Transformations tangent to the identity.

Stable pieces of manifolds.

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ABSTRACT. Let Φ be a germ of analytic transformation from ($\mathbf{C}^p$, 0) to ($\mathbf{C}^p$, 0), tangent to the identity, and let V be a non degenerate characteristic direction of Φ at the origin. We can naturally associate to V a (p-1,p-1)-matrix A, the class of similarity of which is invariant by changes of coordinates. When A has d eigenvalues, $0 \leq d \leq p-1$, with strictly positive real parts, we prove the existence of an invariant piece of analytic manifold M of dimension d+1, with the origin in its boundary, in which every point is attracted to the origin in the direction V. If moreover Φ is a global isomorphism of $\mathbf{C}^p$, the set of points attracted to the origin in the direction V is isomorphic to $\mathbf{C}^{d+1}$.

1. Introduction.

Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ tangent to the identity, given by convergent power series in a neighborhood of 0 as

$$X_1 = X + P_2(X) + P_3(X) + \cdots, \tag{1.1}$$

where for $h \in \mathbf{N}$, P_h is a homogeneous polynomial function of degree h from $\mathbf{C}^p$ to $\mathbf{C}^p$. For sake of simplicity, we assume that P_2 is not identically zero. We will say in chapter 6 how the results can be extended to the case where $X_1 = X + P_k(X) + P_{k+1}(X) + \cdots$, with $P_k \neq 0, k > 2$.

We are interested in the behavior of the iterates $X_n = \Phi^n(X)$ of points X close to the origin. Before stating the results of the following paper, we need some definitions.

Definition 1.1. Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ tangent to the identity. Let [ ] denote the image of a vector of $\mathbf{C}^p\backslash\{0\}$ in the projective space and let V be a vector in $\mathbf{C}^p\backslash\{0\}$. We say that the iterates $\{X_n\}$ of X by Φ *converges to the origin in the direction* V, if we have

$$\lim_{n\to\infty} X_n = 0 \quad \text{and} \quad \lim_{n\to\infty} [X_n] = [V] \quad (\text{in } \mathbf{P}^{p-1}_{\mathbf{C}}).$$

Definition 1.2. Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ tangent to the identity, a characteristic direction is a direction V of

Mathematics subject classification: 32H50, 32H02.

Keywords and phrases. Iteration. Diffeomorphisms tangent to the identity.

$\mathbf{C}^p\backslash\{0\}$ such that $P_2(V) = \lambda V$ for some $\lambda \in \mathbf{C}$. A non degenerate characteristic direction is a characteristic direction V such that $P_2(V) \neq 0$.

The interest of the definition above comes from the following property (see [H] Prop. 2.3) :

Proposition 1.3. *Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$, tangent to the identity. If the iterates of some point X are attracted to the origin in a direction V, then V is a characteristic direction.*

Let now V be a non degenerate characteristic direction. We can choose new coordinates (x, y) in $\mathbf{C}^p$ with x in $\mathbf{C}$ and y in $\mathbf{C}^{p-1}$ such that $V = (1, u_0)$. We express the transformation as

$$(1.2) \qquad \begin{cases} x_1 = x + p_2(x,y) + p_3(x,y) + \cdots \\ y_1 = y + q_2(x,y) + q_3(x,y) + \cdots \end{cases},$$

where $x_1, x, \{p_i(x,y)\}_{i\geq 2}$, have their values in $\mathbf{C}$, and $y_1, y, \{q_i(x,y)\}_{i\geq 3}$, have their values in $\mathbf{C}^{p-1}$.

The condition of non degeneracy implies that $p_2(1, u_0) \neq 0$, and V is a characteristic direction if and only if u_0 is a solution of the system of $p-1$ polynomial equations of degree 3

$$(1.3) \qquad r(u) := q_2(1,u) - p_2(1,u)u = 0\,.$$

Definition 1.4. Let $V = (1, u_0)$ be a non degenerate characteristic direction, *the matrix associated to* V is the $(p-1) \times (p-1)$-matrix

$$(1.4) \qquad A = \frac{r'(u_0)}{p_2(1,u_0)} \quad .$$

In [H] (Prop. 2.4), it is proved that the class of similarity of A is an invariant and that the effect of a linear transformation of the type

$$\begin{cases} x = X + O(\|Y\|) \\ y = SY \end{cases},$$

with S an invertible $(n-1, n-1)$ matrix is to change A in $S^{-1}AS$. In particular, the eigenvalues of A are invariants associated to the characteristic direction V and we can choose the coordinates so that A is in a Jordan canonical form.

Let us choose the coordinates such that $u_0 = 0$, and $(p_2(1,0), q_2(1,0)) = (-1, 0)$, hence $A = -r'(0)$. After the blow-up $y = ux$, the transformation can be written

$$(1.5) \qquad \begin{cases} x_1 = x\big(1 - x + O(\|u\|x, x^2)\big) \\ u_1 = (I - xA)u + O(\|u\|^2 x, x^2) \end{cases},$$

where I is the identity matrix of $\mathbf{C}^{p-1}$.

Definition 1.5. A subset S of $\mathbf{C}^p$ is called a piece of manifold of dimension d with the origin in its boundary, if there exists a bounded open set D in $\mathbf{C}^d$, with a piecewise-$\mathcal{C}^1$ boundary, ∂D such that $0 \in \partial D$, and a (1-1) function $\Psi : \overline{D} \to \mathbf{C}^p$ such that Ψ is analytic in D, piecewise-$\mathcal{C}^1$ up to the boundary, $\Psi(\overline{D}) = \overline{S}$ and $\Psi(0) = 0$.

In this article we prove the following result.

Theorem 1.6. *Let Φ be a germ of transformation of $(\mathbf{C}^p, 0)$ tangent to the identity. Let V be a non degenerate characteristic direction. Assume that the eigenvalues of the matrix A associated to V are divided into two sets $\{\lambda_j\}_{1\le j\le q}$ and $\{\mu_k\}_{1\le k\le \ell}$ in such a way that for some positive real number α, we have*

$$\begin{array}{ll} \Re e\, \lambda_j > \alpha > 0 & \text{for } j = 1, \dots, q\,, \\ \Re e\, \mu_k < \alpha & \text{for } k = 1, \dots, \ell\,. \end{array}$$

Let d_j be the multiplicity of λ_j for $j = 1, \dots, q$. Let d be defined by $d := d_1 + \cdots + d_q$. Let E be the sum of the generalized eigenvectorspaces associated to the λ_j's. Then there exists an invariant piece of analytic manifold of dimension $d+1$, with 0 in the boundary, tangent to $\mathbf{C}\,V + E$ at 0, and where every point is attracted to the origin in the direction V.

Since we can take for α any positive number bigger than the real part of every eigenvalue of A (hence $d = 0$), Theorem 1.6 is an extension of the main result of [H] :

Theorem 1.7. *Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$ tangent to the identity. For every non degenerate characteristic direction V of Φ, there exists an invariant piece of curve, tangent to V at 0, attracted to the origin in the direction V.*

Remarks 1.8.

- There is another proof of Theorem 1.7 which has been given by Ecalle [E] by using his theory of resurgence. The proof of Theorem 1.7 given in [H] uses only analytic tools.
- Theorem 1.6 is a stable manifold theorem. When Φ is replaced by Φ^{-1}, we get, in the same way, stable pieces of manifold for Φ^{-1}, so unstable pieces of manifold for Φ. Let us compare this situation with the classical stable manifold theorems for a transformation of type

$$X_1 = H(x) = exp(-B)X + \cdots\,.$$

In this last case, when H is replaced by H^{-1}, the matrix B is replaced by $-B$. In the case of a transformation tangent to the identity, the matrix

A associated to a non degenerate characteristic direction is the same for Φ and for Φ^{-1} ; the stable and the unstable pieces of manifold are both tangent to the same vector space. But generally, they do not glue together (see Example (3.2) of [H]).

When A has all its eigenvalues with strictly positive real parts, we prove the following statement:

Theorem 1.9. *Let Φ be a germ of transformation of $(\mathbf{C}^p, 0)$ tangent to the identity. Let V be a non degenerate characteristic direction. Assume that the real parts of all the eigenvalues of the matrix A associated to V are strictly positive.*

Then there is an invariant domain D, with 0 in its boundary, in which every point is attracted to the origin in the direction V, and such that the restriction of Φ to D is conjugate to the translation

$$\left(\frac{1}{z_1} = \frac{1}{z} + 1\,,\ U_1 = U\right)\ ,\ \textit{with} \quad (z, U) \in \mathbf{C} \times \mathbf{C}^{p-1}.$$

The general situation can be described as follows: Let Φ be a germ of transformation tangent to the identity. Let V be a non degenerate characteristic direction, and let A be the matrix associated to V. If A has exactly d eigenvalues with strictly positive real parts, there exist coordinates (x, u, v), defined in a sector, where x, u, v , have respectively their values in $\mathbf{C}$, $\mathbf{C}^d$ and $\mathbf{C}^{p-d-1}$, such that the stable piece of manifold is defined by $v = 0$. Then in $v = 0$, Theorem 1.9 applies.

We also prove the following theorems:

Theorem 1.10. *Let Φ be a global isomorphism of $\mathbf{C}^p$. Let 0 be a fixed point of Φ where Φ is tangent to the identity. Let V be a non degenerate characteristic direction at 0. Assume that the matrix A associated to V has exactly d eigenvalues, counted with multiplicities, with strictly positive real parts, and no eigenvalue with real part equal to 0. Let $\Omega_{(0,V)}$ be defined by*

$$(1.6) \qquad \Omega_{(0,V)} = \{\, X \in \mathbf{C}^p \backslash \{0\};\ \lim_{n\to\infty} X_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} [X_n] = [V]\,\}.$$

Then $\Omega_{(0,V)}$ is isomorphic to $\mathbf{C}^{d+1}$.

Remark. B. Weickert [W] has given special examples of isomorphisms of $\mathbf{C}^2$, tangent to the identity at a fixed point 0 with an attractive invariant domain of 0 isomorphic to $\mathbf{C}^2$. It turns out that the domain of his example is the set $\Omega_{(0,V)}$ for a non degenerate characteristic direction V associated to a positive eigenvalue.

When the matrix A has some eigenvalues with real part equal to 0, it is necessary to give a more precise definition of the set $\Omega_{(0,V)}$. For this we use the blow-up $y = ux$, and the coordinates x, u of (1.5). If X_n converges to the origin in the direction $V = (1,0)$, the coordinate $u_n = \dfrac{y_n}{x_n}$ is defined for n big enough.

Theorem 1.11. *Let Φ be a global isomorphism of $\mathbf{C}^p$. Let 0 be a fixed point of Φ where Φ is tangent to the identity. Let $V = (1,0)$ be a non degenerate characteristic direction at 0. Let $\alpha > 0$ be a positive number such that the eigenvalues $\{\lambda_j\}_{1\le j\le d} \cup \{\mu_k\}_{1\le k\le l}$, of the matrix A associated to V satisfy*

$$\begin{array}{ll} \Re e\,\lambda_j > \alpha > 0 & \text{for } j = 1,\dots,d\,, \\ \Re e\,\mu_k \le 0 & \text{for } k = 1,\dots,l\,. \end{array}$$

Let $\Omega_{(0,V)}$ be defined by:

$$\Omega_{(0,V)} = \{X \in \mathbf{C}^p\backslash\{0\};\ \lim_{n\to\infty} X_n = 0\,,\ \lim_{n\to\infty}[X_n] = [V] \tag{1.7}$$

$$\text{and } \lim_{n\to\infty} x_n^{-\alpha} u_n = 0\}.$$

Then the definition of $\Omega_{(0,V)}$ does not depend on α and $\Omega_{(0,V)}$ is isomorphic to $\mathbf{C}^{d+1}$.

