
It is easy to see that the loss of $h^{1 / 2}$ in (3.38) arises purely from the approximation of $\mu$ in (3.32) and the approximation of $K$.

Except for the treatment of the surface tension term, this analysis carries over to the threedimensional case without difficulty. Numerically, the approximation of the surface tension given by (3.14) can be extended in a natural way to three dimensions, as is done in [14]. However, the numerical analysis of this approximation still raises a number of technical issues.

## 4. APPENDIX

This section presents the proofs of general results that are not restricted to two-fluid flows. We start with Theorem 2.1. Recall its statement as follows.

Theorem 2.1. Let $\omega$ be a bounded Lipschitz domain of $\mathbb{R}^{n}, n \geq 2$. Then $\mathcal{D}(\bar{\omega})^{n \times n} \times \mathcal{D}(\bar{\omega})$ is dense in $W$ :

$$
W=\left\{(L, p) \in L^{2}(\omega)^{n \times n} \times L^{2}(\omega) ; \nabla \cdot(-\mu L+p I) \in L^{2}(\omega)^{n}, \nabla(\operatorname{tr} L) \in L^{2}(\omega)^{n}\right\} .
$$

Proof. Let us sketch the proof; its steps are classical.

1. Let us consider first the case where $\omega$ is star-shaped with respect to the origin. The same result for a domain that is star-shaped with respect to an arbirary interior point is easily obtained by translation. Let $\theta \geq 1$ be a parameter that will tend to 1 , and set $\omega_{\theta}=\theta \omega$. Then $\bar{\omega} \subset \omega_{\theta}$ and we can"dilate" $L$ and $p$ by

$$
L_{\theta}(\mathbf{x})=L\left(\frac{\mathbf{x}}{\theta}\right), \quad p_{\theta}(\mathbf{x})=p\left(\frac{\mathbf{x}}{\theta}\right), \quad \forall \mathbf{x} \in \omega_{\theta}
$$

Furthermore, a standard argument shows that

$$
\lim _{\theta \rightarrow 1}\left\|\left(L_{\theta}, p_{\theta}\right)-(L, p)\right\|_{W}=0
$$

2. Now that $L$ and $p$ are extended outside $\omega$, they can be regularized by convolution with a sequence of standard mollifiers $\rho_{\epsilon}$, where $\epsilon>0$ is bounded by an adequate function of $\theta$, e.g., $\epsilon<(\theta-1) / 2$. Choosing for instance $\theta=1+(1 / N)$, keeping $N$ fixed for the moment, set

$$
L_{\epsilon, N}=\rho_{\epsilon} * L_{\theta}, \quad p_{\epsilon, N}=\rho_{\epsilon} * p_{\theta} .
$$

Then properties of the convolution imply that

$$
L_{\epsilon, N} \in \mathcal{D}(\bar{\omega})^{n \times n}, \quad p_{\epsilon, N} \in \mathcal{D}(\bar{\omega})
$$

and

$$
\lim _{\epsilon \rightarrow 1}\left\|\left(L_{\epsilon, N}, p_{\epsilon, N}\right)-\left(L_{N}, p_{N}\right)\right\|_{W}=0
$$

This readily gives the density of $\mathcal{D}(\bar{\omega})^{n \times n} \times \mathcal{D}(\bar{\omega})$ in $W$.
3. As any Lipschitz domain is a finite union of star-shaped domains, this density extends to a Lipschitz domain.

Remark. Recall that $\mathcal{D}_{s}(\bar{\omega})^{n \times n}$ is the subspace of the symmetric tensors of $\mathcal{D}(\bar{\omega})^{n \times n}$. Then the above proof shows that $\mathcal{D}_{s}(\bar{\omega})^{n \times n} \times \mathcal{D}(\bar{\omega})^{n \times n}$ is dense in $W_{s}$, defined in (2.7).

