Non-uniform Hyperbolicity and Universal Bounds for S-unimodal Maps

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Dedicated to the memory of Wiesław Szlenk

Abstract

An S-unimodal map f is said to satisfy the Collet-Eckmann condition if the lower Lyapunov exponent at the critical value is positive. If the infimum of the Lyapunov exponent over all periodic points is positive then f is said to have a uniform hyperbolic structure. We prove that an S-unimodal map satisfies the Collet-Eckmann condition if and only if it has a uniform hyperbolic structure. The equivalence of several non-uniform hyperbolicity conditions follows. One consequence is that an S-unimodal map has an absolutely continuous invariant probability measure with exponential decay of correlations if and only if the Collet-Eckmann condition is satisfied. The proof uses new universal bounds that hold for any S-unimodal map without periodic attractors.

The theory of uniformly hyperbolic maps is rich and well understood. Many results are known on the existence of invariant measures, rates of mixing, structural stability and smoothness of conjugacies, amongst others [36, 31, 34]. The methods of the uniform theory break down when studying the dynamics of smooth interval maps, since interval maps with non-trivial dynamics have critical points. This renders uniform hyperbolicity impossible. Nonetheless, a number of results reminiscent of the uniform theory have been proved in recent years for interval maps satisfying non-uniform hyperbolicity conditions. These results generally require uniform expansion (positive Lyapunov exponents) on certain dynamically significant subsets of the interval, for example at the critical values or on the periodic points.

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The simplest interval maps with critical points are unimodal maps [7, 5]. A unimodal map is said to satisfy the Collet-Eckmann condition if the lower Lyapunov exponent is positive at the critical value [6, 7]. Such maps are known to exhibit a lot of hyperbolic behavior: an S-unimodal map satisfying the Collet-Eckmann condition has positive Lyapunov exponent at Lebesgue almost every point of the interval [28]. It has a uniform hyperbolic structure on the set of periodic points (a uniform hyperbolic structure for short): the infimum of the Lyapunov exponent over the set of periodic points is positive [25, 28].

An S-unimodal map satisfying the Collet-Eckmann condition has a unique invariant probability measure which is absolutely continuous with respect to Lebesgue measure (an *acip* for short) [6, 26, 7]. This measure has exponential decay of correlations [17, 38]. If two Collet-Eckmann maps are topologically conjugate then the conjugacy is Hölder continuous. Indeed, some renormalization is bi-Hölder conjugate to a tent-map (a uniformly expanding model) [29].

It is a remarkable fact that many of these hyperbolicity properties are actually equivalent to the Collet-Eckmann condition:

Theorem A *Suppose f is S-unimodal. Then the following are equivalent:*

1. *f satisfies the Collet-Eckmann condition.*
2. *f has an acip with exponential decay of correlations.*
3. *Some renormalization of f is bi-Hölder conjugate to a tent-map with positive topological entropy.*
4. *f has a uniform hyperbolic structure (on the periodic points).*
5. *f satisfies the backward Collet-Eckmann condition.*
6. *f has exponential decay of cylinders.*
7. *f is semi-uniformly expanding.*
8. *f has a central hyperbolic structure.*
9. *f is uniformly expanding on the c-level.*
10. *f is centrally expanding.*

The last five conditions are technical. The definitions of these properties can be found in the following places: the Collet-Eckmann condition in section 6; exponential decay of correlations and also the terms used in (3) in section 8; uniform hyperbolic structure, exponential decay of cylinders, central hyperbolic structure, uniform expansion on the c-level and central expansion in section 5;

the backward Collet-Eckmann condition and semi-uniform expansion in section 7. Note that we do not claim in theorem A that the various growth/decay rates are the same.

The main difficulty is to prove the implication ($4 \Rightarrow 1$) in theorem A:

Theorem B Suppose f is S-unimodal and has a uniform hyperbolic structure. Then f satisfies the Collet-Eckmann condition.

This is proved in section 6. The rest of theorem A follows readily from this fundamental result and some known or easily proved implications. For example, an S-unimodal map satisfying the Collet-Eckmann condition was known to have an acip with exponential decay of correlations [38, 17]. It is not hard to show that if an S-unimodal map has an acip with exponential decay of correlations, then it has a uniform hyperbolic structure, see section 8. Therefore the equivalence of the Collet-Eckmann condition and exponential decay of correlations follows from theorem B. This "sandwich" technique of showing that a property is implied by the Collet-Eckmann condition, and that it implies a uniform hyperbolic structure is a powerful way of proving equivalence to the Collet-Eckmann condition.

The acip of an S-unimodal map satisfying the Collet-Eckmann condition is known to be strongly stochastically stable (given some mild additional hypotheses) [1], satisfies central limit theorems [17, 38] and is the limit of the measures obtained by distributing mass over the periodic orbits in inverse proportion to their multipliers [17]. We do not know if any of these properties are equivalent to the Collet-Eckmann condition: we were unable to prove that they imply uniform hyperbolic structure. Probably some conditions are needed on convergence rates.

Theorem A is additionally interesting since the Collet-Eckmann condition is conjectured to be a topological invariant of S-unimodal maps [37, 12]. If so, then, for example, the property of having an acip with exponential decay of correlations would be topologically invariant. The topological invariance of the Collet-Eckmann condition is related to the so-called rigidity conjecture for S-unimodal maps: that any two such maps that are topologically conjugate are quasisymmetrically conjugate [20]. Indeed, the Collet-Eckmann condition is known to be invariant under quasisymmetric conjugacy [33]. It follows from theorem A that the Collet-Eckmann condition is even invariant under Hölder conjugacy. The "sandwich" technique described above clearly gives a strategy for proving the topological invariance of the Collet-Eckmann condition: find a topological condition implied by Collet-Eckmann, and show that it implies uniform hyperbolic structure.[1]

[1] The first author and Feliks Przytycki have recently announced a proof of the topological invariance of the Collet-Eckmann condition for S-unimodal maps using this method [30].

Collet-Eckmann maps are known to be abundant in the real quadratic family $Q_a : x \mapsto 4ax(1-x)$: the set of parameters for which Q_a satisfies the Collet-Eckmann condition (i.e. has positive lower Lyapunov exponent at the critical value) is of positive Lebesgue measure [15, 2, 7]. On the other hand, the set of parameters a for which Q_a has a hyperbolic attracting periodic orbit is open and dense [10]. For these maps the lower Lyapunov exponent at the critical value is negative. There also exist parameters a for which the lower Lyapunov exponent at the critical value is zero. We conjecture that Q_a has non-zero lower Lyapunov exponent for Lebesgue almost every value of a.[2] The Collet-Eckmann condition is also abundant in a topological sense: it is known that there exists a set $E \subseteq [0, \log 2]$ of full Lebesgue measure with the property that any S-unimodal map with topological entropy in E satisfies the Collet-Eckmann condition [32, 7].

The proof of theorem A relies heavily on new universal derivative and distortion bounds that hold for any S-unimodal map without periodic attractors. They should also be useful for proving other rigidity results. In appendix B we show that they imply the classical real a priori bounds for renormalizable maps.

The interaction between our results and those recently obtained in the complex plane by Jacek Graczyk and Stas Smirnov [9] is particularly interesting (we received their preprint while preparing this paper). For example, combining their results and ours enables one to prove that the exterior of the Julia set of a real quadratic map with critical point in the Julia set is a Hölder domain if and only if the Collet-Eckmann condition is satisfied. We conjecture that this is also true for complex quadratic maps. Graczyk and Smirnov prove their results for rational maps of the Riemann sphere. They show a number of consequences of the Collet-Eckmann condition and that the Collet-Eckmann condition is equivalent to the backward Collet-Eckmann condition in the Riemann sphere (taking all inverse images of the critical points in the Riemann sphere). Our backward Collet-Eckmann condition is somewhat weaker, since we only put hypotheses on real inverse images of the critical point, and indeed our major difficulty is in a sense due to a lack of hypotheses on inverse images off the real line.

Proof of theorem A. The proof of theorem A is distributed as follows: for ($1 \Leftrightarrow 2$) and ($1 \Leftrightarrow 3$), see theorem 8.5. For ($1 \Leftrightarrow 4$), see theorem 6.11. For ($1 \Leftrightarrow 7$) see theorem 7.7. For ($1 \Rightarrow 10$), see the proof of theorem 6.11. For ($2 \Rightarrow 4$), see lemma 8.2. For ($3 \Rightarrow 4$), see lemma 8.4. For ($3 \Rightarrow 6$), see the proof of lemma 8.4. For ($4 \Leftrightarrow 5$) and ($4 \Leftrightarrow 7$), see theorem 7.7. By definition, ($4 \Rightarrow 8$). For ($4 \Leftrightarrow 6 \Leftrightarrow 8 \Leftrightarrow 9 \Leftrightarrow 10$), see theorem 5.3. For ($5 \Leftrightarrow 7$), see theorem 7.7.

[2] A weaker conjecture is that Q_a has a hyperbolic periodic attractor or an acip for Lebesgue almost every a. While every Collet-Eckmann S-unimodal map has an acip, there exist S-unimodal maps with acips which do not satisfy the Collet-Eckmann condition [4].

The implication ($9 \Rightarrow 5$) is clear from the minimum principle, lemma A.6(1).

Some parts of theorem A were already known. For ($1 \Rightarrow 2$), see [17, 38]; for ($1 \Rightarrow 3$), see [29]; for ($1 \Rightarrow 4$), see [28]; for ($4 \Leftrightarrow 5$), see [25]; for ($10 \Leftrightarrow 9 \Rightarrow 6 \Rightarrow 4$), see [25, 28].

Wiesław Szlenk (1935-1995). Wiesław Szlenk was my (Tomasz Nowicki's) PhD supervisor. He taught me dynamical systems theory, introduced me to the subject of interval dynamics and suggested the Collet-Eckmann $\Rightarrow$ backward Collet-Eckmann conjecture as the subject of my thesis.

Wiesław was one of the founders of dynamical systems theory in Poland, a bright and broad-minded mathematician with a lot of ideas which he shared with his students, a wonderful and patient teacher, and a friend who left us only too early.

It is the pleasure of both authors to dedicate this paper to him. We prove the inverse of the conjecture, and close the circle.

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Structure of the paper. In section 1 we define the terms and notation used throughout the paper.

Section 2 describes the combinatorics of S-unimodal maps. We introduce cutting and co-cutting times ($\mathcal{S}$- and $\mathcal{T}$-times), and prove their important combinatorial properties.

Throughout the paper we use a number of universal distortion and derivative bounds. The universal distortion bounds are proved in section 3 and the universal derivative bounds in section 4.

In section 5 we prove the equivalence of uniform hyperbolic structure with central expansion, exponential decay of cylinders, central hyperbolic structure and uniform expansion on the c-level.

Section 6 is the core of the paper and depends heavily on the preceding sections. In it we prove that uniform hyperbolic structure is equivalent to the Collet-Eckmann condition. The proof is quite involved, so we have summarized the strategy at the beginning of the section.

Using the tools developed in the previous sections, it is relatively straightforward to prove that the Collet-Eckmann condition is equivalent to an a priori stronger condition: semi-uniform expansion. We do this in section 7, where we also prove the equivalence of uniform hyperbolic structure with the backward

Collet-Eckmann condition.

In section 8 we prove the equivalence of exponential correlation decay, bi-Hölder conjugacy with a tent-map and the Collet-Eckmann condition. In fact, we just show that these first two conditions imply a uniform hyperbolic structure; they were already known to be implied by the Collet-Eckmann condition.

In appendix A we list a number of important properties of S-unimodal maps and prove some miscellaneous technical results.

In appendix B we derive the classical real a priori bounds for renormalizable maps from the distortion theorem of section 3.

1 Definitions and notation

We use $[a, b]$ to denote the smallest interval containing a and b, so $[a, b] = [b, a]$. By (a, b) we mean the interior of $[a, b]$. We denote the Lebesgue measure of a measurable set I by $|I|$. The derivative of a function f is written Df.

Definition. A *unimodal map* (see [7, 5]) is a C^1 endomorphism f of $[0, 1]$ such that $f(0) = f(1) = 0$ and, for some $c \in (0, 1)$, Df is positive on $[0, c)$ and negative on $(c, 1]$. The point c is called the *critical point* of f; clearly c is unique and $Df(c) = 0$.

Note. It follows from the definition that $Df(x) \neq 0$ for $x \in [0, 1] \setminus \{c\}$.

By $x_n \equiv f^n(x)$ we mean the n'th iterate of x under f; in particular $c_1 = f(c)$.

Definition. Let f be unimodal. We define $\tau : [0, 1] \to [0, 1]$ by $\tau(c) = c$ and otherwise by: $f(\tau(x)) = f(x)$ but $\tau(x) \neq x$. If $\tau(x) = 1 - x$ for all $x \in [0, 1]$, then we say that f is *symmetric*.

The involution τ takes $x \in [0, 1]$ to its dynamically symmetric point.

Definition. A unimodal map is said to have a *non-flat critical point* if there exists $\ell > 1$ and a C^1 positive function $M : [0, 1] \to \mathbb{R}^+$ such that $|Df(x)| = M(x)|x - c|^{\ell - 1}$ for all $x \in [0, 1]$.

Notes.

1. We require M to be C^1 in order to satisfy the hypotheses of [17] (for theorem 8.1, exponential decay of correlations). We only need M to be positive and continuous otherwise.

2. For the critical point to be non-flat, it is enough for f to be C^2 and $D^2 f(c) \neq 0$. If f is C^∞, then it is enough for some derivative to be non-zero at c.

3. If the critical point of f is non-flat, then there exist $L > 1$ and $\ell > 1$ such that
$$\frac{|x-c|^{\ell-1}}{L} \le |Df(x)| \le L|x-c|^{\ell-1} \text{ for all } x \text{ in } [0,1]. \tag{1}$$
Integrating (1) gives
$$\frac{|x-c|^{\ell}}{\ell L} \le |f(x)-f(c)| \le L\frac{|x-c|^{\ell}}{\ell} \text{ for all } x \text{ in } [0,1]. \tag{2}$$

Definition. An *S-unimodal map* is a C^2 unimodal map satisfying the following conditions:

1. $|Df|^{-1/2}$ is convex on $(0,c)$ and on $(c,1)$.
2. The critical point of f is non-flat.
3. $Df(0) > 1$.

Notes.

1. We require C^2 smoothness because we use Mañé's theorem (lemma A.3) and non-existence of wandering intervals (lemma A.2). We only need C^1 smoothness otherwise.
2. The *Schwarzian derivative* [7] of a C^3 map $g : [0,1] \to \mathbb{R}$ is $S(g)(x) \equiv D^3g(x)/Dg(x) - 3(D^2g(x)/Dg(x))^2/2$. If $S(g)(x) \le 0$ for all $x \in (0,1)$ such that $Dg(x) \ne 0$, then we say that g has *non-positive Schwarzian derivative*. If f is C^3 then condition 1 is equivalent to f having non-positive Schwarzian derivative since $S(g) = -2|Dg|^{1/2}D^2|Dg|^{-1/2}$.
3. Requiring $Df(0) > 1$ avoids some technical problems.

Definition. Let f be an endomorphism of $[0,1]$. The *ω-limit set* of $x \in [0,1]$ is
$$\omega(x) = \bigcap_{m=0}^{\infty} \overline{\{f^n(x) \mid n \ge m\}}. \tag{3}$$
A periodic point of f is a *periodic attractor* if its basin of attraction has non-empty interior, c.f. [7, 8]. The *basin of attraction* consists of those points in $[0,1]$ with ω-limit set equal to the orbit of the periodic point.

Notes.

1. If the critical point c of a unimodal map f is periodic, say of period p, then it is a periodic attractor because $|Df^p(c)| = 0$.

2. If x is periodic, say of period p, and $|Df^p(x)| = 1$, then x may be a periodic attractor according to the definition given here. This is in contrast with the definition $|Df^p(x)| < 1$ given in [8]. See also lemma A.8.

Definition. Let f be a C^1 endomorphism of $[0,1]$ and $x \in [0,1]$. The *lower Lyapunov exponent* at x is

$$\lambda_-(x) = \liminf_{n\to\infty} \frac{\log|Df^n(x)|}{n}. \tag{4}$$

The *upper Lyapunov exponent* at x is

$$\lambda_+(x) = \limsup_{n\to\infty} \frac{\log|Df^n(x)|}{n}. \tag{5}$$

If these coincide, then their common value $\lambda(x)$ is the *Lyapunov exponent* at x.

2 Combinatorics

Classically, one analyses the combinatorics of a unimodal map in terms of its kneading sequence [23, 5]. We will use an alternative method based on cutting and co-cutting times. The two forms of analysis are formally equivalent, but in practice the geometrical nature of the (co-)cutting times makes them the natural tool to work with. Cutting times were introduced by Hofbauer in the context of the so-called Hofbauer tower [13]. Co-cutting times were explicitly introduced by Sands [32]; they had already been implicitly used by Hofbauer and Keller [14]. See also [4].

Definition. Let f be unimodal and suppose c is not periodic. For $n \geq 0$ we denote by $M_n = [\alpha_n, \omega_n]$ the maximal interval of monotonicity of f^n around c_1. We put $\alpha_n = \inf M_n$ and $\omega_n = \sup M_n$, so

$$\alpha_n = \inf\{x \in [0, c_1] \mid f^n\big|_{[x,c_1]} \text{ is monotone}\} \tag{6}$$

and

$$\omega_n = \sup\{x \in [c_1, 1] \mid f^n\big|_{[c_1,x]} \text{ is monotone}\}. \tag{7}$$

1. If $f^n(\alpha_n) = c_1$ then we say that n is a *cutting time*, sometimes said an $\mathcal{S}$*-time*. The set of cutting times of f is $\mathcal{S} = \{n > 0 \mid f^n(\alpha_n) = c_1\}$.

2. If $f^n(\omega_n) = c_1$ then we say that n is a *co-cutting time*, sometimes said a $\mathcal{T}$*-time*. The set of co-cutting times of f is $\mathcal{T} = \{n > 0 \mid f^n(\omega_n) = c_1\}$.

See figure 1. Alternatively, $n \in \mathcal{S}$ if and only if $n > 0$ and $\alpha_n \neq \alpha_{n-1}$; $n \in \mathcal{T}$ if and only if $n > 0$ and $\omega_n \neq \omega_{n-1}$.

We enumerate the cutting times as $\mathcal{S} = \{S_0, S_1, \ldots\}$ where $S_0 < S_1 < \cdots$, and the co-cutting times as $\mathcal{T} = \{T_0, T_1, \ldots\}$ where $T_0 < T_1 < \cdots$.

Define a_n by: $f(a_n) = \alpha_n$ and $a_n < c$.

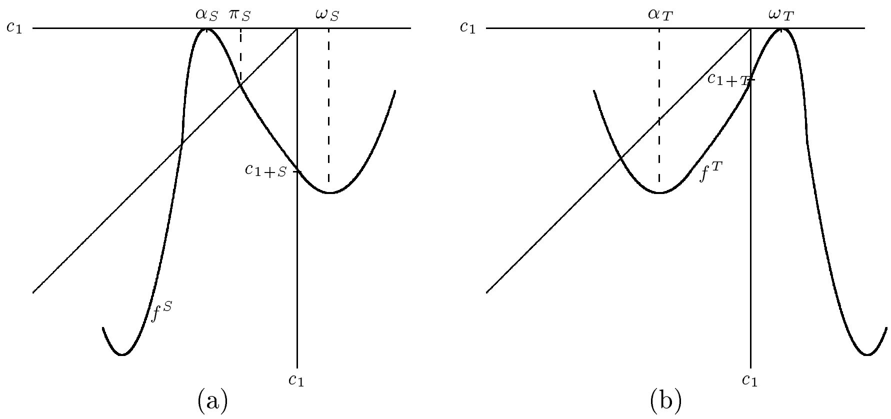

Figure 1: Iterates of f near c_1: (a) A cutting time. (b) A co-cutting time. Note that the axes are centered at (c_1, c_1).

Notes.

1. There may be iterates which are neither cutting nor co-cutting times.
2. Near c_1, an $\mathcal{S}$-iterate of f is decreasing and a $\mathcal{T}$-iterate is increasing, so $\mathcal{S} \cap \mathcal{T} = \emptyset$.
3. If $n > 0$ and $c_1 \in f^n(M_n)$, then $n \in \mathcal{S} \cup \mathcal{T}$.

