Dominos and the Gaussian free field.

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Abstract

We define a scaling limit of the height function on the domino tiling model (dimer model) on simply-connected regions in $\mathbb{Z}^2$ and show that it is the "massless free field", a Gaussian process with independent coefficients when expanded in the eigenbasis of the Laplacian.

Résumé

On définit une "limite d'échelle" pour la fonction d'hauteur dans le modèle des dimères dans $\mathbb{Z}^2$. Nous montrons que la limite est un "champ libre gaussien", un processus stochastique gaussien dont les coefficients, dans la base des fonctions propres du laplacien, sont indépendentes.

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1 Introduction

A **domino tiling** of a polyomino P in $\mathbb{Z}^2$ is a tiling of P with 2×1 and 1×2 rectangles. For a polyomino P let $\mu=\mu(P)$ denote the uniform measure on the set of all domino tilings of P.

Let $U\subset\mathbb{R}^2$ be a Jordan domain with smooth boundary. We study uniform random domino tilings of polyominos P_ϵ in $\epsilon\mathbb{Z}^2$ which approximate U (and using dominos which are $2\epsilon\times\epsilon$ and $\epsilon\times 2\epsilon$ rectangles).

A domino tiling of a polyomino P_ϵ in $\epsilon\mathbb{Z}^2$ can be thought of as a random map from $\epsilon\mathbb{Z}^2\cap P_\epsilon$ to $\mathbb{Z}$ in the following way. Let $V_\epsilon=\epsilon\mathbb{Z}^2\cap P_\epsilon$ be the set of lattice points in the polyomino P_ϵ. Let $h\colon V_\epsilon\to\mathbb{Z}$ be a function which has the property that around every lattice square of P_ϵ the four values of h are 4 consecutive integers h_0, h_0+1, h_0+2, h_0+3, with the values on any two adjacent boundary vertices of P_ϵ differing by 1. The set of such functions h (up to additive constants and a global sign change) is in bijection with the set of domino tilings of P_ϵ: dominos cross exactly those edges whose h-difference is 3. The function h associated to a tiling is called its **height function** [13]. See Figure 1. Note that the height function takes values in $\mathbb{Z}$, not in $\epsilon\mathbb{Z}$.

Our aim is to prove that in the limit as $\epsilon\to 0$ the height function on a random tiling of P_ϵ tends to a random (generalized) function which has a succinct description in terms of the eigenbasis of the Laplacian operator on U.

Theorem 1.1 *Let U be a Jordan domain with smooth boundary in $\mathbb{R}^2$. For each $\epsilon>0$ sufficiently small let P_ϵ be a Temperleyan polyomino approximating U as described below. Let h_ϵ be the height of a random domino tiling of P_ϵ and $\bar h_\epsilon$ be its mean value. Then as ϵ tends to 0, $h_\epsilon-\bar h_\epsilon$ tends weakly in distribution to $4/\sqrt{\pi}$ times the "massless free field" F on U, in the sense that for any smooth function ϕ on U, the random variable $\sum_{x\in V_\epsilon}\phi(x)(h_\epsilon(x)-\overline{h_\epsilon(x)})$ tends in distribution to $\frac{4}{\sqrt{\pi}}\int_U\phi F$.*

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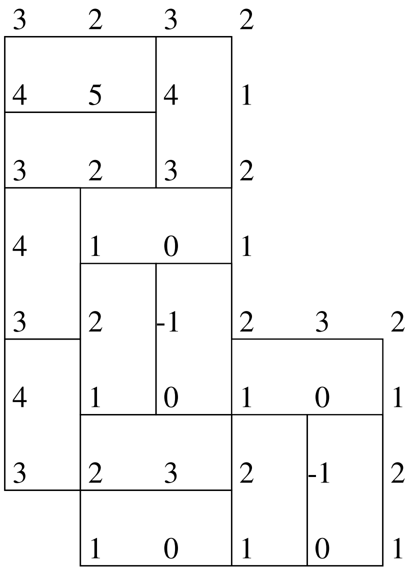

Figure 1: Height function of a domino tiling.

