Exact inverse of the Toeplitz matrix with singular rational symbol

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Abstract

This paper presents the exact an exact computation of the Toeplitz matrice inverse with a symbol $f = |1-\chi|^2 \frac{|Q|^2}{|P|^2}$ where P and Q are trigonometric polynomials whose zeros are outside a closed centered at zero disc with radius $R > 1$. From this result, two precise asymptotic expansions of the entries and of the trace are derived.

Keywords

Toeplitz matrices; Green's kernel; singular symbol ; random walks.
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1 Introduction

Let f be a real positive valued function, defined on the one dimensional torus $\mathbb{T}$ belonging to $L^1(\mathbb{T})$. Such a function allows to define a family of truncated Toeplitz operators $\{T_N(f)\}_{N\in\mathbb{N}}$ whose matrices in the basis $\{\chi^n = e^{in\theta}\}_{n\in\mathbb{Z}}$ still denoted by $\{T_N(f)\}$ are given by

$$T_N(f) = \left(\hat{f}(m-n)\right)_{0\le m,n\le N}$$

where $\hat{f}(s)$ is the s-th Fourier's coefficient of f. $T_N(f)$ is also called the Toeplitz matrix (of order $N+1$) associated to the *symbol* f. Many works are devoted to the asymptotic expansion of the entries of $T_N(f)^{-1}$ when N tends to $+\infty$ and this papers present various points of view. In order to clarify the goal of this work, we give some short examples.
When the symbol is $|1-\chi|^{2p} f_1$, where $p \in \mathbb{N}$ and f_1 is a positive function, then if f_1 is sufficiently smooth, we have:

$$T_N(|1-\chi|^{2p} f_1)^{-1}_{[Nx]+1,[Ny]+1} = \frac{1}{f_1(1)} N^{2p-1} G_p(x,y) + o(N^{2p-1}),$$

G_p being the Green kernel associated to an differential operator of order $2p$ on $[0,1]$ and to some boundary conditions. This result can be found in [4], the case $p = 1$, already dealed in [3] and [6] is here generalised. Notice that in [6], the symbol f is of type $f = 1 - \Phi$ where Φ is the characteristic function associated to a discrete random variable X defined on a probabilised space (Ω, τ, p) :

$$\Phi = \sum_{-\infty}^{+\infty} c_k \chi^k, \text{ with } c_k = p(\{X = k\}).$$

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In this probabilistic frame, the entries a_{ij} of the inverse of $T_N(f)$ have the following interpretation: the sequence of random variables $Z_n = X_0 + X_1 + \ldots + X_n$ where the $X_i, i > 0$ are identically distributed, independent copies of the random variable X, defines a random walk on $[0, N]$. Then a_{ij} is the expected number of visits at j with $Z_0 = i$ before leaving the interval $[0, N]$ for the first time. [7] and [8] can be consulted for this question. That shows how interesting it is to obtain not only asymptotic results but also exact expressions of the inverse's entries. It is precisely the aim of the present work when the symbol is of type

$$f = |1 - \chi|^2 \frac{|Q|^2}{|P|^2}, \tag{1}$$

are P and Q two trigonometric polynomials without zero on $\mathbb{T}$. Our main result is

Theorem 1 *Let P and Q be two trigonometric polynomials without zero on $\mathbb{T}$ respectively of degrees n_1 and n_2. We assume that Q has zeros outside of a closed disc centered in 0 with radius $R > 1$ such that the analytic series*

$$\frac{P}{Q} = \sum_{u=0}^{\infty} \beta_u z^u$$

has a convergence radius $\geq R$. Let $f = |1 - \chi|^2 \frac{|Q|^2}{|P|^2}$. Then

$T_N(f)$ is an inversible operator and we have for $0 \leq k, l \leq N + 1 - (n_1 - n_2)$,

$$\big(T_N(f)\big)^{-1}_{k+1,l+1} = a_{kl} - \frac{d(k)\bar{d}(l)}{N + 2 + \mathcal{A}(P, Q)} \tag{2}$$

with

$$a_{kl} = \sum_{s'=0}^{l} \sum_{s=0}^{k} \bar{\beta}_{k-s} \beta_{l-s'} \min(s + 1, s' + 1),$$

$$d(k) = -(k + 1) \frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)} \sum_{u=0}^{k} \bar{\beta}_u + \frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)} \sum_{u=0}^{k} u\bar{\beta}_u,$$

$$\mathcal{A}(P, Q) = 4\Re\Big(\frac{Q'(1)}{Q(1)} - \frac{P'(1)}{P(1)}\Big). \tag{3}$$

If $n_2 = 0$, that is if Q is a constant polynomial, then one obtains a more intrinsic formula, even if it means restrict the domain of k and l. That is the contain of the following corollary.

Corollary 1 *For a symbol of type $f = \frac{|1-\chi|^2|}{|P|^2}$, and if $n_1 \leq k \leq N - n_1$, the term $d(k)$ of the theorem can be expressed in the following manner:*

$$d(k) = -(k + 1)\bar{P}(1) + \frac{P'(1)\bar{P}(1)}{P(1)}.$$

Corollary 2 *With the same notations and hypotheses, an formula identical to (2) holds, where the asymptotic expansion of $d(k)$ is given by*

$$d(k) = -(k + 1)\Big(\frac{P(1)}{Q(1)} + o\Big(\frac{1}{R^k}\Big)\Big) + \frac{P(1)}{Q(1)}\Big(\frac{P'(1)}{\bar{P}(1)} - \frac{Q'(1)}{\bar{Q}(1)}\Big) + o\Big(\frac{1}{R^k}\Big).$$

