Non-equivalence of symplectic capacities for open sets with restricted contact type boundary

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Abstract

We show in these notes that all symplectic capacities on the standard symplectic space are not equivalent, even when considered on the class of open sets with restricted contact type boundary. Namely, we construct a family of starshaped domains with displacement energy bounded from below and volume as small as we may like.

Keywords: Symplectic capacities, starshaped domains, Reinhardt domains

Résumé:

Le but de ces notes est de montrer que toutes les capacités symplectiques ne sont pas équivalentes, même si l'on considère uniquement des ouverts dont le bord est de type contact restreint. On construit pour cela une famille d'ouverts étoilés dont l'énergie de déplacement est minorée et dont le volume tend vers 0.

Mots-clef: Capacités symplectique, domaines étoilés, domaines de Reinhardt

AMS classification: 53C15 , 52A30

1 Intoduction and main result

In all this paper, we will assume that $n \geq 2$ and we will consider the standard symplectic space

$$\left(\mathbb{R}^{2n}\ ,\ \omega = \sum_{k=1}^{n} dx_k \wedge dy_k\right)\ ,$$

which will be identified with $\mathbb{C}^n$, setting $z_k = x_k + iy_k$. Let λ_0 be the standard primitive of ω, given by

$$\lambda_0 = \frac{1}{2}\sum_{k=1}^{n} x_k \wedge dy_k - y_k \wedge dx_k\ . \tag{1.1}$$

We will write $Symp_{2n}$ for the group of symplectomorphisms of $(\mathbb{R}^{2n}, \omega)$:

$$Symp_{2n} = \{\phi \in Diff(\mathbb{R}^{2n})\ /\ \phi^*\omega = \omega\}\ . \tag{1.2}$$

To each (time-dependant) Hamiltonian $H \in C^\infty(S^1 \times \mathbb{R}^{2n})$, one can associate a Hamiltonian vector field X_H on $\mathbb{R}^{2n}$ given by

$$i_{X_H}\omega = -dH(t, \cdot) \tag{1.3}$$

where $i_{X_H}\omega$ denotes the contraction of ω by X_H. We will assume that this vector field is integrable on $\mathbb{R}^{2n}$, which occurs in particular for H compactly supported. The flow ϕ_t^H is called the Hamiltonian flow of H, and is a symplectic isotopy. Time-1 maps of such flows form a subgroup of $Symp_{2n}$ called the group of compactly supported Hamiltonian diffeomorphisms:

$$Ham_{2n} = \{\phi_1^H\ /\ H \in C_0^\infty(S^1 \times \mathbb{R}^{2n})\} \subset Symp_{2n}\ . \tag{1.4}$$

1.1 Symplectic capacities

Symplectic capacities were initiated by Gromov in his seminal paper ([Gro85]). An axiomatic definition of capacities was given by Ekeland and Hofer in [EH90] as a measure of the symplectic size of a subset of $\mathbb{R}^{2n}$:

Definition 1.1 *A symplectic capacity on $(\mathbb{R}^{2n}, \omega)$ associates to each subset U of $\mathbb{R}^{2n}$ a number $c(U) \in [0, \infty]$ satisfying :*

1. *$U \subset V \Rightarrow c(U) \leq c(V)$ (monotonicity).*
2. *$c(\phi(U)) = \mid \alpha \mid c(U)$ for $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\phi^*\omega = \alpha\omega$ (homogeneity and symplectic invariance).*
3. *$c(B^{2n}(r)) = c(B^2(r) \times \mathbb{C}^{n-1}) = \pi r^2$ (non-triviality and normalization).*

The first two properties are related to the so-called symplectic embedding problem : given two open sets U and V, can one send U into V by an ambient symplectomorphism? Any symplectic capacity gives an obstruction to this problem, namely that the capacity of V should be at least equal to the capacity of U.

The motivation for the last property is to avoid trivial volume-related invariant, and to imply the celebrated non-squeezing theorem of Gromov ([Gro85]) : there is no symplectic embedding of a ball of radius R into a symplectic cylinder of radius $r < R$. We will denote by $Z(r)$ this symplectic cylinder

$$Z(r) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ / \ \mid z_1 \mid < r\} \ . \tag{1.5}$$

We will be essentially interested in bounded open sets: given a capacity defined for these sets, one can define the capacity of an open set by exhaustion, and the capacity of a set as the infimum of the capacities of its open neighbourhoods.