For the definition of Temperleyan polyominos see below. The **massless free field** F on U is a random variable taking values in the space of distributions [1] which are continuous linear functionals on the space of C^1 functions on U (with a C^1-norm). For background on the massless free field see [10]. It can be defined as follows: let $\{f_i\}_{i\geq 1}$ be an L^2-orthonormal eigenbasis for the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on U with Dirichlet boundary conditions (that is, $f_i \equiv 0$ on ∂U). Let λ_i be the eigenvalue of f_i. Then

$$F = \sum_{i\geq 1} \frac{c_i f_i}{(-\lambda_i)^{1/2}}, \tag{1}$$

where the c_i are i.i.d. Gaussian random variables of mean 0 and variance 1. Here this expression is interpreted as the generalized function F satisfying, for any C^1 function ϕ,

$$\int_U \phi F = \sum_{i\geq 1} \frac{c_i}{(-\lambda_i)^{1/2}} \int_U \phi f_i,$$

series which converges almost surely. The expression (1) does not define a function since the series diverges almost everywhere.

Remarks.

1. The above theorem describes the limiting value of $h_\epsilon - \bar{h}_\epsilon$. The limiting average value $\bar{h} = \lim \bar{h}_\epsilon$ was computed in [5]: it is a harmonic function whose boundary values are given by $\frac{2}{\pi}$ times the angle of turning of the boundary tangent counterclockwise from a fixed basepoint. (Regarding the choice of basepoint, see the definition of "Temperleyan" polyomino below.)

2. Theorem 1.1 has a well-known one-dimensional analog: Let X be the sets of random maps h from $0, \frac{1}{n}, \frac{2}{n} \dots, 1$ to $\mathbb{Z}$ satisfying $h(0) = h(1) = 0$ and $|h(\frac{i+1}{n}) - h(\frac{i}{n})| = 1$. A random element of X, when divided by $\sqrt{n}$, converges to a random function known as the "Brownian bridge" [8]. In the eigenbasis

[1] Henceforth we will refer to these objects as "generalized functions" to avoid confusion.

of the one-dimensional Laplacian $\frac{\partial^2}{\partial x^2}$ the coefficients of the Brownian bridge are again independent Gaussians. One difference between the one-dimensional case and Theorem 1.1, however, is that the height function h of Theorem 1.1 is *unnormalized*. It is therefore all the more surprising that the integer-valued function h of Theorem 1.1 converges to a continuous-valued object.

3. An important open problem is to compute the distribution of the height function on a non-simply connected domain, even an annulus. In particular for an annulus the distribution of the height difference between the two boundary components (in the limit $\epsilon \to 0$) is unknown although it was shown in [5] to depend only on the conformal modulus of the annulus.

4. Temperley [11] gave a bijection between the **uniform spanning tree process** on subgraphs of $\mathbb{Z}^2$ and domino tilings. The function h of Theorem 1.1 corresponds under this bijection to the "winding number" of the branches of a spanning tree [5], as first conjectured by I. Benjamini. As it is an open question to show that a scaling limit exists for the uniform spanning tree process [1, 9], one might hope that the reconstruction of the tree from its winding numbers, which is possible for $\epsilon > 0$, also works in the limit $\epsilon = 0$. So far this remains an open problem.

5. The result of Theorem 1.1 depends strongly on the choice of boundary conditions for the approximating polyominos P_ϵ. For even slight generalizations of these boundary conditions our methods will not work: see [5] for a discussion of this issue.

6. When the region U is a rectangle $U = [0, a] \times [0, b]$, the orthonormal eigenvectors of Δ with Dirichlet boundary conditions are $\frac{4}{ab} \sin \frac{\pi j x}{a} \sin \frac{\pi k y}{b}$, where j, k are positive integers. So in this case the massless free field has independent Fourier coefficients.

7. Most of the work to prove Theorem 1.1 was done in [5], where we proved Proposition 2.2, below.

If we consider the massless free field F to be a continuous linear functional on the space of smooth 2-forms on U (rather than on the space of smooth functions on U) then F is **conformally invariant**, in the following sense.

Proposition 1.2 *Let ω be a smooth 2-form on U and let $f: V \to U$ be a conformal bijection. Let F_U, F_V be the massless free fields on U and V respectively. Let $X = \int_U F_U(z)\omega(z)$ and $Y = \int_V F_V(z) f^*\omega(z)$, where $f^*\omega$ is the pullback of ω to V. Then the random variables X and Y are equal in distribution.*

For the proof see section 4.