Corollary 3 (Trace theorem) *With always the same hypotheses and notations we obtain the following asymptotic expansion of the the trace of* $T_N(f)^{-1}$ *:*

$$\mathrm{Tr}\big(T_N(f)^{-1}\big) = \frac{N^2}{6}\big|\frac{P}{Q}(1)\big|^2 + N\left(\Big(\frac{2+\mathcal{A}}{3}\Big)\big|\frac{P}{Q}(1)\big|^2 + \mathcal{B} - C(P,Q)\right) + O(1),$$

where

$$C(P,Q) = \sum_{u=0}^{+\infty}\sum_{u'=0}^{+\infty}\bar{\beta}_u\beta_{u'}\max(u,u'),$$

$$\mathcal{B}(P,Q) = \Re\Big(\frac{P'}{\bar{P}}(1) - \frac{Q'}{\bar{Q}}(1)\Big),$$

and where $\mathcal{A}$ *is given by (3).*

Corollary 4 (Sum of terms) *With the same hypotheses that previously we have*

$$\sum_{k=1}^{N}\sum_{l=0}^{N}(T_N(f))^{-1}_{k,l} = \big|\frac{P}{Q}(1)\big|^2\left(\frac{1}{12}N^3 + N^2\left(\frac{1+\mathcal{A}}{2} + \mathcal{B}(P,Q)\right)\right) + O(N).$$

We have now to have in mind that the formula giving the term $T_N(f)^{-1}_{k+1,l+1}$ is a little different that the equality 2 when k and l are $> N-(n_1-n_2)$. Let us state this fact in the following proposition.

Proposition 1 *If* $k,l < N-(n_1-n_2)$ *then*

$$\big(T_N(f)\big)^{-1}_{k+1,l+1} = a_{kl} - \frac{d(k)\bar{d}(l)}{N+2+\mathcal{A}(P,Q)} + c_{k,l},$$

where $c(k,l) = O(1)$.

Remark 1 *We have an exact computation of* $c_{k,l}$ *given in the corpus of the proof.*

2 proof of the theorem, the corollary 1 and the proposition.

2.1 An inversion formula

Notations :

$$\begin{aligned} H^+ &= \Big\{h \in L^2(\mathbb{T});\quad \widehat{h}(s) = 0,\quad s<0\Big\}, \\ H^- &= L^2(\mathbb{T}) \ominus H^+ \ . \end{aligned}$$

H^- is the orthogonal subspace of H^+ in $L^2(\mathbb{T})$. π_+ et π_- are the orthogonal projections from $L^2(\mathbb{T})$ on H^+ et H^-. The assumptions on f are originated from the following decomposition theorem (Grenander and Szegö [2]):

> Let be $f \in L^1(\mathbb{T})$ positive on $\mathbb{T}$ almost everywhere and such that $\mathrm{Log}\, f \in L^1(\mathbb{T})$.
> Then: $\exists g \in H^+$ such that: $f = |g|^2$

When we are in the previous case, we can define Hankel operators H_{Φ_N} and $H^*_{\Phi_N}$ associated to this decomposition Let

$$\Phi_N = \frac{g}{\bar{g}}\chi^{N+1} \qquad (\chi = e^{i\theta}).$$

Then

$$\begin{aligned} H_{\Phi_N} &: H^+ \to H^-, \quad H_{\Phi_N}(\psi) = \pi_-\left(\Phi_N\psi\right), \\ H^*_{\Phi_N} &: H^- \to H^+, \quad H^*_{\Phi_N}(\varphi) = \pi_+(\bar{\Phi}_N\varphi). \end{aligned}$$

The hypothesis of the decomposition (1) gives : $\left\|H^*_{\Phi_N}H_{\Phi_N}\right\| < 1$. A demonstration of this last result can be found in [5]. At last we denote by $\mathcal{P}_N$ the complex vector space $\mathrm{vect}\{1, \chi, \ldots, \chi^N\}$.

Theorem 2 (Inversion formula) *Let f be a symbol satisfying:*

$$f > 0, f = g\bar{g}, \quad g \in H^\infty = H^+ \cap L^\infty(\mathbb{T}), \quad g^{-1} \in H^\infty.$$

Then

(i) $T_N(f) \in \mathrm{Aut}(\mathcal{P}_N)$

(ii) if p is an analytic polynomial, of degree less or equal at N, we get :

$$T_N(f)^{-1}(p) = \frac{1}{g}\pi_+\left(\frac{p}{\bar{g}}\right) - \frac{1}{g}\pi_+\left(\Phi_N(I - H^*_{\Phi_N}H_{\Phi_N})^{-1}\pi_+\left[\bar{\Phi}_N\,\pi_+\left(\frac{p}{\bar{g}}\right)\right]\right).$$

In [9] Widom use widely this formula to study the asymptotic behaviour of Toeplitz determinants with singular symbol It can be stated in term of an algebric formula based on operator inversion, this result is proved by Kozak [1]. The formulation used here can be find in Seghier's paper [5].

Remark 2 *In the following work we aim to get inversion formula with a singular symbol f, that has zero on the torus. In this case the previous formula cannot be applied directly. Parametrization of symbol (regularization by a positive parameter less than one) allows us to deal with a regular family of symbols. An asymptotic expansion of the entries of the inverse can be obtained when N goes to infinity and parameter goes to one. The first consequence is a exact expansion theorem:*

2.2 The proof

This part is devoted to the presentation of exact expression concerning the entries of $(T_N(f))^{-1}$ whith $f = |1-\chi|^2\frac{|Q|^2}{|P|^2}$,where P and Q are trigonometric polynomials respectively of degrees n_1 and n_2 with zeros outside of a closed centered in 0 disc with radius $R > 1$. The inverse formula stated in theorem 2 will be used in the following way: We have a positive symbol f,with decomposition $f = g\bar{g}$ where $g = (1-\chi)\frac{Q}{P}$. Noticing that $g, g^{-1} \in H^\infty$,write us

$$[T_N(f)^{-1}]_{k,l} = (T_1)_{k,l} - (T_2)_{k,l}$$

where

$$(T_1)_{k,l} = \langle \pi_+(\chi^k/\bar{g}), \pi_+(\chi^l/\bar{g})\rangle$$

and

$$(T_2)_{k,l} = \langle (I - H^*_{\Phi_N} H_{\Phi_N})^{-1} \pi_+ \bar{\Phi}_N \pi_+ (\chi^k/\bar{g}), \pi_+ \bar{\Phi_N} \pi_+ (\chi^l/\bar{g}) \rangle.$$

$(T_1)_{k,l}$ and $(T_2)_{k,l}$ will be called respectively first term and second term. We consider the regularised symbol $f_r(\chi) = |1 - r\chi|^2 \frac{|P(\chi)|^2}{|Q(\chi)^2|}$ with $0 < r < 1$.

