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R E M E R C I E M E N T S

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Je remercie Yves Meyer qui m'a proposé comme second sujet une application des opérateurs pseudo-différentiels au problème de Cauchy pour les opérateurs strictement hyperboliques ; ainsi j'ai pu découvrir une belle théorie, et m'exercer dans l'Analyse.

Enfin je remercie Rene Thom qui a eu la double tâche d'être Rapporteur et Président. Ses travaux ont toujours été une source d'inspiration pour moi et je continue à y puiser beaucoup d'idées.

INTRODUCTION

Cette thèse sur travaux comporte une version revue et corrigée de ma thèse de 1977 à l'Université de Warwick, ainsi que les publications qui contiennent des resultats qui ne sont pas contenus dans la thèse de Warwick. Il y a six parties distinctes.

1. Interprétations topologiques des conditions de Whitney, Journées Singulières de Dijon, juin 1978, Astérisque 59-60, 1979, 233-248.

Dans cet article nous donnons un aperçu historique de la théorie des stratifications et nous décrivons la plupart des resultats dans les cinq premiers chapitres de la thèse de Warwick.

2. Whitney stratifications : faults and detectors, Thèse, Université de Warwick, 1977, 93 pages.

Cette thèse a quatre chapitres. Dans Chapitre 0 nous donnons les définitions de stratification, conditions (a) et (b) de Whitney, et les théorèmes d'existence sur les ensembles semi-algébriques, semi-analytiques et sous-analytiques, ainsi que les conséquences principales des conditions de Whitney (stabilité de transversalité et trivialité topologique locale).

Le chapitre 1 est sur la condition (a) de Whitney. Nous montrons que (a) est en fait équivalente à la stabilité de transversalité à une stratification. Nous considérons la condition (t) qu'une transversale à une strate soit localement transverse aux autres strates et nous montrons que (t) équivaut (a) pour les strates sous-analytiques, mais que les deux conditions sont distinctes en général. Nous donnons une réciproque à un théorème de T. C. Kuo en montrant que s'il n'y a qu'un type topologique d'intersection avec les transversales à une strate, alors on a (t) .

Nous montrons dans §3 que (a) équivaut la condition que tout feuilletage de classe C^1 transverse à une strate soit transverse aux autres strates ; en plus les feuilletages de classe C^2 ne suffisent pas. D'une façon analogue, dans §5 (Chapitre 2) on montre que (b) équivaut l'assertion qu'un voisinage tubulaire de classe C^1 définit des sphères transverses aux strates voisines.

Dans §4 du Chapitre 1 on donne, pour la condition de Thom sur les applications stratifiées, les analogues des resultats déjà démontrés pour (a).

Dans §6 nous étudions le comportement de la condition (b) quand on coupe avec des ailes génériques (variétés lisses contenant une strate). On montre que si (b) est satisfaite après avoir coupe avec une aile générique de codimension k , alors la dimension de l'ensemble des limites de vecteurs sécants orthogonaux pour lesquels on n'a pas (b) est moins que k . Nous montrons que sous une hypothèse sur la dimension de l'ensemble des limites d'espaces tangents, la condition (b) passe aux intersections avec les ailes génériques.

La §7 est sur la différence entre (b) et les conditions de régularité proposées par J.-L. Verdier et T.-C. Kuo ; on le précise en donnant des contre-exemples algébriques et semialgébriques.

Pour conclure on donne des computations qui explicitent les entiers positifs a , b , c , d pour que la partie lisse de $\{y^a = t^b x^c + x^d\}$ satisfasse les conditions de Whitney lelong l'axe Ot , dans les deux cas de $\mathbb{R}^3$ et $\mathbb{C}^3$. Ces computations servent comme une source de contre-exemples.

Le contenu des publications citées au-dessous fait partie de la thèse de Warwick.

(i) A transversality property weaker than Whitney (a)-regularity, Bull. of the London Math. Soc. 8 , 1976, 225-228. (Voir §2)

(ii) Counterexamples in stratification theory : two discordant horns, Proceedings of the Nordic Summer School 1976, ed. P. Holm, Sijthoff & Noordhoff, 1977, 679-686. (Voir §7)

(iii) Stability of transversality to a stratification implies Whitney (a)-regularity, Inventiones Math. 50, 1979, 273-277. (Voir §1)

(iv) Geometric versions of Whitney regularity for smooth stratifications, Annales Scientifiques de l'Ecole Normale Supérieure, 1979, 45 -46 . (Voir §§3,5)

(v) (avec A. Kambouchner) Whitney (a)-faults which are hard to detect, Annales Scientifiques de l'Ecole Normale Supérieure, 1979, 46 -46 . (Voir §§2,3)

En outre je suis en train de rédiger les deux articles suivants:

(vi) Transverse transversals and homeomorphic transversals. (Ceci contient une partie de § 2.)

(vii) (avec V. Navarro Aznar) Whitney regularity and generic wings (qui contient les resultats de §6).

3. Geometric versions of Whitney regularity, Math. Proc. Cambridge Phil. Soc. 80, 1976, 99-101.

Dans cet article on montre dans le cas semi-analytique que la condition (b) équivaut la condition que chaque voisinage tubulaire d'une strate définit des sphères transverses aux autres strates. La démonstration donne un résultat plus précis que la démonstration du cas C^1 : on peut prendre des voisinages tubulaires semi-analytiques.

4. (avec H. Brodersen) Whitney (b)-regularity is weaker than Kuo's ratio test for real algebraic stratifications, Mathematica Scandinavia,45,1979,27-34.

On donne des exemples simples de (r)-défauts (b)-réguliers et une recette pour produire d'autres. On précise que la condition (w) de Verdier équivaut à dire que tout champ de vecteurs rugueux tangent à une strate se prolonge en un champ rugueux tangent aux autres strates.

5. Partial results on the topological invariance of the multiplicity of a complex hypersurface, Séminaire A'Campo-MacPherson, Paris VII, mars 1977.

Nous avons décrit dans cet exposé tous les résultats connus sur une question de Zariski : est-ce que la multiplicité d'une hypersurface complexe est un invariant du type à homéomorphisme près de l'hypersurface comme germe plongé dans $\mathbb{C}^n$?

6. <u>Multiplicity is a C^1 invariant, Orsay preprint, mars 1977.</u>

Nous montrons qu'on a une réponse affirmative à la question de Zariski quand on remplace " type à homéomorphisme près " par " type à difféomorphisme près " .

Société Mathématique de France
Astérisque 59-60 (1978) p. 233-248

INTERPRÉTATIONS TOPOLOGIQUES DES CONDITIONS DE WHITNEY

David TROTMAN

0. Introduction

L'importance des conditions de régularité locale imposées sur les stratifications, que Whitney a introduites en 1965 ([35], [36]), est bien-connue. Elles se sont montrées utiles dans le théorème de stabilité topologique de Thom et Mather ([3], [14], [23]), aussi dans les théorèmes de Lefschetz démontrés par Lê Dũng Tràng et Hamm [4], dans la construction des classes caractéristiques des variétés singulières par MacPherson, M.-H. Schwartz et Brasselet ([12], [16]), et dans la classification des singularités et des systèmes dynamiques.

Parce qu'elles sont génériques et qu'elles ont des conséquences frappantes — équimultiplicité [5] et trivialité topologique [13] — elles sont importantes dans la théorie de l'équisingularité des variétés analytiques complexes. De plus elles sont naturelles dans une telle théorie, au moins dans le cas des hypersurfaces, pour lesquelles (b) équivaut à μ^*-constant (voir les travaux de Teissier [18], [19] et de Briançon et Speder [1], [17]).

Je vais décrire ici pourquoi elles sont naturelles dans la topologie différentielle : (1) on peut les exprimer d'une manière " géométrique " sans mention de suites, ni de limites de vecteurs ou plans, et (2) la condition

(a) est <u>précisément</u> celle dont on a besoin pour que la transversalité à la stratification soit une propriété <u>stable</u> .

Finalement je parlerai de la relation de la condition (a) avec d'autres conditions qui sont équivalentes à la condition (a) dès qu'on peut utiliser le lemme de sélection des courbes (par exemple pour les stratifications semi- ou sous-analytiques) , cependant plus faibles dans le cas général, mais intéressantes parce qu'elles sont très faciles à visualiser .

1. Évolution historique des conditions d'incidence régulière.

Je veux rappeler les premières parutions des définitions et résultats concernant les conditions d'incidence régulière imposées sur les stratifications.

1957 Whitney [34] — Décomposition de toute variété algébrique réelle en un nombre fini de sous-variétés lisses. On dit qu'on a une " manifold collection " (parfois aussi " sub-manifold complex ") .

1960 Thom [20] — Stratification : une partition d'un sous-ensemble fermé de $\mathbb{R}^n$ en une réunion de sous-variétés connexes différentiables (les strates) , telles que l'adhérence de chaque strate soit la réunion de cette strate et d'un nombre fini d'autres strates (de dimensions plus petites) .

Incidence régulière : Pour toute strate Y , il existe une rétraction C^1 $\pi_Y : T_Y \longrightarrow Y$ définie sur un voisinage tubulaire T_Y de Y , telle que si

$Y \subset \partial X$, alors $\pi_Y|_{X \cap T_Y}$ est une submersion.

1964 Thom [21] Dans le cas des ensembles semi-algébriques, l'incidence régulière ci-dessus est remplacée par

(t) Pour toute sous-variété S transverse à Y en y , il existe un voisinage U de y (dans $\mathbb{R}^n$) tel que S soit transverse à X dans U.

1964 Whitney [35] Une stratification est <u>régulière</u> si pour toutes strates adjacentes X , Y , avec $Y \subset \partial X$, et pour tout $y \in Y$, les conditions suivantes sont satisfaites.

(a) Pour toute suite $\{x_i\} \in X$ convergeant vers y , telle que $\{T_{x_i}X\}$ a une limite τ , on a $T_yY \subset \tau$.

(b') Pour toute suite $\{x_i\} \in X$ convergeant vers y , telle que $\{T_{x_i}X\}$ a une limite τ , et que $\left\{\frac{x_i - \pi_Y(x_i)}{|x_i - \pi_Y(x_i)|}\right\}$ a une limite λ , avec π_Y une rétraction C^1 sur Y , on a $\lambda \subset \tau$.

Whitney remarque que (a) implique (t) , et que (a) et (b') sont préservées par les difféomorphismes de classe C^1 .

1965 Whitney [36] Toute variété analytique complexe (ou réelle) admet une stratification régulière.
Introduction de la condition suivante.

(b) Pour toutes suites $\{x_i\} \in X$, $\{y_i\} \in Y$ convergeant vers y, telles que $\{T_{x_i}X\}$ a une limite τ, et $\left\{\frac{x_i - y_i}{|x_i - y_i|}\right\}$ a une limite λ, on a $\lambda \subset \tau$.

Remarque : (b) est équivalente à la conjonction de (a) et (b').

Il est évident que (b) implique (b') pour toute π_Y. D'autre part (b) implique (a) parce que pour tout vecteur $v \in T_yY$, et toute suite $\{x_i\} \in X$, on peut choisir $\{y_i\}$ sur Y approchant y dans la direction de v assez lentement pour que $\frac{x_i - y_i}{|x_i - y_i|}$ tende vers v.

Réciproquement, si (a) est vraie, et (b') est vraie pour une π_Y donnée, on trouve (b) en décomposant le vecteur λ (dans la définition de (b)) en la somme de deux vecteurs, l'un dans T_yY, et l'autre dans $T_y(\pi_Y^{-1}(y))$.

1965 Thom [22] La condition (b) sur un couple de strates X, Y avec $Y \subset \partial X$ implique l'invariance topologique locale : près de chaque point y de Y on a un homéomorphisme entre $\overline{X}$ et $Y \times (\pi_Y^{-1}(y) \cap \overline{X})$.

Conditions géométriques.

Soit (U,ϕ) une carte C^1 pour Y en y,

$$\phi : (U, U \cap Y, y) \longrightarrow (\mathbb{R}^n, \mathbb{R}^m \times 0^{n-m}, 0).$$

Nous avons une rétraction C^1,

$$\pi_\phi = \phi^{-1} \circ \pi_m \circ \phi : U \longrightarrow (Y \cap U),$$

et une fonction tubulaire C^1,

$$\rho_\phi = \rho \circ \phi : U \longrightarrow \mathbb{R}^+,$$

où $\pi_m(x_1,\ldots,x_n) = (x_1,\ldots,x_m,0,\ldots,0)$ et $\rho(x_1,\ldots,x_n) = \sum_{i=m+1}^{n} x_i^2$.

Dans l'article [22] de Thom sont démontrées les implications suivantes (page 10).

La condition (a) pour le couple (X,Y) en y, avec $y \in Y \subset \overline{X} - X$, implique

(a_s) Pour toute carte C^1 (U,ϕ) pour Y en y, il existe un voisinage V de y, $V \subset U$, tel que $\pi_\phi|_{V \cap X}$ soit une submersion.

La condition (b) pour le couple (X,Y) en y implique

(b_s) Pour toute carte C^1 (U,ϕ) pour Y en y, il existe un voisinage V de y, $V \subset U$, tel que $(\pi_\phi, \rho_\phi)|_{V \cap X}$ soit une submersion.

1965 Feldman [2] Les applications différentiables d'une variété N à une variété M, qui sont transverses à chaque strate d'une stratification (a)-régulière d'un fermé de M, forment un ouvert dense de $C^\infty(N,M)$ dans la topologie forte (ou fine). Ceci a des corollaires intéressants en géométrie différentielle.

1965 Lojasiewicz [11] Stratification (b)-régulière des ensembles semi-analytiques.

1971 Kuo [9] Introduction de la condition (r) : " ratio test " .

(r) est strictement plus forte que la condition (b) dans le cas semi-analytique, mais n'est qu'un invariant C^2 : elle n'est pas préservée par les diffeomorphismes C^1 — voir [26] et [27] .

1973 Hironaka [6] Stratification (b)-régulière des ensembles sous-analytiques.

1974 Wall [32] Conjectures : $(a_s) \Longleftrightarrow (a)$, $(b_s) \Longleftrightarrow (b)$.

1976 Verdier [31] Introduction de la condition (w) , qui est une condition générique et implique la trivialité rugeuse locale : on a plus de contrôle sur les homéomorphismes trivialisants que avec (b) .

(w) est strictement plus forte que (r) et donc plus forte que (b) dans le cas semi-analytique (ou sous-analytique). Comme (r) , elle n'est que préservée par les difféomorphismes C^2 et pas par les difféomorphismes C^1 , même pour les strates algébriques (voir [26] et [27]).

1978 Kuo [10] Soit $Y = \bar{X} - X \subset \mathbb{R}^n$ (pas seulement $Y \subset \bar{X} - X$). La condition (a) pour (X,Y) en $y \in Y$ implique

(h^∞) Le type topologique du germe en y de l'intersection avec X d'une sous-variété S de classe C^∞ telle que $y \in S$, $S \pitchfork Y$ en y , et $\dim S = n - \dim Y$, est indépendant du choix de S .

Dans la suite je vais parler de plusieurs résultats démontrés dans ma thèse [27] : les réciproques aux implications $(a) \Longrightarrow (a_s)$ et $(b) \Rightarrow (b_s)$ de Thom (1965) , la réciproque du théorème de Feldman (1965) , et finalement une réciproque partielle au théorème de Kuo (1978) dans les cas où le lemme de sélection des courbes est utilisable.

2. Détecteurs de (a)- et (b)-défauts.

Langage : Quand une condition d'équisingularité E n'est pas satisfaite en un point d'une stratification il est naturel d'appeler ce point un

E-défaut. Très souvent, pour démontrer qu'une condition E_1 implique une condition E_2, on suppose qu'on a un E_2-défaut et on en déduit qu'on a forcément un E_1-défaut.

Les résultats suivants font partie de ma thèse [27] ; les démonstrations seront publiées dans [29] .

Théorème A : (a) équivaut à (a_s) .

Théorème B : (b) équivaut à (b_s) .

Corollaire : Les conditions (a) et (b) sont invariantes par difféomorphisme C^1 .

Comment démontrer le Théorème A :

On considère une formulation de (a_s) suggérée par Dennis Sullivan. Soient X , Y des sous-variétés C^1 de $\mathbb{R}^n$, et $y \in Y \subset \overline{X} - X$. On dit que (X,Y) est $(\mathcal{F}^k)$-régulier en y si

> $(\mathcal{F}^k)$ Pour tout feuilletage $\mathcal{F}$ de classe C^k transverse à Y en y , il existe un voisinage U de y tel que $\mathcal{F}$ est transverse à X dans U .

