Transient random walks on 2d-oriented lattices

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Abstract:

We study the asymptotic behavior of the simple random walk on oriented versions of $\mathbb{Z}^2$. The considered lattices are not directed on the vertical axis but unidirectional on the horizontal one, with random orientations whose distributions are generated by a dynamical system. We find a sufficient condition on the smoothness of the generation for the transience of the simple random walk on a.e. such oriented lattices, and as an illustration we provide a wide class of examples of inhomogeneous or correlated distributions of the orientations. For ergodic dynamical systems, we also prove a strong law of large numbers and, in the particular case of i.i.d. orientations, we solve an open problem and prove a functional limit theorem in the space $\mathcal{D}([0,\infty[,\mathbb{R}^2)$ of càdlàg functions, with an unconventional normalization.

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1 Introduction

The use of random walks as a tool in mathematical physics is now well established and they have been for example widely used in classical statistical mechanics to study critical phenomena (see [4]). It has been recently observed that analogous methods in quantum statistical mechanics require the study of random walks on oriented lattices, due to the intrinsic non commutative character of the (quantum) world (see e.g. [3, 10]). Although random walks in random and non-random environments have been intensively studied for many years, only a few results on random walks on oriented lattices are known. The recurrence versus transience properties of simple random walks on oriented versions of $\mathbb{Z}^2$ are studied in [2] when the horizontal lines are unidirectional towards a random or deterministic direction. An interesting behavior of this model is that, depending on the orientation, the walk could be either recurrent or transient. In the deterministic "alternate" case, for which the orientations of horizontal lines are alternated, i.e. oriented rightwards at one level and leftwards at the following level, the recurrence of the simple random walk is proved, whereas the transience naturally arises when the orientations are all identical in infinite regions. More surprisingly, it is also proved that the recurrent character of the simple random walk on $\mathbb{Z}^2$ is lost when the orientations are i.i.d. with zero mean.

In this paper, we study more general models and focus on spatially inhomogeneous or dependent distributions of the orientations. To do so, we introduce lattices for which the distribution of the horizontal orientation is generated by a dynamical system. We prove that the transience of the simple random walk still holds under some smoothness conditions on this generation and detail examples and counterexamples for various standard dynamical systems. For ergodic dynamical systems, we also prove a strong law of large numbers and, in the case of i.i.d. orientations, a functional limit theorem with an unconventional normalization due to the random character of the environment of the walk, solving an open question of [2].

Our paper is organized as follows: the description of our model and the results are stated in Section 2, Section 3 is devoted to the proofs of the transience property for dynamical orientations and of the limit theorems, while illustrative examples of such dynamical orientations are given in Section 4.

2 Model and results

2.1 Dynamically oriented lattices

Let $S = (E, \mathcal{A}, \mu, T)$ be a dynamical system where $(E, \mathcal{A}, \mu)$ is a probability space and T is an invertible transformation of E preserving the measure μ. This dynamical system is used to introduce inhomogeneity or dependencies in the distribution of the random orientations. To do so, we use a function f defined on E with values in $[0, 1]$ and we ask $\int_E f d\mu = 1/2$ to avoid trivialities. By *orientations*, we mean a random field $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}} \in \{-1, +1\}^{\mathbb{Z}}$, or equivalently a family ϵ of $\{-1, +1\}$-valued random variables $\epsilon_y,\ y \in \mathbb{Z}$, and we distinguish two different approaches to introduce its distribution.

2.1.1 Quenched case:

It describes spatially inhomogeneous distributions of the orientation. We define for fixed $x \in E$ the quenched law $\mathbb{P}_T^{(x)}$ to be the product probability measure on $(\{-1, +1\}^{\mathbb{Z}}, \mathcal{F} =$

$\mathcal{P}(\{-1,+1\})^{\otimes\mathbb{Z}})$ with one-dimensional marginals given by:

$$\begin{aligned}\mathbb{P}_T^{(x)}(\epsilon_y=+1) &= f(T^y x)\\ &= 1-\mathbb{P}_T^{(x)}(\epsilon_y=-1).\end{aligned}$$

To simplify, we have used the same notation for the quenched law and its marginals, which should be written $\mathbb{P}_{T,y}^{(x)}$ with $\mathbb{P}_T^{(x)}=\otimes_y\mathbb{P}_{T,y}^{(x)}$. This quenched case is thus an extension of the i.i.d. case of [2], with independent but not necessarily identically distributed random variables: when $f\equiv\frac{1}{2}$, these orientations are the Rademacher random variables of [2] but if the function f is not constant, the sequence of random variables $(\epsilon_y)_{y\in\mathbb{Z}}$ is not stationary under $\mathbb{P}_T^{(x)}$. Let us remark that these random variables can be viewed as the increments of a dynamic random walk (see [5, 6] for further details).

2.1.2 Annealed case:

We consider now μ-averages over all $x\in E$ and the distribution of ϵ to be the probability law $\mathbb{P}_\mu$ on $(\{-1,+1\}^{\mathbb{Z}},\mathcal{P}(\{-1,+1\})^{\otimes\mathbb{Z}})$ defined for all $A\in\mathcal{F}$ by

$$\mathbb{P}_\mu[\epsilon\in A]=\int_E\mathbb{P}_T^{(x)}[\epsilon\in A]d\mu(x).$$

The one dimensional marginals are thus given for all $y\in\mathbb{Z}$ by

$$\mathbb{P}_\mu[\epsilon_y=+1]=\int_E f(T^yx)d\mu(x)=\int_E f(x)d\mu(x)=\frac{1}{2}$$

and the hypothesis $\int_E f d\mu=\frac{1}{2}$ has been taken to get $\mathbb{E}_\mu[\epsilon_y]=0$. The T-invariance of μ implies the translation-invariance of $\mathbb{P}_\mu$ but this latter is not a product measure in general. For example, one compute easily, for $y\neq y'\in\mathbb{Z}$,

$$\mathrm{Cov}_\mu[\epsilon_y\epsilon_{y'}]=4\int_E f(x)f(T^{y'-y}x)d\mu(x)-1 \tag{2.1}$$

providing thus an explicit relation between the correlations of the orientation and those of μ, defined when it exists for the function f and for all $y\in\mathbb{Z}$ by

$$\begin{aligned}C_\mu^f(y) &:= \int_E f(x)\cdot f\circ T^y(x)d\mu(x)-\int_E f(x)d\mu(x)\cdot\int_E f\circ T^y(x)d\mu(x) \qquad (2.2)\\ &= \int_E f(x)f(T^y(x))d\mu(x)-\frac{1}{4}.\end{aligned}$$

Thus we get

$$\mathrm{Cov}_\mu(\epsilon_0\epsilon_y)=4\,C_\mu^f(y). \tag{2.3}$$

This annealed case leads in Section 4 to another extension of the transience property of the i.i.d. case where this time, on the contrary to the quenched case, independence is dropped but translation-invariance is kept.

