# MASS TRANSPORTATION IN GROUPS OF TYPE H

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ABSTRACT. We give a solution of the optimal transport problem in groups of type $H$ when the cost function is the square of either the Carnot-Carathéodory or Korányi distance. We prove in both cases existence, uniqueness and give a characterization of the optimal transport map. In the case of the Carnot-Carathéodory distance we also prove that the optimal transport is the limit of the optimal transports in natural Riemannian approximations, introducing in particular so-called $cc$-exponential coordinates in the group that may be recovered through the limit of the usual Riemannian exponential maps in the approximating Riemannian manifolds. This extends results already known for the Heisenberg groups.

The purpose of the present paper is to extend to groups of type $H$ equipped with either their Carnot-Carathéodory or Korányi metric the solution to the mass transportation problem that was obtained in [4] for the Heisenberg groups. Groups of type $H$ have been introduced in [17] in connection with the study of the fundamental solutions of the sublaplacian in nilpotent Lie groups and constitute in many ways a natural generalization of the Heisenberg groups.

The original mass transportation problem of Monge has been the source of numerous developments in the recent years, mainly due to the impressive variety of its applications (see e.g. [1], [8], [21], [25] and the references therein, the list is far from being exhaustive). The general mathematical formulation can be stated as follows. Given two Borel probability measures $\mu$, $\nu$ on a topological space $X$ and a measurable cost $c(x,y)$, one wants to minimize

$$\psi \mapsto \int_X c(x,\psi(x))\, d\mu(x)$$

among all transport maps $\psi : X \to X$ that push $\mu$ forward to $\nu$, $\psi_{\#}\mu = \nu$, meaning that $\nu(B) = \mu(\psi^{-1}(B))$ for all Borel sets $B$.

The situation we are interested in is when the space is a group $G$ of type $H$ and the cost is given by either $c(x,y) = d(x,y)^2/2$, where $d$ denotes the Carnot-Carathéodory distance or $c(x,y) = d_g(x,y)^2/2$ where $d_g$ denotes the Korányi distance. In both cases the main ingredients in [4], where $G$ was an Heisenberg group, were based on a careful analysis of the differentiability of the distance function. With the help of these analytic tools one can study in a natural way the $c$-superdifferential of a locally Lipschitz $c$-concave map (see Section 3 for the terminology and definitions), showing in particular that its $c$-superdifferential is up to a null set contained in a graph. This leads in a standard way to the existence and uniqueness of the optimal transport map through nowadays standard arguments from the general theory of optimal transportation. We shall closely follow the same strategy here, the main point being to analyze in the greater generality of groups of type $H$ the differentiability properties of both distances, the Carnot-Carathéodory and Korányi ones. The differences are mainly technical and come essentially from the fact that the dimension of the center of the group is larger than one (when this dimension is one, we are in the Heisenberg case).

The case of the Carnot-Carathéodory distance will occupy the major part of this paper. In that case the main arguments rely on the characterization and explicit computation of the

minimal geodesics. Necessary conditions for a curve to be a minimal geodesic in a group of type $H$ have been given in [18]. A refined analysis has been done in [22]. This was mainly used there to prove the Pansu-differentiability of $d_0$, the Carnot-Carathéodory distance to the origin, outside of the center of the group. We complete here the study very much in the spirit of [4]. On one hand we quantify precisely the non Pansu-differentiability of $d_0$ on the center and we compute on the other hand the derivatives of $d_0$ along the directions that belong to the center.

We also give a complete description of minimal geodesics between any two points and will interpret them as the limit of minimal geodesics in natural Riemannian approximations. We may then introduce accordingly so-called $cc$-exponential coordinates which will explicitly reflect this phenomenon. Going back to the problem of optimal mass transportation, we actually prove that the optimal transport map can be written through these $cc$-exponential coordinates as a suitable $cc$-gradient of a $c$-concave potential. Similarly to what was obtained in [4] we show furthermore that one can recover the optimal transport map as the limit of the optimal transport maps in the Riemannian approximations, as well as the $cc$-gradient of the associated $c$-concave potential as the limit of the gradients of the associated potentials in the approximations.

The issues are of a quite different nature when considering the Korányi distance induced by the natural homogeneous norm of the group. This norm has a fairly simple expression and is smooth outside of the origin. The main argument consists then in deriving, mainly from routine computations of the derivatives of the norm along any direction, intrinsic $BV$ estimates. This allows to make use of results of [2] about intrinsic $BV$ functions and to eventually get the required description of the $c$-superdifferential of a locally Lipschitz $c$-concave map.

The paper is organized as follows. In Section 1 we recall the definition of groups of type $H$ together with that of the Carnot-Carathéodory and Korányi distances and we introduce various notions of differentiability to be used later. Section 2 is devoted to the study of the minimal geodesics and of the differentiability properties of the Carnot-Carathéodory distance. The main theorem about the structure of the $c$-superdifferential of a locally Lipschitz $c$-concave map when the cost is given by $c(x,y) = d(x,y)^2/2$ is proved in Section 3. Section 4 is more specifically devoted to the problem of optimal mass transportation related to this cost. Existence, uniqueness and characterization of the optimal transport map will be proved in §4.1. Riemannian approximations are discussed in §4.2. Finally the case of a cost given by $c(x,y) = d_g(x,y)^2/2$ is studied in Section 5.

## 1. Preliminaries

1.1. **Groups of type $H$.** Let $\mathcal{G}$ be a real Lie algebra equipped with a scalar product and which can be decomposed in a non-trivial orthogonal direct sum, $\mathcal{G} = \mathcal{V} \oplus \mathcal{Z}$, where $[\mathcal{V},\mathcal{V}] \subset \mathcal{Z}$ and $[\mathcal{V},\mathcal{Z}] = [\mathcal{Z},\mathcal{Z}] = \{0\}$. We define the linear map $J : \mathcal{Z} \to \operatorname{End}\mathcal{V}$ by

$$\langle J(Z)X, X' \rangle = \langle Z, [X,X'] \rangle \tag{1}$$

for all $X$, $X' \in \mathcal{V}$ and $Z \in \mathcal{Z}$.

We say that $\mathcal{G}$ is of type $H$ if for all $Z \in \mathcal{Z}$,

$$J(Z)^2 = -\|Z\|^2 I. \tag{2}$$

Note that in such a case $\mathcal{Z}$ is the center of $\mathcal{G}$ and $[\mathcal{V},\mathcal{V}]=\mathcal{Z}$ so that $\mathcal{G}$ is a stratified Lie algebra of step two. Note also that (1) implies that $J(Z)^t=-J(Z)$ for all $Z\in\mathcal{Z}$ and then it follows from (2) that

$$\|J(Z)X\|=\|Z\|\|X\| \tag{3}$$

for all $X\in\mathcal{V}$ and $Z\in\mathcal{Z}$. For more details we refer to [17] where Lie algebra of type $H$ have been introduced, see also e.g. [6] or [18].

A Lie group $G$ is said to be a group of type $H$ if it is connected, simply connected and if its Lie algebra is of type $H$. The exponential map being in such a case a diffeomorphism from $\mathcal{G}$ onto $G$ ([9]), we can write an element in $G$ as $x=\exp(X+Z)$ where $X\in\mathcal{V}$ and $Z\in\mathcal{Z}$ and the group law, given by the Campbell-Hausdorff formula, has the form

$$\exp(X+Z).\exp(X'+Z')=\exp(X+X'+Z+Z'+[X,X']/2)$$

for all $X$, $X'\in\mathcal{V}$ and $Z$, $Z'\in\mathcal{Z}$.

A Lie algebra $\mathcal{G}$ of type $H$ can be equipped with a natural family of dilations $\delta_\rho$, $\rho>0$, defined by

$$\delta_\rho(X+Z)=\rho\,X+\rho^2\,Z$$

for all $X\in\mathcal{V}$ and $Z\in\mathcal{Z}$. For $\rho<0$, we set $\delta_\rho=-\delta_{|\rho|}$. They induce dilations on the group that we shall still denote by $\delta_\rho$,

$$\delta_\rho(\exp(X+Z))=\exp(\delta_\rho(X+Z)).$$

We set $n=\dim\mathcal{G}$ and $m=\dim\mathcal{V}$. We fix an orthonormal basis $(X_1,\dots,X_m)$ of $\mathcal{V}$ and an orthonormal basis $(X_{m+1},\dots,X_n)$ of $\mathcal{Z}$. When needed, we shall also interpret the $X_j$'s (as well as any vector in $\mathcal{G}$) as left invariant vector fields. We set $e_j=\exp X_j$.

If $\mathcal{L}$ denotes the $n$-dimensional Lebesgue measure on $\mathcal{G}$, the push-forward $\lambda=\exp_\#\mathcal{L}$ of $\mathcal{L}$ through the exponential map is a bi-invariant Haar measure on $G$. When speaking of a property that holds a.e. on $G$, without referring to a specific measure, this will always mean $\lambda$-a.e.

1.2. **Carnot-Carathéodory and Korányi distances.** Let $G$ be a group of type $H$. An absolutely continuous curve $\gamma:[0,T]\to G$ is said to be horizontal if there exist measurable functions $b_1,\dots,b_m:[0,T]\to\mathbb{R}$ such that

$$\dot\gamma(t)=\sum_{j=1}^m b_j(t)\,X_j(\gamma(t))$$

for a.e. $t\in[0,T]$. We interpret here $X_1,\dots,X_m$ as left invariant vector fields. If moreover $\sum_j b_j(t)^2\le 1$ for a.e. $t\in[0,T]$, then $\gamma$ is said to be sub-unit. In other words, the differential of the exponential map at a point $Y\in\mathcal{G}$ being given by $I-\mathrm{ad}(Y)/2$ (see e.g. [15]) and hence $\dot\gamma(t)=\dot X(t)+\dot Z(t)-[X(t),\dot X(t)]/2$ whenever $\gamma(t)=\exp(X(t)+Z(t))$ for some curves $X(t)$ and $Z(t)$ in $\mathcal{V}$ and $\mathcal{Z}$ respectively (here we also identify $\mathcal{V}$ and $\mathcal{Z}$ with their tangent spaces), the condition for $\gamma$ to be horizontal can be rephrased as

$$\dot Z-[X,\dot X]/2=0$$

a.e. on $[0,T]$ and the curve is sub-unit if moreover $\|\dot X\|\le 1$ a.e. on $[0,T]$.

Since the family $(X_1, \dots, X_m)$ generates $\mathcal{G}$ as a Lie algebra, it follows from Chow's theorem that any two points $x$, $y$ in $G$ can be joined by a sub-unit curve. Then the Carnot-Carathéodory distance between $x$ and $y$ is defined as

$$d(x,y) = \inf\{T > 0\,;\, \text{there exists a sub-unit curve } \gamma : [0,T] \to G \\ \text{such that } \gamma(0) = x,\ \gamma(T) = y\}.$$

We refer to e.g. [5] or [14] for more details about Carnot-Carathéodory spaces.

On the other hand a group $G$ of type $H$ has a natural gauge defined by

$$\|\exp(X+Z)\| = (\|X\|^4 + 16\|Z\|^2)^{1/4}$$

for all $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$. By [7], this gauge satisfies $\|x.y\| \leq \|x\| + \|y\|$ for all $x$, $y \in G$, and one then defines the so-called Korányi distance between any two points $x$, $y \in G$ by

$$d_g(x,y) = \|x^{-1}.y\|.$$

Both distances $d$ and $d_g$ are natural distances on $G$. They are left invariant and homogeneous with respect to the dilations of the group, that is,

$$d(x.y, x.z) = d(y,z) \quad \text{and} \quad d(\delta_\rho(y), \delta_\rho(z)) = \rho\, d(y,z),$$

for all $x$, $y$ and $z \in G$ and all $\rho > 0$, and similarly for $d_g$. Both of them induce on $G$ the usual Euclidean topology when identifying $G$ with $\mathcal{G}$ via the exponential map and they turn out to be bilipschitz equivalent. In particular the space of Lipschitz functions from $(G,d)$ into $(\mathbb{R}, |.|)$ coincides with that of Lipschitz functions from $(G,d_g)$ into $(\mathbb{R}, |.|)$ hence we shall never specify in the sequel the distance, $d$ or $d_g$, the group is endowed with when speaking about Lipschitz functions.