(a_s) équivaut à $(\mathcal{F}^1)$.

On remarque d'abord que $\pi_\phi|_{X \cap U}$ est une submersion si et seulement si les fibres de π_ϕ sont transverses à X dans U .

Donc, étant donnée $(\mathcal{F}^1)$, on trouve (a_s) parce que les fibres de la rétraction C^1 π_ϕ sont les feuilles d'un feuilletage C^1 transverse à Y et de codimension égale à la dimension de Y.

Étant donnée (a_s) on trouve $(\mathcal{F}^1)$ en prenant une rétraction dont les fibres sont contenues dans les feuilles du feuilletage $\mathcal{F}$.

<u>$(\mathcal{F}^1)$ implique (a):</u>

On suppose que (a) n'est pas satisfaite pour le couple (X,Y) en $y \in Y \subset \overline{X} - X$. On construit un feuilletage C^1 transverse à Y en y, mais qui n'est pas transverse à X en chaque point d'une sous-suite de la suite $\{x_i\}$ avec limite y. Le feuilletage sera appelé un <u>détecteur</u> du (a)-défaut. Pour le construire on part d'un feuilletage $\mathcal{F}_o$ par des hyperplans parallèles à $\lim T_{x_i}X$, et autour de chaque point d'une sous-suite $\{x_{i_k}\}$ de $\{x_i\}$ on remplace $\mathcal{F}_o$ par un feuilletage proche — on ajoute des " rides " telles que la tangente en x_{i_k}, à la feuille qui passe par x_{i_k}, contienne $T_{x_{i_k}}X$, et donc ce nouveau feuilletage n'est pas transverse à X près de y: c'est à dire que $(\mathcal{F}^1)$ n'est pas satisfaite.

Par le même genre d'argument on montre que (b_s) implique (b), cette fois en prenant un feuilletage de $\mathbb{R}^n - Y$ par des cylindres (les fibres d'une fonction tubulaire ρ_ϕ).

Pour X, Y semianalytiques on peut se restreindre à des difféomorphismes avec leurs graphes semianalytiques. (Pour voir celà il suffit de lire attentivement [24] et [25].)

Les feuilletages C^2 transverses ne sont pas des détecteurs effectifs pour les (a)-défauts : $(\mathcal{F}^2)$ n'implique pas $(\mathcal{F}^1)$. Un contre-exemple a été construit en collaboration avec Anne Kambouchner (voir [8] et [27]). Le même contre-exemple donne un (b)-défaut qui n'est pas mis en évidence par les voisinages tubulaires C^2 : la condition (b_s^2), qui est simplement la condition (b_s) limitée à des difféomorphismes ϕ de classe C^2, est satisfaite.

3. La condition (a) et la stabilité de la transversalité à une stratification.

Le théorème énoncé ci-dessous explique l'importance de la condition (a) de Whitney si on s'intéresse aux propriétés de stabilité.

Théorème : Soit Σ une stratification localement finie d'un fermé V d'une variété M de classe C^1. Les conditions suivantes sont équivalentes.

(1) Σ est (a)-régulière,

(2) pour toute variété N de classe C^1, $\{z \in J^1(N,M) : z \pitchfork \Sigma\}$ est un ouvert de $J^1(N,M)$,

(3) pour toute variété N de classe C^1, $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ est un ouvert de $C^1(N,M)$ dans la topologie C^1 forte,

(4) il existe une variété N de classe C^1, avec

$$1 \leq \dim M - \dim N \leq \max\left(1\,,\ \min_{S \in \Sigma} \dim S\right)$$

telle que $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ est un ouvert de $C^1(N,M)$ dans la topologie C^1 forte.

(1) $\Longrightarrow$ (3) a été démontré par Feldman en 1965 (voir §2 ci-dessus). (1) $\Longleftrightarrow$ (2) a été démontré par Wall [33] ; (1) $\Longrightarrow$ (3) en découle parce que (3) est une conséquence immédiate de (2) par la définition même de la topologie C^1 forte (voir [7], [15]).

L'implication (4) $\Longrightarrow$ (1) est nouvelle. Pour les détails de sa démonstration voir [27] ou [28] . La démonstration utilise d'une façon non-triviale le fait qu'un sous-ensemble de $C^k(N,M)$ ($0 \leq k \leq \infty$) qui est fermé dans la topologie C^k faible , a la propriété de Baire dans la topologie C^k forte : ce théorème est démontré par Morlet [15] et Hirsch [7] .

On a le même théorème en remplaçant partout C^1 par C^k , parce que le problème se réduit à un étude des 1-jets.

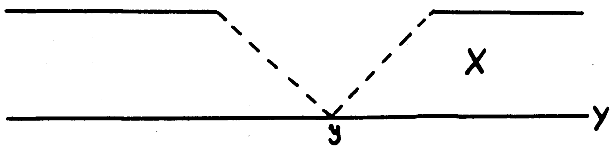
On sait que $\{f \in C^k(N,M) : f \pitchfork \Sigma\}$ est toujours dense dans $C^k(N,M)$ muni de la topologie C^k forte $(1 \leq k \leq \infty)$, par application répétée du théorème de transversalité de Thom (voir [7]). Donc les applications transverses à une stratification forment un ouvert dense si et seulement si la stratification est (a)-régulière.

Avec la topologie C^1 faible les applications transverses à une stratification, même d'un sous-ensemble compact, ne forment un ouvert que si la variété source est compacte (dans ce cas la topologie faible est la même que la topologie forte). En effet un voisinage ouvert dans la topologie faible ne donne aucun contrôle en dehors d'un compact dans la variété source. A ce sujet il faut signaler les erreurs dans chaque partie (a), (b), et (c) de l'exercice 8 , page 83 de [7] .

4. Transversales homéomorphes et transversales transverses.

Dans §2 j'ai énoncé le théorème de Kuo : (a) implique (h^∞) . En suivant la démonstration de ce résultat [10], on voit que (a) implique (h^2) (dans la définition de (h^∞) on peut prendre des sous-variétés de classe C^2), mais il n'est pas clair que (a) implique (h^1) (c'est à dire qu'on puisse prendre des sous-variétés de classe C^1) , parce que la démonstration utilise un champ de vecteurs dans un éclatement.

Il est clair que l'hypothèse $Y = \overline{X} - X$ est nécessaire à cause d'exemples comme



Je me suis posé le problème de considérer une réciproque au théorème de Kuo : est-ce que (h^1) implique (a) ? En effet une telle réciproque n'existe pas en général à cause des exemples que j'ai construit pour montrer que (t) n'implique pas (a) (voir [8] , [24] , [26] , [27]) . Dans l'exemple le plus simple ([26] , [27]) on construit un (a)-défaut en plaçant une suite de bosses sur une courbe tangente à Y , telle que les sous-variétés transverses à Y en y " ne le voient pas " .

Evidemment on obtient ainsi un (a)-défaut qui satisfait la condition (h^1) — d'avoir les <u>transversales homéomorphes</u>. (Ceci suggère que (t) , c'est d'avoir les <u>transversales transverses</u> .)

Maintenant il est naturel de se demander si peut-être (t) et (h^1) sont équivalentes. Je peux montrer que (h^1) implique (t) . Plus généralement, soit (h^k_s) la condition que les transversales à Y de classe C^k et de dimension s aient les germes en y de leurs intersections avec X tous homéomorphes, et soit (t^k_s) la condition que ces transversales soient transverses à X près de y ($1 \leq k \leq \infty$, $\operatorname{codim} Y \leq s \leq n$) .

<u>Théorème</u> : <u>Soient</u> X , Y <u>sous-variétés disjointes de</u> $\mathbb{R}^n$ <u>de classe</u> C^k , <u>et</u> $y \in Y$, $1 \leq k \leq \infty$. <u>Alors</u>

$$(h^k_s) \ \underline{\text{implique}} \ (t^k_s) \ \underline{\text{si}} \begin{cases} k = 1 \\ \text{ou} \\ k > 1 \ \text{ et } \ s > \operatorname{codim} X . \end{cases}$$

La restriction sur s quand $k > 1$ est nécessaire ; la raison est essentiellement parce qu'on ne peut pas trouver une petite C^2-perturbation d'une parabole près de son sommet qui l'applique sur sa tangente au sommet.

Les détails de la démonstration du théorème ci-dessus se trouvent dans [27] et vont paraître dans [30] .

Comme corollaire on obtient le résultat suivant.

Théorème : *Pour les strates sous-analytiques*, (h^1) *implique* (a) .

Démonstration : On applique le théorème ci-dessus et le fait que (t) implique (a) dans le cas des sous-analytiques (on le montre pour les semi-analytiques dans [24] en utilisant le lemme de sélection des courbes, qui est valable pour les semi-analytiques par [11] , et la même démonstration marche pour les sous-analytiques en citant le lemme de sélection d'Hironaka [6] ; voir [27]) .

Donc pour le cas des ensembles sous-analytiques on a les implications :

(a)	$\Longleftrightarrow$	C^1-transversales transverses	$\Longleftarrow$	C^1-transversales homéomorphes
$\Downarrow$		$\Downarrow$		
C^2-transversales homéomorphes	$\Longrightarrow$	C^2-transversales transverses		

Il n'est pas difficile à voir qu'il faut (h^1) et pas seulement (h^2) pour obtenir (a) (voir [27, Note 2.8] , ou [30]) .

Je finis avec une conjecture naturelle d'après la discussion ci-dessus.

Conjecture : $(t^k_s) \Longrightarrow (h^k_s)$ — *transversales transverses impliquent transversales homéomorphes.*

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WHITNEY STRATIFICATIONS : FAULTS AND DETECTORS

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TABLE OF CONTENTS

CHAPTER 0. INTRODUCTION

This work deals with properties of Whitney (a)- and (b)-regularity. The regularity conditions prescribe the local behaviour of limits of tangent spaces to smooth manifolds, which are usually strata of a stratification. So, first, what is a stratification ?

A <u>stratification</u> Σ of a subset V of a C^1 manifold M is a partition of V into connected C^1 submanifolds, called the <u>strata</u> of Σ . Σ is <u>locally finite</u> if each point of V has a neighbourhood meeting only finitely many strata.

<u>Example</u> 0.1. V a connected C^1 submanifold of M . There is a trivial stratification of V with just one stratum.

<u>Example</u> 0.2. V the underlying space of a linearly embedded simplicial complex. There is a natural stratification whose strata are the interiors of the simplices of the complex.

<u>Example</u> 0.3. V an analytic variety in $\mathbb{R}^n$. Let $S(V)$ be the set of points where V is not a submanifold of maximal dimension. Write $S^2(V) = S(S(V))$, etc. Suppose r is the smallest integer such that $S^{r+1}(V) = \emptyset$. Let $\mathcal{C}(A)$ denote the set of connected components of a set A . Then

$$\mathcal{C}(V-S(V)) \amalg \mathcal{C}(S(V)-S^2(V)) \amalg \dots \amalg \mathcal{C}(S^{r-1}(V)-S^r(V)) \amalg \mathcal{C}(S^r(V))$$

defines a locally finite stratification of V called the <u>full partition by dimension</u> (by Whitney in [46]).

The Whitney conditions

Let X, Y be disjoint C^1 submanifolds of a C^1 manifold M and let y be a point in $Y \cap \overline{X}$.

X is (a)-regular over Y at y if,

(a) Given a sequence of points $\{x_i\}$ in X tending to y, such that $T_{x_i}X$ tends to τ, then $T_yY \subset \tau$.

X is (b)-regular over Y at y if,

(b) Given sequences $\{x_i\}$ in X, $\{y_i\}$ in Y, both tending to y, such that $T_{x_i}X$ tends to τ, and the unit vector in the direction of $\overrightarrow{x_i y_i}$ tends to λ, then $\lambda \subset \tau$.

These conditions were first defined by Whitney in [45] and [46]. Accounts of them have been given by Thom in [35] and [36], by Mather in [21] and [22], by Wall in [43] and [44], and by Gibson and Wirthmüller in [7].

Following Thom, we say that X is (b')-regular over Y at y if, for some C^1 local retraction π associated to a C^1 tubular neighbourhood of Y near y (see §5),

(b') Given a sequence $\{x_i\}$ in X tending to y, such that $T_{x_i}X$ tends to τ and the unit vector in the direction of $\overrightarrow{x_i\pi(x_i)}$ tends to λ, then $\lambda \subset \tau$.

(b) clearly implies (b') for any π. Also (b) implies (a), since given any vector v in T_yY and any sequence $\{x_i\}$ in X we can choose $\{y_i\}$ in Y coming in to y in the direction of v so slowly that $\overrightarrow{x_i y_i}/|\overrightarrow{x_i y_i}|$ tends to v (see Mather [21]). Conversely, if (a) holds and (b') holds for some π, we arrive at (b) by decomposing the vector λ into the sum of two vectors, one in $T_y Y$ and one in $T_y(\pi^{-1}(y))$. (Compare Wall [43]) To sum up,

(0.4) $$\boxed{(b') + (a) \iff (b)}$$

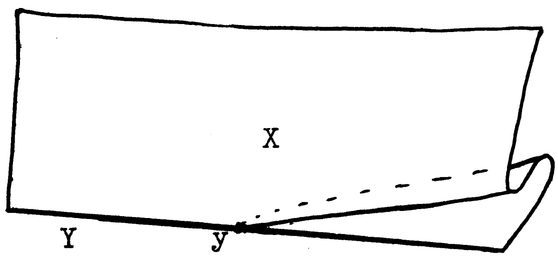
We shall make frequent use of this equivalence.

A stratification Σ is <u>(a)-regular</u> if, for each pair of strata X, Y and at every point $y \in Y \cap \overline{X}$, X is (a)-regular over Y at y. Similarly, we speak of (b)-regular stratifications. We call a locally finite (b)-regular stratification a <u>Whitney stratification</u>.

<u>Example 1</u>. (0.1) is trivially a Whitney stratification since there is only one stratum, and (a)- and (b)-regularity are conditions on a pair of strata.

<u>Example 2</u>. The stratification in (0.2) defined by a linearly embedded simplicial complex is a Whitney stratification by the next example.

<u>Example 3</u>. Let $\overline{X}$ be a C^1 submanifold-with-boundary of a C^1 manifold M, with interior X and boundary Y. Then X is (b)-regular over Y, since <u>(b)-regularity is invariant under C^1 diffeomorphism</u> (see Corollary 5.3), and $\mathbb{R}^p \times (0,\infty)^q \times 0^r$ is (b)-regular over $\mathbb{R}^p \times 0^{q+r}$ in $\mathbb{R}^{p+q+r}$. (b)-regularity is far from being a topological invariant.



Pictured is a topological manifold-with-boundary $\overline{X}$, with interior X a C^1 manifold and boundary Y a line, such that X is not (b)-regular over Y at y: we say the pair (X,Y) has a (b)-fault at y (see below).

Example 4. The stratification defined in (0.3) by the full partition by dimension of an analytic variety is not necessarily a Whitney stratification. We give the standard examples.

1) $V \equiv \{y^2 = t^2x^2 + x^3\} \subset \mathbb{R}^3$.

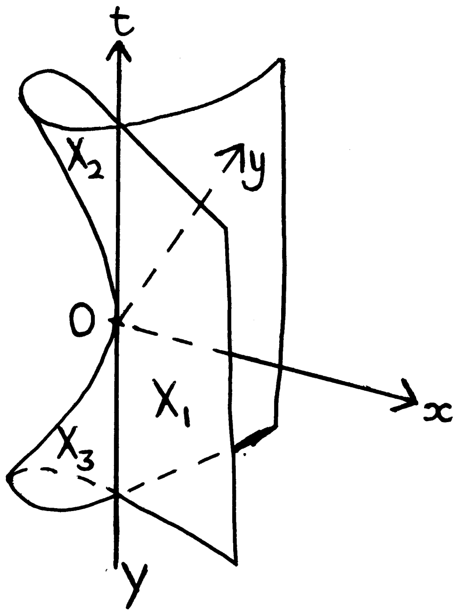
Let Y be the t-axis, and X be $V - Y$.

Then set $X_1 = X \cap \{x > 0\}$,

$X_2 = X \cap \{x < 0\} \cap \{t > 0\}$,

$X_3 = X \cap \{x < 0\} \cap \{t < 0\}$.

X_1 is (b)-regular over Y at 0, but X_2 and X_3 are not (b)-regular over Y at 0. However all three are (a)-regular over Y at 0. The reader may check that $X_1 \amalg X_2 \amalg X_3 \amalg Y$ is the full partition by dimension of V.