Lemma 2.1 *Suppose f is S-unimodal and has no periodic attractors. Then*

1. *$\mathcal{S}$ is infinite and $S_0 = 1$.*
2. *$\mathcal{T} = \emptyset$ if and only if $c_1 = 1$. If $c_1 \neq 1$ then $\mathcal{T}$ is infinite and $T_0 = \inf\{n > 0 \mid \omega_n \neq 1\}$.*

Proof. The proof is left to the interested reader (use the fact that f has no homtervals, see lemma A.2). $\square$

Definition. Let f be unimodal and suppose $n \geq 0$. We define $K(n) = \{x \in [0,1] \mid x_n \in [x, \tau(x)]$ and $x \in [x_i, \tau(x_i)]$ for all $0 < i < n\}$.

Notes.

1. The sets $K(n)$ were introduced by Guckenheimer [11, lemma D, page 144].
2. The $K(n)$ are dynamically symmetric: $\tau(K(n)) = K(n)$.

Lemma 2.2 *Suppose f is unimodal and c is not periodic. If $S \in \mathcal{S}$ then $a_S \in \operatorname{int} K(S)$.*

Proof. We have $f^S(a_S) = c$ because $f^S(\alpha_S) = c_1$, so $f^S(a_S) \in (a_S, \tau(a_S))$. It is easy to see that $f^i(a_S) \notin \{a_S, \tau(a_S)\}$ for any $0 < i < S$ since otherwise c would be periodic. If $f^i(a_S) \in (a_S, \tau(a_S))$ for some $0 < i < S$ then we also obtain a contradiction: $f^{S-i+1}\big|_{[a_S,c]}$ is not monotone because $f^{S-i}(f^i(a_S)) = c$, so $f^{S-i}\big|_{[\alpha_S,c_1]}$ (and therefore $f^S\big|_{[\alpha_S,c_1]}$) is not monotone, but $f^S\big|_{[\alpha_S,c_1]}$ is monotone by definition. Thus $a_S \in (f^i(a_S), \tau(f^i(a_S)))$ for all $0 < i < S$ and $a_S \in \operatorname{int} K(S)$. □

Definition. Let f be unimodal and suppose c is not periodic. If $S \in \mathcal{S}$, then $f^S(\alpha_S) = c_1$ and $f^S\big|_{[\alpha_S,c_1]}$ is decreasing. There therefore exists a unique point $\pi_S \in (\alpha_S, c_1)$ such that $f^S(\pi_S) = \pi_S$, see figure 1(a). The point π_S is called the *central periodic point of f^S*.

Notes.

1. The point π_S is an orientation reversing fixed point of f^S.
2. If $T \in \mathcal{T}$ and f has no periodic attractors, then there is no analogous periodic point for f^T. Indeed, $f^T(x) < x$ for all $x \in [\alpha_T, \omega_T]$, c.f. figure 1(b), since otherwise f would have a periodic attractor.

Lemma 2.3 *Suppose f is unimodal and $n > 0$. If c is not periodic then $Df^n(x) \neq 0$ for all $x \in K(n)$. If $x \in \partial K(n)$ then either $x_n = x$ or $\tau(x)_n = \tau(x)$.*

Proof. See [5, lemma II.5.6, page 118]. If $x \in K(n)$ and $Df^n(x) = 0$, then $x_i = c$ for some $0 \le i < n$. Hence $x = c$ since $x \in [x_i, \tau(x_i)]$, and $x_n = c$ since $x_n \in [x, \tau(x)]$. This implies that c is periodic.

If $x \in \partial K(n)$ then $x_m \in \{x, \tau(x)\}$ for some $0 < m \le n$ (this includes the case $x \in \{0,1\}$). Taking $m > 0$ minimal such that $x_m \in \{x, \tau(x)\}$, one easily obtains a contradiction if $n \neq 0 \bmod m$. In other words, $x_n = x$ or $x_n = \tau(x)_n = \tau(x)$. □

Note. In particular, if c is not periodic then $c \notin K(n)$.

Lemma 2.4 *Suppose f is unimodal, c is not periodic and $S \in \mathcal{S}$. Denote by $K^+(S)$ the connected component of $f(K(S))$ containing α_S. Then $\pi_S = \sup K^+(S)$.*

Proof. If $S \in \mathcal{S}$ then $a_S \in \operatorname{int} K(S)$ by lemma 2.2. Now $K(S)$ is dynamically symmetric and, by lemma 2.3, $c \notin K(S)$, so $\partial f(K(S)) = f(\partial K(S))$. If $x \in \partial K(S)$ then $f^S(x) \in \{x, \tau(x)\}$ by lemma 2.3, so $f^S(f(x)) = f(x)$ which shows that $f^S(y) = y$ for all $y \in \partial f(K(S))$. If $y = \sup K^+(S)$ then $\alpha_S = f(a_S) < y < c_1$ and $f^S(y) = y$. Therefore $y = \pi_S$. □

Lemma 2.5 *Suppose f is S-unimodal and has no periodic attractors. If $m > n > 0$ then $\operatorname{int} K(m) \cap \operatorname{int} K(n) = \emptyset$.*

Proof. Suppose $m > n > 0$, $\operatorname{int} K(m) \cap \operatorname{int} K(n) \neq \emptyset$ and let U be a connected component of $\operatorname{int} K(m) \cap \operatorname{int} K(n)$. It is easy to see that either $f^n(x) = x$ for all $x \in U$ or $f^n(x) = \tau(x)$ for all $x \in U$. In both cases it follows that f^n has a line of fixed points, and in particular a fixed point x with $|Df^n(x)| = 1$. By lemma A.8, f has a periodic attractor. □

Lemma 2.6 *Suppose f is S-unimodal and has no periodic attractors. If $\bar{S}, S \in \mathcal{S}$ and $\bar{S} < S$, then $c \leq \alpha_{\bar{S}} < \pi_{\bar{S}} < \alpha_S < \pi_S < c_1$. Moreover, $\alpha_S \to c_1$ as $S \in \mathcal{S} \to \infty$.*

Proof. Take $\bar{S}, S \in \mathcal{S}$ and suppose $\bar{S} < S$. It follows from the definitions that $c \leq \alpha_{\bar{S}} < \pi_{\bar{S}} < c_1$ and $\alpha_{\bar{S}} < \alpha_S < \pi_S < c_1$. We must show that $\pi_{\bar{S}} < \alpha_S$. We have $\alpha_{\bar{S}} \in \operatorname{int} K^+(\bar{S})$ and $\alpha_S \in \operatorname{int} K^+(S)$ by lemma 2.2. It follows from lemma 2.5 that $K^+(\bar{S}) \cap \operatorname{int} K^+(S) = \emptyset$. Since $\alpha_{\bar{S}} < \alpha_S$ we have $K^+(\bar{S}) < \alpha_S$. By lemma 2.4, $\pi_{\bar{S}} = \sup K^+(\bar{S})$ and therefore $\pi_{\bar{S}} < \alpha_S$. This proves that $c \leq \alpha_{\bar{S}} < \pi_{\bar{S}} < \alpha_S < \pi_S < c_1$.

Define $\alpha_\infty = \sup_{S \in \mathcal{S}} \alpha_S$. If $\alpha_\infty < c_1$ then it follows from the definitions that $[\alpha_\infty, c_1]$ is a homterval. This contradicts the fact that f has no homtervals, see lemma A.2. Therefore $\alpha_S \to c_1$ as $S \in \mathcal{S} \to \infty$. □

Lemma 2.7 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $K > 0$ such that $|\alpha_S - c_1| \leq K|\pi_S - c_1|$ for all $S \in \mathcal{S}$.*

Proof. C.f. the proof of [27, proposition 12]. By lemma A.9 there exists $K_0 > 0$ such that $|Df^n(x)| \geq K_0 |Df^n(\tau(x))|$ for every $x \in [0,1]$ and $n \geq 0$. Put $K = L^2((K_0^{-1} + 1)^\ell - 1) + 1$ where the non-flatness terms $L > 0$ and $\ell > 1$ come from equation 1.

Suppose $S \in \mathcal{S}$ and define $p_S = f^{S-1}(\pi_S)$, so $f(p_S) = \pi_S$ and $f^S(p_S) = p_S$. If $p_S < c$ then put $\bar{a}_S = a_S$ and otherwise put $\bar{a}_S = \tau(a_S)$; set $J = [p_S, \bar{a}_S]$. Due to the definition of $\bar{a}_S$ we have $c \notin J$. Clearly $f^S(J) = [c, p_S]$ and, by lemma 2.4, $J \subseteq K(S)$. Let us show that $|f^S(J)| \geq K_0|J|$.

By lemma 2.3, $Df^S(x) \neq 0$ for $x \in K(S)$. If $x \in \partial K(S)$ then, by lemma 2.3, $f^S(x) = x$ or $f^S(\tau(x)) = \tau(x)$. Since f has no periodic attractors we have respectively $|Df^S(x)| \geq 1$ or $|Df^S(\tau(x))| \geq 1$. In any case we have $|Df^S(x)| \geq K_0$. Lemma A.7 with $I = [0,1]$, $g = f^S$ and $E = K(S)$ thus shows that $|Df^S(x)| \geq K_0$ for all $x \in K(S)$. Therefore $|f^S(J)| \geq K_0|J|$.

Integrating (1) over J yields

$$|\pi_S - \alpha_S| \leq \frac{L\left(|\bar{a}_S - c|^\ell - |p_S - c|^\ell\right)}{\ell}. \tag{8}$$

From (2) we obtain $|\pi_S - c_1| \geq |p_S - c|^\ell/(\ell L)$. These and $|f^S(J)| \geq K_0|J|$, or in other words $|p_S - c| \geq K_0|p_S - \bar{a}_S|$, give

$$\begin{aligned} \frac{|\alpha_S - c_1|}{|\pi_S - c_1|} &\leq L^2\left(\left(\frac{|\bar{a}_S - p_S|}{|p_S - c|} + 1\right)^\ell - 1\right) + 1 && (9)\\ &\leq L^2((K_0^{-1} + 1)^\ell - 1) + 1 \equiv K. \ \Box && (10) \end{aligned}$$

Definition. Let f be unimodal and suppose c is not periodic.

1. If $n > S_0$ then we define $\mathcal{S}(n) = \max\{S \in \mathcal{S} \mid S < n\}$.
2. If $n > T_0$ then we define $\mathcal{T}(n) = \max\{T \in \mathcal{T} \mid T < n\}$.

Note. If $S_i \in \mathcal{S}$ then $S_i = \mathcal{S}(S_{i+1})$. If $T_i \in \mathcal{T}$ then $T_i = \mathcal{T}(T_{i+1})$.

We will systematically factorize $\mathcal{S}$- and $\mathcal{T}$-iterates into compositions of other $\mathcal{S}$- and $\mathcal{T}$-iterates using lemma 2.8 below. The asymmetry between $\mathcal{S}$- and $\mathcal{T}$-iterates comes from the fact that f has a *maximum* at c.

Lemma 2.8 *Suppose f is S-unimodal and has no periodic attractors. Then*

1. *If $S_0 < S \in \mathcal{S}$, then $S - \mathcal{S}(S) \in \mathcal{S}$ and $f^{S-\mathcal{S}(S)}(c_1) = f^S(\alpha_{S-1})$. Moreover, $S \leq 2\mathcal{S}(S)$.*
2. *If $T_0 < T \in \mathcal{T}$, then $T - \mathcal{T}(T) \in \mathcal{S}$ and $f^{T-\mathcal{T}(T)}(c_1) = f^T(\omega_{T-1})$.*
3. *If $T_0 < S \in \mathcal{S}$, then $S - \mathcal{T}(S) \in \mathcal{S}$ and $f^{S-\mathcal{T}(S)}(c_1) = f^S(\omega_S) < f^S(c_1)$. See figure 2.*
4. *If $T \in \mathcal{T}$, then $T - \mathcal{S}(T) \in \mathcal{S}$ and $f^{T-\mathcal{S}(T)}(c_1) = f^T(\alpha_T) < f^T(c_1)$. See figure 3.*

Proof.

1. Put $\bar{S} = S - \mathcal{S}(S)$, so $f^S = f^{\bar{S}} \circ f^{\mathcal{S}(S)}$. Now $f^{S-1}\big|_{[\alpha_{\mathcal{S}(S)}, c_1]}$ is monotone and $f^S(\alpha_S) = c_1$. It follows that $f^S\big|_{[\alpha_{\mathcal{S}(S)}, \alpha_S]}$ is monotone and therefore $f^{\bar{S}}\big|_{f^{\mathcal{S}(S)}[\alpha_{\mathcal{S}(S)}, \alpha_S]} = f^{\bar{S}}\big|_{[f^{\mathcal{S}(S)}(\alpha_S), c_1]}$ is monotone. Moreover $f^{\bar{S}}(f^{\mathcal{S}(S)}(\alpha_S)) = f^S(\alpha_S) = c_1$, which shows that $\alpha_{\bar{S}} = f^{\mathcal{S}(S)}(\alpha_S)$ since $f^{\mathcal{S}(S)}(\alpha_S) < c_1$. Since $f^{\bar{S}}(\alpha_{\bar{S}}) = c_1$, we deduce that $\bar{S} \in \mathcal{S}$. Note that $f^{\bar{S}}(c_1) = f^{\bar{S}}(f^{\mathcal{S}(S)}(\alpha_{\mathcal{S}(S)})) = f^S(\alpha_{\mathcal{S}(S)}) = f^S(\alpha_{S-1})$.

In order to see that $S \leq 2\mathcal{S}(S)$, consider $f^{\mathcal{S}(S)}[\alpha_{\mathcal{S}(S)}, c_1] = [c_{1+\mathcal{S}(S)}, c_1]$. If $c_{1+\mathcal{S}(S)} \geq \alpha_{\mathcal{S}(S)}$ then $f^{\mathcal{S}(S)}$ maps $[\alpha_{\mathcal{S}(S)}, c_1]$ homeomorphically into itself, implying that f has a periodic attractor. Therefore $c_{1+\mathcal{S}(S)} < \alpha_{\mathcal{S}(S)}$. It follows that $f^{\mathcal{S}(S)}\big|_{[c_{1+\mathcal{S}(S)}, c_1]} = f^{2\mathcal{S}(S)}\big|_{[\alpha_{\mathcal{S}(S)}, c_1]}$ is not monotone, which proves that $S \leq 2\mathcal{S}(S)$.

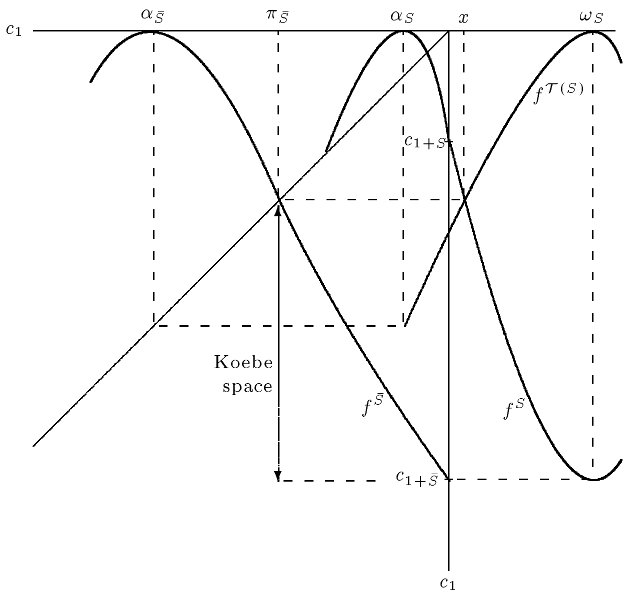

Figure 2: Decomposition $f^S = f^{\bar S} \circ f^{\mathcal{T}(S)}$ where $S \in \mathcal{S}$ and $\bar S = S - \mathcal{T}(S)$.

2. This is proved similarly to the first part of 1.

3. Put $\bar S = S - \mathcal{T}(S)$, so $f^S = f^{\bar S} \circ f^{\mathcal{T}(S)}$. Now $f^S\big|_{[\alpha_S,\omega_S]}$ is monotone and therefore $f^{\bar S}\big|_{f^{\mathcal{T}(S)}[\alpha_S,\omega_S]} = f^{\bar S}\big|_{[f^{\mathcal{T}(S)}(\alpha_S),c_1]}$ is monotone. Moreover $f^{\bar S}(f^{\mathcal{T}(S)}(\alpha_S)) = f^S(\alpha_S) = c_1$, which shows that $\alpha_{\bar S} = f^{\mathcal{T}(S)}(\alpha_S)$ since $f^{\mathcal{T}(S)}(\alpha_S) < c_1$. Since $f^{\bar S}(\alpha_{\bar S}) = c_1$, we deduce that $\bar S \in \mathcal{S}$. Note that $f^{\bar S}(c_1) = f^{\bar S}(f^{\mathcal{T}(S)}(\omega_S)) = f^S(\omega_S) < f^S(c_1)$ because f^S is decreasing.

4. This is proved similarly to 3. □

Notes.

1. In the light of property 1, we can define the so-called *kneading function* Q by $S_i - S_{i-1} = S_{Q(i)}$ [14, 13, 4]. There is also a co-kneading function, defined similarly [4]. We will not use either of these functions.

2. The full map $x \to 4x(1-x)$ has $S_i = i+1$ and so $Q(i) = 0$. The Feigenbaum map [7] has $S_i = 2^i$ and $Q(i) = i-1$. The Fibonacci map [21, 18, 7] has S_i equal to the $i+1$'th Fibonacci number and $Q(i) = i-2$.

3. There is no inequality comparable to $S \leq 2\mathcal{S}(S)$ for $\mathcal{T}$-times. Indeed, it is possible for T_1 to be larger than an arbitrary multiple of T_0.

Definition. Let f be unimodal and suppose $n > 0$. If $c_{1+j} < c_{1+n}$ for all $0 < j < n$ then we say that n *is a closest return to* c_1.

Lemma 2.9 *Suppose f is S-unimodal and has no periodic attractors. If $n > 0$ is a closest return to c_1, then $n \in \mathcal{S} \cup \mathcal{T}$.*

Proof. Clearly $\sup f^n(M_n) = c_{1+j}$ for some $0 \leq j < n$, c.f. figure 1. As $c_1 \in M_n$ we have in particular that $c_{1+n} = f^n(c_1) \leq c_{1+j}$. This is in contradiction with n being a closest return to c_1 unless $j = 0$, which is the same thing as $c_1 \in f^n(M_n)$. Therefore $n \in \mathcal{S} \cup \mathcal{T}$. □

Lemma 2.10 *Suppose f is S-unimodal and has no periodic attractors. Then*

1. *If $T_0 < S \in \mathcal{S}$ then there exists a unique $x \in (\pi_S, \omega_S)$ such that $f^S(x) = f^{\mathcal{T}(S)}(x)$. Moreover, $f^S(x) = \pi_{S-\mathcal{T}(S)}$. If S is a closest return to c_1, then $x \in (c_1, \omega_S)$. See figure 2.*

2. *If $T \in \mathcal{T}$ then there exists a unique $x \in (\alpha_T, \omega_T)$ such that $f^T(x) = f^{\mathcal{S}(T)}(x)$. Moreover, $f^T(x) = \pi_{T-\mathcal{S}(T)}$. If T is a closest return to c_1, then $x \in (\alpha_T, c_1)$.*

Proof.
1. The existence of $x \in (\pi_S, \omega_S)$ satisfying $f^S(x) = f^{\mathcal{T}(S)}(x)$ follows from the inequalities $f^{\mathcal{T}(S)}(\pi_S) < \pi_S = f^S(\pi_S)$ (see note 2 after the definition of π_S) and $f^S(\omega_S) < f^{\mathcal{T}(S)}(\omega_S) = c_1$. It is unique because f^S and $f^{\mathcal{T}(S)}$ are monotone. By lemma 2.9, $\bar{S} \equiv S - \mathcal{T}(S) \in \mathcal{S}$, and by its proof $f^{\mathcal{T}(S)}(x) \in (\alpha_{\bar{S}}, c_1)$. Since $f^{\bar{S}}(f^{\mathcal{T}(S)}(x)) = f^S(x) = f^{\mathcal{T}(S)}(x)$, this shows that $f^{\mathcal{T}(S)}(x) = \pi_{\bar{S}}$. If S is a closest return to c_1, then $f^{\mathcal{T}(S)}(c_1) < f^S(c_1)$ and it follows that $x \in (c_1, \omega_S)$.
2. This is proved similarly to 1. □

3 Universal distortion bounds

The Koebe principle (see lemma A.6) shows that the distortion on a branch of f^n is bounded except near the end-points, the space to the end-points being measured in the image. In order to use the Koebe principle, one must estimate this space. We will do this using the following theorem, via its corollaries. See the next section for examples.