2 Background and preliminaries

2.1 Temperleyan polyominos and approximation

Define the (i, j)-lattice square in $\mathbb{Z}^2$ to be the lattice square whose lower left corner is (i, j). A lattice square is said to be *even* if the coordinates of its lower left corner are even. A **polyomino** is a union of lattice squares which is bounded by a simple closed lattice curve. A polyomino is **even** if all of its *corner squares* are even, where by corner squares we mean those lattice squares adjacent to the corners and containing the interior angle bisector at the corner. In particular note that an edge of an even polyomino P' has odd length if its two extremities are both concave or both convex corners; if the extremities consist of one concave and one convex corner the edge length is even. Let P be a polyomino obtained from an even polyomino P' by removing one lattice square b adjacent to its boundary, where b is of the same parity as the corners of P'. Such a polyomino is called **Temperleyan**, and the removed square b is called its **root**. In Figure 1, the polyomino is Temperleyan with root the lower left (removed) square.

All Temperleyan polyominos have domino tilings [5, section 7]. The term *Temperleyan* comes from the bijection due to Temperley between the set of spanning trees of a rectangle in $\mathbb{Z}^2$ and the set of domino tilings of a rectangular region with a corner removed [11]. This bijection was generalized in [3] and further in [7].

Let U be a smooth Jordan domain with a marked point $b \in \partial U$. For each $\epsilon > 0$ let P_ϵ be a Temperleyan polyomino in $\epsilon\mathbb{Z}^2$ approximating U as follows. The boundary of P_ϵ lies within $O(\epsilon)$ of ∂U, and the counterclockwise boundary path of P_ϵ points locally into the same half-space as the (directed) tangent to ∂U which it is near. Furthermore the root b_ϵ of P_ϵ should be within $O(\epsilon)$ of b.

2.2 Green's functions

Let U be a Jordan domain with basepoint $b \in \partial U$. The Green's function with Dirichlet boundary conditions, or simply Dirichlet Green's function, $g_D(z_1, z_2)$, is defined to be the unique function (of z_2) satisfying $\Delta g_D(z_1, z_2) = \delta_{z_1}(z_2)$ (the Dirac delta), and which is zero when $z_2 \in \partial U$, where the Laplacian is with respect to the second variable. This function is well-defined and when z_2 is near z_1 has the form $g_D(z_1, z_2) = \frac{1}{2\pi} \log|z_2 - z_1| + O(1)$.

The Dirichlet Green's function has the following simple expression in the basis of eigenfunctions of the Laplacian on U:

Lemma 2.1

$$g_D(z_1, z_2) = \sum_{i \geq 1} \frac{f_i(z_1) f_i(z_2)}{\lambda_i}.$$

Proof. Since the eigenbasis $\{f_i\}$ of Δ is an orthonormal basis for $L^2(U)$, it suffices to show that for each i, $\langle f_i(z_2), g_D(z_1, z_2)\rangle = \frac{f_i(z_1)}{\lambda_i}$. But

$$\begin{aligned}
\langle f_i(z_2), g_D(z_1, z_2)\rangle &= \frac{1}{\lambda_i}\langle \lambda_i f_i(z_2), g_D(z_1, z_2)\rangle \\
&= \frac{1}{\lambda_i}\langle \Delta f_i(z_2), g_D(z_1, z_2)\rangle \\
&= \frac{1}{\lambda_i}\langle f_i(z_2), \Delta g_D(z_1, z_2)\rangle \\
&= \frac{1}{\lambda_i}\langle f_i(z_2), \delta_{z_1}(z_2)\rangle \\
&= \frac{1}{\lambda_i} f_i(z_1).
\end{aligned}$$

□

We will also need to define the Neumann Green's function g_N. On a bounded domain, the Green's function with Neumann boundary conditions does not exist. However we can define (see below) the function which is the "difference of two Neumann Green's functions": for points $z_1, z_1' \in U$ define the difference of Neumann Green's functions $g_N(z_1, z_2) - g_N(z_1', z_2)$ to be the function of z_2 satisfying $\Delta(g_N(z_1, z_2) - g_N(z_1', z_2)) = \delta_{z_1}(z_2) - \delta_{z_1'}(z_2)$, where the Laplacian is with respect to the second variable, and which satisfies $\frac{\partial}{\partial \hat{n}}(g_N(z_1, z_2) - g_N(z_1', z_2)) = 0$, that is, the derivative normal to the boundary with respect to the second variable is zero. This function is well-defined up to an additive constant, and we set the constant so that $g(z_1, b) - g(z_1', b) = 0$.

