COUNTER EXAMPLE TO THE ISODIAMETRIC INEQUALITY IN H-TYPE GROUPS

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ABSTRACT. We prove that in a H-type group equipped with either its Carnot-Carathéodory, gauge or $\|.\|_\infty$-distance and with its Haar measure, the isodiametric inequality does not hold. As a consequence one gets that the Hausdorff measure and the spherical Hausdorff measure of dimension Q, where Q is the homogeneous dimension of the group, do not coincide.

1. INTRODUCTION

The classical isodiametric inequality in the euclidean space $\mathbb{R}^n$ equipped with the n-dimensional Lebesgue measure asserts that the balls maximize the measure among all sets with prescribed diameter. This gives non trivial information about the relation between the metric structure of the space and the measure it is endowed with since it is in general not always possible to include a given set in a ball with equal diameter. This inequality is also known as a useful tool when comparing the Hausdorff and spherical Hausdorff measures of dimension n and showing in particular that these two measures coincide.

The question of the validity of a suitable isodiametric inequality and of its role when comparing Hausdorff and spherical Hausdorff measures can be more generally formulated in other metric measure spaces. We will be mostly concerned in this paper by the case of a H-type group G equipped with either its Carnot-Carathéodory, gauge or $\|.\|_\infty$-distance d and with its Haar measure λ. If Q denotes the homogeneous dimension of the group, one knows that λ is a Q-uniform measure, that is, there exists $\omega(d) > 0$ such that $\lambda(B) = \omega(d)\ \mathrm{diam}(B)^Q$ for all balls B (see Section 2 for precise definitions and statements). Then the isodiametric inequality would read as $\lambda(A) \leq \omega(d)\ \mathrm{diam}(A)^Q$ for all Borel set A. The main result of this paper will be the construction of a counter example to this isodiametric inequality. We shall indeed construct from a ball B a set A that strictly contains B but still has the same diameter than B so that $\lambda(A) > \lambda(B) = \omega(d)\ \mathrm{diam}(A)^Q$. This will be done in Section 3.

The reason for focussing in this paper on H-type groups comes from the main initial motivation which was to handle the case of the Carnot-Carathéodory distance. This distance is defined with respect to the length of so-called sub-unit curves and there is in general no explicit formula for computing the distance between any two points. It turns however out that, for the particular class of Carnot-Carathéodory spaces composed by H-type groups, one can determine explicitly the equations of the minimal geodesics and this can be used to derive concrete geometric features induced by the Carnot-Carathéodory metric (see Section 2, in particular Lemma 1, and also the argument of the proof of Theorem 1). When considering the other distances for which there are explicit formulas, the construction of the counter example follows from more crude computations and can be extended to more general situations as e.g. Carnot groups equipped with the analogue of the $\|.\|_\infty$-distance, see Remark 1.

Next, as a general matter of fact, it turns out that in a homogeneous group of homogeneous dimension Q equipped with a homogeneous distance and with a Haar measure, the validity of the isodiametric inequality is equivalent to the equality between the Hausdorff measure $\mathcal{H}^Q$ and the spherical Hausdorff measure $\mathcal{S}^Q$. This relies on quite classical covering arguments and might be quite classical as well. For sake of completeness we shall however give some details about these arguments in Section 4. More specifically we will express the ratio between $\mathcal{H}^Q$ and $\mathcal{S}^Q$ in terms of a so-called isodiametric constant $C_I(d) \in (0,1]$ where the equality $C_I(d) = 1$ characterizes the validity of the isodiametric inequality. All together it will in particular follow that for a H-type group of homogeneous dimension Q equipped with either its Carnot-Carathéodory, gauge or $\|.\|_\infty$-distance, the Hausdorff and spherical Hausdorff measures of dimension Q do not coincide.

2. H-type groups

We begin with the definition of H-type groups together with natural structures attached to them, namely dilations and Haar measure. These groups are special examples of connected, simply connected and nilpotent Lie groups of step two. They have been introduced by Kaplan in [13] as a natural generalization of the Heisenberg groups (see Example 1 below for the definition of the Heisenberg groups).

Let $\mathcal{G}$ be a real Lie algebra equipped with a scalar product and which can be decomposed in a non-trivial orthogonal direct sum, $\mathcal{G} = \mathcal{V} \oplus \mathcal{Z}$, where $[\mathcal{V}, \mathcal{V}] \subset \mathcal{Z}$ and $[\mathcal{V}, \mathcal{Z}] = [\mathcal{Z}, \mathcal{Z}] = \{0\}$. We define the linear map $J : \mathcal{Z} \to \operatorname{End} \mathcal{V}$ by $\langle J(Z)X, X' \rangle = \langle Z, [X, X'] \rangle$ for all $X,\ X' \in \mathcal{V}$ and $Z \in \mathcal{Z}$. We say that $\mathcal{G}$ is a H-type algebra if for all $Z \in \mathcal{Z}$,

$$J(Z)^2 = -\|Z\|^2 \operatorname{Id}. \tag{1}$$

Note that in such a case $\mathcal{Z}$ is the center of $\mathcal{G}$ and $[\mathcal{V}, \mathcal{V}] = \mathcal{Z}$ so that $\mathcal{G}$ is a stratified Lie algebra of step two. Note also that $\dim(\mathcal{V})$ must then be even. For more details about H-type algebras we refer to [13], see also e.g. [3] or [14].