Lemma 3.1 *If $x, y > 0$ and $y^2 \geq x/(1+x)$ then either $y \geq x$ or both $x, y \geq 1/\phi$. Here $\phi = (1+\sqrt{5})/2$, the golden ratio.*

Proof. If $y \leq x$ then $y^2 \geq y/(1+y)$ which implies $y^2 + y - 1 \geq 0$. Therefore $x \geq y \geq 1/\phi$. □

Theorem 3.2 (A priori space) *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\delta > 0$ such that $|c_{1+S} - \pi_S| \geq \delta\, |\pi_S - c_1|$ for all $S \in \mathcal{S}$. Here $\delta \geq \min\{1/\phi, c/(1-c)\}$ where $\phi = (1+\sqrt{5})/2$, the golden ratio.*

Proof. If $c_1 = 1$ then the result is trivial (c.f. below), so suppose $c_1 < 1$. Take $N \geq 0$ such that $S_N < T_0 < S_{N+1}$. Define $\delta_S = |c_{1+S} - \pi_S|/|\pi_S - c_1|$ for $S \in \mathcal{S}$ and put $\delta = \min\{1/\phi, \delta_{S_0}, \ldots, \delta_{S_N}\}$. We shall prove that $\delta_S \geq \delta$ for all $S \in \mathcal{S}$ by showing that $\delta_S \geq \min\{1/\phi, \delta_{S-\mathcal{T}(S)}\}$ for all $S \in \mathcal{S}$ such that $S > S_N$. Indeed, if $S > S_N$ then $S - \mathcal{T}(S) < S$, and $\delta_S \geq \delta$ follows inductively.

So take $S \in \mathcal{S}$ and suppose $S > S_N$. By lemma 2.8(3), $S - \mathcal{T}(S) = \bar{S}$ for some $\bar{S} \in \mathcal{S}$ and $f^{\bar{S}}(c_1) = f^S(\omega_S) < f^S(c_1)$, see figure 2. Let us show that either $\delta_S \geq \delta_{\bar{S}}$ or both $\delta_S, \delta_{\bar{S}} \geq 1/\phi$, which implies the claimed inequality $\delta_S \geq \min\{1/\phi, \delta_{\bar{S}}\}$.

Cross-ratio expansion (lemma A.4(2)) with $I = [\alpha_S, \omega_S]$, $J = [\pi_S, c_1]$ and $g = f^S$ gives

$$\frac{|c_{1+S} - \pi_S|}{|\pi_S - c_1|}\,\frac{|c_{1+\bar{S}} - c_1|}{|\omega_S - \alpha_S|} \geq \frac{|c_{1+\bar{S}} - c_{1+S}|}{|\omega_S - c_1|}\,\frac{|\pi_S - c_1|}{|\alpha_S - \pi_S|}. \tag{11}$$

Using $|\omega_S - \alpha_S| \geq |\omega_S - c_1|$ and rearranging, we obtain

$$\delta_S = \frac{|c_{1+S} - \pi_S|}{|\pi_S - c_1|} \geq \frac{|c_{1+\bar{S}} - c_{1+S}|}{|c_{1+\bar{S}} - c_1|}\,\frac{|\pi_S - c_1|}{|\alpha_S - \pi_S|}. \tag{12}$$

Case 1. $|\pi_{\bar{S}} - c_1| \geq |c_{1+S} - c_1|$. Then $|c_{1+\bar{S}} - c_{1+S}| \geq |c_{1+\bar{S}} - \pi_{\bar{S}}|$ since $c_{1+\bar{S}} < \pi_{\bar{S}} < c_{1+S} < c_1$. Note that $c_{1+S} < \alpha_S$ since otherwise f^S would map $[\alpha_S, c_1]$ diffeomorphically into itself, and f would have a periodic attractor. Therefore $|\alpha_S - \pi_S| \leq |c_{1+S} - \pi_S|$ since $c_{1+S} < \alpha_S < \pi_S$. Substituting into (12) yields

$$\delta_S \geq \frac{|c_{1+\bar{S}} - \pi_{\bar{S}}|}{|c_{1+\bar{S}} - c_1|}\,\frac{|\pi_S - c_1|}{|c_{1+S} - \pi_S|} = \frac{\delta_{\bar{S}}}{\delta_S(1+\delta_{\bar{S}})}, \tag{13}$$

which implies $\delta_{\bar{S}}^2 \geq \delta_{\bar{S}}/(1+\delta_{\bar{S}})$. By lemma 3.1, either $\delta_S \geq \delta_{\bar{S}}$ or both $\delta_S, \delta_{\bar{S}} \geq 1/\phi$.

Case 2. $|\pi_{\bar{S}} - c_1| \leq |c_{1+S} - c_1|$. By lemma 2.6 we have $\pi_{\bar{S}} < \alpha_S < \pi_S$, so $|\alpha_S - \pi_S| \leq |\pi_{\bar{S}} - \pi_S|$. Substituting into (12) gives

$$\frac{|c_{1+S} - \pi_S|}{|\pi_S - c_1|} \geq \frac{|c_{1+\bar{S}} - c_{1+S}|}{|c_{1+\bar{S}} - c_1|}\,\frac{|\pi_S - c_1|}{|\pi_{\bar{S}} - \pi_S|}. \tag{14}$$

Now $|c_{1+S} - \pi_S| = |c_{1+S} - c_1| - |\pi_S - c_1|$ and $|c_{1+\bar{S}} - c_{1+S}| = |c_{1+\bar{S}} - c_1| - |c_{1+S} - c_1|$ since $c_{1+\bar{S}} < c_{1+S} < \pi_S$. Substituting these into (14) and solving for $|c_{1+S} - c_1|$, we obtain

$$\frac{|c_{1+S} - c_1|}{|\pi_S - c_1|} \geq \frac{|c_{1+\bar{S}} - c_1|}{|\pi_{\bar{S}} - c_1|} \Big/ \left(\frac{|c_{1+\bar{S}} - c_1|}{|\pi_{\bar{S}} - c_1|}\,\frac{|\pi_{\bar{S}} - \pi_S|}{|\pi_{\bar{S}} - c_1|} + \frac{|\pi_S - c_1|^2}{|\pi_{\bar{S}} - c_1|^2}\right), \tag{15}$$

or in other words,

$$1+\delta_S \geq \frac{1+\delta_{\bar{S}}}{(1+\delta_{\bar{S}})(1-\xi)+\xi^2} \tag{16}$$

where $\xi = |\pi_S - c_1|/|\pi_{\bar{S}} - c_1|$. Since $\pi_{\bar{S}} < \pi_S < c_1$ and, by assumption, $|\pi_{\bar{S}} - c_1| \leq |c_{1+S} - c_1|$, we have $1 \geq \xi \geq 1/(1+\delta_S)$. The denominator of (16) is a positive quadratic function of ξ, so it assumes its maximal value at one of the endpoints of $[1/(1+\delta_S), 1]$. If this is at $\xi = 1$ then $\delta_S \geq \delta_{\bar{S}}$. If it is at $\xi = 1/(1+\delta_S)$ then $\delta_{\bar{S}}^2 \geq \delta_{\bar{S}}/(1+\delta_{\bar{S}})$, so by lemma 3.1 either $\delta_S \geq \delta_{\bar{S}}$ or both $\delta_S, \delta_{\bar{S}} \geq 1/\phi$.

This proves that $\delta_S \geq \min\{1/\phi, \delta_{S-\mathcal{T}(S)}\}$ for all $S \in \mathcal{S}$ such that $S > S_N$, and therefore that $\delta_S \geq \min\{1/\phi, \delta_{S_0}, \dots, \delta_{S_N}\} \equiv \delta$ for all $S \in \mathcal{S}$.

Let us show that $\min\{\delta_{S_0}, \dots, \delta_{S_N}\} \geq \min\{c/(1-c), 1\}$. Suppose $S \in \{S_0, \dots, S_N\}$, so $f^S|_{[\alpha_S, 1]}$ is monotone. Now from $\pi_S > c$ we obtain $\pi_S/(1-\pi_S) \geq c/(1-c)$, so lemma A.6(2) with $I = [\pi_S, 1]$, $J = [\pi_S, c_1]$ and $g = f^S$ yields

$$\delta_S \geq \min\{|Df^S(\pi_S)|, c/(1-c)\} \geq \min\{c/(1-c), 1\}. \tag{17}$$

Thus $\delta \geq \min\{1/\phi, c/(1-c)\}$. □

Notes.

1. If $c \geq 1/\phi^2 \approx 0.38$ then $\delta \geq 1/\phi$. In particular, $\delta \geq 1/\phi$ if f is symmetric.
2. For any S-unimodal map, $1/\phi$ is an *asymptotic* lower bound for δ. That is, $|c_{1+S} - \pi_S|/|\pi_S - c_1| \geq 1/\phi - \epsilon$ if S is large enough, for any $\epsilon > 0$. Indeed, we can conjugate f to an S-unimodal map g with $c_g = 0.5$ using a Möbius transformation. Since $\pi_S(f) \to c_1(f)$ as $S \to \infty$ and the conjugating Möbius transformation is approximately linear near $c_1(f)$, we obtain $|c_{1+S}(f) - \pi_S(f)|/|\pi_S(f) - c_1(f)| \geq 1/\phi - \epsilon$ from $|c_{1+S}(g) - \pi_S(g)|/|\pi_S(g) - c_1(g)| \geq 1/\phi$ if S is large enough (here we suppose that $c_{1+S}(f) \approx c_1(f)$ in addition to $\pi_S(f) \approx c_1(f)$; the desired inequality holds trivially otherwise).
3. We do not know if $1/\phi$ is the best possible lower bound.
4. In appendix B we derive the classical real a priori bounds for renormalizable maps from theorem 3.2.
5. Results similar to theorem 3.2 have been proved for example by Blokh and Lyubich [3, lemma 1, section 4], and Levin and van Strien [19, lemma 6.4]. Levin and van Strien obtain a lower bound of 0.6, close to our $1/\phi = 0.618\ldots$, but, like Blokh and Lyubich, require additional hypotheses.

Corollary 3.3 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\delta > 0$ such that*

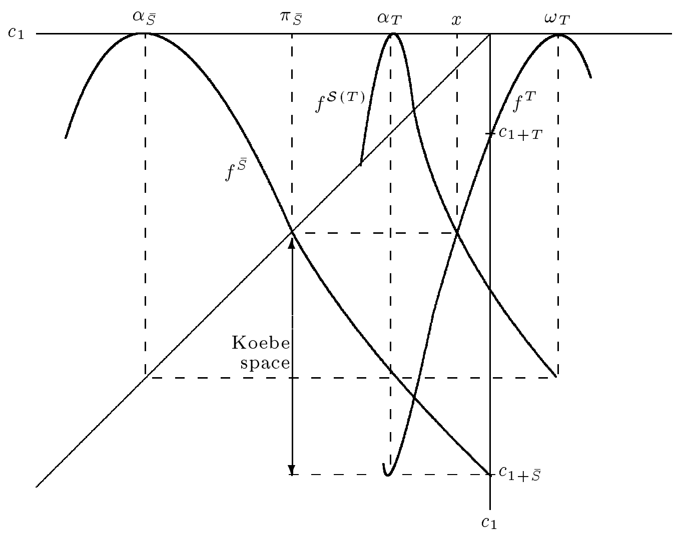

Figure 3: Decomposition $f^T = f^{\bar S} \circ f^{\mathcal{S}(T)}$ where $T \in \mathcal{T}$ and $\bar S = T - \mathcal{S}(T)$.

1. *If $T_0 < S \in \mathcal{S}$ then $|f^S(\omega_S) - f^S(x)| \geq \delta\, |f^S(x) - c_1|$ where x is the unique point in (π_S, ω_S) for which $f^S(x) = f^{\mathcal{T}(S)}(x)$; see lemma 2.10(1).*

2. *If $T \in \mathcal{T}$ then $|f^T(\alpha_T) - f^T(x)| \geq \delta |f^T(x) - c_1|$ where x is the unique point in (α_T, ω_T) for which $f^T(x) = f^{\mathcal{S}(T)}(x)$; see lemma 2.10(2).*

Here $\delta > 0$ is the same as in theorem 3.2.

Proof. We have $|c_{1+S} - \pi_S| \geq \delta |\pi_S - c_1|$ for all $S \in \mathcal{S}$ by theorem 3.2.

1. If $T_0 < S \in \mathcal{S}$ then $S - \mathcal{T}(S) \in \mathcal{S}$ by lemma 2.8(3); put $\bar S = S - \mathcal{T}(S)$. We have $f^S(\omega_S) = c_{1+\bar S}$ by lemma 2.8(3) and $f^S(x) = \pi_{\bar S}$ by lemma 2.10. Therefore $|f^S(\omega_S) - f^S(x)| = |c_{1+\bar S} - \pi_{\bar S}| \geq \delta |\pi_{\bar S} - c_1| = \delta |f^S(x) - c_1|$.

2. If $T \in \mathcal{T}$ then $T - \mathcal{S}(T) \in \mathcal{S}$ by lemma 2.8(4); put $\bar S = T - \mathcal{S}(T)$. We have $f^T(\alpha_T) = c_{1+\bar S}$ by lemma 2.8(4) and $f^T(x) = \pi_{\bar S}$ by lemma 2.10. Therefore $|f^T(\alpha_T) - f^T(x)| = |c_{1+\bar S} - \pi_{\bar S}| \geq \delta |\pi_{\bar S} - c_1| = \delta |f^T(x) - c_1|$, see figure 3. $\square$

Corollary 3.4 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\delta > 0$ such that*

1. *If $T_0 < S \in \mathcal{S}$ and S is a closest return to c_1, then $|f^S(\omega_S) - f^S(c_1)| \geq \delta\,|f^S(c_1) - c_1|$.*

2. *If $T \in \mathcal{T}$ is a closest return to c_1 then $|f^T(\alpha_T) - f^T(c_1)| \geq \delta\,|f^T(c_1) - c_1|$.*

Here $\delta > 0$ is the same as in theorem 3.2.

Proof.

1. We have $|f^S(\omega_S) - f^S(x)| \geq \delta|f^S(x) - c_1|$ by corollary 3.3(i) and $x \in (c_1, \omega_S)$ by lemma 2.10. Therefore $|f^S(\omega_S) - f^S(c_1)| \geq |f^S(\omega_S) - f^S(x)| \geq \delta|f^S(x) - c_1| \geq \delta|f^S(c_1) - c_1|$ since f^S is decreasing on $[\alpha_S, \omega_S]$.

2. We have $|f^T(\alpha_T) - f^T(x)| \geq \delta|f^T(x) - c_1|$ by corollary 3.3(ii) and $x \in (\alpha_T, c_1)$ by lemma 2.10. Therefore $|f^T(\alpha_T) - f^T(c_1)| \geq |f^T(\alpha_T) - f^T(x)| \geq \delta|f^T(x) - c_1| \geq \delta|f^T(c_1) - c_1|$ since f^T is increasing on $[\alpha_T, \omega_T]$. $\square$

4 Universal derivative bounds

In this section we prove that any S-unimodal map without periodic attractors has derivative bounded away from zero at closest returns to c_1 (proposition 4.4), see the definition below. This is stronger than having a lower bound for closest returns to c (which was already known, and is proved like lemma 7.5), the point being that if $n > 0$ is a closest return to c_1, then, more or less, c_n is closer to c than any previous iterate, so $|Df(c_n)|$ is smaller than at any previous time. Now $|Df^n(c_1)| = |Df(c_n)| \cdots |Df(c_1)|$ contains this very small factor, so could be expected to be very small itself. We show that it is bounded below.

From this lower bound, we derive a lower bound for $|Df^n(c_1)|$, for an arbitrary $n > 0$, that is subexponentially decreasing in n (proposition 4.9). Without additional restrictions on the map one cannot hope to do much better, since examples of S-unimodal maps without periodic attractors are known for which $\liminf_{n\to\infty} |Df^n(c_1)| = 0$, see [4].

Definition. Let f be unimodal. If there exists $\alpha > 0$ such that $|Df^n(c_1)| \geq \alpha$ whenever $n > 0$ is a closest return to c_1, then we say that f *has derivative bounded away from zero at closest returns to c_1*.

Lemma 4.1 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $K > 0$ such that if $S \in \mathcal{S}$ is a closest return to c_1 then $|Df^S(c_1)| \geq |Df^S(\pi_S)|/K$.*

Proof. If $c_1 = 1$ then the result is trivial (the only closest return to c_1 is $n = 1$) so suppose $c_1 < 1$. Take $N \geq 0$ such that $S_N < T_0 < S_{N+1}$ and put $K = \max\{|Df^{S_0}(\pi_{S_0})|/|Df^{S_0}(c_1)|, \ldots, |Df^{S_N}(\pi_{S_N})|/|Df^{S_N}(c_1)|, (1+\delta)^2/\delta^2\}$, where $\delta > 0$ is as in theorem 3.2.

Suppose $S \in \mathcal{S}$ is a closest return to c_1. If $S < T_0$ then $|Df^S(c_1)| \geq |Df^S(\pi_S)|/K$ by the definition of K. Otherwise $S > T_0$ and $|f^S(\omega_S) - f^S(c_1)| \geq \delta|f^S(c_1) - c_1| \geq \delta|f^S(c_1) - \pi_S|$ by corollary 3.4. The one-sided Koebe principle (lemma A.6(4)) with $I = [\omega_S, \pi_S]$, $x = c_1$ and $g = f^S$ then gives $|Df^S(c_1)| \geq \delta^2|Df^S(\pi_S)|/(1+\delta)^2 \geq |Df^S(\pi_S)|/K$. □

Corollary 4.2 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\alpha > 0$ such that if $S \in \mathcal{S}$ is a closest return to c_1 then $|Df^S(c_1)| \geq \alpha$.*

Proof. Since $|Df^S(\pi_S)| \geq 1$ we have $|Df^S(c_1)| \geq 1/K \equiv \alpha$ by lemma 4.1. □

Note. If f is symmetric, then the proof gives $\alpha \geq 1/\phi^4$. With a more careful analysis, one can show that $\alpha \geq 1/\phi^2$.

Lemma 4.3 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\alpha > 0$ such that if $T \in \mathcal{T}$ is a closest return to c_1 then $|Df^T(c_1)| \geq \alpha$.*

Proof. Take $\delta > 0$ as in theorem 3.2 and put $\alpha = \delta/(1+\delta)^3$. If $T \in \mathcal{T}$ is a closest return to c_1 then $|f^T(\alpha_T) - f^T(c_1)| \geq \delta|f^T(c_1) - c_1|$ by corollary 3.4. Therefore there exists $y \in [\alpha_T, c_1]$ such that $|f^T(y) - c_1| = (1+\delta)|f^T(c_1) - c_1|$. Now $|y - c_1| \leq |f^T(y) - c_1|$ (otherwise f would have a periodic attractor) so $|f^T(y) - f^T(c_1)|/|y - c_1| \geq \delta/(1+\delta)$. By the mean-value theorem there exists $z \in [y, c_1]$ with $|Df^T(z)| \geq \delta/(1+\delta)$. Since $|f^T(c_1) - c_1|/|f^T(c_1) - f^T(z)| \geq 1/\delta$, the one-sided Koebe principle (lemma A.6(4)) with $I = [\omega_T, z]$, $x = c_1$ and $g = f^T$ gives $|Df^T(c_1)| \geq |Df^T(b)|/K(1/\delta) \geq \delta/((1+\delta)K(1/\delta)) = \alpha$. □

Note. If f is symmetric, then the proof gives $\alpha \geq 1/\phi^4$. With a more careful analysis, one can show that $\alpha \geq 1/\phi^3$.

Proposition 4.4 (Closest return lower bound) *Suppose f is S-unimodal and has no periodic attractors. Then f has derivative bounded away from zero at closest return to c_1.*

Proof. If $n > 0$ is a closest return to c_1 then $n \in \mathcal{S} \cup \mathcal{T}$ by lemma 2.9. The result therefore follows from corollary 4.2 and lemma 4.3. □

Note. If f is symmetric, one can show that the lower bound is at least $1/\phi^3$.