Next, we prove Proposition 2.3.
Proposition 2.3. There exists a constant $\beta>0$ such that

$$
\forall q \in M, \quad \sup _{\mathbf{v} \in X} \frac{1}{\|\nabla \mathbf{v}\|_{L^{2}(\Omega)}} \int_{\Omega} q \nabla \cdot \mathbf{v} d \mathbf{x} \geq \beta\|q\|_{L^{2}(\Omega)}
$$

Proof. The inf-sup condition is well known when $q$ has zero mean value in $\Omega$, which is not the case here. There are several ways for turning this difficulty. One of them consists in splitting $q$ into

$$
q=q^{*}+\bar{q}
$$

where

$$
\bar{q}=\frac{1}{|\Omega|} \int_{\Omega} q d \mathbf{x}
$$

and $q^{*}$ now has zero mean value. On one hand, it is well known that there exists $\mathbf{v}^{*} \in H_{0}^{1}(\Omega)^{2}$ such that

$$
\begin{equation*}
\nabla \cdot \mathbf{v}^{*}=q^{*}, \quad\left\|\nabla \mathbf{v}^{*}\right\|_{L^{2}(\Omega)} \leq \frac{1}{\beta^{*}}\left\|q^{*}\right\|_{L^{2}(\Omega)} \tag{4.1}
\end{equation*}
$$

with a constant $\beta^{*}>0$ that depends on $\Omega$ only.
On the other hand, we choose

$$
\overline{\mathbf{v}}=|\Omega| \bar{q} \varrho \mathbf{n}
$$

where $\mathbf{n}$ is the unit exterior normal to $\Gamma_{\text {out }}\left(\right.$ i.e., $\left.\mathbf{n}=(1,0)^{t}\right), \varrho$ is a smooth non-negative function that vanishes identically in a neighborhood of $\partial \Omega$ except in the neighborhood of $\Gamma_{\text {out }}$. Its trace on $\Gamma_{\text {out }}$ has compact support and satisfies

$$
\begin{equation*}
\int_{\Gamma_{\text {out }}} \varrho d s=1 . \tag{4.2}
\end{equation*}
$$

This choice of $\overline{\mathbf{v}}$ is justified by the fact that

$$
\int_{\Omega} \bar{q} \nabla \cdot \overline{\mathbf{v}} d \mathbf{x}=\|\bar{q}\|_{L^{2}(\Omega)}^{2}
$$

Then (2.19) is established by the technique of Boland and Nicolaides (cf. [22]) which consists of associating with $q$ an adequate linear combination of $\overline{\mathbf{v}}$ and $\mathbf{v}^{*}: \mathbf{v}=\gamma \mathbf{v}^{*}+\overline{\mathbf{v}}$. Thus, we have

$$
\begin{equation*}
\int_{\Omega} q \nabla \cdot \mathbf{v} d \mathbf{x}=\gamma\left\|q^{*}\right\|_{L^{2}(\Omega)}^{2}+\int_{\Omega} q^{*} \nabla \cdot \overline{\mathbf{v}} d \mathbf{x}+\|\bar{q}\|_{L^{2}(\Omega)}^{2} \tag{4.3}
\end{equation*}
$$

Now,

$$
\int_{\Omega} q^{*} \nabla \cdot \overline{\mathbf{v}} d \mathbf{x}=|\Omega| \int_{\Omega} q^{*} \nabla \cdot(\bar{q} \varrho \mathbf{n}) d \mathbf{x}=|\Omega| \bar{q} \int_{\Omega} q^{*} \nabla \varrho \cdot \mathbf{n} d \mathbf{x},
$$

and the Cauchy-Schwartz inequality gives,

$$
\int_{\Omega} q^{*} \nabla \cdot \overline{\mathbf{v}} d \mathbf{x} \geq-|\Omega|^{\frac{1}{2}}\left\|q^{*}\right\|_{L^{2}(\Omega)}\|\bar{q}\|_{L^{2}(\Omega)}\|\nabla \varrho\|_{L^{2}(\Omega)}
$$

Replacing it in (4.3) and choosing $\gamma=|\Omega|\|\nabla \varrho\|_{L^{2}(\Omega)}^{2}$, we obtain,

$$
\begin{equation*}
\int_{\Omega} q \nabla \cdot \mathbf{v} d \mathbf{x} \geq C_{1}\left(\left\|q^{*}\right\|_{L^{2}(\Omega)}^{2}+\|\bar{q}\|_{L^{2}(\Omega)}^{2}\right) \tag{4.4}
\end{equation*}
$$

