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$\mathbf{H}^1$–PROJECTION INTO THE SET OF CONVEX FUNCTIONS : A SADDLE-POINT FORMULATION

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Abstract. We investigate numerical methods to approximate the projection-operator from H_0^1 into the set of convex functions. We introduce a new formulation of the problem, based on gradient fields. It leads in a natural way to an infinite-dimensional saddle-point problem, which can be shown to be ill-posed in general. Existence and uniqueness of a saddle point is obtained for a Lagrangian defined in suitable spaces. This well-posed formulation does not lead to an implementable algorithm. Yet, numerical experiments based on a discretization of the first formulation exhibit a good behaviour.

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1. Introduction

Variational problems subject to a convexity constraint have recently received some attention, since such problems arise for instance in economics (see [6]) as well as in Newton's problem of the body of least resistance (see [2], [5]).

The numerical analysis of those problems has not been settled yet, even in the simplest cases. Our aim here is to give a numerical method to compute the H_o^1 projection of a given (non convex) function into the cone of convex functions.

Our approach is based here on gradient fields ; this is done by remarking that L^2 fields which derive from a convex potential are characterized by the nonpositivity of a certain kernel operator $\mathbf{p} \mapsto C\mathbf{p} \in L^2(\Omega^2)$. We then aim to replace our initial projection problem by a saddle-point one. The main difficulty in this natural approach is that $C^*(L^2_+)$ is not closed, so that the existence of a saddle-point has to be investigated very carefully. More precisely, we have to define dual variables and express a dual problem in suitable functional spaces.

2. Formulations of the problem

2.1. Standard formulation

We consider a bounded, convex domain $\Omega \subset \mathbb{R}^N$. The space $\mathcal{V} = H_0^1(\Omega)$ is endowed with the L^2 norm of the gradient. Let u_0 be a function in $\mathcal{V}$. The problem we consider is : find the projection of u_0 into the set of

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convex functions, which can be expressed

$$(I)\ :\left\{\begin{array}{l}\text{Find } u\in\mathcal{V}_c \text{ such that}\\ J(u)=\min\limits_{v\in\mathcal{V}_c} J(v)=\min\limits_{v\in\mathcal{V}_c}\dfrac{1}{2}\displaystyle\int_\Omega |\nabla u-\nabla u_0|^2\,,\\ \mathcal{V}_c=\{u\in\mathcal{V}\,,\ \ -u(\mathbf{y})+u(\mathbf{x})+\nabla u(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})\le 0\quad a.e. \text{ in } \Omega\times\Omega\}\,.\end{array}\right. \tag{1}$$

Let us first check that this problem is well–posed :

Proposition 2.1. *Problem (I) admits a unique solution u_I.*

Proof. The set of feasible functions $\mathcal{V}_c$ is obviously convex. Let us show that it is closed in $\mathcal{V}=H_0^1(\Omega)$. We consider a sequence (u_n) in $\mathcal{V}_c$ which converges to $u\in\mathcal{V}$. As u_n and ∇u_n converge in L^2 to u and ∇u, respectively, they (or at least subsequences $u_{n'}$ and $\nabla u_{n'}$) converge also almost everywhere, so that

$$-u(\mathbf{y})+u(\mathbf{x})+\nabla u(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})=\lim_{n'\to+\infty}-u_{n'}(\mathbf{y})+u_{n'}(\mathbf{x})+\nabla u_{n'}(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})\le 0\quad a.e. \text{ in } \Omega\times\Omega, \tag{2}$$

and therefore $u\in\mathcal{V}_c$. As problem (I) consists in minimizing a strictly convex functional over a closed, convex set, it admits a unique solution $u_I\in\mathcal{V}_c$. □

Remark 2.1. *We could have used the fact that the set of convex functions is locally compact, and therefore closed, in $H_0^1(\Omega)$.*

2.2. New formulation

Let us introduce the space $\mathcal{A}=L^2(\Omega)^2$ and the continuous, linear operator

$$\Phi\ :\ \left\{\begin{array}{lcl}\mathcal{A} & \longrightarrow & \mathcal{V}\\ \mathbf{p} & \longmapsto & \Phi\mathbf{p}=\triangle_o^{-1}\nabla\cdot\mathbf{p}\end{array}\right. \tag{3}$$

where $\triangle_o^{-1}$ maps any $f\in H^{-1}(\Omega)$ onto u, solution of

$$\left\{\begin{array}{rcll}\triangle u & = & f & \text{in}\quad \Omega\\ u & = & 0 & \text{on}\quad \partial\Omega\end{array}\right. \tag{4}$$

The kernel of Φ is

$$\mathrm{N}(\Phi)=\left\{\mathbf{p}\in\mathcal{A};\,,\ \ \int_\Omega \mathbf{p}\cdot\nabla w=0\quad \forall w\in\mathcal{V}\right\}, \tag{5}$$

and its range is $\mathrm{R}(\Phi)=\mathcal{V}$. The adjoint operator of Φ is

$$f\in H^{-1}(\Omega)\longmapsto \Phi^\star f=\nabla\triangle_o^{-1}f\in L^2(\Omega). \tag{6}$$

We now introduce $\mathbf{p}_0=\nabla u_0\in\mathcal{A}$, so that $u_0=\Phi\mathbf{p}_0$, and we define

$$\tilde{J}(\mathbf{p})=\frac{1}{2}\int_\Omega |\mathbf{p}-\mathbf{p}_0|^2\,. \tag{7}$$

The new problem is :

$$(II)\ :\ \begin{cases} \text{Find } \mathbf{p} \in \mathcal{A}_c \text{ such that} \\ \tilde{J}(\mathbf{p}) = \min\limits_{\mathbf{q}\in\mathcal{A}_c} \tilde{J}(\mathbf{q}) \\ \mathcal{A}_c = \{\mathbf{p} \in \mathcal{A}\ ,\ \ -\Phi\mathbf{p}(\mathbf{x}_2) + \Phi\mathbf{p}(\mathbf{x}_1) + \mathbf{p}(\mathbf{x}_1)\cdot(\mathbf{x}_2 - \mathbf{x}_1) \le 0 \quad a.e. \text{ in } \Omega\times\Omega\}\,, \end{cases} \tag{8}$$

Let us first establish the well-posedness of this new formulation:

Proposition 2.2. *Problem (II) admits a unique solution* $\mathbf{p}_{II}$*.*

Proof. As Φ is continuous $\mathcal{A} \longrightarrow L^2$ (it is even compact), a proof similar to that of proposition 2.1 establishes the closedness of $\mathcal{A}_c$ in $\mathcal{A}$. As $\tilde{J}$ is strictly convex, problem (II) admits a unique solution $\mathbf{p}_{II} \in \mathcal{A}_c$. □

Proposition 2.3. *Problems (I) and (II) are equivalent,* i.e. *:*

(i) *If* $\mathbf{p} \in \mathcal{A}$ *is a solution of problem (II), then* $u = \Phi\mathbf{p}$ *solves problem (I), and* $\mathbf{p} = \nabla u$*.*
(ii) *If* $u \in \mathcal{V}$ *is a solution of problem (I), then* $\mathbf{p} = \nabla u$ *solves problem (II).*

Proof. (i) Let $\mathbf{p} \in \mathcal{A}$ be a solution of problem (II), and $u = \Phi\mathbf{p}$. There exists a measurable set $\omega \subset \Omega$, with $|\omega| = |\Omega|$, such that

$$-u(\mathbf{y}) + u(\mathbf{x}) + \mathbf{p}(\mathbf{x})\cdot(\mathbf{y} - \mathbf{x}) \le 0 \quad \forall(\mathbf{x},\mathbf{y}) \in \omega\times\omega. \tag{9}$$

For every $\mathbf{x} \in \omega$, we introduce the affine function

$$F_{\mathbf{x}}\ :\ \mathbf{y} \longmapsto u(\mathbf{x}) + \mathbf{p}(\mathbf{x})\cdot(\mathbf{y} - \mathbf{x}). \tag{10}$$

As $F_{\mathbf{y}}(\mathbf{y}) = u(\mathbf{y})$ and $u(\mathbf{y}) \ge F_{\mathbf{x}}(\mathbf{y})$ for every $\mathbf{x} \in \omega$, the function u can be written (ω is considered here an index set for the family $(F_{\mathbf{x}})$)

$$u(\mathbf{y}) = \sup_{\mathbf{x}\in\omega} F_{\mathbf{x}}(\mathbf{y}), \tag{11}$$

therefore u is convex and, for any $\mathbf{x} \in \omega$,

$$\nabla F_{\mathbf{x}} = \mathbf{p}(\mathbf{x}) \in \partial u(\mathbf{x}) \text{ (subdifferential of u).} \tag{12}$$

As u is convex, the previous identity (12) implies

$$\mathbf{p} = \nabla u \quad a.e. \text{ in } \Omega. \tag{13}$$

As the solution u_I of problem (I) is unique, u is necessarily this solution (if not, $\tilde{J}(\nabla U_I) < \tilde{J}(\mathbf{p})$, with u_I convex, which is impossible).

(ii) Let u be a solution of problem (I). The field $\mathbf{p} = \nabla u$, which is such that $u = \Phi\mathbf{p}$, is in $\mathcal{A}_c$. As any $\mathbf{q} \in \mathcal{A}_c$ is the gradient of a function of $H^1_0(\Omega)$ (see *(i)*), $\mathbf{p}$ is necessarily the solution of problem (II). □

Remark 2.2. *The key point in the previous proof is the equivalence :*

$$\mathbf{p} \in \mathcal{A}_c \Longleftrightarrow \exists u \in \mathcal{V}_c \ \ s.t. \ \ \mathbf{p} = \nabla u. \tag{14}$$

3. Saddle-point formulation

We introduce

$$\mathcal{B} = L^2(\Omega\times\Omega)\ ,\ \ \mathcal{B}_+ = \{\lambda \in \mathcal{B}\ ,\ \ \lambda(\mathbf{x}_1,\mathbf{x}_2) \ge 0 \text{ a.e. in } \Omega\times\Omega\}\,. \tag{15}$$

3.1. Formal saddle-point formulation in $\mathcal{A} \times \mathcal{B}$

A natural way to establish a saddle-point formulation is to introduce the mapping C_0 defined on $\mathcal{A}$ into $\mathcal{B}$ by

$$C_0\mathbf{p}(\mathbf{x}_1, \mathbf{x}_2) = -\Phi\mathbf{p}(\mathbf{x}_2) + \Phi\mathbf{p}(\mathbf{x}_1) + \mathbf{p}(\mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1). \tag{16}$$

Formally, the saddle-point problem can be stated as follows: Find $(\mathbf{p}, \lambda) \in \mathcal{A} \times \mathcal{B}_+$, saddle-point for the Lagrangian

$$\mathcal{L}_0(\mathbf{p}, \lambda) = \frac{1}{2} \int |\mathbf{p} - \mathbf{p}_0|^2 + \iint \lambda(\mathbf{x}_1, \mathbf{x}_2)\, C_0\mathbf{p}\, d\mathbf{x}_1\, d\mathbf{x}_2, \tag{17}$$

i.e.

$$\mathcal{L}_0(\mathbf{p}, \lambda) = \inf_{\mathbf{q}} \sup_{\mu \geq 0} \mathcal{L}_0(\mathbf{q}, \mu) = \sup_{\mu \geq 0} \inf_{\mathbf{q}} \mathcal{L}_0(\mathbf{q}, \mu). \tag{18}$$

As for the existence of a saddle-point, we would like to use a property like the one established in [4] (proposition 2.4, p. 176). A key assumption is a coercivity-like property which reduces here to

$$\lim_{\|\mu\| \to +\infty, \mu \in \mathcal{B}_+} \|C_0^\star \mu\| = +\infty. \tag{19}$$

Let us show that this fundamental assumption is **not** verified. We can express $C_0^\star$:

$$C_0^\star \mu(\mathbf{x}) = -\int \mu(\mathbf{x}, \mathbf{y})(\mathbf{y} - \mathbf{x})\, d\mathbf{y} + \Phi^\star \left(\int (\mu(\mathbf{x}, \mathbf{y}) - \mu(\mathbf{y}, \mathbf{x}))\, d\mathbf{y} \right). \tag{20}$$

We consider the case $N = 1$, we introduce a neighbourhood of the diagonal of $\Omega \times \Omega =]0,1[\times]0,1[$

$$D_\varepsilon = \{(\mathbf{x}, \mathbf{y}) \in \Omega \times \Omega\ ,\ \ |\mathbf{y} - \mathbf{x}| < \varepsilon\}, \tag{21}$$

and we define χ_ε as the characteristic function of D_ε. Then a straightforward calculation leads to

$$\lim_{\varepsilon \to 0} \frac{\|C_0^\star \chi_\varepsilon\|}{\|\chi_\varepsilon\|} = 0, \tag{22}$$

which contradicts (19).