2. Preliminary results.

Let Φ be a germ of analytic transformation from $(\mathbf{C}^p,0)$ to $(\mathbf{C}^p,0)$, tangent to the identity. Let $V = (-1,0)$ be a non degenerate characteristic direction. After the blow-up $y = ux$, the transformation is given as in (1.5) by:

$$\begin{cases} x_1 = x\big(1 - x + O(\|u\|x, x^2)\big) \\ u_1 = (I - xA)u + O(\|u\|^2 x, x^2) \end{cases} . \tag{2.1}$$

In what follows, we use the notation $O_1()$ instead of $O()$ with the convention, for any integers $q \ge 0$ and $k \ge 1$:

$$O_1(\|u\|^q x^k) := O(\|u\|^q x^k |\log x|^p), \tag{2.2}$$

for some unspecified integer $p \ge 0$.

Using Theorem 1.7, we can assume that in a domain

$$\Delta_{r,\rho} = \{(x,u)/|x - r| < r, \|u\| < \rho\}, \tag{2.3}$$

with $r > 0$, $\rho > 0$ the transformation Φ is written in the form

$$\text{(2.4)} \qquad \begin{cases} x_1 = f(x,u) = x - x^2 + O(\|u\|x^2) + O_1(|x|^3) \\ u_1 = g(x,u) = (I - xA)u + O_1(\|u\|^2 x, \|u\|x^2)) \end{cases},$$

where f and g are analytic functions in $\Delta_{r,\rho}$. The coordinates are chosen in such a way that $u = 0$ is the invariant curve.

The reason why logarithms appear in (2.4) comes from the proof of Theorem 1.7 given in [H]. The first step of this proof ([H] Prop. 3.5) consists in showing that there is a formal invariant solution $u = \tilde{u}()$ of the equation

$$\tilde{u}(f(x, \tilde{u}(x)) = g(x, \tilde{u}(x)),$$

with notations of (2.4). This solution is of the following type: if A has no eigenvalue in $\mathbf{N}$, $\tilde{u}()$ is a formal series in x; if A has the eigenvalues $\{h_1, \dots, h_k\}$ in $\mathbf{N}$, $\tilde{u}()$ is a formal series in x and in $\{x^{h_i}(\log x)^\tau\}_{1\leq\tau\leq r_i}$, where r_i is the multiplicity of the eigenvalue h_i . Hence, in $\Re e\, x > 0$, we can change the coordinate u in $u - P_N(x)$, with P_N a polynomial in x or, possibly in x and in $\{x^{h_i}(\log x)^\tau\}$, of total degree N (by convention the degree of $x^s(\log x)^t$ is s), with $P_N(0) = 0$.

Remark 2.1. A consequence of the construction above is that for a given positive integer s, there is only a finite set E_s of positive integers t such that a term $x^s(\log x)^t$ can occur in the formal series $\tilde{u}(x)$.

After the change of coordinate u in $u - P_N(x)$, with P_N described above, the transformation Φ is given in $\Re e\, x > 0$ by

$$\text{(2.5)} \qquad \begin{cases} x_1 = x\big(1 - x + O(\|u\|x) + O_1(x^2)\big) \\ u_1 = (I - xA)u + O(\|u\|^2 x) + O_1(\|u\|x^2, x^{N+1}) \end{cases},$$

with the convention (2.2). For N big enough, the process described in [H] lemma 4.5 , is converging in some $\Delta_{r,\rho}$ (2.3), for r, ρ small enough, and gives a solution $u = O_1(x^N)$.

Hence, we can conclude that in (2.4), we have

$$\text{(2.6)} \qquad \begin{cases} x_1 = f(x,u) = x - x^2 + F(x,u) \\ u_1 = g(x,u) = (I - xA)u + G(x,u) \end{cases},$$

with

$$F(x,u) = O(\|u\|x^2) + O_1(x^3) \quad ,$$

$$G(x,u) = O(\|u\|^2 x) + O_1(\|u\|x^2)),$$

and

$$F(x,u) = \sum_{1\le s\le N, t\in E_s} a_{s,t}(u)x^s(\log x)^t + O_1(x^{N+1}), \tag{2.7}$$

$$G(x,u) = \sum_{1\le s\le N,\, t\in E_s} b_{s,t}(u)x^s(\log x)^t + O_1(\|u\|x^{N+1}).$$

where $a_{s,t}(u)$ and $b_{s,t}(u)$ are holomorphic functions in $\|u\| \le \rho$ and E_s denote the finite set (see Remark 2.1):

$$E_s = \{t \in \mathbf{N} \setminus |a_{s,t}(u)| + |b_{s,t}(u)| \not\equiv 0\}. \tag{2.8}$$

Remark. Since $P_N(0) = 0$, the terms $O(\|u\|x^2)$ in F and $O(\|u\|^2 x)$ in G are not affected by the transformation of u in $u - P_N(x)$.

When we are under the hypothesis of Theorem 1.6, we will use more convenient notations. Let us call (x, U) the coordinates in $\mathbf{C} \times \mathbf{C}^{p-1}$ where the non degenerate direction is $V = (1, 0)$. Assume that the matrix associated to V is in Jordan form. We split U in $(u, v) \in \mathbf{C}^q \times \mathbf{C}^\ell$ so that in

$$\Delta_{r,\rho} = \{(x,U)/|x-r| < r, \|U\| < \rho\}, \tag{2.9}$$

the transformation Φ can be written

$$\begin{cases} x_1 = f(x,u,v) = x - x^2 + F(x,u,v) \\ u_1 = g(x,u,v) = (I_q - xA)u + G(x,u,v) \\ v_1 = h(x,u,v) = (I_\ell - xB)v + H(x,u,v) \end{cases}, \tag{2.10}$$

where we have:

- A and B are square matrices in Jordan form. Here $M := \mathrm{Diag}(A, B)$ is the matrix associated to V.
- I_q and I_ℓ, $(q + \ell = p - 1)$, are respectively the $q \times q$ and the $\ell \times \ell$ identity matrices.
- The eigenvalues $\{\lambda_j\}_{1\le j\le q}$ of A and the eigenvalues $\{\mu_k\}_{1\le k\le \ell}$ of B verify

$$\begin{array}{lll} \Re e\,\lambda_j > \alpha > 0 & \text{for} & j = 1, \dots, q, \\ \Re e\,\mu_k < \alpha & \text{for} & k = 1, \dots, \ell. \end{array} \tag{2.11}$$

- the functions F, G and H verify the conditions

$$\begin{array}{rcl} F(x,U) & = & O(\|U\|x^2) + O_1(x^3), \\ G(x,U) & = & O_1(\|U\|x^2, \|U\|^2 x) \\ H(x,U) & = & O_1(\|U\|x^2, \|U\|^2 x) \end{array}. \tag{2.12}$$

The next proposition shows that we can get better estimates on H.

Proposition 2.2. *Let Φ be a germ of transformation tangent to the identity, with $V = (1,0)$ a non degenerate characteristic direction such that the matrix associated to V verifies the properties (2.11). Then, for every integers $m, k \geq 2$, we can choose coordinates $(x, U) = (x, u, v)$ in $\Delta_{r,\rho}$ (2.9) as above, where moreover H satisfies the condition*

$$H(x,u,0) = O(|x|\|u\|^m + |x|^k\|u\|). \tag{2.13}$$

Proof. Let us write the transformation Φ in $\Delta_{r,\rho}$ as in (2.10)

$$\begin{cases} x_1 = f(x,u,v) = x - x^2 + F(x,u,v) \\ u_1 = g(x,u,v) = (I_q - xA)u + G(x,u,v) \\ v_1 = h(x,u,v) = (I_\ell - xB)v + H(x,u,v) \end{cases} .$$

The estimates (2.7) show that $H(x,u,0)$ can be written for an arbitrary k

$$H(x,u,0) = \sum_{1 \leq s \leq k-1, t \in E_s} c_{s,t}(u) x^s (\log x)^t + O_1(\|u\||x|^k), \tag{2.14}$$

where E_s (2.8) is finite and each $c_{s,t}(u)$ is a holomorphic function in $\{\|u\| \leq \rho\}$ of order greater than 1. We will prove Proposition 2.2 by showing that there exists a change of coordinates of the type $\tilde{v} = v - \lambda(x,u)$, for which $c_{s,t}(u) = O(\|u\|^m)$ for each $s \leq k$. We prove it by induction on s, t, and the order d of $c_{s,t}(u)$ in the following way:

Lemma 2.3. *With the notations of Proposition 2.2, let s be the smallest integer and for that s, let t be the biggest integer in E_s for which there exist terms $c_{s,t}(u)$ of order smaller than m. Then there exists a homogeneous polynomial function $P(u)$ of degree d, with values in $\mathbf{C}^\ell$, such that, after the transformation of v in*

$$\tilde{v} := v - x^{s-1}(\log x)^t P(u),$$

$c_{s,t}(u)$ is of order greater than d.

Proof. We can write

$$c_{s,t}(u) = Q(u) + O(\|u\|^{d+1}),$$

where $Q(u) = (Q_1(u), Q_2(u), \ldots, Q_\ell(u))$ is a homogeneous polynomial function of degree d, with values in $\mathbf{C}^\ell$. An elementary computation shows that the transformation

$$\tilde{v} = v - x^{s-1}(\log x)^t P(u)$$

cancels the term $Q(u)x^s(\log x)^t$ if and only if P is such that

$$P\big((I_q - xA)u\big) - (I_\ell - xB)P(u) - (s-1)xP(u) = xQ(u) + O(\|u\|x^2).$$

Solving this last relation is equivalent to solving the linear system in the coefficients of P obtained by identifying the terms of same degrees in

$$\langle \operatorname{grad} P_i\,,\, Au\,\rangle - \big(B - (s-1)I_\ell)P(u)\big)_i = -Q_i(u), \qquad 1 \le i \le \ell\,. \tag{2.15}$$

To prove that such a square linear system has a unique solution P, is equivalent to prove that the relations

$$\langle \operatorname{grad} P_i\,,\, Au\,\rangle - \big((B - (s-1)I_\ell)P(u)\big)_i = 0, \qquad 1 \le i \le \ell\,, \tag{2.16}$$

imply $P = 0$.

Since B is in a Jordan canonical form, (2.16) is reduced to

$$\frac{\partial P_i}{\partial u_1}(Au)_1 + \ldots + \frac{\partial P_i}{\partial u_q}(Au)_q - ((\mu_i - s + 1))P_i + \varepsilon_{i,i+1}P_{i+1}) = 0, \tag{2.17}$$

for $1 \le i \le \ell$, where $\varepsilon_{i,i+1} = 0$ or 1 if $i < \ell$ and $\varepsilon_{\ell,\ell+1} = 0$.

By decreasing induction on i from $i = \ell$ to 1, the assertion is reduced to the following one: let R be a homogeneous polynomial of degree d such that

$$\frac{\partial R}{\partial u_1}(Au)_1 + \ldots + \frac{\partial R}{\partial u_q}(Au)_q - (\mu_i - s + 1)R = 0, \tag{2.18}$$

then $R = 0$.