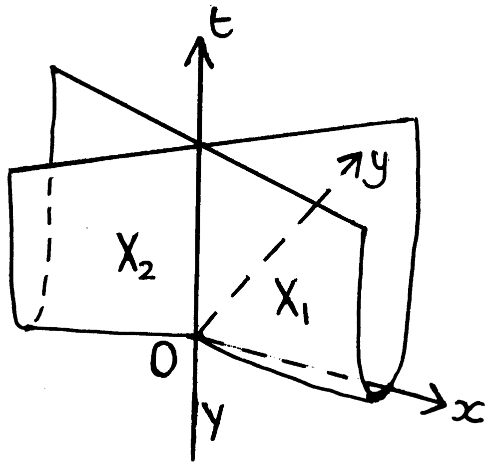


2) $V \equiv \{y^2 = tx^2\} \subset \mathbb{R}^3$.

Let Y be the t-axis, and X be $V - Y$.

Then set $X_1 = X \cap \{x > 0\}$, $X_2 = X \cap \{x < 0\}$.

X_1 and X_2 are neither (a)-regular over Y at 0, but are both (b')-regular over Y. Again $X_1 \amalg X_2 \amalg Y$ is the full partition by dimension of V.



The fact that we do not get a Whitney stratification from the full partition by dimension of an analytic variety is only a minor handicap because of the following theorem.

Theorem(Whitney [45],[46]) : Every analytic variety admits an analytic Whitney stratification.

This is proved by showing that every locally finite analytic stratification (i.e. whose strata are locally analytic manifolds) admits an analytic Whitney stratification as a refinement : this is because (b)-regularity is generic — the set of points where (b) fails for a pair (X,Y) of analytic strata is contained in the complement of an open dense subset of Y .

The class of sets for which (b)-regularity is generic has been extended by Lojasiewicz [18] and Hironaka [12]. See also Hardt [10] and Gabrielov's thesis.

Definition : A subset of $\mathbb{R}^n$ which is globally (resp. locally at each point of $\mathbb{R}^n$) a finite union of subsets each of the form $\{ f_i = 0 , g_j > 0 \mid i=1,..,p; j=1,...,q \}$ where the $\{f_i\}$, $\{g_j\}$ are polynomial (resp. analytic) functions on $\mathbb{R}^n$, is called semialgebraic (resp. semianalytic) .

Theorem (Lojasiewicz [18]) : Every semianalytic set admits an analytic stratification, and every analytic stratification of a semianalytic set admits an analytic Whitney stratification as a refinement.

A more accessible proof, for semialgebraic sets, was given by Wall [43].

Definition : A subanalytic set in $\mathbb{R}^n$ is the image of a semianalytic set in $\mathbb{R}^m$, some m , by a proper analytic map $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

Theorem (Hironaka [12]) : Every subanalytic set admits an analytic stratification, and every analytic stratification of a subanalytic set admits an analytic Whitney stratification as a refinement.

So far we have discussed the existence of Whitney stratifications. Among the most important applications of Whitney regularity are the consequences of the following results.

Theorem A : Let Σ be a locally finite stratification of a closed subset of a C^1 manifold M . Σ is (a)-regular $\iff$ the set of maps transverse to Σ is open in $C^1(N,M)$ for all C^1 manifolds N .

See §1 for a precise statement and proof of Theorem A .

Theorem B : A Whitney stratification is locally topologically trivial.

Theorem B was conjectured by Thom and proved by Mather [21] .

Neither Theorem A nor Theorem B makes use of analyticity. However in most of the work done either on the Whitney conditions themselves — as in Speder's thesis [29], and Teissier's study of the equisingularity of hypersurfaces [30] , [31], and the equimultiplicity theorem of Hironaka [11] — or using the Whitney conditions as tools — as in the proof of the topological stability theorem [7] , and the Lefschetz hyperplane theorems of Hamm and Lê [9] , and the extensions of characteristic class theory to singular varieties by MacPherson [19 , 20], and M.-H. Schwartz [26] — extensive use of the special properties of analytic varieties has been made. And it was for complex analytic hypersurfaces that Zariski demanded a theory of equisingularity [49,50] .

This thesis can be thought of as a study of aspects of the theory of equisingularity of smooth stratified sets, the plans of which were drawn in Thom's "Ensembles et morphismes stratifiés" [36] . When there are improvements in the case of subanalytic sets we give them; and we make special mention of any relations with complex hypersurfaces.

With Theorem B in mind, we make all our counterexamples topological manifolds-with-boundary, hence topologically trivial, whenever possible. This shows well the great difference in the nature of the results found here, and those obtained for complex hypersurfaces, for which topological triviality has fairly strong consequences, including (a)-regularity.

The basic local situation is as follows : let X and Y be C^1 submanifolds (and, when appropriate, subanalytic subsets) of $\mathbb{R}^n$, with $Y \subset \overline{X} - X$. Y is the base stratum, and X the attaching stratum. When X is (b)-regular over Y at 0 in Y, we will say that the pair (X,Y) is (b)-regular at 0, or that $(X,Y)_0$ is (b)-regular. When $(X,Y)_0$ is not (b)-regular, we say that $(X,Y)_0$ is a (b)-fault : we justify this term below.

Faults and detectors :

When some equisingularity condition E is not satisfied at a point of a stratification, it is natural to call the point an E-fault (so retaining the geological terminology). Many proofs showing that one equisingularity condition implies another are by reductio ad absurdum : we suppose that the second condition fails, and then we show that the first condition necessarily fails as well. When we can do this we say we have detected the fault (the point where the second condition fails). In the same way counterexamples to implications between equisingularity conditions tend to be faults which are not detectable in some given way. Most of the results given in this thesis consist of taking an equisingularity condition E and deciding whether possible detectors are effective or ineffective in detecting every E-fault. We hope that this will clarify and motivate the point of view taken throughout.

CHAPTER 1. WHITNEY (a)-REGULARITY

We begin by showing that (a)-regularity is precisely the condition to impose on a stratification in order that the maps transverse to the stratification form an open set, i.e. that transversality be stable, as well as being generic (the transverse maps always form a dense set). (a)-regularity was introduced by Whitney in [45] as a sufficient condition for this to be true ; at the time it was thought that (t)-regularity (defined in § 2) was the condition required, and that (a) was only useful in that it implied (t) (see the introduction to [45]). This is true in the analytic case, since then (t) and (a) are equivalent as proved in Theorem 2.5 below (and [37]), but we give examples (2.1 and 2.4) showing that (t) is in general weaker than (a) . (a) is necessary and sufficient for openness : the sufficiency was proved in detail by E. A. Feldman in [5] and we prove necessity here in Theorem 1.1. The only difficulty in the proof is to find a transverse map with a given transverse 1-jet at a given point : for this we show that in a suitably chosen Baire subspace of the space of maps containing the given jet at the given point, transverse maps are dense.

Example 2.1 , showing (t) to be weaker than (a) in the smooth case, has (a) failing for a sequence on a curve (in the ambient space) tangent to the base stratum, thus defining an (a)-fault not detectable by transverse submanifolds. To show that the property that the (a)-fault be given by sequences tangent to the base stratum does not characterise those (a)-faults which are not detectable by transverse submanifolds, we give a second example (2.4) which uses a basic semialgebraic object called a "barrow", which is defined in 2.3 . We then prove, in Theorem 2.5 , that (t) is equivalent to (a) when curve selection is available, and obtain as a consequence in this case the conjecture of C. T. C. Wall [43] that (a)-regularity be equivalent to the condition that the fibres of a

C^1 retraction onto the base stratum be transverse to the attaching stratum for all retractions. We prove this conjecture in general as Theorem 3.3 after rephrasing the conjecture to read "do transverse C^1 foliations detect (a)-faults ? " Example 3.6 shows, using the barrows of 2.3 , that transverse C^2 foliations do not detect all (a)-faults.

To complete §2 we discuss results relating to a theorem of T.-C. Kuo , that (a)-regularity implies that transversals to the base stratum have germs at 0 of their intersection with the attaching stratum, of a single topological type, and we prove a partial converse to Kuo's theorem.

Finally in §4 we describe the analogues of the results proved here about (a)-regularity of stratified sets for the (a_f) condition on stratified morphisms.

1. (a)-regularity and stability of transverse maps

C^k topologies

First we briefly define the weak and strong C^k topologies on the space of C^k mappings between two C^k manifolds $(1 \leq k \leq \infty)$.

A thorough treatment of these topologies is given in Hirsch's book "Differential Topology" [13] . Other versions are given by Morlet [24] , Feldman [5] , and Golubitsky and Guillemin [8] .

Let N , P be C^k manifolds. $C^k(N,P)$ denotes the set of C^k mappings from N to P , $J^k(N,P)$ denotes the bundle of k-jets associated to such mappings, and $j^k : C^k(N,P) \longrightarrow C^o(N,J^k(N,P))$ is the associated jet map. The map $j^k f : N \longrightarrow J^k(N,P)$ is called the k-jet prolongation of f .

A basis for the weak C^k topology on $C^k(N,P)$ is given by taking all sets of the form $\{f \in C^k(N,P) : j^k f(K) \subset U\}$ where K is a compact subset of N , and U is an open subset of $J^k(N,P)$.

A basis for the <u>strong C^k topology</u> (also known as the Whitney C^k topology) on $C^k(N,P)$ is given by taking all sets of the form $\{f \in C^k(N,P) : j^k f(N) \subset U\}$ where U is an open subset of $J^k(N,P)$.

If N is compact these topologies are clearly the same.

<u>Transversality</u>

We shall use the notation $\pitchfork$ for "is transverse to" .

If X , Y are C^1 submanifolds of a C^1 manifold M ,

$$X \pitchfork Y \text{ at } m \iff T_m X + T_m Y = T_m M$$

$$X \pitchfork Y \iff X \pitchfork Y \text{ at } m , \ \forall\, m \in X \cap Y$$

If $f : N \longrightarrow M$ is a C^1 map,

$$f \pitchfork X \text{ at } n \iff T_{f(n)}X + (df)_n(T_n N) = T_{f(n)}M \quad \text{or} \quad f(n) \notin X$$

$$f \pitchfork X \iff f \pitchfork X \text{ at } n , \ \forall\, n \in f^{-1}(X)$$

If $z \in J^1(N,M)$ is a 1-jet, and $f \in C^1(N,M)$ is a map representing z (at $n \in N$)

$$z \pitchfork X \iff f \pitchfork X \text{ at } n$$

We say X is transverse to a stratification Σ , and write $X \pitchfork \Sigma$, when $X \pitchfork S$ $\forall$ strata S of Σ .

We say X is transverse to a foliation $\mathfrak{F}$ of M at x , and write $X \pitchfork \mathfrak{F}$ at x, when X is transverse at x to the leaf of $\mathfrak{F}$ through x .

We say a foliation $\mathfrak{F}$ of a submanifold X is transverse at x to a foliation $\mathfrak{G}$ of a submanifold Y , and write $\mathfrak{F} \pitchfork \mathfrak{G}$ at x , when the leaf of $\mathfrak{F}$ through x is transverse at x to the leaf of $\mathfrak{G}$ through x . (This requires that X be transverse to Y at x .)

Now we are in a position to state Theorem A of the introduction.

Theorem 1.1 Let Σ be a locally finite stratification of a closed subset V of a C^1 manifold M. Then the following conditions are equivalent :

(1) Σ is (a)-regular,

(2) for every C^1 manifold N, $\{z \in J^1(N,M) : z \pitchfork \Sigma\}$ is open in $J^1(N,M)$,

(3) for every C^1 manifold N, $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ is open in $C^1(N,M)$ with the strong C^1 topology,

(4) there is some integer r, $1 \leq r \leq \max(1, \min_{S \in \Sigma}(\dim S))$, and some C^1 manifold N with $\dim N = \dim M - r$, for which $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ is open in $C^1(N,M)$ with the strong C^1 topology.

Notes 1.2 (i) (1) $\Longleftrightarrow$ (2) is proved by Wall [44]. In fact he asserts that (2) implies that V is closed, which is not quite true. Consider the case where $V = M -$ pt., and Σ has a single stratum.

(ii) (1) $\Longrightarrow$ (3) is implicit in Thom [34] (1964) and explicit in [35,36], but see the discussion in §2 below. It was proved by Feldman [5], who describes Σ as cohesive if Σ is (a)-regular, and now appears as Exercise 15 at the end of Chapter 3 of Hirsch's "Differential Topology" [13]. Feldman's proof went unnoticed by several specialists in the theory to the extent that a very short false proof of (1) $\Longrightarrow$ (3) appeared several times (see the discussion and counterexample in §2), and in 1975, D. W. Bass [1] wrote "there seems to be no published proof of this". This was probably due to Feldman's use of the term "cohesive" before "(a)-regular" came into common usage ; also his proof appeared as a technical lemma in a paper on immersion theory rather than in a paper on stratification theory. Observe also that before the term "stratification" was accepted people talked of "submanifold complex" and "manifold collection".

(iii) We have the same theorem replacing C^1 everywhere by C^k $(1 \leq k \leq \infty)$, as the problem reduces to a study of 1-jets.

(iv) The set of C^k maps transverse to Σ $(1 \leq k \leq \infty)$ is dense in $C^k(N,M)$ with the strong C^k topology by applying Thom's transversality theorem countably often as in [8] or [13], even without applying (a)-regularity. Thus if Σ is (a)-regular, the maps transverse to Σ in $C^k(N,M)$ form an open dense set in the strong C^k topology (C^1-open implies C^k-open).

(v) If each stratum is closed, then it follows from the result that for a closed submanifold W of M, $\{f \in C^k(N,M) : f \pitchfork W\}$ is open (see [8] or [13]), that $\{f \in C^k(N,M) : f \pitchfork \Sigma\}$ is open. But we do not assume the strata are closed (only that $V = |\Sigma|$ is closed) and in almost every situation of interest they will not be closed.

<u>Proof of Theorem 1.1</u> : (2) implies (3) by definition of the strong topology. That (3) implies (4) is immediate. We shall prove that (1) implies (2), and that (4) implies (1), which will establish the equivalences.

<u>(1) implies (2)</u> :

Suppose (2) is not satisfied for some C^1 manifold N. Then there is a 1-jet $z \in J^1(N,M)$, with $z \pitchfork \Sigma$ and a sequence $\{z_n\} \in J^1(N,M)$ such that z_n tends to z as n tends to ∞, but for all n, z_n is not transverse to Σ. Let ν, μ denote the maps $J^1(N,M) \longrightarrow N$, $J^1(N,M) \longrightarrow M$, taking source and target respectively. Let $x = \nu(z)$, $x_n = \nu(z_n)$, $y = \mu(z)$, $y_n = \mu(z_n)$. Since $z \pitchfork \Sigma$ and $z_n \not\pitchfork \Sigma$, for all sufficiently large n we have that $y_n \neq y$. Also clearly $y_n \in V$ for all n. Since V is closed, and since $y_n \longrightarrow y$ ($n \longrightarrow \infty$ we have that $y \in V$. Let S be the stratum of Σ containing y. Since Σ is locally finite, we can suppose (by taking a subsequence) that for all n, y_n belongs to the same stratum S'. $S' \neq S$ since S is a C^1 submanifold. Thus $y \in S \cap (\overline{S'} - S')$ and S' is (a)-regular over S by the hypothesis (1).

Now by means of a chart for M at y we can identify all the tangent spaces (and their subspaces) to M at points near y, with $\mathbb{R}^m$ (and its subspaces), where $m = \dim M$.

Let P_n (resp. P) denote the vector subspace of $\mathbb{R}^m$ determined by the jet z_n (resp. z) $\forall$ n. By choosing a further subsequence we can suppose that the dimension of P_n is constant for all n. (It is possible however that the dimension of P is less than that of P_n.) Because grassmannians are compact we may suppose by taking more subsequences that $\{P_n\}$ tends to a limit P_∞ and $\{T_{y_n} S'\}$ tends to a limit τ. Then $P \subseteq P_\infty$, and, since S' is (a)-regular over S, $T_y S \subseteq \tau$.

$z \pitchfork \Sigma$ means that $P \pitchfork T_y S$, and so $P_\infty \pitchfork \tau$. Then $\exists\, \varepsilon > 0$ such that if $d(P_\infty, Q) < \varepsilon$ $(Q \in G^m_{\dim P_\infty})$, and $d(\tau, T) < \varepsilon$ $(T \in G^m_{\dim S'})$, then $Q \pitchfork T$ (transversality is an open condition on vector subspaces). Now choose n_1 such that $\forall\, n \geq n_1$, $d(P_\infty, P_n) < \varepsilon$, and n_2 such that $\forall\, n \geq n_2$, $d(\tau, T_{y_n} S') < \varepsilon$. Then $\forall\, n \geq \max(n_1, n_2)$, $P_n \pitchfork T_{y_n} S'$, i.e. $z_n \pitchfork \Sigma$, contradicting the choice of $\{z_n\}$, and proving that (1) implies (2).