2.1.3 Lattices

We use these dynamic random variables to build our *dynamically oriented* lattices. These lattices are oriented versions of $\mathbb{Z}^2$: the vertical lines are not oriented and the horizontal ones are unidirectional, the orientation at a level $y \in \mathbb{Z}$ being given by the random variable ϵ_y (say right if the value is $+1$ and left if it is -1). More formally we give the

Definition 2.4 (Dynamically oriented lattices) *Let $\epsilon = (\epsilon_y)_{y\in\mathbb{Z}}$ be a sequence of random variables defined as previously. The* dynamically oriented lattice $\mathbb{L}^\epsilon = (\mathbb{V}, \mathbb{A}^\epsilon)$ *is the (random) directed graph with (deterministic) vertex set $\mathbb{V} = \mathbb{Z}^2$ and (random) edge set $\mathbb{A}^\epsilon$ defined by the condition that for $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{Z}^2$, $(u, v) \in \mathbb{A}^\epsilon$ if and only if*

1. *either $v_1 = u_1$ and $v_2 = u_2 \pm 1$*
2. *or $v_2 = u_2$ and $v_1 = u_1 + \epsilon_{u_2}$.*

2.2 Simple random walk on $\mathbb{L}^\epsilon$

We consider the usual simple random walk $M = (M_n)_{n\in\mathbb{N}}$ on $\mathbb{L}^\epsilon$. For every realization ϵ, it is a $\mathbb{Z}^2$-valued Markov chain $\mathbb{M}$ defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, whose ($\epsilon$-dependent) transition probabilities are defined for all $(u, v) \in \mathbb{V} \times \mathbb{V}$ by

$$\mathbb{P}[M_{n+1} = v | M_n = u] = \begin{cases} \frac{1}{3} & \text{if (u, v)} \in \mathbb{A}^\epsilon \\ \\ 0 & \text{otherwise.} \end{cases}$$

Its transience is proved in [2] for almost every orientation ϵ when they are given by a sequence of independent Rademacher random variables $(\epsilon_y)_{y\in\mathbb{Z}}$. We generalize this result in this dynamical context when the orientations are either *annealed* or *quenched*.

Theorem 2.5 *Assume that*

$$\int_E \frac{1}{\sqrt{f(1-f)}}\, d\mu < \infty \tag{2.6}$$

then

1. *in the annealed case, for $\mathbb{P}_\mu$-a.e. orientation ϵ, the simple random walk on dynamically oriented lattice $\mathbb{L}^\epsilon$ is transient.*
2. *in the quenched case, for μ-a.e. $x \in E$, for $\mathbb{P}_T^{(x)}$-a.e. realization of the orientation ϵ, the simple random walk on the dynamically oriented lattice $\mathbb{L}^\epsilon$ is transient.*

Let us mention that non-invertible transformations T of the space E can also be considered in the following

Theorem 2.7 *Assume that T is not invertible but that the distribution of the orientations $(\epsilon_y)_{y\in\mathbb{Z}}$ have one-dimensional marginals defined by*

$$\mathbb{P}_T^{(x)}(\epsilon_y = +1) = f(T^{|y|}x) \tag{2.8}$$

$$= 1 - \mathbb{P}_T^{(x)}(\epsilon_y = -1). \tag{2.9}$$

Then conclusions of Theorem 2.5 still hold.

It is worth noting that in the annealed case the measure $\mathbb{P}_\mu$ is not stationary anymore, and we illustrate this result by the example of Manneville-Pomeau maps of the interval in Section 4.

2.3 Limit theorems in the ergodic case

Let us assume that the dynamical system $S = (E, \mathcal{A}, \mu, T)$ defined in Section 2.1 is ergodic. Then, it is not difficult to prove that in this particular setting, the annealed measure $\mathbb{P}_\mu$ is also ergodic and the following theorem holds

Theorem 2.10 (Strong law of large numbers) *The random walk on the lattice $\mathbb{L}^\epsilon$ has $\tilde{\mathbb{P}}_\mu$-almost surely zero speed, i.e.*

$$\lim_{n\to+\infty} \frac{M_n}{n} = (0,0) \quad \tilde{\mathbb{P}}_\mu\text{-almost surely.} \tag{2.11}$$

2.3.1 Functional limit theorem for i.i.d. orientations

We also answer in this paper to an open question of [2] and obtain a functional limit theorem with a suitable and unconventional normalization. We establish that the study of the simple random walk on dynamically oriented graph $\mathbb{L}^\epsilon$ is closely related to another model called *simple random walks in random sceneries* defined for every $n \geq 1$ by

$$Z_n = \sum_{k=0}^{n} \epsilon_{Y_k}$$

where $(Y_k)_{k\geq 0}$ is the simple random walk on $\mathbb{Z}$ starting from 0.
Let us consider a standard Brownian motion $(B_t)_{t\geq 0}$ and denote by $(L_t(x))_{t\geq 0}$ its corresponding local time at $x \in \mathbb{R}$. Moreover, we introduce a pair of independent Brownian motions $Z_+(x), Z_-(x), x \geq 0$. We assume these processes to be defined on one probability space and to be independent of each other so that the following process is well-defined for all $t \geq 0$:

$$\Delta_t = \int_0^\infty L_t(x) dZ_+(x) + \int_0^\infty L_t(-x) dZ_-(x). \tag{2.12}$$

It has been proved by Kesten *et al* ([9]) that this process has a continuous version which is self-similar with index $\frac{3}{4}$ and has stationary increments.

Theorem 2.13 [Kesten and Spitzer (1979)]

$$\Big(\frac{1}{n^{3/4}} Z_{[nt]}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} (\Delta_t)_{t\geq 0} \tag{2.14}$$

where $\overset{\mathcal{D}}{\Longrightarrow}$ stands for convergence in the space of càdlàg functions $\mathcal{D}([0,\infty),\mathbb{R})$ endowed with the Skorohod topology.

We introduce a real constant $m = \frac{1}{2}$, defined later as the mean of some geometric random variables related to the behavior of the walk in the horizontal direction[1]. Using Theorem 2.13, we shall prove

[1]Our results are in fact valid for similar model for which $m \neq \frac{1}{2}$ corresponding to non symmetric nearest neighbors random walks. Of course the transience is not at all suprising in this case, but getting the limit theorems can be of interest.

Theorem 2.15 (Functional limit theorem)

$$\Big(\frac{1}{n^{3/4}}M_{[nt]}\Big)_{t\geq 0} \overset{\mathcal{D}}{\Longrightarrow} \frac{m}{(1+m)^{3/4}}(\Delta_t, 0)_{t\geq 0} \tag{2.16}$$

where $\overset{\mathcal{D}}{\Longrightarrow}$ *stands for convergence in the space of càdlàg functions* $\mathcal{D}([0,\infty),\mathbb{R}^2)$ *endowed with the Skorohod topology.*

We conjecture the following local limit theorem for the random walk M_n.