A Lie group G is said to be a H-type group if it is connected, simply connected and if its Lie algebra is of H-type. The exponential map being in such a case a diffeomorphism from $\mathcal{G}$ onto G ([6]), we can write an element in G as $x = \exp(X + Z)$ where $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$ and the group law is then given by the Campbell-Hausdorff formula and has the form

$$\exp(X + Z).\exp(X' + Z') = \exp(X + X' + Z + Z' + [X, X']/2)$$

for all $X,\ X' \in \mathcal{V}$ and $Z,\ Z' \in \mathcal{Z}$.

Next a H-type algebra $\mathcal{G}$ can be equipped with a natural family of dilations $\delta_\rho,\ \rho > 0$, defined by

$$\delta_\rho(X + Z) = \rho\, X + \rho^2\, Z$$

for all $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$. They induce dilations on the group that we shall still denote by δ_ρ,

$$\delta_\rho(\exp(X + Z)) = \exp(\delta_\rho(X + Z)).$$

Finally, if $\mathcal{L}$ denotes the n-dimensional Lebesgue measure on $\mathcal{G}$ where $n = \dim(\mathcal{G})$, then the push-forward $\lambda = \exp_\# \mathcal{L}$ of $\mathcal{L}$ through the exponential map is a bi-invariant Haar measure on G. It is also not hard to see that λ is homogeneous of degree Q with respect to the dilations where $Q = \dim \mathcal{V} + 2 \dim \mathcal{Z}$ is the homogeneous dimension of the group, that is,

$$\lambda(\delta_\rho(F)) = \rho^Q\, \lambda(F)$$

for all Borel set F and all $\rho > 0$ ([6]). As a convention, when speaking, with a slight abuse of terminology, of *the* Haar measure of the group, we shall always refer here to λ.

Example 1. As already mentioned, H-type groups generalize the definition of the Heisenberg groups. The Heisenberg algebra corresponds to the case where $\dim(\mathcal{Z}) = 1$ so that $\mathcal{Z} \simeq \mathbb{R}$. Then the linear map J is entirely determined by $J(1)$ and if $\dim(\mathcal{V}) = 2n$ for some $n \in \mathbb{N}^*$, one can represent $J(1)$ in a suitable orthonormal basis as

$$J(1) = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

Hence the corresponding Heisenberg group $\mathbb{H}^n$ may be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ and the group law is then given for $[x,t]$, $[x',t'] \in \mathbb{R}^{2n} \times \mathbb{R}$ by

$$[x,t].[x',t'] = [x + x', t + t' + \frac{1}{2}\sum_{i=1}^{n}(x_{n+i}x'_i - x'_{n+i}x_i)].$$

The dilations are defined by $\delta_\rho([x,t]) = [\rho\, x, \rho^2\, t]$ and the Haar measure coincides with the usual Lebesgue measure on $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$.

Next there are many ways to define natural distances on a H-type group , that is, distances d which induce on the group G the usual euclidean topology when identifying G with its Lie algebra $\mathcal{G}$ via the exponential map, that are left invariant, that is, $d(x.y, x.z) = d(y,z)$ for all x, y and $z \in G$, and homogeneous of degree one with respect to the dilations, that is, $d(\delta_\rho(y), \delta_\rho(z)) = \rho\, d(y,z)$ for all y, $z \in G$ and all $\rho > 0$. Any distance with these properties will be called a homogeneous distance. We begin with some common features of such distances that will be used in this paper.

First it turns out that any two homogeneous distances are bi-lipschitz equivalent. It follows in particular that if d is a homogeneous distance on G then (G,d) is a complete and locally compact metric space (it is indeed enough to check this property for a single particular natural distance and this will be clear for the gauge or $\|.\|_\infty$-distance to be defined below).

Next, if d is a homogeneous distance, it follows from the left invariance and homogeneity of the metric and of the measure λ that

$$\lambda(B) = \omega(d)\ \mathrm{diam}(B)^Q \tag{2}$$

for all ball B where $\omega(d) = \lambda(B_1)/\mathrm{diam}(B_1)^Q$ and B_1 denotes the closed unit ball centered at 0. This is actually true more generally in any homogeneous group equipped with a homogeneous distance and with a Haar measure. One can also note that for H-type groups (or more generally Carnot groups) equipped with a homogeneous distance, one always has $\mathrm{diam}(B) = 2\ \mathrm{radius}(B)$ for all balls B. (Here, once a metric is given, balls are always understood as defined with respect to this fixed distance).

We define now more specifically the three homogeneous distances we shall consider here. We begin with the so-called Carnot-Carathéodory distance. We fix an orthonormal basis $(X_1, \ldots, X_m)$ of $\mathcal{V}$. When needed, we shall also interpret X_j (as well as any vector in $\mathcal{G}$) as a left invariant vector field. Then we say that an absolutely continuous curve $\gamma : [0,T] \to G$ is sub-unit if there exist measurable functions $b_1, \ldots, b_m : [0,T] \to \mathbb{R}$ such that $\sum_j b_j(t)^2 \leq 1$ and

$$\dot{\gamma}(t) = \sum_{j=1}^{m} b_j(t)\, X_j(\gamma(t))$$

for a.e. $t \in [0,T]$. In other words, the differential of the exponential map at a point $Y \in \mathcal{G}$ being given by $I - \mathrm{ad}(Y)/2$ (see e.g. [12]) and hence $\dot\gamma(t) = \dot X(t) + \dot Z(t) - [X(t), \dot X(t)]/2$ whenever $\gamma(t) = \exp(X(t) + Z(t))$ for some curves $X(t)$ and $Z(t)$ in $\mathcal{V}$ and $\mathcal{Z}$ respectively (here we also identify $\mathcal{V}$ and $\mathcal{Z}$ with their tangent spaces), the conditions for γ to be a sub-unit curve can be rephrased as $\dot Z - [X, \dot X]/2 = 0$ and $\|\dot X\| \le 1$ a.e. on $[0,T]$.