<u>(4) implies (1)</u> :

Suppose that Σ is not (a)-regular. Then there is a point y in V contained in a stratum Y of Σ $(\dim Y \geq 1)$, and a sequence of points $\{x_i\}$ of V in a stratum X of Σ such that $x_i \to y$ as $i \to \infty$, and $T_{x_i} X \to \tau$ as $i \to \infty$, and there is a vector $v \in T_y Y$ such that $v \notin \tau$. Let E be the 1-dimensional subspace of $T_y M$ spanned by v. Choose a basis for $T_y M$ such that

$$T_y Y = E \oplus W_1 \oplus T_1$$

$$\tau = T_1 \oplus T_2$$

$$T_y M = E \oplus W_1 \oplus W_2 \oplus T_1 \oplus T_2$$

where T_1, T_2, W_1, W_2 are vector subspaces of $T_y M$ and T_1, W_1, W_2 are perhaps empty. Then find a subspace H of $T_y M$ with $\dim H = m - r$ ($= \dim N$), such that $T_2 \oplus W_2 \subseteq H \subseteq T_1 \oplus T_2 \oplus W_1 \oplus W_2$ (this is possible since $1 \leq r \leq \dim Y$). Then $H + T_y Y = T_y M$, but $H + \tau \neq T_y M$. Let $p \in N$, and define

$$\mathfrak{D}_H = \{ f \in C^1(N,M) : f(p) = y \ , \ (df)_p(T_p N) = H \}.$$

<u>**Lemma 1.3**</u>: $\exists\, g \in \mathfrak{D}_H$ <u>such that</u> $g \pitchfork \Sigma$.

Choose a chart (W,ψ) for N at p such that $g|_W$ is an embedding (if $g \in \mathfrak{D}_H$, $(dg)_p$ has maximal rank), and choose a chart (U,ϕ) for M at y such that $g(W) \subset U$. Then it is not hard, since we have reduced the problem to one for $C^1(\mathbb{R}^{m-r}, \mathbb{R}^m)$, to construct, for each i such that $x_i \in U$, an f_i in $C^1(N,M)$ such that,

(i) $f_i|_{N-W} = g|_{N-W}$,

(ii) $f_i|_W$ is an embedding,

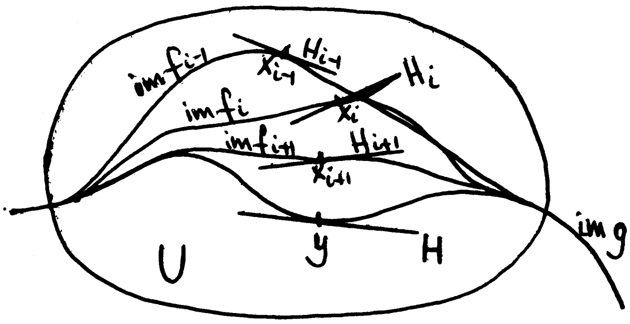
(iii) $f_i(W) \subset U$, $f_i(p) = x_i$,

(iv) $(df_i)_p(T_pN) = H_i \subseteq T_{x_i}X \oplus W_1 \oplus W_2$,

for i sufficiently large, where we have considered W_1, W_2 as subspaces of $T_{x_i}M$.

(v) $H_i \to H$ $(i \to \infty)$,

(vi) $f_i \to g$ $(i \to \infty)$ in the strong C^1 topology.



Then for each sufficiently large i, f_i is not transverse to X at x_i, since $E \not\subset H_i + T_{x_i}X$, i.e. f_i is not transverse to Σ. But by the lemma, $\lim f_i = g$ is transverse to Σ, thus we have a contradiction to the hypothesis of (4) that the set of maps transverse to Σ is open in $C^1(N,M)$, completing the proof that (4) implies (1).

<u>Proof of lemma 1.3</u> : Choose charts (U,ϕ) for M at y, (W,ψ) for N at p, and a C^1 map $h : N \to M$ such that $h(W) \subset U$, $h|_W$ is an embedding, $h(p) = y$, and $(dh)_p(T_pN) = H$. Let $W' \subset W$ be an open set containing p, with compact closure $\overline{W'} \subset W$. Then $\exists\ \delta > 0$ such that if $f \in \mathcal{V}_{\delta,\overline{W'}}(h)$, which is $\{f \in C^1(N,M) : |j^1f(x) - j^1h(x)| < \delta\ \forall x \in \overline{W'}\}$, then $f|_{W'}$ is an embedding (see [13], Chapter 2, Lemma 1.3). Let $\overline{\mathcal{V}_{\delta,\overline{W'}}(h)}$ denote the weak C^1 closure of the weakly open set $\mathcal{V}_{\delta,\overline{W'}}(h)$, and let $\mathcal{E}_H = \mathfrak{D}_H \cap \overline{\mathcal{V}_{\delta/2,\overline{W'}}(h)}$. Then $\mathcal{E}_H$ is weakly C^1 closed in $C^1(N,M)$. For, consider any limit point f_0 of a convergent sequence in $\mathfrak{D}_H$ with the weak C^1 topology. Clearly $f_0(p) = y$

and $(df_0)_p(T_pN) \subseteq H$; however the inclusion can be strict : the rank of f can drop at p . But if $f_0 \in \overline{\mathcal{V}_{\delta/2,\overline{W}'}(h)} \subset \mathcal{V}_{\delta,\overline{W}'}(h)$, f_0 has maximal rank at p since $f_0|_{W'}$ is an embedding by choice of δ . Thus $(df_0)_p(T_pN) = H$, and $f_0 \in \mathcal{D}_H$. Hence $\mathcal{E}_H$ is weakly C^1 closed. Now we quote

<u>Theorem 1.4</u> : <u>Any weakly C^k closed subspace of $C^k(N,M)$ is a Baire space in the strong C^k topology</u> $(1 \leq k \leq \infty)$.

<u>Proof</u>. See [13] , Chapter 2, Theorem 4.4 , or [24] .

Using this result we can now apply the usual procedure of the Thom transversality theorem (as in [8] , or [13]) to prove that $\{f \in \mathcal{E}_H : f \pitchfork \Sigma\}$ is strongly dense in $\mathcal{E}_H$. Cover each stratum S of Σ by countably many compact coordinate discs $\{K^S_\alpha\}^{S\in\Sigma}_{\alpha\in A}$ such that if $y \in K^Y_{\alpha(y)}$ then no other K^Y_α contains y , and if $f \in \mathcal{E}_H$, then $f(\overline{W}') \cap K^Y_{\alpha(y)} = y$. Now verify that for each S and each α , $\{f \in \mathcal{E}_H : f \pitchfork S \text{ on } K^S_\alpha\}$ is open and dense in $\mathcal{E}_H$ with the strong C^1 topology. The proof of this is a local argument near K^S_α and goes through as for the standard proof in $C^1(N,M)$ by the choice of $K^Y_{\alpha(y)}$. (Given $f \in \mathcal{E}_H$, f not transverse to Y on $K^Y_{\alpha(y)}$, we can find an arbitrarily small perturbation of f to a map $g \in \mathcal{E}_H$ which is transverse to Y on $K^Y_{\alpha(y)}$, and such that $g|_{\overline{W}'} = f|_{\overline{W}'}$.) Because there are countably many strata (Σ being assumed locally finite) , and because $\mathcal{E}_H$ is a Baire space in the strong C^1 topology (Theorem 1.4) , we deduce that

$$\{f \in \mathcal{E}_H : f \pitchfork S \text{ on } K^S_\alpha , \forall \alpha , \forall S\} = \{f \in \mathcal{E}_H : f \pitchfork \Sigma\}$$

is strongly dense in $\mathcal{E}_H$. Since $\mathcal{E}_H \neq \emptyset$, as $h \in \mathcal{E}_H$, we have shown the existence of some g in $\mathcal{E}_H$, and hence in $\mathcal{D}_H$, with $g \pitchfork \Sigma$. This completes the proof of Lemma 1.3 .

<u>Notes on the proof</u> : 1. It is not clear if $\mathcal{D}_H$ is a Baire space . This is the reason for introducing $\mathcal{E}_H$ in the proof of Lemma 1.3 . Certainly $\mathcal{D}_H$ is

not weakly closed, since the rank at p of a limit map may be less than the rank of the maps of a sequence in $\mathcal{D}_H$, convergent in $C^1(N,M)$.

2. The proof of (4) implies (1) shows that if there is a C^1 manifold N with $\{f \in C^1(N,M) : f \pitchfork \Sigma\}$ open, then Σ is (a)-regular over the strata of dimension $\geq \dim M - \dim N$.

2. <u>(a)-regularity and transverse submanifolds</u>

Consider the following condition on a pair of adjacent strata (X,Y) at a point $0 \in Y \cap (\overline{X} - X)$, with X, Y C^1 submanifolds of $\mathbb{R}^n$.

(t) **Given a C^1 submanifold S of $\mathbb{R}^n$ transverse to Y at 0, there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that S is transverse to X in U.**

If (t) is satisfied for $(X,Y)_0$ we say X <u>is (t)-regular over</u> Y <u>at</u> 0. If X is (t)-regular over Y for each point in $Y \cap (\overline{X} - X)$ we say X is (t)-regular over Y. If each pair of adjacent strata of a stratification are (t)-regular, then Σ is a <u>(t)-regular stratification</u>.

Since spanning is an open condition, it follows at once that (a) implies (t) The false argument referred to above to prove (1) implies (3) of Theorem 1.1 is

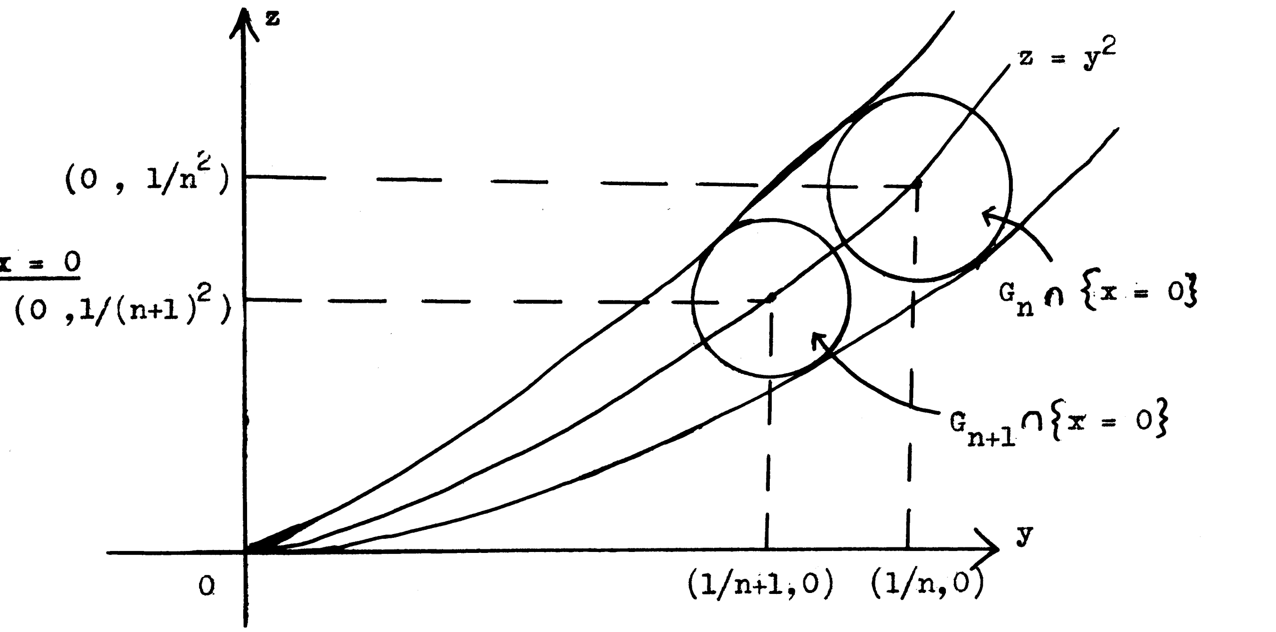
$$\Big[\text{(a) implies (t)}\Big] \quad \text{implies} \quad \Big[\underset{\substack{\parallel\!\!\!\mid \\ (1)}}{\text{(a)}} \text{ implies } \underbrace{\text{openness of transverse maps}}_{(3)}\Big]$$

This suggests that (t) implies the openness of transverse maps, which is false in general, although true in the case of subanalytic strata (or any situation where the curve selection lemma is available), as proved in Theorem 2.5 below. Thom, in [34] mentioned that (t) implied that the transverse maps formed an open set in the semialgebraic case. In [35] he used this to deduce that (a) implies that the transverse maps are open, again using analyticity. The mistake first occurs in [36] where he repeats the argument, but does not assume analyticity. The error was then copied by Wall [41] , Trotman [37] , and Chenciner [4] . Although [37] contains an example showing that (t) does not imply (a) , I did not then realise that (a) was equivalent to the openness of transverse maps, and missed the fact that the example there was actually a counterexample to (t) implies openness. A fortuitous remark by E. Bierstone at Oslo in August 1976 led to the recognition of the counterexample which follows.

Example 2.1. A (t)-regular stratification which is not (a)-regular [39] .

Let (x, y, z) be coordinates in $\mathbb{R}^3$. Take Y to be the y-axis, and let $X = \left(\bigcup_{n=1}^{\infty} \{f_n = 0 , g_n \leq 0\}\right) \cup \left(\bigcap_{n=1}^{\infty} \{x = 0 , g_n \geq 0 , z > 0\}\right)$ where $\{g_n \leq 0\}$ defines the cylinder G_n of radius $1/3n(n+1)$ with axis the line $\{y = 1/n , z = 1/n^2\}$ and where $\{f_n = 0\}$ defines the surface F_n obtained from $\{x = ((y^2 + z^2) - \tfrac{1}{2})^2\}$ by translating the origin to $(0, 1/n , 1/n^2)$ and reducing by a factor of $3n(n+1)/\sqrt{2}$ so that F_n intersects ∂G_n exactly where $\{x = 0\}$ is tangent to F_n.

Figure: x = 0



X is a C^1 submanifold and is semialgebraic on the complement of the origin. The normal vector to X at the point

$$x_n = \left(1/24\sqrt{2}n(n+1) , (1/n) + 1/3\sqrt{2}n(n+1) , 1/n^2 \right)$$

is $(2 : 1 : 0)$ for all n . Hence the limit as n tends to ∞ is $(2 : 1 : 0)$

and (a) fails. (For (a) to hold, all limits of normals would have to be of the form $(c_1 : 0 : c_2)$, where c_1, c_2 are not both zero.)

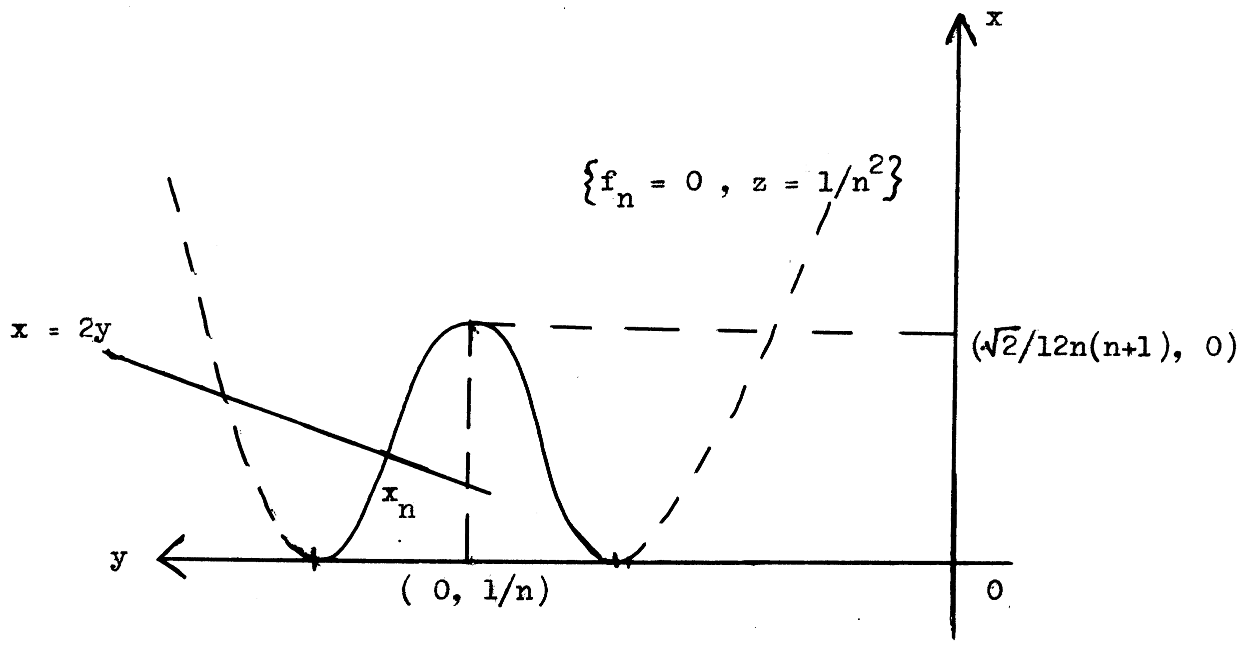


Figure: $z = 1/n^2$

(t) holds since any submanifold transverse to Y will intersect X near Y only at points near which X is defined by $\{x = 0\}$. Hence the stratification Σ of R^3 defined by $\{Y, X, \mathbb{R}^3 - (X \cup Y)\}$ is (t)-regular.