Conjecture 2.17 (Local limit theorem) *There exists a constant* $C > 0$ *such that as* $n \to +\infty$,

$$\mathbb{P}[M_n = (0,0)] \sim Cn^{-5/4}. \tag{2.18}$$

3 Proofs

3.1 Vertical and horizontal embeddings of the simple random walk

The simple random walk M defined on $(\Omega, \mathcal{B}, \mathbb{P})$ can be decomposed into vertical and horizontal embeddings by projection to the corresponding axis. These embeddings will carry the interesting asymptotic properties of the walk. The vertical one is a simple random walk $Y = (Y_n)_{n\in\mathbb{N}}$ on the line and we define for all $n \in \mathbb{N}$ and $y \in \mathbb{Z}$ its *local time* at level y to be

$$\eta_n(y) = \sum_{k=0}^{n} \mathbf{1}_{Y_k = y}.$$

The horizontal embedding is a random walk with $\mathbb{N}$-valued geometric jumps. More formally, a doubly infinite family $(\xi_i^{(y)})_{i\in\mathbb{N}^*, y\in\mathbb{Z}}$ of independent geometric random variables of parameter $p = \frac{1}{3}$ (and mean $m = \frac{1}{2}$) is given and one defines the embedded horizontal random walk $X = (X_n)_{n\in\mathbb{N}}$ by $X_0 = 0$ and for $n \geq 1$,

$$X_n = \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

with the convention that the last sum is zero when $\eta_{n-1}(y) = 0$. Of course, the walk M_n does not coincide with (X_n, Y_n) but these objects are closely related: define for all $n \in \mathbb{N}$

$$T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

to be the instant just after the random walk M has performed its n^{th} vertical move. A direct and useful consequence of this decomposition is the following result (see [2])

Lemma 3.19 *1.* $M_{T_n} = (X_n, Y_n),\ \forall n \in \mathbb{N}$.

2. For a given orientation ϵ*, the transience of* $(M_{T_n})_{n\in\mathbb{N}}$ *implies the transience of* $(M_n)_{n\in\mathbb{N}}$.

3.2 Proof of the transience of the simple random walk

Independent of the orientation ϵ, the vertical walk Y is known to be recurrent and its asymptotic behavior is rather well controlled. The transience is due to the behavior of the embedded horizontal random walk X and to exploit it and prove Theorem 2.5, we introduce a partition of Ω between typical or untypical paths of Y.

In all this proof, for any $i \in \mathbb{N}$, δ_i is a strictly positive real number and we write $d_{n,i} = n^{\frac{1}{2}+\delta_i}$. Define the sets

$$A_n = \left\{\omega \in \Omega; \max_{0\le k\le 2n} |Y_k| < n^{\frac{1}{2}+\delta_1}\right\} \cap \left\{\omega \in \Omega; \max_{y\in\mathbb{Z}} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\right\}$$

and

$$B_n = \left\{\omega \in A_n; \left|\sum_{y\in\mathbb{Z}} \epsilon_y \eta_{2n-1}(y)\right| > n^{\frac{1}{2}+\delta_3}\right\}.$$

We prove the transience in the annealed case, but the proof in fact contains this of the quenched case. We first consider the joint measures $\tilde{\mathbb{P}}_\mu = \mathbb{P} \otimes \mathbb{P}_\mu$ (annealed case) or $\tilde{\mathbb{P}}_T^{(x)} = \mathbb{P} \otimes \mathbb{P}_T^{(x)}$ (quenched case) and prove that

$$\tilde{\mathbb{P}}_\mu[M_{T_n} = (0,0) \text{ i.o.}] = 0 \tag{3.20}$$

or, using Lemma 3.19,

$$\sum_{n\in\mathbb{N}} \tilde{\mathbb{P}}_\mu[X_{2n} = 0; Y_{2n} = 0] \ < \ \infty. \tag{3.21}$$

By definition

$$\sum_{n\in\mathbb{N}} \tilde{\mathbb{P}}_\mu[X_{2n} = 0; Y_{2n} = 0] = \int_E \sum_n \mathbb{P}[\mathbb{P}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0]] d\mu(x)$$

and we first decompose $\tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0]$ into

$$\tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; A_n^c] + \tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; B_n] + \tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n]. \tag{3.22}$$

Some results of the i.i.d. case of [2] still hold uniformly in x and in particular we can prove using standard techniques the following

Lemma 3.23 *For every $x \in E$,*

$$\sum_{n\in\mathbb{N}} \tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; A_n^c] \ < \ \infty.$$

The second term of (3.22) is also a generic term of convergent series due to the untypical character of the paths in B_n. Again from [2] with standard techniques, we have the

Lemma 3.24 *For every $x \in E$,*

$$\sum_{n\in\mathbb{N}} \tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; B_n] \ < \ \infty.$$

Now, we denote (independent) σ-algebras generated by this vertical walk Y and the orientation ϵ by

$$\mathcal{F} = \sigma(Y_n,\ n \in \mathbb{N}) \quad \text{and} \quad \mathcal{G} = \sigma(\epsilon_y,\ y \in \mathbb{Z}).$$

Then

$$\begin{aligned} p_n^{(x)} &= \tilde{\mathbb{P}}_T^{(x)}[X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n] \\ &= \mathbb{E}\Big[\mathbf{1}_{Y_{2n}=0}\mathbb{E}\big[\mathbf{1}_{A_n\setminus B_n}\mathbb{E}\big[\mathbf{1}_{X_{2n}=0}\big|\mathcal{F}\vee\mathcal{G}\big]\big|\mathcal{F}\big]\Big] \end{aligned}$$

where $\mathbb{E}$ stands for the expectation under the measure $\tilde{\mathbb{P}}_T^{(x)}$. To prove the theorem, it remains to show that

$$\int_E \left(\sum_{n\in\mathbb{N}} p_n^{(x)}\right)\ d\mu(x) < \infty. \tag{3.25}$$

It is well known that for the simple random walk Y, there exists $C > 0$ s.t.

$$\mathbb{P}[Y_{2n} = 0] \sim C \cdot n^{-\frac{1}{2}},\ n \to +\infty \tag{3.26}$$

and we can prove as in [2] the

Lemma 3.27 *On the set $A_n \setminus B_n$, we have,*

$$\tilde{\mathbb{P}}_T^{(x)}\big[X_{2n} = 0\big|\mathcal{F}\vee\mathcal{G}\big] = \mathcal{O}\Big(\sqrt{\frac{\ln n}{n}}\Big) \tag{3.28}$$

uniformly in $x \in E$.