In the remainder of this section we prove the existence of a lower bound for $|Df^n(c_1)|$ which decreases subexponentially in n (proposition 4.9). The same technique will be used in section 6 to prove that if an S-unimodal map has a uniform hyperbolic structure and satisfies the Collet-Eckmann condition at closest returns to c_1, then it satisfies the Collet-Eckmann condition (c.f. lemmas 6.9 and 6.10). Note the similarities between the sets $N(n)$ used here and the so-called Guckenheimer sets $K(n)$ used in sections 2 and 7.

Definition. Let f be unimodal and suppose $n \geq 0$. We define $N(n) = \{x \in [0, c_1] \mid x_n \leq x \text{ and } x_i \leq x_n \text{ for all } 0 < i < n\}$.

Lemma 4.5 *Suppose f is unimodal and $n > 0$. Then $Df^n(x) \neq 0$ for all $x \in \operatorname{int} N(n)$. If $x \in \partial N(n)$ then either $x_i = x_n$ for some $0 \leq i < n$ and $x \in N(i)$, or $x = c_1$ and n is a closest return to c_1.*

Proof. If $x \in \operatorname{int} N(n)$ then $x < c_1$ and $x_i \leq x$ for all $0 \leq i \leq n$. If $Df^n(x) = 0$ then $x_i = c$ for some $0 \leq i < n$, so $x_{i+1} = c_1 > x$, a contradiction.

If $x \in \partial N(n)$ then either $x_i = x_n$ for some $0 \leq i < n$ (this includes the case $x = 0$), or, if $x_i \neq x_n$ for all $0 \leq i < n$, we have $x = c_1$, in which case $c_{1+j} < c_{1+n}$ for all $0 < j < n$ since $x \in N(n)$ and $x_i \neq x_n$ for all $0 \leq i < n$. If $x_i = x_n$ for some $0 \leq i < n$ then clearly $x \in N(i)$. □

Lemma 4.6 *Suppose f is unimodal and $x \in [0, c_1]$. If $n \geq 0$ and $x_i \leq x$ for all $0 < i \leq n$, then there exists $m \geq 0$ and non-negative integers $i_0, \ldots, i_m$ such that $0 = i_0 < i_1 < \cdots < i_m = n$, $x_n = x_{i_m} \leq \cdots \leq x_{i_1} \leq x_{i_0} = x$ and $x_{i_j} \in N(i_{j+1} - i_j)$ for all $0 \leq j < m$.*

Proof. Define $\hat{\imath}_0, \hat{\imath}_1, \ldots$ by $\hat{\imath}_0 = n$ and $\hat{\imath}_{j+1} = \max\{0 \leq k < \hat{\imath}_j \mid x_k \geq x_{\hat{\imath}_j}\}$. We have $\hat{\imath}_m = 0$ for some $m \geq 0$ by the hypothesis that $x_i \leq x$ for all $0 < i \leq n$. Take $i_j = \hat{\imath}_{m-j}$, where $0 \leq j \leq m$. □

Lemma 4.7 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\alpha > 0$ such that $\inf_{N(n)} |Df^n| \geq \alpha$ for all $n \geq 0$.*

Proof. By proposition 4.4, f has derivative bounded away from zero at closest returns to c_1. In other words, there exists $1 \geq \alpha > 0$ such that $|Df^n(c_1)| \geq \alpha$ whenever $n > 0$ is a closest return to c_1. We shall show that $\inf_{N(n)} |Df^n| \geq \alpha$ for all $n \geq 0$ by complete induction on n.

Case 1. $n = 0$. Then $N(n) = [0, c_1]$ and $|Df^n(x)| = 1 \geq \alpha$ for all $x \in N(n)$.

Case 2. $n > 0$. Take $x \in \partial N(n)$. By lemma 4.5, either $x_i = x_n$ for some $0 \leq i < n$ and $x \in N(i)$, or $x = c_1$ and n is closest return to c_1. Suppose $x_i = x_n$ for some $0 \leq i < n$ and $x \in N(i)$. Then $|Df^i(x)| \geq \alpha$ by the inductive hypothesis and $|Df^{n-i}(x_i)| \geq 1$ because $f^{n-i}(x_i) = x_i$ (f has no periodic attractors). Thus $|Df^n(x)| \geq \alpha$. If $x = c_1$ and n is a closest return to c_1, then $|Df^n(x)| \geq \alpha$ by hypothesis. Therefore $|Df^n(x)| \geq \alpha$ for all $x \in \partial N(n)$. It follows from lemma A.7 that $\inf_{N(n)} |Df^n| \geq \alpha$.

This proves that $\inf_{N(n)} |Df^n| \geq \alpha$ for all $n \geq 0$. □

Lemma 4.8 *Suppose f is S-unimodal and has no periodic attractors. Then for every $0 \leq \rho < 1$ there exists $K > 0$ such that if $x \in [0, c_1]$, $n \geq 0$ and $x_i \leq x$ for all $0 < i \leq n$, then $|Df^n(x)| \geq K\rho^n$.*

Proof. By lemma 4.7, there exists $1 \geq \alpha > 0$ such that $\inf_{N(n)} |Df^n| \geq \alpha$ for all $n \geq 0$. Take any $0 \leq \rho < 1$ and choose $N \geq 1$ large enough that $\rho^N \leq \alpha$. The critical point of f is not periodic since f has no periodic attractors. It follows that there exists a convex neighborhood U of c such that if $x \in f(U)$ and $x_i \in f(U)$ for some $i > 0$, then $i \geq N$. By Mañé's lemma (lemma A.3) there exist $K_1 > 0$ and $\lambda_1 > 1$ such that if $x \in [0,1]$, $n \geq 0$ and $x_i \notin U$ for all $0 \leq i < n$, then $|Df^n(x)| \geq K_1 \lambda_1^n$. Put $K = \alpha K_1$. Suppose $x \in [0, c_1]$, $n \geq 0$ and $x_i \leq x$ for all $0 < i \leq n$. We shall show that $|Df^n(x)| \geq K\rho^n$.

By lemma 4.6, there exists $m \geq 0$ and non-negative integers $i_0, \ldots, i_m$ such that $0 = i_0 < i_1 < \cdots < i_m = n$, $x = x_{i_0} \geq x_{i_1} \geq \cdots \geq x_{i_m} = x_n$ and $x_{i_j} \in N(i_{j+1} - i_j)$ for all $0 \leq j < m$. In particular, $|Df^{i_{j+1}-i_j}(x_{i_j})| \geq \alpha$ for all $0 \leq j < m$.

Case 1. $x_{i_j} \notin f(U)$ for all $0 \leq j < m$. Then $x_i \notin f(U)$ for all $0 \leq i \leq n$. This follows from the definition of the sets $N(i_{j+1} - i_j)$. Thus $x_i \notin U$ for all $0 \leq i < n$, which implies that $|Df^n(x)| \geq K_1 {\lambda_1}^n \geq K\rho^n$.

Case 2. $x_{i_j} \in f(U)$ for some $0 \leq j < m$. Take $s = \max\{0 \leq j < m \mid x_{i_j} \in f(U)\}$. We have

$$|Df^n(x)| = |Df^{i_{s+1}-i_s}(x_{i_s})|\,|Df^{n-i_{s+1}}(x_{i_{s+1}})| \prod_{0 \leq j < s} |Df^{i_{j+1}-i_j}(x_{i_j})|. \tag{18}$$

If $0 \leq j < s$ then $x_{i_j} \in f(U)$ and $x_{i_{j+1}} = f^{i_{j+1}-i_j}(x_{i_j}) \in f(U)$, so $i_{j+1} - i_j \geq N$ from the choice of U. But then $|Df^{i_{j+1}-i_j}(x_{i_j})| \geq \alpha \geq \rho^{i_{j+1}-i_j}$ by the choice of N. If $s < j < m$ then $x_{i_j} \notin f(U)$, so as in case 1 we have $|Df^{n-i_{s+1}}(x_{i_{s+1}})| \geq K_1 \lambda_1^{n-i_{s+1}} \geq K_1 \rho^{n-i_{s+1}}$. Finally, $|Df^{i_{s+1}-i_s}(x_{i_s})| \geq \alpha \geq \alpha\rho^{i_{s+1}-i_s}$. Thus $|Df^n(x)| \geq \alpha\rho^{i_{s+1}-i_s} K_1 \rho^{n-i_{s+1}} \rho^{i_s} \geq K\rho^n$. □

Proposition 4.9 (Subexponential lower bound) *Suppose f is S-unimodal and has no periodic attractors. Then for every $0 \leq \rho < 1$ there exists $K > 0$ such that $|Df^n(c_1)| \geq K\rho^n$ for all $n \geq 0$.*

Proof. Take $x = c_1$ in lemma 4.8. □

Notes.

1. The point is that $\rho < 1$ can be chosen arbitrarily close to 1. The price to pay is that $K > 0$ may then be very small.

2. An alternative formulation of the proposition is: if an S-unimodal map has negative lower Lyapunov exponent at c_1, then it has a periodic attractor. It was already known that a negative upper Lyapunov exponent at c_1 implies a periodic attractor. Indeed, if an S-unimodal map has no periodic attractors then there is a lower bound for $|Df^n(c_1)|$ at closest returns to c as mentioned at the start of this section. Therefore $\lambda^+(c_1) \geq 0$.

Corollary 4.10 (Nowicki [25]) *Suppose f is S-unimodal and has no periodic attractors. Then there exist $K > 0$ and $M > 1$ such that $|c_{1+n} - c_1| \geq KM^{-n}$ for all $n > 0$. In fact $M = (\sup_{[0,1]} |Df|)^{\ell/(\ell-1)} > 1$, where $\ell > 1$ is the order of the critical point.*

Proof. It is enough to consider closest returns to c_1. By proposition 4.4, there exists $\alpha > 0$ such that $|Df^n(c_1)| \geq \alpha$ for every $n > 0$ which is a closest return to c_1. By non-flatness, there exists $C > 0$ such that $|Df(x)| \leq C|f(x) - c_1|^{(\ell-1)/\ell}$ for all $x \in [0,1]$. Put $L = \sup_{[0,1]} |Df| > 1$ and $K = (L\alpha/C)^{\ell/(\ell-1)} > 0$.

If $n > 0$ is a closest return to c_1, then $\alpha \leq |Df^n(c_1)| \leq |Df(c_n)|\, L^{n-1} \leq CL^{n-1}|c_{1+n} - c_1|^{(\ell-1)/\ell}$. Rearranging, one obtains $|c_{1+n} - c_1| \geq KM^{-n}$. □

5 Hyperbolic structure and central expansion

In this section we prove that if f has a uniform hyperbolic structure, then it is centrally expanding (see below for the definition). This result plays a key role in proving that the Collet-Eckmann condition is satisfied. Firstly, it implies the non-existence of almost-saddle-node cascades (lemma 6.6). Secondly, it enables us to prove that closest return $\mathcal{T}$-branches have large average slope (lemma 6.3).

We also take the opportunity to introduce a number of useful if somewhat technical hyperbolicity conditions, and to prove that they are equivalent to having a uniform hyperbolic structure (theorem 5.3).

Definition. Let f be unimodal and suppose $\lambda > 1$. If $|Df^n(x)| \geq \lambda^n$ for every $n > 0$ and $x \in [0,1]$ such that $f^n(x) = x$, then we say that f has a *λ-hyperbolic structure*. If f has a λ-hyperbolic structure for some $\lambda > 1$, then we say that f has a *uniform hyperbolic structure*, sometimes said a *uniform hyperbolic structure on the periodic points* [25].

Note. Uniform hyperbolic structure is equivalent to

$$\inf\{\lambda(x) \mid x \in [0,1] \text{ and } f^n(x) = x \text{ for some } n > 0\} > 0. \tag{19}$$

The Lyapunov exponent $\lambda(x)$ was defined in section 1.

Definition. Let f be S-unimodal. If f has no periodic attractors and there exist $K > 0$ and $\lambda > 1$ such that $|c_{1+S} - c_1| \geq K\lambda^S|\alpha_S - c_1|$ for all $S \in \mathcal{S}$, then we say that f is *centrally expanding*.

Notes.

1. One can manipulate the constant λ in order to cover small iterates and get rid of the constant K.

2. Central expansion is equivalent to $|c_S - c| \geq K_1 {\lambda_1}^S |a_S - c|$ for all $S \in \mathcal{S}$, where $\lambda_1 = \lambda^{1/\ell}$ and K_1 depends on K, ℓ and non-flatness constants. This last is the usual definition of central expansion [25, 28]. See section 2 for the definition of a_S.

3. On one side of c (the side where c_S lies) one can get rid of K_1 by manipulating λ_1, however the estimate on the other side may be influenced by the nonsymmetry. If f is symmetric then one can get rid of K_1 totally.

4. The presence of the constant K_1 in the non-symmetric case creates a number of technical difficulties for the proof of some parts of theorem 5.3. These parts of theorem 5.3 are not used in the proof of theorem B. We treat the symmetric case here and refer to [28] for the general case.

Definition. Let f be S-unimodal. If f has no periodic attractors and there exists $\lambda > 1$ such that $|Df^S(\pi_S)| \geq \lambda^S$ for all $S \in \mathcal{S}$, then we say that f has a *central hyperbolic structure.*

Note. If f has a uniform hyperbolic structure then it certainly has a central hyperbolic structure. Therefore the central hyperbolic structure condition is a priori weaker than uniform hyperbolic structure.

Lemma 5.1 *Suppose f is S-unimodal and has a central hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that $|c_{1+S} - \pi_S| \geq K\lambda^S |\pi_S - c_1|$ for all $S \in \mathcal{S}$.*

Proof. Take $\lambda_1 > 1$ such that $|Df^S(\pi_S)| \geq \lambda_1^S$ for all $S \in \mathcal{S}$. By proposition 4.9 there exists $K_1 > 0$ such that $|Df^n(c_1)| \geq K_1 {\lambda_1}^{-n/2}$ for all $n \geq 0$.

If $S \in \mathcal{S}$ then we have

$$\frac{|c_{1+S} - \pi_S|}{|\pi_S - c_1|} \geq \sqrt{|Df^S(\pi_S)|\,|Df^S(c_1)|} \geq \sqrt{K_1}\,\sqrt[4]{\lambda_1}^{\,S} \tag{20}$$

by lemma A.6(3) with $I = [\pi_S, c_1]$ and $g = f^S$. Take $K = \sqrt{K_1}$ and $\lambda = \sqrt[4]{\lambda_1}$. $\square$

Proposition 5.2 *Suppose f is S-unimodal and has a central hyperbolic structure. Then f is centrally expanding.*

Proof. We must show that there exist $K > 0$ and $\lambda > 1$ such that $|c_{1+S} - c_1| \geq K\lambda^S |\alpha_S - c_1|$ for all $S \in \mathcal{S}$. By lemma 5.1 there exist $K_1 > 0$ and $\lambda > 1$ such that $|c_{1+S} - \pi_S| \geq K\lambda^S |\pi_S - c_1|$ for all $S \in \mathcal{S}$. By lemma 2.7 there exists $K_2 \geq 1$ such that $|\alpha_S - c_1| \leq K_2 |\pi_S - c_1|$ for all $S \in \mathcal{S}$. Take $K = K_1/K_2$. If $S \in \mathcal{S}$ then

$$\frac{|c_{1+S} - c_1|}{|\alpha_S - c_1|} \geq \frac{|c_{1+S} - \pi_S|}{K_2 |\pi_S - c_1|} \geq \frac{K_1}{K_2}\lambda^S, \tag{21}$$

which proves central expansion. $\square$

Definition. Let f be unimodal. If there exist $K > 0$ and $\lambda > 1$ such that $|f^n(x) - f^n(y)| \geq K\lambda^n |x - y|$ for all $n > 0$ and $x, y \in [0, 1]$ for which $f^n(x) = c$ and $f^n|_{[x,y]}$ is monotone, then we say that f is *uniformly expanding on the c-level.*

Definition. Let f be unimodal. If there exist $K > 0$ and $\lambda > 1$ such that $|I| \leq K\lambda^{-n}$ for every $n > 0$ and interval $I \subseteq [0, 1]$ for which $f^n|_I$ is monotone, then we say that f has *exponential decay of cylinders.*

Theorem 5.3 *Suppose f is S-unimodal. Then the following are equivalent:*

1. *f is centrally expanding.*
2. *f is uniformly expanding on the c-level.*
3. *f has exponential decay of cylinders.*
4. *f has a uniform hyperbolic structure.*
5. *f has a central hyperbolic structure.*

Proof. The implications $1 \Leftrightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ were already known [25, 28]. Proposition 5.2 ($5 \Rightarrow 1$) was not known, and completes the circle of ideas. We sketch the other proofs for completeness.

($1 \Rightarrow 2$). We recall the proof in the symmetric case which shows the essential ideas without struggling too much with unpleasant constants. We refer to [28] for the general case.

Since f is symmetric, we may assume that there is a $\lambda > 1$ such that $|c_S - c|/|a_S - c| \geq \lambda^S$ for any $S \in \mathcal{S}$ (without the leading constant, see the remarks above). We will also suppose that $1 < \lambda \leq |Df(0)|$. We proceed by induction and prove that for any $n \geq 0$:

$$\text{If } f^n|_{[x,y]} \text{ is monotone and } f^n(x) = c, \text{ then } |x_n - y_n| \geq K\lambda^n |x - y|. \quad (22)$$

The value of $K > 0$ is chosen small enough that (22) holds for $n = 0, 1$, and small enough to cover the constants that appear below. Note that these constants do not accumulate because of the symmetry assumption.

Suppose $n > 1$ and that (22) holds for all $0 \leq k < n$. First we prove (22) when y is maximal, i.e. $Df^n(y) = 0$ or $y \in \{0, 1\}$. In the latter case, the maximal interval $[v, y] \ni x$ of monotonicity of f^n has the property that $f^n(v) = c_1$. The branch of f^n over $[x, y]$ is extendable, and we obtain $|x_n - y_n| \geq \text{const } |Df(0)|^n |x - y| \geq K\lambda^n |x - y|$, which proves (22).

Otherwise $Df^n(y) = 0$, so $f^k(y) = c$ for some $0 \leq k < n$. Clearly $f^k|_{[x,y]}$ is monotone, so by the inductive assumption $|x_k - y_k| \geq K\lambda^k |x - y|$. Moreover,

f^{n-k} is monotone on $f^k[x,y] \equiv [z,c]$, with $f^{n-k}(z) = f^n(x) = c$. Therefore $n-k = S \in \mathcal{S}$, $z = a_S$ or $z = \tau(a_S)$, and

$$|x_n - y_n| = |c_S - c| \geq \lambda^{n-k}|z-c| = \lambda^{n-k}|x_k - y_k| \geq K\lambda^n|x-y| \qquad (23)$$

by central expansion. This proves (22) in this case.

If y is not maximal then we deduce (22) from the maximal case using lemma A.6(2).

($2 \Rightarrow 1$). By definition.

($2 \Rightarrow 3$). First note that if $n > 0$ and z are such that $f^n(z) = c$ and $f^n|_{[c,z]}$ is monotone, then $n = S_i \in \mathcal{S}$ and $z = a_S$ or $z = \tau(a_S)$. Therefore $f^{2n+1}|_{[c,z]}$ is not monotone, since $S_{i+1} \leq 2S_i$ (lemma 2.8(1)).

If $[x,y]$ is maximal with $f^n(x) = c$ and $f^n|_{[x,y]}$ monotone, then $f^{2n+1}|_{[x,y]}$ is not monotone. Indeed, let $0 \leq k < n$ be such that $f^k(y) = c$. Then $n-k \in \mathcal{S}$ and $f^{2(n-k)+1}|_{[x_k,y_k]}$ is not monotone as shown above, so $f^{2n-k+1}|_{[x,y]}$ is not monotone.

So suppose that $n > 0$ and $f^n|_I$ is monotone. We may assume that $I = [a,b]$ is maximal. Then by the above there exists $m > n/2$ such that $f^m|_I$ is monotone and $f^m(a) = c$ or $f^m(b) = c$. Therefore, by uniform expansion on the c-level, $|I| \leq \lambda^{-m}/K \leq \lambda^{-n/2}/K$.