with $C_{1}=\frac{1}{2} \min \left(|\Omega|\|\nabla \varrho\|_{L^{2}(\Omega)}^{2}, 1\right)$.
Now, taking into account (4.1), the definition of $\overline{\mathbf{v}}$ and the above choice of $\gamma$, we derive

$$
\|\nabla \mathbf{v}\|_{L^{2}(\Omega)} \leq C_{2}\|q\|_{L^{2}(\Omega)}
$$

where

$$
C_{2}=\|\nabla \varrho\|_{L^{2}(\Omega)}\left(\frac{1}{\left(\beta^{*}\right)^{2}}\|\nabla \varrho\|_{L^{2}(\Omega)}^{2}+\frac{1}{|\Omega|}\right)^{1 / 2}
$$

and the inf-sup condition follows with $\beta=C_{1} / C_{2}$.
Finally, we prove Proposition 3.2.

Proposition 3.2. Let $\mathcal{T}_{h}$ satisfy (3.1) and the assumptions of Section 3.A. There exists a constant $\beta^{\star}>0$ such that the following inf-sup condition holds:

$$
\forall q_{h} \in M_{h}, \quad \sup _{\mathbf{v}_{h} \in X_{h}} \frac{1}{\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}(\Omega)}} \int_{\Omega} q_{h} \nabla \cdot \mathbf{v}_{h} d \mathbf{x} \geq \beta^{\star}\left\|q_{h}\right\|_{L^{2}(\Omega)} .
$$

Proof. The proof proceeds in three steps. All constants in this proof are independent of $h$ and of the index $j$ of the macro-elements.

1. To simplify, we drop the subscript $T$ of the macro-elements. We construct an auxiliary approximation operator $P_{h} \in \mathcal{L}\left(X ; X_{h}\right)$ such that

$$
\begin{gather*}
\forall \mathbf{v} \in X, \quad \forall 1 \leq j \leq R, \int_{\mathcal{O}_{j}} \operatorname{div}\left(P_{h}(\mathbf{v})-\mathbf{v}\right) d \mathbf{x}=0,  \tag{4.5}\\
\forall \mathbf{v} \in X, \quad\left|P_{h}(\mathbf{v})\right|_{H^{1}(\Omega)} \leq C_{1}|\mathbf{v}|_{H^{1}(\Omega)}, \tag{4.6}
\end{gather*}
$$

with a constant $C_{1}$ independent of $\mathbf{v}$. Here is the construction. Let $F_{h}$ denote the set of the "sides" of $\mathcal{O}_{j}$, for $1 \leq j \leq R$. Since the condition

$$
\forall F \in F_{h}, \quad \int_{F}\left(P_{h}(\mathbf{v})-\mathbf{v}\right) d s=\mathbf{0},
$$

is sufficient for (4.5), we choose

$$
P_{h}(\mathbf{v})=R_{h}(\mathbf{v})+\mathbf{c}_{h}(\mathbf{v}),
$$

where $R_{h} \in \mathcal{L}\left(X ; X_{h}\right)$ is the Scott and Zhang [21] regularization operator, that we can construct so that $\left.R_{h}(\mathbf{v})\right|_{T} \in P_{1}^{2}$, and $\mathbf{c}_{h}(\mathbf{v}) \in X_{h}$ is a correction of the form

$$
\mathbf{c}_{h}(\mathbf{v})=\sum_{F \in F_{h}} \mathbf{c}_{F}(\mathbf{v}),
$$

where

$$
\mathbf{c}_{F}(\mathbf{v})=\mathbf{c} \varphi_{F},
$$

with

$$
\begin{equation*}
\mathbf{c}=\frac{1}{|F|} \int_{F}\left(\mathbf{v}-R_{h}(\mathbf{v})\right) d s \tag{4.7}
\end{equation*}
$$

and $\varphi_{F}$ is the piecewise $P_{1}$ basis function that takes the value one at the vertex $\mathbf{a}_{F}$ shared by the two segments of $F$ and zero at all other vertices. It is easy to see that with this definition, $P_{h}(\mathbf{v})$ satisfies (4.5).