Near z_1 we have $g_N(z_1, z_2) - g_N(z_1', z_2) = \frac{1}{2\pi}\log|z_2 - z_1| + c_1(z_1, z_1') + O(z_2 - z_1)$. Let $\hat{g}_N(z_1, z_2) - \hat{g}_N(z_1', z_2)$ be the harmonic conjugate (with respect to the second variable) of $g_N(z_1, z_2) - g_N(z_1', z_2)$. This function is multiply-valued, increasing by 1 when z_2 turns counterclockwise around z_1 and by -1 when z_2 turns counterclockwise around z_1'. The function $\tilde{g}_N(z_1, z_2) - \tilde{g}_N(z_1', z_2) := g_N(z_1, z_2) - g_N(z_1', z_2) + i(\hat{g}_N(z_1, z_2) - \hat{g}_N(z_1', z_2))$ is analytic in z_2 (except at z_1 and z_1') and is the *analytic Neumann Green's function*. It is also multiply-valued.

We can define the exterior derivative of $\tilde{g}_N$ with respect to the first variable as $d\tilde{g}_N(z_1, z_2) = F_0(z_1, z_2)dx_1 + iF_1(x, z_2)dy_1$, where $z_1 = x_1 + iy_1$ and F_0, F_1 are defined by taking limits (for δ real)

$F_0(z_1,z_2) = \lim_{\delta\to 0} \frac{\tilde g_N(z_1+\delta,z_2)-\tilde g_N(z_1,z_2)}{\delta}$ and $iF_1(z_1,z_2) = \lim_{\delta\to 0} \frac{\tilde g_N(z_1+i\delta,z_2)-\tilde g_N(z_1,z_2)}{\delta}$. These functions F_0, F_1 are well-defined (single-valued) and vanish at $z_2 = b$.

As examples of these functions, on the upper half-plane $\mathbb{H}$ with $b = \infty$ we have

$$g_D(z_1,z_2) = \frac{1}{2\pi}\log\left|\frac{z_2-z_1}{z_2-\bar z_1}\right|, \tag{2}$$

and

$$\tilde g_N(z_1,z_2) - \tilde g_N(z_1',z_2) = \frac{1}{2\pi}\log\frac{(z_2-z_1)(z_2-\bar z_1)}{(z_2-z_1')(z_2-\overline{z_1'})}. \tag{3}$$

Note that when $z_2 \in \mathbb{R}$ the imaginary part of (3) is constant (in fact vanishes); this implies that the real part has Neumann boundary conditions. For a more general Jordan domain V, let f be a Riemann map from V to the upper half-plane sending b (the base point of V) to ∞. Then the Dirichlet Green's function on V is $g_D^V(z_1,z_2) = g_D^{\mathbb{H}}(f(z_1), f(z_2))$, and the analytic Neumann Green's function is defined similarly $\tilde g_N^V(z_1,z_2) - \tilde g_N^V(z_1',z_2) = \tilde g_N^{\mathbb{H}}(f(z_1),f(z_2)) - \tilde g_N^{\mathbb{H}}(f(z_1'),f(z_2))$. One can in fact take this to be the definition of the Green's functions on V.

2.3 Moment formula

For a region U with basepoint $b \in \partial U$, define the functions $F_+(z_1,z_2)$ and $F_-(z_1,z_2)$ by

$$-4d\tilde g_N(z_1,z_2) = F_+(z_1,z_2)dz_1 + F_-(z_1,z_2)d\overline{z_1}, \tag{4}$$

where d is exterior differentiation with respect to the first variable. Then $F_+(z_1,b) = 0 = F_-(z_1,b)$ (in terms of the functions F_0, F_1 of the previous section we have $F_\pm = -2(F_0 \pm F_1)$).