2.2.1 Computation of the First Term of Inversion Formula.

Computation of $\pi_+(\chi^k/\bar{g})$. We get

$$\pi_+(\chi^k/\bar{g}) = \sum_{u=0}^{\infty} \bar{\beta}_u \pi_+(\chi^{k-u}/(1 - r\bar{\chi}))$$

But

$$\pi_+\left(\frac{\chi^{k-u}}{1 - r\chi}\right) = \begin{cases} \bar{\chi} \frac{\chi^{k-u+1} - r^{k-u+1}}{1 - r\bar{\chi}} & \text{if } u \leq k, \\ 0 & \text{if } u > k. \end{cases}$$

Then

$$\begin{aligned} \pi_+(\chi^k/\bar{g}) &= \sum_{u=0}^{k} \bar{\beta}_u \bar{\chi} \frac{\chi^{k-u+1} - r^{k-u+1}}{1 - r\bar{\chi}} \\ &= \sum_{s=0}^{k} \bar{\beta}_{k-s} \bar{\chi} \frac{\chi^{k-u+1} - r^{k-u+1}}{1 - r\bar{\chi}}. \end{aligned}$$

The first term of inversion formula is given by

$$\begin{aligned} (T_1)_{kl} &= \langle \pi_+(\chi^k/\bar{g}), \pi_+(\chi^l/\bar{g}) \rangle \\ &= \sum_{s'=0}^{l} \sum_{s=0}^{k} \bar{\beta}_{k-s} \beta_{l-s'} \langle (\chi^{s+1} - r^{s+1})/(1 - r\bar{\chi}), (\chi^{s'+1} - r^{s'+1})/(1 - r\bar{\chi}) \rangle. \end{aligned}$$

And $\langle (\chi^{s+1} - r^{s+1})/(1 - r\bar{\chi}), (\chi^{s'+1} - r^{s'+1})/(1 - r\bar{\chi}) \rangle = r^{|s-s'|} \left(1 - r^{2\min(s+1,s'+1)}/(1 - r^2)\right)$. Indeed, the previous scalar product can be expressed as:

$$\langle \chi^{s+1}(1 - (r\bar{\chi})^{s+1})/(1 - r\bar{\chi}), (\chi^{s'+1}(1 - (r\bar{\chi})^{s'+1})/(1 - r\bar{\chi}) \rangle.$$

And an elementary computation gives then $r^{s-s'}(1 - r^{2(s'+1)})/(1 - r^2)$ si $s \geq s'$, and we get

$$(T_1)_{kl} = \sum_{s=0}^{k} \sum_{s'=0}^{l} \bar{\beta}_{k-s} \beta_{l-s'} r^{|s-s'|} \frac{1 - r^{2\min(s+1,s'+1)}}{1 - r^2}.$$

2.2.2 Computation of the Second Term of Inversion Formula.

We begin with the computation of $\pi_+ \bar{\Phi}_N \pi_+ \left(\frac{\chi^k}{\bar{g}}\right) = x_k$. Here are introduced two notations :

$$g_1 = \frac{P}{Q} \text{ and } \frac{g_1}{\bar{g_1}} = \frac{\bar{P}Q}{P\bar{Q}} = \sum_{u=-m}^{\infty} \gamma_u \chi^u \text{ where } m = n_1 - n_2.$$

Lemma 1 *Vector* $\frac{1}{1-r\chi}$ *is eigenvector for* $(I - H^*_{\Phi_N} H_{\Phi_N})^{-1}$.

Proof : Vector $\frac{1}{1-r\chi}$ is eigenvector for $(I-H^*_{\Phi_N} H_{\Phi_N})^{-1}$ if $\frac{1}{1-r\chi}$ is eigenvector for $\left(H_{\Phi^*_N} H_{\Phi_N}\right)$. According to the following computation:

$$\pi_- \left(\chi^{N+1} \frac{g_1(\chi)}{\bar{g}_1(\chi)} \frac{1-r\chi}{1-r\bar{\chi}} \frac{1}{1-r\chi} \right) = \pi_- \left(\frac{g_1(\chi)}{\bar{g}_1(\chi)} \frac{\chi^{N+1}}{1-r\bar{\chi}} \right) =$$

$$\pi_- \left(\sum_{u \geq -m} \frac{\gamma_u \chi^{u+N+1}}{1-r\bar{\chi}} \right) = \sum_{u \geq -m} \gamma_u \frac{r^{u+N+2}\bar{\chi}}{1-r\bar{\chi}} = \frac{g_1(r)}{\bar{g}_1(1/r)} \frac{\bar{\chi} r^{N+2}}{1-r\bar{\chi}}.$$