By Euler formula, we have $R = \dfrac{1}{d}\{\dfrac{\partial R}{\partial u_1}u_1 + \ldots + \dfrac{\partial R}{\partial u_q}u_q\}$. Let C_i be the matrix $C_i := A - \frac{\mu_i - s+1}{d}I_q$. By hypothesis (2.11) the eigenvalues $\{\nu_1, \cdots, \nu_q\}$ of C_i have positive real parts. So we have just to verify that, if C is a matrix in Jordan form with eigenvalues of positive real parts, and R a homogeneous polynomial, we have

$$\langle \operatorname{grad} R\,,\, Cu\,\rangle = 0 \qquad \Longrightarrow \qquad R = 0. \tag{2.19}$$

We prove Property (2.19) by a double induction on the dimension q and the degree d of R. For $q = 1$, it is clearly true for any degree. The result is also easy to prove, if $d = 1$, for any q. We complete the proof by showing that when $q > 1$ and $d > 1$, if the property is true for $(q-1, d)$ and $(q, d-1)$, it is also true for (q, d). Let us assume that we have $\langle \operatorname{grad} R\,,\, Cu\,\rangle = 0$ for R a homogeneous polynomial of degree d in q variables. By the induction hypothesis, $R(u_1, \cdots, u_{q-1}, 0) = 0$. So $R(u) = u_q S(u)$ for S a homogeneous polynomial of degree $d-1$ in q variables. After division by u_q, the hypothesis on R becomes

$$\langle \operatorname{grad} S\,,\, Cu\,\rangle + \nu_q S = 0,$$

with $\Re e\, \nu_q > 0$. Using again Euler formula, this relation can also be written $\langle \operatorname{grad} S\,,\, C'u\,\rangle = 0$, with $C' = C + \frac{\nu_q}{d} I$. So $R = 0$ by the induction hypothesis. This concludes the proof of Lemma 2.3 □

To prove Proposition 2.2, we apply Lemma 2.3 for the same integers s, t until we get $c_{s,t}(u)$ of order m. Now either $E_s = \emptyset$ or the biggest term t' in E_s is smaller than t. We apply the same proof until $E_s = \emptyset$. Then we replace s by $s+1$ until $s+1 = k$. The proof of Proposition (2.2) is complete. □

Remarks 2.4.

(i) It is clear from the proof of Proposition 2.2 that we can get

$$H(x,u,0) = O(|x|\|u\|^m + |x|^k\|u\|)\,, \tag{2.20}$$

with a $O(\ )$ instead of $O_1(\ )$. We have just to push the computation a little further.

(ii) The proof of Proposition 2.2 depends on Property (2.19). Hence (2.15) is a Cramer's system in the coefficients of P, for every positive integer s, if, instead of relation (2.2), the matrix $M = \mathrm{Diag}(A, B)$ satisfies the following property: the distinct eigenvalues of M and 1 are $\mathbf{Z}$ independent, i.e., if $\{\lambda'_k\}_{1\le k\le \nu}$ are the distinct eigenvalues of M, we must have

$$\forall\, j = (j_k)_{0\le k\le\nu} \in \mathbf{Z}^{\nu+1}, \quad j_0 + \sum j_k \lambda'_k = 0 \implies j = 0. \tag{2.21}$$

This hypothesis implies that A has no eigenvalue equal to 0, and that this property remains true for $C = A - \frac{\mu_i - s + 1}{d} I_q$. This remark will be used later.

Lemma 2.5. *Let Φ be a germ of transformation of $(\mathbf{C}^p, 0)$ tangent to the identity, with the non degenerate characteristic direction $V = (1,0)$ and written in the form (2.4). The coordinate x can be chosen in such a way that the first relation (2.4) becomes*

$$x_1 = \frac{x}{1+x} + \tilde F(x,u),$$

with the estimate

$$\tilde F(x,u) = O(\|u\| x^2). \tag{2.22}$$

Proof. This is a consequence of Fatou's work [F1], since the transformation

$$x_1 = x - x^2 + F(x,0)$$

is conjugate to $\dfrac{1}{x_1} = \dfrac{1}{x} + 1$ in a domain of type $\{\Re e\, \dfrac{1}{x} > R\}$, for R big enough, and of the estimate (2.12) on F. $\square$

The next proposition and its corollaries concern the case of a transformation tangent to the identity written in the form (2.4)

$$\begin{cases} x_1 = f(x,u) = x - x^2 + O(\|u\|x^2) + O_1(x^3) \\ u_1 = g(x,u) = (I - xA)u + 0(\|u\|^2 x) + O_1(\|u\|x^2) \end{cases},$$

when A has all its eigenvalues with strictly positive real parts. Since they play an important role in what follows, although some of these results were already used in [H], we give and prove here a more precise statement.

Proposition 2.6. *Let Φ be a germ of transformation of $(\mathbf{C}^p, 0)$ tangent to the identity, with the non degenerate characteristic direction $V = (1,0)$ and let A be the matrix associated to V. Assume that all the eigenvalues of A verify*

$$\Re e\, \lambda_j > \alpha > 0, \qquad j = 1, \ldots, p-1.$$

Then, if γ, s, ρ are small enough positive constants, every point of the sector

$$\mathcal{S}_{\gamma,s,\rho} := \{(x,u) \in \mathbf{C} \times \mathbf{C}^{p-1}; |\Im m\, x| \le \gamma\ \Re e\, x,\ |x| \le s,\ \|u\| \le \rho\}$$

is attracted to the origin in the direction V and $x_n \sim \dfrac{1}{n}$. Moreover we have $|x_n| \le \dfrac{2}{n}$ and

$$\|u_n\||x_n|^{-\alpha} \le \|u\||x|^{-\alpha}\,. \tag{2.23}$$

Proof. By a linear change of coordinate in u, we can assume that the matrix A is almost diagonal, namely that the coefficients above the diagonal of A are equal to zero or ε_0, with $\varepsilon_0 > 0$, close to zero and small compared to α.

By Lemma 2.5 and Theorem 1.7, we can assume that Φ is given by

$$\begin{cases} x_1 = f(x,u) = \dfrac{x}{1+x} + F(x,u) \\ u_1 = g(x,u) = (I - xA)u + G(x,u) \end{cases}. \tag{2.24}$$

with F and G analytic functions in some domain

$$\Delta_{r,\rho} = \{(x,u)/|x - r| < r, \|u\| \le \rho\},$$

such that

$$\begin{aligned} F(x,u) &= O(\|u\||x|^2) \\ G(x,u) &= O_1(\|u\||x|^2 + \|u\|^2|x|))\,. \end{aligned} \tag{2.25}$$

Let Δ_j be defined, for $j = 1, \ldots, p-1$, by

$$\Delta_j = \{x \in \mathbf{C}; |1 - \lambda_j x| \leq 1\}.$$

Δ_j is a disc centered at $\dfrac{1}{\lambda_j}$, with radius $\dfrac{1}{|\lambda_j|}$.

Since $\Re e\, \lambda_j > 0$ for $j = 1, \ldots, p-1$, there exist positive constants γ, s such that the sector

$$(2.26) \qquad A_{\gamma,s} = \{x \in \mathbf{C}; |\Im m\, x| \leq \gamma\ \Re e\, x,\ \ |x| \leq s\}.$$

is contained in the intersection of all the Δ_j's and of $D_r = \{x \in \mathbf{C}/|x - r| < r\}$.

Assume $\|u\| \leq \rho$. Let c be a positive constant such that for every j, $\Re e\, \lambda_j > c > \alpha$. If γ, s and ρ are small enough, for $(x, u) \in \mathcal{S}_{\gamma,s,\rho}$, where

$$(2.27) \qquad \mathcal{S}_{\gamma,s,\rho} := A_{\gamma,s} \times \{\|u\| \leq \rho\},$$

we have in fact

$$(2.28) \qquad \|u_1\| \leq \|u\|(1 - c|x|).$$

and the $\{\|u_n\|\}$ are bounded by ρ. ¿From (2.24) and (2.25), we deduce that

$$(2.29) \qquad \frac{1}{x_1} = \frac{1}{x} + 1 + O(\|u\|).$$

So (2.28) and (2.29) show that $A_{\gamma,s} \times \{\|u\| \leq \rho\}$ is a stable domain, and that $x_n = O(\frac{1}{n})$. Inequality (2.28) then implies that $\lim\limits_{n \to \infty} u_n = 0$. From (2.29), we get

$$\frac{1}{nx_n} = \frac{1}{nx} + 1 + \frac{1}{n} \sum_{k=0}^{n} O(\|u_k\|),$$

hence $x_n \sim \dfrac{1}{n}$. We also get, from $\|u_n\| \leq \rho$, $|x_n| < \dfrac{2}{n}$, for ρ small enough.

From

$$(2.30) \qquad x_1^{-\alpha} = x^{-\alpha}(1 + \alpha x + O_1(\|u\|x, x^2)),$$

we deduce that, for $|x|$, $\|u\|$ and γ small enough, we have

$$|x_1|^{-\alpha} \leq |x|^{-\alpha}(1 + c'|x|),$$

for some c' such that $c > c' > \alpha$. This inequality, with (2.28), implies

$$\|u_1\|\, |x_1|^{-\alpha} \leq \|u\|\, |x|^{-\alpha}.$$

As a consequence, for $(x,u) \in \mathcal{S}_{\gamma,s,\rho}$, (x_n, y_n) converges to the origin in the direction V and $\|u_n\| = \|\frac{y_n}{x_n}\|$ converges to 0 faster than $\dfrac{1}{n^\alpha}$, when n goes to infinity . $\square$

Corollary 2.7. *Under the hypothesis of Proposition 2.6, if (x,u) is in $\mathcal{S}_{\gamma,s,\rho}$, we have*

$$\lim_{n\to\infty} \|u_n\| |x_n|^{-\alpha} = 0 \, .$$

Proof. We apply Proposition 2.6 to α' such that

$$\Re e\, \lambda_j > \alpha' > \alpha > 0, \qquad j = 1, \ldots, p-1.$$

We get $\|u_n\| |x_n|^{-\alpha} \le \|u\| |x|^{-\alpha'} |x_n|^{\alpha' - \alpha}$. $\square$

Corollary 2.8. *Under the hypothesis of Proposition 2.6, if (x,u) is in $\mathcal{S}_{\gamma,s,\rho}$, for γ, s, ρ small enough, for each real number $\mu > 1$, for each integer q, there exists a constant $C_{\mu,q}$ such that*

$$\sum_{n=0}^{\infty} |x_n|^\mu \big|\log x_n\big|^q \le C_{\mu,q} |x|^{\mu-1} \big|\log |x|\,\big|^q.$$

Proof. From (2.29), we have

$$\frac{1}{x_1} = \frac{1}{x} + 1 + O(\|u\|)$$

so that

$$\frac{1}{x_n} = \frac{1}{x} + n + O(n\rho).$$

Hence

$$|\frac{x}{x_n}|^2 = |1 + nx + O(n\rho x)|^2 = 1 + 2n \Re e\, x(1 + O(\rho)) + |nx|^2(1 + O(\rho)).$$

Since $\Re e\, x > 0$ and $|\Im m\, x| < \gamma \, \Re e\, x$, this implies, for ρ small enough,

$$|x_n| \le \frac{|x|}{\sqrt{1 + |\frac{n}{2}x|^2}}.$$

So

$$|x_n|^\mu |\log x_n|^q \le |x_n|^\mu |\log 2|x_n|\,|^q \le \frac{|x|^\mu}{(1 + |\frac{n}{2}x|^2)^{\frac{\mu}{2}}} |\log \frac{2|x|}{\sqrt{1 + |\frac{n}{2}x|^2}}|^q.$$

Hence

$$\sum_{n=0}^{\infty}|x_n|^{\mu}|\log x_n|^q \leq |x|^{\mu-1}\int_0^{\infty}|\log\frac{2|x|}{\sqrt{1+|\frac{t}{2}x|^2}}|^q\frac{1}{(1+|\frac{t}{2}x|^2)^{\frac{\mu}{2}}}|x|dt$$

$$=2|x|^{\mu-1}\int_0^{\infty}|\log\frac{2|x|}{\sqrt{1+s^2}}|^q\frac{1}{(1+s^2)^{\frac{\mu}{2}}}ds.$$