Remark 3.1. *Note that, in the finite-dimensional case, the criterium (19) is equivalent to ker($C^\star$)$\cap \mathcal{B}_+ = \{0\}$, which is easy to verify. Any reasonable space-discretized version of the saddle-point problem will therefore admit a saddle-point. Nevertheless, the previous remark suggests a deterioration of any numerical algorithm based on a direct discretization of the Lagrangian (18), as the mesh step size tends to* 0.

3.2. Weighting of the constraints

The space of "feasible" states $\mathcal{A}_c$ can be written

$$\mathcal{A}_c = \left\{ \mathbf{p} \in \mathcal{A}\ ,\ \ \frac{-\Phi\mathbf{p}(\mathbf{x}_2) + \Phi\mathbf{p}(\mathbf{x}_1) + \mathbf{p}(\mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)}{\theta(\mathbf{x}_1, \mathbf{x}_2)} \leq 0 \quad a.e. \text{ in } \Omega \times \Omega \right\}, \tag{23}$$

where θ is any positive function of $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega \times \Omega$. The choice of θ does not change the definition of $\mathcal{A}_c$, but it may affect the existence or non-existence of a saddle-point to the corresponding Lagrangian, and the convergence properties of the corresponding numerical algorithm. We shall focus on a family of weighting functions based on

$$\theta_\alpha(\mathbf{x}_1,\mathbf{x}_2) = |\mathbf{x}_2-\mathbf{x}_1|^\alpha, \quad \alpha \in \left]\frac{N}{2}, 1+\frac{N}{2}\right[. \tag{24}$$

We introduce the family of mappings (C_α)

$$C_\alpha \; : \; \mathcal{A} \longrightarrow \mathcal{B} \tag{25}$$

$$\mathbf{p} \longmapsto C\mathbf{p} \tag{26}$$

where $C_\alpha \mathbf{p}$ is defined by

$$C_\alpha\mathbf{p}(\mathbf{x}_1,\mathbf{x}_2) = \frac{-\Phi\mathbf{p}(\mathbf{x}_2)+\Phi\mathbf{p}(\mathbf{x}_1)+\mathbf{p}(\mathbf{x}_1)\cdot(\mathbf{x}_2-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|^\alpha}. \tag{27}$$

Lemma 3.1. *Let $\alpha = s+\frac{N}{2}$ with $s \in]0,1[$. C_α is a linear continuous mapping from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Let $\mathbf{p} \in \mathcal{A}$, since $\Phi\mathbf{p} \in H_0^1$ and since the injection $H_o^1 \subset W^{s,2}$ is continuous (see [1]), there exists a constant c_1 such that

$$||\mu_1||_{\mathcal{B}} \le c_1||\Phi||\,||\mathbf{p}||_{\mathcal{A}} \tag{28}$$

where

$$\mu_1(\mathbf{x}_1,\mathbf{x}_2) = \frac{\Phi\mathbf{p}(\mathbf{x}_2)-\Phi\mathbf{p}(\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|^\alpha} \tag{29}$$

On the other hand, we define

$$\mu_2(\mathbf{x}_1,\mathbf{x}_2) = \frac{\mathbf{p}(\mathbf{x}_1)\cdot(\mathbf{x}_2-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|^\alpha}. \tag{30}$$

Since $N+2(s-1)<N$

$$||\mu_2||^2_{L^2(\Omega^2)} \le \int_{B(0,R)} \frac{d\mathbf{x}}{|\mathbf{x}|^{N+2(s-1)}}\,||\mathbf{p}||^2_{L^2(\Omega)^2} \tag{31}$$

with R such that $\Omega-\Omega \subset B(0,R)$. Hence the Lemma is proved. □

Remark 3.2. *The statement of Lemma 3.1 is false if $s=1$ as the following counter-example shows. If the statement were true for $s=1$, taking successively $\mathbf{p}=\mathbf{e}_i$, $i=1,...,N$ with $(\mathbf{e}_i)$ the canonical basis of $\mathbb{R}^N$ and taking the sum of the squares of the L^2 norms of the corresponding $\Phi\mathbf{p}$ we would obtain $|\mathbf{x}_1-\mathbf{x}_2|^{-N} \in L^1(\Omega\times\Omega)$, hence a contradiction.*

3.3. Existence of a saddle-point in $\mathcal{A}\times\mathcal{K}$

We take $\alpha \in]N/2, 1+N/2[$, so that C_α is properly defined from $\mathcal{A}$ to $\mathcal{B}$. We introduce $\mathcal{K} = \overline{C^*_\alpha(\mathcal{B}_+)}$ (the closure of $C^*_\alpha(\mathcal{B}_+)$ in $\mathcal{A}$). Notice that $\mathcal{K}$ does not depend on α (see remark 3.3). We define then for all $(\mathbf{p},\mathbf{q}) \in \mathcal{A}^2$ and all $\lambda \in \mathcal{B}$:

$$\mathcal{L}(\mathbf{p},\mathbf{q}) = \frac{1}{2}\int |\mathbf{p}-\mathbf{p}_o|^2\,d\mathbf{x} + \int \mathbf{q}\cdot\mathbf{p}\,d\mathbf{x} \tag{32}$$

and

$$\mathcal{L}_\alpha(\mathbf{p},\lambda) = \mathcal{L}(\mathbf{p}, C_\alpha^*\lambda) = \frac{1}{2}\int |\mathbf{p}-\mathbf{p}_o|^2 + \iint \lambda(\mathbf{x}_1,\mathbf{x}_2)\, C_\alpha \mathbf{p}(\mathbf{x}_1,\mathbf{x}_2)\, d\mathbf{x}_1\, d\mathbf{x}_2. \tag{33}$$

Using those Lagrangians, our projection problem (II) is equivalent to

$$\begin{cases} \text{Find } \tilde{\mathbf{p}} \in \mathcal{A} \text{ such that} \\ \sup\limits_{\mu\in\mathcal{B}_+} \mathcal{L}_\alpha(\tilde{\mathbf{p}},\mu) = \inf\limits_{\mathbf{p}\in\mathcal{A}} \sup\limits_{\mu\in\mathcal{B}_+} \mathcal{L}_\alpha(\mathbf{p},\mu) = \inf\limits_{\mathbf{p}\in\mathcal{A}} \sup\limits_{\mathbf{q}\in\mathcal{K}} \mathcal{L}(\mathbf{p},\mathbf{q}). \end{cases} \tag{34}$$

We shall then replace problem (II) by: find a saddle-point of $\mathcal{L}$ in $\mathcal{A}\times\mathcal{K}$.