So that

$$H_{\Phi^*_N} H_{\Phi_N} \left(\frac{1}{1-r\chi} \right) = \pi_+ \left(\bar{\chi}^{N+1} \frac{\bar{g}_1(\chi)}{g_1(\chi)} \frac{1-r\bar{\chi}}{1-r\chi} \frac{\bar{\chi}}{1-r\bar{\chi}} r^{N+2} \frac{g_1(r)}{\bar{g}_1(1/r)} \right) =$$

$$= r^{N+2} \frac{g_1(r)}{\bar{g}_1(1/r)} \pi_+ \left(\sum_{u \geq -m}^{\infty} \bar{\gamma}_u \frac{\bar{\chi}^{u+N+2}}{1-r\chi} \right) = r^{N+2} \frac{g_1(r)}{\bar{g}_1(1/r)} \left(\sum_{u \geq -m}^{\infty} \bar{\gamma}_u \frac{r^{u+N+2}}{1-r\chi} \right) =$$

$$= r^{N+2} \frac{g_1(r)}{\bar{g}_1(1/r)} \left(\sum_{u \geq -m}^{\infty} \bar{\gamma}_u r^u \right) \frac{1}{1-r\chi} = r^{2(N+2)} \frac{g_1(r)}{\bar{g}_1(1/r)} \frac{\bar{g}_1(r)}{g_1(1/r)} \frac{1}{1-r\chi} = r^{2(N+2)} \frac{1}{1-r\chi} G(r),$$

noticing

$$\frac{g_1(r)}{\bar{g}_1(1/r)} \frac{\bar{g}_1(r)}{g_1(1/r)} = G(r).$$

Observe that $G(1) = 1$ and $G'(1) = \mathcal{A}(P) = 4\Re\left(\frac{Q'(1)}{Q(1)} - \frac{P'(1)}{P(1)}\right)$. Indeed, one obtains with a straightforward computation

$$G'(1) = 2\frac{g_1(1)\bar{g_1}'(1) + g_1'(1)\bar{g_1}(1)}{|g_1|^2(1)} = 4\Re(\frac{g_1'(1)}{g_1(1)})$$

and

$$\frac{g_1'(1)}{g_1(1)} = \frac{Q'(1)}{Q(1)} - \frac{P'(1)}{P(1)}.$$

Put $x_k = x'_k + y_k$, where x'_k and y_k are respectively proportional and orthogonal to the vector $\frac{1}{1-r\chi}$ We get

$$\pi_+(\chi^k/\bar{g}) = \sum_{s=0}^{k} \bar{\beta}_{k-s} \bar{\chi}(\chi^{s+1} - r^{s+1})/(1-r\bar{\chi}).$$

and

$$\bar{\Phi}_N \pi_+ \left(\frac{\chi^k}{\bar{g}} \right) = \bar{\chi}^{(N+2)} \sum_{u \geq -m} \bar{\gamma}_u \, \bar{\chi}^u (1-r\bar{\chi})/(1-r\chi) \left(\sum_{s=0}^{k} \bar{\beta}_{k-s} (\chi^{s+1} - r^{s+1})/(1-r\bar{\chi}) \right),$$

finally

$$\pi_+ \bar{\Phi}_N \pi_+ \left(\frac{\chi^k}{\bar{g}} \right) = \sum_{u \geq -m} \pi_+ \left(\bar{\chi}^{N+2+u} \bar{\gamma}_u \sum_{s=0}^{k} \bar{\beta}_{k-s} (\chi^{s+1} - r^{s+1})/(1-r\chi) \right)$$

expanding the previous expression we get:

$$\sum_{s=0}^{k}\bar{\beta}_{k-s}\left(\sum_{u=-m}^{-N-2+s+1}\bar{\gamma}_u(\chi^{-(N+2+u)+s+1}-r^{N+2+u+s+1})/(1-r\chi)\right.$$

$$\left.+\sum_{u\geq\max(-m,-N-2+s+1)}\bar{\gamma}_u(r^{N+2+u-s-1}-r^{N+2+u+s+1})/(1-r\chi)\right).$$

Then, if $p=N+2+u$

$$x_k=\sum_{s=0}^{k}\bar{\beta}_{k-s}\left(\sum_{p=N+2-m}^{s+1}\bar{\gamma}_{p-(N+2)}\frac{\chi^{p-s-1}-r^{p+s+1}}{1-r\chi}+\sum_{p=s+2}^{\infty}\bar{\gamma}_{p-(N+2)}\frac{r^{p-s-1}-r^{p+s+1}}{1-r\chi}\right).$$

Put $x_k=x'_k+y_k$ with $x'_k=(A_{k,N,r})/(1-r\chi)$ where

$$y_k=\sum_{s=0}^{k}\bar{\beta}_{k-s}\sum_{p=N+2-m}^{s+1}\bar{\gamma}_{p-(N+2)}\frac{\bar{\chi}^{s+1-p}-r^{s+1-p}}{1-r\chi}$$

and

$$\begin{aligned}A_{N,k,r} &= \sum_{s=0}^{k}\bar{\beta}_{k-s}\Big(r^{s+1}\sum_{p=N+2-m}^{s+1}\bar{\gamma}_{p-(N+2)}(r^{-p}-r^{p})\\ &+ \Big(\sum_{p>s+1}\bar{\gamma}_{p-(N+2)}r^{p}(r^{-(s+1)}-r^{s+1})\Big).\end{aligned}$$

A direct computation gives

$$\left\langle\frac{1}{1-r\chi},y_k\right\rangle=y_k(r)=0.$$

The announced decomposition is thus obtained.

Remark 3 *At this stage of the proof, it's important to notice that for* $0\leq k\leq N-(n_1-n_2)=N-m$, *the term* y_k *is* 0. *In fact, in this case, the index* $p-(N+2)$ *is less than* $-m$: $p-(N+2)\leq(k+1)-(N+2)<-m$.