Since the family $(X_1, \dots, X_m)$ generates $\mathcal{G}$ as a Lie algebra, it follows from Chow's theorem that any two points x, y in G can be joined by a sub-unit curve. Then the Carnot-Carathéodory distance between x and y is defined as

$$d_c(x,y) = \inf\{T > 0\,;\, \text{there exists a sub-unit curve } \gamma : [0,T] \to G \\ \text{such that } \gamma(0) = x,\ \gamma(T) = y\}.$$

One can easily check from this very definition that d_c is left invariant and homogeneous of degree one with respect to the dilations. It also turns out that d_c induces the euclidean topology so that d_c is a homogeneous distance on G. We refer to e.g. [2] or [11] for more details about Carnot-Carathéodory spaces.

The argument we shall use to contradict the isodiametric inequality when G is equipped with d_c will use a specific geometric feature of minimal geodesics stated in Lemma 1 below. First we recall that (G, d_c) is a complete and locally compact metric length space hence any two points can be joined by a minimal geodesic ([10]). By minimal geodesics we will always mean here minimal geodesics with constant and unit speed, that is, curves $\gamma : [0,T] \to G$ such that $d_c(\gamma(t), \gamma(t')) = |t - t'|$ for all t, $t' \in [0,T]$.

Lemma 1. *Let $z \in \exp \mathcal{Z}$ and $\gamma : [0,T] \to G$ be a minimal geodesic starting from z. Then $\gamma(t) \notin \exp \mathcal{Z}$ for all $t \in (0,T)$.*

In other words a minimal geodesic starting from the center of the group can meet the center only at its extremal points. This will follow from an explicit description of the minimal geodesics. In the Heisenberg groups they have been first studied by Gaveau ([8], see also e.g. [15] or [1] and the references therein). For a H-type group necessary conditions for a curve to be a minimal geodesic can be found in [14] and a first step to a complete characterization has been given in [16]. We shall complete here the description. We set

$$P = \{(a, W, Y, T) \in (0, +\infty) \times \mathcal{V} \times \mathcal{Z} \times (0, +\infty);\ \|W\| = 1,\ \|Y\| = 1\}$$

and

$$m(a) = a - \sin(a), \quad \mu(a) = \frac{1 - \cos(a)}{a - \sin(a)}.$$

We say that a curve γ is a curve with parameter (a, W, Y, T) if $\gamma : [0,T] \to G$ is defined by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$X(t) = \frac{T}{a}\,(\exp(\frac{at}{T} J(Y)) - I)W, \quad Z(t) = \frac{1}{2}\,\frac{T^2}{a^2}\, m(\frac{at}{T})\, Y. \tag{3}$$

We say that γ is an horizontal segment (with constant and unit speed) if $\gamma : [0,T] \to G$ is defined by $\gamma(t) = \exp(tX)$ for some $T \in (0, +\infty)$ and some $X \in \mathcal{V}$ with $\|X\| = 1$. We have the following two propositions.

Proposition 1 ([14]). *A minimal geodesic in (G, d_c) starting from 0 must be either a curve with parameter (a, W, Y, T) for some $(a, W, Y, T) \in P$ or an horizontal segment.*

Proposition 2 ([16]). *All curves with parameter $(a, W, Y, T) \in P$ where $a \in (0, 2\pi]$ and all horizontal segments are minimal geodesics.*

To complete the picture it thus remains to study the case of curves with parameter $(a, W, Y, T) \in P$ where $a > 2\pi$.

Proposition 3. *Curves with parameter $(a, W, Y, T) \in P$ with $a > 2\pi$ are not minimal geodesics.*

The argument of the proof given below generalizes the situation of complete Riemannian manifolds where, whenever c is a geodesic and there exists another distinct geodesic between $c(t)$ and $c(t')$ with the same length than c, then c cannot be minimal on any bigger interval $[t, t' + \varepsilon]$. This gives a simpler alternative to the more involved argument used in [1] in the case of the Heisenberg groups.

Proof. By contradiction assume that some curve γ with parameter $(a, W, Y, T) \in P$ with $a > 2\pi$ is a minimal geodesic. We choose $\hat{W} \in \mathcal{V}$ with $\|\hat{W}\| = 1$ and $\hat{W} \neq W$ and let c denote the curve with parameter $(a, \hat{W}, Y, T) \in P$. We set $t_0 = 2\pi T/a$. Since $\|Y\| = 1$, it follows from (1) that $\exp(2\pi J(Y)) - \mathrm{Id} = (\cos(2\pi) - 1)\,\mathrm{Id} + \sin(2\pi)J(Y) = 0$ so that $\gamma(t_0) = c(t_0)$. Then we consider the curve $l : [0, T] \to G$ given by $l_{|[0,t_0]} = c_{|[0,t_0]}$ and $l_{|[t_0,T]} = \gamma_{|[t_0,T]}$. We claim that l is not a minimal geodesic. Indeed on one hand, all curves with parameter in P and all horizontal segments being smooth curves, it follows from Proposition 1 that any minimal geodesic starting from 0 is of class C^∞ (we note that this is also true for all minimal geodesics by left invariance of the distance and because the group law is of polynomial type, see also [17], [18] where this is shown to hold more generally under the so-called strong bracket generating hypothesis and [9] where this is extended to all Carnot groups of step 2). On the other hand l is clearly not smooth at t_0. Indeed if $l(t) = \exp(X(t) + Z(t))$, we have