1.2 Examples

One can in particular define the smallest and the greatest possible capacities

Definition 1.2 *The symplectic radius of U is*

$$\underline{c}(U) = \sup\{\pi r^2 \ / \ \textit{There exists } \phi \in Symp_{2n} \textit{ with } \phi\Big(B^{2n}(r)\Big) \subset U\} \ .$$

The cylindrical capacity of U is

$$\overline{c}(U) = \inf\{\pi r^2 \ / \ \textit{There exists } \phi \in Symp_{2n} \textit{ with } \phi(U) \subset Z(r)\} \ .$$

Both are symplectic capacities, according to the non-squeezing theorem. Moreover, it follows immediately from the normalization that any capacity c satisties

$$\underline{c}(U) \leq c(U) \leq \overline{c}(U) \ . \tag{1.6}$$

In particular, if U lies "between" a ball and a cylinder of the same radii, *i.e.* if there exists $r > 0$ and and ϕ, ψ in $Symp_{2n}$ such that

$$\phi\Big(B(r)\Big) \subset U \subset \psi\Big(Z(r)\Big) \tag{1.7}$$

then $\underline{c}(U) = \overline{c}(U)$, and all symplectic capacities take the same value for U. This is the case for any ellipsoïd

$$E(r_1, \ldots, r_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ / \ \sum_{k=1}^{n} \frac{\mid z_k \mid^2}{r_k^2} < 1\} \text{ with } r_1 \leq \ldots \leq r_n$$

and any capicity must have value

$$c\Big(E(r_1, \ldots, r_n)\Big) = \pi r_1^2 \ . \tag{1.8}$$

Several symplectic capacities can be defined, using different approaches. For a nice introduction to these subjects, see for instance [Vit89], [HZ94] and [Sik]. Let us just mention one of them. Following Hofer ([Hof90]), one can define the displacement energy of U as follows:

Definition 1.3 *Let $\phi \in Ham_{2n}$. The energy of ϕ is the number*

$$e(\phi) = \inf\{\max H - \min H \ / \ H \textit{ satisfies } \phi_1^H = \phi\} \ .$$

The displacement energy of U is the number

$$d(U) = \inf\{e(\phi) \ / \ \phi \in Ham_{2n} \textit{ satisfies } \phi(U) \cap U = \emptyset\} \ .$$

This defines a symplectic capacity.

1.3 The equivalence problem

As we saw before (1.6), all symplectic capacities satisfy

$$\underline{c}(U) \le c(U) \le \overline{c}(U) \ .$$

A natural question is the problem of the equivalence of all capacities:

Problem 1.4 *Does there exist a constant $C_n > 0$ such that*

$$\overline{c}(U) \le C_n \cdot \underline{c}(U)$$

for each (bounded) open set U in $\mathbb{R}^{2n}$?

The answer to this problem is in general negative, as shows the following example. First of all, if U contains the ball $\phi(B(r))$, its volume is at least $\sigma_n r^{2n}$,where σ_{2n} denotes the volume of the unit ball in $\mathbb{R}^{2n}$. Therefore

$$\underline{c}(U) \le \pi \Big(\frac{vol(U)}{\sigma_{2n}}\Big)^{\frac{1}{n}} \ . \tag{1.9}$$

On the other hand, Chekanov proved in [Che] that the displacement energy of a lagrangian submanifold is positive, *i.e.*

Theorem 1.5 *If U is a neighbourhhod of a compact lagrangian submanifold L, then*

$$d(U) \ge \tilde{w}_D(L) > 0$$

with

$$\tilde{w}_D(L) = \sup_{J \in \mathcal{J}} \inf_{u \in Hol_D(J,L)} \int_D u^*\omega$$

where $\mathcal{J}$ denotes the space of ω-calibrated almost complex structures on $\mathbb{R}^{2n}$ and $Hol_D(J,L)$ the space of J-holomorphic disks with boundary in L.