To get the corollary, we develop the second member of the inequality

$$|\log\frac{2|x|}{\sqrt{1+s^2}}|^q \leq (|\log|x||+|\log\frac{\sqrt{1+s^2}}{2}|)^q,$$

we multiply by $\dfrac{1}{(1+s^2)^{\frac{\mu}{2}}}$ and we integrate. □

3. Proof of Theorem 1.6.

Let Φ be a germ of transformation tangent to the identity, with $V=(1,0)$ a non degenerate characteristic direction. Under the hypothesis of Theorem 1.6, we can assume that Φ is described, as in (2.10) by equations

$$\left\{\begin{array}{l} x_1=f(x,u,v)=x-x^2+F(x,u,v)\\ u_1=g(x,u,v)=(I_q-xA)u+G(x,u,v)\\ v_1=h(x,u,v)=(I_r-xB)v+H(x,u,v)\end{array}\right. ,\tag{3.1}$$

with A and B satisfying conditions (2.11), and with F, G, H analytic in some $\Delta_{r,\rho}=\{(x,U)/|x-r|<r,\ \|U\|<\rho\}$, and verifying conditions (2.12). By Proposition 2.2, we can assume that we have, for m,k, arbitrary fixed integers

$$H(x,u,0)=O(|x|\|u\|^m+|x|^k\|u\|).\tag{3.2}$$

We want to show that there exists a function $v=\varphi(x,u)$, analytic in some sector $S_{\gamma,s,\rho}$ (2.27), such that for

$$\left\{\begin{array}{l} x_1=f(x,u,\varphi(x,u))\\ u_1=g(x,u,\varphi(x,u))\end{array}\right. ,\tag{3.3}$$

we have

$$\varphi(x_1,u_1)=h(x,u,\varphi(x,u)).\tag{3.4}$$

The function φ will be obtained as a fixed point of an operator T that we describe below. We use the same kind of ideas as for the proof of Theorem 1.7 in [H]. We first transform the third relation (3.1)

$$v_1 = (I_r - xB)v + H(x,u,v),$$

by setting

$$w = x^{-B} v \tag{3.5}$$

where x^{-B} is the matrix $x^{-B} = \exp(-B \log x)$ (well defined since x remains in $\Re e\, x > 0$). Let H_1 be the function defined by the formula

$$w - w_1 = x^{-B} H_1(x,u,v). \tag{3.6}$$

By the definitions of x_1 and u_1 in (3.1), we have

$$x_1^{-B} = x^{-B}(I + xB + O_1(\|U\|x, x^2));$$

and

$$w_1 = (I + O_1(\|U\|x, x^2))w + x^{-B}(I + O_1(x))H(x,u,v).$$

So H_1 satisfies the same estimates as H :

$$H_1(x,u,v) = O(|x|\|U\|^2) + O_1(|x|^2\|U\|), \tag{3.7}$$

and

$$H_1(x,u,0) = O(|x|\|u\|^m + |x|^k\|u\|). \tag{3.8}$$

The invariance (3.4) of φ is then equivalent to

$$x^{-B}\varphi(x,u) - x_1^{-B}\varphi(x_1,u_1) = x^{-B} H_1(x,u,\varphi(x,u)). \tag{3.9}$$

3.1. Definition of the operator T..

Let $\{(x_n, u_n)\}$ be the iterates of the transformation

$$\begin{cases} x_1 = f(x,u,\varphi(x,u)) = x - x^2 + F(x,u,\varphi(x,u)) \\ u_1 = g(x,u,\varphi(x,u)) = (I_q - xA)u + G(x,u,\varphi(x,u)) \end{cases} \tag{3.10}$$

with f and g given in (3.1) and $\varphi(.,.)$ chosen in a set $\mathcal{F}$ (to be defined later) of functions, analytic in some $\mathcal{S}_{\gamma,s,\rho}$. Let $T\varphi$ be defined by

$$T\varphi(x,u) = x^B \sum_{n=0}^{\infty} x_n^{-B} H_1(x_n, u_n, \varphi(x_n,u_n)). \tag{3.11}$$

We will define a closed set $\mathcal{F}$, containing $\varphi_0 = 0$, in the Banach space of analytic bounded functions $\varphi : \mathcal{S}_{\gamma,s,\rho} \to \mathbf{C}^r$ such that T is well defined, continuous and contracting from $\mathcal{F}$ to $\mathcal{F}$. So T will have a unique fixed point in $\mathcal{F}$. This fixed point will clearly be a solution of (3.9).

Proposition 3.2. *Let m and k be integers such that H_1 satisfies the estimate (3.8). There is a positive constant K_0 such that*

$$\|\varphi(x,u)\| \leq K_0(\|u\|^m + |x|^{k-1}\|u\|) \tag{3.12}$$

implies that the series (3.11) is uniformly converging on the invariant set (see (2.23))

$$\mathcal{S}_{\gamma,s,\rho} \cap \{\|u\||x|^{-\alpha} \leq 1\},$$

and that we also have

$$\|T\varphi(x,u)\| \leq K_0(\|u\|^m + |x|^{k-1}\|u\|). \tag{3.13}$$

Proof. Using Corollary 2.7, we see that there is no loss of generality in assuming that $\|u\||x|^{-\alpha}$ is bounded by 1. Let $\beta < \alpha$ be a number such that all the eigenvalues of B satisfy the inequality $\Re e\, \mu_k < \beta$. This condition implies

$$\|x^B {x_n}^{-B}\| \leq C_1 |\frac{x_n}{x}|^{-\beta}. \tag{3.14}$$

¿From Proposition 2.6, we know that, for γ, s, ρ small enough, if $(x,u) \in \mathcal{S}_{\gamma,s,\rho}$,

$$|x_n| \leq \frac{2}{n}, \quad \|u_n\| \leq \|u\||x|^{-\alpha}|x_n|^{\alpha}. \tag{3.15}$$

By (3.7) and (3.8) there exist constants K_1 and K_2 depending only on H_1, such that, for some integer q,

$$\begin{aligned} \|H_1(x,u,v)\| \leq {} & K_1(|x|\|u\|^m + |x|^k\|u\|) + \\ & K_2\left[\|v\||x|^2|\log x|^q + \|v\|^2|x| + \|u\|\|v\||x|\right]. \end{aligned} \tag{3.16}$$

Assume that Inequality (3.12) is true with some K (instead of K_0 to be determined later). Then for $v = \varphi(x,u)$, the three terms in (3.16) containing $\|v\|$ are bounded by

$$O(\|u\| + |x||\log|x||^q)(|x|\|u\|^m + |x|^k\|u\|).$$

So, if s and ρ are small enough, $H_1(x,u,\varphi(x,u))$ satisfies almost the same estimates in $S_{\gamma,s,\rho}$ as $H_1(x,u,0)$, namely

$$\|H_1(x,u,\varphi(x,u))\| \leq (K_1+1)(|x|\|u\|^m + |x|^k\|u\|).$$

Hence, using (3.15), we get for the series (3.11),

$$\sum_{n=0}^{\infty} \|(\frac{x_n}{x})^{-B} H_1(x_n, u_n, \varphi(x_n, u_n))\|$$

$$\leq (K_1+1) \sum_{n=0}^{\infty} |\frac{x_n}{x}|^{-\beta} (|x_n| \|u_n\|^m + |x_n|^k \|u_n\|)$$

$$\leq (K_1+1) \sum_{n=0}^{\infty} |\frac{x_n}{x}|^{-\beta} (|x_n|^{m\alpha+1} \|u\|^m |x|^{-m\alpha} + |x_n|^{k+\alpha} \|u\| |x|^{-\alpha}).$$

Since $\alpha > \beta$, the series is normally convergent on the set $\|u\||x|^{-\alpha} \leq 1$. Corollary 2.8 then shows that there exists a constant K_0 depending only on H_1 such that

$$\|T\varphi(x,u)\| \leq K_0(\|u\|^m + |x|^{k-1}\|u\|). \tag{3.17}$$

To conclude the proof of Proposition 3.2, we choose $K = K_0$. □

Let $\mathcal{F}_0$ be the set of analytic functions defined on $S_{\gamma,s,\rho}$ such that condition (3.12) is satisfied with the constant K_0 given by Proposition 3.1. Then T sends $\mathcal{F}_0$ to $\mathcal{F}_0$. Since we also want T to be contracting, we must impose more restrictions. We first reduce the domain of definition of the functions in $\mathcal{F}_0$.

3.3. Choice of the domain of definition $\mathcal{D}$.

In what follows, instead of $S_{\gamma,s,\rho}$ (2.4), we take as domain of definition for the functions φ

$$\mathcal{D} := S_{\gamma,s,\rho} \cap \{(x,u); \|u\||x|^{-\alpha} \leq 1\}, \tag{3.18}$$

and $\mathcal{F}_0$ is the set of holomorphic functions $\varphi : \mathcal{D} \to \mathbf{C}^r$ such that condition (3.12) is satisfied.

We will prove an analogous of Proposition 3.2 for the partial derivatives of φ.

Proposition 3.4. *Let φ be in the set $\mathcal{F}_0$. There are positive constants K_1, K_2 such that*

$$\begin{cases} |\dfrac{\partial\varphi}{\partial x}| \leq K_1(\|u\|^m |x|^{-1} + |x|^{k-2}\|u\|) \\ |\dfrac{\partial\varphi}{\partial u}| \leq K_2(\|u\|^{m-1} + |x|^{k-1}) \end{cases}, \tag{3.19}$$

imply the same inequalities for $\|\frac{\partial T\varphi}{\partial x}\|$ *and* $\|\frac{\partial T\varphi}{\partial u}\|$.

To prove this proposition, we have to bound the two sums

$$\sum_{n=0}^{\infty}\|\frac{\partial}{\partial x}\{(\frac{x_n}{x})^{-B}H_1(x_n,u_n,\varphi(x_n,u_n))\}\| \tag{3.20}$$

and

$$\sum_{n=0}^{\infty}\|\frac{\partial}{\partial u}\{(\frac{x_n}{x})^{-B}H_1(x_n,u_n,\varphi(x_n,u_n))\}\|. \tag{3.21}$$

Here we need to control the partial derivatives $|\frac{\partial x_n}{\partial x}|$, $\|\frac{\partial u_n}{\partial x}\|$, $\|\frac{\partial x_n}{\partial u}\|$, $\|\frac{\partial u_n}{\partial x}\|$. This is done in the next lemma.

Lemma 3.5. *Let* $\delta=\min\{\alpha,1\}$*, and let* ε *be a positive number,* $\varepsilon<\delta$*. Then, for* γ,s,ρ *small enough, we have the following inequalities in* $\mathcal{D}$ *(3.18):*

$$|\frac{\partial x_n}{\partial x}|\le|\frac{x_n}{x}|^{1+\delta-2\varepsilon},\qquad \|\frac{\partial u_n}{\partial x}\|\le\frac{\|u\||x_n|^{\delta-\varepsilon}}{|x|^{1+\delta-2\varepsilon}}, \tag{3.22}$$

and

$$\|\frac{\partial x_n}{\partial u}\|\le\frac{|x_n|^{1+\delta-2\varepsilon}}{|x|^{\delta-\varepsilon}},\qquad \|\frac{\partial u_n}{\partial u}\|\le|\frac{x_n}{x}|^{\delta-\varepsilon}. \tag{3.23}$$

.

Proof. Since

$$|\frac{\partial x_1}{\partial x}|=|(1-2x+o(x))|\le|\frac{x_1}{x}|^{2-\varepsilon}\quad,\quad\|\frac{\partial u_1}{\partial x}\|\le K\|u\|,$$

and

$$\|\frac{\partial x_1}{\partial u}\|\le K|x|^2\quad,\quad\|\frac{\partial u_1}{\partial u}\|\le|1-\alpha x|,$$

for γ,s,ρ small enough, the inequalities (3.22) and (3.23) are true for $n=1$. We prove the result by induction on n.