($3 \Rightarrow 4$). Suppose $f^n(x) = x$ and take any $\epsilon > 0$. Clearly f has no periodic attractors, so $|Df^n(x)| > 1$. Choose an interval I_0 containing x such that $f^n|_{I_0}$ is a diffeomorphism onto its image, $\sup_{I_0}|Df^n| \leq |Df^n(x)|^{1+\epsilon}$ and $I_0 \subseteq f^n(I_0)$. Take $m > 0$ and define $I_m = (f^n|_{I_0})^{-m}(I_0)$.

Clearly $\sup_{I_m}|Df^{mn}| \leq |Df^n(x)|^{m(1+\epsilon)}$, so $|I_m| \geq |I_0|/|Df^n(x)|^{m(1+\epsilon)}$. Note that $f^{mn}|_{I_m}$ is monotone, so from exponential decay of cylinders we obtain $|I_m| \leq K\lambda^{-mn}$. Thus $|Df^n(x)|^{1+\epsilon} \geq (|I_0|/K)^{1/m}\lambda^n$. Since $m > 0$ and $\epsilon > 0$ were arbitrary, this yields $|Df^n(x)| \geq \lambda^n$.

($4 \Rightarrow 5$). By definition. $\square$

6 The Collet-Eckmann condition

In this section we prove that an S-unimodal map f satisfies the Collet-Eckmann condition if and only if it has a uniform hyperbolic structure (theorem 6.11). Since it was already known that the Collet-Eckmann condition implies the presence of a uniform hyperbolic structure [28], we relegate the proof of this part to the end of the section. The rest of the section is devoted to proving that a uniform hyperbolic structure implies the Collet-Eckmann condition.

We start by proving that the Collet-Eckmann condition is satisfied at closest returns to c_1. A closest return to c_1 is either an $\mathcal{S}$-time or a $\mathcal{T}$-time. In the $\mathcal{S}$ case we simply compare the derivative at c_1 with that at the periodic point π_S, see lemma 6.1. The $\mathcal{T}$ case is more difficult since there is no periodic point to compare with.

The idea in the $\mathcal{T}$ case is to find places on the c_1-branch of f^T where the slope is exponentially big, and then to pull this derivative down the branch to c_1, c.f. figure 5. Derivative pull-back works because there is an identifiable region of the c_1-branch of f^T on which f^T is an approximately-affine rescaling of $x \mapsto x^\ell$. Here ℓ is the order of the critical point of f. Points in this region are said to be well-placed in f^T.

A key observation is that one can calculate the relationship between the derivative of f^T at a well-placed point x and at its inverse image y under the c_1-branch of f^T, in essence $|Df^T(y)| \gtrapprox |Df^T(x)|^{1/\ell}$ (see lemma 6.5): if the derivative at x is exponentially big, then so is the derivative at y, but with the loss of a factor ℓ in the rate. Taking successive pull-backs of the derivative at x in this way until we reach c_1, we obtain $|Df^T(c_1)| \gtrapprox |Df^T(x)|^{1/\ell^n}$, where n is the number of backward iterates needed to reach c_1. This will be exponentially big if the derivative at x is exponentially big and the number of backward iterates n is bounded independently of T.

It turns out that the slope of f^T is exponentially big whenever there is an intersection between the central branch of f^T and a previous $\mathcal{S}$-branch (lemma 6.3). Since one can always find a previous $\mathcal{S}$-branch which intersects f^T in the well-placed region, there always exists a well-placed point x with exponentially big derivative (lemma 6.4). We apply derivative pull-back to this point.

Maps without an upper bound for the number of iterates needed to go from one end of a c_1 $\mathcal{T}$-branch to the other are said to suffer from *almost-saddle-node cascades*. Using proposition 5.2 of the previous section, it is easy to show that this does not occur for S-unimodal maps with a uniform hyperbolic structure (lemma 6.6). The number of pull-backs n being indeed bounded independently of T, we obtain in this way the required exponentially large lower bound for the derivative at c_1 at closest $\mathcal{T}$ return times to c_1: the Collet-Eckmann condition is satisfied at closest returns to c_1.

Once we know that the Collet-Eckmann condition is satisfied at closest returns to c_1, it is relatively straightforward to prove that f satisfies the Collet-Eckmann condition at all times. This is done in lemmas 6.9 and 6.10.

Definition. Let f be unimodal. If there exist $K > 0$ and $\lambda > 1$ such that $|Df^n(c_1)| \geq K\lambda^n$ for all $n > 0$, then we say that f satisfies the *Collet-Eckmann condition*, see [6, C1, page 14].

Note. The Collet-Eckmann condition is equivalent to $\lambda_-(c_1) > 0$.

In section 5 it was shown that uniform hyperbolic structure is equivalent to the a priori weaker condition of central hyperbolic structure. In what follows we refer to whichever of uniform hyperbolic structure or central hyperbolic structure seems most appropriate.

Lemma 6.1 (Closest $\mathcal{S}$ returns) *Suppose f is S-unimodal and has a central hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that if $S \in \mathcal{S}$ is a closest return to c_1 then $|Df^S(c_1)| \geq K\lambda^S$.*

Proof. This follows from lemma 4.1. □

That is, the Collet-Eckmann condition is satisfied at closest $\mathcal{S}$ returns to c_1.

In the case of a closest $\mathcal{T}$ return to c_1, we first show that there is exponentially big slope wherever the c_1-branch of f^T is crossed by a previous $\mathcal{S}$-branch. This is done in lemma 6.3.

Lemma 6.2 *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\epsilon > 0$ such that if $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$ and $S < T$, then $|f^T(c_1) - f^S(\omega_S)| \geq \epsilon\, |f^S(\omega_S) - c_1|$.*

Proof. By theorem 3.2, there exists $\delta > 0$ such that $|c_{1+S} - \pi_S| \geq \delta\, |\pi_S - c_1|$ for all $S \in \mathcal{S}$. If $c_1 = 1$ then $\mathcal{T} = \emptyset$ and there is nothing to prove, so suppose $c_1 < 1$ which implies $c_2 > 0$. Put $\epsilon = \min\{\delta/(1+\delta), c_2/c_1\} > 0$. Suppose $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$ and $S < T$.

If $S < T_0$ then $\omega_S = 1$ by lemma 2.1, so $f^S(\omega_S) = 0$ and

$$\frac{|f^T(c_1) - f^S(\omega_S)|}{|f^S(\omega_S) - c_1|} \geq \frac{c_2}{c_1} \geq \epsilon \tag{24}$$

since $f^T(c_1) \in [c_2, c_1]$. If $S > T_0$ then, by lemma 2.8(3), $S - \mathcal{T}(S) \in \mathcal{S}$ and $f^S(\omega_S) = f^{S-\mathcal{T}(S)}(c_1)$. Put $\bar{T} = \mathcal{T}(S)$ and $\bar{S} = S - \mathcal{T}(S)$.

By lemma 2.10(1) there exists $x \in (\pi_S, \omega_S)$ such that $f^S(x) = f^{\bar{T}}(x)$ and $f^S(x) = \pi_{\bar{S}}$. We have $f^T(c_1) > \max\{f^S(c_1), f^{\bar{T}}(c_1)\}$ since T is a closest return to c_1 and $\bar{T} < S < T$; it follows that $f^T(c_1) \geq \pi_{\bar{S}}$. Now $|f^{\bar{S}}(c_1) - \pi_{\bar{S}}| \geq \delta\, |\pi_{\bar{S}} - c_1|$, so

$$\begin{aligned} |f^S(\omega_S) - c_1| &= |f^{\bar{S}}(c_1) - \pi_{\bar{S}}| + |\pi_{\bar{S}} - c_1| & (25)\\ &\geq (1+\delta)\, |\pi_{\bar{S}} - c_1| \geq (1+\delta)\, |f^T(c_1) - c_1|. & (26) \end{aligned}$$

Therefore

$$\frac{|f^T(c_1) - f^S(\omega_S)|}{|f^S(\omega_S) - c_1|} \geq 1 - \frac{|f^T(c_1) - c_1|}{|f^S(\omega_S) - c_1|} \geq \frac{\delta}{1+\delta} \geq \epsilon \tag{27}$$

as required. □

Lemma 6.3 (Average slope estimate) *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ with the following property: if $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$, $S < T$ and $f^S(x) = f^T(x)$ for some $x \in (\alpha_T, c_1)$, then*

$$\frac{|f^T(x) - f^T(c_1)|}{|x - c_1|} \geq K\lambda^T. \tag{28}$$

Proof. Take $\lambda_1 > 1$ such that f has a λ_1-hyperbolic structure. By proposition 5.2, f is centrally expanding: there exist $K_2 > 0$ and $\lambda_2 > 1$ such that $|c_{1+S} - c_1| \geq K_2\lambda_2^S|\alpha_S - c_1|$ for all $S \in \mathcal{S}$. By lemma 6.2, there exists $\epsilon > 0$ such that if $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$ and $S < T$, then $|f^T(c_1) - f^S(\omega_S)| \geq \epsilon\,|f^S(\omega_S) - c_1|$. Put $\rho = {\lambda_1}^{-1/4}$. By proposition 4.9, there exists $1 \geq K_0 > 0$ such that $|Df^n(c_1)| \geq K_0\rho^n$ for all $n \geq 0$. By lemma 4.3, there exists $\alpha > 0$ such that if $T \in \mathcal{T}$ is a closest return to c_1 then $|Df^T(c_1)| \geq \alpha$. Put $K = \min\{\epsilon K_2/2, \sqrt{\alpha K_0}, \epsilon\sqrt{\alpha}/(2+\epsilon)\}$ and $\lambda = \min\{\sqrt[4]{\lambda_1}, \sqrt[2]{\lambda_2}\}$. Suppose $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$, $S < T$ and $f^S(x) = f^T(x)$ for some $x \in (\alpha_T, c_1)$.

Now $f^{T-S}(f^S(x)) = f^S(x)$, so $f^S(x)$ is periodic of period $T - S$. Therefore $|Df^{T-S}(f^S(x))| \geq {\lambda_1}^{T-S}$ by the λ_1-hyperbolic structure. Note that $\pi_S < x < c_1$ since $f^S(x) = f^T(x) < x$, c.f. the note after lemma 2.1.

Case 1. $S \leq T/2$. Now $|Df^S(\pi_S)| \geq 1$ and $|Df^S(c_1)| \geq K_0\rho^S$, so we have $|Df^S(x)| \geq \min\{|Df^S(\pi_S)|, |Df^S(c_1)|\} \geq K_0\rho^S$ by lemma A.6(1). It follows that

$$|Df^T(x)| = |Df^{T-S}(f^S(x))|\,|Df^S(x)| \geq K_0{\lambda_1}^{T-S}\rho^S \geq K_0{\lambda_1}^{T/2} \tag{29}$$

since $S \leq T/2$. Hence by lemma A.6(3),

$$\frac{|f^T(x) - f^T(c_1)|}{|x - c_1|} \geq \sqrt{|Df^T(x)|\,|Df^T(c_1)|} \geq \sqrt{\alpha K_0}\,{\lambda_1}^{T/4} \geq K\lambda^T. \tag{30}$$

Case 2. $S \geq T/2$ and $|f^T(x) - f^T(c_1)| \geq |f^T(c_1) - f^S(\omega_S)|/2$. Then

$$\begin{aligned} |f^T(x) - f^T(c_1)| &\geq \frac{|f^T(c_1) - f^S(\omega_S)|}{2} && (31)\\ &\geq \epsilon\frac{|f^S(\omega_S) - c_1|}{2} \geq \epsilon\frac{|c_{1+S} - c_1|}{2}. && (32)\end{aligned}$$

Thus

$$\frac{|f^T(x) - f^T(c_1)|}{|x - c_1|} \geq \frac{\epsilon}{2}\frac{|c_{1+S} - c_1|}{|\alpha_S - c_1|} \geq \frac{\epsilon K_2{\lambda_2}^S}{2} \geq K\lambda^T. \tag{33}$$

Case 3. $S \geq T/2$ and $|f^T(x) - f^T(c_1)| \leq |f^T(c_1) - f^S(\omega_S)|/2$. Then

$$|f^S(\omega_S) - f^S(x)| = |f^S(\omega_S) - f^T(x)| \tag{34}$$

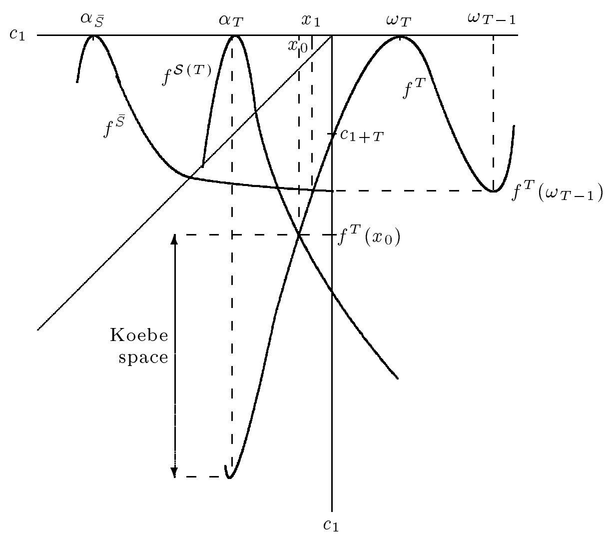

Figure 4: Case 2 of lemma 6.4. The point x_1 is well-placed in f^T.

$$
\begin{aligned}
&= |f^S(\omega_S) - f^T(c_1)| - |f^T(x) - f^T(c_1)| && (35)\\
&\geq \frac{|f^T(c_1) - f^S(\omega_S)|}{2} && (36)\\
&\geq \epsilon\frac{|f^S(\omega_S) - c_1|}{2} \geq \epsilon\frac{|f^S(x) - f^S(\pi_S)|}{2}. && (37)
\end{aligned}
$$

The one-sided Koebe principle (lemma A.6(4)) with $I = [\omega_S, \pi_S]$ and $g = f^S$ yields $|Df^S(x)| \geq |Df^S(\pi_S)|/K(\epsilon/2) \geq {\lambda_1}^S/K(\epsilon/2)$, where $K(\epsilon/2) = (2+\epsilon)^2/\epsilon^2$. As in case 1, $|Df^T(x)| = |Df^{T-S}(f^S(x))|\,|Df^S(x)| \geq \epsilon^2{\lambda_1}^T/(2+\epsilon)^2$. Therefore

$$\frac{|f^T(x) - f^T(c_1)|}{|x - c_1|} \geq \sqrt{|Df^T(x)|\,|Df^T(c_1)|} \geq \frac{\epsilon\sqrt{\alpha}}{2+\epsilon}{\lambda_1}^{T/2} \geq K\lambda^T \tag{38}$$

by lemma A.6(3). □

Definition. Let f be S-unimodal and suppose f has no periodic attractors. If $\delta > 0$, $T \in \mathcal{T}$ and $x \in [\alpha_T, c_1]$ are such that $|f^T(\omega_{T-1}) - c_1| \geq \delta\,|f^T(x) - c_1|$ and $|f^T(\alpha_T) - f^T(x)| \geq \delta\,|f^T(x) - c_1|$, then we say that x *is δ-well-placed in f^T*.

Lemma 6.4 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $\delta > 0$, $K > 0$ and $\lambda > 1$ with the following property: if $T \in \mathcal{T}$*

is a closest return to c_1 then there exists $x \in (\alpha_T, c_1)$ such that x is δ-well-placed in f^T and $|Df^T(x)| \geq K\lambda^T$.

Proof. The idea is that one can always find a previous $\mathcal{S}$-branch that intersects the c_1-branch of f^T in the well-placed region, see figure 4. By corollary 3.3(2) there exists $1 \geq \delta > 0$ such that if $T \in \mathcal{T}$ then $|f^T(\alpha_T) - f^T(x_0)| \geq \delta\, |f^T(x_0) - c_1|$ where x_0 is the unique point in (α_T, ω_T) for which $f^T(x_0) = f^{\mathcal{S}(T)}(x_0)$, see lemma 2.10(2). By lemma 6.3 there exist $K > 0$ and $\lambda > 1$ such that if $T \in \mathcal{T}$ is a closest return to c_1, $S \in \mathcal{S}$, $S < T$ and $f^S(x) = f^T(x)$ for some $x \in (\alpha_T, c_1)$, then $|f^T(x) - f^T(c_1)| \geq K\lambda^T |x - c_1|$.

Suppose $T \in \mathcal{T}$ is a closest return to c_1. Let x_0 be the unique point in (α_T, ω_T) for which $f^T(x_0) = f^{\mathcal{S}(T)}(x_0)$. In fact $x_0 \in (\alpha_T, c_1)$ because T is a closest return to c_1, see lemma 2.10(2).

Case 1. $f^T(x_0) \geq f^T(\omega_{T-1})$. By lemma 6.3, $|f^T(x_0) - f^T(c_1)| \geq K\lambda^T |x_0 - c_1|$, so there exists $x \in (x_0, c_1)$ with $|Df^T(x)| \geq K\lambda^T$ by the mean-value theorem. We claim that x is δ-well-placed in f^T. Indeed, $|f^T(\omega_{T-1}) - c_1| \geq |f^T(x_0) - c_1| \geq |f^T(x) - c_1| \geq \delta\, |f^T(x) - c_1|$ since $\delta \leq 1$ and

$$\frac{|f^T(\alpha_T) - f^T(x)|}{|f^T(x) - c_1|} \geq \frac{|f^T(\alpha_T) - f^T(x_0)|}{|f^T(x_0) - c_1|} \geq \delta \tag{39}$$

since $f^T(x_0) < f^T(x) < c_1$.

Case 2. $f^T(x_0) \leq f^T(\omega_{T-1})$. Then $\omega_{T-1} \neq 1$, so $T > T_0$. By lemma 2.8(2), $T - \mathcal{T}(T) = \bar{S}$ for some $\bar{S} \in \mathcal{S}$ and $f^T(\omega_{T-1}) = f^{\bar{S}}(c_1)$. Since $\bar{S} < T$ we have $f^{\bar{S}}(c_1) < f^T(c_1)$ because T is closest return to c_1. Now $f^{\bar{S}}(\alpha_T) > f^{\bar{S}}(c_1) = f^T(\omega_{T-1}) \geq f^T(x_0) > f^T(\alpha_T)$ because $f^{\bar{S}}\big|_{[\alpha_T, c_1]}$ is decreasing and $f^T\big|_{[\alpha_T, c_1]}$ is increasing. Thus $f^{\bar{S}}(x_1) = f^T(x_1)$ for some (unique) $x_1 \in (\alpha_T, c_1)$.

By lemma 6.3, $|f^T(x_1) - f^T(c_1)| \geq K\lambda^T |x_1 - c_1|$, so there exists $x \in (x_1, c_1)$ such that $|Df^T(x)| \geq K\lambda^T$ by the mean-value theorem. We claim that x is δ-well-placed in f^T. Indeed, $|f^T(\omega_{T-1}) - c_1| \geq |f^T(x) - c_1| \geq \delta\, |f^T(x) - c_1|$ since $f^T(\omega_{T-1}) = f^{\bar{S}}(c_1) \leq f^{\bar{S}}(x_1) = f^T(x_1) \leq f^T(x) \leq c_1$ and $\delta \leq 1$. Moreover,

$$\frac{|f^T(\alpha_T) - f^T(x)|}{|f^T(x) - c_1|} \geq \frac{|f^T(\alpha_T) - f^T(x_0)|}{|f^T(x_0) - c_1|} \geq \delta \tag{40}$$

since in $f^T(x_0) \leq f^T(\omega_{T-1}) \leq f^T(x) \leq c_1$. $\square$

The definition of x being well-placed implies that $f^{T-1}[\alpha_T, \omega_{T-1}]$ is a scaled neighborhood of $f^{T-1}[x, \omega_T]$, so f^{T-1} is approximately affine on $[x, \omega_T]$ by the Koebe principle, lemma A.6(5). This means that we can treat f^T as being an approximately-affine rescaling of f on the well-placed region. Now f is itself an approximately affine rescaling of $x \to x^\ell$, so one expects the derivative of f^T to have similar properties to the derivative of an affine rescaling of $x \to x^\ell$. This is the content of the following lemma:

Lemma 6.5 (Derivative pull-back) *Suppose f is S-unimodal, has critical order $\ell > 1$ and has no periodic attractors. Then for every $\delta > 0$ there exists $K > 1$ with the following property: if $T \in \mathcal{T}$, $x \in [\alpha_T, c_1]$ and x is δ-well-placed in f^T, then*

$$|Df^T(y)|^{\ell/(\ell-1)} \geq \frac{|Df^T(x)|^{1/(\ell-1)}}{K} - K\,|Df^T(c_1)|^{1/(\ell-1)}. \tag{41}$$

Here $y = \left(f^T\big|_{[\alpha_T,\omega_T]}\right)^{-1}(x)$.