$$\dot{X}(t) = J(Y) \exp(\frac{at}{T} J(Y))\hat{W}$$

for all $t \in (0, t_0)$ hence

$$\lim_{t \to t_0^-} \dot{X}(t) = J(Y) \exp(2\pi J(Y))\hat{W} = J(Y)\hat{W}.$$

Similarly we have

$$\lim_{t \to t_0^+} \dot{X}(t) = J(Y)W$$

and the claim follows since $J(Y)$ is invertible and $\hat{W} \neq W$. Hence l cannot be a minimal geodesic and thus $d_c(0, l(T)) < T$ (note that l is a sub-unit curve). But $l(T) = \gamma(T)$ and γ was assumed to be minimal hence $d_c(0, \gamma(T)) = T$ and the contradiction follows. □

We now turn to the proof of Lemma 1. Let $z \in \exp \mathcal{Z}$. By left invariance, any minimal geodesic starting from z is of the form $z.\gamma$ where γ is a minimal geodesic starting from 0. And we have that $z.\gamma$ meets $\exp \mathcal{Z}$ if and only if γ meets $\exp \mathcal{Z}$. On the other hand we know from Proposition 1, Proposition 2 and Proposition 3 that γ is a curve with parameter (a, W, Y, T) for some $(a, W, Y, T) \in P$ with $0 < a \leq 2\pi$ or an horizontal segment. Obviously an horizontal segment does not meet $\exp \mathcal{Z}$ except at its starting point. Next, since $\|Y\| = 1$, it follows from (1) that $\exp(bJ(Y)) = \cos(b)\,\mathrm{Id} + \sin(b)J(Y)$ for all $b \in \mathbb{R}$. Hence

$$\begin{aligned}(\exp(bJ(Y)) - \mathrm{Id})\,(\exp(-bJ(Y)) - \mathrm{Id}) &= 2\,\mathrm{Id} - (\exp(bJ(Y)) + \exp(-bJ(Y))) \\ &= 2(1 - \cos(b))\,\mathrm{Id}\end{aligned}$$

and $\exp(bJ(Y)) - \mathrm{Id}$ is in particular invertible for all $b \in (0, 2\pi)$. If $t \in (0, T)$, we have $at/T \in (0, 2\pi)$ and it then follows that $X(t) \neq 0$ whenever $X(t)$ is given by (3). Hence $\gamma(t) \notin \exp \mathcal{Z}$ as claimed.

We will also consider two other homogeneous distances. One of them is the so-called gauge (or Korányi) distance d_g defined with respect to the homogeneous norm

$$\|\exp(X + Z)\| = (\|X\|^4 + 16\,\|Z\|^2)^{1/4}$$

for all $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$. By [4], this norm satisfies $\|x.y\| \leq \|x\| + \|y\|$ for all x, $y \in G$, and one then defines the distance $d_g(x, y)$ between any two points x, $y \in G$ by

$$d_g(x, y) = \|x^{-1}.y\|.$$

The second one, that we shall call $\|.\|_\infty$-distance, is defined by

$$d_\infty(x, y) = \|x^{-1}.y\|_\infty.$$

where

$$\|\exp(X + Z)\|_\infty = \max(\|X\|, 2\,\|Z\|^{1/2})$$

for all $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$. A routine computation shows that $\|x.y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ for all x, $y \in G$ and then d_∞ defines a homogeneous distance on G as well.

3. Counter example to the isodiametric inequality

We prove in this section that in a H-type group equipped with either the distance d_c, d_g or d_∞, the isodiametric inequality fails to hold.

Theorem 1. *Let G be a H-type group , λ be the Haar measure of the group and d be one of the distances d_c, d_g, d_∞. Then there exists a closed set A such that $\lambda(A) > \omega(d)\ \mathrm{diam}(A)^Q$.*

Recall (2) for the definition of $\omega(d)$. Note that since two locally finite Borel regular Haar measures of the group only differ by a multiplicative constant there is here no loss of generality when considering the specific Haar measure λ.

Proof. We start from the closed unit ball B_1 and we first prove that there exists a point $p \in \partial B_1$ such that $d(p, x) < \mathrm{diam}(B_1)$ for all $x \in B_1$. The conclusion will then easily follow from this geometric observation. We distinguish three cases corresponding to each distance.

Case 1. When $d = d_c$, we start from a point $p \in \exp \mathcal{Z} \cap \partial B_1$. The claim will follow from the fact that there is no minimal geodesic starting from z that can pass through the center of the ball B_1 (except possibly at its terminal point). If $x \in B_1$, $x \neq 0$, we let $\gamma_1 : [0, T_1] \to G$ be a minimal geodesic between p and 0 and $\gamma_2 : [0, T_2] \to G$ be a minimal geodesic between 0 and x. Then we define the curve $\gamma : [0, T_1 + T_2] \to G$ by $\gamma(t) = \gamma_1(t)$ if $t \in [0, T_1]$ and $\gamma(t) = \gamma_2(t - T_1)$ if $t \in [T_1, T_1 + T_2]$. It follows from Lemma 1 that γ cannot be a minimal geodesic between p and x since $\gamma(T_1) = 0 \in \exp \mathcal{Z}$. Since γ is sub-unit it follows that $d_c(p, x) < T_1 + T_2 = d_c(p, 0) + d_c(0, x) \leq 2$ where the equality follows from the minimality of γ_1 and γ_2. If $x = 0$ we obviously have $d_c(p, x) = 1 < 2$.