Proposition 3.1. *$\mathcal{L}$ has a unique saddle-point $(\tilde{\mathbf{p}},\tilde{\mathbf{q}}) \in \mathcal{A}\times\mathcal{K}$:*

$$\mathcal{L}(\tilde{\mathbf{p}},\tilde{\mathbf{q}}) = \inf_{\mathbf{p}\in\mathcal{A}} \sup_{\mathbf{q}\in\mathcal{K}} \mathcal{L}(\mathbf{p},\mathbf{q}) = \sup_{\mathbf{q}\in\mathcal{K}} \inf_{\mathbf{p}\in\mathcal{A}} \mathcal{L}(\mathbf{p},\mathbf{q}) \tag{35}$$

where $\tilde{\mathbf{p}} \in \mathcal{A}_c$ is the solution of problem (II), and $\tilde{\mathbf{q}}$ is the solution of the projection problem

$$\inf_{\mathbf{q}\in\mathcal{K}} \|\mathbf{q}-\mathbf{p}_0\|_{\mathcal{A}}. \tag{36}$$

Proof. Fix $\mathbf{q}\in\mathcal{K}$, one has

$$I(\mathbf{q}) = \inf_{\mathbf{p}\in\mathcal{A}} \mathcal{L}(\mathbf{p},\mathbf{q}) = -\frac{1}{2}\|\mathbf{q}\|_{\mathcal{A}}^2 + \langle \mathbf{q}\,,\mathbf{p}_0\rangle \tag{37}$$

so that the supremum of I over $\mathcal{K}$ is attained at a unique $\tilde{\mathbf{q}}$ which is simply the projection of $\mathbf{p}_0$ into $\mathcal{K}$. This actually proves the other statements of the proposition (see chapter VI of [4] for details). ☐

Remark 3.3. *The dual problem of the projection problem (II) (project $\mathbf{p}_0$ into $\mathcal{A}_c$) is the other projection problem : project $\mathbf{p}_0$ into $\mathcal{K}$. Identifying $\mathcal{A}$ with its dual space, note also that $\mathcal{K}$ is equal to the polar cone of $\mathcal{A}_c$*

$$\mathcal{K} = \{\mathbf{q}\in\mathcal{A},\ \langle \mathbf{q}\,,\mathbf{p}\rangle \le 0,\ \textit{for all } \mathbf{p}\in\mathcal{A}_c\}. \tag{38}$$

In particular $\mathcal{K}$ does not depend on α.

Remark 3.4. *The duality relation*

$$\inf_{\mathbf{p}\in\mathcal{A}} \sup_{\mu\in\mathcal{B}_+} \mathcal{L}_0(\mathbf{p},\mu) = \sup_{\mu\in\mathcal{B}_+} \inf_{\mathbf{p}\in\mathcal{A}} \mathcal{L}_0(\mathbf{p},\mu) \tag{39}$$

holds but the supremum in the rightmost member may not be attained i.e. $\tilde{\mathbf{q}}$ obtained in Proposition 3.1 need not in general be of the form $C_\alpha^*\lambda$ with $\lambda\in\mathcal{B}_+$.

Remark 3.5. *Finally note that the solution of the dual problem $\tilde{\mathbf{q}}$ (hence the projection of $\mathbf{p}_0$ into $\mathcal{K}$) does not depend on the exponent $s\in(0,1)$ since $\tilde{\mathbf{q}} = \mathbf{p}_0 - \tilde{\mathbf{p}}$.*

Remark 3.6. *The previous argument shows that $\mathcal{A} = L^2(\Omega)^2 = \mathcal{A}_c + \mathcal{K}$. Now note that this implies that $C^*(\mathcal{B}_+)$ is not closed. Indeed, assume on the contrary $\mathcal{K} = C^*(\mathcal{B}_+)$, so that $\mathcal{A} = \mathcal{A}_c + C^*(\mathcal{B}_+)$. Firstly, elements of $\mathcal{A}_c$ are monotone fields so that $\mathcal{A}_c \subset L^\infty_{\mathrm{loc}}$. Secondly, since $\alpha = s + N/2 < 1 + N/2$, adapting slightly the proof of Lemma 3.1, one can prove $C^*(\mathcal{B}_+) \subset L^p(\Omega)^2$ for some $p>2$. Then we would have $\mathcal{A} = L^2(\Omega)^2 = L^\infty_{\mathrm{loc}}(\Omega)^2 + L^p(\Omega)^2$, hence a contradiction.*

Remark 3.7. *The variational inequalities of the dual problem indeed imply* $\langle \mathbf{p}^*, \mathbf{q}^* \rangle = 0$ *which is a classical* complementary slackness *or* exclusion condition *since* $\mathbf{q}^*$ *can naturally be interpreted as a Kuhn and Tucker multiplier associated to the constraint* $C_\alpha \mathbf{p} \leq 0$. *It can be shown conversely that* $(\mathbf{p}^*, \mathbf{q}^*)$ *is a saddle-point of* $\mathcal{L}$ *if and only if the following conditions are satisfied:*

$$(\mathbf{p}^*, \mathbf{q}^*) \in \mathcal{A}_c \times \mathcal{K}, \ \mathbf{p}_0 = \mathbf{p}^* + \mathbf{q}^*, \ \textit{and} \ \langle \mathbf{p}^*, \mathbf{q}^* \rangle = 0. \tag{40}$$

4. Numerical experiments in the two–dimensional case

Although the proper definition of C_α requires $\alpha < 1 + N/2$ in the continuous case, and although, even in that case, the existence of a saddle-point in $\mathcal{A} \times \mathcal{B}_+$ is not established, numerical experiments exhibit a good behaviour of the Uzawa algorithm applied to the Lagrangian $\mathcal{L}_\alpha$ (see (33)), with $\alpha = 2$. We are not able to explain the discrepancy between theoretical and numerical aspects.