Hence, the transience of the simple random walk is a direct consequence of the following

Proposition 3.29 *It is possible to choose $\delta_1, \delta_2, \delta_3 > 0$ such that there exists $\delta > 0$ and*

$$\int_E \tilde{\mathbb{P}}_T^{(x)}\big[A_n \setminus B_n\big|\mathcal{F}\big]\ d\mu(x) = \mathcal{O}\big(n^{-\delta}\big). \tag{3.30}$$

Proof. We have to estimate, on the event A_n, the conditional probability

$$\tilde{\mathbb{P}}_T^{(x)}[|\sum_{y\in\mathbb{Z}} \zeta_y| \le d_{n,3}\big|\mathcal{F}]$$

where $\zeta_y = \epsilon_y\eta_{2n-1}(y), y \in \mathbb{Z}$. Let G be a centered Gaussian random variable with variance $d_{n,3}^2$, (conditionally on $\mathcal{F}$) independent of the random variables ζ_y's. Clearly,

$$\tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [0, d_{n,3}]\big|\mathcal{F}\big] = \frac{\tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [0, d_{n,3}]; 0 \le G \le d_{n,3}\big|\mathcal{F}\big]}{\tilde{\mathbb{P}}_T^{(x)}[0 \le G \le d_{n,3}\big|\mathcal{F}]}$$

where $\tilde{\mathbb{P}}_T^{(x)}[0 \le G \le d_{n,3}\big|\mathcal{F}] = c$ is a strictly positive real number independent of n. Since G is independent of the random variables ζ_y's and using the symmetry of the Gaussian distribution, we have

$$\tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [0, d_{n,3}]; 0 \le G \le d_{n,3}\big|\mathcal{F}\big] = \tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [0, d_{n,3}]; -d_{n,3} \le G \le 0\big|\mathcal{F}\big].$$

Consequently, we obtain

$$\tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [0, d_{n,3}]\big|\mathcal{F}\big] \leq \frac{1}{c}\tilde{\mathbb{P}}_T^{(x)}[|\sum_y \zeta_y + G| \leq d_{n,3}\big|\mathcal{F}\big].$$

In the same way, we get

$$\tilde{\mathbb{P}}_T^{(x)}\big[\sum_y \zeta_y \in [-d_{n,3}, 0]\big|\mathcal{F}\big] \leq \frac{1}{c}\tilde{\mathbb{P}}_T^{(x)}\big[|\sum_y \zeta_y + G| \leq d_{n,3}\big|\mathcal{F}\big]$$

and then, we have the following inequality

$$\tilde{\mathbb{P}}_T^{(x)}\big[|\sum_y \zeta_y| \leq d_{n,3}\big|\mathcal{F}\big] \leq \frac{2}{c}\ \tilde{\mathbb{P}}_T^{(x)}\big[|\sum_y \zeta_y + G| \leq d_{n,3}\big|\mathcal{F}\big].$$

From Plancherel's formula, we deduce that there exists a constant $C > 0$ such that

$$\tilde{\mathbb{P}}_T^{(x)}\big[|\sum_y \zeta_y + G| \leq d_{n,3}\big|\mathcal{F}\big] \leq C \cdot d_{n,3} \cdot I_n(x) \tag{3.31}$$

where

$$I_n(x) = \int_{-\pi}^{\pi} \mathbb{E}\big[\big[e^{it\sum_{y\in\mathbb{Z}} \epsilon_y \eta_{2n-1}(y)}\big|\mathcal{F}\big]\big] e^{-t^2 d_{n,3}^2/2} dt.$$

To use that for $td_{n,3}$ small enough, $e^{-t^2 d_{n,3}^2/2}$ dominates the term under the expectation, we split the integral in two parts. For $b_n = \frac{n^{\delta_2}}{d_{n,3}}$, we write

$$I_n(x) = I_n^1(x) + I_n^2(x)$$

with

$$I_n^1(x) = \int_{|t|\leq b_n} \mathbb{E}\big[e^{it\sum_{y\in\mathbb{Z}} \epsilon_y \eta_{2n-1}(y)}\big|\mathcal{F}\big] e^{-t^2 d_{n,3}^2/2} dt$$
$$I_n^2(x) = \int_{|t|> b_n} \mathbb{E}\big[e^{it\sum_{y\in\mathbb{Z}} \epsilon_y \eta_{2n-1}(y)}\big|\mathcal{F}\big] e^{-t^2 d_{n,3}^2/2} dt.$$

To control the integral $I_n^2(x)$, we write

$$\begin{aligned} |I_n^2(x)| &\leq C\int_{|t|>b_n} e^{-t^2 d_{n,3}^2/2} dt = \frac{C}{d_{n,3}} \int_{|s|>n^{\delta_2}} e^{-s^2/2} ds \\ &\leq 2\frac{C}{d_{n,3}} n^{-\delta_2} e^{-n^{2\delta_2}/2} \end{aligned}$$

to get

$$|I_n^2(x)| = \mathcal{O}\big(e^{-n^{2\delta_2}/2}\big).$$

uniformly in $x \in E$.

Lemma 3.32 *For $\delta_3 > 2\delta_2$,*

$$\int_E |I_n^1(x)|\ d\mu(x) = \mathcal{O}\big(n^{-\frac{3}{4}+\frac{\delta_1}{2}}\big).$$

Proof. From the definition of the orientations $(\epsilon_y)_y$, an explicit formula for the characteristic function $\phi_{\epsilon_y}^{(x)}$ of the random variable ϵ_y can be given and we deduce that

$$\begin{aligned} |\phi_{\epsilon_y}^{(x)}(u)|^2 &= \cos^2(u) + (2f(T^y x) - 1)^2 \sin^2(u) \\ &= 1 - 4f(T^y x)(1 - f(T^y x)) \sin^2(u) \end{aligned}$$

and by independence of the ϵ's we get

$$|I_n^1(x)| \leq \int_{|t| \leq b_n} |\prod_y \phi_{\epsilon_y}^{(x)}(\eta_{2n-1}(y)t)| \, dt.$$

Denote for all $y \in \mathbb{Z}, n \in \mathbb{N}$, $p_{n,y} = \frac{\eta_{2n-1}(y)}{2n}$, $C_n = \{y : \eta_{2n-1}(y) \neq 0\}$. We use Hölder's inequality to get

$$|I_n^1(x)| \leq \prod_y \Big[\Big(\int_{|t| \leq b_n} |\phi_{\epsilon_y}^{(x)}(\eta_{2n-1}(y)t)|^{1/p_{n,y}} dt \Big)^{p_{n,y}} \Big]. \tag{3.33}$$

Now, using the fact that we work on A_n, we choose $\delta_3 > 2\delta_2$ in order to have $b_n \eta_{2n-1}(y) \to 0$ uniformly in y when n goes to infinity. Using that $\sin(x) \geq \frac{2}{\pi} x$ for $x \in [0, \frac{\pi}{2}]$ and $\exp(-x) \geq 1 - x$, one has

$$\begin{aligned} |I_n^1(x)| &\leq \prod_{y \in C_n} \left(\frac{1}{\eta_{2n-1}(y)} \int_{|v| \leq b_n \eta_{2n-1}(y)} \exp\left(-\frac{16}{p_{n,y} \pi^2} f(T^y x)(1 - f(T^y x)) v^2 \right) \, dv \right)^{p_{n,y}} \\ &\leq \prod_{y \in C_n} \left(\frac{c \mathbf{1}_{f(T^y x)(1-f(T^y x))>0}}{\sqrt{2n \eta_{2n-1}(y) f(T^y x)(1 - f(T^y x))}} \right)^{p_{n,y}} \quad (\text{ with } \quad c = \pi^{3/2}/4) \\ &= c \exp\big[-\frac{1}{2} \sum_{y \in C_n} p_{n,y} \log(2n \eta_{2n-1}(y)) \big] \cdot \prod_{y \in C_n} \left(\frac{\mathbf{1}_{f(T^y x)(1-f(T^y x))>0}}{\sqrt{f(T^y x)(1 - f(T^y x))}} \right)^{p_{n,y}}. \end{aligned}$$