As far as (4.6) is concerned, we have on one hand,

$$
\left|\varphi_{F}\right|_{H^{1}\left(\mathcal{O}_{j}\right)} \leq C_{2} .
$$

On the other hand, let us fix once and for all a reference macro-element, $\hat{\mathcal{O}}$ (union of four reference triangles), image of $\mathcal{O}_{j}$ by a continuous mapping $\mathcal{F}_{j}$, affine in each $T$ of $\mathcal{O}_{j}$. Then reverting to $\hat{\mathcal{O}}$, and applying the trace theorem there, we obtain

$$
\left|\mathbf{c}_{F}\right| \leq C_{3}\left\|\left(\mathbf{v}-R_{h}(\mathbf{v})\right) \circ \mathcal{F}_{j}\right\|_{L^{2}(\hat{F})} \leq C_{4}\left\|\left(\mathbf{v}-R_{h}(\mathbf{v})\right) \circ \mathcal{F}_{j}\right\|_{H^{1}(\hat{\mathcal{O}})} \leq C_{5}|\mathbf{v}|_{H^{1}\left(\Delta_{\mathcal{O}_{j}}\right)},
$$

where $\Delta_{\mathcal{O}_{j}}$ is the macro-element used for the definition of $R_{h}(\mathbf{v})$ in $\mathcal{O}_{j}$. Then (4.6) follows readily from these inequalities.
2. Now, let us prove a discrete inf-sup condition locally in each $\mathcal{O}_{j}$ between the following spaces:

$$
\begin{gathered}
\tilde{X}_{h}=\left\{\mathbf{v}_{h} \in X_{h} ; \forall T \in \mathcal{T}_{h},\left.\mathbf{v}_{h}\right|_{T}=\mathbf{c}_{T} b_{T}, \mathbf{c}_{T} \in \mathbb{R}^{2}\right\}, \\
\tilde{M}_{h}=\left\{\tau_{h}\left(q_{h}\right) ; q_{h} \in M_{h}\right\},
\end{gathered}
$$

where $b_{T}$ is the "bubble" function and, for all $q_{h} \in M_{h}$ and all $j$,

$$
\left.\tau_{h}\left(q_{h}\right)\right|_{\mathcal{O}_{j}}=\left.q_{h}\right|_{\mathcal{O}_{j}}-\frac{1}{\left|\mathcal{O}_{j}\right|} \int_{\mathcal{O}_{j}} q_{h} d \mathbf{x} \in L_{0}^{2}\left(\mathcal{O}_{j}\right)
$$

We associate with any $q_{h} \in \tilde{M}_{h}$ the function $\mathbf{v}_{h} \in \tilde{X}_{h}$ defined in each $T$ by

$$
\left.\mathbf{v}_{h}\right|_{T}=\left.\left(\nabla q_{h}\right)\right|_{T} b_{T},
$$

and to simplify, we denote $\left.\left(\nabla q_{h}\right)\right|_{T}$ by $\nabla q_{T}$. Then, on one hand,

$$
-\int_{\mathcal{O}_{j}} q_{h} \operatorname{div} \mathbf{v}_{h} d \mathbf{x}=\sum_{T \in \mathcal{O}_{j}}\left|\nabla q_{T}\right|^{2} \int_{T} b_{T} d \mathbf{x}=C_{6}|T| \sum_{T \in \mathcal{O}_{j}}\left|\nabla q_{T}\right|^{2},
$$

where

$$
C_{6}=\frac{1}{|\hat{T}|} \int_{\hat{\lambda}} \hat{\lambda}_{1} \hat{\lambda}_{2} \hat{\lambda}_{3} d \hat{\mathbf{x}}
$$

On the other hand,

$$
\left|\mathbf{v}_{h}\right|_{H^{1}\left(\mathcal{O}_{j}\right)} \leq C_{7}\left(\sum_{T \in \mathcal{O}_{j}} \frac{|T|}{\rho_{T}^{2}}\left|\nabla q_{T}\right|^{2}\right)^{1 / 2} \leq C_{7} \frac{1}{\rho_{j}}\left(\sum_{T \in \mathcal{O}_{j}}|T|\left|\nabla q_{T}\right|^{2}\right)^{1 / 2},
$$

where

$$
\rho_{j}=\inf _{T \in \mathcal{O}_{j}} \rho_{T} .
$$

Hence

$$
\begin{equation*}
-\frac{\int_{\mathcal{O}_{j}} q_{h} \operatorname{div} \mathbf{v}_{h} d \mathbf{x}}{\left|\mathbf{v}_{h}\right|_{H^{1}\left(\mathcal{O}_{j}\right)}} \geq C_{8} \rho_{j}\left(\sum_{T \in \mathcal{O}_{j}}|T|\left|\nabla q_{T}\right|^{2}\right)^{1 / 2}=C_{8} \rho_{j}\left|q_{h}\right|_{H^{1}\left(\mathcal{O}_{j}\right)} . \tag{4.8}
\end{equation*}
$$