Now we verify explicitly that the set of maps transverse to Σ is not open. The mapping h in $C^1(S^2, \mathbb{R}^3)$ defined by inclusion of the sphere of radius 1 and tangent $\{2x + y = 0\}$ at 0 and with centre at $(-1/\sqrt{5}, -2/\sqrt{5}, 0)$ is transverse to the stratification, but for each n the mapping h_n defined by inclusion of the unit sphere with tangent at x_n the plane

$$\{2x + y = (5 + 12\sqrt{2}(n+1))/(12\sqrt{2}n(n+1))\}$$

and with 0 in the bounded component of $\mathbb{R}^3 - h_n(S^2)$, is not transverse to X at x_n. Since $\{h_n\}$ tends to h in the weak C^1 topology, which is also the strong C^1 topology (since S^2 is compact), the set of mappings transverse to Σ is not open in $C^1(S^2, \mathbb{R}^3)$.

Thus (t) cannot replace (a) in the statement of Theorem 1.1.

Note that by smoothing near each circle $\{x = 0 , g_n = 0\}$, X can be made into a C^∞ submanifold of $\mathbb{R}^3$, with the normal vector to X at each x_n as before, for all n , thus producing a C^∞ counterexample.

Construction 2.2 (Hills, or Round Barrows)

The example above used a simple construction of a C^1 semialgebraic hill which will prove useful as a building block for both examples and proofs of theorems. Consider the curve $\{x = (y^2 - 1)^2\}$ in $\mathbb{R}^2$: it has tangent parallel to the y-axis for $y = \pm 1$.

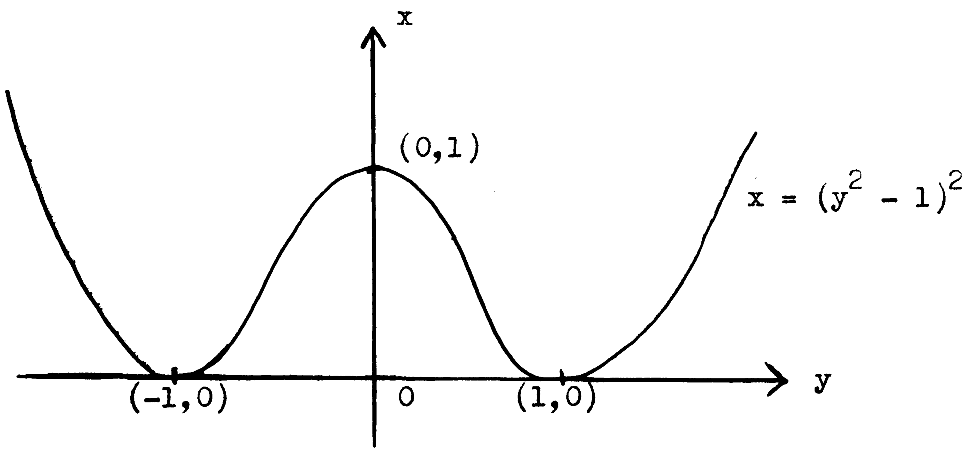


Figure : Hill of dimension one

Rotating in $\mathbb{R}^3$ about the x-axis, and cutting around the circle $\{y^2 + z^2 = 1 , x = 0\}$ and then inserting in the plane $\{x = 0\}$ with the disc $\{y^2 + z^2 \leq 1 , x = 0\}$ removed, gives a C^1 semialgebraic manifold. The vital property of the curve $\{x = (y^2 - 1)^2\}$ which will be used again and again is that in the region $\{y^2 \leq 1\}$ the tangent to the curve is furthest from $\{x = 0\}$ when $y = \pm 1/\sqrt{3}$, and at the points $(4/9 , \pm 1/\sqrt{3})$ the normal is $(1 ; \pm 8/3\sqrt{3})$.

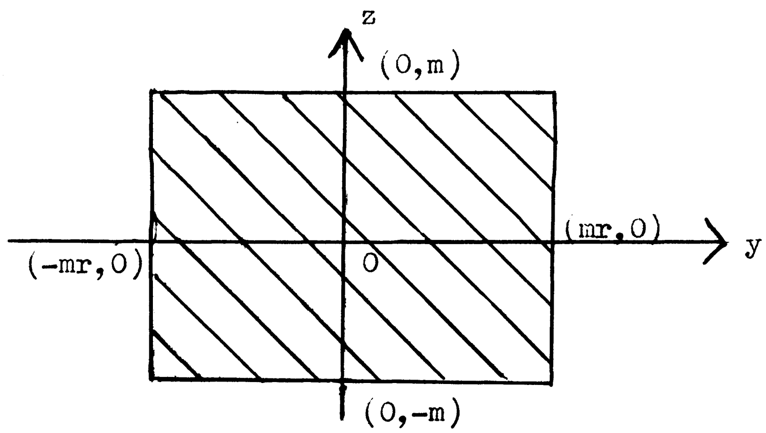
Construction 2.3 (Long Barrows)

Consider the surface in $\mathbb{R}^3$ with coordinates x , y , z ,

$$m^7 r^3 x = (m^2 - z^2)^2 (m^2 r^2 - y^2)^2$$

where $m , r \in [0 , \infty)$. The normal to the surface at (x , y , z) is

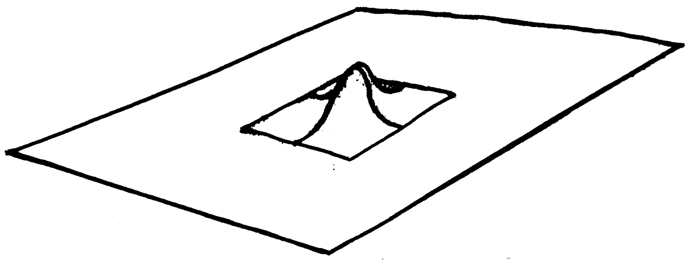
$$(m^7r^3 : 4(m^2 - z^2)^2(m^2r^2 - y^2)y : 4(m^2r^2 - y^2)^2(m^2 - z^2)z) .$$



On $\{z^2 = m^2, x = 0\}$ and $\{y^2 = m^2r^2, x = 0\}$ the normal is $(1 : 0 : 0)$, and thus we can cut along these lines to obtain the surface

$$B(m,r) \equiv \{m^7r^3x = (m^2 - z^2)^2(m^2r^2 - y^2)^2 , z^2 \leqslant m^2 , y^2 \leqslant m^2r^2\}$$

and we can insert $B(m,r)$ in the plane $\{x = 0\}$ with a rectangle $\{x = 0 , z^2 \leqslant m^2 , y^2 \leqslant m^2r^2\}$ removed, to give a C^1 semialgebraic manifold.



At (mrx , mry , mz) for $z^2 \leqslant 1 , y^2 \leqslant 1$, the normal is now $(1 : 4y(1 - z^2)^2(1 - y^2) : 4rz(1 - z^2)(1 - y^2)^2)$. Thus as m varies $B(m,r)$ varies in size, but the tangent structure (that is the set of points in $P^2(\mathbb{R})$ defined by the normals or tangents to the surface) remains the same. But as r varies the normals change, and as r tends to 0 the normals tend to lie in the arc of lines $\{(1 : \frac{8\lambda}{3\sqrt{3}} : 0) : \lambda \in [-1 , 1]\}$.

We call this surface $B(m,r)$ a <u>(long) barrow of magnitude</u> m , <u>ratio</u> r , <u>with axis</u> Oz , <u>and centre</u> O , <u>and base</u> yOz . The axis, centre, and base will always be specified. Calculation shows that for $r < \sqrt{3}/4$, the normal to the surface is furthest from $(1 : 0 : 0)$ when $y = \pm mr/\sqrt{3}$ and $z = 0$, and at these points , $(4mr/9 , \pm mr/\sqrt{3}, 0)$, the normal is $(1 : \pm 8/3\sqrt{3} : 0)$. (Compare Construction 2.2)

<u>Linguistic Note</u> : The term <u>barrow</u> is used because of the resemblance of the surface to the ancient burial mounds called by that name in England, when r is small.

<u>Example 2.4</u> : This will show that the phenomenon that (t) be strictly weaker than (a) is not solely due to the possibility of (a)-faults given by sequences tangent to the base stratum as in Example 2.1 : that is, it is not true that (a) holds for those sequences on curves with limiting direction not tangent to the base stratum.

In $\mathbb{R}^3$ with coordinates (x , y , z) let Y be the y-axis, and let X be $(\bigcup_{n=1}^{\infty} \{f_n = 0 , g_n \leq 0\}) \cup (\bigcap_{n=1}^{\infty} \{x = 0 , g_n \geq 0 , z > 0\})$ where $\{f_n = 0\}$ is the equation defining the barrow $B(m_n, r_n)$ with centre $(0 , 1/n , 1/n)$ and axis $\{x = 0 , z + y = 2/n\}$, with base in the plane $\{x = 0\}$, and $\{g_n \leq 0\}$ defines the interior of the rectangular base of the barrow. X is a C^1 manifold, and is semialgebraic on the complement of the origin in $\mathbb{R}^3$. We choose $\{(m_n, r_n)\}_{n=1}^{\infty}$ such that,

(1) r_n tends to 0 as n tends to ∞,

(2) the barrows are pairwise disjoint (in particular m_n tends to 0) ,

(3) m_n tends to 0 fast enough so that the n^{th} barrow $B(m_n, r_n)$ is contained in the 2-sphere with centre $(0 , 1/n , 1/n)$ and radius $1/2n^2$ (so $m_n = 1/4n^2$ will do).

By (1) the set of limiting normals is exactly $\{(1 : (4\sqrt{2}/3\sqrt{3})\lambda : (4\sqrt{2}/3\sqrt{3})\lambda) : 0 \leq |\lambda| \leq 1\}$. (Cf. Construction 2.3) Thus (a) fails, since for (a) to hold <u>all</u> limiting normals must be of the form $(c_1, 0, c_2)$.

By (3) the set of barrows is contained in the horn which is tangent to $\{z = y , x = 0\}$ and which intersects the plane $\{z + y = 2t\}$ in a circle of radius t^2 . Hence a C^1 submanifold S transverse to Y at 0 intersects infinitely many barrows only if $\{z = y , x = 0\} \subset T_0S$. But then S will be transverse to all barrows in some neighbourhood of 0 . For, suppose S were nontransverse to infinitely many barrows ; then N_0S would be one of the limiting $(1 : (4\sqrt{2}/3\sqrt{3})\lambda : (4\sqrt{2}/3\sqrt{3})\lambda)$. But $\{z = y , x = 0\} \subset T_0S$, and S is transverse to $\{x = 0 , z = 0\}$ at 0 , thus N_0S is of the form $(\mu : \nu : -\nu)$ with $\nu \neq 0$, which is not a limiting normal to X .

Thus we have shown that (t) holds and that (a) fails along sequences which are not tangent to Y .

As in example 2.1 , by smoothing near the base of each barrow we obtain a C^∞ example.

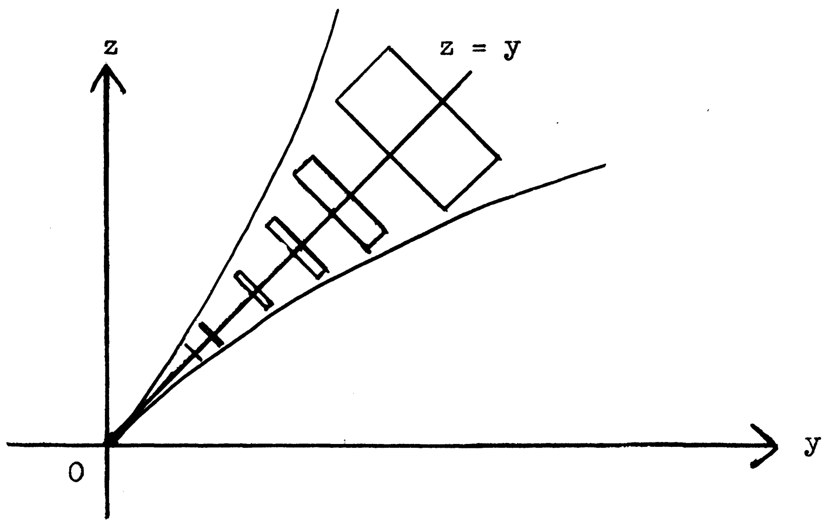


Figure : x = 0 .

Now we shall prove that (t) and (a) are equivalent in the subanalytic case. Precisely, we have the following result.

Theorem 2.5 : Let X , Y be C^1 submanifolds of R^n and let $0 \in Y \cap (\overline{X} - X)$, and let X be a subanalytic set. Then X is (a)-regular over Y at 0 if and only if for every semianalytic C^1 submanifold S transverse to Y at 0 there is some neighbourhood U of 0 in which S is transverse to X .

The proof will depend upon two technical lemmas which we display for future reference.

<u>Curve Selection Lemma 2.6</u> : <u>Let</u> U <u>be a subanalytic subset of the analytic space</u> A , <u>and let</u> $0 \in \overline{U}$. <u>Then there is an analytic arc</u>

$$\alpha : [0, 1] \longrightarrow A$$

<u>such that</u> $\alpha(0) = 0$, $\alpha(t) \in U$ <u>if</u> $t \neq 0$.

<u>Proofs of Lemma 2.6</u> : (1) Subanalytic U : Hironaka [12, Proposition 3.9].
(2) Semianalytic U : Lojasiewicz [18, page 103].
(3) Semialgebraic U : Milnor [23, Chapter 2].
(Of course, (1) implies (2) , and (2) implies (3).)

<u>Lemma 2.7</u> : <u>Let</u> X^m <u>be a</u> C^1 <u>submanifold of</u> $\mathbb{R}^n$, <u>and a subanalytic subset of</u> $\mathbb{R}^n$. <u>Then</u> $\{(x, T_xX) : x \in X\}$ <u>is a subanalytic subset of</u> $\mathbb{R}^n \times G^n_m(\mathbb{R})$.

<u>Proof</u> : See Verdier [40, Lemma 1.6] .

Lemma 2.7 , with semianalytic replacing subanalytic each time, follows after partition into real analytic manifolds from the proof of Whitney [47] for complex analytic varieties. A short proof of Lemma 2.7 , with semialgebraic replacing subanalytic each time, appears in Gibson [6, page 30] .

<u>Proof of Theorem 2.5</u> : <u>Only if</u> - this is immediate since spanning (and hence transversality) is an open condition.

<u>If</u> - Suppose (a) fails. Thus there is a unit vector $v \in T_0Y$, a sequence $\{x_i\} \in X$ such that x_i tends to 0 , and $T_{x_i}X$ tends to a limit τ , and $v \notin \tau$.

Choose $\varepsilon > 0$ and $i_0 \in \mathbb{N}$ such that $\forall\ i \geq i_0$, $d(v, T_{x_i}X) > \varepsilon$, where $d(v, P)$ denotes the distance between $P \in G^n_m(\mathbb{R})$ and the endpoint of the unit vector v , both considered as subspaces of $\mathbb{R}^n$ at 0 .