¿From the definition of Φ, we get

$$|\frac{\partial x_{n+1}}{\partial x}|\le|(1-2x_n+o(x_n)||\frac{\partial x_n}{\partial x}|+K|x_n|^2\|\frac{\partial u_n}{\partial x}\|. \tag{3.24}$$

and

$$\|\frac{\partial u_{n+1}}{\partial x}\|\le K\|u_n\||\frac{\partial x_n}{\partial x}|+|1-\alpha x_n|\|\frac{\partial u_n}{\partial x}\|. \tag{3.25}$$

So, by the hypothesis of induction and (3.24),

$$|\frac{\partial x_{n+1}}{\partial x}| \leq |\frac{x_n}{x}|^{1+\delta-2\varepsilon}(1-2\Re e x_n + o(x_n) + K\|u\||x_n|^{1+\varepsilon}),$$

which is smaller than $|\frac{x_{n+1}}{x}|^{1+\delta-2\varepsilon}$ since $1+\delta-2\varepsilon < 2$. Using the hypothesis of induction, the inequality $\|u_n\| \leq \|u\||x|^{-\alpha}|x_n|^{\alpha}$ and (3.25), we get

$$\|\frac{\partial u_{n+1}}{\partial x}\| \leq \frac{\|u\|}{|x|^{1+\delta-2\varepsilon}}|x_n|^{\delta-\varepsilon}(1-\alpha\Re e x_n + o(x_n) + K|x|^{-\alpha}|x_n|^{1+\alpha-\varepsilon}),$$

which is smaller than $\frac{\|u\|}{|x|^{1+\delta-2\varepsilon}}|x_{n+1}|^{\delta-\varepsilon}$ since $\delta-\varepsilon < \alpha$.

We have proved the inequality (3.22) for $n+1$. The same argument gives the inequality (3.23). □

Proof of Proposition 3.4. We just sketch the proof which is similar to the proof of Proposition 3.2. Assume that inequalities (3.19) are satisfied for two arbitrarily given constants K and K' (instead of K_1 and K_2). If we develop and bound the partial derivatives (3.20) and (3.21), by using the hypothesis, Lemma 3.5 and Corollary 2.8, we find that $\|\frac{\partial T\varphi}{\partial x}\|$ and $\|\frac{\partial T\varphi}{\partial u}\|$ satisfy the inequalities (3.19) with constants K_1 and K_2, depending only on H_1 (and not on K and K'). So Proposition 3.2 is true for these constants K_1 and K_2. □

3.6. Definition of $\mathcal{F}$.

Let m and k be integers such that H_1 satisfies conditions (3.8). Let F_0 be the Banach space of holomorphic functions φ on $\mathcal{S}_{\gamma,s,\rho}$ such that

$$\|\varphi\|_0 := \sup_{x,u}\{\frac{|\varphi(x,u)|}{(\|u\|^m + |x|^{k-1}\|u\|)}\}, \tag{3.26}$$

is bounded, endowed with the norm $\|\varphi\|_0$. We define $\mathcal{F}$ as the closed subset of F_0 of functions such that the inequalities (3.12) and (3.19) are satisfied with the constants K_0, K_1, K_2 given by Propositions 3.2 and 3.4.

End of the proof of Theorem 1.6.

Propositions 3.2 and 3.4 and the definition of $\mathcal{F}$ prove that T maps $\mathcal{F}$ into $\mathcal{F}$. To conclude the proof of Theorem 1.6, we have to show that T is a contraction. So we have to bound for φ and ψ in $\mathcal{F}$, the variation

$$S := \|\sum_{n=0}^{\infty}(\frac{x_n}{x})^{-B}H_1(x_n,u_n,\varphi(x_n,u_n)) - \sum_{n=0}^{\infty}(\frac{x'_n}{x})^{-B}H_1(x'_n,u'_n,\psi(x'_n,u'_n))\|,$$

where (x_n, u_n) and (x'_n, u'_n) are the iterates of (x, u) by the transformations (3.10) respectively associated to φ and ψ. We can bound S by the sum of S_1 and S_2 with

$$S_1 := \| \sum_{n=0}^{\infty} (\frac{x_n}{x})^{-B} H_1(x_n, u_n, \varphi(x_n, u_n)) - \sum_{n=0}^{\infty} (\frac{x_n}{x})^{-B} H_1(x_n, u_n, \psi(x_n, u_n)) \| ,$$

and

$$S_2 := \| \sum_{n=0}^{\infty} (\frac{x_n}{x})^{-B} H_1(x_n, u_n, \psi(x_n, u_n)) - \sum_{n=0}^{\infty} (\frac{x'_n}{x})^{-B} H_1(x'_n, u'_n, \psi(x'_n, u'_n)) \| .$$

The first term S_1 is easily bounded. By (3.16), we have for some integer q,

$$S_1 \leq K \sum_{n=0}^{\infty} |\frac{x_n}{x}|^{-\beta} (|x_n|^2 |\log x_n|^q + \|u_n\| |x_n|)(\|u_n\|^m + \|u_n\| |x_n|^{k-1}) \|\varphi - \psi\|_0 .$$

Using Corollary 2.8, and the inequality $\|u_n\| \leq \|u\| |\frac{x_n}{x}|^{\alpha}$, we get

$$S_1 \leq K'(|x| \, |\log x|^q + \|u\|)(\|u\|^m + \|u\| |x|^{k-1}) \|\varphi - \psi\|_0 . \tag{3.27}$$

To bound the term S_2, we must estimate the dependence of the sequence $\{(x_n, u_n)\}$ on φ in the definition (3.10). It is given by the following lemma.

Lemma 3.7. *Let $\delta = \min\{\alpha, 1\}$. Let ε be a positive number, such that $\varepsilon < \delta$ and $\Re e \, \lambda_j > \alpha + \varepsilon$ for every eigenvalue λ_j of A. Let φ and ψ be in $\mathcal{F}$. Let $\{(x_n, u_n)\}$ and $\{(x'_n, u'_n)\}$ be the iterates of the transformations (3.10) respectively associated to φ and ψ. Then for γ, s, ρ small enough, we have the following bounds in $S_{\gamma,s,\rho}$:*

$$|x_n - x'_n| \leq |x_n|^{1+\delta-\varepsilon} |x|^{-\delta} (\|u\|^m + \|u\| |x|^{k-1}) \|\varphi - \psi\|_0 , \tag{3.28}$$

and

$$\|u_n - u'_n\| \leq |x_n|^{\delta} |x|^{-\delta} (\|u\|^m + \|u\| |x|^{k-1}) \|\varphi - \psi\|_0 . \tag{3.29}$$

Proof. We use the notations: Δx_n for $|x_n - x'_n|$, Δu_n for $\|u_n - u'_n\|$ and $\Delta\varphi(x, u)$ for $(\|u\|^m + \|u\| |x|^{k-1}) \|\varphi - \psi\|_0$. ¿From the definition of x_1 and u_1, we have

$$\Delta x_1 \leq K |x|^2 \Delta\varphi(x, u) , \quad \Delta u_1 \leq K(|x|^2 + |x| \|u\|) \Delta\varphi(x, u) .$$

So the inequalities are true for $n = 1$ if s, ρ are small enough. We assume the property true for n and we prove it for $n+1$. Since x_n and x'_n are booth equivalent to $\frac{1}{n}$, we have $x'_n = x_n + o(x_n)$. From the definition of Φ, we get

$$\Delta x_{n+1} \leq \Delta x_n |1 - 2x_n + o(x_n)| + K|x_n|^2 \Delta u_n + K|x_n|^2 \Delta\varphi(x_n, u_n)$$

and

$$\Delta u_{n+1} \leq K\|u_n\|\Delta x_n + |1 - (\alpha+\varepsilon)x_n + o(x_n)|\Delta u_n$$
$$+K(|x_n|^2 + \|u_n\||x_n|)\Delta\varphi(x_n, u_n).$$

Using the inequality $\|u_n\| \leq \|u\||\dfrac{x_n}{x}|^\alpha$, we get

$$\Delta\varphi(x_n, u_n) \leq (\|u\|^m + \|u\||x|^{k-1})\|)(|\frac{x_n}{x}|^{m\alpha} + |\frac{x_n}{x}|^{\alpha+k-1})\|\varphi - \psi\|_0\,,$$

and, since $|\dfrac{x_n}{x}|^\gamma \leq |\dfrac{x_n}{x}|^\delta$ when $\delta < \gamma$,

$$|x|^\delta \Delta\varphi(x_n, u_n) \leq 2|x_n|^\delta \Delta\varphi(x, u)\,.$$

The proof is now similar to the proof of Lemma 3.5. We use the inequalities (3.28) and (3.29) at rank n to bound $|x|^\delta \dfrac{\Delta x_{n+1}}{\Delta\varphi}$ and $|x|^\delta \dfrac{\Delta u_{n+1}}{\Delta\varphi}$. We get

$$|x|^\delta \frac{\Delta x_{n+1}}{\Delta\varphi} \leq |1 - \varepsilon x_n + o(x_n)||x_{n+1}|^{1+\delta-\varepsilon} + K|x_n|^{2+\delta} + 2K|x_n|^{2+\delta}$$

$$= |1 - \varepsilon x_n + o(x_n)||x_{n+1}|^{1+\delta-\varepsilon} \leq |x_{n+1}|^{1+\delta-2\varepsilon}\,,$$

and

$$|x|^\delta \frac{\Delta u_{n+1}}{\Delta\varphi} \leq K\|u_n\||x_n|^{1+\delta-\varepsilon} + |1 - \varepsilon x_n + o(x_n)||x_{n+1}|^\delta$$

$$+K(|x_n| + \|u_n\|)|x_n|^{1+\delta} = |1 - \varepsilon x_n + o(x_n)||x_{n+1}|^\delta \leq |x_{n+1}|^\delta\,.$$

For the last inequality, we use the fact that $\|u_n\||x_n|^{-\alpha} = o(1)$, so

$$\|u_n\||x_n|^{1+\delta-\varepsilon} = o(|x_n|^{1+\alpha+\delta-\varepsilon}) = |x_n|^\delta o(|x_n|)\,,$$

if ε is small enough. This concludes the proof of Lemma 3.7. □

We can now give a bound for S_2. We have

$$S_2 \leq \sum_{n=0}^{\infty} K_1 |\frac{x_n}{x}|^{-\beta-\varepsilon}(\|u_n\|^m + |x_n|^{k-1}\|u_n\|)\Delta x_n$$

$$+K_2|\frac{x_n}{x}|^{-\beta-\varepsilon}(\|u_n\|^{m-1}|x_n|+|x_n|^k)\Delta u_n\,.$$

Using Lemma 3.7, Corollary 2.8 and the fact that $\|u_n\||x_n|^{-\varepsilon}=o(1)$, we get then

$$S_2\leq K(\|u\|^{m-1}+|x_n|^{k-1})(\|u\|^m+\|u\||x|^{k-1})\|\varphi-\psi\|_0.$$

So T is contracting and the proof of Theorem 1.6 is complete. □

¿From Theorem 1.6, we deduce the corollary:

Corollary 3.8. *Let Φ be a germ of transformation of $(\mathbf{C}^p,0)$ tangent to the identity. Let V be a non degenerate characteristic direction. Let $\{\lambda_j\}$ be the eigenvalues of the matrix A associated to V which have strictly positive real parts and let the distinct level values of the $\{\Re e\,\lambda_j\}$'s be*

$$\alpha_1>\alpha_2>\ldots>\alpha_h>0.$$

Then there is an increasing sequence

$$M_1\subset M_2\subset\ldots\subset M_h$$

of h invariant pieces of analytic manifolds, defined in some sector, which are attracted to the origin in the direction V; for every i, $1\leq i\leq h$, the dimension of M_i is equal to one plus the sum of the multiplicity of the λ_j's such that $\Re e\,\lambda_j\geq\alpha_i$, and is tangent at the origin to the sum of Cv and of the generalized eigenspaces associated to the corresponding λ_j's.