Proof. Choose $L \geq 1$ such that $f \in S(\ell, L)$ and suppose $\delta > 0$. Put $\tau = (1+\delta/L^2)^{1/\ell} - 1 > 0$. We take $K = \ell(L^2 K(\tau))^{\ell/(\ell-1)}$, where $K(\tau) = (1+\tau)^2/\tau^2$ comes from lemma A.6. Suppose $T \in \mathcal{T}$, $x \in [\alpha_T, c_1]$ and x is δ-well-placed in f^T.

We first show that $f^{T-1}[\alpha_T, \omega_{T-1}]$ is a τ-scaled neighborhood of $f^{T-1}[x, \omega_T]$. Indeed, from non-flatness, equation 2,

$$|f^T(\omega_{T-1}) - c_1| \leq L\,|f^{T-1}(\omega_{T-1}) - c|^\ell/\ell, \tag{42}$$

$$|f^T(x) - c_1| \geq |f^{T-1}(x) - c|^\ell/(L\ell) \tag{43}$$

and, integrating (1) over $[f^{T-1}(\alpha_T), f^{T-1}(x)]$,

$$|f^T(\alpha_T) - f^T(x)| \leq L\,\big|\,|f^{T-1}(\alpha_T) - c|^\ell - |f^{T-1}(x) - c|^\ell\,\big|/\ell. \tag{44}$$

Therefore

$$\frac{|f^{T-1}(\omega_{T-1}) - f^{T-1}(\omega_T)|}{|f^{T-1}(\omega_T) - f^{T-1}(x)|} \geq \left(\frac{|f^T(\omega_{T-1}) - c_1|}{L^2|f^T(x) - c_1|}\right)^{1/\ell} \geq \left(\frac{\delta}{L^2}\right)^{1/\ell} \geq \tau \tag{45}$$

since x is δ-well-placed in f^T (note that $f^{T-1}(\omega_T) = c$ since $f^T(\omega_T) = c_1$). Moreover,

$$\left(1 + \frac{|f^{T-1}(\alpha_T) - f^{T-1}(x)|}{|f^{T-1}(x) - c|}\right)^\ell - 1 \geq \frac{|f^T(\alpha_T) - f^T(x)|}{L^2|f^T(x) - c_1|} \geq \frac{\delta}{L^2} \tag{46}$$

from which it follows that

$$\frac{|f^{T-1}(\alpha_T) - f^{T-1}(x)|}{|f^{T-1}(x) - c|} \geq \left(1 + \frac{\delta}{L^2}\right)^{1/\ell} - 1 = \tau. \tag{47}$$

Thus $f^{T-1}[\alpha_T, \omega_{T-1}]$ is a τ-scaled neighborhood of $f^{T-1}[x, \omega_T]$.

Put $y = \left(f^T\big|_{[\alpha_T,\omega_T]}\right)^{-1}(x)$. We have $|f^{T-1}(x) - f^{T-1}(c_1)|/|x - c_1| = |Df^{T-1}(z)|$ for some $z \in [x, c_1]$. Also, $|x - c_1| = |f^T(y) - c_1| \leq L\,|f^{T-1}(y) - c|^\ell/\ell$ from non-flatness, equation 2. Therefore

$$L\,|f^{T-1}(y) - c|^\ell/\ell \geq |c_1 - x| = \frac{|f^{T-1}(x) - c| - |f^{T-1}(c_1) - c|}{|Df^{T-1}(z)|} \tag{48}$$

where we have used the fact that $f^{T-1}(c_1) \in [f^{T-1}(x), c]$.

For any $z_1 \in [0,1]$ such that $Df^{T-1}(z_1) \neq 0$, we obtain $|f^{T-1}(z_1)-c|^{\ell-1}/L \leq |Df^T(z_1)|/|Df^{T-1}(z_1)| \leq L\,|f^{T-1}(z_1) - c|^{\ell-1}$ from non flatness. In particular,

$$|f^{T-1}(x) - c| \geq \left(\frac{|Df^T(x)|}{L|Df^{T-1}(x)|}\right)^{1/(\ell-1)}, \tag{49}$$

$$|f^{T-1}(y) - c| \leq \left(\frac{L|Df^T(y)|}{|Df^{T-1}(y)|}\right)^{1/(\ell-1)} \tag{50}$$

and

$$|f^{T-1}(c_1) - c| \leq \left(\frac{L|Df^T(c_1)|}{|Df^{T-1}(c_1)|}\right)^{1/(\ell-1)}. \tag{51}$$

Substituting into (48) yields

$$|Df^T(y)|^{\ell/(\ell-1)} \geq \ell\left(\beta\frac{|Df^T(x)|^{1/(\ell-1)}}{L^{2\ell/(\ell-1)}} - \gamma\frac{|Df^T(c_1)|^{1/(\ell-1)}}{L^2}\right), \tag{52}$$

where

$$\beta = \frac{|Df^{T-1}(y)|^{\ell/(\ell-1)}}{|Df^{T-1}(z)|\,|Df^{T-1}(x)|^{1/(\ell-1)}} \tag{53}$$

and

$$\gamma = \frac{|Df^{T-1}(y)|^{\ell/(\ell-1)}}{|Df^{T-1}(z)|\,|Df^{T-1}(c_1)|^{1/(\ell-1)}}. \tag{54}$$

Since $f^{T-1}[\alpha_T, \omega_{T-1}]$ is a τ-scaled neighborhood of $f^{T-1}[x, \omega_T]$, we deduce from lemma A.6(5) that $1/K(\tau) \leq |Df^{T-1}(z_1)|/|Df^{T-1}(z_2)| \leq K(\tau)$ for all $z_1, z_2 \in [x, \omega_T]$. It follows that $\beta \geq K(\tau)^{-\ell/(\ell-1)}$ and $\gamma \leq K(\tau)^{\ell/(\ell-1)}$. Thus $|Df^T(y)|^{\ell/(\ell-1)} \geq |Df^T(x)|^{1/(\ell-1)}/K - K\,|Df^T(c_1)|^{1/(\ell-1)}$ where $K = \ell(L^2K(\tau))^{\ell/(\ell-1)}$. $\square$

Lemma 6.6 (No cascades) *Suppose f is S-unimodal and has a central hyperbolic structure. Then there exists $N > 1$ such that $T_{i+1} \leq NT_i$ for all $T_i \in \mathcal{T}$.*

Proof. By corollary 4.10, there exist $1 \geq K_0 > 0$ and $M > 1$ such that $|c_{1+n} - c_1| \geq K_0 M^{-n}$ for all $n > 0$. By proposition 5.2, f is centrally expanding: there exist $1 \geq K_1 > 0$ and $\lambda_1 > 1$ such that $|c_{1+S} - c_1| \geq K_1 {\lambda_1}^S |\alpha_S - c_1|$ for all $S \in \mathcal{S}$, which implies $|\alpha_S - c_1| \leq {\lambda_1}^{-S}/K_1$ for all $S \in \mathcal{S}$ since $|c_{1+S} - c_1| \leq 1$. By lemma 2.8(1), $S_{j+1} \leq 2S_j$ for all $j \geq 0$. Put $N = 2 + 2\log(M/(K_0K_1))/\log\lambda_1$.

Suppose $T_i \in \mathcal{T}$. By lemma 2.8(2), $T_{i+1} - T_i = S_j$ where clearly $S_j \in \mathcal{S}$ is minimal with $\alpha_{S_j} \in (c_{1+T_i}, c_1)$. If $j > 0$ then

$$\begin{aligned} K_0 M^{-T_i} &\leq |c_{1+T_i} - c_1| \leq |\alpha_{S_{j-1}} - c_1| && (55)\\ &\leq \frac{{\lambda_1}^{-S_{j-1}}}{K_1} \leq \frac{{\lambda_1}^{-S_j/2}}{K_1} && (56)\\ &= \frac{{\lambda_1}^{-(T_{i+1}-T_i)/2}}{K_1}. && (57) \end{aligned}$$

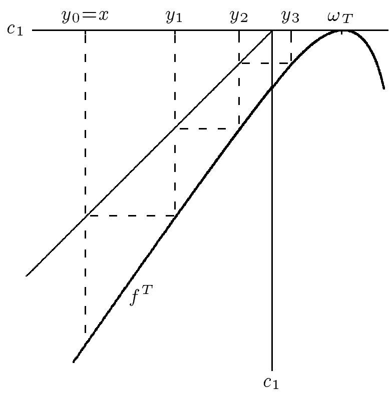

Figure 5: Pull-back of x down a $\mathcal{T}$-branch.

This gives $T_{i+1} \leq -2\log(K_0 K_1)/\log\lambda + T_i(1+2\log M/\log\lambda_1) \leq NT_i$. If $j = 0$ then $T_{i+1} = 1 + T_i \leq NT_i$. $\square$

One difficulty remains in showing that f satisfies the Collet-Eckmann condition at closest $\mathcal{T}$ returns to c_1: the term $-K\,|Df^T(c_1)|^{1/(\ell-1)}$ in equation 41. If this term was not present then the result would follow immediately from the preceding lemmas. It follows readily nonetheless:

Lemma 6.7 (Closest $\mathcal{T}$ returns) *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that if $T \in \mathcal{T}$ is a closest return to c_1, then $|Df^T(c_1)| \geq K\lambda^T$.*

Proof. If $\mathcal{T} = \emptyset$ then there is nothing to prove, so suppose $\mathcal{T} \neq \emptyset$. By lemma 6.6 there exists $N > 1$ such that $T_{i+1} \leq NT_i$ for all $i \geq 0$. By lemma 6.4 there exist $\delta > 0$, $1 \geq K_1 > 0$ and $\lambda_1 > 1$ such that if $T \in \mathcal{T}$ is a closest return to c_1, then there exists $x \in (\alpha_T, c_1)$ such that x is δ-well-placed in f^T and $|Df^T(x)| \geq K_1{\lambda_1}^T$. By lemma 6.5 there exists $K_0 > 1$ such that if $T \in \mathcal{T}$, $x \in [\alpha_T, c_1]$ and x is δ-well-placed in f^T, then $|Df^T(y)|^{\ell/(\ell-1)} \geq |Df^T(x)|^{1/(\ell-1)}/K_0 - K_0\,|Df^T(c_1)|^{1/(\ell-1)}$, where $y = \left(f^T\big|_{[\alpha_T,\omega_T]}\right)^{-1}(x)$. We put $K = K_1/(2{K_0}^2\,(2K_0)^{N/\ell})^{\ell-1} > 0$ and $\lambda = {\lambda_1}^{1/\ell^N} > 1$. Suppose $T \in \mathcal{T}$ is a closest return to c_1 and write $T = T_i$ where $i \geq 0$.

By lemma 6.4, there exists $x \in (\alpha_T, c_1)$ such that x is δ-well-placed in f^T and $|Df^T(x)| \geq K_1{\lambda_1}^T$. Take $n \geq 1$ minimal such that $f^{nT}(c_1) \leq x$. We claim that $n \leq N$. Indeed, $f^{nT}\big|_{[c_1,\omega_T]}$ is monotone, so $T_{i+1} \geq nT$. On the other hand, $T_{i+1} \leq NT$ by corollary 6.6, which proves the claim.

For $0 \le j \le n$, define $y_j = \left(f^T|_{[\alpha_T,\omega_T]} \right)^{-j}(x)$. Note that $x = y_0 < y_1 < \cdots < y_{n-1} < c_1 \le y_n$, see figure 5. Therefore $y_0, \ldots, y_{n-1}$ are δ-well-placed in f^T because x is.

Case 1. $|Df^T(y_j)| \ge (2{K_0}^2)^{l-1}|Df^T(c_1)|$ for all $0 \le j < n$.

Then $|Df^T(y_{j+1})|^{\ell/(\ell-1)} \ge |Df^T(y_j)|^{1/(\ell-1)}/(2K_0)$ for all $0 \le j < n$ since

$$|Df^T(y_{j+1})|^{\ell/(\ell-1)} \ge \frac{|Df^T(y_j)|^{1/(\ell-1)}}{K_0} - K_0\,|Df^T(c_1)|^{1/(\ell-1)} \tag{58}$$

by lemma 6.5. In other words, $|Df^T(y_j)| \ge |Df^T(y_{j-1})|^{1/\ell}/(2K_0)^{(\ell-1)/\ell}$ for all $0 < j \le n$. It follows that

$$|Df^T(y_j)| \ge \frac{|Df^T(x)|^{1/\ell^j}}{(2K_0)^{j(\ell-1)/\ell}} \ge \frac{{K_1}^{1/\ell^j}\,{\lambda_1}^{T/\ell^j}}{(2K_0)^{j(\ell-1)/\ell}} \ge \frac{K_1{\lambda_1}^{T/\ell^N}}{(2K_0)^{N(\ell-1)/\ell}} \ge K\lambda^T \tag{59}$$

for all $0 \le j \le n$ since $0 < K_1 \le 1$ and $n \le N$. By lemma A.6(3), $|Df^T(c_1)| \ge \min\{|Df^T(y_{n-1})|, |Df^T(y_n)|\} \ge K\lambda^T$.

Case 2. $|Df^T(y_j)| < (2{K_0}^2)^{\ell-1}|Df^T(c_1)|$ for some $0 \le j < n$.

Take $0 \le m < n$ minimal such that $|Df^T(y_m)| < (2{K_0}^2)^{\ell-1}|Df^T(c_1)|$. For $0 \le j < m$ we have $|Df^T(y_j)| \ge (2{K_0}^2)^{\ell-1}|Df^T(c_1)|$, so as above $|Df^T(y_j)| \ge K_1{\lambda_1}^{T/\ell^N}/(2K_0)^{N(\ell-1)/\ell}$ for all $0 \le j \le m$. Therefore

$$|Df^T(c_1)| > \frac{|Df^T(y_m)|}{(2{K_0}^2)^{\ell-1}} \ge \frac{K_1{\lambda_1}^{T/\ell^N}}{(2{K_0}^2\,(2K_0)^{N/\ell})^{\ell-1}} \ge K\lambda^T. \tag{60}$$

That is, $|Df^T(c_1)| \ge K\lambda^T$ in both cases. □

Definition. Let f be unimodal. If there exist $K > 0$ and $\lambda > 1$ such that $|Df^n(c_1)| \ge K\lambda^n$ whenever $n > 0$ is a closest return to c_1 then we say that f *satisfies the Collet-Eckmann condition at closest returns to* c_1.

Proposition 6.8 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then f satisfies the Collet-Eckmann condition at closest returns to c_1.*

Proof. If $n > 0$ is a closest return to c_1 then $n \in \mathcal{S} \cup \mathcal{T}$ by lemma 2.9. The result therefore follows from lemmas 6.1 and 6.7. □

It remains to be shown that f satisfies the Collet-Eckmann condition. In section 4 we showed that if the derivative is bounded below at closest returns to c_1 then there is a subexponentially decaying lower bound for $|Df^n(c_1)|$ for all $n > 0$. Using the same techniques, we now prove that if f has a uniform hyperbolic structure (and therefore satisfies the Collet-Eckmann condition at closest returns to c_1), then f satisfies the Collet-Eckmann condition.

Lemma 6.9 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that $\inf_{N(n)} |Df^n| \geq K\lambda^n$ for all $n \geq 0$.*

Proof. By proposition 6.8, f satisfies the Collet-Eckmann condition at closest returns to c_1. Take $1 \geq K > 0$ and $\lambda > 1$ such that f has a λ-hyperbolic structure and $|Df^n(c_1)| \geq K\lambda^n$ whenever $n > 0$ is a closest return to c_1. We shall show that $\inf_{N(n)} |Df^n| \geq K\lambda^n$ for all $n \geq 0$ by complete induction on n.

Case 1. $n = 0$. Then $N(n) = [0, c_1]$ and $|Df^n(x)| = 1 \geq K\lambda^n$ for all $x \in N(n)$.

Case 2. $n > 0$. Take $x \in \partial N(n)$. By lemma 4.5, either $x_i = x_n$ for some $0 \leq i < n$ and $x \in N(i)$, or $x = c_1$ and n is a closest return to c_1. Suppose $x_i = x_n$ for some $0 \leq i < n$ and $x \in N(i)$. Then $|Df^i(x)| \geq K\lambda^i$ by the inductive hypothesis and $|Df^{n-i}(x_i)| \geq \lambda^{n-i}$ from the λ-hyperbolic structure, since $f^{n-i}(x_i) = x_i$. Thus $|Df^n(x)| \geq K\lambda^n$. If $x = c_1$ and n is a closest return to c_1, then $|Df^n(x)| \geq K\lambda^n$ by hypothesis. Therefore $|Df^n(x)| \geq K\lambda^n$ for all $x \in \partial N(n)$. It follows from lemma A.7 that $\inf_{N(n)} |Df^n| \geq K\lambda^n$.

This proves that $\inf_{Nf(n)} |Df^n| \geq K\lambda^n$ for all $n \geq 0$. □

Lemma 6.10 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that if $x \in [0, c_1]$, $n \geq 0$ and $x_i \leq x$ for all $0 < i \leq n$, then $|Df^n(x)| \geq K\lambda^n$.*

Proof. By lemma 6.9, there exist $K_1 > 0$ and $\lambda_1 > 1$ such that $\inf_{N(n)} |Df^n| \geq K_1{\lambda_1}^n$ for all $n \geq 0$. Choose $N \geq 1$ large enough that $K_1\lambda_1^N \geq \sqrt{\lambda_1}^N$. The critical point of f is not periodic since f has no periodic attractors. It follows that there exists a convex neighborhood U of c such that if $x \in f(U)$ and $x_i \in f(U)$ for some $i > 0$, then $i \geq N$. By Mañé's lemma (lemma A.3) there exist $K_2 > 0$ and $\lambda_2 > 1$ such that if $x \in [0, 1]$, $n \geq 0$ and $x_i \notin U$ for all $0 \leq i < n$, then $|Df^n(x)| \geq K_2{\lambda_2}^n$. Put $K = K_1K_2$ and $\lambda = \min(\sqrt{\lambda_1}, \lambda_2)$. Suppose $x \in [0, c_1]$, $n \geq 0$ and $x_i \leq x$ for all $0 < i \leq n$. We shall show that $|Df^n(x)| \geq K\lambda^n$.

By lemma 4.6, there exists $m \geq 0$ and non-negative integers $i_0, \dots, i_m$ such that $0 = i_0 < i_1 < \cdots < i_m = n$, $x = x_{i_0} \geq x_{i_1} \geq \cdots \geq x_{i_m} = x_n$ and $x_{i_j} \in N(i_{j+1} - i_j)$ for all $0 \leq j < m$. In particular, $|Df^{i_{j+1}-i_j}(x_{i_j})| \geq K_1\lambda_1^{i_{j+1}-i_j}$ for all $0 \leq j < m$.

Case 1. $x_{i_j} \notin f(U)$ for all $0 \leq j < m$. Then $x_i \notin f(U)$ for all $0 \leq i \leq n$. This follows from the definition of the sets $N(i_{j+1} - i_j)$. Thus $x_i \notin U$ for all $0 \leq i < n$, which implies that $|Df^n(x)| \geq K_2{\lambda_2}^n \geq K\lambda^n$.