Let $h_0(x) = h(x) - \bar h(x)$.

Proposition 2.2 ([5]) *Under the hypotheses of Theorem 1.1, let $z_1,\dots,z_k$ be distinct points of U, and $\gamma_1,\dots,\gamma_k$ disjoint paths running from the boundary of U to $z_1,\dots,z_k$ respectively. Let $h(z_1)$ denote the height of a point of P_ϵ lying within $O(\epsilon)$ of z_1. Then*

$$\lim_{\epsilon\to 0}\mathbb{E}(h_0(z_1)\cdots h_0(z_k)) = \sum_{\varepsilon_1,\dots,\varepsilon_k\in\{\pm 1\}} \varepsilon_1\cdots\varepsilon_k \int_{\gamma_1}\cdots\int_{\gamma_k} \det_{i,j\in[1,k]}\Big(F_{\varepsilon_i,\varepsilon_j}(z_i,z_j)\Big) dz_1^{(\varepsilon_1)}\cdots dz_k^{(\varepsilon_k)}, \tag{5}$$

where $dz_j^{(1)} = dz_j$ and $dz_j^{(-1)} = d\overline{z_j}$, and

$$F_{\varepsilon_i,\varepsilon_j}(z_i,z_j) = \begin{cases} 0 & \text{if } i=j \\ F_+(z_i,z_j) & \text{if } (\varepsilon_i,\varepsilon_j) = (1,1) \\ F_-(z_i,z_j) & \text{if } (\varepsilon_i,\varepsilon_j) = (-1,1) \\ \overline{F_-(z_i,z_j)} & \text{if } (\varepsilon_i,\varepsilon_j) = (1,-1) \\ \overline{F_+(z_i,z_j)} & \text{if } (\varepsilon_i,\varepsilon_j) = (-1,-1). \end{cases}$$

3 Proof of Theorem 1.1

When U is the upper half plane with basepoint at ∞, the derivative of the analytic Neumann Green's function is (see (3))

$$d\tilde g_N(z_1,z_2) = \frac{dz_1}{2\pi(z_1-z_2)} + \frac{d\overline{z_1}}{2\pi(\overline{z_1}-z_2)}.$$

Thus from (4) we have $F_+(z_1,z_2) = \frac{2}{\pi(z_2-z_1)}$ and $F_-(z_1,z_2) = \frac{2}{\pi(z_2-\overline{z_1})}$.

Let $p, q \in U$. From Proposition 2.2 we have $\lim_{\epsilon \to 0} \mathbb{E}(h_0(p)h_0(q)) =$

$$= \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & F_+(z_1,z_2) \\ F_+(z_2,z_1) & 0 \end{vmatrix} dz_1 dz_2 - \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & F_-(z_1,z_2) \\ \overline{F_-(z_2,z_1)} & 0 \end{vmatrix} d\overline{z_1} dz_2 -$$

$$- \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & \overline{F_-(z_1,z_2)} \\ F_-(z_2,z_1) & 0 \end{vmatrix} dz_1 d\overline{z_2} + \int_{\gamma_1,\gamma_2} \begin{vmatrix} 0 & \overline{F_+(z_1,z_2)} \\ \overline{F_+(z_2,z_1)} & 0 \end{vmatrix} d\overline{z_1} d\overline{z_2}.$$

Plugging in for $F_\pm$ gives

$$\begin{aligned}\lim_{\epsilon \to 0} \mathbb{E}(h_0(p)h_0(q)) &= -\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{1}{(z_2-z_1)^2} dz_1 dz_2 + \frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{1}{(z_2-\overline{z_1})^2} d\overline{z_1} dz_2 + \\ &\quad + \frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{1}{(\overline{z_2}-z_1)^2} dz_1 d\overline{z_2} - \frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2} \frac{1}{(\overline{z_2}-\overline{z_1})^2} d\overline{z_1} d\overline{z_2} \\ &= \frac{8}{\pi^2} \operatorname{Re} \log\left(\frac{\overline{p}-q}{p-q}\right). \end{aligned} \tag{6}$$

Note that this is $-\frac{16}{\pi} g_D(p,q)$ where g_D is the Dirichlet Green's function on U (see (2)).