4. Proof of Theorem 1.9.

We assume now that Φ is a germ of transformation of $(\mathbf{C}^p,0)$ tangent to the identity, that $V=(1,0)$ is a non degenerate characteristic direction and that all the eigenvalues of the matrix A associated to V verify

$$\Re e\,\lambda_j>\alpha>0\quad,\quad 1\leq j\leq p-1\,. \tag{4.1}$$

Let $\lambda_1,\lambda_2,\ldots,\lambda_h$, be the distinct eigenvalues of A ordered in such a way that their real parts $\alpha_1,\alpha_2,\ldots,\alpha_h$, satisfy $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_h>\alpha$. We choose coordinates in which A is in Jordan form, almost diagonal, i.e., with the non zero coefficients above the diagonal equal to $\varepsilon>0$, with ε small compared to α. Let A_j be the matrices such that

$$A=\mathrm{Diag}\{A_1,A_2,\ldots,A_h\}, \tag{4.2}$$

where for each $j = 1, 2, \ldots, h$, the eigenvalues of A_j are equal to λ_j. We define for $\Re e\, x > 0$ the matrix x^{-A} by

$$x^{-A} = \exp(-A \log x) = \mathrm{Diag}\{x^{-A_1}, x^{-A_2}, \ldots, x^{-A_h}\}.$$

The transformation can be written

$$\begin{cases} x_1 = x - x^2 + F(x,u) \\ u_1^1 = (I_1 - xA_1)u^1 + G^1(x,u) \\ u_1^2 = (I_2 - xA_2)u^2 + G^2(x,u) \\ \qquad\vdots \\ u_1^h = (I_h - xA_h)u^h + G^h(x,u) \end{cases}, \tag{4.3}$$

where $u = (u^1, u^2, \ldots, u^h)$ is the splitting of the coordinates in $\mathbf{C}^{p-1}$ associated to the splitting (4.1) of A and the I_j's are unit matrices of suitable dimensions, and for each j, $G^j(x,u)$ is such that

$$G^j(x,u) = O(\|u\|^2|x|) + O_1(\|u\||x|^2)\,.$$

When $j = 1, 2, \ldots, h-1$ we introduce for $u = (u^1, u^2, \ldots, u^h) \in \mathbf{C}^{p-1}$, the notations $u^{\leq j}$ and $u^{>j}$ with the meaning

$$u^{\leq j} := (u^1, u^2, \ldots, u^j), \qquad u^{>j} := (u^{j+1}, u^{j+2}, \ldots, u^h). \tag{4.4}$$

Let k be any positive integer. The estimates (2.7) show that, for every j such that $1 \leq j \leq h$, we have

$$G^j(x,u) = P_k^j(x,u) + O_1(\|u\||x|^N), \tag{4.5}$$

where

$$P_k^j(x,u) = \sum_{1 \leq s \leq N, t \in E_s} x^s (\log x)^t c_{s,t}(u),$$

(see (2.7) and (2.8) for the definition of the finite set E_s) . The functions $c_{s,t}(u)$ are analytic in $u \leq \rho$.

Proposition 4.1. *For any positive integers k and m, there exist local coordinates such that we have (4.5) and moreover*

$$\|P_k^j(x, (0, u^{>j}))\| = O(|x|\|u^{>j}\|^m) \tag{4.6}$$

for $1 \leq j \leq h$.

Proof. We want to cancel in $P_k^j(x, (0, u^{>j}))$ the terms in $u^{>j}$ of degree smaller than m. This will be done by successive changes of coordinates of type $\tilde{u}^j =$

$u^j - q_j(x, u^{>j})$ where the q_j's are polynomials in $(x, \log x, u^{>j})$, of positive order in $u^{>j}$. Hence if the property is satisfied for $j = 1, 2, \ldots, j_0$, changing the variables u^j for $j > j_0$ will not affect the property for the j_0 first variables. So we can do the construction by induction on j, starting with $j = 1$, and, to define the change of coordinates on u^j, we can consider only the transformation on $u^{\geq j}$ with $u^{<j} = 0$ and forget what happens to the first $j-1$ variables.

Let $v = u^{>j}$, and let B_j be the matrix

$$B_j = \mathrm{Diag}\{A_{j+1}, \ldots, A_r\}.$$

We have to prove a statement analog to Proposition 2.2 but, with the notations of Proposition 2.2, we want to make a change of variable of type $\tilde{u} = u - \lambda(x, v)$ such that

$$\tilde{G}(x, 0, v) = O(|x|\|v\|^m + |x|^k\|v\|). \tag{4.7}$$

instead of the same condition on $\tilde{H}(x, u, 0)$. The situation is different because the roles of $A(= A_j)$ and $B = B_j$ are exchanged.

If the eigenvalues of the whole matrix A satisfy the condition of independence (2.21) of Remark 2.4 (ii), the same proof than in Proposition 2.2 applies and we have finished.

We have now to look at the more complicated case where condition (2.21) is not satisfied.

Let us try to imitate the proof of Proposition 2.2: Let $Q(v)x^s(\log x)^t$ be the term of lowest degree (s, t, d) in $P_k^j(x, (0, v))$, with $Q(v)$ a homogeneous polynomial of degree d in $v = u^{>j}$. If we make the transformation

$$\tilde{u}^j = u^j - x^{s-1}(\log x)^t P(v),$$

the term $Q(v)x^s(\log x)^t$ vanishes if P is solution of

$$P\big((I_q - xB)v\big) - (I_r - xA_j)P(v) - (s-1)xP(v) = xQ(v) + O(\|v\|x^2).$$

Here r and q are the respective dimensions of u^j and v.

Since A_j is in Jordan form, by decreasing induction on the indices of the components of u^j, this leads to solve for the r components P_i of P an equation of type

$$\frac{\partial P_i}{\partial v_1}(Bv)_1 + \ldots + \frac{\partial P_i}{\partial v_q}(Bv)_q - (\lambda_j - s + 1)P_i = \tilde{Q}_i(v). \tag{4.8}$$

By Euler formula, this equation is equivalent to

$$\frac{\partial P_i}{\partial v_1}(Cv)_1 + \ldots + \frac{\partial P_i}{\partial v_q}(Cv)_q = \tilde{Q}_i(v)\,, \tag{4.9}$$

with $C = B - \frac{\lambda_j - s+1}{d} I$.

We solve these equations line by line, for monomials of type cv^j, $j = (j_1, \ldots, j_q) \in \mathbf{N}^q$ as second members of the equations (4.9).

For $j \in \mathbf{N}^q$, let us define the weight $w(j)$ of j by

$$w(j) = j_1 + 2j_2 + \cdots + qj_q.$$

We remark that for $P_i(v) = av^j$ and $\tilde{Q}_i(v) = cv^j$, equation (4.8) is reduced, modulo terms of bigger weight to

$$a(\nu_1 j_1 + \nu_2 j_2 + \cdots + \nu_q j_q) = c,$$

where $\{\nu_1, \cdots, \nu_q\}$ are the eigenvalues of C. So if $\nu_1 j_1 + \nu_2 j_2 + \cdots + \nu_q j_q \neq 0$, we can find a solution of the equation. If $\nu_1 j_1 + \nu_2 j_2 + \cdots + \nu_q j_q = 0$, we make the change of variable

$$\tilde{u}^j = u^j - av^j x^{s-1} (\log x)^{t+1},$$

The terms in $v^j x^{s-1} (\log x)^{t+1}$ and in $v^j x^s (\log x)^{t+1}$ in the first member vanish, and the equation is reduced to $a(t+1) = -c$. We have introduced new terms in $x^s (\log x)^{t+1}$, but this is not a problem, since it can happen only a finite number of times. So proceeding by induction on the weight of j, for a given degree d, we have to solve the case of j of maximal weight. This means that $v^j = v_q^d$. In this case, equation (4.8) is reduced to $a\nu_q d = c$, so we can find c if $\nu_q \neq 0$. If $\nu_q = 0$, we proceed as before with the change of variable

$$\tilde{u}^j = u^j - av_q^d x^{s-1} (\log x)^{t+1}.$$

Proposition 4.1 is thus proved. $\square$

Corollary 4.2. *Under the hypothesis (4.1), for every integers m and N, we can assume that the transformation is described by the equations (4.2) with estimates*

$$G^{\leq j}(x, (0, u^{>j})) = O(|x| \|u^{>j}\|^m + |x|^N \|u^{>j}\|). \tag{4.10}$$

for each $j = 1, 2, \ldots, h$.

Corollary 4.3. *Let ε be a small positive number, $\varepsilon < \alpha_h$. Under the hypothesis of Theorem 1.9, assume that the coordinates are chosen in such a way that*

- *the coefficients above the diagonal of A are equal to zero or ε_0, with $\varepsilon_0 > 0$, close to zero and small compared to ε,*

– *Condition (4.10) is satisfied with m and N such that*

$$m\alpha_h - \alpha_1 \geq 1 \text{ and } N + \alpha_h - \alpha_1 \geq 2. \tag{4.11}$$

Then, for every j, and for every x in $\mathcal{S}_{\gamma,s,\rho}$, we have, for γ, s, ρ small enough,

$$\|u_n^{\leq j}\| \leq |x_n|^{\alpha_j - \varepsilon}(\|u^{\leq j}\||x|^{-\alpha_j+\varepsilon} + K|x|), \tag{4.12}$$

for some constant K.

Proof. By Proposition 2.6, we know that, for $\alpha = \alpha_h - \varepsilon$, we have

$$\|u_n\| \leq |x_n|^{\alpha}\|u\||x|^{-\alpha}.$$

Hence from (4.10) and (4.11), we get

$$|x_{n+1}^{-\alpha_j+\varepsilon}|\|G^{\leq j}(x,(0,u_n^{>j}))\| \leq K_1|x_n|^2.$$

The hypothesis on ε_0 and ε imply, by the same proof as in Proposition 2.6, since the eigenvalues of $A_j - (\alpha_j - \varepsilon)I_j$ have positive real parts, that if x is in $\mathcal{S}_{\gamma,s,\rho}$, for γ, s, ρ small enough, we have

$$\|I_j - \big(A_j - (\alpha_j - \varepsilon)I_j\big)x + o(x)\| \leq 1.$$

We have

$$|x_{n+1}^{-\alpha_j+\varepsilon}|\|u_{n+1}^{\leq j}\| \leq \|I_j - \big(A_j - (\alpha_j-\varepsilon)I_j\big)x + o(x)\||x_n^{-\alpha_j+\varepsilon}|\|u_n^{\leq j}\|$$

$$+|x_{n+1}^{-\alpha_j+\varepsilon}|\|G^{\leq j}(x,(0,u_n^{>j}))\|.$$

So setting $V_n^{\leq j} = |x_n^{-\alpha_j+\varepsilon}|\|u_n^{\leq j}\|$, we get

$$V_{n+1}^{\leq j} \leq V_n^{\leq j} + K_1|x_n|^2.$$

Using the fact that in $\mathcal{S}_{\gamma,s,\rho}$, $|x_{n+1}| \leq |x_n|(1 - c|x_n|)$ for some $c, 0 < c < 1$, so

$$|x_n|^2 \leq c^{-1}(|x_n| - |x_{n+1}|),$$

we get for some positive constant K,

$$V_{n+1}^{\leq j} + K|x_{n+1}| \leq V_n^{\leq j} + K|x_n| \leq \ldots \leq V^{\leq j} + K|x|.$$

Hence Inequality (4.12) is proved. $\square$

Corollary 4.4. *Under the hypothesis of Corollary 4.3, for γ, s, ρ small enough, if (x,u) is in $\mathcal{S}_{\gamma,s,\rho}$, $\|u_n^{\leq j}\||x_n|^{-\alpha_j+\varepsilon}$ converges to zero as n goes to infinity.*

Proof. The preceding corollary proves that $\|u_n^{\leq j}\||x_n|^{-\alpha_j+\varepsilon_1}$ is bounded for ε_1 such that $\varepsilon_0 \ll \varepsilon_1 < \varepsilon$, so $\|u_n^{\leq j}\||x_n|^{-\alpha_j+\varepsilon}$ converges to zero. □

Using Corollary 4.4, there is no loss of generality in assuming that the coordinates are chosen as in Corollary 4.3 and that (x,u) is in $\mathcal{D}$, where

$$\mathcal{D} := \{(x,u) \in \mathcal{S}_{\gamma,s,\rho};\, \|u^{\leq j}\||x|^{-\alpha_j+\varepsilon} \leq 1\}. \tag{4.13}$$

The following lemma will almost conclude the proof of Theorem 1.9.