Lemma 2 *The second term is given by:*

$$\left\langle\left(I-H_{\Phi_N^*}H_{\Phi_N}\right)^{-1}x'_k,x'_l\right\rangle+\langle y_k,y_l\rangle\,.$$

Proof : To compute $H_{\Phi_N}(y_k)=\pi_-(\Phi_N y_k)$, we have to evaluate, using linearity of projection, the quantity:

$$\pi_-\left(\left(\sum_{u\geq-m}\gamma_u\chi^u\chi^{N+1}\frac{1-r\chi}{1-r\bar{\chi}}\right)\left(\frac{\bar{\chi}^{s+1-p}-r^{s+1-p}}{1-r\chi}\right)\right)$$

or, equivalently

$$\pi_-\left(\left(\sum_{u\geq-m}\gamma_u\frac{\chi^{N+1+u}}{1-r\bar{\chi}}\right)\left(\bar{\chi}^{s+1-p}-r^{s+1-p}\right)\right);$$

with $-p+s+1>0,\ N+2+u>0,\ N+2+u-p+s+1>0$. Or:

$$\begin{aligned}\pi_-\left(\left(\chi^{N+2+u-p+s}-r^{-p+s+1}\chi^{N+1+u}\right)/(1-r\bar{\chi})\right) &= \\ \bar{\chi}\left(\left(r^{N+2+u-p+s+1}-r^{-p+s+1}r^{N+2+u}\right)/(1-r\bar{\chi})\right) &= 0.\end{aligned}$$

Hence the second term is :

$$\left\langle \left(I-H_{\Phi_N^*}H_{\Phi_N}\right)^{-1}(x'_k)+y_k,(x'_l+y_l)\right\rangle.$$

We conclude with the lemma 1 □

We are able now to get the announced result by direct computations. The function $\dfrac{1}{1-r\chi}$ being an eigenvector of $H^*_{\Phi_N}H_{\Phi_N}$ associated eigenvalue $r^{2(N+2)}G(r)$ we get

$$\left\langle \left(I-H^*_{\Phi_N}H_{\Phi_N}\right)^{-1}(x'_k),(x'_l)\right\rangle = \frac{A_{k,N,r}\overline{A_{l,N,r}}}{1-r^2}\,\frac{1}{1-r^{2(N+2)}G(r)}.$$

We have to compute $\langle y_k,y_l\rangle$ in term of :

$$\left\langle \frac{\chi^{-p+s+1}-r^{-p+s+1}}{1-r\chi},\frac{\chi^{-p'+s'+1}-r^{-p'+s'+1}}{1-r\chi}\right\rangle.$$

Using the following equalities of scalar products

$$\left\langle \frac{\chi^v-r^v}{1-r\bar{\chi}},\frac{\chi^{v'}-r^{v'}}{1-r\chi}\right\rangle = \left\langle \frac{\chi^v-r^v}{1-r\bar{\chi}}\frac{1-r\bar{\chi}}{1-r\chi},\frac{\chi^{v'}-r^{v'}}{1-r\bar{\chi}}\frac{1-r\bar{\chi}}{1-r\chi}\right\rangle = \left\langle \frac{\chi^v-r^v}{1-r\bar{\chi}},\frac{\chi^{v'}-r^{v'}}{1-r\bar{\chi}}\right\rangle,$$

we get $r^{v-v'}\frac{1-r^{2(v'+1)}}{1-r^2}$ if $v'\le v$. Therefore

$$\langle y_k,y_l\rangle = \sum_{s=0}^{k}\sum_{s'=0}^{l}\bar{\beta}_{k-s}\beta_{l-s}\sum_{p=N+2-m}^{s+1}\sum_{p'=N+2-m}^{s'+1}\bar{\gamma}_{p-(N+2)}\gamma_{p'-(N+2)}r^{|p-p'+s'-s|}r^2\frac{1-r^{2\inf(s-p+1,s'-p'+1)}}{1-r^2}.$$

With the notation $\langle y_k,y_l\rangle = c_{k,l,r}$ we obtain the following $\left[(T_Nf_r)^{-1}\right]_{k,l}$ pour $r<1$:

$$\begin{aligned}\left[(T_Nf_r)^{-1}\right]_{k+1,l+1} &= \sum_{s=0}^{k}\sum_{s'=0}^{l}\bar{\beta}_{k-s}\beta_{l-s}r^{|s-s'|}\frac{1-r^{2\min(s+1,s'+1)}}{1-r^2} \\ &+ \frac{A_{k,N,r}\overline{A_{l,N,r}}}{1-r^2}\frac{1}{1-r^{2(N+2)}G(r)}+c_{k,l,r}.\end{aligned}$$

Observee that

$$\lim_{r\to 1}\frac{A_{N,k,r}}{1-r^2}=\sum_{s=0}^{k}\bar{\beta}_{k-s}\left(\sum_{p=0}^{s+1}\bar{\gamma}_{p-(N+2)}p+(s+1)\sum_{p>s+1}\bar{\gamma}_{p-(N+2)}\right),$$

Finally we get

$$\lim_{r\to 1}\left[(T_N f_r)^{-1}\right]_{k+1,l+1} = a_{k,l} - \frac{d(k)\bar{d}(l)}{N+2+\mathcal{A}(P)} + c_{k,l}.$$

where $a_{k,l}, d(k), d(l), c_{k,l}, \mathcal{A}(P)$ are given by the following expressions:

$$\mathcal{A}(P,Q) = 4\Re\Big(\frac{Q'(1)}{Q(1)} - \frac{P'(1)}{P(1)}\Big),$$

$$\begin{aligned} c_{k,l} &= \lim_{r\to 1} c_{k,l,r} = \\ &= \sum_{s=0}^{k}\sum_{s'=0}^{l} \bar{\beta}_{k-s}\beta_{l-s'} \sum_{p=N+2-m}^{s+1}\ \sum_{p'=N+2-m}^{s'+1} \bar{\gamma}_{p-(N+2)}\gamma_{p'-(N+2)} \inf(s-p+1, s'-p'+1), \end{aligned} \tag{4}$$

$$a_{k,l} = a_{kl} = \sum_{s'=0}^{l}\sum_{s=0}^{k} \bar{\beta}_{k-s}\beta_{l-s'} \min(s+1, s'+1),$$

$$d(k) = \sum_{s=0}^{k}\left[\bar{\beta}_{k-s}\Big(\sum_{p=1}^{s+1} p\bar{\gamma}_{p-(N+2)} - (s+1)\sum_{p=s+2}^{\infty}\bar{\gamma}_{p-(N+2)}\Big)\right] \tag{5}$$