4.1. **Space-discretization**

We introduce a conforming triangulation T_h of Ω, and we define the corresponding dual mesh $\mathcal{T}_h$ as the set of cells delimited by segments joining centers of adjacent triangles (see figure 1). We denote by N the number of vertices of T_h (= number of cells of $\mathcal{T}_h$). We introduce approximation spaces of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}_h = \{\mathbf{p}_h \in \mathcal{A}, \ \mathbf{p}_h \text{ is constant on each cell of } \mathcal{T}_h\} \tag{41}$$

$$\mathcal{B}_h = \{\lambda_h \in \mathcal{B}, \ \lambda_h \text{ is constant on each cell product of } \mathcal{T}_h \times \mathcal{T}_h\} \tag{42}$$

$$\mathcal{V}_h = \left\{u_h \in C^0\left(\overline{\Omega}\right) \cap H_0^1(\Omega), \ u_h \text{ is affine on each triangle of } T_h\right\}. \tag{43}$$

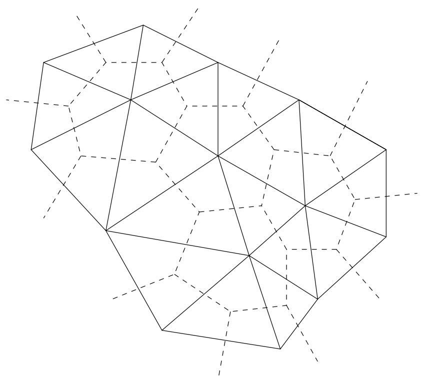

Fig. 1 : Primal and dual meshes.

We define the discrete version of Φ as

$$\Phi_h : \mathcal{A}_h \longrightarrow \mathcal{V}_h \tag{44}$$

$$\mathbf{p}_h \longmapsto u_h \text{ solution of} \tag{45}$$

$$\int_\Omega \nabla u_h \cdot \nabla v_h = {}_{H_0^1}\langle v_h, \nabla \cdot \mathbf{p}_h \rangle_{H^{-1}} \quad \forall v_h \in \mathcal{V}_h. \tag{46}$$

Note that the right hand side of the variational formulation can be expressed

$$ {}_{H_0^1}\langle v_h, \nabla \cdot \mathbf{p}_h \rangle_{H^{-1}} = -\int_\Omega \nabla v_h \cdot \mathbf{p}_h, \tag{47}$$

which can be computed explicitely (both v_h and $\mathbf{p}_h$ are piecewise constant).

4.2. **Matrix formulation**

Let C be the $N(N-1)\times 3N$ matrix expressing the weighted constraints in terms of $(u_h, \mathbf{p}_h)$ values (the first N columns correspond to nodal values $u_h(\mathbf{x}_i)$, the following N columns to cell-values for the first coordinate of $\mathbf{p}_h$, and the last N columns to cell-values for the second coordinate of $\mathbf{p}_h$). A row of C expresses

$$\frac{-u_h(\mathbf{x}_j)+u_h(\mathbf{x}_i)+\mathbf{p}_h(\mathbf{x}_i)\cdot(\mathbf{x}_j-\mathbf{x}_i)}{|\mathbf{x}_j-\mathbf{x}_i|^2}\leq 0, \tag{48}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ are two (distinct) mesh vertices. We introduce the matrix R associated to the Laplace operator relatively to $\mathcal{V}_h$ (rigidity matrix), M the mass matrix relatively to $\mathcal{A}_h$, and G the matrix associated to the mapping

$$\mathbf{p}_h \longmapsto \nabla\cdot\mathbf{p}_h, \tag{49}$$

where $\nabla\cdot\mathbf{p}_h$ is considered as an element of the dual of $\mathcal{V}_h$ (according to equation (47)). The discrete constraints expressed in terms of $\mathbf{p}_h$ only can be written

$$C\begin{bmatrix} R^{-1}G \\ I \end{bmatrix}\mathbf{p}_h \leq 0, \tag{50}$$

where R^{-1} stands for the discrete inverse of the Laplace operator with Dirichlet boundary conditions.

4.3. **Uzawa algorithm**

Let $\rho > 0$ be given, and let $\mathbf{p}_{0,h}$ be the projection of $\mathbf{p}_0$ into $\mathcal{A}_h$. We define Π^+ as the projection on the positive quadrant of $\mathbb{R}^{N(N-1)}$

$$\Pi^+(\xi) = (\max(\xi_i, 0))_{1\leq i\leq N(N-1)}\,. \tag{51}$$

An iteration of the Uzawa consists in the three steps :

$$\begin{aligned} \mu_h^k &= \begin{bmatrix} G^T R^{-1} & I \end{bmatrix} C^T \lambda_h^k, \\ \mathbf{p}_h^k &= \mathbf{p}_{0,h} - M^{-1}\mu_h^k, \\ \lambda_h^{k+1} &= \Pi^+\left(\lambda_h^k + \rho\, C\begin{bmatrix} R^{-1}G \\ I \end{bmatrix}\mathbf{p}_h^k\right). \end{aligned} \tag{52}$$

4.4. **Numerical experiments**

We present here some computations based on a 50×50 mesh of $\Omega =]0,1[\times]0,1[$. For the sake of clarity, we present projections into the set of concave functions. Initial and projected functions are represented in a coarse mesh (30×30) in figure 2.

As the Lagrange multipliers are defined on the product domain $\mathcal{T}_h\times\mathcal{T}_h\subset\mathbb{R}^4$, it is difficult to represent them globally. In order to give an idea of the behaviour of λ_h^k, we chose to represent

$$\Lambda_h^k = \Lambda_h^k(\mathbf{x}) = \max_{\mathbf{y}\in\Omega}\lambda_h^k(\mathbf{x},\mathbf{y}). \tag{53}$$

Both approximate field and Lagrange multiplier are represented at different steps of the algorithm, for three different initial functions (see Figs. 3,4, and 5).