Case 2. When $d = d_g$ the conclusion follows from a direct computation. We choose $p = \exp(Z_p)$ for some $Z_p \in \mathcal{Z}$ such that $\|Z_p\| = 1/4$. Then if $x = \exp(X + Z) \in B_1$ with $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$, we have $\exp(Z_p)^{-1}.\exp(X + Z) = \exp(X + Z - Z_p)$ hence

$$\begin{aligned} d_g(p, x)^4 &= \|X\|^4 + 16\,\|Z - Z_p\|^2 \\ &= \|X\|^4 + 16\,\|Z\|^2 + 16\,\|Z_p\|^2 - 32\,\langle Z_p, Z\rangle \leq 2\,(1 + 16\,\|Z\|\,\|Z_p\|) \leq 4, \end{aligned}$$

because $x \in B_1$ hence $\|X\|^4 + 16\,\|Z\|^2 \leq 1$ and $\|Z\| \leq 1/4$. It follows that $d_g(p,x) \leq \sqrt{2} < 2$ as wanted.

Case 3. When $d = d_\infty$ the argument is similar to the previous one starting once again from $p = \exp(Z_p)$ with $Z_p \in \mathcal{Z}$ such that $\|Z_p\| = 1/4$. Then if $x = \exp(X+Z) \in B_1$ with $X \in \mathcal{V}$ and $Z \in \mathcal{Z}$, we have $\|X\| \leq 1$ and $16\,\|Z - Z_p\|^2 = 16\,\|Z\|^2 + 16\,\|Z_p\|^2 - 32\,\langle Z_p, Z\rangle \leq 2\,(1 + 16\,\|Z\|\,\|Z_p\|) \leq 4$. Hence $d_\infty(p,x) \leq \sqrt{2} < 2$ as wanted.

To conclude let $p \in \partial B_1$ be such that $d(p,x) < \mathrm{diam}(B_1)$ for all $x \in B_1$. By continuity of $x \mapsto d(p,x)$ and compactness of B_1, we get that $\sup_{x\in B_1} d(p,x) < \mathrm{diam}(B_1)$. Since $q \mapsto \sup_{x\in B_1} d(q,x)$ is 1-Lipschitz hence continuous, it follows that one can find a nonempty closed ball B centered at p such that $\sup_{x\in B_1} d(q,x) < \mathrm{diam}(B_1)$ for all $q \in B$. Considering a smaller ball if necessary, one can also assume that $\mathrm{diam}(B) \leq \mathrm{diam}(B_1)$. Then we set $A = B_1 \cup B$. Since $p \in \partial B_1$, A clearly contains a small nonempty ball that does not meet B_1 hence $\lambda(A) > \lambda(B_1) = \omega(d)\,\mathrm{diam}(B_1)^Q$. On the other hand we have that $d(q,q') \leq \sup_{x\in B_1} d(q,x) < \mathrm{diam}(B_1)$ for all $q \in B$ and all $q' \in B_1$ and it easily follows that $\mathrm{diam}(A) = \mathrm{diam}(B_1)$ hence $\lambda(A) > \omega(d)\,\mathrm{diam}(A)^Q$ as wanted. □

Remark 1. One can extend the previous argument to Carnot groups equipped with the analogue of the $\|.\|_\infty$-distance. More precisely a Carnot group G is a connected, simply connected Lie group whose Lie algebra $\mathcal{G}$ admits a stratification,

$$\mathcal{G} = \oplus_{j=1}^k V_j, \quad [V_1, V_j] = V_{j+1}, \quad V_k \neq \{0\}, \quad V_j = \{0\} \text{ if } j > k.$$

The group may be identified via the exponential map with $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ where $m_j = \dim(V_j)$. Then the dilations are given by $\delta_\rho(x_1, \ldots, x_k) = (\rho x_1, \ldots, \rho^k x_k)$. One defines a homogeneous distance d_∞ on G setting $d_\infty(x,y) = \|x^{-1}.y\|_\infty$ where $\|(x_1,\ldots,x_k)\|_\infty = \max\{\beta_j \|x_j\|_{\mathbb{R}^{m_j}}^{1/j},\ j = 1,\ldots,k\}$ and the β_j's are suitable positive constants that depend on the group structure and are chosen in such a way that $\|x.y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ (see e.g. [7] for more details about this kind of distances). Then starting from a point $p \in \partial B_1$ of the form $p = (0,\ldots,0,x_p)$ with $\beta_k^k \|x_p\|_{\mathbb{R}^{m_k}} = 1$, one easily extends the argument of Case 3 above and a counter example to the isodiametric inequality follows.

4. Comparision between Hausdorff and spherical Hausdorff measures

We recall in this section some relationships between various natural measures that can be defined on a H-type group equipped with a homogeneous distance, namely the Haar measure and the Hausdorff and spherical Hausdorff measures of dimension Q where Q is the homogeneous dimension of the group. In connection with the previous section we shall in particular be interested in the role of the isodiametric inequality when comparing the Hausdorff and spherical Hausdorff measures. In our present context the validity of the isodiametric inequality turns indeed out to be equivalent to the equality between these two measures. As already mentioned, the main arguments we use here are based on a classical covering theorem and the results we shall state might be quite classical as well. However for sake of completeness we shall give some few details about these arguments in this section. We also stress that the main results (Theorem 2 and Theorem 4) can be easily extended, with the same arguments and proofs, to the more general setting of homogeneous groups equipped with a homogeneous distance.