1.3. **Notions of differentiability.** We begin with the notion of Pansu-differentiability. It can be more generally defined in any Carnot group and is essentially due to P. Pansu (see [20]). We say that a map $L : G \to \mathbb{R}$ is homogeneous if $L(\delta_\rho(x)) = \rho\, L(x)$ for all $x \in G$ and $\rho > 0$. Let $\Omega$ denote an open subset of $G$.

**Definition 1.** *A map $f : \Omega \to \mathbb{R}$ is said to be Pansu-differentiable at $x \in \Omega$ if there exists an homogeneous group homomorphism $L : G \to \mathbb{R}$ such that*

$$\lim_{y \to x} \frac{f(y) - f(x) - L(x^{-1}.y)}{d(x,y)} = 0.$$

*The map $L$ is then unique and will be denoted by $D_G f(x)$.*

Note that this definition is independent from the choice of the Carnot-Carathéodory distance rather than that of the Korányi one.

**Definition 2.** *Let $1 \leq j \leq m$ be fixed. We say that a map $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ along the vector field $X_j$ if the map $\rho \mapsto f(x.\delta_\rho(e_j))$ is differentiable at $\rho = 0$ and we then denote by $X_j f(x)$ the corresponding derivative.*

Note that this definition coincides with the usual one whenever $f$ is smooth. We define similarly the right and left derivatives along the vector fields $X_j$.

**Definition 3.** *Let $1 \leq j \leq m$, $f : \Omega \to \mathbb{R}$ and $x \in \Omega$ be fixed. We define $X_j^+ f(x)$, resp. $X_j^- f(x)$, as the right, resp. left, derivative of $\rho \mapsto f(x.\delta_\rho(e_j))$ at $\rho = 0$ whenever they exist.*

When a map $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ along $X_j$ for all $1 \leq j \leq m$, we set

$$\nabla_{\mathrm{G}} f(x) = \sum_{j=1}^{m} X_j f(x)\, X_j,$$

and we shall then simply say that $\nabla_{\mathrm{G}} f(x)$ exists. The next proposition gives the connection between Pansu-differentiability and derivatives along $X_1, \ldots, X_m$.

**Proposition 1.** *If $f : \Omega \to \mathbb{R}$ is Pansu-differentiable at $x \in \Omega$ then $\nabla_{\mathrm{G}} f(x)$ exists and*

$$D_G f(x)(\exp(X + Z)) = \langle \nabla_{\mathrm{G}} f(x), X \rangle$$

*for all $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$.*

Another important tool in the present paper will be the differentiability of a function along the center.

**Definition 4.** *A map $f : \Omega \to \mathbb{R}$ is said to be $\mathcal{Z}$-differentiable at $x \in \Omega$ if the map $Z \in \mathcal{Z} \mapsto f(x.\exp Z)$ is differentiable at $Z = 0$ in the classical usual sense when one endows $\mathcal{Z}$ with the Euclidean structure induced by that of $\mathcal{G}$. We then denote by $\nabla_{\mathcal{Z}} f(x)$ the corresponding gradient.*

In other words, $f : \Omega \to \mathbb{R}$ is $\mathcal{Z}$-differentiable at $x \in \Omega$ with $\mathcal{Z}$-gradient $\nabla_{\mathcal{Z}} f(x)$ if and only if

$$\lim_{Z \to 0, Z \in \mathcal{Z}} \frac{f(x.\exp Z) - f(x) - \langle \nabla_{\mathcal{Z}} f(x), Z \rangle}{\|Z\|} = 0.$$

If a map $f : \Omega \to \mathbb{R}$ is differentiable along $X_j$ for all $1 \leq j \leq m$ and $\mathcal{Z}$-differentiable at $x \in \Omega$, we define its *cc*-gradient as

$$\nabla_{\mathrm{cc}} f(x) = \nabla_{\mathrm{G}} f(x) + \nabla_{\mathcal{Z}} f(x),$$

and we shall then simply say that $\nabla_{\mathrm{cc}} f(x)$ exists.

## 2. Carnot-Carathéodory distance - Minimal geodesics

We give in this section a complete description of minimal geodesics in groups of type $H$ equipped with their Carnot-Carathéodory distance and use their explicit features to study the differentiability properties of the Carnot-Carathéodory distance to the origin. One of the main useful technical tool will be the introduction of so-called spherical coordinates related to this distance. Thus we let $G$ be a group of type $H$ equipped with its Carnot-Carathéodory distance $d$ as introduced in Section 1. The Carnot-Carathéodory distance to the origin $d_0$ is defined by $d_0(x) = d(0, x)$ for all $x \in G$.

By minimal geodesics we will always mean in this section minimal geodesics with constant and unit speed, that is, curves $\gamma : [0, T] \to G$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t$, $t' \in [0, T]$. It turns out that $(G, d)$ is a complete and locally compact metric length space hence any two points can be joined by a minimal geodesic ([13]). We set

$$m(a) = a - \sin(a), \quad \mu(a) = \frac{1 - \cos(a)}{a - \sin(a)},$$

and

$$P = \{(a, W, Y, T) \in (0, +\infty) \times \mathcal{V} \times \mathcal{Z} \times (0, +\infty);\ \|W\| = 1,\ \|Y\| = 1\}.$$

**Definition 5.** *Given $(a, W, Y, T) \in P$, we say that a curve $\gamma$ is a curve with parameter $(a, W, Y, T)$ if $\gamma : [0, T] \to G$ is defined by $\gamma(t) = \exp(X(t) + Z(t))$ with*

$$X(t) = \frac{T}{a}\,(\exp(\frac{at}{T}J(Y)) - I)W, \quad Z(t) = \frac{1}{2}\,\frac{T^2}{a^2}\,m(\frac{at}{T})\,Y.$$

Note that if $Y \in \mathcal{Z}$ with $\|Y\| = 1$ then it follows from (2) that for all $a \in \mathbb{R}$,

$$\exp(aJ(Y)) = \cos(a)\,I + \sin(a)\,J(Y). \tag{4}$$

In particular $\exp(2k\pi J(Y)) - I = 0$ for all $k \in \mathbb{Z}$. On the other hand,

$$\begin{aligned}(\exp(aJ(Y)) - I)\,(\exp(aJ(-Y)) - I) &= 2\,I - (\exp(aJ(Y)) + \exp(-aJ(Y))) \\ &= 2(1 - \cos(a))\,I,\end{aligned}$$

hence, if $a \neq 2k\pi$ for all $k \in \mathbb{Z}$, then $\exp(aJ(Y)) - I$ is an invertible linear map with

$$(\exp(aJ(Y)) - I)^{-1} = \frac{1}{2(1 - \cos(a))}\,(\exp(aJ(-Y)) - I). \tag{5}$$

In particular,

$$\begin{cases}\gamma \text{ is a curve with parameter } (a, W, Y, T) \in P \text{ so that } \gamma(T) \in \exp \mathcal{Z} \text{ if and only if} \\ a = 2k\pi \text{ for some } k = 1, 2, \dots\end{cases} \tag{6}$$

Next let $\gamma$ be a curve with parameter $(a, W, Y, T) \in P$ and let $\gamma(t) = \exp(X(t) + Z(t))$ with $X(t)$ and $Z(t)$ as in Definition 5. We have

$$X(T) = \frac{T}{a}\,(\exp(aJ(Y)) - I)W = \frac{T}{a}\,((\cos(a) - 1)W + \sin(a)\,J(Y)W)$$

by (4). Then, remembering that $\langle J(Y)W, W\rangle = 0$ and $\|W\| = \|J(Y)W\| = 1$ by (1) and (3), we get

$$\|X(T)\| = \frac{T}{a}\,\sqrt{2(1 - \cos(a))}. \tag{7}$$

On the other hand, we have

$$\|Z(T)\| = \frac{1}{2}\,\frac{T^2}{a^2}\,m(a) > 0 \quad \text{and} \quad Y = \frac{Z(T)}{\|Z(T)\|}. \tag{8}$$

Note that this implies in particular that

$$\gamma(T) \in G \setminus \exp \mathcal{V}. \tag{9}$$

Finally it follows from (7) and (8) that

$$\mu(a) = \frac{\|X(T)\|^2}{4\|Z(T)\|}. \tag{10}$$

**Definition 6.** *We say that a curve $\gamma$ is an horizontal segment (with constant and unit speed) if $\gamma : [0, T] \to G$ is defined by $\gamma(t) = \exp(tX)$ for some $T \in (0, +\infty)$ and some $X \in \mathcal{V}$ with $\|X\| = 1$.*

Minimal geodesics in the Heisenberg groups have been first studied by Gaveau ([11], see also [4] and the references therein). For groups of type $H$ one knows from [18] that a minimal geodesic starting from 0 must be a curve with parameter $(a, W, Y, T)$ for some $(a, W, Y, T) \in P$ or an horizontal segment. Then it has been proved in [22] that curves with parameter $(a, W, Y, T) \in P$ where $a \in (0, 2\pi]$ and all horizontal segments are minimal geodesics, and that

curves with parameter $(2k\pi, W, Y, T) \in P$ for some $k = 2, 3, \dots$ are not minimal geodesics. More precisely, setting

$$\overline{P} = \{(a, W, Y, T) \in (0, 2\pi) \times \mathcal{V} \times \mathcal{Z} \times (0, +\infty);\ \|W\| = 1,\ \|Y\| = 1\},$$

we have the following description.

**Theorem 1** ([22]). *Any curve with parameter $(a, W, Y, T) \in P$ with $a \in (0, 2\pi]$ and any horizontal segment is a minimal geodesic. Furthermore, we have:*

*i) if $x \in \exp \mathcal{Z} \setminus \{0\}$, $x = \exp Z$ for some $Z \in \mathcal{Z} \setminus \{0\}$, then there is exactly a one parameter family of curves that are minimal geodesics from 0 to $x$. This family is composed by all curves with parameter $(2\pi, W, Z/\|Z\|, 2\sqrt{\pi\|Z\|})$, $W \in \mathcal{V}$, $\|W\| = 1$,*

*ii) if $x \in \exp \mathcal{V} \setminus \{0\}$, then there is a unique minimal geodesic from 0 to $x$ which is the horizontal segment between 0 and $x$,*

*iii) if $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$, $x = \exp(X + Z)$ with $X \in \mathcal{V} \setminus \{0\}$ and $Z \in \mathcal{Z} \setminus \{0\}$, then there is a unique curve with parameter $(a, W, Y, T) \in \overline{P}$ from 0 to $x$. The parameters are uniquely given by the following relations,*

$$\mu(a) = \frac{\|X\|^2}{4\|Z\|}, \quad W = \frac{1}{\sqrt{2(1 - \cos(a))}} (\exp(aJ(-Y)) - I) \frac{X}{\|X\|},$$
$$Y = \frac{Z}{\|Z\|}, \qquad T = \frac{a}{\sqrt{2(1 - \cos(a))}} \|X\|.$$

The relations giving the parameter $(a, W, Y, T)$ in *iii)* follow from the various remarks after Definition 5. The uniqueness follows noting that $\mu_{|(0,2\pi)} : (0, 2\pi) \to (0, +\infty)$ is a decreasing $C^1$ diffeomorphism, hence equation (10) has a unique solution $a \in (0, 2\pi)$. In general there are finitely many other solutions of (10) strictly larger than $2\pi$ and to complete the description of minimal geodesics one must now prove that curves with parameter $(a, W, Y, T) \in P$ where $a > 2\pi$ are not minimal geodesics. The argument generalizes the situation for complete Riemannian manifolds where whenever $c$ is a geodesic and there exists another distinct geodesic between $c(t)$ and $c(t')$ with the same length than $c$ then $c$ cannot be minimal on any bigger interval $[t, t' + \varepsilon]$. This gives a simpler alternative to the more involved argument used in [4] in the Heisenberg groups.