Consequently, any tubular neigbourhood of L has "large" displacement energy, thus "large" cylindrical capacity, but it can have arbitrarily small volume, thus arbitrarily small symplectic radius. This implies the non-equivalence of symplectic capacities in general, *i.e.* we established

Theorem 1.6 *For each $\varepsilon > 0$, there exists a bounded open set $U_\varepsilon \subset \mathbb{R}^{2n}$ such that*

$$\overline{c}(U_\varepsilon) \ge 1 \ \text{ and } \ \underline{c}(U_\varepsilon) < \varepsilon \ .$$

A quite different answer holds for convex sets: Viterbo noticed in [Vit] that

Theorem 1.7 *There exists a constant γ_n such that, for any convex set C in $\mathbb{R}^{2n}$ we have*

$$\overline{c}(C) \le \gamma_n \cdot \underline{c}(C)$$

where $\gamma_n \le 2n$ for a general convex set, and $\gamma_n \le \sqrt{2n}$ if C is centrally symmetric.

Indeed, the John ellipsoïd E for C satifies

$$\gamma_n^{-1} E \subset C \subset E \text{ and } \overline{c}(E) = \underline{c}(E) \ .$$

This proves the equivalence of all symplectic capacities for convex sets in $\mathbb{R}^{2n}$.

Remark 1.8 *It has been conjectured for a long time that the constant γ_n should be equal to 1:*

Conjecture 1.9 $\underline{c}(C) = \overline{c}(C)$ *for a convex set C.*

It means that C lies (in the symplectic sense) between a ball and a cylinder of the same radii. This conjecture sounds very hard.

As convexity is not symplectically invariant, it is natural to search for a larger class of open sets for which the equivalence holds. For simplicity, we will assume that U is a regular domain, *i.e.* a bounded open set with smooth boundary ∂U, such that $0 \in U$. We will refer as a defining function for U to any function F satisfying:

- F is positive and smooth,
- 1 is a regular value of F,
- $\partial U = F^{-1}(1)$ and $U = F^{-1}([0, 1[)$.

Following Rabinowitz, we will call here "starshaped domain" any regular domain which is strictly starshaped with respect to the origin. In other words, let F be a defining function for U. The starshaped condition means

$$dF_z \cdot z > 0 \text{ for all } z \in \partial U \ . \tag{1.10}$$

It is equivalent to say that ∂U is transverse to the vector field $\eta_0(z) = \frac{1}{2}z$, which satisfies $i_{\eta_0}\omega = \lambda_0$. This gives rise to the following generalization of the starshaped condition, which is itself a generalization of strict convexity for regular domains. It was introduced by Weinstein, and it is a symplectically invariant condition.

Definition 1.10 *An hypersurface Σ is said to be of contact type if there exists a neighbourhood U of Σ and a vector field η such that*

$$L_\eta \omega = \omega \ \textit{in } U \ \textit{and } \eta \pitchfork \Sigma$$

where L denotes the Lie derivative. Σ is said to be of restricted contact type if the vector field η is globally conformal, i.e. if

$$L_\eta \omega = \omega \ \textit{on } \mathbb{R}^{2n} \ \textit{and } \eta \pitchfork \Sigma \ .$$

It follows then from the fact that η_0 is globally conformal that, given a regular domain U:

$$\begin{aligned} & U \text{ stricly convex} \\ \Rightarrow\ & U \text{ starshaped} \\ \Rightarrow\ & \partial U \text{ of restricted contact type} \\ \Rightarrow\ & \partial U \text{ of contact type} \end{aligned}$$

but all these inclusions are strict:

- For the first one, the simplest counterexample is due to Siburg (see for instance [HZ94]): using the Ekeland-Hofer capacity, one can construct a starshaped Reinhardt domain (see below) which looks like a full Bordeaux bottle and which is not symplectomorphic to a convex domain (the example of theorem 1.12 below will also provide such a counterexample, due to theorem 1.7).
- For the second one, Laudenbach showed how to attach handles to a sphere in order to obtain a restricted contact type hypersurface diffeomeorphic to a product of spheres. Its interior has restricted contact type boundary, but it is not homeomorphic to a ball, therefore not starshaped.
- For the last one, consider a lagrangian submanifold L. According to Weinstein's theorem, one can easily construct a family of tubular neighbourhoods of L with contact type boundary. If L is not exact, such a neighbourhood does not have a restricted contact type boundary.