We keep the same notations as before and let G be H-type group and λ be the Haar measure on G as defined in Section 2. We fix a homogeneous distance d on G. If Q is the

homogeneous dimension of the group the Hausdorff measure $\mathcal{H}^Q$ and the spherical Hausdorff measure $\mathcal{S}^Q$ are defined in the classical way using Carathéodory's construction. If $A \subset G$,

$$\mathcal{H}^Q(A) = \lim_{\rho \downarrow 0} \mathcal{H}^Q_\rho(A)$$

where $\mathcal{H}^Q_\rho(A) = \inf\{\sum_{i=0}^\infty \operatorname{diam}(A_i)^Q;\ A \subset \cup_{i=0}^\infty A_i,\ \operatorname{diam}(A_i) \le \rho\}$,

$$\mathcal{S}^Q(A) = \lim_{\rho \downarrow 0} \mathcal{S}^Q_\rho(A)$$

where $\mathcal{S}^Q_\rho(A) = \inf\{\sum_{i=0}^\infty \operatorname{diam}(A_i)^Q;\ A \subset \cup_{i=0}^\infty A_i,\ A_i \text{ is a ball with } \operatorname{diam}(A_i) \le \rho\}$.

The normalization adopted here may not be the classical one, where one adds the Q-dimensional Lebesgue measure of the euclidean ball with unit diameter in $\mathbb{R}^Q$ as a multiplicative factor, but it has the advantage that $\mathcal{H}^Q$ and $\mathcal{S}^Q$ depend here only on the intrinsic homogeneous distance d.

We recall that we have set $\omega(d) = \lambda(B_1)/\operatorname{diam}(B_1)^Q$ where B_1 is the closed unit ball so that $\lambda(B) = \omega(d)\ \operatorname{diam}(B)^Q$ for all ball B. All the three measures λ, $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are locally finite Borel regular Haar measures on G hence they only differ from a positive and finite multiplicative factor. First, as a classical matter of fact, it follows from the left invariance and homogeneity and from the normalization that the ratio between λ and $\mathcal{S}^Q$ is exactly given by $\omega(d)$.

Theorem 2. *We have* $\lambda = \omega(d)\,\mathcal{S}^Q$.

We first recall the covering theorem which is the central tool for the proof of this result as well as for the comparision between $\mathcal{H}^Q$ and $\mathcal{S}^Q$. It can be found, in a more general setting, in [5, Chapter 2.8].

Let $\mathcal{F}$ be a family of closed subsets of G.

We say that $\mathcal{F}$ covers a set A finely if for all $a \in A$ and all $\varepsilon > 0$, there exists $F \in \mathcal{F}$ such that $a \in F$ and $\operatorname{diam}(F) \le \varepsilon$.

We say that $\mathcal{F}$ is adequate for a set A if for any open set U there exists a countable disjointed subfamily $\tilde{\mathcal{F}}$ of $\mathcal{F}$ such that

$$\cup_{F \in \tilde{\mathcal{F}}} F \subset U \quad \text{and} \quad \lambda((U \cap A) \setminus \cup_{F \in \tilde{\mathcal{F}}} F) = 0.$$

(Note that we can replace here λ by $\mathcal{S}^Q$ or $\mathcal{H}^Q$.)

If $F \in \mathcal{F}$, we define $\hat{F} = \cup\{T \in \mathcal{F};\ T \cap F \neq \emptyset \text{ and } \operatorname{diam}(T) \le 2\ \operatorname{diam}(F)\}$.

Theorem 3 ([5]). *We assume that all sets in $\mathcal{F}$ are bounded and that $\lambda(F) > 0$ for all $F \in \mathcal{F}$. Then if $\mathcal{F}$ covers A finely and if there exists $\tau \ge 1$ such that $\lambda(\hat{F}) \le \tau\lambda(F)$ for all $F \in \mathcal{F}$, then $\mathcal{F}$ is adequate for A.*

We turn now to the proof of Theorem 2. Since λ and $\mathcal{S}^Q$ are both locally finite Borel regular Haar measures on G it is sufficient to check that $\lambda(U) = \omega(d)\,\mathcal{S}^Q(U)$ where U is an open subset of G with $\lambda(U)$, and hence $\mathcal{S}^Q(U)$, finite. Then let $\varepsilon > 0$ be fixed and let $(B_i)_{i=0}^{+\infty}$ be a family of balls such that $U \subset \cup_{i=0}^{+\infty} B_i$ and

$$\sum_{i=0}^{+\infty} \operatorname{diam}(B_i)^Q \le \mathcal{S}^Q(U) + \varepsilon.$$

We have

$$\lambda(U) \leq \sum_{i=0}^{+\infty} \lambda(B_i) = \omega(d) \sum_{i=0}^{+\infty} \operatorname{diam}(B_i)^Q \leq \omega(d)\,(\mathcal{S}^Q(U) + \varepsilon).$$