**Proposition 2.** *Curves with parameter $(a, W, Y, T) \in P$ with $a > 2\pi$ are not minimal geodesics.*

*Proof.* By contradiction assume that some curve $\gamma$ with parameter $(a, W, Y, T) \in P$ with $a > 2\pi$ is a minimal geodesic. We choose $\hat{W} \in \mathcal{V}$ with $\|\hat{W}\| = 1$ and $\hat{W} \neq W$ and let $c$ denote the curve with parameter $(a, \hat{W}, Y, T) \in P$. We set $t_0 = 2\pi T/a$. By the remark following (4) we have that $\gamma(t_0)$ and $c(t_0)$ coincide (and both belong to $\exp \mathcal{Z}$). Then we consider the curve $l : [0, T] \to G$ given by $l_{|[0,t_0]} = c_{|[0,t_0]}$ and $l_{|[t_0,T]} = \gamma_{|[t_0,T]}$. We claim that $l$ is not a minimal geodesic. Indeed on one hand, all curves with parameter in $P$ and all horizontal segments being smooth curves, any minimal geodesic in $(G, d)$ is of class $C^\infty$ (see also [23], [24] where this is shown to hold more generally under the so-called strong bracket generating hypothesis and [12] where this is extended to all Carnot groups of step 2). On the other hand $l$ is clearly not smooth at $t_0$. Indeed if $l(t) = \exp(X(t) + Z(t))$, we have

$$\dot{X}(t) = J(Y) \exp(\frac{at}{T} J(Y)) \hat{W}$$

for all $t \in (0, t_0)$ hence

$$\lim_{t \to t_0^-} \dot{X}(t) = J(Y) \exp(2\pi J(Y)) \hat{W} = J(Y) \hat{W}.$$

Similarly we have

$$\lim_{t \to t_0^+} \dot{X}(t) = J(Y) W$$

and the claim follows since $\hat{W} \neq W$. Hence $l$ cannot be a minimal geodesic and thus $d_0(l(T)) < T$ (note that $l$ is a sub-unit curve). But $l(T) = \gamma(T)$ and $\gamma$ was assumed to be minimal hence $d_0(\gamma(T)) = T$ and the contradiction follows. □

All together we get immediately Corollary 1 below. This will be one of the key tool when studying Riemannian approximations for $(G, d)$, see §4.2 and especially Lemma 8 there.

**Corollary 1.** *Let $x \in G \setminus \exp \mathcal{Z}$. Then there exists a unique minimal geodesic from 0 to $x$. This minimal geodesic is given by ii) or iii) in Theorem 1 whenever $x \in \exp \mathcal{V} \setminus \{0\}$ or $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ respectively.*

We now turn to the $\mathcal{Z}$-differentiability of $d_0$ outside of the center of the group. Most of the arguments to follow in the rest of this section are close to those of [4]. There are some differences though that arise mainly from the fact that in general the dimension of the center is larger than one (when $\dim \mathcal{Z} = 1$ then $\mathcal{G}$ is the Heisenberg algebra) and that points in $\exp \mathcal{V}$ must often be considered separately.

**Lemma 1.** *Let $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ be fixed and assume that $x$ is the end point of a curve with parameter $(a, W, Y, T) \in P$. For $h \in \mathcal{Z}$, let $x_h = x.\exp h$ and assume that there is some curve with parameter $(a_h, W_h, Y_h, T_h) \in P$ from 0 to $x_h$. Then, if $a_h - a = o(1)$, we have*

$$T_h = T + \frac{a}{T} \langle Y, h \rangle + o(\|h\|).$$

*Proof.* We set $\varepsilon_h = a_h - a$. We have $\varepsilon_h = o(1)$ by assumption. First, since $x \notin \exp \mathcal{Z}$, it follows from (6) that $a \neq 2k\pi$ for all $k = 1, 2...$ and then $a_h \neq 2k\pi$ for all $k = 1, 2...$ provided $\|h\|$ is small enough. Next let $X \in \mathcal{V} \setminus \{0\}$ and $Z \in \mathcal{Z} \setminus \{0\}$ be such that $x = \exp(X + Z)$. We have $x_h = \exp(X + Z + h)$ with $X \in \mathcal{V}$ and $Z + h \in \mathcal{Z}$. Since $x$ and $x_h$ are the end points of the curves with parameter respectively $(a, W, Y, T) \in P$ and $(a_h, W_h, Y_h, T_h) \in P$, it follows from (7) that

$$\frac{T}{a} \sqrt{1 - \cos(a)} = \frac{T_h}{a_h} \sqrt{1 - \cos(a_h)}$$

and hence

$$T_h = T \frac{f(a_h)}{f(a)} = T + T \frac{f'(a)}{f(a)} \varepsilon_h + o(\varepsilon_h) \tag{11}$$

where

$$f(a) = \frac{a}{\sqrt{1 - \cos(a)}}.$$

On the other hand, we have (see (8))

$$Z + h = \frac{1}{2} \frac{T_h^2}{a_h^2} m(a_h) Y_h$$

with

$$Y_h = \frac{Z+h}{\|Z+h\|} = \frac{Z+h}{\|Z\|} - \frac{1}{\|Z\|}\,\langle \frac{Z}{\|Z\|}, h\rangle\,\frac{Z}{\|Z\|} + o(\|h\|)$$

and

$$\begin{aligned}\frac{T_h^2}{a_h^2}\,m(a_h) &= \frac{T_h^2}{a_h^2}\,(1-\cos(a_h))\,\frac{1}{\mu(a_h)} = \frac{T^2}{a^2}\,(1-\cos(a))\,\frac{1}{\mu(a_h)}\\ &= \frac{T^2}{a^2}\,\frac{(1-\cos(a))}{\mu(a)}\,(1-\frac{\mu'(a)}{\mu(a)}\,\varepsilon_h + o(\varepsilon_h)).\end{aligned}$$

Remembering that

$$\|Z\| = \frac{1}{2}\,\frac{T^2}{a^2}\,m(a) = \frac{1}{2}\,\frac{T^2}{a^2}\,\frac{(1-\cos(a))}{\mu(a)},$$

and that $Y = Z/\|Z\|$, we get

$$Z+h = (1-\frac{\mu'(a)}{\mu(a)}\,\varepsilon_h + o(\varepsilon_h))\,(Z+h-\langle Y,h\rangle\,Y + o(\|h\|)).$$

It follows that

$$\begin{aligned}\langle Y,h\rangle\,Y + o(\|h\|) &= -\frac{\mu'(a)}{\mu(a)}\,\varepsilon_h\,Z + o(\varepsilon_h)\\ &= -\frac{1}{2}\,\frac{T^2}{a^2}\,\frac{\mu'(a)}{\mu(a)}\,m(a)\,\varepsilon_h\,Y + o(\varepsilon_h)\end{aligned}$$

and hence

$$\varepsilon_h \underset{\|h\|\to 0}{\sim} -2\,\frac{a^2}{T^2}\,\frac{\mu(a)}{\mu'(a)}\,\frac{1}{m(a)}\,\langle Y,h\rangle.$$

Going back to (11) we get

$$T_h = T - 2\,\frac{a^2}{T}\,\frac{f'(a)}{f(a)}\,\frac{\mu(a)}{\mu'(a)}\,\frac{1}{m(a)}\,\langle Y,h\rangle + o(\|h\|).$$

We have

$$\frac{f'(a)}{f(a)} = \frac{2(1-\cos(a)) - a\sin(a)}{2a(1-\cos(a))} = \frac{2(1-\cos(a)) - a\sin(a)}{2a\,m(a)\,\mu(a)},$$

$$\mu'(a) = \frac{a\sin(a) - 2(1-\cos(a))}{m(a)^2},$$

hence,

$$\frac{f'(a)}{f(a)}\,\frac{\mu(a)}{\mu'(a)}\,\frac{1}{m(a)} = -\frac{1}{2a},$$

and we finally get

$$T_h = T + \frac{a}{T}\,\langle Y,h\rangle + o(\|h\|)$$

as wanted. □

Before we deduce from Lemma 1 the $\mathcal{Z}$-differentiability of $d_0$ on $G\backslash\exp\mathcal{Z}$, we give explicitly some easy consequences of Theorem 1 that will be particularly useful in the sequel. First,

since when a curve $\gamma : [0,T] \to G$ is a minimal geodesic starting from 0, we have $d_0(\gamma(T)) = T$, one can get explicit expressions for $d_0$, namely,

$$d_0(\exp(X+Z)) = \begin{cases} 2\sqrt{\pi\|Z\|} & \text{if } X = 0, \\ \|X\| & \text{if } Z = 0, \\ \dfrac{a\,\|X\|}{\sqrt{2(1-\cos(a))}} = \dfrac{a\,\sqrt{2\|Z\|}}{\sqrt{a-\sin(a)}} & \text{if } X \neq 0 \text{ and } Z \neq 0. \end{cases} \tag{12}$$

where $a \in (0, 2\pi)$ is given by $\mu(a) = \|X\|^2/4\|Z\|$. Next point *iii)* in Theorem 1 can be used as a starting point for the definition of suitable spherical coordinates related to the Carnot-Carathéodory distance on $G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$.

**Theorem 2** ([22]). *The map $\Phi : \overline{P} \to G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ defined by $\Phi(a, W, Y, T) = \exp(X + Z)$ where*

$$X = \frac{T}{a}\,(\exp(aJ(Y)) - I)W, \quad Z = \frac{1}{2}\,\frac{T^2}{a^2}\,m(a)\,Y,$$

*is a $C^1$ diffeomorphism (in the classical usual sense when $G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ is identified with $\mathbb{R}^m \setminus \{0\} \times \mathbb{R}^{n-m} \setminus \{0\}$). For any $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ we will call $(a, W, Y, T) = \Phi^{-1}(x)$ the spherical coordinates of $x$.*

In other words, $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ has spherical coordinates $(a, W, Y, T) \in \overline{P}$ if and only if $x$ is the end point of the minimal geodesic starting from 0 with parameter $(a, W, Y, T) \in \overline{P}$. In particular, $d_0(\Phi(a, W, Y, T)) = T$ for all $(a, W, Y, T) \in \overline{P}$.

**Lemma 2.** *The map $d_0$ is $\mathcal{Z}$-differentiable on $G \setminus \exp \mathcal{Z}$ with continuous $\mathcal{Z}$-gradient there. Namely, if $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ has spherical coordinates $(a, W, Y, T) \in \overline{P}$, we have*

$$\nabla_{\mathcal{Z}}\, d_0(x) = \frac{a}{T}\,Y \tag{13}$$

*and, if $x \in \exp \mathcal{V} \setminus \{0\}$,*

$$\nabla_{\mathcal{Z}}\, d_0(x) = 0. \tag{14}$$

*Proof.* First let $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ be fixed and let $(a, W, Y, T) \in \overline{P}$ denote its spherical coordinates. For $h \in \exp \mathcal{Z}$ with $\|h\|$ small enough, we have $x_h = x.\exp h \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and if $(a_h, W_h, Y_h, T_h) \in \overline{P}$ denote its spherical coordinates, the curve $\gamma_h$ with parameter $(a_h, W_h, Y_h, T_h)$ connects 0 to $x_h$. Moreover, since $\Phi$ is a $C^1$ diffeomorphism, we have $a_h - a = o(1)$. Then Lemma 1 together with the fact that the curve with parameter $(a, W, Y, T)$ and $\gamma_h$ are minimal geodesics imply that

$$d_0(x.\exp h) - d_0(x) = T_h - T = \frac{a}{T}\,\langle Y, h\rangle + o(\|h\|),$$

that is, $d_0$ is $\mathcal{Z}$-differentiable at $x$ with $\mathcal{Z}$-gradient

$$\nabla_{\mathcal{Z}}\, d_0(x) = \frac{a}{T}\,Y.$$

Next if $x = \exp X$ with $X \in \mathcal{V} \setminus \{0\}$ and $x_h = x.\exp h$ with $h \in \mathcal{Z} \setminus \{0\}$, we have $x_h \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and then, if $(a_h, W_h, Y_h, T_h) \in \overline{P}$ denote its spherical coordinates,

$$d_0(x_h) = \frac{a_h}{\sqrt{2(1-\cos(a_h))}}\,\|X\|$$

with $\mu(a_h) = \|X\|^2/4\|h\|$ thanks to (12). In particular $\lim_{\|h\|\to 0} \mu(a_h) = +\infty$ hence $a_h = o(1)$ and then,

$$d_0(x_h) = \|X\| + o(a_h). \tag{15}$$

On the other hand, we also have, once again by (12),

$$d_0(x_h) = \frac{a_h}{\sqrt{a_h - \sin(a_h)}} \sqrt{2\|h\|}.$$

Remembering that $\lim_{\|h\|\to 0} d_0(x_h) = d_0(x) = \|X\|$ and doing an asymptotic expansion of the right-hand side, we get

$$a_h \underset{\|h\|\to 0}{\sim} \frac{12}{\|X\|^2} \|h\|.$$

Going back to (15), it follows

$$d_0(x_h) = d_0(x) + o(\|h\|),$$

hence $d_0$ is $\mathcal{Z}$-differentiable at $x$ with vanishing $\mathcal{Z}$-gradient.