As equivalence holds for convex sets but not for general open sets, one can reformulate problem 1.4 as follows

Problem 1.11 *For which class $\mathcal{C}$ of open sets in $\mathbb{R}^{2n}$ does there exist a constant $C_n > 0$ such that*

$$\overline{c}(U) \leq C_n \cdot \underline{c}(U)$$

for each U in $\mathcal{C}$?

The example above (theorem 1.6) shows that the equivalence does not hold for open sets with contact type boundary. Indeed, given a lagrangian submanifold L, consider a family of tubular neighbourhoods of L with contact type boundaries, whose volume goes to zero. According to theorem 1.5, their displacement energy is bounded from below, and we deduce that $\underline{c}$ is not equivalent to $\overline{c}$ in this case.

This example cannot be turned to an example with restricted contact type boundary, since L cannot be exact. However, the main result of this paper is

Theorem 1.12

For each $\varepsilon > 0$, there exists a starshaped domain $U_\varepsilon \subset \mathbb{R}^{2n}$ such that $d(U_\varepsilon) \geq \pi$ and $vol(U_\varepsilon) < \varepsilon$.

As a corollary, one can conclude that the equivalence of all symplectic capacities does not generalize to the class of starshaped domains, since

$$\overline{c}(U_\varepsilon) \geq \pi \text{ and } \lim_{\varepsilon \to 0} \underline{c}(U_\varepsilon) = 0 \ .$$

Therefore, it doesn't generalize to the class of open sets with restricted contact type boundary. This leaves open the problem of the "symplectic" generalization of convexity for which the equivalence still holds.

The domains U_ε in theorem 1.12 will be constructed as Reinhardt domains in $\mathbb{C}^n$, which will be the object of the rest of this paper.

2 Reinhardt domains

2.1 Definitions

Let us consider the n-torus:

$$\mathbb{T}_n = S^1 \times \ldots \times S^1 \text{ (} n \text{ times).}$$

The standard action of $\mathbb{T}_n$ on $\mathbb{C}^n$ is given by

$$(\theta_1, \ldots, \theta_n) \cdot (z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \ . \tag{2.1}$$

Definition 2.1 *A subset X of $\mathbb{C}^n$ is called a Reinhardt domain if X is invariant under the standard action of $\mathbb{T}_n$ on $\mathbb{C}^n$.*

We can thus parametrize such domains by their intersection with $\mathbb{R}^n_+ \subset \mathbb{C}^n$ in the following way:

Definition 2.2 *Let S be any subset of $\mathbb{R}^n$. The (quadratic) Reinhardt domain associated to S is the domain $\mathcal{R}(S)$ in $\mathbb{C}^n$ defined by*

$$\mathcal{R}(S) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ / \ (\mid z_1 \mid^2, \ldots, \mid z_n \mid^2) \in S\} \ .$$

We consider S in $\mathbb{R}^n$ (instead of $\mathbb{R}^n_+$) and we take the square of the moduli in order to get a nice sufficient condition for the regularity of $\mathcal{R}(S)$ (see lemma 2.5 below). For any $S \subset \mathbb{R}^n$, let $S^+ = S \cap \mathbb{R}^n_+$. One has of course

$$\mathcal{R}(S^+) = \mathcal{R}(S) \ . \tag{2.2}$$

With these notations, theorem 1.12 is a corollary of the following

Proposition 2.3 *There exists a family $(V_\varepsilon)_{\varepsilon>0}$ of subsets of $\mathbb{R}^n$ such that $\mathcal{R}(V_\varepsilon)$ is a starshaped domain in $\mathbb{C}^n$ satisfying*

$$d(\mathcal{R}(V_\varepsilon)) \geq \pi \ \textit{and} \ \lim_{\varepsilon \to 0} vol(\mathcal{R}(V_\varepsilon)) = 0 \ .$$

This will be proved in section 2.2 below. In striking contrast with proposition 2.3, we will also prove that convex Reinhardt domains satisfy the conjecture 1.9:

Proposition 2.4 *If U is a convex Reinhardt domain in $\mathbb{C}^n$, then*

$$\underline{c}(U) = \overline{c}(U) \ .$$

This will be proved in section 2.3 below. It can be seen as an improvement of the constant γ_n in theorem 1.7 when U possesses more symmetries. Comparing those two results, we see that from the point of view of symplectic geometry, convex Reinhardt domains are always "round", whereas starshaped Reinhardt domains can be very "tortuous".