The vector $\mathbf{p} = (p_{n,y})_{y \in C_n}$ defines a probability measure on C_n and we have

$$\begin{aligned} -\frac{1}{2} \sum_{y \in C_n} p_{n,y} \log(2n \eta_{2n-1}(y)) &= -\log 2n - \frac{1}{2} \sum_{y \in C_n} p_{n,y} \log p_{n,y} \\ &= -\log 2n + \frac{1}{2} H(\mathbf{p}) \end{aligned}$$

where $H(\cdot)$ is the entropy of the probability vector $\mathbf{p}$, always bounded by $\log(\mathrm{card}(C_n))$. We thus have on the set A_n,

$$|I_n^1(x)| \leq c \exp\left[-\log 2n + \frac{1}{2} \log(2d_{n,1}) \right] \prod_{y \in C_n} \left(\frac{\mathbf{1}_{\{f(T^y x)(1-f(T^y x))>0\}}}{\sqrt{f(T^y x)(1 - f(T^y x))}} \right)^{p_{n,y}}.$$

By applying Hölder's inequality and the fact that T preserves the measure μ, we get

$$\begin{aligned} \int_E |I_n^1(x)| \, d\mu(x) &\leq C \cdot n^{-\frac{3}{4} + \frac{\delta_1}{2}} \int_E \prod_{y \in C_n} \left(\frac{\mathbf{1}_{\{f(T^y x)(1-f(T^y x))>0\}}}{\sqrt{f(T^y x)(1 - f(T^y x))}} \right)^{p_{n,y}} d\mu(x) \\ &\leq C \cdot n^{-\frac{3}{4} + \frac{\delta_1}{2}} \prod_{y \in C_n} \left[\int_E \left(\frac{\mathbf{1}_{\{f(T^y x)(1-f(T^y x))>0\}}}{\sqrt{f(T^y x)(1 - f(T^y x))}} \right) d\mu(x) \right]^{p_{n,y}} \\ &= C \cdot n^{-\frac{3}{4} + \frac{\delta_1}{2}} \int_E \frac{1}{\sqrt{f(x)(1 - f(x))}} \, d\mu(x). \end{aligned}$$

Now, using (3.31), write with the usual notation $d_{n,3} = n^{\frac{1}{2}+\delta_3}$:

$$\int_E \tilde{\mathbb{P}}_T^{(x)}[A_n \setminus B_n|\mathcal{F}]\ d\mu(x) \leq C \cdot d_{n,3} \int_E \left(|I_n^1(x)| + |I_n^2(x)|\right)\ d\mu(x)$$

and consider $\delta_3 > 2\delta_2$. By the previous lemmata, we have

$$d_{n,3} \cdot \int_E |I_n^1(x)|\ d\mu(x) = \mathcal{O}\big(n^{-\frac{1}{4}+\delta_3+\frac{\delta_1}{2}}\big),\ d_{n,3} \cdot \int_E |I_n^2(x)|\ d\mu(x) = \mathcal{O}\big(e^{-n^{2\delta_2}/2}\big)$$

and the proposition follows by choosing $\delta_i, i = 1, 2, 3$ small enough. ∎

Combining Equations (3.25), (3.26), (3.28) and (3.30), we obtain (3.25) and then (3.21). By Borel-Cantelli's Lemma, we get (3.20):

$$\tilde{\mathbb{P}}_\mu\big[M_{T_n} = (0,0) \text{ i.o.}\big] = \mathbb{P}_\mu\big[\mathbb{P}\big[\mathrm{M_{T_n}} = (0,0) \text{ i.o.}\big]\big] = 0$$

and thus for $\mathbb{P}_\mu$-almost every orientation ϵ,

$$\mathbb{P}\big[M_{T_n} = (0,0) \text{ i.o.}\big] = 0.$$

This proves that $(M_{T_n})_{n\in\mathbb{N}}$ is transient for $\mathbb{P}_\mu$-almost every orientation ϵ, and by Lemma 3.19, the $\mathbb{P}_\mu$-almost sure transience of the simple random walk on the annealed oriented lattice. The proof of point 2. of the theorem (transience in the quenched case) is contained in the proof of point 1. but can also be recovered from it as follows. By Fubini-Tonnelli's theorem,

$$\sum_n \tilde{\mathbb{P}}_\mu\big[M_{T_n} = (0,0)\big] < \infty \Longrightarrow \sum_n \int_E \tilde{\mathbb{P}}_T^{(x)}\big[M_{T_n} = (0,0)\big] d\mu(x) < \infty$$

$$\Longrightarrow \int_E \sum_n \tilde{\mathbb{P}}_T^{(x)}\big[M_{T_n} = (0,0)\big] d\mu(x) < \infty \Longrightarrow \text{For } \mu\text{−a.e. x}, \sum_{\mathrm{n}} \tilde{\mathbb{P}}_{\mathrm{T}}^{(\mathrm{x})}\big[\mathrm{M_{T_n}} = (0,0)\big] < \infty$$

$$\Longrightarrow \text{For } \mu - \text{a.e. x},\ \tilde{\mathbb{P}}_{\mathrm{T}}^{(\mathrm{x})}\big[\mathrm{M_{T_n}} = (0,0) \text{ i.o.}\big] = 0$$

$$\Longrightarrow \text{For } \mu\text{−a.e. x},\ \mathbb{P}_{\mathrm{T}}^{(\mathrm{x})}\big[\mathbb{P}\big[\mathrm{M_{T_n}} = (0,0) \text{ i.o.}\big]\big] = 0$$

and thus the $\mathbb{P}_T^{(x)}$-almost sure transience on the quenched dynamically oriented lattice holds:

$$\text{For } \mu\text{−a.e. x}, \text{For } \mathbb{P}_{\mathrm{T}}^{(\mathrm{x})}\text{−a.e. orientation } \epsilon,\ \mathbb{P}\big[\mathrm{M_n} = (0,0) \text{ i.o.}\big] = 0.$$

The proof of Theorem 2.7 follows the same lines.

3.3 Proofs of the limit theorems

3.3.1 Strong law of large numbers

We still use the decomposition of the simple random walk along the vertical and horizontal axis introduced in Section 3.1, and first prove the following

Lemma 3.34 (SLLN for the embedded random walk)

$$\lim_{n\to+\infty} \frac{M_{T_n}}{n} = (0,0) \quad \tilde{\mathbb{P}}_\mu\text{-almost surely.} \tag{3.35}$$

Proof.