To complete the proof it remains to show that $\nabla_{\mathcal{Z}} d_0$ is continuous on $G \setminus \exp \mathcal{Z}$. Since $\Phi$ is a $C^1$ diffeomorphism, $\nabla_{\mathcal{Z}} d_0$ is clearly continuous on $G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$. Let $x \in \exp \mathcal{V} \setminus \{0\}$ be fixed, $x = \exp X$ for some $X \in \mathcal{V} \setminus \{0\}$, and let $x_k$ be a sequence in $G \setminus \exp \mathcal{Z}$ converging to $x$. Note that $\nabla_{\mathcal{Z}} d_0$ is locally bounded on $G \setminus \exp \mathcal{Z}$, hence, to prove the continuity of $\nabla_{\mathcal{Z}} d_0$ at $x$, we only need to prove that $\nabla_{\mathcal{Z}} d_0(x)$ is the only possible limit for $\nabla_{\mathcal{Z}} d_0(x_k)$. Thus, up to a subsequence, assume that $\nabla_{\mathcal{Z}} d_0(x_k)$ converges. If $x_k \in \exp \mathcal{V} \setminus \{0\}$ for infinitely many $k$'s, then one can extract a subsequence $x_{k_l} \in \exp \mathcal{V} \setminus \{0\}$ and we get

$$\lim_{k\to+\infty} \nabla_{\mathcal{Z}} d_0(x_k) = \lim_{l\to+\infty} \nabla_{\mathcal{Z}} d_0(x_{k_l}) = \lim_{l\to+\infty} 0 = \nabla_{\mathcal{Z}} d_0(x).$$

Otherwise $x_k \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ for all $k$ large enough. Let $(a_k, W_k, Y_k, T_k) \in \overline{P}$ denote the spherical coordinates of $x_k$. We have $\mu(a_k) = \|X_k\|^2/4\|Z_k\|$ whenever $x_k = \exp(X_k + Z_k)$ with $X_k \in \mathcal{V} \setminus \{0\}$ and $Z_k \in \mathcal{Z} \setminus \{0\}$. Since $x_k$ converges to $x$, we have

$$\lim_{k\to+\infty} X_k = X \quad \text{and} \quad \lim_{k\to+\infty} Z_k = 0,$$

hence $\lim_{k\to+\infty} \mu(a_k) = +\infty$ which eventually implies that $\lim_{k\to+\infty} a_k = 0$. Next we have

$$\lim_{k\to+\infty} T_k = \lim_{k\to+\infty} d_0(x_k) = d_0(x) > 0.$$

Remembering that $Y_k$ is a unit vector in $\mathcal{Z}$, it follows that

$$\lim_{k\to+\infty} \nabla_{\mathcal{Z}} d_0(x_k) = \lim_{k\to+\infty} \frac{a_k}{T_k} Y_k = 0 = \nabla_{\mathcal{Z}} d_0(x),$$

and this concludes the proof. □

We now turn to the Pansu-differentiability of $d_0$. The situation outside of the center has already been studied in [22].

**Theorem 3** ([22]). *The map $d_0$ is Pansu-differentiable with continuous Pansu-differential on $G \setminus \exp \mathcal{Z}$. Moreover, if $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and $(a, W, Y, T) \in \overline{P}$ denote its spherical coordinates, then*

$$\nabla_{\mathrm{G}} d_0(x) = J(Y) \exp(a\, J(Y)) W = -\sin(a)\, W + \cos(a)\, J(Y) W, \tag{16}$$

*and if* $x = \exp X$ *with* $X \in \mathcal{V} \setminus \{0\}$, *then*

$$\nabla_G d_0(x) = \frac{X}{\|X\|}. \tag{17}$$

We will also need here to know what happens on the center for the derivatives of $d_0$ along the vector fields $X_j$, $1 \leq j \leq m$. It turns out that $d_0$ is singular on $\exp \mathcal{Z}$ and one of the key ingredients to get the existence and uniqueness of an optimal transport map that we have in mind will be the explicit computation of the right and left derivatives of $d_0$ along the $X_j$'s (see Lemma 5).

**Lemma 3.** *Let* $x = \exp Z$ *with* $Z \in \mathcal{Z} \setminus \{0\}$ *be fixed. For all* $1 \leq j \leq m$, *we have*

$$X_j^+ d_0(x) = -1 \quad \text{and} \quad X_j^- d_0(x) = 1.$$

*Proof.* Let $1 \leq j \leq m$ and $x = \exp Z$ with $Z \in \mathcal{Z} \setminus \{0\}$ be fixed. We define $f : \mathbb{R} \to \mathbb{R}$ by $f(s) = d_0(x.\delta_s(e_j))$. For all $s \neq 0$ we have $x.\delta_s(e_j) \notin \exp \mathcal{Z}$, hence we know from Theorem 3 that $X_j d_0(x.\delta_s(e_j))$ exists and, by definition,

$$\begin{aligned} X_j d_0(x.\delta_s(e_j)) &= \lim_{h \to 0} \frac{d_0(x.\delta_s(e_j).\delta_h(e_j)) - d_0(x.\delta_s(e_j))}{h} \\ &= \lim_{h \to 0} \frac{d_0(x.\delta_{s+h}(e_j)) - d_0(x.\delta_s(e_j))}{h} = f'(s). \end{aligned}$$

In particular, $f$ is differentiable away from 0 and to prove the lemma, it will be sufficient to show that

$$\lim_{s \to 0^+} f'(s) = -1 \quad \text{and} \quad \lim_{s \to 0^-} f'(s) = 1.$$

Noting that $f'$ is bounded on $\mathbb{R}^*$ ($f$ is indeed 1-Lipschitz), let us prove that the only possible limit for $f'(s)$ when $s \to 0^+$ is $-1$. For all $s > 0$, we have $x.\delta_s(e_j) = \exp(sX_j + Z) \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and we then let $(a_s, W_s, Y_s, T_s) \in \overline{P}$ denote its spherical coordinates. Up to a subsequence, one can always assume that $f'(s)$, $a_s$, $W_s$ and $Y_s$ converge when $s \to 0^+$ to some limits say $l \in [-1, 1]$, $a \in [0, 2\pi]$, $W \in \mathcal{V}$ with $\|W\| = 1$ and $Y \in \mathcal{Z}$ with $\|Y\| = 1$ respectively. Note that if $T = d_0(x)$, we always have

$$\lim_{s \to 0^+} T_s = \lim_{s \to 0^+} d_0(x.\delta_s(e_j)) = T > 0.$$

We have $a > 0$. Indeed, if by contradiction $a = 0$, we would have, remembering the expression of the exponential coordinates of $x.\delta_s(e_j)$ in terms of the spherical ones,

$$Z = \lim_{s \to 0^+} \frac{1}{2} \frac{T_s^2}{a_s^2} m(a_s) Y_s = \lim_{s \to 0^+} \frac{T_s^2}{2} \frac{a_s - \sin(a_s)}{a_s^2} Y_s = 0,$$

which gives a contradiction since $Z \neq 0$ by assumption. Next we prove that we actually have $a = 2\pi$. To see this, we use now the expression of the exponential coordinate in $\mathcal{V}$ in terms of the spherical ones and pass to the limit. Since $a > 0$, we have

$$\lim_{s \to 0^+} sX_j = \lim_{s \to 0^+} \frac{T_s}{a_s} (\exp(a_s J(Y_s)) - I) W_s = \frac{T}{a} (\exp(aJ(Y)) - I) W,$$

hence,

$$(\exp(aJ(Y)) - I) W = 0$$

and the claim follows from the remarks after (4). Next, for all $1 \leq k \leq m$, we have

$$\langle sX_j, X_k \rangle = \frac{T_s}{a_s} \left( (\cos(a_s) - 1) \langle W_s, X_k \rangle + \sin(a_s) \langle J(Y_s) W_s, X_k \rangle \right).$$

It follows that if $k \neq j$ and $s > 0$ is small enough, so that $a_s \in (\pi, 2\pi)$, then

$$\langle J(Y_s) W_s, X_k \rangle = \frac{1 - \cos(a_s)}{\sin(a_s)} \langle W_s, X_k \rangle,$$

and hence,

$$\langle J(Y) W, X_k \rangle = \lim_{s \to 0^+} \langle J(Y_s) W_s, X_k \rangle = 0.$$

Since thanks to (3), $\|J(Y)W\| = 1$, we get that one must have $|\langle J(Y)W, X_j \rangle| = 1$. We have

$$\langle sX_j, X_j \rangle \underset{\varepsilon_s \to 0^+}{\sim} - \frac{T}{2\pi} \langle J(Y)W, X_j \rangle \varepsilon_s$$

whenever $a_s = 2\pi - \varepsilon_s$ with $\varepsilon_s > 0$ and $\varepsilon_s = o(1)$ when $s \to 0^+$. On the other hand $\langle sX_j, X_j \rangle = s > 0$ for all $s > 0$. It follows that $\langle J(Y)W, X_j \rangle < 0$ and finally

$$\langle J(Y)W, X_j \rangle = -1.$$

To conclude we go back now to $f'(s)$. Remembering that $f'(s) = X_j d_0(x.\delta_s(e_j))$ and using (16), we get that $\lim_{s \to 0^+} f'(s)$ exists with

$$\lim_{s \to 0^+} f'(s) = \lim_{s \to 0^+} - \sin(a_s) \langle W_s, X_j \rangle + \cos(a_s) \langle J(Y_s) W_s, X_j \rangle = \langle J(Y)W, X_j \rangle = -1$$

as wanted. The fact that $\lim_{s \to 0^-} f'(s) = 1$ can be proved exactly in the same way. □

## 3. Superdifferential and c-concave maps when $c(x, y) = d(x, y)^2/2$

We first recall the general definitions of $c$-concavity and $c$-superdifferential relative to a given map $c : G \times G \to \mathbb{R}$ where $G$ denotes a fixed group of type $H$. Similar definitions can more generally be given in any abstract space.

**Definition 7.** *We say that a function $\varphi : G \to \mathbb{R}$ is c-concave if*

$$\varphi(x) = \inf_{y \in Y} c(x, y) + \phi(y) \qquad \forall x \in G$$

*for some suitable nonempty set $Y \subset G$ and $\phi : Y \to \mathbb{R} \cup \{+\infty\}$, $\phi \not\equiv +\infty$.*

**Definition 8.** *Let $\varphi : G \to \mathbb{R}$ be a function. The c-superdifferential $\partial_c \varphi(x)$ of $\varphi$ at $x \in G$ is defined by*

$$\partial_c \varphi(x) = \{ y \in G;\ c(x, y) - \varphi(x) \leq c(z, y) - \varphi(z) \text{ for all } z \in G \}.$$

*and the c-superdifferential $\partial_c \varphi$ of $\varphi$ by*

$$\partial_c \varphi = \{ (x, y) \in G \times G;\ y \in \partial_c \varphi(x) \}.$$

For the rest of this section we restrict ourselves to the case $c(x, y) = d(x, y)^2/2$ where $d$ is the Carnot-Carathéodory distance on $G$. The case where $c(x, y) = d_g(x, y)^2/2$ will be studied in Section 5. We shall frequently use the notation $c_y(x) = c(x, y)$. The main result in this section will be the description of the $c$-superdifferential of a $c$-concave and locally Lipschitz map as the graph of minus its $cc$-gradient up to a null set, through some suitable system of coordinates, see Theorem 4. This gives the analogue of Theorem 4.4 in [4] and we shall follow

here the same strategy of proof. For sake of completeness and clarity we shall however give quite extensively the technical modifications when needed.

The study of $c$-superdifferentials is naturally related to the differentiability properties of the function. We first investigate the $\mathcal{Z}$-differentiability of $c$-concave maps.