One has still to prove that: $\lim_{r\to 1}(T_N f_r)^{-1}_{k+1,l+1} = (T_N f)^{-1}_{k+1,l+1}$. Put

$$(T_N f_r)^{-1}(T_N f) = (T_N f_r)^{-1}(T_N f_r) + (T_N f_r)^{-1}(T_N(f-f_r)).$$

With fixed N we have $\lim_{r\to 1}(T_N(f-f_r)) = 0$, then $\lim_{r\to 1}(T_N f_r)^{-1}(T_N f) = I_N$. To obtain the definitive proof, it remains to show that the expession of $d(k)$ is as the announced one and that $c_{k,l}$ in fact 0. Taking in account the hypothesis $0 \le k \le N+1+(n_2-n_1)$ the following inequalities hold in the second summation of the equation (5) :

$$s+1-(N+2) \le k+1-N-2 = k-N-1 \le N+(n_2-n_1)-N-1,$$

and so $s+1-(N+2) \le (n_2-n_1)-1$ so that $\gamma_{s+1-(N+2)} = 0$ and

$$d(k) = -\sum_{s=0}^{k}\bar{\beta}_{k-s}(s+1)\sum_{p=s+2}^{\infty}\bar{\gamma}_{p-(N+2)}.$$

Noticing that $\gamma_{(s+2)-(N+2)} = \gamma_{s-N}$ and $-N \le s-N \le k-N < n_2-n_1$, one can express the third summation of (5) in this manner :

$$\sum_{p=s+2}^{\infty}\bar{\gamma}_{p-(N+2)} = \overline{\left(\frac{\bar{P}Q}{P\bar{Q}}\right)}(1) = \frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)}.$$

Weighing it all up, we are allowed to get the desired equation :

$$\begin{aligned} d(k) &= -\frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)}\sum_{s=0}^{k}(s+1)\bar{\beta}_{k-s} = \\ &= -\frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)}\sum_{u=0}^{k}(k+1-u)\bar{\beta}_{u} = \\ &= -\frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)}(k+1)\sum_{u=0}^{k}\bar{\beta}_{u} + \frac{P(1)\bar{Q}(1)}{\bar{P}(1)Q(1)}\sum_{u=0}^{k}u\bar{\beta}_{u}. \end{aligned}$$

In the polynomial case, for $k \geq n_1$, we get

$$d(k) = -\sum_{s=0}^{k}\bar{\beta}_{k-s}\frac{P(1)}{\bar{P}(1)} = -\sum_{s=0}^{m}\bar{\beta}_{k-s}(s+1)\frac{P(1)}{\bar{P}(1)},$$

or equivalently

$$d(k) = -(k+1)P(1) + \frac{\bar{P}'(1)P(1)}{\bar{P}(1)}$$

The arguments of the remark 3 shows that the coefficients $c_{k,l}$ are zero as soon as k or l is $\leq N-m$. On the contrary, when k and l are $> N-m$ then the summation of the formula (4) contains at the very most m^2 terms : only the β_u with $0 < u < m$ and the γ_v whith $-m < v < 0$ exist. This proves the proposition: in this case $c_{k,l} = O(1)$. □

3 Proof of the trace theorem

The principle of this proof is to compute successively the terms of the trace coming from the first term of the inversion formula and those ones coming from the second term.

3.1 Contribution of the first term

With k as in the main theorem the first term is :

$$\begin{aligned} a_{kk} &= \sum_{s=0}^{k}\sum_{s'=0}^{k}\bar{\beta}_{k-s}\beta_{k-s'}\min(s+1, s'+1) = \sum_{u=0}^{k}\sum_{u'=0}^{k}\bar{\beta}_{u}\beta_{u'}\min(k-u+1, k-u'+1) \\ &= \tau_1(k) - \sigma_1(k), \end{aligned}$$

if we set :

$$\tau_1(k) = (k+1)\sum_{u=0}^{k}\sum_{u'=0}^{k}\bar{\beta}_{u}\beta_{u'}, \text{ and}$$

$$\tau_2(k) = \sum_{u=0}^{k}\sum_{u'=0}^{k}\bar{\beta}_{u}\beta_{u'}\max(u, u').$$

Let us examine $\tau_1(k)$:

$$\frac{\tau_1(k)}{k+1} = \sum_{u=0}^{\infty}\sum_{u'=0}^{\infty} \bar{\beta}_u\beta_{u'} - \underbrace{\sum_{u=k+1}^{\infty}\sum_{u'=k+1}^{\infty} \bar{\beta}_u\beta_{u'}}_{\sigma_1(k)}$$
$$- \underbrace{\sum_{u=0}^{k+1}\sum_{u'=k+1}^{\infty} \bar{\beta}_u\beta_{u'}}_{\mu_1(k)} - \underbrace{\sum_{u=k+1}^{\infty}\sum_{u'=0}^{k+1} \bar{\beta}_u\beta_{u'}}_{\nu_1(k)}$$

A direct computation leads to the following estimations :

$$\sum_{u=0}^{\infty}\sum_{u'=0}^{\infty} \bar{\beta}_u\beta_{u'} = \big|\frac{P}{Q}(1)\big|^2,$$
$$\big|\sigma_1(k)\big| \leq \frac{a^2}{R^{2(k+1)}}$$

where a is a constant term. Similarly, there exists constants reals numbers b and c such that

$$\big|\mu_1(k)\big| \leq \frac{b}{R^{2(k+1)}} \text{ and } \big|\nu_1(k)\big| \leq \frac{c}{R^{2(k+1)}}.$$