Now let $p_1, \ldots, p_k$ be distinct points in the upper half plane U. By symmetry of h_0, if k is odd the moment $\mathbb{E}(h_0(p_1)\cdots h_0(p_k))$ is zero. We therefore assume k is even. In (5), the matrix has ij entry

$$F_{\varepsilon_i,\varepsilon_j}(z_i,z_j) = \frac{2}{\pi(z_j^{(\varepsilon_j)} - z_i^{(\varepsilon_i)})}.$$

Such a matrix has a simple determinant:

Lemma 3.1 *For k even let M be the $k \times k$ matrix $M = (m_{ij})$ with $m_{ii} = 0$ and $m_{ij} = \frac{1}{x_j - x_i}$ when $i \neq j$. Then*

$$\det(M) = \sum \frac{1}{(x_{\sigma(1)} - x_{\sigma(2)})^2 (x_{\sigma(3)} - x_{\sigma(4)})^2 \cdots (x_{\sigma(k-1)} - x_{\sigma(k)})^2}, \tag{7}$$

where the sum is over all $(k-1)!!$ possible pairings $\{\{\sigma(1),\sigma(2)\}, \ldots, \{\sigma(k-1),\sigma(k)\}\}$ of $\{1,\ldots,k\}$.

This lemma also appears in [4].

Proof. The proof is by induction on k. The formula clearly holds when $k = 2$. For $k > 2$, the determinant is a rational function of x_1 with a double pole at $x_1 = x_2$; we can write

$$\det(M) = \frac{c_{-2}}{(x_1 - x_2)^2} + \frac{c_{-1}}{(x_1 - x_2)} + c_0 + O(x_1 - x_2).$$

The coefficient c_{-1} is zero since the determinant is even under the exchange of x_1 and x_2 (exchange the first two rows and exchange the first two columns). The coefficient c_{-2} is the determinant of M_{12}, the matrix obtained from M by deleting the first two rows and columns. Therefore the right and left-hand sides of (7) both represent rational functions (in each variable) with the same poles and residues; hence they differ by a constant. This constant is zero by homogeneity. $\square$

Combining the lemma with Proposition 2.2 gives the following.

Proposition 3.2 *Let U be a Jordan domain with smooth boundary. Let $p_1, \ldots, p_k \in U$ (with k even) be distinct points. We have*

$$\lim_{\epsilon \to 0} \mathbb{E}(h_0(p_1)\cdots h_0(p_k)) = \left(-\frac{16}{\pi}\right)^{k/2} \sum_{\textit{pairings } \sigma} g_D(p_{\sigma(1)}, p_{\sigma(2)}) \cdots g_D(p_{\sigma(k-1)}, p_{\sigma(k)}).$$

Proof. When U is the upper half plane this follows by combining Proposition 2.2 with Lemma 3.1 and the calculation (6). For arbitrary U, equation (6) shows that $\mathbb{E}(h_0(p_1)h_0(p_2)) = -\frac{16}{\pi}g_D^U(p_1,p_2)$ (where g_D^U is the Dirichlet Green's function on U) by conformal invariance of the height moments and of g_D. This completes the proof. $\square$

The proof of Theorem 1.1 is completed as follows. Let $f_{n_1},\ldots,f_{n_k}$ be (not necessarily distinct) eigenvectors of Δ with Dirichlet boundary conditions. Let $C_{n_j}^{(\epsilon)}$ be the real-valued random variable $C_{n_j}^{(\epsilon)} = \epsilon^2\sum_{x\in V_\epsilon} h_0(x)f_{n_j}(x)$, where the sum is over the vertices V_ϵ of P_ϵ, and $f_{n_j}(x)$ is f_{n_j} evaluated at the vertex x. We have

$$\lim_{\epsilon\to 0}\mathbb{E}(C_{n_1}^{(\epsilon)}\cdots C_{n_k}^{(\epsilon)}) =$$