**Lemma 4.** *Let $\varphi$ be a $c$-concave map. Then $\varphi$ is $\mathcal{Z}$-differentiable a.e. on $G$. Moreover we have $\| \nabla_{\mathcal{Z}} \varphi(x)\| \leq 2\pi$ for a.e. $x \in G$.*

*Proof.* In this proof $\mathcal{Z}$ will be regarded as endowed with the Euclidean structure induced by that of $\mathcal{G}$. First, for any fixed $x \in G$, we define $f_x : \mathcal{Z} \to \mathbb{R}$ by

$$f_x(Z) = c_x(\exp Z) = d_0(x^{-1}.\exp Z)^2/2.$$

If $x \in G \setminus \exp \mathcal{Z}$, then $x^{-1}.\exp Z \in G \setminus \exp \mathcal{Z}$ for all $Z \in \exp \mathcal{Z}$ and it follows from Lemma 2 that $f_x$ is differentiable on $\mathcal{Z}$ with gradient

$$\nabla f_x(Z) = d_0(x^{-1}.\exp Z)\ \nabla_{\mathcal{Z}} d_0(x^{-1}.\exp Z).$$

Moreover, it also follows from (13) and (14) that $\|d_0(y)\ \nabla_{\mathcal{Z}} d_0(y)\| \leq 2\pi$ for all $y \in G \setminus \exp \mathcal{Z}$ and we eventually get that $f_x$ is $2\pi$-Lipschitz. If $x \in \exp \mathcal{Z}$, $x = \exp Z_x$ for some $Z_x \in \mathcal{Z}$, then $x^{-1}.\exp Z \in \exp \mathcal{Z}$ for all $Z \in \mathcal{Z}$ and it follows from (12) that $f_x(Z) = 2\pi \|Z - Z_x\|$ and thus $f_x$ is $2\pi$-Lipschitz as well.

Now let $\varphi$ be a $c$-concave map with $\varphi(x) = \inf_{y \in Y} c(x,y) + \phi(y)$ as in Definition 7. Given $x \in \exp \mathcal{V}$, we set $\varphi_x : \mathcal{Z} \to \mathbb{R}$,

$$\varphi_x(Z) = \varphi(x.\exp Z) = \inf_{y \in x^{-1}.Y} f_y(Z) + \phi(x.y),$$

so that $\varphi_x$ can be written as an infimum of $2\pi$-Lipschitz functions and hence is still $2\pi$-Lipschitz. In particular we get from Rademacher's theorem that, for all $x \in \exp \mathcal{V}$, $\varphi_x$ is differentiable a.e. on $\mathcal{Z}$. Finally, since $\nabla_{\mathcal{Z}} \varphi(x) = \nabla \varphi_{\exp X}(Z)$ whenever $x = \exp(X + Z)$ with $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$, the conclusion follows from Fubini's theorem. □

We turn now to the study of $c$-superdifferentials.

**Lemma 5.** *Let $\varphi : G \to \mathbb{R}$ and $x \in G$ be such that $X_j \varphi(x)$ exists for some $1 \leq j \leq m$. Then $(\partial_c \varphi(x) \setminus \{x\}) \cap x.\exp \mathcal{Z} = \emptyset$.*

*Proof.* This follows from Lemma 3 arguing exactly in the same way as in [4, Lemma 4.7]. □

**Lemma 6.** *Let $\varphi : G \to \mathbb{R}$ and $x \in \pi_0(\partial_c \varphi)$ be such that $\nabla_{\mathrm{cc}} \varphi(x)$ exists. Then $\partial_c \varphi(x)$ is a singleton. Furthermore we have $\partial_c \varphi(x) = \{x\}$ if and only if $\nabla_{\mathrm{G}} \varphi(x) = 0$. When $\nabla_{\mathrm{G}} \varphi(x) \neq 0$, let $y$ denote the unique point in $\partial_c \varphi(x)$. If $\nabla_{\mathcal{Z}} \varphi(x) = 0$, then*

$$y = x.\exp(-\nabla_{\mathrm{G}} \varphi(x)).$$

*Otherwise $x^{-1}.y \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and $y = x.\Phi(a, W, Y, T)$ where $(a, W, Y, T) \in \overline{P}$ is given by*

$$a = \| \nabla_{\mathcal{Z}} \varphi(x)\|, \qquad W = -J\Big(\frac{\nabla_{\mathcal{Z}} \varphi(x)}{\| \nabla_{\mathcal{Z}} \varphi(x)\|}\Big)\frac{\nabla_{\mathrm{G}} \varphi(x)}{\| \nabla_{\mathrm{G}} \varphi(x)\|},$$

$$Y = -\frac{\nabla_{\mathcal{Z}} \varphi(x)}{\| \nabla_{\mathcal{Z}} \varphi(x)\|}, \qquad T = \| \nabla_{\mathrm{G}} \varphi(x)\|.$$

Here $\pi_0 : G \times G \to G$ denotes the canonical projection on the first factor.

*Proof.* Let $\varphi : G \to \mathbb{R}$ and $x \in \pi_0(\partial_c\varphi)$ be such that $\nabla_{\rm cc}\,\varphi(x)$ exists. First we assume that one can find $y \in \partial_c\varphi(x) \setminus \{x\}$. We know from Lemma 5 that $y^{-1}.x \in G \setminus \exp \mathcal{Z}$ and hence $d_0$ is Pansu-differentiable and $\mathcal{Z}$-differentiable at $y^{-1}.x$ by Theorem 3 and Lemma 2. Then it follows that $\nabla_{\rm cc}\,c_y(x)$ exists with

$$\nabla_{\rm cc}\,c_y(x) = d_0(y^{-1}.x)\ \nabla_{\rm cc}\,d_0(y^{-1}.x).$$

On the other hand, by definition of $\partial_c\varphi(x)$, the map $z \mapsto c_y(z) - \varphi(z)$ is minimal at $z = x$ and, since $\nabla_{\rm G}\,\varphi(x)$ and $\nabla_{\mathcal{Z}}\,\varphi(x)$ exist by assumption, we get that

$$\nabla_{\rm G}\,c_y(x) = \nabla_{\rm G}\,\varphi(x) \quad \text{and} \quad \nabla_{\mathcal{Z}}\,c_y(x) = \nabla_{\mathcal{Z}}\,\varphi(x). \tag{18}$$

If $\nabla_{\mathcal{Z}}\,\varphi(x) = 0$, we get $\nabla_{\mathcal{Z}}\,d_0(y^{-1}.x) = 0$ and it follows from (13) and (14) that one must then have $y^{-1}.x \in \exp\mathcal{V} \setminus \{0\}$, $y^{-1}.x = \exp X$ for some $X \in \mathcal{V} \setminus \{0\}$. Then, thanks to (12) together with (17), we get that $\nabla_{\rm G}\,c_y(x) = X$, hence $X = \nabla_{\rm G}\,\varphi(x)$ and thus

$$y = x.\exp(-\nabla_{\rm G}\,\varphi(x)).$$

Note that since $X \neq 0$, we have in particular $\nabla_{\rm G}\,\varphi(x) \neq 0$. On the other hand, if $\nabla_{\mathcal{Z}}\,\varphi(x) \neq 0$, then $\nabla_{\mathcal{Z}}\,d_0(y^{-1}.x) \neq 0$ and it follows once again from (13) and (14) that $y^{-1}.x \in G \setminus (\exp\mathcal{V} \cup \exp\mathcal{Z})$. We then denote by $(a, W, Y, T) \in \overline{P}$ its spherical coordinates. Going back to $\nabla_{\rm G}\,\varphi$ and $\nabla_{\mathcal{Z}}\,\varphi$, we get from (18), (16) and (13) that

$$\nabla_{\rm G}\,\varphi(x) = T\,J(Y)\exp(aJ(Y))W,$$
$$\nabla_{\mathcal{Z}}\,\varphi(x) = a\,Y.$$

Hence,

$$a = \|\nabla_{\mathcal{Z}}\,\varphi(x)\|, \qquad Y = \frac{\nabla_{\mathcal{Z}}\,\varphi(x)}{\|\nabla_{\mathcal{Z}}\,\varphi(x)\|},$$
$$T\,\exp(aJ(Y))W = -J(Y)\,\nabla_{\rm G}\,\varphi(x)$$

(remember (2) for the last equality). Taking into account (4), the fact that $\langle J(Y)W, W\rangle = 0$ and that $\|J(Y)W\| = \|W\| = 1$ (see (1) and (3)), we note that $\|\exp(aJ(Y))W\| = 1$ and thus it follows that

$$T = \|\nabla_{\rm G}\,\varphi(x)\| \quad \text{and} \quad \exp(aJ(Y))W = -J\Big(\frac{\nabla_{\mathcal{Z}}\,\varphi(x)}{\|\nabla_{\mathcal{Z}}\,\varphi(x)\|}\Big)\frac{\nabla_{\rm G}\,\varphi(x)}{\|\nabla_{\rm G}\,\varphi(x)\|}.$$

In particular, since $T > 0$, we have $\nabla_{\rm G}\,\varphi(x) \neq 0$. Finally, one recovers the spherical coordinates of $x^{-1}.y$ given in the statement noting that the rule for computing the inverse of an element in spherical coordinates is given by

$$\Phi(a, W, Y, T)^{-1} = \Phi(a, \exp(aJ(Y))W, -Y, T).$$

At this point we have in particular proved that whenever $\partial_c\varphi(x)\setminus\{x\} \neq \emptyset$ then $\partial_c\varphi(x)\setminus\{x\}$ is a singleton and $\nabla_{\rm G}\,\varphi(x) \neq 0$. To conclude the proof, it is thus sufficient to show that if $x \in \partial_c\varphi(x)$ then $\nabla_{\rm G}\,\varphi(x) = 0$. For all $1 \leq j \leq m$, we have

$$X_j c_x(x) = \lim_{\rho\to 0}\frac{d_0(\delta_\rho(e_j))^2}{2\,\rho} = \lim_{\rho\to 0}\frac{\rho}{2} = 0,$$

hence $\nabla_{\rm G}\,c_x(x) = 0$. Then, if $x \in \partial_c\varphi(x)$, we get that $\nabla_{\rm G}\,\varphi(x) = \nabla_{\rm G}\,c_x(x) = 0$ as wanted. □

We conclude now this section with the description of the $c$-superdifferential of $c$-concave and locally Lipschitz maps. Before we state the result we introduce some notations. Given $\hat{X} \in \mathcal{V}$ and $\hat{Z} \in \mathcal{Z}$ with $\|\hat{Z}\| < 2\pi$, we set

$$\exp_{cc}(\hat{X}+\hat{Z}) = \begin{cases} \exp \hat{X} & \text{if } \hat{Z}=0, \\ \Phi(\|\hat{Z}\|, -J(\frac{\hat{Z}}{\|\hat{Z}\|})\frac{\hat{X}}{\|\hat{X}\|}, \frac{\hat{Z}}{\|\hat{Z}\|}, \|\hat{X}\|) & \text{if } \hat{X}\neq 0 \text{ and } 0<\|\hat{Z}\|<2\pi, \\ 0 & \text{if } \hat{X}=0. \end{cases}$$

Note that if $x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z})$ and if $(a, W, Y, T) \in \overline{P}$ denote its spherical coordinates, then $x = \exp_{cc}(\hat{X}+\hat{Z})$ with

$$\hat{X} = T\, J(Y)W, \qquad \hat{Z} = a\, Y,$$

and it turns out that

$$(\hat{X}, \hat{Z}) \mapsto \exp_{cc}(\hat{X}+\hat{Z})$$

is a diffeomorphism from $\mathcal{C} = \{(\hat{X}, \hat{Z}) \in \mathcal{V} \times \mathcal{Z};\ \hat{X} \neq 0,\ \|\hat{Z}\| < 2\pi\}$ onto $G \setminus \exp \mathcal{Z}$.

These somehow Carnot-Carathéodory exponential coordinates (*cc*-exponential for short) will be helpful to interpret formally the $c$-superdifferential of a $c$-concave and locally Lipschitz map as the graph of minus its *cc*-gradient, at least up to a null set. A more significant justification for the introduction of these *cc*-exponential coordinates will be given later when studying Riemannian approximations for $(G, d)$. We will indeed prove that these coordinates arise in a natural way through the limit of the usual Riemannian exponential maps in the approximating manifolds (see Lemma 8). One can also note that if $l : [0,1] \to G$ with $l(s) = \exp(X(s) + Z(s))$, $X(s) \in \mathcal{V}$, $Z(s) \in \mathcal{Z}$, is a minimal geodesic in $(G, d)$ from 0 to some $x$ with *cc*-exponential coordinates $(\hat{X}, \hat{Z}) \in \mathcal{C}$ with $\hat{Z} \neq 0$ and spherical coordinates $(a, W, Y, T) \in \overline{P}$, then the *cc*-exponential coordinates of $x$ are related to the velocity of $l$ through the relation $\dot{X}(0) = T\, J(Y)W = \hat{X}$ and the fact that $l$ is an horizontal curve. If $x \in \exp \mathcal{V} \setminus \{0\}$, then $\dot{X}(0) = \hat{X}$. Here the minimal geodesic is taken as parameterized on $[0,1]$ and thus with constant speed $T$.