Now, let us examine $\tau_2(k)$. The principle of its decomposition is similar :

$$\tau_2(k) = \sum_{u=0}^{\infty}\sum_{u'=0}^{\infty} \bar{\beta}_u\beta_{u'}\max(u,u') - \underbrace{\sum_{u=k+1}^{\infty}\sum_{u'=k+1}^{\infty} \bar{\beta}_u\beta_{u'}\max(u,u')}_{\sigma_2(k)}$$
$$- \underbrace{\sum_{u=k+1}^{\infty}\sum_{u'=0}^{k+1} \bar{\beta}_u\beta_{u'}\max(u,u')}_{\mu_2(k)} - \underbrace{\sum_{u=0}^{k+1}\sum_{u'=k+1}^{\infty} \bar{\beta}_u\beta_{u'}\max(u,u')}_{\nu_2(k)}.$$

Clearly the term $\sum_{u=0}^{\infty}\sum_{u'=0}^{\infty} \bar{\beta}_u\beta_{u'}\max(u,u')$ is a real number depending only on the polynomials P and Q. For the second, we have directly

$$\sigma_2(k) = \sum_{u=k+1}^{\infty} u\bar{\beta}_u \sum_{u'=k+1}^{u} \bar{\beta}_{u'} + \sum_{u=k+1}^{\infty} \bar{\beta}_u \sum_{u'=u+1}^{\infty} u'\beta_{u'}.$$

Then for a contant d, the following majoration holds : $\big|\sigma_2(k)\big| \leq \frac{d}{R^{k+1}}$. The terms $\mu_2(k)$ and $\nu_2(k)$ are of the same order. It remains to stydy only one of this two numbers, for example $\nu_2(k)$:

$$\big|\nu_2(k)\big| = \sum_{u=0}^{k+1}\sum_{u'=k+1}^{\infty} \bar{\beta}_u\beta_{u'}\max(u,u') = \sum_{u=0}^{k+1} \bar{\beta}_u \sum_{u'=k+1}^{\infty} u'\beta_{u'}$$

and so

$$\big|\nu_2(k)\big| \leq \sum_{u=0}^{k+1} |\bar{\beta}_u| \sum_{u=k+1}^{\infty} \frac{u'}{R^u}$$

A straightforward computation leads then to the majoration

$$\big|\nu_2(k)\big| \leq e\ \frac{k+1}{R^k} \text{ for a constant } \ e.$$

In the same manner :

$$\big|\mu_2(k)\big| \leq f\ \frac{k+1}{R^k} \text{ for a constant } \ f.$$

We are able now to compute the contribution, denoted T_1, of the first term to the trace :

$$T_1 = \sum_{k=0}^{N} \tau_1(k) - \sum_{k=0}^{N} \tau_2(k).$$

A short computation shows that

$$\sum_{k=0}^{N} \tau_1(k) = \frac{(N+1)(N+2)}{2}\big|\frac{P}{Q}(1)\big|^2 + R_1 \text{ where } R_1 \text{ is bounded}$$

$$\textit{and}$$

$$\sum_{k=0}^{N} \tau_2(k) = (N+1)C(P,Q) + R_2 \text{ where } R_1 \text{ is bounded.}$$

so that $T_1 = \frac{N^2}{2}\big|\frac{P}{Q}(1)\big|^2 + N\Big(\frac{3}{2}\big|\frac{P}{Q}(1)\big|^2 - C(P,Q)\Big) + O(1)$.

3.2 Contribution of the second term

This contribution, denoted T_2, is given by

$$T_2 = -\frac{1}{N+2+\mathcal{A}} \sum_{k=0}^{N} |d(k)|^2.$$

Let us denote n_0 the integer part of the real number $N^{1/\alpha}$ where α is a positiv integer having to be determined later. We can whrite now :

$$T_2 = -\frac{1}{N+2+\mathcal{A}} \underbrace{\sum_{k=n_0}^{N} |d(k)|^2}_{S_2(N)} - \frac{1}{N+2+\mathcal{A}} \underbrace{\sum_{k=0}^{n_0-1} |d(k)|^2}_{S_1(N)}.$$

We obtain then

$$S_2(N) = \big|\frac{P}{Q}(1)\big|^2 \left(\frac{(N+1)(N+2)(2N+3)}{6} - (N+1)(N+2)\Re\Big(\frac{P'}{\bar{P}}(1) - \frac{Q'}{\bar{Q}}(1)\Big) + O(n_0^3)\right).$$

This term being divised by N, the rest $\frac{O(n_0^3)}{N+2+\mathcal{A}}$ is of order smaller than a constant if one takes $\alpha > 3$. Furthermore a straitforward computation shows that $S_1(N)$ is bounded. The contribution of the second term is then :

$$T_2 = -\frac{1}{N+2+\mathcal{A}}\big|\frac{P}{Q}(1)\big|^2 \left(\frac{(N+1)(N+2)(2N+3)}{6}\big|\frac{P}{Q}(1)\big|^2 - \right.$$
$$\left. - (N+1)(N+2)\Re\Big(\frac{P'}{\bar{P}}(1) - \frac{Q'}{\bar{Q}}(1)\Big)\right) + O(1),$$

that we stil write

$$T_2 = \Big|\frac{P}{Q}(1)\Big|^2 \left(-\frac{N^2}{3} - N\big(1 - \Re\Big(\frac{P'}{\bar{P}}(1) - \frac{Q'}{\bar{Q}}(1)\Big) + \frac{2+\mathcal{A}}{6} + O(1)\right).$$

We can conclude at last :

$$\mathrm{Tr}\big(T_N(f)^{-1}\big) = T_1 + T_2 = \frac{N^2}{6}\Big|\frac{P}{Q}(1)\Big|^2 + N\left(\Big|\frac{P}{Q}(1)\Big|^2\Big[\frac{2+\mathcal{A}}{3} + \Re\Big(\frac{P'}{\bar{P}}(1) - \frac{Q'}{\bar{Q}}(1)\Big)\Big] - C(P,Q)\right) + O(1).$$

This achieves the proof of the trace theorem, taking in account the proposition.