Lemma 4.5. *Under the hypothesis of Theorem 1.9, assume that the coordinates are chosen as in Corollary 4.3 with ε satisfying moreover $3\varepsilon < \inf(\alpha_1, 1)$. Then the sequence $\{x_n^{-A}u_n\}$ converges normally on $\mathcal{D}$ (4.13).*

Proof. For each j, we will bound $\|x_{n+1}^{-A_j}u_{n+1}^j - x_n^{-A_j}u_n^j\|$ in $\mathcal{D}$ by a convergent numerical series. We have, with notations (4.4), for some integer q,

$$\begin{aligned}
&\|x_{n+1}^{-A_j}u_{n+1}^j - x_n^{-A_j}u_n^j\| \\
&\qquad\qquad \leq K(\|x_n^{-A_j}\|\|u_n^{\leq j}\|\left(\|u_n\||x_n| + |x_n|^2|\log x_n|^q\right)) \\
&\qquad\qquad +K'\|x_n^{-A_j}\|\|G^j\left(x,(0,u_n^{>j})\right)\|
\end{aligned}$$

Since we have $\|u_n^{\leq j}\| \leq |x_n|^{\alpha_j-\varepsilon}$ in $\mathcal{D}$, and using $\|x_n^{-A_j}\| \leq |x_n|^{-\alpha_j-\varepsilon}$, we get

$$\begin{aligned}
&\|x_{n+1}^{-A_j}u_{n+1}^j - x_n^{-A_j}u_n^j\| \\
&\leq K_1|x_n|^{-\alpha_j-\varepsilon}(|x_n|^{1+\alpha+\alpha_j-\varepsilon} + |x_n|^{2+\alpha_j-\varepsilon}|\log x_n|^q) + K_2|x_n|^2.
\end{aligned}$$

Hence, we have for some constant K,

$$\|x_{n+1}^{-A_j}u_{n+1}^j - x_n^{-A_j}u_n^j\| \leq K(|x_n|^{1+\alpha_1-3\varepsilon} + |x_n|^{2-2\varepsilon}|\log x_n|^q).$$

This series is normally converging. So, we can define in $\mathcal{D}$ a holomorphic bounded function

$$H(x,u) = \sum_{n=0}^{+\infty}(x_{n+1}^{-A}u_{n+1} - x_n^{-A}u_n)\,. \tag{4.14}$$

Moreover, by Corollary 2.8, we have

$$\|H(x,u)\| \leq K(|x|^{\alpha_1-3\varepsilon} + |x|^{1-2\varepsilon}|\log x|^q) \leq K(|x|^{\alpha_1-3\varepsilon} + |x|^{1-3\varepsilon}. \tag{4.15}$$

Therefore

$$U(x,u) := x^{-A}u + H(x,u) = \lim_{n\to+\infty} x_n^{-A}u_n \tag{4.16}$$

is an invariant function. The main term near the origin is $x^{-A}u$ and (4.16) implies that $\{U(x,u)=c\}$, $c\in\mathbf{C}$, are complex invariant analytic curves.

End of the proof of Theorem 1.9.

If we take (x,U) as new coordinates, where U is the invariant function defined by (4.16), the transformation takes the form

$$\begin{cases} x_1 = x - x^2 + \tilde{F}(x,U) \\ U_1 = U \end{cases}, \tag{4.17}$$

with $\tilde{F}$ a function of order greater than 2 in x, and U behaves like a parameter. So using the ideas of Fatou [F2], we can make a change of the coordinate x, depending on U, such that the transformation Φ is given by the equations: $\{\frac{1}{z_1} = \frac{1}{z} + 1\,,\ U_1 = U\}$. □

Corollary 4.6. *Let Φ be a germ of transformation of $(\mathbf{C}^p,0)$ tangent to the identity. Let V be a non degenerate characteristic direction. Assume that the matrix A associated to V has exactly d eigenvalues with strictly positive real part. Let M be the piece of stable manifold of dimension $d+1$ provided by Theorem 1.6 in which every point is is attracted to the origin in the direction V . Then we can choose local coordinates (x,u,v) such that M is described by $v=0$, and the restriction of Φ to M is conjugate to the translation*

$$\left(\frac{1}{z_1} = \frac{1}{z} + 1\,,\ U_1 = U\right).$$

Proof. By Theorem 1.6, there exists local coordinates (x,u,v) defined in a sector $S_{\gamma,s,\rho}$ (2.27), in which M is described by $v=0$. Hence Φ is defined by equations of type (2.10) with F, G, H satisfying (2.12) and $H(x,u,0)=0$. The restriction of Φ to M is then given by equations

$$\begin{cases} x_1 = x - x^2 + F(x,u,0) \\ u_1 = (I - xA)u + G(x,u,0) \end{cases},$$

where A is a matrix which has eigenvalues with strictly positive real parts.

Let α_1 and α_h be respectively the biggest and the smallest real part of the eigenvalues of A. Let m and k be integers such that

$$m\alpha_h - \alpha_1 \geq 1 \text{ and } k + \alpha_h - \alpha_1 \geq 2.$$

We can assume that G has a Taylor expansion (see 4.5)

$$G(x,u,0) = \sum_{1\le s\le k,\, t\in E_s} c_{s,t}(u) x^s (\log x)^t + O(|x|\|u\|^m + |x|^k\|u\|), \tag{4.18}$$

where $c_{s,t}(u)$ is a polynomial of degree less or equal to m. So the proof of Theorem 1.9 applies to the restriction $\Phi(x,u,0)$.

This concludes the proof of the corollary. □

5. Global isomorphisms of $\mathbf{C}^p$.

In all this paragraph, we assume that Φ is a global isomorphism of $\mathbf{C}^p$, that 0 is a fixed point of Φ and that Φ is tangent to the identity at 0.

Definition. Let Φ a global isomorphism of $\mathbf{C}^p$ as above. Let V be a non degenerate characteristic direction at 0. The *attractive set to* $(0,V)$ is the set

$$\Omega_{(0,V)} := \{X \in \mathbf{C}^p,\ X \neq 0\,;\ X_n \to 0,\ [X_n] \to V \text{ in } \mathbf{P}^{p-1}_{\mathbf{C}}\}\,. \tag{5.1}$$

We will study the attractive set $\Omega_{(0,V)}$ when the matrix associated to V has some eigenvalues with positive real parts.

The isomorphism Φ is now given by entire functions

$$X_1 = \Phi(X) = X + P_2(X) + P_3(X) + \cdots.$$

Let V be a non degenerate characteristic direction of Φ at 0. We can choose coordinates x, y, $x \in \mathbf{C}$, $y \in \mathbf{C}^{p-1}$ such that $V = (1,0)$ and that Φ is is defined by

$$\begin{cases} x_1 = x + p_2(x,y) + p_3(x,y) + \dots \\ y_1 = y + q_2(x,y) + q_3(x,y) + \dots \end{cases},$$

with $p_2(1,0) = -1$ and $q_2(1,0) = 0$.

Lemma 5.1. *Let $X = (x,y) \neq 0$ be such that $X_n = (x_n, y_n)$ converges to 0 and $[X_n]$ converges to $[1,0]$. Let γ, s, ρ be arbitrary small positive constants. Then for n big enough, we have $x_n \neq 0$ and X_n is in the sector $\mathcal{S}_{\gamma,s,\rho}$, defined, for $x \neq 0$ and $u = \dfrac{y}{x}$, by*

$$\mathcal{S}_{\gamma,s,\rho} = \{(x,u) \in \mathbf{C}\times\mathbf{C}^{p-1};\, |\Im m\, x| < \gamma\, \Re e\, x,\ |x| < s,\ \|u\| < \rho\}.$$

Proof. We cannot have $x_n = 0$ an infinity of times: if it was so, since $[X_n]$ converges to $[1,0]$, we would also have $(x_n, y_n) = 0$ an infinity of times. But Φ^n being an isomorphism, this implies $X = 0$. We also know (Prop. 2.6) that $x_n \sim \dfrac{1}{n}$, so we have $|\Im m\, x_n| \leq \gamma\ \Re e\, x_n$ for n big enough. We have also $|x_n| \leq s$, $\|u_n\| \leq \rho$ by the hypothesis. $\square$

Hence, we have

$$\Omega_{(0,V)} = \bigcup_n \Phi^{-n}(\Omega_{(0,V)} \cap \mathcal{S}_{\gamma,s,\rho}). \tag{5.2}$$

and we can restrict the study of $\Omega_{(0,V)}$ to $\Omega_{(0,V)} \cap \mathcal{S}_{\gamma,s,\rho}$.

In particular, since $\mathcal{S}_{\gamma,s,\rho}$ does not intersect $x = 0$, we can make the blow-up $y = ux$ and assume that the equations of Φ in this sector are

$$\begin{cases} x_1 = x - x^2 + O(\|u\|x^2, x^3) \\ u_1 = (I_{p-1} - xM)u + O(\|u\|^2 x, x^2) \end{cases}, \tag{5.3}$$

where M is the matrix associated to V. We can make all the change of coordinates in $\mathcal{S}_{\gamma,s,\rho}$ which have been used to prove theorems 1.6 or 1.9.

We will now prove Theorems 1.10 and 1.11. We first deal with the special case where all the eigenvalues of M have strictly positive real parts. In this case, we will apply the results of Chapter 4.

Theorem 5.2. *Let Φ be a global isomorphism of $\mathbf{C}^p$. Let $V = (1,0)$ be a non degenerate characteristic direction of Φ at 0. We assume that all the eigenvalues of the matrix A associated to V have a strictly positive real part. Then the attractive set $\Omega_{(0,V)}$ of $(0,V)$ is a domain isomorphic to $\mathbf{C}^p$.*

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_h$, with $\alpha_1 > \alpha_2 > \cdots > \alpha_h > 0$, be the different values taken by the real parts of the eigenvalues of A. We can assume that A is in Jordan form, almost diagonal,

$$A = \mathrm{Diag}\{A_1, A_2, \cdots, A_h\}$$

where the eigenvalues of A_j have real parts equal to α_j for each j.

Let ε be a positive integer small enough to have

$$\alpha_1 > \alpha_1 - \varepsilon > \alpha_2 > \alpha_2 - \varepsilon > \cdots > \alpha_h > \alpha_h - \varepsilon > 0.$$

We know by Theorem 1.9 that the coordinates $u = (u_1, u_2, \cdots, u_h)$ adapted to the splitting can be chosen in such a way that for $j = 1, 2, \cdots, h$, for n big enough, we have (Corollary 4.4)

$$\|u_n^j\| \leq |x_n|^{\alpha_j - \varepsilon}. \tag{5.4}$$

We also know that on

$$\mathcal{D} = \{(x,u) \in \mathcal{S}_{\gamma,s,\rho}\,;\ \|u^j\| \leq |x|^{\alpha_j - \varepsilon},\ \text{for } j = 1, 2, \cdots, h\}, \tag{5.5}$$

the transformation is conjugate to a translation

$$\{\frac{1}{z_1} = \frac{1}{z} + 1\,,\ U_1 = U\}.$$

The change of coordinates is given by two functions $(z(x,u), U(x,u))$ such that

$$U(x,u) := x^{-A} u + O(|x|^{\eta}), \tag{5.6}$$

for some positive η, and $z(x,u) \sim x$, when $x \to 0$.