2.2 Starshaped Reinhardt domains

We will construct here the family V_ε of proposition 2.3. The first step is to make sure to get a starshaped domain. In this aim, we will consider starshaped domains V in $\mathbb{R}^n$ (in the sense of (1.10)), and we get the following

Lemma 2.5 *Let $V \subset \mathbb{R}^n$ be a starshaped domain. Then $\mathcal{R}(V)$ is a starshaped domain.*

Proof: We have to prove both that $\mathcal{R}(V)$ is regular and starshaped. Let f be a defining function for V, and consider the following smooth function on $\mathbb{C}^n$:

$$F(z_1, \dots, z_n) = f(| z_1 |^2, \dots, | z_n |^2) \ . \tag{2.3}$$

Obviously, we get

$$\mathcal{R}(V) = F^{-1}([0,1[) \ .$$

On the other hand, we compute

$$dF_z \cdot z = 2 df_s \cdot s \text{ with } s = (| z_1 |^2, \dots, | z_n |^2) \ .$$

If $F(z) = 1$, then $f(s) = 1$, and the starshaped condition for V implies:

$$F(z) = 1 \Rightarrow dF_z \cdot z > 0 \ , \tag{2.4}$$

which implies both that F is a defining function for $\mathcal{R}(V)$ and that $\mathcal{R}(V)$ is starshaped, implying the lemma.

One can also compute the volume of $\mathcal{R}(V)$ in terms of V^+:

Lemma 2.6 *Let $V \subset \mathbb{R}^n$. Then*

$$vol(\mathcal{R}(V)) = \pi^n vol(V^+) \ .$$

Proof: We have first to remove those points, where the action of $\mathbb{T}_n$ is not free. Consider

$$V_*^+ = \{s \in V^+ \ / \ s_1 \cdot \ldots \cdot s_n \neq 0\} \ .$$

$\mathcal{R}(V_*^+)$ is then symplectomorphic to $\mathbb{T}_n \times V_*^+$, with symplectic form

$$\Omega = \frac{1}{2} \sum_{k=1}^{n} ds_k \wedge d\theta_k$$

due to the polar change of coordinates (this follows also from the fact that the action of $\mathbb{T}_n$ is hamiltonian and from the properties of the moment map). As the volume form on $\mathbb{T}_n \times V_*^+$ is given by

$$\frac{1}{n!}\Omega^n = -2^{-n} d\theta_1 \wedge \ldots \wedge d\theta_n \wedge ds_1 \wedge \ldots \wedge ds_n$$

if follows that

$$vol(\mathcal{R}(V^+)) = vol(\mathcal{R}(V_*^+)) = \pi^n vol(V_*^+) = \pi^n vol(V^+)$$

which proves the lemma.

Finally, we can estimate the displacement energy of U in terms of V:

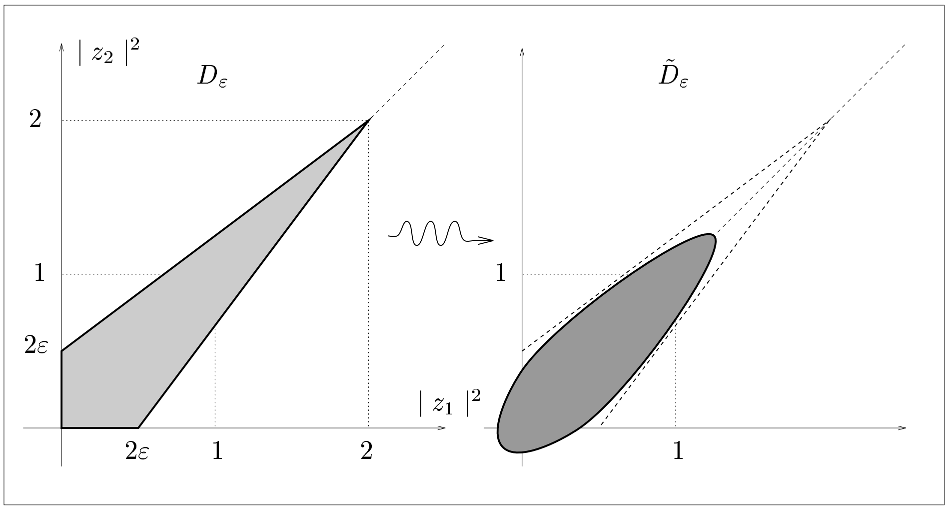

Figure 1: Starshaped domain $\tilde{D}_\varepsilon$

Lemma 2.7 *Let $V \subset \mathbb{R}^n$. If V contains the point $(1, \ldots, 1)$, then*