Since this holds for all $\varepsilon > 0$, it follows that $\lambda(U) \leq \omega(d)\,\mathcal{S}^Q(U)$. Conversely, for $\rho > 0$ fixed, we consider the family $\mathcal{F}_\rho$ composed by all closed balls in G with diameter less than ρ. Then $\mathcal{F}_\rho$ covers U finely. Next, for all $B \in \mathcal{F}_\rho$, we have $\operatorname{diam}(\hat{B}) \leq 5 \operatorname{diam}(B)$ hence $\hat{B}$ is contained in a ball with diameter $10 \operatorname{diam}(B)$ and thus $\lambda(\hat{B}) \leq 10^Q \lambda(B)$. Then it follows from Theorem 3 that one can find a countable disjointed subfamily $\tilde{\mathcal{F}}_\rho \subset \mathcal{F}_\rho$ such that

$$U_\rho \subset U \quad \text{and} \quad \lambda(U \setminus U_\rho) = \mathcal{S}^Q(U \setminus U_\rho) = 0$$

where $U_\rho = \cup_{B \in \tilde{\mathcal{F}}_\rho} B$. Then we get

$$\omega(d)\,\mathcal{S}^Q_\rho(U_\rho) \leq \omega(d) \sum_{B \in \tilde{\mathcal{F}}_\rho} \operatorname{diam}(B)^Q = \sum_{B \in \tilde{\mathcal{F}}_\rho} \lambda(B) = \lambda(U_\rho) = \lambda(U).$$

It follows that

$$\omega(d)\,\mathcal{S}^Q_\rho(U) \leq \omega(d)\,\mathcal{S}^Q_\rho(U \setminus U_\rho) + \omega(d)\,\mathcal{S}^Q_\rho(U_\rho) \leq \omega(d)\,\mathcal{S}^Q(U \setminus U_\rho) + \lambda(U) = \lambda(U)$$

for all $\rho > 0$. Then we let $\rho \downarrow 0$ to get that $\omega(d)\,\mathcal{S}^Q(U) \leq \lambda(U)$ and this concludes the proof.

We turn now to the comparison between $\mathcal{H}^Q$ and $\mathcal{S}^Q$ using a so-called isodiametric constant $C_I(d)$ that will make the connection with the isodiametric inequality. If A is a bounded Borel subset of G with positive diameter, we put

$$C_d(A) = \omega(d)\ \operatorname{diam}(A)^Q / \lambda(A)$$

with the convention that $C_d(A) = +\infty$ whenever $\lambda(A) = 0$. Then we set

$$C_I(d) = \inf\{C_d(A);\ A \text{ is a bounded Borel subset of } G \text{ with } \operatorname{diam}(A) > 0\}.$$

First we note that $C_I(d) > 0$. Indeed otherwise, for each $\varepsilon > 0$, one could find a Borel set A_ε with positive and finite diameter such that $\omega(d)\ \operatorname{diam}(A_\varepsilon)^Q \leq \varepsilon\,\lambda(A_\varepsilon)$. Then we set $r_\varepsilon = \operatorname{diam}(A_\varepsilon)^{-1}$ and $D_\varepsilon = \delta_{r_\varepsilon}(A_\varepsilon)$. We have $\operatorname{diam}(D_\varepsilon) = 1$ hence D_ε is contained in a ball with unit radius and thus $\lambda(D_\varepsilon) \leq \omega(d)\ \operatorname{diam}(B_1)^Q$. On the other hand $\lambda(D_\varepsilon) = r_\varepsilon^Q\,\lambda(A_\varepsilon) \geq \omega(d)/\varepsilon$ and the contradiction follows for ε small enough. Next it turns out that the multiplicative factor between $\mathcal{H}^Q$ and $\mathcal{S}^Q$ is exactly given by the isodiametric constant $C_I(d)$.

Theorem 4. *We have $\mathcal{H}^Q = C_I(d)\,\mathcal{S}^Q$.*

Proof. The proof is very similar to that of Theorem 2. First it is enough to check the equality for open subsets U of G with finite $\mathcal{H}^Q$, and hence $\mathcal{S}^Q$, measure. Then, using verbatim arguments of the proof of Theorem 2, we let $\varepsilon > 0$ be fixed and let $(U_i)_{i=0}^{+\infty}$ be a family of bounded sets such that $U \subset \cup_{i=0}^{+\infty} U_i$ and

$$\sum_{i=0}^{+\infty} \operatorname{diam}(U_i)^Q \leq \mathcal{H}^Q(U) + \varepsilon.$$

We have

$$C_I(d)\,\mathcal{S}^Q(U) \leq \sum_{i=0}^{+\infty} C_d(U_i)\,\mathcal{S}^Q(U_i) = \sum_{i=0}^{+\infty} \operatorname{diam}(U_i)^Q \leq \mathcal{H}^Q(U) + \varepsilon,$$

where the equality follows from Theorem 2 and the definition of $C_d(U_i)$. Since this holds for all $\varepsilon > 0$, it follows that $C_I(d)\,\mathcal{S}^Q(U) \le \mathcal{H}^Q(U)$.