**Theorem 4.** *Let $\varphi$ be a $c$-concave and locally Lipschitz map. Then $\partial_c\varphi(x)$ is a singleton for a.e. $x \in G$. More explicitly we have $\lambda(\pi_0(\partial_c\varphi \setminus \Gamma)) = 0$ where $\Gamma$ is the graph of the measurable function $\psi$ defined for a.e. $x \in G$ by $\psi(x) = x.\exp_{cc}(-\nabla_{cc}\varphi(x))$.*

*Proof.* Since $\varphi$ is locally Lipschitz, one knows that $\varphi$ is Pansu-differentiable a.e. on $G$ ([20]). By $c$-concavity and Lemma 4 it also follows that $\varphi$ is $\mathcal{Z}$-differentiable a.e. on $G$, hence $\nabla_{cc}\varphi$ exists a.e. on $G$. Then the theorem is an automatic consequence of Lemma 6 together with the definition of the map $\exp_{cc}$ (see [4, Theorem 4.4] for more details). □

## 4. Optimal Mass Transportation with a cost $c(x,y) = d(x,y)^2/2$

We consider in this section the problem of optimal mass transportation in a group $G$ of type $H$ equipped with its Carnot-Carathéodory distance.

4.1. **Existence and uniqueness.** Recall from the introduction that we are given two Borel probability measures $\mu$, $\nu$ on $G$ and that we want to minimize

$$\psi \mapsto \int_G c(x, \psi(x))\, d\mu(x)$$

among all transport maps $\psi : G \to G$ such that $\psi_{\#}\mu = \nu$. Throughout this section the cost function is fixed and given by $c(x, y) = d(x, y)^2/2$.

A major difficulty comes from the fact that the condition $\psi_{\#}\mu = \nu$ is genuinely non linear with respect to $\psi$ and the class of transport maps is in general not closed with respect to any reasonable weak topology. Kantorovich proposed to consider instead the larger class $\Pi(\mu, \nu)$ of transport plannings, that is, probability measures $\boldsymbol{\gamma}$ in $G \times G$ such that $\pi_{0\#}\boldsymbol{\gamma} = \mu$ and $\pi_{1\#}\boldsymbol{\gamma} = \nu$, $\pi_0$, $\pi_1 : G \times G \to G$ being the canonical projections on the first and on the second factor, and to minimize

$$\boldsymbol{\gamma} \mapsto \int_{G\times G} c(x, y)\, d\boldsymbol{\gamma}(x, y)$$

in the class of transport plannings. The new constraint on transport plannings being now linear, existence of generalized solutions holds under quite general assumptions and in particular in our present setting.

Going back to the original problem one can note that any transport map $\psi$ induces via the map $(Id \times \psi)(x) = (x, \psi(x))$ a transport planning with the same cost, defined by $(Id \times \psi)_{\#}\mu$. Then solving the original problem reduces in showing that one optimal transport planning is induced by some map $\psi$. Moreover one can easily recover from the linearity of the Kantorovich's formulation that the optimal transport map is unique provided one knows that any optimal transport planning is induced by a transport map. It turns out that this approach naturally leads through a well developed duality theory to the study of $c$-concave maps and their $c$-superdifferential, see e.g. [3], [10], [21], [25] but the list is not exhaustive. Then the main result about existence, uniqueness and characterization of an optimal transport map, Theorem 5 below, follows essentially from Theorem 4 exactly in the same way as in [4, Theorem 5.1] to which we refer for complete arguments.

**Theorem 5.** *Let $\mu$ and $\nu$ be two Borel probability measures on $G$. Assume that $\mu$ is absolutely continuous with respect to $\lambda$ and that*

$$\int_G d_0(x)^2\, d\mu(x) + \int_G d_0(y)^2\, d\nu(y) < +\infty. \tag{19}$$

*Then there exists a unique optimal transport planning from $\mu$ to $\nu$, and this planning is induced by a map $\psi$. If $\operatorname{spt}\nu$ is compact, $\psi$ is given by*

$$\psi(x) = x.\exp_{cc}(-\nabla_{\mathrm{cc}}\,\varphi(x)) \text{ for } \mu\text{-a.e. } x \in G \tag{20}$$

*for some $c$-concave and locally Lipschitz map $\varphi$. Conversely, if $\psi$ is representable as in (20) for some map $\varphi$ such that*

$$\nabla_{\mathrm{cc}}\,\varphi(x) \text{ exists and } \varphi(x) = \min_{y\in G} c(x, y) - \varphi^c(y) \text{ for } \mu\text{-a.e. } x \in G \tag{21}$$

*and if*

$$\int_G (d_0(x)^2 + d_0(\psi(x))^2)\, d\mu(x) < +\infty \tag{22}$$

*then $\psi$ is the optimal transport map between $\mu$ and $\nu = \psi_{\#}\mu$.*

Here $\varphi^c(y) = \inf_{x\in G} c(x,y) - \varphi(x)$ is the $c$-transform of $\varphi$.

4.2. **Riemannian approximations.** Let $G$ be as before a fixed group of type $H$ equipped with its Carnot-Carathéodory distance $d$. For any $\varepsilon > 0$, let $M_\varepsilon$ be the Riemannian manifold with coincides with $G$ as a manifold and is equipped with the Riemannian structure where the metric at the point $x$ has

$$X_1, \dots, X_m, X^\varepsilon_{m+1}, \dots, X^\varepsilon_n,$$

as an orthonormal basis. Here $X^\varepsilon_j = \varepsilon X_j$ for $m+1 \le j \le n$ and $X_1, \dots, X_m, X^\varepsilon_{m+1}, \dots, X^\varepsilon_n$ are regarded as left invariant vector fields. We denote by $d_\varepsilon$ the induced Riemannian distance. This distance is left invariant. Note that $M_\varepsilon$ and $G$ coincide as topological spaces.

We have $d_\varepsilon \le d_{\varepsilon'} \le d$ for all $0 < \varepsilon' < \varepsilon$ and it turns out that $(G, d)$ can be recovered as the limit of $(M_\varepsilon, d_\varepsilon)$ in the sense that

$$d(x,y) = \sup_{\varepsilon>0} d_\varepsilon(x,y) \tag{23}$$

for all $x$, $y \in G$ ([16]).

In view of the transport problem we fix as in Theorem 5 two Borel probability measures $\mu$, $\nu$ on $G$ with $\mu$ absolutely continuous with respect to $\lambda$ and satisfying (19). We set $c_\varepsilon(x,y) = d_\varepsilon(x,y)^2/2$ and, as before, $c(x,y) = d(x,y)^2/2$. It follows from (19) and (23) that

$$\int_{M_\varepsilon} d_\varepsilon(x)^2 \, d\mu(x) + \int_{M_\varepsilon} d_\varepsilon(y)^2 \, d\nu(y) < +\infty.$$

It has been proved in [19] that under these assumptions there exists a unique optimal transport map $\psi_\varepsilon$ from $\mu$ to $\nu$ relative to the cost $c_\varepsilon$. It turns out that the optimal transport map relative to the cost $c$ is the limit of $\psi_\varepsilon$. The proof can be achieved in the same way as in [4, Theorem 6.2]

**Theorem 6.** *Let $\mu$ and $\nu$ be two Borel probability measures on $G$ with $\mu$ absolutely continuous with respect to $\lambda$ and satisfying* (19). *Then the optimal transport maps $\psi_\varepsilon$ from $\mu$ to $\nu$ relative to the cost $c_\varepsilon$ converge in $\mu$-measure as $\varepsilon$ goes to 0 to the optimal transport map $\psi$ from $\mu$ to $\nu$ relative to the cost $c$.*

R. McCann gives moreover a characterization of the optimal transport map $\psi_\varepsilon$ in terms of some $c_\varepsilon$-concave potential whose gradient -and its norm- specifies the direction -and how far- to move the mass geodesically in $M_\varepsilon$, namely,

$$\psi_\varepsilon(x) = \exp^\varepsilon_x(-\nabla^\varepsilon \varphi_\varepsilon(x)) \tag{24}$$

for $\mu$-a.e. $x \in M_\varepsilon$, where $\exp^\varepsilon_x$ is the Riemannian exponential in $M_\varepsilon$,

$$\nabla^\varepsilon \varphi_\varepsilon(x) = \sum_{j=1}^m X_j\varphi_\varepsilon(x)\, X_j + \sum_{j=m+1}^n X^\varepsilon_j \varphi_\varepsilon(x)\, X^\varepsilon_j \in T_x M_\varepsilon,$$

$\varphi_\varepsilon$ is a Kantorovich potential, that is, a $c_\varepsilon$-concave function such that $\boldsymbol{\gamma} \in \Pi(\mu,\nu)$ is an optimal transport planning if and only if

$$c_\varepsilon(x,y) = \varphi_\varepsilon(x) + \varphi^c_\varepsilon(y) \qquad \boldsymbol{\gamma}\text{-a.e.} \tag{25}$$

and where it turns moreover out that the curve $s \in [0,1] \mapsto \exp^\varepsilon_x(-s\nabla^\varepsilon\varphi_\varepsilon(x))$ is a minimal geodesic in $(M_\varepsilon, d_\varepsilon)$ for $\mu$-a.e. $x$. Note that by left invariance, one then has

$$\exp^\varepsilon_x(-\nabla^\varepsilon\varphi_\varepsilon(x)) = x.\exp^\varepsilon_0(-\nabla^\varepsilon\varphi_\varepsilon(x))$$

where, with a slight abuse of notation, $\nabla^\varepsilon\varphi_\varepsilon(x)$ showing up the right-hand side is given by

$$\nabla^\varepsilon\varphi_\varepsilon(x) = \sum_{j=1}^{m} X_j\varphi_\varepsilon(x)\, X_j(0) + \sum_{j=m+1}^{n} X_j^\varepsilon\varphi_\varepsilon(x)\, X_j^\varepsilon(0) \in T_0M_\varepsilon.$$

Following the notations of §1.3 we set

$$\nabla_{\mathrm{G}}\,\varphi_\varepsilon(x) = \sum_{j=1}^{m} X_j\varphi_\varepsilon(x)\, X_j, \qquad \nabla_{\mathcal{Z}}\,\varphi_\varepsilon(x) = \sum_{j=m+1}^{n} X_j\varphi_\varepsilon(x)\, X_j,$$

so that

$$\nabla^\varepsilon\varphi_\varepsilon(x) = \nabla_{\mathrm{G}}\,\varphi_\varepsilon(x) + \varepsilon^2\ \nabla_{\mathcal{Z}}\,\varphi_\varepsilon(x).$$

We shall now prove that any limit point of the $\varphi_\varepsilon$', suitably normalized, gives rise to a $c$-concave potential associated to the optimal transport map $\psi$ as in (20) together with convergence of the corresponding gradients.

**Theorem 7.** *Let $\mu$ and $\nu$ be two Borel probability measures on $G$ with $\mu$ absolutely continuous with respect to $\lambda$, $\operatorname{spt}\nu$ compact and $\int_G d_0(x)^2\, d\mu(x) < +\infty$. For $\varepsilon > 0$, let $\varphi_\varepsilon : M_\varepsilon \to \mathbb{R}$ be a $c_\varepsilon$-concave dual potential as in (25), normalized in such a way that $\varphi_\varepsilon(0) = 0$. Then $(\varphi_\varepsilon)_{\varepsilon>0}$ is relatively compact with respect to the local uniform convergence in $(G,d)$ as $\varepsilon$ goes to 0. Moreover any limit point $\varphi$ is locally Lipschitz, c-concave and satisfies (21). Next, if $\psi$ denotes the optimal transport map from $\mu$ to $\nu$ relative to the cost c, then $\psi$ coincides with $x \mapsto x.\exp_{\mathrm{cc}}(-\nabla_{\mathrm{cc}}\,\varphi(x))$ $\mu$-a.e. and*

$$\nabla_{\mathrm{G}}\,\varphi_\varepsilon \longrightarrow \nabla_{\mathrm{G}}\,\varphi \quad \text{in } L^p_{loc}(G,d\mu) \text{ for all } p \in [1,+\infty),$$
$$\nabla_{\mathcal{Z}}\,\varphi_\varepsilon \longrightarrow \nabla_{\mathcal{Z}}\,\varphi \quad \mu\text{-a.e. on } \{x \in G\,;\ \psi(x) \neq x\}.$$

An important tool in the proof of Theorem 7 comes from the fact that the *cc*-exponential map $\exp_{\mathrm{cc}}$ can be interpreted as the limit (in some suitable sense) of the exponential maps $\exp_0^\varepsilon$ in $M_\varepsilon$. We first recall the explicit expressions of the geodesics starting from 0 in $M_\varepsilon$. For simplicity of notations we set $\exp^\varepsilon = \exp_0^\varepsilon$.