We have $M_{T_n} = (X_n, Y_n)$ and since[2] $(Y_n)_{n\geq 0}$ is a simple random walk,

$$\lim_{n\to+\infty} \frac{Y_n}{n} = 0 \quad \tilde{\mathbb{P}}_\mu\text{-almost surely.} \tag{3.36}$$

So it is enough to prove that $(\frac{X_n}{n})$ converges almost surely to 0. Introduce the random variable

$$Z_n = \sum_{k=0}^{n-1} \epsilon_{Y_k} = \sum_{y\in\mathbb{Z}} \epsilon_y \eta_{n-1}(y).$$

Under the probability measure $\tilde{\mathbb{P}}_\mu$, the stationary sequence $(\epsilon_{Y_k})_{k\geq 0}$ is ergodic ([7]), so from Birkhoff's theorem, as n tends to infinity,

$$\frac{Z_n}{n} \to \mathbb{E}[\epsilon_0] = 0 \quad \text{almost surely.} \tag{3.37}$$

Clearly,

$$X_n - mZ_n = \sum_{y\in\mathbb{Z}} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m).$$

Let r be an even integer.

$$\mathbb{E}[(X_n - mZ_n)^r]$$

$$= \sum_{y_1\in\mathbb{Z},\dots y_r\in\mathbb{Z}} \mathbb{E}\left[\epsilon_{y_1}\dots\epsilon_{y_r} \sum_{i_1=1}^{\eta_{n-1}(y_1)} \cdots \sum_{i_r=1}^{\eta_{n-1}(y_r)} \mathbb{E}[(\xi_{i_1}^{(y_1)} - m)\dots(\xi_{i_r}^{(y_r)} - m)|\mathcal{F}\vee\mathcal{G}]\right].$$

The $\xi_i^{(y)}$'s are independent of the vertical walk and the orientations; moreover, the random variables $\xi_i^{(y)} - m, i \geq 1, y \in \mathbb{Z}$ are i.i.d. and centered, so the summands are non zero if and only if $i_1 = \dots = i_r$ and $y_1 = \dots = y_r$. Then,

$$\mathbb{E}[(X_n - mZ_n)^r] = n\mathbb{E}[(\xi_1^{(0)} - m)^r] := nm_r \qquad \text{(say)}.$$

Let $\delta > 0$. By Tchebychev's inequality,

$$\begin{aligned}\mathbb{P}\Big[\left|\frac{X_n - mZ_n}{n}\right| \geq \epsilon\Big] &\leq \frac{1}{\delta^r n^r}\mathbb{E}[(X_n - mZ_n)^r] \\ &\leq \frac{m_r}{\delta^r n^{r-1}}.\end{aligned}$$

We choose $r = 4$ and thus from Borel-Cantelli Lemma, we deduce that $\frac{X_n - mZ_n}{n}$ converges almost surely to 0 as n goes to infinity. ∎

To get the result for the simple random walk $(M_n)_n$, we use the

[2]For the vertical walk Y, the result is also true $\mathbb{P}$ but we need to consider the law of ϵ because, among other reasons, this is not the case for X for which the SLLN is valid for $\mathbb{P}_\mu$-a.e. ϵ only.

Lemma 3.38 *The sequence of random variables* $(\frac{T_n}{n})_{n\geq 1}$ *converges* $\tilde{\mathbb{P}}_\mu$*-a.s. to* $(1+m)$ *as* $n \to +\infty$.

Proof. Let us remark that

$$T_n = n + \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m) + m \sum_{y\in\mathbb{Z}} \eta_{n-1}(y).$$

Now, reasoning as in the proof of Lemma 3.34,

$$\begin{aligned}
\mathbb{E}\Big[\Big(\sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m)\Big)^3\Big] &= \sum_{y_1\in\mathbb{Z},\dots,y_3\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i_1=1}^{\eta_{n-1}(y_1)} \dots \sum_{i_3=1}^{\eta_{n-1}(y_3)} (\xi_{i_1}^{(y_1)} - m)\dots(\xi_{i_3}^{(y_3)} - m)\Big] \\
&= \sum_{y_1\in\mathbb{Z},\dots,y_3\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i_1=1}^{\eta_{n-1}(y_1)} \dots \sum_{i_3=1}^{\eta_{n-1}(y_3)} \mathbb{E}[(\xi_{i_1}^{(y_1)} - m)\dots(\xi_{i_3}^{(y_3)} - m)|\mathcal{F}\vee\mathcal{G}]\Big] \\
&= m_3 \sum_{y_1\in\mathbb{Z},\dots,y_3\in\mathbb{Z}} \mathbb{E}\Big[\sum_{i_1=1}^{\eta_{n-1}(y_1)} \dots \sum_{i_3=1}^{\eta_{n-1}(y_3)} \mathbf{1}_{i_1=\dots=i_3}\mathbf{1}_{y_1=\dots=y_3}\Big] \\
&= m_3 \sum_{y\in\mathbb{Z}} \mathbb{E}[\eta_{n-1}(y)] = m_3 n \quad (\text{where } \; m_3 = \mathbb{E}[(\xi_1^{(0)} - m)^3]).
\end{aligned}$$

From Tchebychev's inequality and Borel-Cantelli Lemma again, we deduce that, as $n \to +\infty$,

$$\frac{1}{n} \sum_{y\in\mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)} - m) \to 0 \text{ a.s..}$$

Using the fact that $\sum_{y\in\mathbb{Z}} \eta_{n-1}(y) = n$, we deduce the lemma. ∎

Let us prove now the almost sure convergence of the sequence $(\frac{M_n}{n})_{n\geq 1}$ to $(0,0)$. Since the sequence $(T_n)_{n\geq 1}$ is strictly increasing, for every $n \geq 1$, there exists a non-decreasing sequence of integers sequence $(U_n)_n$ such that

$$T_{U_n} \leq n < T_{U_n+1}.$$

From the definition of the embedding, and if we denote $M_n = (M_n^{(1)}, M_n^2)$,

$$M_n^{(1)} \in [\min(M_{T_{U_n}}^{(1)}, M_{T_{U_n+1}}^{(1)}), \max(M_{T_{U_n}}^{(1)}, M_{T_{U_n+1}}^{(1)})]$$

and

$$M_n^{(2)} = M_{T_{U_n}}^{(2)}.$$

The (sub-)sequence $(U_n)_{n\geq 1}$ is nondecreasing and $\lim_{n\to+\infty} U_n = +\infty$, then by combining Lemmata 3.34 and 3.38, we get that as $n \to +\infty$,

$$\frac{M_{T_{U_n}}}{T_{U_n}} \to (0,0) \; \tilde{\mathbb{P}}_\mu \text{ a.s.} \tag{3.39}$$

Now,

$$\left|\frac{M_n^{(1)}}{n}\right| \leq \max\left(\left|\frac{M_{T_{U_n}}^{(1)}}{n}\right|, \left|\frac{M_{T_{U_n+1}}^{(1)}}{n}\right|\right) \leq \max\left(\left|\frac{M_{T_{U_n}}^{(1)}}{T_{U_n}}\right|, \left|\frac{M_{T_{U_n+1}}^{(1)}}{T_{U_n}}\right|\right) \tag{3.40}$$

and

$$\left|\frac{M_n^{(2)}}{n}\right| = \left|\frac{M_{T_{U_n}}^{(2)}}{T_{U_n}}\right| \cdot \frac{T_{U_n}}{n} \leq \left|\frac{M_{T_{U_n}}^{(2)}}{T_{U_n}}\right|.$$

From (3.39), we deduce the almost sure convergence of the coordinates to 0 and then this of the sequence $(\frac{M_n}{n})_{n\geq 1}$ to (0,0) as $n \to \infty$.