Case 2. $x_{i_j} \in f(U)$ for some $0 \leq j < m$. Take $s = \max\{0 \leq j < m \mid x_{i_j} \in f(U)\}$. We have

$$|Df^n(x)| = |Df^{i_{s+1}-i_s}(x_{i_s})|\,|Df^{n-i_{s+1}}(x_{i_{s+1}})| \prod_{0 \leq j < s} |Df^{i_{j+1}-i_j}(x_{i_j})|. \quad (61)$$

If $0 \leq j < s$ then $x_{i_j} \in f(U)$ and $x_{i_{j+1}} = f^{i_{j+1}-i_j}(x_{i_j}) \in f(U)$, so $i_{j+1} - i_j \geq N$ from the choice of U. But then $|Df^{i_{j+1}-i_j}(x_{i_j})| \geq \sqrt{\lambda_1}^{\,i_{j+1}-i_j}$ by the choice of N. If $s < j < m$ then $x_{i_j} \notin f(U)$, so as in case 1 we have $|Df^{n-i_{s+1}}(x_{i_{s+1}})| \geq K_2 \lambda_2^{n-i_{s+1}}$. Finally, $|Df^{i_{s+1}-i_s}(x_{i_s})| \geq K_1 {\lambda_1}^{i_{s+1}-i_s}$. Thus $|Df^n(x)| \geq K_1 {\lambda_1}^{i_{s+1}-i_s} K_2 {\lambda_2}^{n-i_{s+1}} \sqrt{\lambda_1}^{\,i_s} \geq K\lambda^n$. $\square$

Theorem B Suppose f is S-unimodal and has a uniform hyperbolic structure. Then f satisfies the Collet-Eckmann condition.

Proof. Take $x = c_1$ in lemma 6.10. $\square$

Theorem 6.11 *Suppose f is S-unimodal. Then f satisfies the Collet-Eckmann condition if and only if f has a uniform hyperbolic structure.*

Proof. If f has a uniform hyperbolic structure then it satisfies the Collet-Eckmann condition by theorem B.

The converse was already known [28] and is proved as follows: take $K > 0$ and $\lambda > 1$ such that $|Df^n(c_1)| \geq K\lambda^n$ for all $n > 0$. We show that f is centrally expanding. Now f has no periodic attractors, since if it did, then by Singer's theorem (lemma A.1) the orbit of c would converge to the periodic attractor $\{x_0, \ldots, x_{n-1}\}$. However f could not then satisfy the Collet-Eckmann condition because $|Df^n(x)| \leq 1$.

Suppose $S \in \mathcal{S}$. We have $|c_{1+S} - \pi_S| \geq |\pi_S - c_1| \sqrt{|Df^S(\pi_S)|\,|Df^S(c_1)|}$ by lemma A.6(3). Hence $|c_{1+S} - \pi_S| \geq \sqrt{K}\sqrt{\lambda}^{\,S} |\pi_S - c_1|$ since $|Df^S(\pi_S)| \geq 1$. That is, the conclusion of lemma 5.1 holds. It follows that f is centrally expanding as in the proof of proposition 5.2. Therefore f has a uniform hyperbolic structure by theorem 5.3. $\square$

7 Semi-uniform expansion

In this section we prove that uniform hyperbolic structure is equivalent to a property which is a priori stronger than the Collet-Eckmann condition: semi-uniform expansion (see below for the definition). This extends the results of the last section. We also prove the equivalence of uniform hyperbolic structure and the backward Collet-Eckmann (this was already known [25]).

Definition. Let f be unimodal. If there exist $K > 0$ and $\lambda > 1$ such that $|Df^n(x)| \geq K\lambda^n \min_{0 \leq i < n} |Df(x_i)|$ for every $x \in [0,1]$ and $n > 0$ then we say that f is *semi-uniformly expanding.*

Note. S-unimodal Misiurewicz maps were already known to be semi-uniformly expanding [7, theorem 6.3, page 261].

Definition. Let f be unimodal. If f has no periodic attractors and there exist $K > 0$ and $\lambda > 1$ such that $|Df^n(x)| \geq K\lambda^n$ for every $n > 0$ and $x \in [0,1]$ for which $f^n(x) = c$, then we say that f satisfies the *backward Collet-Eckmann condition* [6, C2, page 14].

Lemma 7.1 *Suppose f is unimodal and semi-uniformly expanding. Then the critical point of f is not periodic.*

Proof. Suppose c is periodic and take $n > 0$ minimal such that $c_n = c$. If $x \neq c$ is close enough to c, then $\min_{0 \leq j \leq n} |Df(x_j)| = |Df(x_n)|$. This holds because $Df(c) = 0$ implies that $|x_n - c|$ is much smaller than $|x - c|$ if x is close enough to c. By semi-uniform expansion, $|Df^{n+1}(x)| \geq K\lambda^{n+1}|Df(x_n)|$. Therefore $|Df^n(x)| = |Df^{n+1}(x)|/|Df(x_n)| \geq K\lambda^{n+1}$ and taking $x \to c$ gives $|Df^n(c)| \geq K\lambda^{n+1} \neq 0$, a contradiction. This shows that c is not periodic. □

Lemma 7.2 *Suppose f is unimodal and semi-uniformly expanding. Then*

1. *f satisfies the Collet-Eckmann condition.*
2. *f has a uniform hyperbolic structure.*
3. *f satisfies the backward Collet-Eckmann condition.*

Proof. As shown in lemma 7.1, c is not periodic.

1. Take $n \geq 0$. If $x \neq c$ is close enough to c, then $\min_{0 \leq j \leq n} |Df(x_j)| = |Df(x)|$. We have $|Df^{n+1}(x)| \geq K\lambda^{n+1}|Df(x)|$ by semi-uniform expansion, so $|Df^n(x_1)| = |Df^{n+1}(x)|/|Df(x)| \geq K\lambda^{n+1}$. Taking $x \to c$, we obtain $|Df^n(c_1)| \geq K\lambda^{n+1}$. Therefore f satisfies the Collet-Eckmann condition.

2. Suppose $x \in [0,1]$, $n > 0$ and $f^n(x) = x$. For any $m > 0$ we have $|Df^n(x)|^m = |Df^{mn}(x)| \geq K\lambda^{mn} \min_{0 \leq j < n} |Df(x_j)|$ by semi-uniform expansion. Now $c \notin \{x, \ldots, x_{n-1}\}$, so $\min_{0 \leq j < n} |Df(x_j)| > 0$. Taking $m \to \infty$, we deduce that $|Df^n(x)| \geq \lambda^n$. Thus f has a λ-hyperbolic structure.

3. Suppose $x \in [0,1]$, $n > 0$ and $f^n(x) = c$. If $y \neq x$ is close enough to x, then $\min_{0 \leq j \leq n} |Df(y_j)| = |Df(y_n)|$. Now $|Df^{n+1}(y)| \geq K\lambda^{n+1}|Df(y_n)|$ by semi-uniform expansion, so $|Df^n(y)| = |Df^{n+1}(y)|/|Df(y_n)| \geq K\lambda^{n+1}$. Taking $y \to x$, we obtain $|Df^n(x)| \geq K\lambda^{n+1}$. Therefore f satisfies the backward Collet-Eckmann condition. □

We now prove that an S-unimodal map with a uniform hyperbolic structure is semi-uniformly expanding. The set $K(i_{j+1} - i_j)$ was defined in section 2.

Lemma 7.3 *Suppose f is unimodal and $x \in [0,1]$. If $n \geq 0$ and $x_n \in [x_i, \tau(x_i)]$ for all $0 \leq i < n$, then there exists $m \geq 0$ and non-negative integers $i_0, \ldots, i_m$ such that $0 = i_0 < i_1 < \cdots < i_m = n$, $[x_{i_m}, \tau(x_{i_m})] \subseteq \cdots \subseteq [x_{i_0}, \tau(x_{i_0})]$ and $x_{i_j} \in K(i_{j+1} - i_j)$ for all $0 \leq j < m$.*

Proof. Define $i_0, i_1, \ldots$ by $i_0 = 0$ and $i_{j+1} = \min\{k > i_j \mid x_k \in [x_{i_j}, \tau(x_{i_j})]\}$. We have $i_m = n$ for some $m \geq 0$ by the hypothesis that $x_n \in [x_i, \tau(x_i)]$ for all $0 \leq i < n$. □

Lemma 7.4 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that $\inf_{K(n)} |Df^n| \geq K\lambda^n$ for all $n \geq 0$.*

Proof. Take $\lambda > 1$ such that f has a λ-hyperbolic structure. By lemma A.9 there exists $K > 0$ such that $|Df^n(x)| \geq K|Df^n(\tau(x))|$ for all $n \geq 0$ and $x \in [0,1]$. From the λ-hyperbolic structure we know that c is not periodic.

If $n \geq 0$ then $Df^n(x) \neq 0$ for all $x \in K(n)$ by lemma 2.3. It follows from lemma A.7 that $\inf_{K(n)} |Df^n| = \inf_{\partial K(n)} |Df^n|$. If $x \in \partial K(n)$ then either $f^n(x) = x$ or $f^n(\tau(x)) = \tau(x)$ by lemma 2.3, so $|Df^n(x)| \geq \lambda^n$ or $|Df^n(\tau(x))| \geq \lambda^n$. In either case, $|Df^n(x)| \geq K\lambda^n$. This shows that $\inf_{K(n)} |Df^n| \geq K\lambda^n$. □

Lemma 7.5 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then there exist $K > 0$ and $\lambda > 1$ such that if $x \in [0,1]$, $n \geq 0$ and $x_n \in [x_i, \tau(x_i)]$ for all $0 \leq i < n$, then $|Df^n(x)| \geq K\lambda^n$.*

Proof. By lemma 7.4, there exist $K_1 > 0$ and $\lambda_1 > 1$ such that $\inf_{K(n)} |Df^n| \geq K_1{\lambda_1}^n$ for all $n \geq 0$. Take $N \geq 1$ large enough that $K_1{\lambda_1}^N \geq \sqrt{\lambda_1}^N$. The critical point of f is not periodic since f has no periodic attractors. It follows that there exists a convex, symmetric (i.e. $\tau(U) = U$) neighborhood of c such that if $x \in U$ and $x_i \in U$ for some $i > 0$, then $i \geq N$. By Mañé's lemma (lemma A.3) there exist $K_2 > 0$ and $\lambda_2 > 1$ such that if $x \in [0,1]$, $n \geq 0$ and $x_i \notin U$ for all $0 \leq i < n$, then $|Df^n(x)| \geq K_2{\lambda_2}^n$. Put $K = K_2$ and $\lambda = \min\{\sqrt{\lambda_1}, \lambda_2\}$. Suppose $x \in [0,1]$, $n \geq 0$ and $x_n \in [x_i, \tau(x_i)]$ for all $0 \leq i < n$. We shall show that $|Df^n(x)| \geq K\lambda^n$.

By lemma 7.3, there exists $m \geq 0$ and non-negative integers $i_0, \ldots, i_m$ such that $0 = i_0 < i_1 < \cdots < i_m = n$, $[x_{i_m}, \tau(x_{i_m})] \subseteq \cdots \subseteq [x_{i_0}, \tau(x_{i_0})]$ and $x_{i_j} \in K(i_{j+1} - i_j)$ for all $0 \leq j < m$. In particular, $|Df^{i_{j+1}-i_j}(x_{i_j})| \geq K_1{\lambda_1}^{i_{j+1}-i_j}$ for all $0 \leq j < m$.

Case 1. $x_{i_j} \notin U$ for all $0 \leq j < m$. Then $x_i \notin U$ for all $0 \leq i < n$. This follows from the definition of the sets $K(i_{j+1} - i_j)$. Thus $|Df^n(x)| \geq K_2{\lambda_2}^n \geq K\lambda^n$.

Case 2. $x_{i_j} \in U$ for some $0 \leq j < m$. Take $s = \min\{0 \leq j < m \mid x_{i_j} \in U\}$. We have

$$|Df^n(x)| = |Df^{i_s}(x)| \prod_{s \leq j < m} |Df^{i_{j+1}-i_j}(x_{i_j})|. \tag{62}$$

If $s \le j < m$ then $x_{i_j} \in U$ and $x_{i_{j+1}} = f^{i_{j+1}-i_j}(x_{i_j}) \in U$, so $i_{j+1} - i_j \ge N$ from the choice of U. But then $|Df^{i_{j+1}-i_j}(x_{i_j}) \ge \sqrt{\lambda}^{i_{j+1}-i_j}$ by the choice of N. If $0 \le j < s$ then $x_{i_j} \notin U$, so as in case 1 we have $|Df^{i_s}(x)| \ge K_2 {\lambda_2}^{i_s}$. Thus $|Df^n(x)| \ge K_2 {\lambda_2}^{i_s} \sqrt{\lambda_1}^{n-i_s} \ge K\lambda^n$. $\square$

Proposition 7.6 *Suppose f is S-unimodal and has a uniform hyperbolic structure. Then f is semi-uniformly expanding.*

Proof. By lemmas 6.10 and 7.5, there exist $1 \ge K_1 > 0$ and $\lambda_1 > 1$ such that (1) if $x \in [0, c_1]$, $n \ge 0$ and $x_i \le x$ for all $0 < i \le n$, then $|Df^n(x)| \ge K_1 {\lambda_1}^n$, and (2) if $x \in [0,1]$, $n \ge 0$ and $x_n \in [x_i, \tau(x_i)]$ for all $0 \le i < n$, then $|Df^n(x)| \ge K_1 {\lambda_1}^n$. It follows from equation 1 that there exists $K_2 > 0$ such that $\inf_{[0,1]\setminus(x,\tau(x))} |Df| \ge |Df(x)|/K_2$ for all $x \in [0,1]$. Put $K = {K_1}^2 K_2/\lambda_1$ and $\lambda = \lambda_1$. Suppose $x \in [0,1]$ and $n > 0$. We shall show that $|Df^n(x)| \ge K\lambda^n \min_{0\le j<n} |Df(x_j)|$.

Take $0 \le i \le n$ maximal such that $x_i \in [x_j, \tau(x_j)]$ for all $0 \le j < i$. That is, such that x has its final closest return to c at time i. Then $|Df^i(x)| \ge K_1 {\lambda_1}^i$ by lemma 7.5. If $i = n$ then $|Df^n(x)| \ge K_1 {\lambda_1}^n$. If $i < n$ then, by the definition of i, $x_j \le x_{i+1}$ for all $i+1 \le j \le n$, so $|Df^{n-i}(x_{i+1})| \ge K_1 {\lambda_1}^{n-i}$ by lemma 6.10. That is, $|Df^n(x)| = |Df^i(x)|\,|Df(x_i)|\,|Df^{n-i}(x_{i+1})| \ge {K_1}^2 {\lambda_1}^n |Df(x_i)|/\lambda_1$. This last inequality also holds if $i = n$.

Clearly $|Df(x_i)| \ge K_2 \min_{0\le j<n} |Df(x_j)|$. Thus

$$|Df^n(x)| \ge K_1^2 K_2 \lambda_1^n \min_{0\le j<n} |Df(x_j)|/\lambda_1 \ge K\lambda^n \min_{0\le j<n} |Df(x_j)|. \tag{63}$$

In other words, f is semi-uniformly expanding. $\square$

Theorem 7.7 *Suppose f is S-unimodal. Then the following are equivalent:*

1. *f has a uniform hyperbolic structure.*
2. *f satisfies the backward Collet-Eckmann condition.*
3. *f is semi-uniformly expanding.*
4. *f satisfies the Collet-Eckmann condition.*

Proof.

$(1 \Rightarrow 2)$. Apply lemma 7.5 to those $x \in [0,1]$ and $n > 0$ for which $f^n(x) = c$. This implication was already known [25].

$(1 \Rightarrow 3)$. This is proposition 7.6.

$(1 \Rightarrow 4)$. This is theorem B.

$(2 \Rightarrow 1)$. This implication was already known [25]. It follows easily from the fact that the inverse images of c are dense in $[0,1]$, see the note after lemma A.2. The reasoning is similar to that used to prove $3 \Rightarrow 4$ in theorem 5.3.

$(3 \Rightarrow 1, 3 \Rightarrow 2, 3 \Rightarrow 4)$. This is lemma 7.2.

$(4 \Rightarrow 1)$. This was proved in theorem 6.11. $\square$

8 Correlation decay and Hölder conjugacy

In this section we prove that the Collet-Eckmann condition is satisfied if and only if f has an acip with exponential decay of correlations, and also if and only if some renormalization of f is bi-Hölder conjugate to a tent-map with positive topological entropy. In both cases the "only if" was already known. We prove "if" by showing that both properties imply f has a uniform hyperbolic structure.

Definition. An f-invariant Borel probability measure on $[0,1]$ which is absolutely continuous with respect to Lebesgue measure is said to be an *acip*.

Definition. Let f be a continuous endomorphism of $[0,1]$ and μ an f-invariant Borel probability measure on $[0,1]$. If there exist $C > 0$ and $0 < \rho < 1$ such that

$$\left| \int F \cdot (G \circ f^n) \, d\mu - \int F \, d\mu \int G \, d\mu \right| \leq C \rho^n \operatorname{var}(F) \, ||G||_\infty \tag{64}$$

for all continuous functions $F, G : [0,1] \to \mathbb{C}$ with F of bounded variation, and every $n > 0$, then we say that μ has *exponential decay of correlations under f*. Here $\operatorname{var}(F)$ is the total variation of F on $[0,1]$.

Note. In order for our results to hold, we need the uniformity of the correlation decay in F given by the $\operatorname{var}(F)$ term in (64). We could however let C depend on G.

Definition. Let f be unimodal. A closed subinterval I of $[0,1]$ is said to be *restrictive with period $n > 0$* for f if $c \in I$, the interiors of $I, \ldots, f^{n-1}(I)$ are pairwise disjoint, $f^n(I) \subseteq I$ and $f^n(\partial I) \subseteq \partial I$. Clearly $f^n(\partial I) = \{\inf I\}$ or $f^n(\partial I) = \{\sup I\}$. In the first case we say that the restrictive interval I is *positively oriented*, in the second case that it is *negatively oriented*.

Let $\Lambda : I \to [0,1]$ be the unique surjective affine map which is orientation preserving/reversing according to whether I is positively/negatively oriented respectively. Then $\Lambda \circ f^n \circ \Lambda^{-1}$ is a unimodal map, called a *renormalization of f*.

If $[0,1]$ is the only restrictive interval for f, then f is said to be *non-renormalizable.* If f has finitely many restrictive intervals then f is said to be *finitely renormalizable.* Otherwise f is said to be *infinitely renormalizable.*

See [7] for more information on the theory of renormalization.

Note. An S-unimodal map f satisfying the Collet-Eckmann condition is finitely renormalizable. Indeed, any such f has an acip (see below), and must therefore be finitely renormalizable, see [7, theorem 1.5, page 349]

Definition. Suppose $0 < s \leq 2$. The *tent-map* $T_s : [0,1] \to [0,1]$ is defined by: $T_s(x) = sx$ if $x \leq 1/2$, $T_s(x) = s(1-x)$ if $x \geq 1/2$.

Note. The topological entropy of T_s is equal to $\max\{0, \log s\}$. See [7, chapter II, section 7] for the definition of topological entropy and a proof.

Definition. Let f and g be endomorphisms of $[0,1]$. We say that f and g are *topologically conjugate* if there exists a homeomorphism $h : [0,1] \to [0,1]$ such that $f \circ h = h \circ g$. If h and h^{-1} are both Hölder continuous then we say that f and g are *bi-Hölder conjugate.*

Theorem 8.1 (Young/Keller and Nowicki) *Suppose f is S-unimodal and satisfies the Collet-Eckmann condition. Then f has a unique acip μ, and μ has exponential decay of correlations under f.*

Proof. See [17, 38]. The existence and uniqueness of the acip is well known [6, 26, 28, 7]; Young/Keller and Nowicki prove the exponential decay of correlations. $\square$

Note. If the Collet-Eckmann condition is satisfied then (64) holds for every $n > 0$ and any *measurable* test functions $F, G : [0,1] \to \mathbb{C}$ with F of bounded variation and $||G||_{\ell+\delta} < \infty$ for some $\delta > 0$, see [17]. Here ℓ is the order of the critical point of f.

Lemma 8.2 *Suppose f is S-unimodal and has an acip μ with exponential decay of correlations. Then f has a uniform hyperbolic structure.*

Proof. This proof was suggested to us by Gerhard Keller. Since f has an acip, f has no periodic attractors and is finitely renormalizable. The support of μ equals the cycle of transitive intervals and thus contains the critical point of f in its interior. There exists $K_0 > 0$ such that $\mu(A) \geq K_0|A|$ for every Borel set A contained in the support of μ. See [7, theorem 1.5, page 349; theorem 3.2, page 366] for a proof of these properties.

By lemma A.3, there exist $K_1 > 0$ and $\lambda_1 > 1$ such that $|Df^n(x)| \geq K_1{\lambda_1}^n$ for any $x \in [0,1]$ and $n > 0$ for which $x, \dots, x_{n-1} \notin \operatorname{int}\operatorname{support}(\mu)$. Put $\lambda = \min\{\lambda_1, 1/\rho\} > 1$. Here $0 < \rho < 1$ is the correlation decay rate. Suppose $x \in [0,1]$, $n > 0$ and $f^n(x) = x$. We shall show that $|Df^n(x)| \geq \lambda^n$.