**Lemma 7.** *Let $\varepsilon > 0$ be fixed. Let $v \in T_0M_\varepsilon \simeq \mathcal{G}$ be fixed with $v = \dot X + \dot Z$, $\dot X \in \mathcal{V}$, $\dot Z \in \mathcal{Z}$. For $s \in [0,1]$, we set $\exp^\varepsilon(sv) = \exp(X_\varepsilon(s) + Z_\varepsilon(s))$ with $X_\varepsilon(s) \in \mathcal{V}$, $Z_\varepsilon(s) \in \mathcal{Z}$, and $X = X_\varepsilon(1)$, $Z = Z_\varepsilon(1)$. We also set $a_\varepsilon = \varepsilon^{-2}\,\|\dot Z\|$. If $\dot Z = 0$, we have*

$$X_\varepsilon(s) = s\dot X, \qquad Z_\varepsilon(s) = 0$$

*for all $s \in [0,1]$ and in particular $\dot X = X$ and $Z = 0$. If $\dot Z \neq 0$, then, setting*

$$Y = \frac{\dot Z}{\|\dot Z\|}, \qquad W_\varepsilon = -\frac{1}{a_\varepsilon} J(Y)\dot X,$$

*we have*

$$X_\varepsilon(s) = (\exp(a_\varepsilon s J(Y)) - I)W_\varepsilon, \qquad Z_\varepsilon(s) = (\varepsilon^2 a_\varepsilon s + \frac{1}{2}\,\|W_\varepsilon\|^2 m(a_\varepsilon s))Y$$

*for all $s \in [0,1]$. In particular $Z \neq 0$ and $Y = Z/\|Z\|$.*

*Proof.* These equations can be found in [18]. Alternatively one can argue in the same way as in [4, Proposition 6.6]. □

We come now to the connection between the maps $\exp_{cc}$ and $\exp^\varepsilon$. For $x \in G \setminus \{0\}$, we set

$$a_x = \begin{cases} 0 & \text{if } x \in \exp \mathcal{V}, \\ a & \text{if } x \in G \setminus (\exp \mathcal{V} \cup \exp \mathcal{Z}) \text{ has spherical coordinates } (a, W, Y, T) \in \overline{P}, \\ 2\pi & \text{if } x \in \exp \mathcal{Z}. \end{cases} \tag{26}$$

**Lemma 8.** *Let $(\varepsilon_k)_{k\geq 1}$ be a positive sequence that goes to 0. For each $k \geq 1$ assume that $v_{\varepsilon_k} \in T_0 M_{\varepsilon_k} \simeq \mathcal{G}$, $v_{\varepsilon_k} = \dot{X}_{\varepsilon_k} + \dot{Z}_{\varepsilon_k}$ with $\dot{X}_{\varepsilon_k} \in \mathcal{V}$ and $\dot{Z}_{\varepsilon_k} \in \mathcal{Z}$, is such that the curve $l_{\varepsilon_k} : s \in [0,1] \mapsto \exp^{\varepsilon_k}(s v_{\varepsilon_k})$ is a minimal geodesic in $(M_{\varepsilon_k}, d_{\varepsilon_k})$. Assume also that $\lim_{k\to+\infty} \exp^{\varepsilon_k} v_{\varepsilon_k} = x$ for some $x \in G \setminus \{0\}$. Then we have*

$$\lim_{k\to+\infty} a_{\varepsilon_k} = a_x$$

*where $a_{\varepsilon_k} = \varepsilon_k^{-2} \|\dot{Z}_{\varepsilon_k}\|$ and $a_x$ is given by (26). Moreover, if $x \in G \setminus \exp \mathcal{Z}$, $x = \exp_{cc}(\hat{X} + \hat{Z})$ with $(\hat{X}, \hat{Z}) \in \mathcal{C}$, then*

$$\lim_{k\to+\infty} \dot{X}_{\varepsilon_k} = \hat{X} \quad \text{and} \quad \lim_{k\to+\infty} \varepsilon_k^{-2} \dot{Z}_{\varepsilon_k} = \hat{Z}.$$

*Proof.* For simplicity of notations, we shall replace in this proof all indexes $\varepsilon_k$ by $k$. We set $x_k = \exp^k v_k$, $x_k = \exp(X_k + Z_k)$ with $X_k \in \mathcal{V}$ and $Z_k \in \mathcal{Z}$. First we note that $\|v_k\|_k = d_k(0, x_k)$ by minimality of the curve $l_k$ where $\|.\|_k$ denotes the norm on $T_0 M_k$ induced by the Riemannian structure on $M_k$. We have $\|v_k\|_k^2 = \|\dot{X}_k\|^2 + \varepsilon_k^{-2} \|\dot{Z}_k\|^2$ and $d_k(0, x_k) \leq d(0, x_k)$ hence the two sequences $\|\dot{X}_k\|$ and $\varepsilon_k^{-1} \|\dot{Z}_k\| = \varepsilon_k a_k$ are bounded. Let us prove that we actually have $a_k = O(1)$. By contradiction, assume that $\lim_{k\to+\infty} a_k = +\infty$ along some subsequence (as usual we shall not relabel subsequences for simplicity of notations). Then $\dot{Z}_k \neq 0$ for all $k$ large enough and, with the notations of Lemma 7 applied to $v = v_k$, we have $\|W_k\| = \|\dot{X}_k\|/a_k$ hence $\lim_{k\to+\infty} \|W_k\| = 0$. This implies that $\lim_{k\to+\infty} X_k = 0$. Indeed, we have

$$\|X_k\| = \sqrt{2(1 - \cos(a_k))}\, \|W_k\| \tag{27}$$

(computations are similar to those leading to (7)) and the claim follows. Similarly, we have

$$\|Z_k\| = \varepsilon_k^2 a_k + \frac{1}{2} \|W_k\|^2 m(a_k) = \varepsilon_k^2 a_k + \frac{1}{2} \frac{a_k - \sin(a_k)}{a_k^2} \|\dot{X}_k\|^2, \tag{28}$$

hence $\lim_{k\to+\infty} Z_k = 0$. It follows that $x = \lim_{k\to+\infty} \exp(X_k + Z_k) = 0$ which gives the contradiction.

Then let $\hat{a}$ and $\dot{X}$ be limit points of $a_k$ and $\dot{X}_k$ respectively. We distinguish two cases. First if $x \in G \setminus \exp \mathcal{V}$, we have $\hat{a} \neq 0$. Otherwise, remembering (28), we would get that $\lim_{k\to+\infty} Z_k = 0$ hence $x \in \exp \mathcal{V}$ which gives the contradiction. It follows in particular that $\dot{Z}_k \neq 0$ for all $k$ large enough and, up to a subsequence, one can always assume that $Y_k = \dot{Z}_k / \|\dot{Z}_k\|$ converges to some $Y$. Then, if $l_k(s) = \exp(X_k(s) + Z_k(s))$, we get from Lemma 7 that

$$\lim_{k\to+\infty} X_k(s) = X(s), \qquad \lim_{k\to+\infty} Z_k(s) = Z(s)$$

for all $s \in [0,1]$ where

$$X(s) = (\exp(\hat{a} s J(Y)) - I)\hat{W}, \qquad Z(s) = \frac{1}{2} \|\hat{W}\|^2 m(\hat{a}s)\, Y.$$

with $\widehat{W} = -J(Y)\dot{X}/\hat{a}$. By monotone convergence of $d_k$ to $d$, one gets that the limit curve $l : s \in [0,1] \mapsto \exp(X(s) + Z(s))$ is a minimal geodesic in $(G,d)$. On the other hand, if $T = d(0,x)$, we also have

$$\begin{aligned} T &= \lim_{k\to+\infty} d_k(0,x_k) = \lim_{k\to+\infty} \|v_k\|_k = \lim_{k\to+\infty} (\|\dot{X}_k\|^2 + \varepsilon_k^{-2}\,\|\dot{Z}_k\|^2)^{1/2} \\ &= \lim_{k\to+\infty} a_k\,(\|W_k\|^2 + \varepsilon_k^2)^{1/2} = \hat{a}\,\|\widehat{W}\| = \|\dot{X}\|. \end{aligned}$$

Then, setting $W = \widehat{W}/\|\widehat{W}\|$, we finally get that

$$X(s) = \frac{T}{\hat{a}}\,(\exp(\hat{a}sJ(Y)) - I)W, \qquad Z(s) = \frac{1}{2}\,\frac{T^2}{\hat{a}^2}\,m(\hat{a}s)\,Y,$$

and that the curve $\gamma : [0,T] \to G$ defined by $\gamma(t) = l(t/T)$ is a minimal geodesic from 0 to $x$ with constant and unit speed. Then it follows from the characterization of minimal geodesics (see Theorem 1 and Corollary 1), that one must have $\hat{a} = a_x$. If moreover $x \in G \setminus (\exp\mathcal{V} \cup \exp\mathcal{Z})$ we also get that $(\hat{a}, W, Y, T)$ must coincide with the spherical coordinates of $x$. Then $\dot{X} = T\,J(Y)W$ coincides with $\hat{X}$ by definition of the *cc*-exponential coordinates. Next we have $\hat{a} = \|\hat{Z}\|$ and $Y = \hat{Z}/\|\hat{Z}\|$ also by definition of the *cc*-exponential coordinates, hence,

$$\lim_{k\to+\infty} \varepsilon_k^{-2}\,\dot{Z}_k = \lim_{k\to+\infty} a_k\,Y_k = \hat{a}\,Y = \hat{Z}.$$

If $x \in \exp\mathcal{V} \setminus \{0\}$ then $x = \exp X = \exp_{\mathrm{cc}} X$ for some $X \in \mathcal{V} \setminus \{0\}$ and we have $\hat{a} = 0$. Indeed, if $\dot{Z}_k = 0$ for infinitely many $k$'s, this follows from the very definition of $a_k$. Moreover we get from Lemma 7 that $\dot{X}_k = X$ for these $k$'s and this concludes the proof. Otherwise, we argue by contradiction, and since $x \in \exp\mathcal{V}$, we have $\lim_{k\to+\infty} Z_k = 0$ and it would follow from (28) that $\lim_{k\to+\infty} \dot{X}_k = 0$ and then (remember (27))

$$\lim_{k\to+\infty} \|X_k\| = \lim_{k\to+\infty} \frac{\sqrt{2(1-\cos(a_k))}}{a_k}\,\|\dot{X}_k\| = 0.$$

Thus we eventually would get that $x = \lim_{k\to+\infty} \exp(X_k + Z_k) = 0$ which gives the contradiction. Then, assuming as before that, up to a subsequence, $Y_k$ converges to some $Y$, we have

$$\begin{aligned} X &= \lim_{k\to+\infty} X_k = \lim_{k\to+\infty} (\exp(a_k J(Y_k)) - I)W_k \\ &= \lim_{k\to+\infty} \frac{1-\cos(a_k)}{a_k}\,J(Y_k)\dot{X}_k + \frac{\sin(a_k)}{a_k}\,\dot{X}_k = \dot{X} \end{aligned}$$

and this concludes the proof. □

Theorem 7 follows now from Lemma 8 arguing as in [4, Theorem 6.11].

## 5. Optimal Mass Transportation with a cost $c(x,y) = d_g(x,y)^2/2$

We consider in this section the problem of optimal transport in a group $G$ of type $H$ equipped with the Korányi distance. From now on we thus set $c(x,y) = d_g(x,y)^2/2$. With the same strategy as for the Carnot-Carathéodory distance, existence and uniqueness of the optimal transport will follow as soon as one gets the analogue of Theorem 4 about the $c$-superdifferential of locally Lipschitz and $c$-concave maps. However this will be obtained in the present case for quite different reasons than in the case of the Carnot-Carathéodory distance, the main difference coming from the situation on the center of the group. We begin

with a first description of the $c$-superdifferential of a map $\varphi$ at some point where $\varphi$ is Pansu- and $\mathcal{Z}$-differentiable.