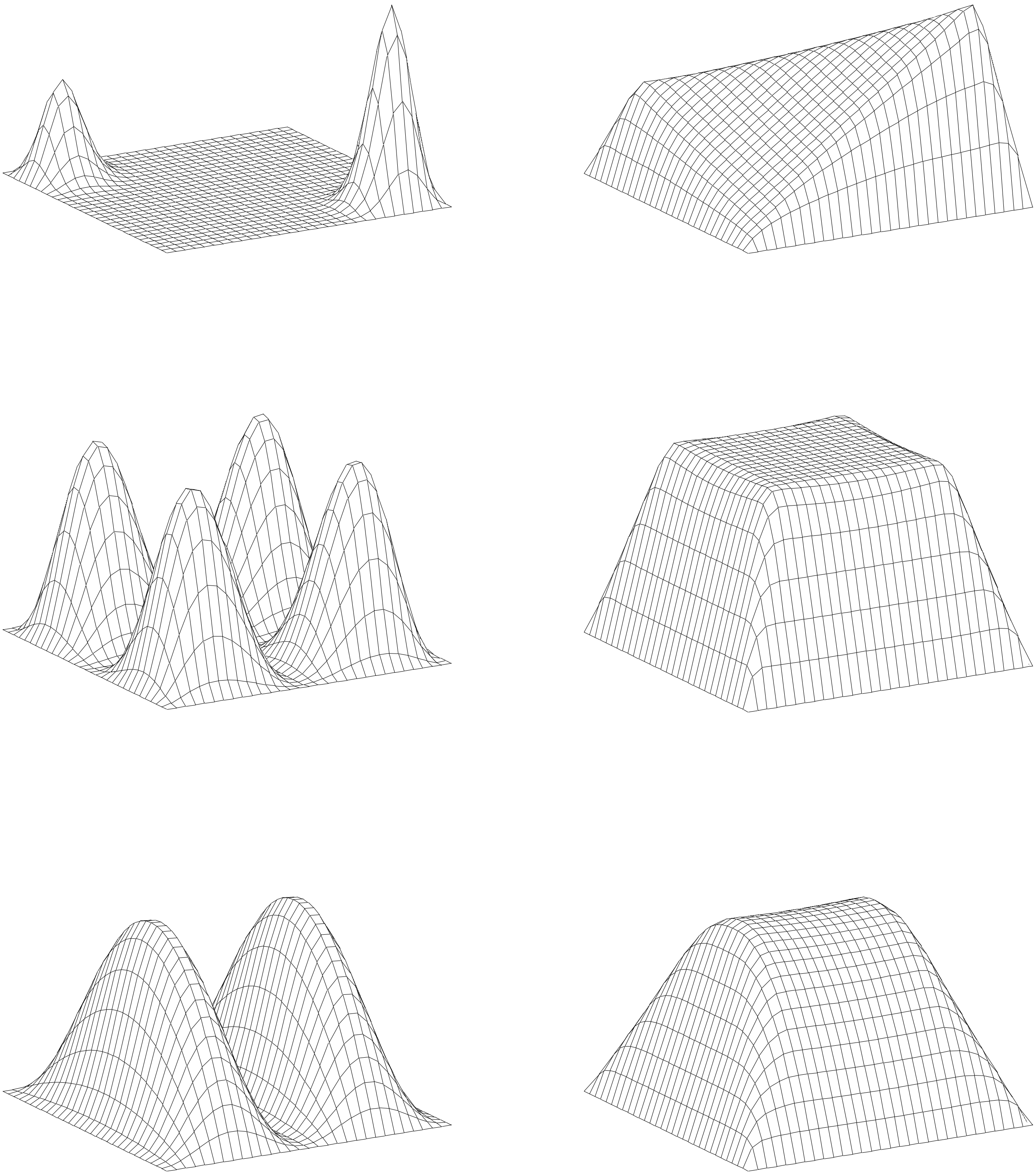

Fig 2 : *Left*: Initial functions u_1, u_2, and u_3. *Right*: Corresponding projected functions.