Let ψ be the map from $\mathcal{D}$ to $\mathbf{C}^p$ defined by

$$\psi(x,u) = (Z := \frac{1}{z},\ U(x,u)).$$

Let τ be the translation $\tau(Z,U) = (Z+1, U)$ in $\mathbf{C}^p$. $\mathcal{D}$ is invariant by Φ and we have

$$\tau \circ \psi = \psi \circ \Phi\,. \tag{5.7}$$

Let W be the image of $\mathcal{D}$ by ψ. For γ small enough and R big enough , the projection $Z(W)$ of W on $\mathbf{C}$ contains the piece of angle

$$\Sigma_{\gamma,R} := \{Z \in \mathbf{C};\ |\Im m\, Z| < \gamma\, \Re e\, Z\,, \quad |Z| > R\}. \tag{5.8}$$

For $Z \in \mathbf{C}$ and $r > 0$, let $P_{(Z,r)}$ be the generalized polydisc

$$P_{(Z,r)} := \{(Z,U) \in \mathbf{C}^p;\ \|U^j\| \leq r \ \text{ for } j = 1, 2, \cdots, h\,\}.$$

¿From the definition (5.5) of $\mathcal{D}$ and the form (5.6) of the function $U(x,u)$, we see that for $Z \in \Sigma_{\gamma,R}$ and R big enough, W contains the generalized polydisc $P_{(Z,\,|Z|^{\varepsilon/2})}$. When $|Z|$ goes to infinity, the fiber of W above Z contains generalized polydiscs $P_{(Z,r)}$ of radius arbitrarily large. So we have

$$\bigcup_{n}^{\infty} \tau^{-n}(W) = \mathbf{C}^p. \tag{5.9}$$

The end of the argument is classical after Fatou [F3] and [F4]:

Since $\mathcal{D} \subset \Omega_{(0,V)}$, and since for n big enough, for every X of $\Omega_{(0,V)}$, X_n is in $\mathcal{D}$ (property (5.4)) , we also have

$$\Omega_{(0,V)} = \bigcup_n \Phi^{-n}(\mathcal{D}). \tag{5.10}$$

So we can extend the isomorphism $\psi : \mathcal{D} \to W$ to

$$\tilde{\psi} : \Omega_{(0,V)} \to \mathbf{C}^p$$

in the following way: for $X \in \Omega_{(0,V)}$, let n be such that $\Phi^n(X)$ is in $\mathcal{D}$, then we define

$$\tilde{\psi}(X) := \tau^{-n} \circ \psi \circ \Phi^n(X).$$

The definition does not depend on n because of (5.7). It is immediate to see that $\tilde{\psi}$ is an injective map. It is surjective because of (5.9). $\square$

Remark 5.3. $\Omega_{(0,V)}$ is a proper open subset of $\mathbf{C}^p$ isomorphic to $\mathbf{C}^p$. That situation can only occur if $p > 1$. Such domains are called Fatou-Bieberbach domains (from Fatou [F3] and [F4] and Bieberbach [Bi]).

Proof of Theorem 1.10

Let us assume now that $M = \{\mathrm{Diag}(A),\ \mathrm{Diag}(B)\}$, where A has eigenvalues with strictly positive real parts, and B has eigenvalues with strictly negative real parts. A is a $k \times k$ matrix, $1 \leq k \leq p-2$ and B is a $h \times h$ matrix, $h := p - 1 - k \geq 1$.

Lemma 5.1 and property (5.2) still apply. Hence, we can choose local coordinates in a sector , such that after blowing up, Φ is written

$$\begin{cases} x_1 = f(x,u,v) = x - x^2 + F(x,u,v) \\ u_1 = g(x,u,v) = (I_k - xA)u + G(x,u,v) \\ v_1 = h(x,u,v) = (I_h - xB)v + H(x,u,v) \end{cases},$$

with F, G, H satisfying conditions (2.3). Using Theorem 1.6, we can assume that $H(x,u,0) = 0$.

If X is in $\Omega_{(0,V)}$, for γ, s, ρ arbitrary small positive numbers, X_n is in $\mathcal{S}_{\gamma,s,\rho}$ for n big enough. Hence, by the above equations, since B has eigenvalues with strictly negative real parts, we have $\|v_{n+1}\| \geq \|v_n\|$ for n big enough. So v_n cannot converge to 0, except if $v_n = 0$. Hence we see that

$$\Omega_{(0,V)} \cap \mathcal{S}_{\gamma,s,\rho} \subset \{v = 0\}.$$

We can then apply the same argument than in Theorem 5.2 to the restriction of Φ to $\mathcal{S}_{\gamma,s,\rho} \cap \{v = 0\}$. $\square$

When B has some eigenvalues with real part equal to 0, we have to give a more precise definition (1.7) of $\Omega_{(0,V)}$.

Proof of Theorem 1.11

The proof is the same than in the previous theorem. Since for n big enough, we have $\|x_{n+1}^{-\alpha} v_{n+1}\| \geq \|x_n^{-\alpha} v_n\|$, we cannot have $x_n^{-\alpha} v_n \to 0$ except if if $v_n = 0$. $\square$

6. Transformations tangent to the identity at order k+1.

We know come to the case where the transformation Φ tangent to the identity has the form

$$X_1 = X + P_{k+1}(X) + P_{k+2}(X) + \cdots,$$

with $P_{k+1} \neq 0, k > 1$. In this case, we say that Φ is *tangent to the identity at order* $k+1$. The situation can be compared to the case in one variable of a transformation

$$z_1 = f(z) = z + a_k z^{k+1} + \cdots,$$

with $a_k \neq 0$. Fatou studies this case in the following way: after a linear transformation, we can assume that $a_k = -\frac{1}{k}$. Hence we get

$$z_1 = z(1 - \frac{1}{k} z^k + \cdots),$$

which gives after raising to the power k,

$$z_1^k = z^k(1 - z^k + O(z^{k+1})).$$

By a polynomial transformation we can get in fact

$$z_1^k = z^k(1 - z^k + O(z^{2k})).$$

so that

$$\frac{1}{z_1^k} = \frac{1}{z^k} + 1 + O(z^k).$$

For r small enough, the domain D_r

$$D_r := \{z \in \mathbf{C}; |z^k - r| < r\}$$

is invariant and attracted to the origin. The domain D_r has k disjoint components, $\{\Pi_h\}_{1\leq h\leq k}$, called "petals". Each petal is invariant and attracted to the origin. Moreover, if z_n converges to the origin, z_n is in one Π_h for n big enough, and the restriction

$$f : \Pi_h \to \Pi_h$$

is conjugate to a translation (see [Be] p. 116 for a modern exposition).

The results stated in Chapter 1 can easily be generalized to the case of a transformation tangent to the identity at order $k+1$ in the following way:

Definitions 6.1. Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$, tangent to the identity at order $k+1$. A *characteristic direction* is a direction $V \in \mathbf{C}^p \setminus 0$ such that $P_{k+1}(V) = \lambda V$ for some constant λ. The characteristic direction V is *non degenerate* if $P_{k+1}(V) \neq 0$.

Let V be a non degenerate characteristic direction. We can assume that the first coordinate of V is not equal to 0. Let us split the coordinates (x, y) in $\mathbf{C} \times \mathbf{C}^{p-1}$ and assume that $V = (1, u_o)$. We get for Φ equations of type

$$\text{(6.1)} \qquad \begin{cases} x_1 = x + p_{k+1}(x,y) + p_{k+2}(x,y) + \cdots \\ y_1 = y + q_{k+1}(x,y) + q_{k+2}(x,y) + \cdots \end{cases},$$

with $p_{k+1}(1, u_o) \neq 0$. If we make the blow-up $y = ux$, we have as in Chapter 1 the transformation

$$\text{(6.2)} \qquad \begin{cases} x_1 = x + p_{k+1}(1,u)x^{k+1} + p_{k+2}(1,u)x^{k+2} + \cdots \\ u_1 = u + r(u)x^k + O(x^{k+1}) \end{cases},$$

with $r : \mathbf{C}^{p-1} \to \mathbf{C}^{p-1}$ the polynomial fonction of degree $k+2$ defined by

$$\text{(6.3)} \qquad r(u) := q_{k+1}(1,u) - p_{k+1}(1,u)u\,.$$

$V = (1, u_0)$ is a characteristic direction, if and only if u_0 is a zero of r and V is a non degenerate characteristic direction, if, moreover, we have $p_{k+1}(1, u_0) \neq 0$.

Let $V = (1, u_0)$ be a non degenerate characteristic direction, we can choose the coordinates so that the transformation is given by

$$\text{(6.4)} \qquad \begin{cases} x_1 = x(1 - \frac{1}{k}x^k + O(\|u\||x|^k + |x|^{2k}) \\ \\ u_1 = (I - x^k A)u + O(\|u\|^2|x|^k + |x|^{k+1}) \end{cases}.$$

Here, the matrix A associated to V is the matrix

$$A = \frac{r'(u_0)}{k p_{k+1}(1, u_0)}.$$

The class of similarity of A is an invariant and the coordinates can be chosen so that A is changed in $S^{-1}AS$, for any invertible matrix S.

By elevating the first relation of (6.3) at the power k, we get

$$\left\{\begin{array}{l} x_1^k = x^k(1 - x^k + O(\|u\||x|^k + |x|^{2k}) \\ u_1 = (I - x^k A)u + O(\|u\|^2|x|^k + |x|^{k+1}) \end{array}\right. . \tag{6.5}$$

So we can use the results of Fatou in one variable, to extend to this case the theorems stated in Chapter 1 , with almost the same proofs,. We get, in particular:

Proposition 6.2. *Let Φ be a germ of analytic transformation from $(\mathbf{C}^p, 0)$ to $(\mathbf{C}^p, 0)$, tangent to the identity at order $k+1$. If the iterates X_n of X converge to the origin in a direction V, then V is a characteristic direction.*

Proposition 6.3. *With the notations of (6.5), let $X_n = (x_n, u_n)$ such that X_n converge to the origin in the direction of the non degenerate characteristic direction V, then we have $x_n^k \sim \dfrac{1}{n}$.*

Theorem 1.6 has the following generalization:

Theorem 6.4. *Let Φ be a germ of transformation of $(\mathbf{C}^p, 0)$ tangent to the identity at order $k+1$. Let V be a non degenerate characteristic direction. Assume that the eigenvalues of the matrix A associated to V are divided into two sets $\{\lambda_j\}_{1\le j\le q}$ and $\{\mu_k\}_{1\le k\le \ell}$ in such a way that for some positive real number α, we have*

$$\begin{array}{ll} \Re e\, \lambda_j > \alpha > 0 & \text{for } j = 1, \dots, q\,, \\ \Re e\, \mu_k < \alpha & \text{for } k = 1, \dots, \ell\,. \end{array}$$

Let d_j be the multiplicity of λ_j for $j = 1, \dots, q$. Let d be defined by $d := d_1 + \cdots + d_q$. Let E be the sum of the generalized eigenvectorspaces associated to the λ_j's. Then there exist k disjoint invariant pieces of analytic manifold of dimension $d+1$, with 0 in the boundary, tangent to $\mathbf{C}V + E$ at 0 and where every point is attracted to the origin in the direction V.

In the same way we get generalizations of theorems 1.10 and 1.11. When Φ an automorphism of $\mathbf{C}^p$, tangent to the identity at order $k+1$ at the origin and with the same hypothesis than in theorems 1.10 and 1.11, The set $\Omega_{(0,V)}$ (see (1.6) or (1.7)), has k connected invariant components and each of them is isomorphic to $\mathbf{C}^{d+1}$.

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