4 Proof of the corollary 4

4.1 Contribution of the first term

By symmetry we can write

$$\sum_{k=0}^{N}\sum_{l=0}^{N} a_{k,l} = 2\Re\left(\sum_{l=0}^{N}\sum_{k=0}^{l-1} a_{k,l}\right) + \sum_{k=0}^{N} a_{k,k}.$$

If $k < l$ we can write

$$\begin{aligned}
a_{k,l} &= \sum_{s=0}^{k}\sum_{s'=0}^{l} \bar{\beta}_{k-s}\beta_{l-s'}\min(s+1, s'+1) \\
&= \sum_{s=0}^{k}\bar{\beta}_{k-s}\sum_{s'=0}^{s}\beta_{l-s'}(s'+1) + \sum_{s=0}^{k}\bar{\beta}_{k-s}(s+1)\sum_{s'=s+1}^{l}\beta_{l-s'} \\
&= \sum_{s=0}^{k}\bar{\beta}_{k-s}\sum_{s'=0}^{s}\beta_{l-s'}(l+1) - \sum_{s=0}^{k}\bar{\beta}_{k-s}\sum_{s'=0}^{s}\beta_{l-s'}(l-s') \\
&+ \sum_{s=0}^{k}\bar{\beta}_{k-s}(k+1)\sum_{s'=s+1}^{l}\beta_{l-s'} - \sum_{s=0}^{k}\bar{\beta}_{k-s}(k-s)\sum_{s'=s+1}^{l}\beta_{l-s'} \\
&= (l+1)\sum_{u=0}^{k}\bar{\beta}_u\sum_{u'=l-k+u}^{l}\beta_{u'} - \sum_{u=0}^{k}\bar{\beta}_u\sum_{u'=l-k+u}^{l}u'\beta_{u'} \\
&+ (k+1)\sum_{u=0}^{k}\bar{\beta}_u\sum_{u'=0}^{l-k+1+u}\beta_{u'} - \sum_{u=0}^{k}u\bar{\beta}_u\sum_{u'=0}^{l-k+1+u}\beta_{u'}
\end{aligned}$$

Lastly we obtain if $k = [Nx]$, $l = [Ny]$, $x < y$

$$a_{k,l} = (k+1)\Big|\frac{P(1)}{Q(1)}\Big|^2 - \overline{\left(\frac{P}{Q}\right)'(1)}\frac{P(1)}{Q(1)} + o(\frac{l}{R^{l-k}}).$$

Hence we have

$$2\Re\left(\sum_{l=0}^{N}\sum_{k=0}^{l-1}a_{k,l}\right) = 2\Re(\sum_{l=0}^{N}\sum_{k=0}^{l-1}((k+1)C_1+C_2)+o(\frac{l}{R^{l-k}})$$
$$= \left(\frac{(2N+1)(N+1)N}{6}+\frac{N(N+1)}{2}\right)C_1+N(N+1)\Re(C_2)+o(1)$$

with

$$C_1 = |\frac{P(1)}{Q(1)}|^2$$

$$C_2 = -\overline{(\frac{P}{Q})'(1)}\frac{P(1)}{Q(1)}$$

and

$$\Re(C_2) = \frac{1}{4}\mathcal{A}C_1.$$

On the other hand we have, using a computation already made in the previous part,

$$\sum_{k=0}^{N}a_{k,k} = \frac{N^2}{2}\left|\frac{P}{Q}(1)\right|^2+N\Big(\frac{3}{2}\left|\frac{P}{Q}(1)\right|^2-C(P,Q)\Big)+O(1).$$

And if S_1 denote the contribution of the first term to the sum we have

$$S_1 = \frac{N^3}{3}C_1+\left(\frac{3}{2}C_1+\Re(C_2)\right)N^2+O(N).$$

4.2 Contribution of the second term

We may now compute the sum

$$S_2 = \left(\sum_{k=1}^{N}d(k)\right)\left(\sum_{l=1}^{N}\bar{d}(l)\right)\frac{1}{N+2+\mathcal{A}}.$$

Using asymptotic expansion of d_k we can write

$$\sum_{k=1}^{N}d(k) = \sum_{k=1}^{N}-(k+1)\frac{P(1)}{Q(1)}+\frac{P(1)}{Q(1)}B(P,Q)+o(\frac{1}{R^k})$$

where $B(P,Q) = \big(\frac{P'(1)}{P(1)}-\frac{Q'(1)}{Q(1)}\big)$. That gives us

$$\sum_{k=1}^{N}d(k) = \frac{P(1)}{Q(1)}\left(-N^2\frac{1}{2}+\left(-\frac{3}{2}+B(P,Q)\right)N\right)+O(1).$$

Then we obtain

$$\left(\sum_{k=1}^{N}d(k)\right)\left(\sum_{l=1}^{N}\bar{d}(l)\right) = C_1\left((\frac{N^4}{4}+\left(\frac{3}{2}-\Re(B(P,Q))\right)N^3\right)+O(N^2).$$

and also

$$S_2 = C_1 \left(\frac{N^4}{4} + \left(\frac{3}{2} - \Re(B(P,Q)) \right) N^3 \right) \left(\frac{1}{N} - \frac{\mathcal{A}+2}{N^2} \right) + O(N)$$

And, lastly, we can write

$$S_2 = C_1 \left(\frac{N^3}{4} + \left(1 - \Re(B(P,Q)) - \frac{\mathcal{A}}{4} \right) N^2 \right) + O(N).$$

4.3 Conclusion.

An easy computation prove

$$S_1 - S_2 = \frac{C_1}{12} N^3 + N^2 C_1 \left(\frac{1+\mathcal{A}}{2} + \Re(B(P,Q)) \right) + O(N),$$

that we have wroten:

$$T_1 - T_2 = \frac{C_1}{12} N^3 + N^2 C_1 \left(\frac{1+\mathcal{A}}{2} + \mathcal{B}(P,Q)) \right) + O(N)$$

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