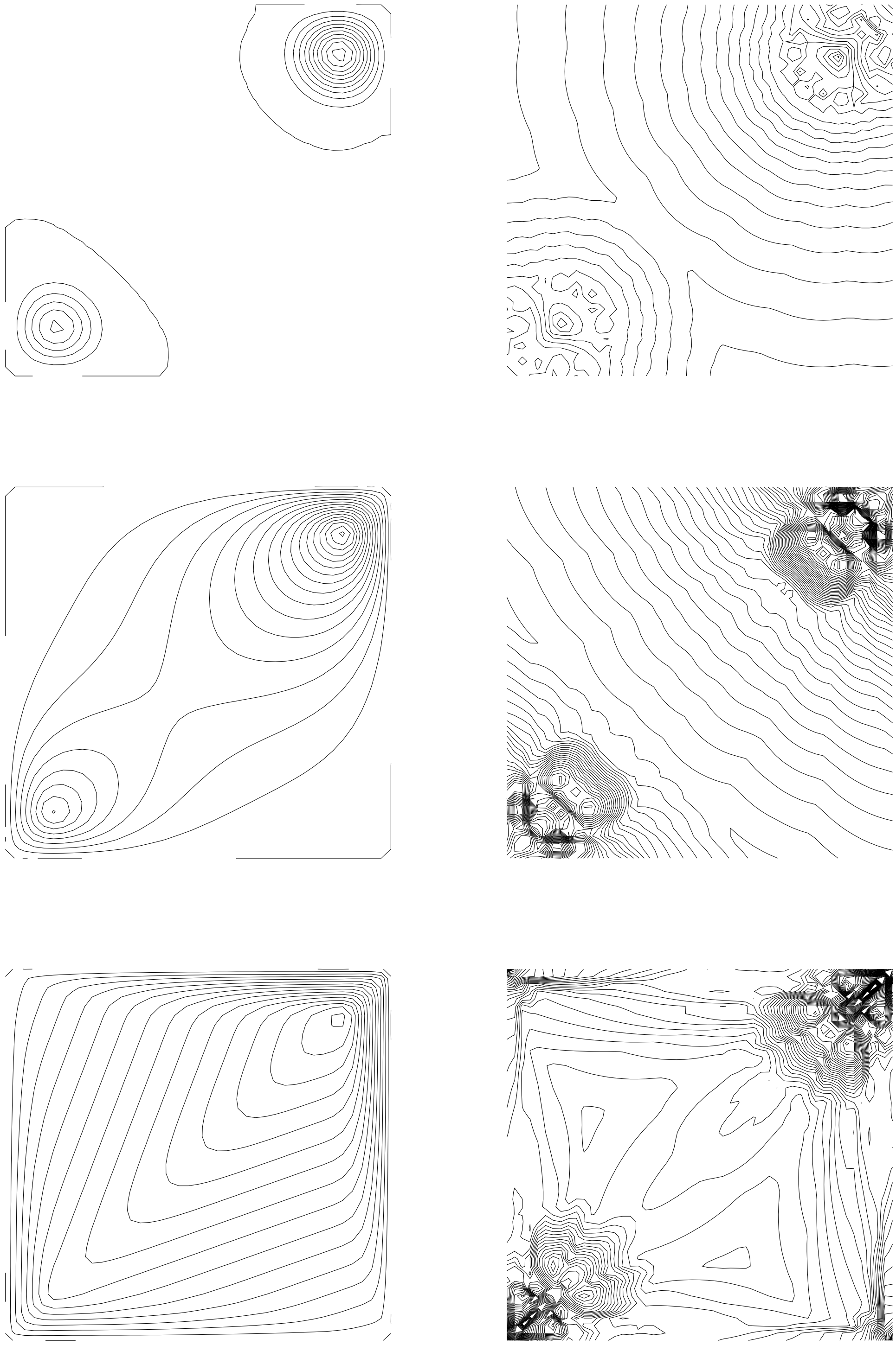

Fig 3 : *Left*: isolines of u_h^k for $k = 0,\ 8,\ 500$. *Right*: isolines of Λ_h^k for $k = 1,\ 8,\ 500$.
$u_1(\mathbf{x}) = \left(e^{-100r_1} + 2e^{-100r_2}\right) x_1(1-x_1)x_2(1-x_2)\ ,\ \ r_i = |\mathbf{x} - \mathbf{x}_i|\ ,\ \mathbf{x}_1 = (0.1, 0.1)\ ,\ \mathbf{x}_2 = (0.9, 0.9).$

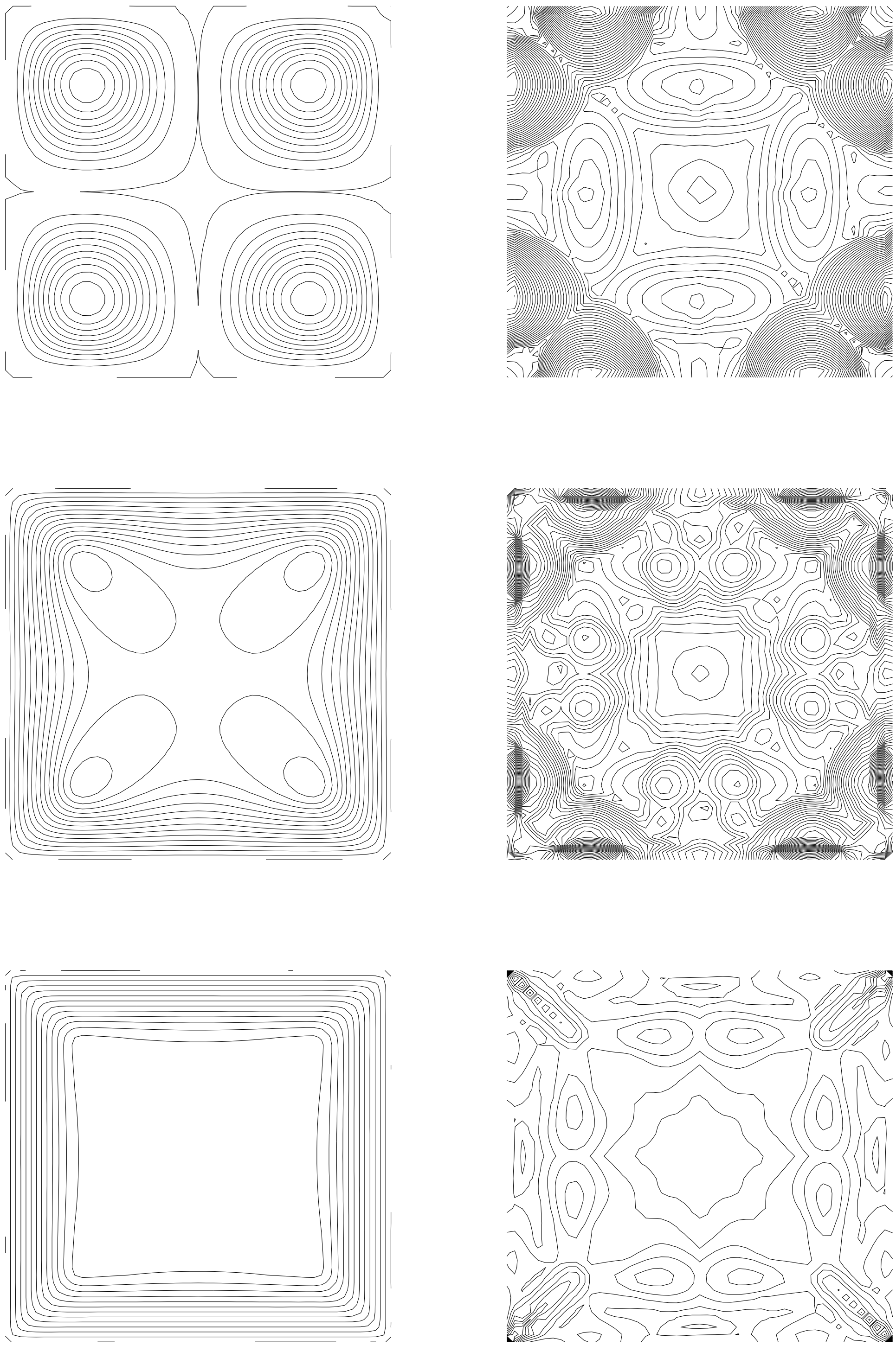

Fig 4 : *Left*: isolines of u_h^k for $k = 0,\, 16,\, 300$ *Right*: isolines of Λ_h^k for $k = 1,\, 2,\, 300$.
$u_2(\mathbf{x}) = (x_1 - 0.5)^2\,(x_2 - 0.5)^2 x_1^2(1 - x_1)^2 x_2^2(1 - x_2)^2$.