The desired inf-sup condition follows from the fact that $\left.q_{h}\right|_{\mathcal{O}_{j}} \in L_{0}^{2}\left(\mathcal{O}_{j}\right)$. Indeed, this implies that (cf. [20]) there exists a constant $C_{9}$, such that

$$
\begin{equation*}
\left\|q_{h}\right\|_{L^{2}\left(\mathcal{O}_{j}\right)} \leq C_{9} h_{j}\left|q_{h}\right|_{H^{1}\left(\mathcal{O}_{j}\right)} \tag{4.9}
\end{equation*}
$$

where

$$
h_{j}=\sup _{T \in \mathcal{O}_{j}} h_{T} .
$$

Then (4.8), (4.9) and the regularity of $\mathcal{T}_{h}$ yield the local inf-sup condition:

$$
\begin{equation*}
-\frac{\int_{\mathcal{O}_{j}} q_{h} \operatorname{div} \mathbf{v}_{h} d \mathbf{x}}{\left|\mathbf{v}_{h}\right|_{H^{1}\left(\mathcal{O}_{j}\right)}} \geq C_{10} \frac{\rho_{j}}{h_{j}}\left\|q_{h}\right\|_{L^{2}\left(\mathcal{O}_{j}\right)} \geq C_{11}\left\|q_{h}\right\|_{L^{2}\left(\mathcal{O}_{j}\right)} \tag{4.10}
\end{equation*}
$$

3. Finally, we construct an approximation operator $\Pi_{h} \in \mathcal{L}\left(X ; X_{h}\right)$ such that

$$
\begin{gather*}
\forall \mathbf{v} \in X, \int_{\Omega} \operatorname{div}\left(\Pi_{h}(\mathbf{v})-\mathbf{v}\right) d \mathbf{x}=0  \tag{4.11}\\
\forall \mathbf{v} \in X, \quad\left|\Pi_{h}(\mathbf{v})\right|_{H^{1}(\Omega)} \leq C_{12}|\mathbf{v}|_{H^{1}(\Omega)} . \tag{4.12}
\end{gather*}
$$

According to Fortin's lemma (cf. for instance [20]), this implies the uniform discrete inf-sup condition. We take

$$
\Pi_{h}(\mathbf{v})=P_{h}(\mathbf{v})+\tilde{\mathbf{c}}_{h}(\mathbf{v})
$$

where $\tilde{\mathbf{c}}_{h}(\mathbf{v}) \in \tilde{X}_{h}$ is the solution in each $\mathcal{O}_{j}$ of

$$
\begin{equation*}
\forall q_{h} \in \tilde{M}_{h}, \int_{\mathcal{O}_{j}} q_{h} \operatorname{div} \tilde{\mathbf{c}}_{h}(\mathbf{v}) d \mathbf{x}=\int_{\mathcal{O}_{j}} q_{h} \operatorname{div}\left(\mathbf{v}-P_{h}(\mathbf{v})\right) d \mathbf{x} \tag{4.13}
\end{equation*}
$$

Owing to (4.10), this problem has a unique solution in $\tilde{V}_{h}^{\perp}$, where

$$
\tilde{V}_{h}=\left\{\mathbf{v}_{h} \in \tilde{X}_{h} ; \forall q_{h} \in \tilde{M}_{h}, \int_{\mathcal{O}_{j}} q_{h} \operatorname{div} \mathbf{v}_{h} d \mathbf{x}=0,1 \leq j \leq R\right\} .
$$

Furthermore, it satisfies the bound

$$
\begin{equation*}
\left|\tilde{\mathbf{c}}_{h}(\mathbf{v})\right|_{H^{1}\left(\mathcal{O}_{j}\right)} \leq \frac{1}{C_{11}}\left\|\operatorname{div}\left(\mathbf{v}-P_{h}(\mathbf{v})\right)\right\|_{L^{2}\left(\mathcal{O}_{j}\right)} \tag{4.14}
\end{equation*}
$$