Define $V_1 = \mathbb{R}^n \times \{P \in G^n_m(\mathbb{R}) : d(v,P) > \varepsilon\} \subset \mathbb{R}^n \times G^n_m(\mathbb{R})$

$V_2 = \{(x, T_xX) : x \in X\} \subset \mathbb{R}^n \times G^n_m(\mathbb{R})$.

V_1 is semialgebraic, and V_2 is subanalytic by Lemma 2.7, since X is assumed to be subanalytic. Semialgebraic sets are subanalytic, and the finite intersection of subanalytic sets is subanalytic (by Hironaka [12]). Hence $V_1 \cap V_2$ is subanalytic and $(0, \tau) \in \overline{V_1 \cap V_2}$ satisfies the hypotheses of the curve selection lemma 2.6. Thus there is an analytic arc in $\mathbb{R}^n \times G^n_m(\mathbb{R})$ (which is an analytic, even algebraic, manifold),

$$\alpha : [0, 1] \longrightarrow \mathbb{R}^n \times G^n_m(\mathbb{R})$$

with $\alpha(0) = (0, \tau)$ and $\alpha(t) \in V_1 \cap V_2$ if $t > 0$.

Write $\alpha_1(t)$ for the $\mathbb{R}^n$-component of $\alpha(t)$; the $G^n_m(\mathbb{R})$-component is $T_{\alpha_1(t)}X$. Let $N_t \in G^n_{n-1}(\mathbb{R})$ denote the normal space at $\alpha_1(t)$ to the C^1 manifold-with-boundary $\alpha_1([0, 1])$, and let the vector v_t be the projection of v into N_t spanning $\langle v_t \rangle \in G^n_1(\mathbb{R})$.

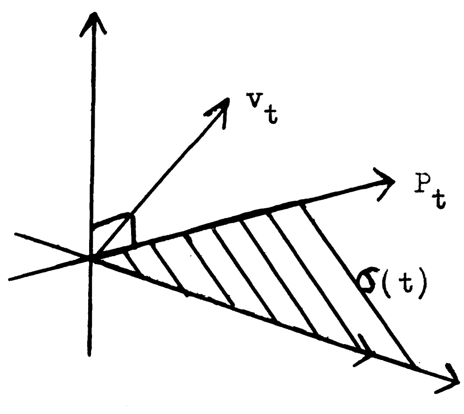
We shall define an analytic arc $\sigma : [0,1] \longrightarrow G^n_{n-2}(\mathbb{R})$ such that

$$\sigma(t) \oplus \langle v_t \rangle = N_t \qquad (*)$$

Then the union of the $\{\sigma(t)\}$, considered as embedded $(n-2)$-planes in $\mathbb{R}^n$ passing through the points $\alpha_1(t)$ defines an analytic manifold-with-boundary S' of dimension $(n-1)$. Reflection in N_0 extends S' to a C^1 manifold S which is a semianalytic subset of $\mathbb{R}^n$, and which is transverse to Y at 0 by $(*)$. However we shall show that no neighbourhood U of 0 exists within which S is transverse to X.

<u>Construction of σ</u> :

Let $P_t = N_t \cap T_{\alpha_1(t)}X \in G^n_{m-1}(\mathbb{R})$. Then $0 \neq v_t \notin P_t$ by definition of $V_1 \cap V_2$. Let $\sigma(t) = P_t \oplus (P_t \oplus \langle v_t \rangle)^{\perp} \in G^n_{n-2}(\mathbb{R})$, where $(\ )^{\perp}$ denotes orthogonal complement in N_t.



<u>Figure</u> : N_t $(n=4, m=2)$.

σ satisfies (*) by construction, and so it only remains to show that S fails to be transverse to X in any given neighbourhood U of 0 . Now there exists some $t_0 \in (0,1]$ such that $U \cap \alpha_1(0,1] \supset \alpha_1(0,t_0]$. But S' (and hence S) is not transverse to X at any point of $\alpha_1(0,1]$. For, if A_t denotes the tangent space to the curve $\alpha_1(0,1]$ at $\alpha_1(t)$,

$$T_{\alpha_1(t)}X = P_t \oplus A_t \subset \sigma(t) \oplus A_t = T_{\alpha_1(t)}S .$$

This completes the proof of Theorem 2.5.

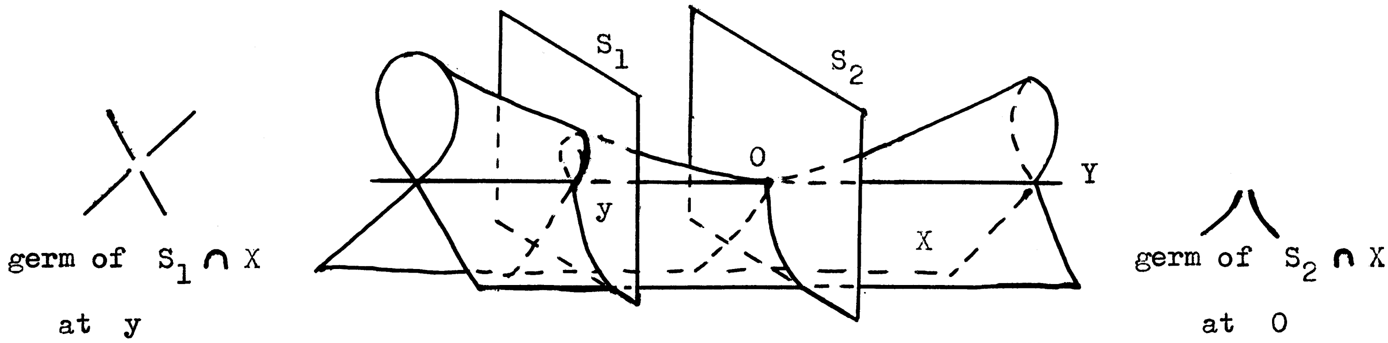
Note 2.8 : Even if X and Y are C^∞ submanifolds we cannot restrict to C^∞, or even C^2 , semianalytic submanifolds S , since (a) may fail only near a cusp of type "$y^2 = x^3$" , each branch of which is a C^1 manifold-with-boundary, but not a C^2 manifold-with-boundary. The same type of example excludes restricting to analytic submanifolds S , although by the proof of 2.5 we can restrict to analytic submanifolds-with-boundary S , since the statement that S be transverse to Y at 0 still makes sense if $0 \in Y \cap \partial S$. The proof of 2.5 also shows that we can restrict to those S which are " ruled submanifolds " , that is a differentiable one-dimensional family of planes of codimension 2 in $\mathbb{R}^n$. Moreover it suffices to consider all submanifolds of some fixed dimension greater than or equal to the codimension of Y , by a small adjustment in the proof (choose $\sigma_1(t) \subset \sigma(t)$, where $\sigma_1(0) + T_0Y = N_0$, $\sigma_1(t) \in G^n_{c-1}(\mathbb{R})$, and $c \geqslant \operatorname{codim} Y$).

T.-C. Kuo has recently proved the following result, which is related to the questions already treated in this section.

Theorem 2.9 (Kuo) : *Let* X , Y *be* C^∞ *submanifolds of* $\mathbb{R}^n$, $Y = \overline{X} - X$ *in some neighbourhood of* Y . *Suppose* X *is* (a)-*regular over* Y *at* $0 \in Y$. *Let* S_1 , S_2 *be* C^∞ *submanifolds transverse to* Y *at* 0 , *with* $\dim S_i = n - \dim Y_i$ (i = 1,2). *Then the germs of* $S_1 \cap X$ *and* $S_2 \cap X$ *at* 0 *are homeomorphic.*

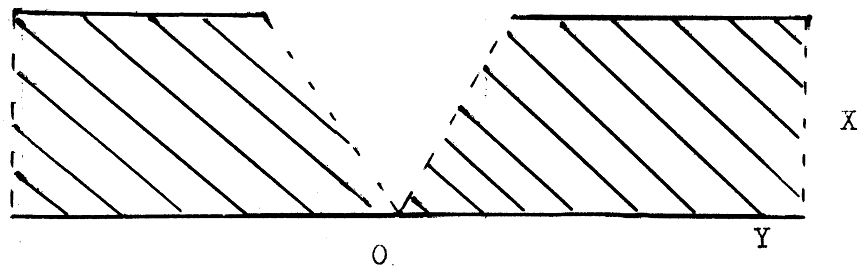
Proof : In [15] .

This is an attractive result since it parallels the Thom-Mather theorem (Theorem B of the introduction) that (b)-regularity implies topological triviality. Explicitly, if X is (b)-regular over Y and S_1 and S_2 are two submanifolds transverse to Y at points y_1 and y_2 in Y (with $y_1 \neq y_2$ allowed) , then the germs of $S_1 \cap X$ at y_1 and $S_2 \cap X$ at y_2 are homeomorphic. This follows from Corollary 10.6 of [21] . (a)-regularity is definitely insufficient for the latter property as shown by the figure below.



Conjecture 2.10 : Theorem 2.9 is true with the weaker hypothesis that X be (t)-regular over Y at 0 . (Added 1980 : we have now proved this [59].)

Observe that the hypothesis $Y = \overline{X} - X$ rather than $Y \subset \overline{X} - X$ is essential in 2.9 and 2.10 , as shown by the next figure.



We might also ask if the converse of Theorem 2.9 is true. However examples 2.1 and 2.4 show that this is not so. In both examples X is not (a)-regular over Y at 0 , but any C^1 submanifold transverse to Y (at 0) intersects

X in a topological open half-line near 0. We do though have a converse to 2.9 if we replace (a)-regularity by (t)-regularity as in Theorem 2.11 below.

<u>Definition</u> : Let X, Y be C^1 submanifolds of $\mathbb{R}^n$, and $0 \in Y \subset \overline{X} - X$. The pair (X, Y) is said to have <u>homeomorphic</u> C^k <u>transversals of dimension</u> s <u>at</u> 0 ($1 \leq k \leq \infty$, $\operatorname{codim} Y \leq s \leq n$) if,

(h^k_s) Given a C^k submanifold S of dimension s transverse to Y at 0, the topological type of the germ of $S \cap X$ at 0 is independent of S.

Theorem 2.9 says that (a) implies ($h^{\infty}_{\operatorname{cod} Y}$). From the proof of 2.9 [15], one sees that (a) implies ($h^{2}_{\operatorname{cod} Y}$), but it is left in doubt whether (a) implies ($h^{1}_{\operatorname{cod} Y}$) since the proof makes use of a (tangent) vector field in a blowing-up.

Write (t^k_s) for condition (t) restricted to those C^1 submanifolds S of class C^k ($1 \leq k \leq \infty$) and dimension s ($\operatorname{codim} Y \leq s \leq n$). Then we have ,

<u>Theorem 2.11</u> : <u>Let</u> X, Y <u>be disjoint</u> C^k <u>submanifolds of</u> $\mathbb{R}^n$, <u>and let</u> $0 \in Y \cap \overline{X}$, <u>with</u> $1 \leq k \leq \infty$. <u>Then</u>

$$(h^k_s) \ \underline{\text{implies}} \ (t^k_s) \ \underline{\text{if}} \begin{cases} k = 1 \\ \text{or} \\ k > 1 \ \underline{\text{and}} \ s > n - \dim X . \end{cases}$$

(David Epstein has given a counterexample showing that the restriction on s when $k > 1$ is necessary.)

<u>Proof</u> : Suppose X is not (t^k_s)-regular over Y at 0. Then there is some C^k submanifold S of dimension s transverse to Y at 0, and an infinite sequence of points x_i in X, tending to 0, such that S and X are not transverse at x_i, for all i.

We are working locally at 0 , so we can suppose that S is the image of a C^k embedding $i_S : (\mathbb{R}^s, 0) \longrightarrow (S, 0) \subset (\mathbb{R}^n, 0)$.

Choose a sequence of pairwise disjoint balls B_i of radius r_i and centre x_i , which are contained in coordinate charts for X , such that $i_S^{-1}(S \cap B_i) = D_i$ is an open subset of $\mathbb{R}^s$, and diffeomorphic to $\mathbb{R}^s$. Let $s_i = i_S^{-1}(x_i)$.

We shall show the existence of a C^k embedding $g : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that,

(I) $g = i_S$ off $\bigcup_{i=1}^{\infty} D_i$,

(II) for all i , $g(\mathbb{R}^s) \cap X \cap B_i$ is not homeomorphic to a manifold of dimension $(s + \dim X - n)$, and is nonempty.

From (I) it follows that i_S and g have the same k-jet at 0 , so that in particular $g(\mathbb{R}^s) = S'$ is transverse to Y at 0 .

<u>Existence of g when $k = 1$</u> :

Finding such a g is particularly simple when $k = 1$.

Fix i , and let ϕ_i be a C^1 diffeomorphism of B_i , fixing x_i , so that $\phi_i(X \cap B_i)$ is affine. By an arbitrarily small C^1-perturbation of i_S near s_i we can change $i_S|_{D_i}$ to a C^1 embedding $g_i : (D_i, s_i) \longrightarrow (B_i, x_i)$, such that there are open neighbourhoods N_i and L_i of s_i in $\mathbb{R}^s$ with $N_i \subset \overline{N}_i \subset L_i \subset \overline{L}_i \subset D_i$, and $g_i|_{D_i - L_i} = i_S|_{D_i - L_i}$, and $\phi_i \circ g_i|_{N_i} = d(\phi_i \circ i_S)(s_i)|_{N_i}$. (We have pushed $\phi_i(S)$ onto its tangent space near x_i.)

Near x_i we now have two affine subspaces $\phi_i(X \cap B_i)$ and $(\phi_i \circ g_i)(N_i)$ which intersect at x_i , but are not transverse at x_i , and hence intersect in an affine subspace of dimension greater than $d = \max(-1, s + \dim X - n)$. Thus $\dim(\phi_i(X \cap B_i) \cap (\phi_i \circ g_i)(D_i))$ is greater than d , and hence

$$(*) \qquad \dim(X \cap g_i(D_i)) > d .$$

In particular $X \cap g_i(D_i)$ is nonempty.

Now define $g : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^n, 0)$ by setting $g|_{D_i} = g_i$ for all i , and g equal to i_S elsewhere.

For g to be a C^1 embedding, it suffices to choose $\{g_i\}$ such that $|j^1(i_S)(s) - j^1(g_i)(s)| < r_i/2^i$ for all $s \in D_i$, for all i.

Then (I) is satisfied by construction, and (*) gives (II).

Existence of g when $k > 1$, and $s > (n - \dim X)$:

Fix i. We shall change $i_S|_{D_i}$ to a C^k embedding $g_i : (D_i, s_i) \longrightarrow (B_i, x_i)$ by an arbitrarily small C^k-perturbation (less than $r_i/2^i$, say) near s_i, so that there are open neighbourhoods N_i and L_i of s_i in R^s with $\bar{N}_i \subset L_i$, and $\bar{L}_i \subset D_i$ such that $g_i|_{D_i - L_i} = i_S|_{D_i - L_i}$, and such that $g_i^{-1}(X) \cap N_i$ is homeomorphic to a cone in $\mathbb{R}^s$, of the form

$$\sum_{j=1}^{s+\dim X-n+1} \varepsilon_j z_j^2 = 0 , \quad \text{where } \varepsilon_j = \pm 1 ;$$

hence $g_i^{-1}(X) \cap N_i$ is not homeomorphic to a topological manifold of dimension $(s + \dim X - n)$, and is nonempty.

The existence of such a g_i follows from the Perturbation Lemma of May (Lemma 1A of his thesis [53] ; Damon has given a detailed proof of a more precise perturbation in Lemma 3.1 of [51]) applied to the C^k embedding i_S at 0, using the hypothesis $s > n - \dim X$. The Perturbation Lemma is stated for C^∞ maps and uses the C^∞ Morse Lemma, however the proof works for C^k maps $(k \geq 2)$, using the C^2 Morse Lemma due to Kuiper ([52] ; Ostrowski [55] and Takens [56] provide different proofs) . Note that the classical proof of the Morse Lemma is only valid for C^3 functions (see [13], Chapter 6, Section 1) .

(I) and (II) now follow for the C^k embedding g defined in terms of i_S and $\{g_i\}$, as in the case of $k = 1$. This completes our proof of the existence of g.

<u>Lemma 2.12 : There is some</u> C^k <u>submanifold</u> S'' <u>of dimension</u> s, <u>with</u> $0 \in S''$, <u>transverse to</u> Y <u>at</u> 0 <u>and transverse to</u> X <u>near</u> 0.

<u>Proof</u> : This proof will be similar to that of Lemma 1.3 .