**Lemma 9.** *Let $\varphi : G \to \mathbb{R}$ and $x \in \pi_0(\partial_c\varphi)$ be such that $\nabla_{\rm cc}\,\varphi(x)$ exists. Then we have, either*

$$\partial_c\varphi(x) \subset G \setminus x.\exp \mathcal{Z}, \tag{$i$}$$

*and then $\partial_c\varphi(x)$ is a singleton, $\partial_c\varphi(x) = \{y\}$, where $y = x.\exp(X+Z)$ with*

$$\begin{aligned} X &= \frac{1}{2}\left(J(\nabla_{\mathcal{Z}}\,\varphi(x)) - \sqrt{4 - \|\nabla_{\mathcal{Z}}\,\varphi(x)\|^2}\, I\right)\nabla_{\rm G}\,\varphi(x), \\ Z &= -\frac{1}{4}\,\frac{\|\nabla_{\rm G}\,\varphi(x)\|^2}{\sqrt{4 - \|\nabla_{\mathcal{Z}}\,\varphi(x)\|^2}}\,\nabla_{\mathcal{Z}}\,\varphi(x), \end{aligned} \tag{29}$$

*or*

$$\partial_c\varphi(x) \subset x.\exp \mathcal{Z}. \tag{$ii$}$$

*Moreover, when $(ii)$ holds and $\partial_c\varphi(x) \setminus \{x\} \neq \emptyset$, we have*

$$\nabla_{\rm G}\,\varphi(x) = 0 \ \text{ and } \ \|\nabla_{\mathcal{Z}}\,\varphi(x)\| = 2. \tag{30}$$

*Proof.* The proof uses verbatim arguments of [4, Lemma 4.11] together with the necessary computational modifications. They rely essentially on explicit computations for the derivatives of the gauge of the group. First, setting as before $c_y(x) = c(x,y)$ for all $x$, $y \in G$, we have that for all $y \in G$, $c_y$ is smooth on $G \setminus \{y\}$ and

$$\nabla_{\rm G}\,c_y(x) = \frac{1}{(\|X\|^4 + 16\|Z\|^2)^{1/2}}\,(\|X\|^2\, I + 4\,J(Z))X, \tag{31}$$

$$\nabla_{\mathcal{Z}}\,c_y(x) = \frac{8}{(\|X\|^4 + 16\|Z\|^2)^{1/2}}\,Z \tag{32}$$

whenever $y^{-1}.x = \exp(X+Z)$, $X \in \mathcal{V}$, $Z \in \mathcal{Z}$, $(X,Z) \neq (0,0)$. In particular it follows that if $y \in G \setminus x.\exp\mathcal{Z}$, then $\nabla_{\rm G}\,c_y(x) \neq 0$ and if $y \in x.\exp\mathcal{Z}$, $y \neq x$, then $\nabla_{\rm G}\,c_y(x) = 0$. Note that, arguing as in Lemma 6, we also have $\nabla_{\rm G}\,c_x(x) = 0$.

Going back to $\partial_c\varphi(x)$ with $x \in \pi_0(\partial_c\varphi)$ such that $\nabla_{\rm cc}\,\varphi(x)$ exists, we get from the definition of the $c$-superdifferential that

$$\nabla_{\rm G}\,c_y(x) = \nabla_{\rm G}\,\varphi(x) \qquad \forall\, y \in \partial_c\varphi(x), \tag{33}$$

$$\nabla_{\mathcal{Z}}\,c_y(x) = \nabla_{\mathcal{Z}}\,\varphi(x) \qquad \forall\, y \in \partial_c\varphi(x) \setminus \{x\}. \tag{34}$$

All together, it follows in particular that $\nabla_{\rm G}\,\varphi(x) \neq 0$ whenever $\partial_c\varphi(x) \setminus x.\exp\mathcal{Z} \neq \emptyset$ and $\nabla_{\rm G}\,\varphi(x) = 0$ whenever $\partial_c\varphi(x) \cap x.\exp\mathcal{Z} \neq \emptyset$ from which we get the exclusive alternative between $(i)$ and $(ii)$.

When $(i)$ holds, the fact that $\partial_c\varphi(x)$ is a singleton together with the expression of $y \in \partial_c\varphi(x)$ follow from the explicit resolution of the system

$$\begin{aligned} \frac{1}{(\|X\|^4 + 16\|Z\|^2)^{1/2}}\,(\|X\|^2\, I + 4\,J(Z))X &= \nabla_{\rm G}\,\varphi(x) \\ \frac{8}{(\|X\|^4 + 16\|Z\|^2)^{1/2}}\,Z &= \nabla_{\mathcal{Z}}\,\varphi(x) \end{aligned}$$

coming from (31), (33) and (32), (34) and where $X$ and $Z$ are the exponential coordinates of $y^{-1}.x$ (note that the exponential coordinates of $x^{-1}.y$ given in the statement are then $-X$

and $-Z$). To solve this system one note in particular that $\|X\|^2 I + 4 J(Z)$ is an invertible linear map with

$$(\|X\|^2 I + 4 J(Z))^{-1} = \frac{1}{\|X\|^4 + 16\|Z\|^2} (\|X\|^2 I - 4 J(Z)).$$

When $(ii)$ holds we have already seen that $\nabla_{\rm G} \varphi(x) = 0$. If moreover $\partial_c\varphi(x) \setminus \{x\} \neq \emptyset$, the fact that $\| \nabla_{\mathcal{Z}} \varphi(x)\| = 2$ follows from (32) and (34) applied to $y \in (\partial_c\varphi(x)\setminus\{x\})\cap x.\exp \mathcal{Z}$. □

When $(ii)$ holds, it may a priori happen that $\partial_c\varphi(x)$ does not reduce to a singleton. To get the description of $\partial_c\varphi$ as a graph outside a negiglible set we shall prove that this happens only on a set of zero measure. As in [4] this will be a consequence of (30) and of Lemma 11 below based on $BV_G$ estimates for $c$-concave maps together with differentiability properties of $BV_G$ functions. We first prove the $\mathcal{Z}$-differentiability and $L^\infty$ bound for the $\mathcal{Z}$-gradient of $c$-concave maps. This is also useful on its own when proving Theorem 8.

**Lemma 10.** *Let $\varphi : G \to \mathbb{R}$ be a $c$-concave map. Then $\varphi$ is $\mathcal{Z}$-differentiable a.e. on $G$. Moreover we have $\| \nabla_{\mathcal{Z}} \varphi(x)\| \leq 2$ for a.e. $x \in G$.*

*Proof.* The proof can be achieved arguing as for Lemma 4. The point is that the map $f_x : \mathcal{Z} \to \mathbb{R}$ defined by $f_x(Z) = c_x(\exp Z)$ is 2-Lipschitz for all $x \in G$. Indeed, when $x \in G \setminus \exp \mathcal{Z}$, then $f_x$ is differentiable on $\mathcal{Z}$ with $\|\nabla f_x(Z)\| \leq 2$ for all $Z \in \mathcal{Z}$ (see (32)), and when $x \in \exp \mathcal{Z}$, $x = \exp Z_x$ for some $Z_x \in \mathcal{Z}$, then $f_x(Z) = 2\|Z - Z_x\|$. □

**Lemma 11.** *Let $\varphi : G \to \mathbb{R}$ be a $c$-concave and locally Lipschitz map. Then we have*

$$\lambda(\{x \in G;\, \nabla_{\rm G} \varphi(x) = 0 \text{ and } \nabla_{\mathcal{Z}} \varphi(x) \neq 0\}) = 0.$$

*Proof.* First since $\varphi$ is locally Lipschitz, it follows, essentially from Pansu's differentiability theorem ([20]), that $X_j\varphi$, in the distributional sense, can be represented by a $L^\infty_{loc}$ function for all $1 \leq j \leq m$. Then we have that, for all $1 \leq j \leq m$, $X_j\varphi \in BV_{G,loc}$, that is, $X_iX_j\varphi$ can be locally represented by a (signed) Radon measure for all $1 \leq i \leq m$ (for more details about the space $BV_G$ of functions of intrinsic bounded variations, we refer to e.g. [2] and the references therein). Indeed, first, direct computations give a constant $C > 0$ such that

$$|X_iX_jc_y(x)| \leq C \tag{35}$$

for all $y \in G$, $x \in G \setminus \{y\}$ and all $1 \leq i, j \leq m$. Then one can argue as in [2, Theorem 4.3]. On one hand one gets that $XX\varphi$ can be locally represented by a Radon measure for all $X \in \mathcal{V}$. In particular considering $X = X_j$ and $X = X_i + X_j$ gives that $(X_iX_j + X_jX_i)\varphi$ can be locally represented by a Radon measure. On the other hand it follows from (35) that $|(X_iX_j - X_jX_i)c_y(x)| \leq 2\,C$ for all $y \in G$ and all $x \in G \setminus \{y\}$ and consequently $(X_iX_j - X_jX_i)\varphi$ can be locally represented by a Radon measure as well. All together we get that $2X_iX_j\varphi = (X_iX_j + X_jX_i)\varphi + (X_iX_j - X_jX_i)\varphi$ can be locally represented by a Radon measure for all $1 \leq i, j \leq m$, hence $X_j\varphi \in BV_{G,loc}$ as claimed.

Then it follows from [2, Theorem 2.2] that, for all $1 \leq j \leq m$, $X_j\varphi$ is approximately differentiable a.e. and the approximate differential is then identically zero on $S = \{x \in G;\, \nabla_{\rm G} \varphi(x) = 0 \text{ and } \nabla_{\mathcal{Z}} \varphi(x) \neq 0\}$. Once again by [2, Theorem 2.2] we get that the measure $[X_i, X_j]\varphi$ has zero absolutely continuous part with respect to $\lambda$ on $S$ for all $1 \leq i, j \leq m$. Since $[\mathcal{V}, \mathcal{V}] = \mathcal{Z}$, we get that $Z\varphi$ can be locally represented by a Radon measure with zero absolutely continuous part with respect to $\lambda$ on $S$ for all $Z \in \mathcal{Z}$. On the other hand, we know from Lemma 10 that $Z\varphi$ is representable by an $L^\infty$ function and coincides as a measure with $\langle \nabla_{\mathcal{Z}} \varphi, Z\rangle\, \lambda$. Hence $\nabla_{\mathcal{Z}} \varphi = 0$ a.e. on $S$ and the lemma follows. □

As for the case of the Carnot-Carathéodory distance we automatically get from the previous lemmata the analogue of Theorem 4 (see also [4]).

**Theorem 8.** *Let $\varphi$ be a c-concave and locally Lipschitz map. Then $\partial_c\varphi(x)$ is a singleton for a.e. $x \in G$. More explicitly we have $\lambda(\pi_0(\partial_c\varphi \setminus \Gamma)) = 0$ where $\Gamma$ is the graph of a measurable function $\psi$ which satisfies that $\psi(x) = x.\exp(X + Z)$ where $X$ and $Z$ are given by (29) for a.e. $x \in G$ such that $\psi(x) \neq x$.*

We conclude with the existence, uniqueness and characterization of the optimal transport map relative to the cost $c(x, y) = d_g(x, y)^2/2$ which follows as for the case of the Carnot-Carathéodory distance from Theorem 8 together with general arguments from the theory of optimal transport.

**Theorem 9.** *Let $\mu$ and $\nu$ be two Borel probability measures on $G$. Assume that $\mu$ is absolutely continuous with respect to $\lambda$ and that*

$$\int_G d_g(0, x)^2 \, d\mu(x) + \int_G d_g(0, y)^2 \, d\nu(y) < +\infty.$$

*Then there exists a unique optimal transport plan from $\mu$ to $\nu$, and this plan is induced by a map $\psi$. If $\operatorname{spt}\nu$ is compact, there exists a c-concave and locally Lipschitz map $\varphi$ such that $\psi(x) = x.\exp(X + Z)$ where $X$ and $Z$ are given by (29) for $\mu$-a.e. $x \in G$ such that $\psi(x) \neq x$. Conversely, if $\psi$ is representable in such a way for some c-concave map $\varphi$ which satisfies that*

$$\nabla_{\mathrm{cc}}\,\varphi(x) \text{ exists and } \varphi(x) = \min_{y \in G} c(x, y) - \varphi^c(y) \text{ for } \mu\text{-a.e. } x \in G$$

*and*

$$\int_G (d_g(0, x)^2 + d_g(0, \psi(x))^2) \, d\mu(x) < +\infty$$

*then $\psi$ is the optimal transport map between $\mu$ and $\nu = \psi_{\#}\mu$.*

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