$$d(\mathcal{R}(V)) \geq \pi \ .$$

Proof: By definition, $\mathcal{R}(V)$ must contain the standard torus

$$T = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ / \ \mid z_1 \mid = \ldots = \mid z_n \mid = 1\}$$

which is a rational lagrangian submanifold, that is

$$[\omega]\Big(\pi_2(\mathbb{R}^{2n}, T)\Big) = \pi\mathbb{Z} \ . \tag{2.5}$$

It follows that $\tilde{w}_D(T) \geq \pi$, and this lemma is an easy corollary of theorem 1.5.

Remark 2.8 *Chekanov's proof of the positivity of $d(T)$ requires the use of Floer homology. Simpler proofs are known in this case: for instance, the one in [Pol93] shows, using holomorphic curves, that*

$$d(T) \geq \frac{1}{2}\pi$$

which is sufficient for proving the non-equivalence.

Proof of proposition 2.3: It suffices now to construct a starshaped domain V_ε in $\mathbb{R}^n$ which contains the point $(1, \ldots, 1)$ with arbitrarily small volume. For $n = 2$, we will just smooth the polygonal domain given by figure 1 as follows. Consider the polygonal domain

$$D_\varepsilon = \left\{ \begin{array}{lc} & s_1 \geq 0 \ \ , \ \ s_2 \geq 0 \\ (s_1, s_2) \in \mathbb{R}^2 \ / & s_2 - (1-\varepsilon)s_1 \leq 2\varepsilon \\ & s_1 - (1-\varepsilon)s_2 \leq 2\varepsilon \end{array} \right\}$$

The key point here is that the slope of the upper line is less than 1, whereas the slope of the lower line is greater than 1. We can thus smooth the domain D_ε

into a starshaped domain with smooth boundary, *i.e.* find such a starshaped domain $\tilde{D}_\varepsilon$ satisfying

$$(1,1) \in \tilde{D}_\varepsilon^+ \subset D_\varepsilon \ .$$

In other words, we can find a smooth homogeneous function $g_\varepsilon : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_+^*$ such that

- $g_\varepsilon(1,1) < 1$
- $g_\varepsilon(t, 2\varepsilon + (1-\varepsilon)t) \geq 1$ for $t \in [0,2]$
- $g_\varepsilon(2\varepsilon + (1-\varepsilon)t, t) \geq 1$ for $t \in [0,2]$
- 1 is a regular value for g_ε.

Consider now the following function on $\mathbb{R}^n \setminus \{0\}$:

$$f_\varepsilon(s_1, \ldots, s_n) = g_\varepsilon\Big(\frac{1}{n-1}\sum_{k=1}^{n-1} s_k, s_n\Big) \ .$$

The function f_ε is smooth, positive and homogeneous, and has 1 as a regular value. Consequently, the open set

$$V_\varepsilon = \{s \in \mathbb{R}^n \ / \ s = 0 \text{ or } f_\varepsilon(s) < 1\}$$

is a (regular) starshaped domain. It contains the point $(1, \ldots, 1)$. The last thing to check is the volume of its positive part. But any point of V_ε^+ satisfies

$$0 \leq \frac{1}{n-1}\sum_{k=1}^{n-1} s_k \leq 2$$

and

$$| \ s_n - \frac{1}{n-1}\sum_{k=1}^{n-1} s_k \ | \leq 2\varepsilon \ ,$$

and we conclude easily

$$vol(V_\varepsilon^+) \leq (2n-2)^{n-1} \cdot 4\varepsilon \ , \tag{2.6}$$

which implies

$$\lim_{\varepsilon \to 0} vol(V_\varepsilon^+) = 0 \ .$$

The domain V_ε is therefore starshaped, contains the point $(1, \ldots, 1)$ and the volume of V_ε^+ goes to 0 with ε. Consider now the Reinhardt domain $\mathcal{R}(V_\varepsilon)$: by lemma 2.5, it is starshaped. By lemma 2.7, its displacement energy is at least π, and by lemma 2.6, its volume goes to 0 with ε. Proposition 2.3 is therefore proved.