Conversely we prove that $\mathcal{H}^Q(U) \le C_d(A)\,\mathcal{S}^Q(U)$ for all bounded Borel set A with positive diameter. First we have $C_d(\overline{A}) \le C_d(A)$ and, by left invariance and homogeneity, $C_d(A_{x,r}) = C_d(A)$ whenever $A_{x,r} = x.\delta_r(A)$ with $x \in G$ and $r > 0$. Hence, taking closure and translating A if necessary, it is enough to prove the claim for closed bounded sets A with positive diameter and containing 0. We also assume that $C_d(A) < +\infty$ otherwise there is nothing to prove. The proof is now similar to the second part of Theorem 2 using coverings with sets of the form $A_{x,r}$, the main point being to check that we are in a position to use Theorem 3. Thus let $\rho > 0$ be fixed. We set

$$\mathcal{A}_\rho = \{A_{x,r};\ x \in G,\ r\,\mathrm{diam}(A) \le \rho\}.$$

The family $\mathcal{A}_\rho$ covers U finely since for all $x \in G$ and all $r > 0$ we have $x \in A_{x,r}$ and $\mathrm{diam}(A_{x,r}) = r\,\mathrm{diam}(A)$. Next, similarly as before, we have $\mathrm{diam}(\hat{A}_{x,r}) \le 5\,\mathrm{diam}(A_{x,r})$ hence $\hat{A}_{x,r}$ is contained in a ball with diameter $10\,\mathrm{diam}(A_{x,r})$ and thus

$$\lambda(\hat{A}_{x,r}) \le 10^Q\,\omega(d)\ \mathrm{diam}(A_{x,r})^Q = 10^Q\,C_d(A)\,\lambda(A_{x,r}).$$

It follows from Theorem 3 that $\mathcal{A}_\rho$ is adequate for U and we conclude now as in Theorem 2. One can find a countable disjointed subfamily $\tilde{\mathcal{A}}_\rho \subset \mathcal{A}_\rho$ such that

$$U_\rho \subset U \quad \text{and} \quad \mathcal{S}^Q(U \setminus U_\rho) = \mathcal{H}^Q(U \setminus U_\rho) = 0$$

where $U_\rho = \cup_{B \in \tilde{\mathcal{A}}_\rho} B$. Then we have

$$\mathcal{H}^Q_\rho(U_\rho) \le \sum_{B \in \tilde{\mathcal{A}}_\rho} \mathrm{diam}(B)^Q = C_d(A) \sum_{B \in \tilde{\mathcal{A}}_\rho} \mathcal{S}^Q(B) = C_d(A)\,\mathcal{S}^Q(U).$$

It follows that

$$\mathcal{H}^Q_\rho(U) \le \mathcal{H}^Q_\rho(U \setminus U_\rho) + \mathcal{H}^Q_\rho(U_\rho) \le \mathcal{H}^Q(U \setminus U_\rho) + C_d(A)\,\mathcal{S}^Q(U) = C_d(A)\,\mathcal{S}^Q(U)$$

for all $\rho > 0$. Then we let $\rho \downarrow 0$ to get that $\mathcal{H}^Q(U) \le C_d(A)\,\mathcal{S}^Q(U)$ as wanted. □

Going back to the isodiametric inequality we get that its validity, that is, the fact that $\lambda(A) \le \omega(d)\ \mathrm{diam}(A)^Q$ for all Borel set A, is equivalent to the equality $C_I(d) = 1$. Indeed since one always has $\mathcal{H}^Q \le \mathcal{S}^Q$ it follows from Theorem 4 that $C_I(d) \le 1$ and on the other hand the isodiametric inequality asserts that $C_I(d) \ge 1$. Thus, for the specific distances d_c, d_g and d_∞, we know from Theorem 1 that the corresponding isodiametric constant is strictly less than 1 and the following corollary easily follows.

Corollary 1. *Let G be a H-type group of homogeneous dimension Q and let d denote either the distance d_c, d_g or d_∞. We have $C_I(d) < 1$ and consequently $\mathcal{H}^Q(U) = C_I(d)\,\mathcal{S}^Q(U) < \mathcal{S}^Q(U)$ for all Borel set U with $0 < \mathcal{S}^Q(U) < +\infty$.*

References

1. L. Ambrosio and S. Rigot, *Optimal mass transportation in the Heisenberg group*, J. Funct. Anal. **208** (2004), no. 2, 261–301.
2. A. Bellaïche, *The tangent space in sub-Riemannian geometry*, in Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 1–78.
3. M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math. **87** (1991), no. 1, 1–41.

4. J. Cygan, *Subadditivity of homogeneous norms on certain nilpotent Lie groups*, Proc. Amer. Math. Soc. **83** (1981), no. 1, 69–70.
5. H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.
6. G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982.
7. B. Franchi, R. Serapioni, and F. Serra Cassano, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geom. Anal. **13** (2003), no. 3, 421–466.
8. B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math. **139** (1977), no. 1-2, 95–153.
9. C. Golé and R. Karidi, *A note on Carnot geodesics in nilpotent Lie groups*, J. Dynam. Control Systems **1** (1995), no. 4, 535–549.
10. M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques [Mathematical Texts], vol. 1, CEDIC, Paris, 1981, Edited by J. Lafontaine and P. Pansu.
11. ———, *Carnot-Carathéodory spaces seen from within*, in Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 79–323.
12. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Applied Mathematics, Vol. XII, Academic Press, New York, 1962.
13. A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258** (1980), no. 1, 147–153.
14. A. Korányi, *Geometric properties of Heisenberg-type groups*, Adv. in Math. **56** (1985), no. 1, 28–38.
15. R. Monti, *Some properties of Carnot-Carathéodory balls in the Heisenberg group*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), no. 3, 155–167 (2001).
16. S. Rigot, *Counter example to the Besicovitch covering property for some Carnot groups equipped with their Carnot-Carathéodory distance*, to appear on Math. Z.
17. R. Strichartz, *Sub-Riemannian geometry*, J. Differential Geom. **24** (1986), no. 2, 221–263.
18. ———, *Corrections to: "Sub-Riemannian geometry" [J. Differential Geom. **24** (1986), no. 2, 221–263]*, J. Differential Geom. **30** (1989), no. 2, 595–596.

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