$$\begin{aligned}
&= \lim_{\epsilon\to 0}\mathbb{E}\left(\sum_{x_1\in V_\epsilon}\epsilon^2 h_0(x_1)f_{n_1}(x_1)\cdots\sum_{x_k\in V_\epsilon}\epsilon^2 h_0(x_k)f_{n_k}(x_k)\right)\\
&= \lim_{\epsilon\to 0}\sum_{x_1\in V_\epsilon}\cdots\sum_{x_k\in V_\epsilon}\epsilon^2 f_{n_1}(x_1)\cdots\epsilon^2 f_{n_k}(x_k)\mathbb{E}\left(h_0(x_1)\cdots h_0(x_k)\right)\\
&= \left(-\frac{16}{\pi}\right)^{k/2}\int_U\cdots\int_U f_{n_1}(x_1)\cdots f_{n_k}(x_k)\sum_\sigma g_D(x_{\sigma(1)},x_{\sigma(2)})\cdots g_D(x_{\sigma(k-1)},x_{\sigma(k)})\\
&= \left(-\frac{16}{\pi}\right)^{k/2}\sum_\sigma\int_U\cdots\int_U f_{n_1}(x_1)\cdots f_{n_k}(x_k)\sum_{m_1,\ldots,m_{k/2}}\frac{f_{m_1}(x_{\sigma(1)})f_{m_1}(x_{\sigma(2)})}{\lambda_{m_1}}\cdots\frac{f_{m_{k/2}}(x_{\sigma(k-1)})f_{m_{k/2}}(x_{\sigma(k)})}{\lambda_{m_{k/2}}}\\
&= \left(\frac{16}{\pi}\right)^{k/2}\sum_\sigma\frac{\delta_{\sigma(1),\sigma(2)}}{(-\lambda_{\sigma(1)})}\cdots\frac{\delta_{\sigma(k-1),\sigma(k)}}{(-\lambda_{\sigma(k-1)})}.
\end{aligned}$$

By Wick's theorem [10], these are exactly the moments for a set of independent Gaussians of mean zero and variances $-\frac{16}{\pi\lambda_i}$. Now to conclude we invoke the following standard probability lemma :

Lemma 3.3 ([2]) *A sequence of (multidimensional) random variables whose moments converge to the moments of a Gaussian, converges itself to a Gaussian.*

This completes the proof.

4 Proof of Proposition 1.2

Since X and Y are Gaussians (each being the sum of Gaussians), and have mean 0, it suffices to compute their variances. But

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_U\int_U\omega(z_1)\omega(z_2)\mathbb{E}(F(z_1)F(z_2))\\
&= \int_U\int_U\omega(z_1)\omega(z_2)g_D^U(z_1,z_2)\\
&= \int_V\int_V f^*\omega(y_1)f^*\omega(y_2)g_D^U(f(y_1),f(y_2))\\
&= \int_V\int_V f^*\omega(y_1)f^*\omega(y_2)g_D^V(y_1,y_2)\\
&= \mathbb{E}(Y^2)
\end{aligned}$$

where we used the conformal invariance of the Green's function, $g_D^U(f(y_1),f(y_2)) = g_D^V(y_1,y_2)$. This completes the proof.

References

[1] M. Aizenman, A. Burchard, C. M. Newman, D. B. Wilson, Scaling limits for minimal and random spanning trees in two dimensions, *Rand. Str. Alg.*, to appear.

[2] P. Billingsley, Probability and measure. Wiley, New York 1979.

[3] R. Burton, R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, *Ann. Probab.* **21** (1993), 1329–1371.

[4] C. Itzykson, J.-M. Drouffe, Théorie statistique des champs I. Editions du C.N.R.S., Paris.

[5] R. Kenyon, Conformal invariance of domino tiling. to appear, *Ann. Probability*

[6] R. Kenyon, Long range properties of spanning trees. to appear, *J. Math. Phys.*

[7] R. Kenyon, J. Propp, D. Wilson, Trees and Matchings, to appear, *El. J. Combinatorics*

[8] P. Levy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris 1965.

[9] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees. to appear, *Isr. J. Math.*

[10] B. Simon, The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton Univ. Press, Princeton, New Jersey (1974).

[11] H. Temperley, Combinatorics: Proceedings of the British Combinatorial Conference 1973, *London Math. Soc. Lecture Notes Series #13*, (1974) 202-204.

[12] G. Tesler, Matchings in graphs on non-orientable surfaces, preprint.

[13] W. P. Thurston, Conway's tiling groups, *Amer. Math. Monthly* **97** (1990),757-773.