Let $\mathcal{E}_S = \{ f \in C^k(S , \mathbb{R}^n) : f(0) = 0 \}$. $\mathcal{E}_S$ is weakly closed in the C^k topology, and thus, by Theorem 1.4 , $\mathcal{E}_S$ is a Baire space in the strong

C^k topology. Now we apply the standard procedure of covering X by countably many coordinate discs $\{K_\alpha\}$, and proving that $\{f \in \mathcal{E}_S : f \pitchfork X \text{ on } K_\alpha\}$ is open and dense in $\mathcal{E}_S$ in the strong C^k topology, for each α, to deduce that $\{f \in \mathcal{E}_S : f \pitchfork X\}$ is dense in $\mathcal{E}_S$.

Choose a weak C^k neighbourhood $\mathcal{V}_{\delta,\overline{V}}(i_S)$ of the (C^k) mapping i_S defined by inclusion of S in $\mathbb{R}^n$, where δ is a positive real number, V is a neighbourhood of 0 in S, with compact closure $\overline{V}$, and if $f \in \mathcal{V}_{\delta,\overline{V}}(i_S)$, $f|_V$ is a C^k embedding transverse to Y at 0 (Lemma 1.3 in Chapter 2 of Hirsch [13] gives δ, V for such a C^1 neighbourhood, and the same δ, V provide an adequate C^k neighbourhood). Then the strong C^k neighbourhood $\mathcal{V}_{\delta,S}(i_S)$ has $\mathcal{V}_\pitchfork = \mathcal{V}_{\delta,S}(i_S) \cap \{f \in \mathcal{E}_S : f \pitchfork X\}$ as a strongly C^k dense subset. For any f in $\mathcal{V}_\pitchfork$, $S'' = f(V)$ satisfies the requirements of Lemma 2.12.

(Recall that $\mathcal{V}_{\delta,\overline{V}}(i_S) = \{f \in C^k(S, \mathbb{R}^n) : |j^k f(z) - j^k i_S(z)| < \delta, \forall z \in \overline{V}\}$.)

Let S'' be given by Lemma 2.12. Then $S'' \cap X$ is either empty in some neighbourhood of 0, or is a topological manifold of dimension $(s + \dim X - n)$. Let S' be given as the image of the embedding g constructed above. Then the germs at 0 of $S' \cap X$ and $S'' \cap X$ are of distinct topological types, by (II), and so (h^k_s) is not satisfied, thus proving Theorem 2.11.

<u>Corollary 2.13</u> : <u>If</u> X <u>is subanalytic and the pair</u> (X, Y) <u>has homeomorphic</u> C^1 <u>transversals of dimension</u> s <u>at</u> 0 <u>for some</u> s, $n - 1 \geq s \geq \operatorname{codim} Y$, <u>then</u> X <u>is</u> (a)-<u>regular over</u> Y <u>at</u> 0.

<u>Proof</u> : Combine Theorem 2.11 with Theorem 2.5, using the remark at the end of Note 2.8 that for any s, $n - 1 \geq s \geq \operatorname{codim} Y$, (t^1_s) implies (a).

Remark : Conjecture 2.10 and Theorem 2.11 are in accord with the general principle of Thom that instability of topological type corresponds to a lack of transversality.

One of the original motivations for this work was the hope of generalising the theorems about equisingularity of families of complex hypersurfaces achieved by Zariski and the French School (led by Teissier). We now explain how the results just described fit in with this idea.

Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0 \times \mathbb{C}^k) \longrightarrow (\mathbb{C}, 0)$ be a complex analytic function such that $Y = 0 \times \mathbb{C}^k$ contains the singular set of F. Let $r : \mathbb{C}^{n+1} \times \mathbb{C}^k \longrightarrow Y$ be an analytic retraction. In [30] we find the following implications :-

(T.E) topological type of $F^{-1}(0) \cap r^{-1}(y)$ is constant as y varies in Y

$\Downarrow$

(μ) the Milnor number $\mu(F^{-1}(0) \cap r^{-1}(y))$ is constant as y varies in Y

$\Downarrow$

(a) $(F^{-1}(0) - Y)$ is (a)-regular over Y

(The first implication is (0.1.4) of [30], and is also sketched on page 68 of [23]. The second implication is (II.3.10) of [30] ; a different proof appears in [16].)

In [31], Teissier denotes by (S.T.E) the condition that (T.E) hold for <u>all</u> such retractions r. Corollary 2.13 can now be thought of as a generalisation of the implication : (S.T.E) implies (a). Also Kuo's Theorem 2.9 has as a direct consequence that (T.E) implies (S.T.E), a result left unsettled in [31] .

The example given by Teissier in the post-script to [31] is instructive . Consider $V \equiv \{y^3 = tx^2 + x^5\}$ in $\mathbb{R}^3$ and let Y be the t-axis, and $X = V - Y$. Then X is topologically trivial over Y , and the topological type of the intersection of X with each plane $\{t = \text{constant}\}$ is constant, so that (T.E) holds for $r : \mathbb{R}^3 \longrightarrow Y$ defined by $(x, y, t) \longmapsto t$. However X is not (a)-regular over Y at 0 , and (X,Y) does not have homeomorphic C^1 transversals of dimension 2 at 0 as is seen from the figure.

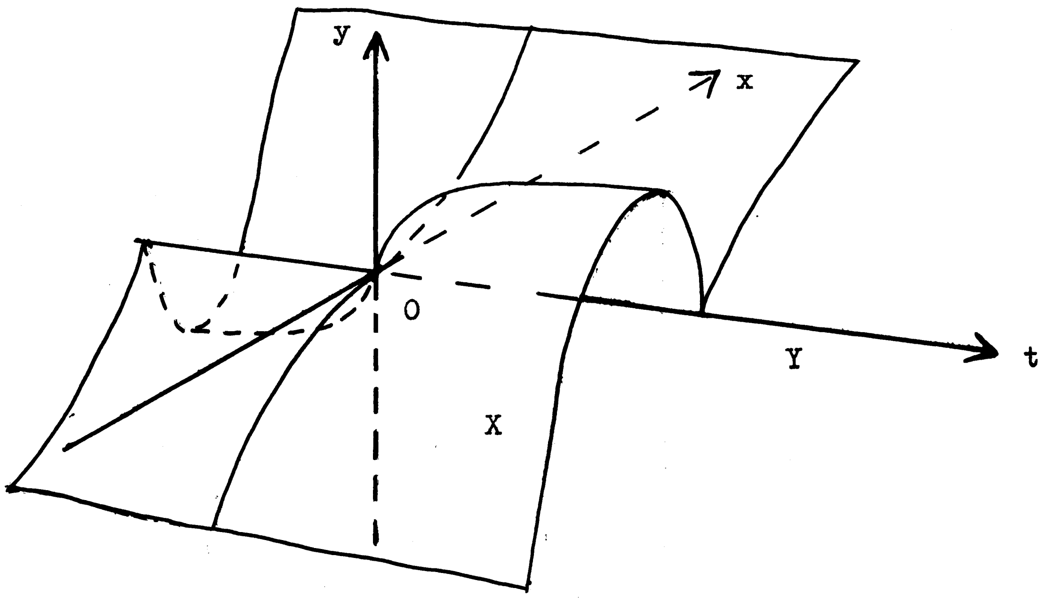


Figure : $y^3 = tx^2 + x^5$.

3. (a)-regularity and transverse foliations

In his paper " Regular Stratifications " [43] C. T. C. Wall noted that if a pair of adjacent strata (X,Y) in $\mathbb{R}^n$ are (a)-regular at 0 in Y then,

(a_s) Given a C^1 local retraction π onto Y defined near 0 , then there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that $\pi|_{X \cap U}$ is a submersion.

He suggested that the converse was also true, and this will be the main result of this section.

First note that $\pi|_{X \cap U}$ is a submersion if and only if the fibres of π are transverse to X in U. Then we see that (a_s) implies (t). For, given a C^1 submanifold S transverse to Y at 0 we can choose a chart at 0 in which S and Y become linear and then take a linear retraction π whose fibres lie in S. If the fibres of π are transverse to X, S will be transverse to X. Thus we obtain,

Corollary 3.1 : *Let* X, Y *be* C^1 *submanifolds of* $\mathbb{R}^n$ *and let* $0 \in Y \subset \overline{X} - X$ *and let* X *be a subanalytic set. Then* X *is* (a)*-regular over* Y *at* 0 *if and only if* X *is* (a_s)*-regular over* Y *at* 0.

Proof : As above, (a) implies (a_s), and (a_s) implies (t). Now apply Theorem 2.5.

Clearly if Y is an analytic manifold we can restrict to C^1 local retractions π whose fibres are semianalytic : further improvements on Corollary 3.1 may be culled from Note 2.8.

Remark 3.2 : In both examples 2.1 and 2.4 we can choose a (linear) retraction π whose fibres are translates (over Y) of a limiting tangent plane for which (a) fails, and these fibres fail to be transverse to X at each point of a sequence tending to 0.

Before we prove that (a_s) implies (a), we give a helpful reformulation of (a_s) suggested by Dennis Sullivan.

$(\mathcal{F}^k)$ Given a C^k foliation $\mathcal{F}$ transverse to Y at 0, there is a neighbourhood U of 0 in $\mathbb{R}^n$ such that $\mathcal{F}$ is transverse to X in U.

It is clear that (a_s) is equivalent to $(\mathfrak{F}^1)$. Given $(\mathfrak{F}^1)$, (a_s) follows since the fibres of a C^1 local retraction define a foliation transverse to Y of codimension the dimension of Y. Given (a_s), $(\mathfrak{F}^1)$ follows by choosing a retraction whose fibres are contained in the leaves of the foliation.

So the question of whether (a_s) implies (a) can be formulated as : <u>do transverse C^1 foliations detect (a)-faults</u> ?

<u>Theorem 3.3 (" Transverse C^1 foliations detect (a)-faults ")</u>

<u>Let</u> X, Y <u>be</u> C^1 <u>submanifolds of</u> $\mathbb{R}^n$, <u>and let</u> $0 \in Y \subset \overline{X} - X$. <u>Then</u> X <u>is</u> (a)-<u>regular over</u> Y <u>at</u> 0 <u>if and only if</u> X <u>is</u> $(\mathfrak{F}^1)$-<u>regular over</u> Y <u>at</u> 0

<u>Proof</u> : We have already established that (a) implies $(\mathfrak{F}^1)$. So suppose that there is an (a)-fault at 0 given by a sequence $\{x_i\} \in X$ tending to 0, with $\tau = \lim T_{x_i} X$, and $T_0 Y \not\subset \tau$.

We shall adjust a codimension 1 foliation by hyperplanes parallel to a hyperplane containing τ so as to be nontransverse to X at infinitely many x_i.

<u>Construction 3.4 (Ripples)</u>

Given a hyperplane $H \in G^n_{n-1}(\mathbb{R})$, a real number $s \in [0,\frac{1}{2}]$, and a real number $r > 0$, we construct a C^1 foliation $\mathfrak{F}^s_H$ of codimension 1 of the ball B^n_r of radius r with centre 0 in $\mathbb{R}^n$ such that

(1) for all $x \in B^n_r - B^n_{\frac{1}{2}r}$, $T_x \mathfrak{F}^s_H = H$,

(2) for all $x \in B^n_{\frac{1}{2}r}$, $d(H, T_x \mathfrak{F}^s_H) \leq s$,

(3) for all $K \in G^n_{n-1}(\mathbb{R})$ such that $d(K,H) = s$, there is a unique $x_K \in B^n_{\frac{1}{2}r}$ such that $T_{x_K} \mathfrak{F}^s_H = K$,

(4) there is a C^1 diffeomorphism $\phi^s_H : B^n_r \longrightarrow B^n_r$ such that $\phi^s_H(\mathfrak{F}^s_H)$ is the trivial foliation $\mathfrak{F}^0_H$ by hyperplanes parallel to H, and such that $\phi^s_H|_{B^n_r - B^n_{\frac{1}{2}r}} = id|_{B^n_r - B^n_{\frac{1}{2}r}}$, and $d\phi^s_H$ tends to the identity uniformly as s tends to 0, i.e. $\forall\ \varepsilon > 0$, $\exists\ s_\varepsilon > 0$ such that

$s < s_\varepsilon$ implies $|d\Phi_H^s(x) - I| < \varepsilon$ for all $x \in B_r^n$.

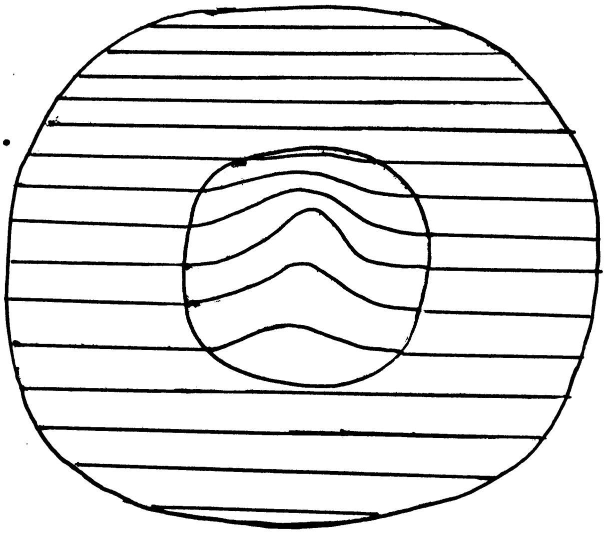


Figure : Foliation with a ripple.

(We shall postpone the verification of Construction 3.4 until after the proof of Theorem 3.3 . The reader may in any case prefer to admit the verification as geometrically evident.)

Choose a one-dimensional subspace $V \subset T_0 Y$ such that $V \not\subset \tau$. Define a hyperplane H by $\tau \oplus (\tau \oplus V)^\perp$, where $(\ )^\perp$ denotes orthogonal complement in $T_0\mathbb{R}^n$.

Since $T_{x_i}X$ tends to τ as i tends to ∞ , there is some i_0 such that $i \geq i_0$ implies $V \not\subset T_{x_i}X$. Then for all $i \geq i_0$ define a hyperplane H_i by $T_{x_i}X \oplus (T_{x_i}X \oplus V)^\perp \subset T_{x_i}\mathbb{R}^n$. Then H_i tends to H as i tends to ∞. Pick $i_1 \geq i_0$ such that $|H_i - H| < \frac{1}{2}$ for $i \geq i_1$.

Now pick an infinite sequence of pairwise disjoint balls $B_{r_i}(x_i)$ with radius r_i and centre x_i . This is possible since 0 is the only accumulation point of $\{x_i\}_{i=1}^\infty$. Then for all i , $0 \notin B_{r_i}(x_i)$.

For all $i \geq i_1$, place inside $B_{r_i}(x_i)$ a " ripple " : a foliated ball $B_i = B_{\frac{1}{2}r_i}(y_i)$ with radius $\frac{1}{2}r_i$, centre y_i , and the foliation $\mathcal{F}_i = \mathcal{F}_H^{|H_i - H|}$ given by Construction 3.4 such that $x_i = x_{H_i}$, i.e. $T_{x_i}\mathcal{F}_i = H_i$. (There are two possible positions for the ripple.) Define a foliation $\mathcal{F}$ on $\mathbb{R}^n$ by the trivial foliation $\mathcal{F}_H$ by hyperplanes parallel to H on $\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)$, together with $\mathcal{F}_i$ on B_i for all $i \geq i_1$. $\mathcal{F}$ will be a C^1 foliation if we can define a C^1 diffeomorphism $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ taking $\mathcal{F}$ onto $\mathcal{F}_H$. Let $\phi|_{\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)}$ = identity, and $\phi|_{B_i} = \phi_H^{|H_i - H|}$ as defined in Construction 3.4. To check that ϕ is a C^1 diffeomorphism it is enough to check that $d\phi(x)$ is continuous at 0 and equal to the identity at 0.

Given $\varepsilon > 0$, (4) of Construction 3.4 gives us an $s_\varepsilon > 0$. Pick $i_2 \geq i_1$ such that $|H_i - H| < s_\varepsilon$ for all $i \geq i_2$. Let $\delta = \min_{\substack{x \in \overline{B}_i \\ i_1 \leq i < i_2}} \{|x|\}$. δ is well-defined and nonzero since $0 \notin \bigcup_{i=i_1}^{i_2 - 1} \overline{B}_i \subset \bigcup_{i=i_1}^{i_2-1} B_{r_i}(x_i)$.

Then $|x| < \delta$ implies $x \notin \bigcup_{i=i_1}^{i_2-1} B_i$, so

$$|d\phi(x) - I| \leq \max_{\substack{x' \in B_i \\ i \geq i_2}} \left\{ \left| d\phi_H^{|H_i - H|}(x') - I \right| \right\}$$

$$< \varepsilon \qquad \text{by (4) of Construction 3.4,}$$

and the choice of s_ε, i_2.