3.3.2 Proof of the functional limit theorem

Proposition 3.41 *The sequence of random processes $n^{-3/4}(X_{[nt]})_{t\geq 0}$ weakly converges in the space $\mathcal{D}([0,\infty[,\mathbb{R})$ to the process $(m\Delta_t)_{t\geq 0}$.*

Proof. Let us first prove that the finite dimensional distributions of $n^{-3/4}(X_{[nt]})_{t\geq 0}$ converge to those of $(m\Delta_t)_{t\geq 0}$ as $n \to \infty$. We can rewrite for every $n \in \mathbb{N}$,

$$X_n = X_n^{(1)} + X_n^{(2)}$$

where

$$X_n^{(1)} = \sum_{y\in\mathbb{Z}} \epsilon_y \Big(\sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)} - m \Big)$$

and

$$X_n^{(2)} = m \sum_{y\in\mathbb{Z}} \epsilon_y \eta_{n-1}(y).$$

Thanks to Theorem 2.13 the finite dimensional distributions of $n^{-3/4}(X_{[nt]}^{(2)})_{t\geq 0}$ converge to those of $(m\Delta_t)_{t\geq 0}$ as $n \to \infty$. To conclude we show that the sequence of random variables $n^{-3/4}(X_n^{(1)})_{n\in\mathbb{N}}$ converges for the L^2-norm to 0 as $n \to +\infty$. We have

$$\mathbb{E}\Big[(X_n^{(1)})^2\Big] = \mathbb{E}\Big[\sum_{x,y\in\mathbb{Z}} \epsilon_x\epsilon_y \sum_{i=1}^{\eta_{n-1}(x)} \sum_{j=1}^{\eta_{n-1}(y)} \mathbb{E}[(\xi_i^{(x)} - m)(\xi_j^{(y)} - m)|\mathcal{F}\vee\mathcal{G}]\Big]$$

From the equality

$$\mathbb{E}[(\xi_i^{(x)} - m)(\xi_j^{(y)} - m)|\mathcal{F}\vee\mathcal{G}] = m^2\delta_{i,j}\delta_{x,y},$$

we obtain

$$n^{-3/2}\mathbb{E}\Big[(X_n^{(1)})^2\Big] = m^2 n^{-3/2} \sum_{x\in\mathbb{Z}} \eta_{n-1}(x) = m^2 n^{-1/2} = o(1).$$

∎

Let us recall that $M_{T_n} = (X_n, Y_n)$ for every $n \geq 1$. The sequence of random processes $n^{-3/4}(Y_{[nt]})_{t\geq 0}$ weakly converges in $\mathcal{D}([0,\infty[,\mathbb{R})$ to 0, thus the sequence of $\mathbb{R}^2-$valued random processes $n^{-3/4}(M_{T_{[nt]}})_{t\geq 0}$ weakly converges in $\mathcal{D}([0,\infty[,\mathbb{R}^2)$ to the process $(m\Delta_t, 0)_{t\geq 0}$. Theorem 2.15 follows from this remark and Lemma 3.38.

4 Examples

The main motivation of this work is the generalization of the transience of the i.i.d. case of [2] to dependent or inhomogeneous orientations. Depending on the original dynamical systems, we obtain various extensions corresponding to well known examples of dynamical systems such that Bernoulli and Markov shifts, Gibbs measures, SRB measures (Manneville-Pomeau maps), rotations on the torus, etc., our framework is very general from this point of view. Nevertheless, to get the transience of the walk, we need to generate the orientations by choosing a suitable function f satisfying (2.6). In some sense, this condition requires the model not to be too close to the deterministic case because to satisfy the condition, f should not be "μ-too often" 0 or 1. We describe now the examples providing extensions of the i.i.d. case to various disordered orientations.

1. **Bernoulli shift**

   The first considered dynamical system S is the *Bernoulli shift* on the product space $E = [0,1]^{\mathbb{Z}}$ endowed with the Borel σ-algebra, the bilatere shift transformation T defined by

   $$\begin{array}{rcl} T : E & \longrightarrow & E \\ x = (x_y)_{y\in\mathbb{Z}} & \longmapsto & (Tx)_y = x_{y+1}, \forall y \in \mathbb{Z}. \end{array}$$

   The product Lebesgue measure $\mu = \lambda^{\otimes\mathbb{Z}}$ of the Lebesgue measure λ on $[0,1]$ is T-invariant and the function f is the projection on the zero coordinate:

   $$\begin{array}{rcl} f : E & \longrightarrow & [0,1] \\ x & \longmapsto & x_0. \end{array}$$

   For all $y \in \mathbb{Z}$, we then have

   $$f \circ T^y(x) = x_y := \xi(y) \in [0,1].$$

   We consider this ξ's as new random variables on E whose independence is inherited from the product structure of μ. The condition

   $$\int_E \frac{d\mu}{\sqrt{f(1-f)}} < \infty$$

   becomes

   $$\int_0^1 \frac{d\lambda(x)}{\sqrt{x(1-x)}} < \infty$$

   and the transience holds in this particular case. In fact, this product form of μ allows in this annealed case another description of the i.i.d. case of [2] and for example one could check with $\xi(y) \equiv \frac{1}{2}$ for all $y \in \mathbb{Z}$ that

   $$\mathrm{Cov}_\mu[\epsilon_0\epsilon_{\mathrm{y}}] = \mathbb{E}_\mu[\epsilon_0\epsilon_{\mathrm{y}}] = 4\mathbb{E}[\xi(0)\xi(\mathrm{y})] - 1 = 0 \tag{4.42}$$

   The result is also valid in the quenched case, for which the distribution of the orientation has an inhomogeneous product form.

2. **Markov shift**

   If one considers a measure μ with correlations, then the same holds for $\mathbb{P}_\mu$. It is the case when one considers a Markovian measure instead of a product one on the space $[0,1]^{\mathbb{Z}}$ with stationary distribution π, whose correlations are given by (2.3).

   The transience of the simple random walk on this particular dynamically oriented lattice holds for $\mathbb{P}_\mu$-a.e. environment as soon as the following condition is satisfied:

$$\int_0^1 \frac{d\pi(x)}{\sqrt{x(1-x)}} < \infty.$$

   It is the case when the usual Lebesgue measure or Lebesgue measure of index p is the invariant measure.

   In the quenched case, there are no correlations by construction and the law of the orientations depend on the measurable transformation only. This case is nevertheless different from this of the Bernoulli shift because the typical set of points x for which the transience holds depend on the measure μ.