Then (4.11) follows from (4.13), (4.5) and the fact that each $q_{h} \in M_{h}$ can be split in $\mathcal{O}_{j}$ into

$$
q_{h}=\tilde{q}_{h}+\frac{1}{\left|\mathcal{O}_{j}\right|} \int_{\mathcal{O}_{j}} q_{h} d \mathbf{x}
$$

where $\tilde{q}_{h} \in \tilde{M}_{h}\left(\mathcal{O}_{j}\right)$. And (4.12) follows from (4.6), (4.14) and the approximation properties of $R_{h}$.

The authors thank the referees for many helpful suggestions.

## References

1. D. Joseph and Y. Renardy, Lubricated transport, drops and miscible liquids, fundamentals of two-fluid dynamics part 2, interdisciplinary applied mathematics series, 4, Springer-Verlag, New York, 1993.
2. P. Saavedra and L. R. Scott, Variational formulation of a model free-boundary problem, Math Comp 57 (1991), 451-475.
3. A. Friedman and F. Reitich, Nonlinear stability of a quasi-static Stefan problem with surface tension: a continuation approach, Ann Scuola Norm Sup Pisa Cl Sci 30 (2001), 341-403.
4. V. A. Solonnikov, Lectures on evolution free boundary problems: classical solutions, in mathematical aspects of evolving interfaces, Eds. colli and rodrigues, lectures notes 1812, Springer-Verlag, Berlin, 2003.
5. V. A. Solonnikov, On some free boundary value problems for the Navier-Stokes equations with moving contact points and lines, Math Annalen 302 (1995), 743-772.
6. V. A. Solonnikov, Problème de frontière libre dans l'écoulement d'un fluide à la sortie d'un tube cylindrique, Asymptotic Analysis, 17 (1998), 135-163.
7. A. Friedman and J. Velázquez, The analysis of coating flows near the contact line, J Diff Eq 119 (1995), 137-208.
8. A. Friedman and J. Velázquez, The analysis of coating flow in a strip, J Diff Eq 121 (1995), 134-182.
9. J. Socolowsky, On the existence and uniqueness of two-fluid channel flows, Mathematical Modelling and Analysis 9 (2004), 67-78.
10. J. Li and Y. Renardy, Numerical study of flows of two immiscible liquids at low reynolds number, SIAM Rev, (2000), 417-439.
11. R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
12. J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.
13. J. L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications, I, Dunod, Paris, 1968.
14. B. Maury, Characteristics ALE method for the 3D Navier-Stokes equations with a free surface, Int Journal of Comp Fluid Dyn 6 (1996), 175-188.
15. I. Babuška, The finite element method with Lagrangian multipliers, Numer Math 20 (1973), 179-192.
16. F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from lagrange multipliers, RAIRO, Anal Num R2 (1974), 129-151.
17. P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978.
18. D. Arnold, F. Brezzi, and M. Fortin, A stable finite element for the Stokes equations, Calcolo 21 (1984), 337-344.
19. R. B. Kellogg and J. E. Osborn, A regularity result for the Stokes problem in a convex polygon, Journal of Funct Anal 21 (1976), 397-431.
20. V. Girault and P. A. Raviart, Finite element methods for Navier-Stokes equations. Theory and Algorithms, SCM 5, Springer-Verlag, Berlin, 1986.
21. L. R. Scott and S. Zhang, Finite element interpolation of non-smooth functions satisfying boundary conditions, Math Comp 54 (1990), 483-493.
22. I.Boland and R. Nicolaides, Stability of finite elements under divergence constraints, SIAM J Numer Anal 20 (1983), 722-731.

[^0]:    Correspondence to: Bertrand Maury, Labratoire de Mathématiques, Université Paris-Sud, 91405 Orsay Cedex, France (e-mail: Bertrand.Maury@math.u-psud.fr)
    Contract grant sponsor: Ecos-Nord
    Contract grant sponsor: FONACIT
    Contract grant sponsor: Université René Descartes
    AMS classification: $65 \mathrm{~N} 15,65 \mathrm{~N} 30,76 \mathrm{D} 07$
    © 2005 Wiley Periodicals, Inc.