2.3 Convex Reinhardt domains

We will now focus on proposition 2.4, *i.e.* we will prove that any convex Reinhardt domain U satisifes $\underline{c}(U) = \overline{c}(U)$. We will not make any regularity hypothesis here, and we will not use definition 2.2. It suffices to deal with the case where U is bounded. Consider the sets

$$V = U \cap \mathbb{R}^n \text{ and } V_+ = U \cap \mathbb{R}^n_+ \ .$$

As U is a Reinhardt domain, we get

$$U = \{z \in \mathbb{C}^n \ / \ (\mid z_1 \mid, \dots, \mid z_n \mid) \in V\} \tag{2.7}$$

and

$$V = \{x \in \mathbb{R}^n \ / \ (\mid x_1 \mid, \dots, \mid x_n \mid) \in V_+\} \ . \tag{2.8}$$

As U is convex, V and V_+ are convex. Moreover, V is symmetric with respect to the origin. Recall the classical definitions of gauge and support function for a symmetric convex set

$$h(v) = \sup_{x \in V} v \cdot x \text{ for } v \in S^{n-1} \tag{2.9}$$

and

$$g(v) = \sup\{\lambda > 0 \ / \ \lambda \cdot v \in V\} \text{ for } v \in S^{n-1} \ . \tag{2.10}$$

Obviously, $g(v) \leq h(v)$ for all $v \in S^{n-1}$. Moreover, a very classical result is the following definition of the minimal "half-width" of a symmetric convex set

$$\Delta(V) = \inf_{v \in S^{n-1}} g(v) = \inf_{v \in S^{n-1}} h(v) \ . \tag{2.11}$$

Indeed, one knows already

$$\Delta = \inf_{v \in S^{n-1}} g(v) \leq \inf_{v \in S^{n-1}} h(v) \ .$$

Consider now a minimal point v_0 for g, and $x_0 = \Delta \cdot v_0$: by definition, V contains the ball $B^n(\Delta)$, so the support plane to V in x_0 has to be orthogonal to v_0. It follows that $h(v_0) = \Delta$, which finishes to prove (2.11).

But now, because of (2.8), V is invariant under any symmetry

$$(x_1, \dots, x_n) \mapsto (\pm x_1, \dots, \pm x_n) \ ,$$

which implies

$$h(\pm x_1, \dots, \pm x_n) = h(x) \text{ for } x \in S^{n-1} \ .$$

Let $S^{n-1}_+ = S^{n-1} \cap \mathbb{R}^n_+$. It follows

$$\Delta = \inf_{v \in S^{n-1}_+} h(v) \ .$$

Let $v_0 \in S^{n-1}_+$ be a minimal point for h: we get by definition

$$\Delta = \sup_{x \in V} v_0 \cdot x \ ,$$

which means

$$V \subset \{x \in \mathbb{R}^n_+ \ / \ v_0 \cdot x \leq \Delta\} \ . \tag{2.12}$$

On the other hand, (2.10) and (2.11) imply as already mentioned

$$B^n(\Delta) \subset V \tag{2.13}$$

Let $v_0 = (r_1, \ldots, r_n)$ be the coordinates of v_0. It follows from (2.7), (2.12) and (2.13) that

$$B^{2n}(\Delta) \subset U \subset \{z \in \mathbb{C}^n \ / \ \sum_{k=1}^{n} r_k \mid z_k \mid \leq \Delta\} \ .$$

But now, we can easily construct a matrix $\phi \in U(n)$ with first row v_0, and we get with definition (1.5)

$$U \subset \{z \in \mathbb{C}^n \ / \ \mid \sum_{k=1}^{n} r_k z_k \mid \leq \Delta\} = \phi^{-1}\Big(\overline{Z}(\Delta)\Big) \ .$$

We have thus found a symplectomorphism ϕ satisfying

$$B^{2n}(\Delta) \subset U \subset \phi^{-1}\Big(\overline{Z}(\Delta)\Big) \ ,$$

which proves proposition 2.4.

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