Case 1. $x \notin \operatorname{int}\operatorname{support}(\mu)$. Then $x, \dots, x_{n-1} \notin \operatorname{int}\operatorname{support}(\mu)$ since otherwise $x = x_n \in \operatorname{int}\operatorname{support}(\mu)$, a contradiction. Therefore $x, \dots, x_{mn-1} \notin \operatorname{int}\operatorname{support}(\mu)$ for any $m > 0$, which implies that $|Df^n(x)|^m = |Df^{mn}(x)| \geq K_1{\lambda_1}^{mn}$ for any $m > 0$. In the limit $m \to \infty$ we obtain $|Df^n(x)| \geq {\lambda_1}^n \geq \lambda^n$.

Case 2. $x \in \operatorname{int}\operatorname{support}(\mu)$. Take any $\epsilon > 0$ and choose a non-trivial interval I_0 containing x small enough that $I_0 \subseteq \operatorname{support}(\mu)$, $\mu(I_0) < 1/4$, $f^n|_{I_0}$ is a diffeomorphism onto its image, $\sup_{I_0}|Df^n| \leq |Df^n(x)|^{1+\epsilon}$ (possible since $|Df^n(x)| > 1$; f has no periodic attractors) and $I_0 \subseteq f^n(I_0)$. Let $G : [0,1] \to \mathbb{R}$ be a C^∞ function such that $0 \leq G \leq 1$, $G = 1$ on I_0 and $\int G\, d\mu \leq 2\mu(I_0)$.

Take $m > 0$ and put $I_m = (f^n|_{I_0})^{-m}(I_0)$; clearly $f^{mn} : I_m \to I_0$ is a diffeomorphism. Let $F_m : [0,1] \to \mathbb{R}$ be a C^∞ function such that $0 \le F_m \le 1$, $F_m = 0$ on $[0,1] \setminus I_m$, $\int F_m \, d\mu \ge \mu(I_m)/2$ and $\mathrm{var}(F_m) \le 4$. With these choices we have

$$
\begin{aligned}
\int F_m \cdot (G \circ f^{mn}) \, d\mu - \int F_m \, d\mu \int G \, d\mu &= \qquad (65) \\
\int F_m \, d\mu \, (1 - \int G \, d\mu) &\ge \qquad (66) \\
\mu(I_m)(1 - 2\mu(I_0))/2 &\ge \qquad (67) \\
\mu(I_m)/4 &\ge K_0 |I_m|/4. \qquad (68)
\end{aligned}
$$

Now $\sup_{I_m} |Df^{mn}| \le |Df^n(x)|^{m(1+\epsilon)}$, so $|I_m| \ge |I_0|/|Df^n(x)|^{m(1+\epsilon)}$. It follows that

$$
\begin{aligned}
\frac{K_0|I_0|}{4|Df^n(x)|^{m(1+\epsilon)}} &\le \left| \int F_m \cdot (G \circ f^{mn}) \, d\mu - \int F_m \, d\mu \int G \, d\mu \right| \qquad (69) \\
&\le C\rho^{mn} \, \mathrm{var}(F_m) \, ||G||_\infty \le 4C\rho^{mn}. \qquad (70)
\end{aligned}
$$

Rearranging, we obtain $|Df^n(x)|^{1+\epsilon} \ge \rho^{-n}(K_0|I_0|/(16C))^{1/m}$. Since $m > 0$ and $\epsilon > 0$ were arbitrary, this yields $|Df^n(x)| \ge \rho^{-n} \ge \lambda^n$. □

Note. The proof only uses C^∞ test functions $F, G : [0,1] \to \mathbb{R}$.

Theorem 8.3 (Nowicki and Przytycki) *Suppose f is S-unimodal and satisfies the Collet-Eckmann condition. Then some renormalization of f is bi-Hölder conjugate to a tent-map with positive topological entropy.*

Proof. See [29]. □

Lemma 8.4 *Suppose f is S-unimodal. If some renormalization of f is bi-Hölder conjugate to a tent-map with positive topological entropy, then f has a uniform hyperbolic structure.*

Proof. Suppose f is not renormalizable. In this case f itself is bi-Hölder conjugate to some tent-map T_s with positive topological entropy, or in other words with $1 < s \le 2$. Every such tent-map clearly has exponential decay of cylinders. Now, exponential decay of cylinders is preserved by Hölder conjugacy, so f also has exponential decay of cylinders. By theorem 5.3, f has a uniform hyperbolic structure. The general case is left to the interested reader (eliminate the possibility of a periodic attractor using lemma A.1; use lemma A.3 to get uniform hyperbolic structure outside the restrictive interval). □

Theorem 8.5 *Suppose f is S-unimodal. Then the following are equivalent:*

1. *f satisfies the Collet-Eckmann condition.*

2. *Some renormalization of f is bi-Hölder conjugate to a tent-map with positive topological entropy.*

3. *f has an acip with exponential decay of correlations.*

Proof. The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are theorems 8.3 and 8.1 respectively. On the other hand, 1 and 2 both imply that f has a uniform hyperbolic structure (lemmas 8.2 and 8.4). Thus $2 \Rightarrow 1$ and $3 \Rightarrow 1$ follow from theorem B. □

A Properties of S-unimodal maps

We make use of the following well-known properties of S-unimodal maps.

Lemma A.1 (Singer [35]) *Suppose f is S-unimodal. Every periodic attractor of f attracts the critical point of f. In particular, f has at most one periodic attractor.*

Proof. See [7, theorem 6.1, page 155] or [5, theorem II.4.1, page 95]. □

Definition. Let f be an endomorphism of $[0, 1]$. A *homterval* is a non-trivial subinterval I of $[0, 1]$ with the property that $f^n|_I$ is monotone for all $n > 0$. A *wandering interval* is a homterval none of the points of which are in the basin of attraction of a periodic attractor.

Lemma A.2 (Guckenheimer [11]) *Suppose f is S-unimodal. Then f has no wandering intervals.*

Proof. See [7, theorem 6.3, page 157]. The proof uses [7, theorem 2.2, page 39] which requires f to be C^2. See also [5, theorem II.5.4, page 116]. □

Note. An S-unimodal map without periodic attractors therefore has no homtervals: the pre-images of the critical point are dense in $[0, 1]$.

Lemma A.3 (Mañé [22], Misiurewicz [24]) *Suppose f is S-unimodal and has no periodic attractors. For each neighborhood U of c there exist $K > 0$ and $\lambda > 1$ such that if $n \geq 0$, $x \in [0, 1]$ and $x_i \notin U$ for any $0 \leq i < n$, then $|Df^n(x)| \geq K\lambda^n$.*

Proof. See [7, page 231, theorem 3.3]. The proof uses [7, theorem 2.2, page 39] which requires f to be C^2. See also [5, theorem II.5.2, page 109]. □

The reason for many of the special properties of S-unimodal maps is that they expand cross-ratios.

Definition. Let N be an interval and suppose $g : N \to \mathbb{R}$ is continuous. If

$$\frac{|g(I)|}{|I|}\frac{|g(J)|}{|J|} \geq \frac{|g(R)|}{|R|}\frac{|g(L)|}{|L|}. \tag{71}$$

for every pair $J \subseteq I$ of bounded non-trivial subintervals of N such that $g|_I$ is strictly monotone and $I \setminus J$ consists of non-trivial intervals L and R, then we say that g is *cross-ratio expanding*, c.f. [5, property R, page 103].

Note. The composition of two cross-ratio expanding maps is clearly cross-ratio expanding.

Lemma A.4 *Suppose N is an interval, $g : N \to \mathbb{R}$ is C^1 and $Dg(x) \neq 0$ for all $x \in \operatorname{int} N$. If $|Dg|^{-1/2}$ is convex on $\operatorname{int} N$, then g is cross-ratio expanding.*

Proof. If g is C^3 then g has non-positive Schwarzian derivative as noted in section 1. Therefore g expands cross-ratios, see [7, Property 4, page 273], c.f. [11, page 135], [5, page 103]. If g is not C^3 then the result follows from an approximation argument. □

Note. It can be shown that $|Dg|^{-1/2}$ is convex if and only if g is cross-ratio expanding.

Corollary A.5 *Suppose f is S-unimodal. Then f^n is cross-ratio expanding for all $n > 0$.*

Definition. Let $U \subseteq V$ be nested intervals with U finite. We say that V is a *τ-scaled neighborhood* of U if $V \setminus U$ consists of two components both of length at least $\tau\,|U|$.

Lemma A.6 *Suppose $I = [a, b]$ is a non-trivial interval and $g : I \to \mathbb{R}$ is C^1 and strictly monotone. If g is cross-ratio expanding, then*

1. *Minimum principle: $|Dg(x)| \geq \min\{|Dg(a)|, |Dg(b)|\}$ for all $x \in I$.*
2. *Variant minimum principle: If $J = [x, y] \subseteq (a, b)$ and $x \neq y$ then*

$$\frac{|g(J)|}{|J|} \geq \min\{\frac{|g(a) - g(y)|}{|a - y|}, \frac{|g(b) - g(y)|}{|b - y|}\}. \tag{72}$$

3. *Square-root lemma: $|g(I)|/|I| \geq \sqrt{|Dg(a)|\,|Dg(b)|}$.*

Suppose $\tau > 0$ and put $K(\tau) = (1 + \tau)^2/\tau^2$.

4. *One-sided Koebe principle: $|Dg(x)| \geq |Dg(b)|/K(\tau)$ for every $x \in I$ for which $|g(x) - g(a)| \geq \tau|g(x) - g(b)|$.*

5. *Koebe principle: If $J \subseteq I$ is an interval such that $g(I)$ contains a τ-scaled neighborhood of $g(J)$, then*

$$\frac{1}{K(\tau)} \leq \frac{|Dg(x)|}{|Dg(y)|} \leq K(\tau) \tag{73}$$

for all $x, y \in J$.

Proof. See [7, chapter IV, section 1]. We sketch the proofs for completeness.

3. Take the limit $J \uparrow I$ in (71).

1. We may suppose that $x \in \operatorname{int} I$. By taking the limit $J \downarrow \{x\}$ in (71) and considering cases one shows that $|Dg(x)| \geq \min\{|g(a) - g(x)|/|a - x|, |g(x) - g(b)|/|x - b|\}$. By the square-root lemma, $|g(a) - g(x)|/|a - x| \geq \sqrt{|Dg(a)|\,|Dg(x)|}$ and $|g(x) - g(b)|/|x - b| \geq \sqrt{|Dg(x)|\,|Dg(b)|}$. The result follows.

2. We may suppose that $x \in (a, y)$. By the minimum principle, $|g(J)|/|J| \geq \min\{|g(a) - g(x)|/|a - x|, |g(b) - g(y)|/|b - y|\}$. If $|g(J)|/|J| \geq |g(a) - g(x)|/|a - x|$ then $|g(J)|/|J| \geq |g(a) - g(y)|/|a - y|$. Otherwise $|g(J)|/|J| \geq |g(b) - g(y)|/|b - y|$.

4. Taking the limit $J \downarrow \{x\}$ in (71) yields

$$|Dg(x)| \geq \frac{|g(x) - g(a)|}{|g(b) - g(a)|} \frac{|b - a|}{|x - a|} \frac{|g(x) - g(b)|}{|x - b|}. \tag{74}$$

Now $|g(x) - g(a)|/|g(b) - g(a)| \geq \tau/(1+\tau)$ and $|b - a| \geq |x - a|$. The square-root lemma with $I = [x, b]$ gives $|g(x) - g(b)|/|x - b| \geq \sqrt{|Dg(x)|\,|Dg(b)|}$. Substituting these into (74) yields $\sqrt{|Dg(x)|} \geq \tau\sqrt{|Dg(b)|}/(1 + \tau)$.

5. Use the one-sided Koebe principle twice. □

Lemma A.7 *Suppose I is a compact interval, $E \subseteq I$ and $g : I \to \mathbb{R}$ is C^1. If g is cross-ratio expanding and $Dg(x) \neq 0$ for all $x \in \operatorname{int} E$, then $\inf_E |Dg| = \inf_{\partial E} |Dg|$.*

Proof. This follows from the minimum principle. □

Lemma A.8 *Suppose f is S-unimodal. If there exists $x \in [0, 1]$ and $n > 0$ such that $x_n = x$ and $|Df^n(x)| \leq 1$, then f has a periodic attractor.*

Proof. If $|Df^n(x)| < 1$ then x is a periodic attractor, so we may suppose that $|Df^n(x)| = 1$, and even, by considering f^{2n}, that $Df^n(x) = 1$. If A is the component of $\{y \in [0, 1] \mid Df^n(y) = 1\}$ containing x, then f^n fixes A and,

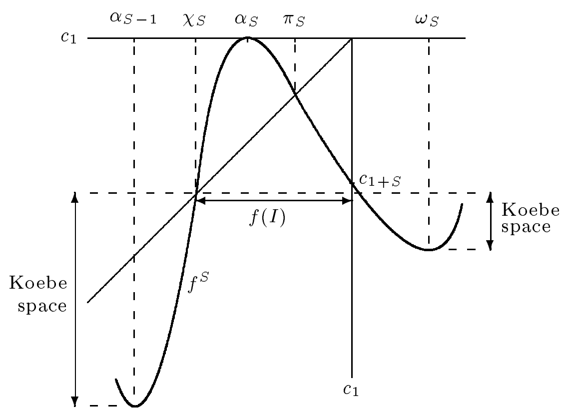

Figure 6: Renormalization near c_1.

because $Df(0) > 1$ and $f(1) = f(0) = 0$, $A \subseteq (0,1)$. Let B be an interval such that $f^n|_B$ is monotone and $B \setminus A$ consists of components L and R.

By the minimum principle, $Df^n \leq 1$ on L or $Df^n \leq 1$ on R, since $Df^n = 1$ on A. Suppose $Df^n \leq 1$ on L and let y be the common boundary point of A and L. Since f^n has no fixed points in L from the maximality of A and the minimum principle, we deduce that $f^{mn}(L) \to y$ as $m \to \infty$. Therefore y is a periodic attractor. □

Note. The point x itself may not be attracting since it might belong to a line of fixed points of f^n. This is incompatible with *negative* Schwarzian derivative, c.f. the proofs of [7, theorem 6.1, page 155] and [5, theorem II.4.1, page 95], but not with our assumptions.

Lemma A.9 *Suppose f is S-unimodal. Then there exists $K > 0$ such that $|Df^n(x)| \geq K|Df^n(\tau(x))|$ for every $x \in [0,1]$ and $n \geq 0$.*

Proof. It is enough to show that $|Df(x)| \geq K|Df(\tau(x))|$. But this follows from equation 1. □

B A priori bounds for renormalizable maps

In this appendix we explain how the classical real a priori bounds for renormalizable maps [7, chapter VI, section 2] can be derived from theorem 3.2.

Definition. Let f be unimodal and suppose c is not periodic. If $S \in \mathcal{S}$ then we define χ_S, the *off-central periodic point of* f^S, by $\chi_S = \inf K^+(S)$. Here $K^+(S)$ is the connected component of $f(K(S))$ containing α_S, defined in lemma 2.4.

Notes.

1. The point χ_S is an orientation preserving fixed point of f^S, see figure 6. Indeed, $f^S(\chi_S) = \chi_S$ since $\chi_S \in \partial f(K(S))$, see the proof of lemma 2.4. If χ_S was orientation reversing then there would clearly exist points $x > \chi_S$ arbitrarily close to χ_S with $x \notin f(K(S))$. But then χ_S could not equal $\inf K^+(S)$.
2. If f is S-unimodal and has no periodic attractors, then f is renormalizable of period $S > 0$ if and only if $S \in \mathcal{S}$ and $c_{1+S} \geq \chi_S$. Moreover, $[\chi_S, c_1]$ equals $f(I)$ where I is the restrictive interval with period S, see figure 6.

The main step in the proof of the a priori bounds is to show that $f^S|_{[\chi_S,c_1]}$ is the composition of a map with distortion independent of S and $x \mapsto x^\ell$, where ℓ is the order of the critical point of f. It is enough to show that there is Koebe space independent of S on the real and virtual sides of the renormalization box $f(I)$, see figure 6 — what we prove here is the existence of this space (proposition B.2).

Indeed, since $f^S = f \circ f^{S-1}$, pulling back the space of proposition B.2 by f gives Koebe space for $f^{S-1}|_{[\chi_S,c_1]}$ that depends only on δ and the non-flatness/non-symmetry of f, c.f. the proof of lemma 6.5. The Koebe principle (lemma A.6(5)) then gives a bound for the distortion of f^{S-1} that depends only on this space. This shows that $f^S|_{[\chi_S,c_1]}$ is the composition of the approximately-affine map $f^{S-1}|_{[\chi_S,c_1]}$ (with distortion depending only on f) and f, which by non-flatness is itself the composition of a map with bounded distortion and $x \mapsto x^\ell$. The a priori bounds, for example the precompactness of renormalizations of f, follow by the usual arguments, see [7].

Lemma B.1 *Suppose f is S-unimodal and has no periodic attractors. If $\bar{S}, S \in \mathcal{S}$ and $\bar{S} < S$, then $\pi_{\bar{S}} \leq \chi_S < \alpha_S$.*

Proof. Clearly $\chi_S < \alpha_S$ since $\alpha_S \in \operatorname{int} K^+(S)$, see lemma 2.2. Now $\pi_{\bar{S}} = \sup K^+(\bar{S})$ and $\chi_S = \inf K^+(S)$, so $\pi_{\bar{S}} \leq \chi_S$ follows from $\alpha_{\bar{S}} < \alpha_S$ and $\operatorname{int} K^+(\bar{S}) \cap \operatorname{int} K^+(S) = \emptyset$ as in the proof of lemma 2.6. □

Proposition B.2 (A priori bounds) *Suppose f is S-unimodal and has no periodic attractors. Then there exists $\delta > 0$ such that $|f^S(\omega_S) - \chi_S| \geq \delta\,|\chi_S - c_1|$ and $|f^S(\alpha_{S-1}) - \chi_S| \geq \delta\,|\chi_S - c_1|$ for all $1 < S \in \mathcal{S}$. If $S > T_0$ then $\delta > 0$ can be taken the same as in theorem 3.2.*

Proof. If $\delta > 0$ is small enough then the result clearly holds for $S \leq T_0$, so suppose $T_0 < S \in \mathcal{S}$ and that $\delta > 0$ is the same as in theorem 3.2.

We have $|f^S(\omega_S) - f^S(x_0)| \geq \delta\,|f^S(x_0) - c_1|$ by corollary 3.3(1), where x_0 is the unique point in (π_S, ω_S) for which $f^S(x_0) = f^{\mathcal{T}(S)}(x_0)$. By lemma 2.10(1), $f^S(x_0) = \pi_{\bar{S}}$ where $\bar{S} = S - \mathcal{T}(S) \in \mathcal{S}$. Now $\bar{S} < S$, so $\pi_{\bar{S}} \leq \chi_S$ by lemma B.1. It follows that $|f^S(\omega_S) - \chi_S| \geq \delta\,|\chi_S - c_1|$.

We have $\hat{S} \equiv S - \mathcal{S}(S) \in \mathcal{S}$ by lemma 2.8(1). Since $f^S(\alpha_{S-1}) < c_1 = f^{\mathcal{S}(S)}(\alpha_{S-1})$ and $f^S(\alpha_S) = c_1 > f^{\mathcal{S}(S)}(\alpha_S)$, there exists a (unique) point x_1 in (α_{S-1}, α_S) for which $f^S(x_1) = f^{\mathcal{S}(S)}(x_1)$. It can be shown similarly to lemma 2.10 and corollary 3.3 that $f^S(x_1) = \pi_{\hat{S}}$ and $|f^S(\alpha_{S-1}) - f^S(x_1)| \geq \delta\,|f^S(x_1) - c_1|$. Now $\hat{S} < S$, so $\pi_{\hat{S}} \leq \chi_S$ by lemma B.1. Therefore $|f^S(\alpha_{S-1}) - \chi_S| \geq \delta\,|\chi_S - c_1|$. $\square$

Note. Proposition B.2 applies to any $S \in \mathcal{S}$ and not just those for which f is renormalizable.

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