3. **Translation-invariant Gibbs measures**

   We consider now a measurable space of the form $E = \Sigma^{\mathbb{Z}}$ where Σ is a finite alphabet and T is again the bilatere shift defined above. We focus on the Ising model for which $\Sigma = \{-1, +1\}$, and the function f used to generate the transition probabilities and to come back in $[0,1]$ is a dyadic transformation. In ergodic theory, Gibbs measures can be defined as equilibrium states or directly in term of an *energy function* $\Psi : E \longrightarrow \mathbb{R}$, regular enough and chosen here to be Hölder continuous (more details can be found in [8]). A Borel probability measure on E is a Gibbs measure for Ψ if for every homeomorphism τ that affects only finitely many coordinates,

$$\tau\Psi = \mu e^{\Psi_\tau}$$

   where

$$\Psi_\tau := \lim_n \Psi_n \circ \tau^{-1} - \Psi_n$$

   and Ψ_n is the restriction of Ψ on $\Sigma^{\{-n,\dots,n\}}$. There exist many equivalent definitions of Gibbs measures in ergodic theory, see [8]. We focus here on the example of the Ising model where the energy function is

$$\Psi(\omega) = -J\omega_0\omega_1$$

   with a coupling $J \in \mathbb{R}$. The Gibbs measures are very different depending on the sign of the coupling; if $J > 0$, the model is said to be ferromagnetic and has positive correlations (one orientation is likely to agree with its neighbors), while in the antiferromagnetic case ($J < 0$) the sign of the correlations can differ. The case $J = 0$ correspond to the i.i.d. case, already known to be a transient case.

   To go back in $[0,1]$, we introduce $f = d \circ \pi$ with $\pi : \{-1,+1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$; $\omega \longmapsto \sigma$ and $\sigma_y = \frac{1+\omega_y}{2}$, and

$$\begin{aligned} f \quad : \quad & \{0,+1\}^{\mathbb{Z}} \longrightarrow [0,1] \\ & \omega \longmapsto \sum_{y\in\mathbb{Z}} \frac{\sigma_y}{2^y}. \end{aligned}$$

Due to the absence of phase transition in this one dimensional model, the mean of f under μ is $\frac{1}{2}$. The condition (2.6) is believed to be true as soon as the energy function is finite range. This would extend in particular the transience of the i.i.d. case to more general models with exponential decays of (positive or negative) correlations.

4. **SRB measures, Manneville-Pomeau maps**

SRB measures provide another source of examples for dependent orientations. When E is the interval $[0,1]$, a measure μ of the dynamical system S is said to be an *SRB* measure if the ergodic sums converge μ-a.s. In particular it has the Bowen boundedness property in the sense that it is close to a Gibbs measure on some increasing cylinder, i.e. there exists a constant $C > 0$ such that for all $x \in [0,1]$ and every $n \geq 1$

$$\frac{1}{C} \leq \frac{\mu(I_{i_1,\ldots,i_n}(x))}{\exp\left(\sum_{k=0}^{n-1} \Phi(T^k(x))\right)} \leq C$$

where $\Phi = -\log|T'|$ and $I_{i_1,\ldots,i_n}$ is the interval of monotonicity for T^n which contains x.

In some cases, it is possible to control the correlations for SRB measures and we detail now an example where our transience result holds, the *Manneville-Pomeau maps*. These maps have been introduced in the 1980's to study intermittency phenomenon in the study of turbulence in chaotic systems ([1] and references therein) and has been recently identified as weakly Gibbsian measures, see [11]. They are expanding interval maps and in this example we describe the original MP map. The measurable E is the unit interval $[0,1]$ and for $\alpha \in]0,1[$ the map is given by

$$\begin{array}{rcl} T : [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & T(x) = x + x^{1+\alpha} \bmod 1. \end{array}$$

The existence of an SRB invariant measure μ has been established by [12]. This measure is absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$ and the following bounds of Radon-Nikodym derivative $h = \frac{d\mu}{d\lambda}$ has been proved (see [11]):

$$\exists C_\star, C^\star > 0 \text{ s.t. } \frac{\mathrm{C}_\star}{\mathrm{x}^\alpha} < \mathrm{h(x)} < \frac{\mathrm{C}^\star}{\mathrm{x}^\alpha}. \tag{4.43}$$

This measure is known to be mixing, and a polynomial decay of correlation has even been proved for Hölder continuous function f:

$$| C_\mu^f(y) | = \mathcal{O}\big(| y |^{-\frac{1}{\alpha+1}} \big) \tag{4.44}$$

The map T is not invertible but we use Theorem 2.15. It remains to find suitable function f who generates orientations for which the simple random walk is transient. By (4.43), a sufficient condition for the condition (2.6) to hold is

$$\int_0^1 \frac{dx}{x^\alpha \sqrt{f(x)(1-f(x))}} < \infty$$

and this is for example true for the function $f(x) = \frac{1}{2}(1 + x - T(x))$ and the choice of an $\alpha < \frac{1}{3}$.

5. **The rotation on the torus**

We consider the dynamical system $S = ([0,1], \mathcal{B}([0,1]), \lambda, T_\alpha)$ where T_α is the rotation on the torus $[0,1]$ with angle $\alpha \in \mathbb{R}$ defined by

$$x \longmapsto x + \alpha \bmod 1$$

and λ is the Lebesgue measure on $[0,1]$. For every function $f : [0,1] \mapsto [0,1]$ such that $\int_0^1 f(x)\, dx = \frac{1}{2}$ and

$$\int_0^1 \frac{dx}{\sqrt{f(x)(1-f(x))}} < \infty,$$

conclusions of Theorem 2.5 hold. Such functions are called *admissible*. Every function uniformly bounded from 0 and 1, with integral $\frac{1}{2}$ is admissible. We also allow functions f to take values 0 and 1: for instance, $f_1(x) = x$ is admissible although $f_2(x) = \cos^2(2\pi x)$ is not. We actually have no explanations about this phenomenon, moreover we do not know the behavior (recurrence or transience) of the simple random walk on the dynamically oriented lattice generated by f_2. Nevertheless, we can construct particular angles and functions for which the random walk on the corresponding lattice is recurrent. Take $\alpha = \frac{1}{2q}$ for q an integer larger or equal to 1 and $f = \mathbf{1}_{[0,1/2[}$, then the lattice we obtain is $\mathbb{Z}^2$ with undirected vertical lines and horizontal strips of height q, alternatively oriented to the left then to the right. The simple random walk on this deterministic and periodic lattice is known to be recurrent, see [2]. A deeper study, in progress, is needed for this particular choice of dynamical system.

5 Comments

We have extended the results of [2] to non-independent or inhomogeneous orientations. In particular, we have proved that the simple random walk is still transient for a large class of models. As the walk can be recurrent for deterministic orientations, it would be interesting to perturb deterministic cases in order to get a full picture of the transience versus recurrence properties and a more systematic study of this problem is in progress. We believe that the functional limit could be extended, at least to other ergodic dynamical systems, but this requires unknown results on random walks on ergodic random sceneries. In the i.i.d. case, Campanino *et al.* have also proved an improvement of the strong law of large number: almost surely, $\frac{Z_n}{n^\beta} \longrightarrow 0$ for all $\beta > \frac{3}{4}$. Together with our functional limit theorem, this suggests that the conjectured local limit theorem (2.18) is true, at least in this i.i.d. context, getting a full picture of "purely random cases", for which the condition on the generation f holds. This work is in progress and we also investigate the limit theorems in more general cases.

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