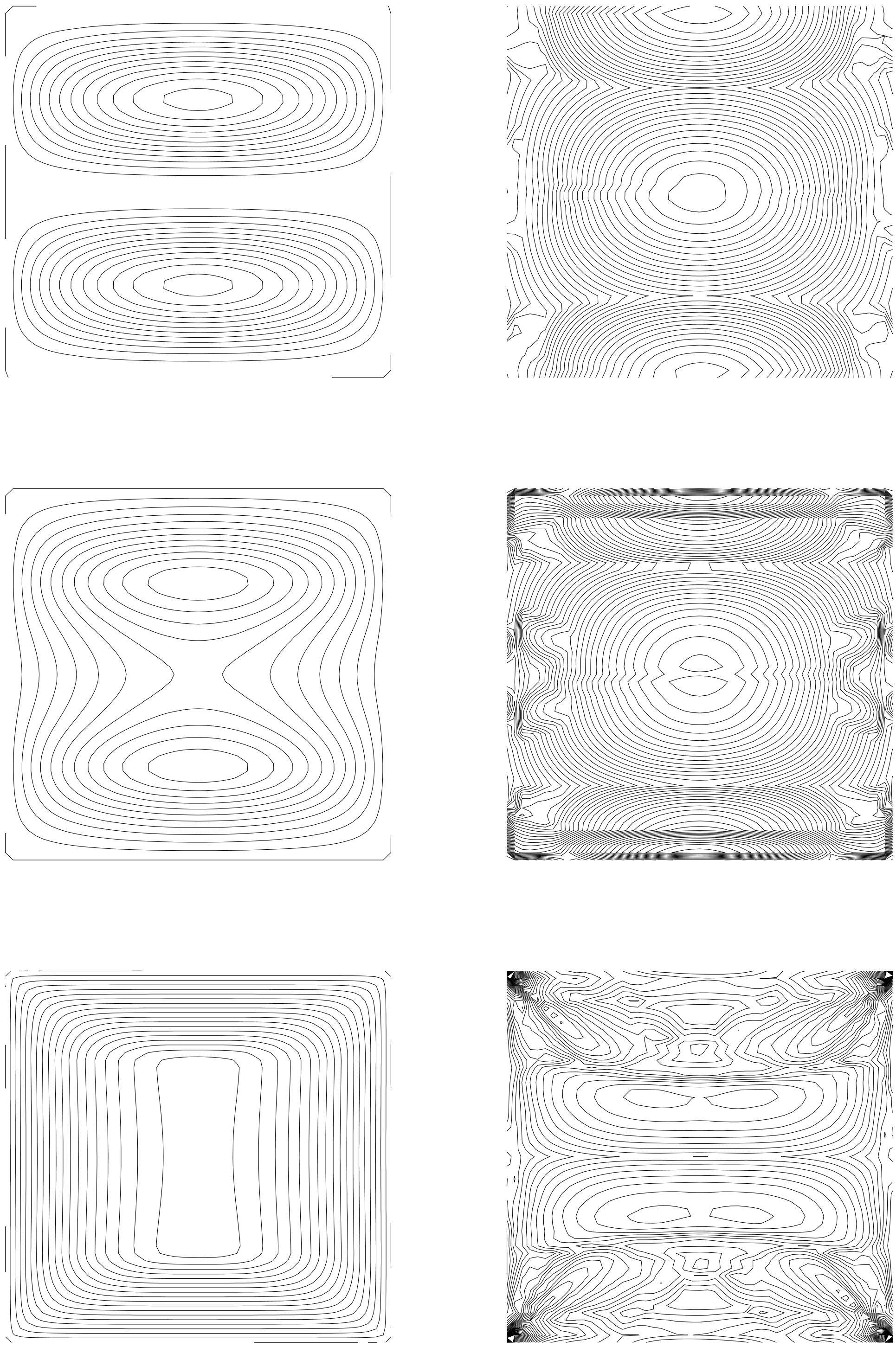

Fig 5 : *Left*: isolines of u_h^k for $k = 0,\ 2,\ 500$. *Right*: isolines of Λ_h^k for $k = 1,\ 2,\ 500$.
$u_3(\mathbf{x}) = u_3(x_1, x_2) = (x_1 - 0.5)^2\,(x_2 - 0.5)^2 x_1^2 (1 - x_1)^2 x_2^2 (1 - x_2)^2$.

5. Conclusion

We presented a new method to minimize a quadratic functional of the gradient over sets of convex functions. The cost and accuracy of this method makes it comparable to the one we introduced in [3], but it presents some new features :

1. The methodology makes it possible to deal with unstuctured meshes, so that general shapes for Ω may be handled.
2. The domain in which Lagrange multipliers are defined can be reduced to a neighbourhood of the diagonal of $\Omega \times \Omega$. From a numerical point of view, it consists in reducing the number of constraints, and consequently the computational cost of an iteration of the Uzawa algorithm. Numerical experiments seem to confirm this remark. The computational costs can be reduced, but the "localization" of the constraints in the neighbourhood of the diagonal in $\Omega \times \Omega$ limits the convergence speed for the Uzawa algorithm. It suggests future formulations, which may be based on local constraints to ensure convergence to the expected solution, and a selection of long range constraints to speed up the convergence of the Uzawa algorithm.

We must add that the algorithm in its present form does not make it possible to handle large number of vertices (60×60 is a maximum on a PC). Improvements have to be made to reduce the computational costs and memory requirements.

From the theoretical point of view, the preliminary analysis which was performed is still incomplete, as there is a gap between continuous discretized formulations.

References

[1] R. A. Adams *Sobolev Spaces*, Academic Press (1975).

[2] G. Buttazzo, V. Ferone, B. Kawohl, *Minimum Problems over Sets of Concave Functions and Related Questions*, Math. Nachrichten, **173** (1993), pp. 71–89.

[3] G. Carlier, T. Lachand-Robert and B. Maury, *A variational approach to variational problems subject to convexity constraint*, Publications du Laboratoire d'Analyse Numérique, Univ. P. & M. Curie, R99040 (1999), to appear in Numerische Mathematik.

[4] I. Ekeland and R. Temam *Convex analysis and variational problems*, Studies in Mathematics and its Applications, vol. 1 (1976).

[5] T. Lachand-Robert, M. A. Peletier, *Newton's problem of the body of minimal resistance in the class of convex developable functions*, to appear in Ann. I.H.P.

[6] J.-C. Rochet, P. Choné. *Ironing, Sweeping and Multidimensional screening*, Econometrica, vol. 66 (1998), pp. 783–826.