Thus $d\phi(x)$ is continuous near 0, and $d\phi(0) = I$ (the identity matrix). Hence $\mathcal{F}$ is a C^1 foliation and $T_0\mathcal{F} = H$, so that $\mathcal{F}$ is transverse to Y at 0 ($V \not\subset H$ by definition of H). But for all $i \geq i_1$, $T_{x_i}\mathcal{F} = T_{x_i}\mathcal{F}_i = H_i$ and $T_{x_i}X \subseteq H_i$, so that $\mathcal{F}$ is nontransverse to X at x_i. This shows that X is not $(\mathcal{F}^1)$-regular over Y at 0, proving Theorem 3.3.

<u>Verification of Construction 3.4</u> : It suffices to take $H = \mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n$ and $n = 2$. For $n > 2$ the calculations are similar.

Consider,
$$\begin{cases} y = \lambda + (1-\lambda^2)^2(x^2 - a^2)^2 & \lambda^2 \leq 1, \ x^2 \leq a^2 \\ y = \lambda & \lambda^2 \leq 1, \ a^2 \leq x^2 \leq 1, \end{cases}$$

with the constant a in $(0,1)$ to be chosen shortly.

We shall prove that this defines a C^1 foliation of $[-1,1]^2$ of codimension 1, with the leaves corresponding to fixed values of λ . (If $n > 2$, take $x_n = \lambda + (1-\lambda^2)^2(\sum_{i=1}^{n-1} x_i^2 - a^2)^2$, et cetera.)

Multiplying by $r/4$ gives a foliation of $[-r/4 , r/4]^2$ which fits into the ball $B_{\frac{1}{2}r}(0)$ and extends trivially to a foliation $\mathcal{F}_a$ of $B_r(0)$ which satisfies (1) . The leaf with normal vector furthest from $(0 : 1)$ is clearly given by $\lambda = 0$, and this normal is $(1 : \mp(8a^3)/(3\sqrt{3}))$ at the points $((4/9)a^4 , \pm a/\sqrt{3})$. (Compare Construction 2.2)

Write $\nu_a = (8a^3)/(3\sqrt{3})$. Then $|(1 : \nu_a) - (1 : 0)| = (\nu_a)/(1+\nu_a^2)^{\frac{1}{2}}$. So , given s , choose a such that

$$\frac{\nu_a^2}{1+\nu_a^2} = s^2 ,$$

$$\text{i.e.} \quad \nu_a^2 = \frac{s^2}{1-s^2} .$$

$$\text{Then} \quad a^6 = \frac{27 s^2}{64(1-s^2)} .$$

With this choice of a , (2) and (3) of 3.4 are satisfied.

Note that for $s \in [0,\frac{1}{2}]$ we have : $a^6 \leq 9/64$ (*) .

Define $\phi_a : [-1,1]^2 \longrightarrow [-1,1]^2$ by

$$\phi_a(x,y) = \begin{cases} (x,y) & a^2 \leq x^2 \leq 1 \\ (x,\ y + (1-y^2)^2(x^2 - a^2)^2) & x^2 \leq a^2 \end{cases}$$

ϕ_a is then a C^1 map. Elementary calculation using (*) shows that ϕ_a is injective. Now

$$d\phi_a(x,y) = \begin{pmatrix} 1 & 0 \\ 4x(x^2-a^2)(1-y^2)^2 & 1 - 4y(1-y^2)(x^2-a^2)^2 \end{pmatrix} \text{ if } x^2 \leq a^2 ,$$

and $d\phi_a(x,y)$ is the identity matrix if $a^2 \leq x^2 \leq 1$.

Calculation using (*) shows that $d\phi_a(x,y)$ is always nonsingular. Thus ϕ_a is a C^1 diffeomorphism of $[-1,1]^2$, which after scalar multiplication by $r/4$

as described above may be extended by the identity to a C^1 diffeomorphism of $B_r(0)$ since $d\phi_a(x,\pm 1)$ is the identity matrix. It defines the foliation.

ϕ^s_H will be the inverse of the resulting diffeomorphism. It only remains to verify (4) of Construction 3.4 , i.e. to show that $d(\phi_a^{-1})$ tends uniformly to the identity matrix as a tends to 0 ; but this follows from the same result for $d\phi_a$, and this in turn follows from the expression above.

Thus we have verified conditions (1) - (4) of Construction 3.4 .

<u>Corollary 3.5</u> : (a)-<u>regularity is a</u> C^1 <u>diffeomorphism invariant</u> .

<u>Proof</u> : $(\mathcal{F}^1)$ is clearly C^1 diffeomorphism invariant .

Having shown that transverse C^1 foliations detect (a)-faults, we give an example of an (a)-fault which is not detectable by transverse C^2 foliations, showing that Theorem 3.3 is sharp. The details of this example were worked out with the help of Anne Kambouchner.

<u>Example 3.6</u> : <u>An</u> (a)-<u>fault not detectable by transverse</u> C^2 <u>foliations</u>.

In $\mathbb{R}^3$ let (x, y, z) be coordinates, and let Y be the y-axis, and let X be $\left(\bigcap_{n=1}^{\infty}\{x = 0\ ,\ g_n \geqslant 0\ ,\ z > 0\}\right) \cup \left(\bigcup_{n=1}^{\infty}\{f_n = 0\ ,\ g_n \leqslant 0\}\right)$, where g_n is a function of y and z and $\{g_n \leq 0\}$ intersects $\{x = 0\}$ in a rectangle of length m_n , width $m_n r_n$, and $\{f_n = 0\}$ defines the barrow B_n of magnitude m_n , ratio r_n , axis $\{x = 0\ ,\ y + \tan(\theta_n)z = (1/2n) + (\tan\theta_n)/2n\}$, and centre $p_n = (0,\ 1/2n\ ,\ 1/2n)$ with base in the plane $\{x = 0\}$.(Cf. 2.3.)

First choose a monotonic decreasing sequence $\{m_n\}$ such that for any choice of θ_n , and any $r_n \leqslant 1$, the barrows are pairwise disjoint (and do not intersect Y). Now let δ_n be the radius of the largest 2-sphere $S^2_\delta(0)$ such that $S^2_\delta(0) \cap B_n \neq \emptyset$ when $r_n = 1$ and θ_n takes all values in $[-\pi/2\ ,\ \pi/2]$.

Then set $r_n = (3\sqrt{3}/8)\delta_n^{\frac{2}{3}}$ and $\theta_n = \sin^{-1}((3\sqrt{3}/8)(\delta_n^{\frac{2}{3}} + \delta_n^{\frac{2}{3}}))$, so defining

B_n completely, and hence specifying X.

(Note that $(3\sqrt{3}/4)\delta_n^{\frac{1}{3}} < 1$, i.e. $\delta_n < 64/81\sqrt{3}$, and so this choice of θ_n is possible for all $n \geq 1$, by the choice of the centre $p_1 = (0, \frac{1}{2}, \frac{1}{2})$ of B_1.)

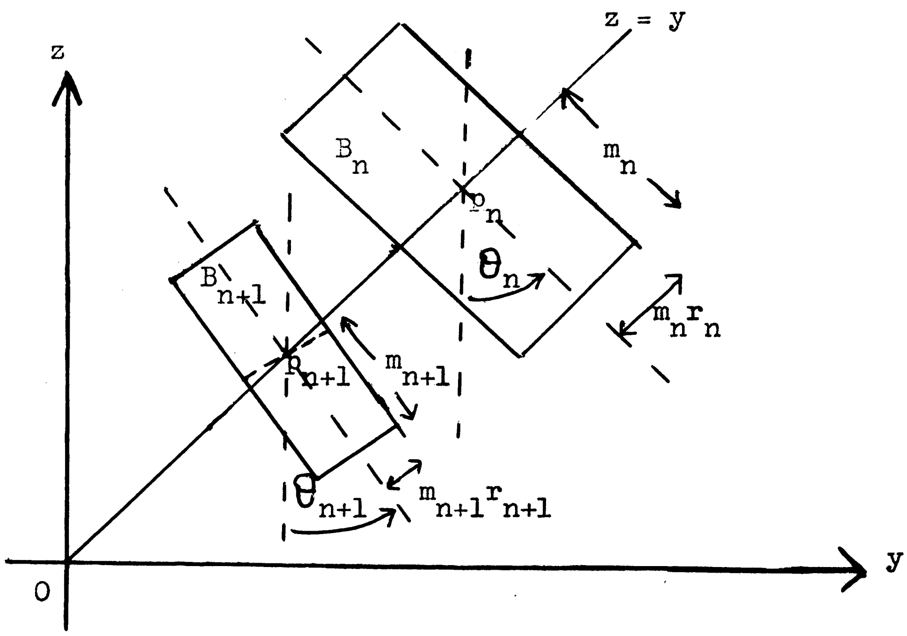


Figure : x = 0

Since $\{\delta_n\}$ is a monotonic decreasing sequence, tending to 0, both $\{r_n\}$ and $\{\theta_n\}$ are monotonis decreasing to 0. Thus (cf. Construction 2.3) the set of limiting normals to X at 0 is $\{(1 : \lambda : 0) : -8/3\sqrt{3} \leq \lambda \leq 8/3\sqrt{3}\}$. Hence (a) fails at 0 for the pair (X,Y).

Suppose $(\mathfrak{F}^2)$ does not hold at 0 for (X,Y). Then there is a C^2 foliation $\mathfrak{F}$ which is transverse to Y at 0 and which is not transverse to X in any neighbourhood of 0. Necessarily $\mathfrak{F}$ is of codimension 1 and $T_0\mathfrak{F}$ (the tangent at 0 to the leaf of $\mathfrak{F}$ passing through 0) must be of the form $(1 : \alpha : 0)$ where $0 < |\alpha| \leq 8/3\sqrt{3}$.

We shall show that there is a constant $C > 0$ and an n_0 such that for all $n \geq n_0$ and for all $p \in B_n$,

$$|N_pX - (1 : \alpha : 0)| > C\delta_n^{\frac{1}{3}} \qquad (*)$$

($N_p X$ is the normal space to X at p.) The proof of $(*)$ will be given later.

Let $\phi : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0)$ denote the C^2 diffeomorphism defining

$\mathcal{F}$ so that the leaves of $\mathcal{F}$ are the images of $\{\mathbb{R}^2 \times w\}_{w \in \mathbb{R}}$. Then $d\phi(0)(\mathbb{R}^2 \times 0)$ is the plane with normal $(1 : \alpha : 0)$.

Since ϕ is C^2, the map $(\mathbb{R}^3, 0) \longrightarrow (GL_3(\mathbb{R}), d\phi(0))$ is C^1 and
$$p \longmapsto d\phi(\phi^{-1}(p))$$
thus there exist $\varepsilon > 0$ and $M > 0$ such that

$$\left| d\phi(\phi^{-1}(p)) - d\phi(0) \right| < M|p| \quad , \text{ for all } p \in B_\varepsilon(0) .$$

It follows at once that

$$\left| (d\phi(\phi^{-1}(p)) - d\phi(0))\Big|_{\mathbb{R}^2 \times 0} \right| < M|p| \text{ , for all } p \in B_\varepsilon(0) \quad ,$$

or in other words that

$$\left| T_p\mathcal{F} - T_0\mathcal{F} \right| < M|p| \text{ , for all } p \in B_\varepsilon(0) .$$

Now, by hypothesis, $\mathcal{F}$ is nontransverse to X at some point of B_n, for infinitely many n, i.e. for infinitely many n, there exists $p \in B_n$ such that $T_p\mathcal{F} = T_pX$. Let $n_1 \geq n_0$ be such that for all $n \geq n_1$, if $p \in B_n$, then $|p| < \varepsilon$. Then for infinitely many $n \geq n_1$, there exists $p \in B_n$ such that $M|p| > |N_pX - (1 : \alpha : 0)|$. But assuming (*) and using the choice of δ_n, we know that for all $n \geq n_0$, and for all $p \in B_n$, $|N_pX - (1 : \alpha : 0)| > C|p|^{\frac{1}{3}}$. These last two inequalities are absurd, since there is some n_2 such that for all $n \geq n_2$, and for all $p \in B_n$, $|p| < (C/M)^{3/2}$, i.e. $M|p| < C|p|^{\frac{1}{3}}$. Thus we obtain a contradiction, showing that $(\mathcal{F}^2)$ holds, and that transverse C^2 foliations cannot detect this (a)-fault.

<u>Proof of (*)</u> : A short calculation shows that for all n the set of normals to B_n (rotated back through θ_n), is contained in

$$\left\{ (1 : \lambda : \mu) : \lambda \in [-8/3\sqrt{3} , 8/3\sqrt{3}] , \mu \in [-8r_n/3\sqrt{3} , 8r_n/3\sqrt{3}] \right\} .$$

It will suffice to establish (*) in the euclidean norm $|\ |_e$ in the usual chart for $P^2(\mathbb{R})$ centred at $(1:0:0)$ given by the homogeneous coordinates $(\nu : \lambda : \mu) \longmapsto (\lambda/\nu , \mu/\nu)$, since this norm is equivalent to the standard one. ($|(1:\lambda':\mu') - (1:\lambda'':\mu'')|_e = ((\lambda' - \lambda'')^2 + (\mu' - \mu'')^2)^{\frac{1}{2}}$.)

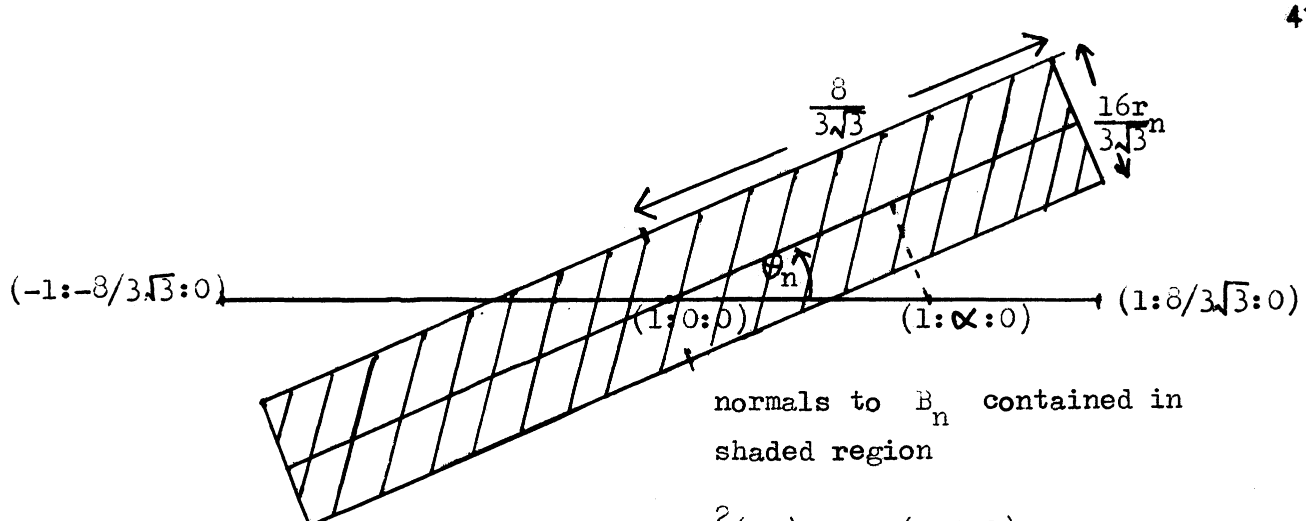


Figure : Chart for $P^2(\mathbb{R})$ at $(1:0:0)$.

It is evident from the figure above and the choice of r_n and θ_n that there exists n' such that for all $n \geq n'$, $(1:\alpha:0)$ is outside the shaded region which contains the normals to B_n . We calculate the minimal distance of $(1:\alpha:0)$ from a normal of B_n . This is clearly $(\alpha \sin\theta_n - 8r_n/3\sqrt{3})$. Thus for all $n \geq n'$ and all $p \in B_n$,

$$\begin{aligned} |N_pX - (1:\alpha:0)|_e &\geq \alpha \sin\theta_n - 8r_n/3\sqrt{3} \\ &= \alpha(3\sqrt{3}/8)(\delta_n^{\frac{1}{3}} + \delta_n^{\frac{2}{3}}) - \delta_n^{\frac{2}{3}} \\ &= \delta_n^{\frac{1}{3}}(\, (3\sqrt{3}\,\alpha/8) - \delta_n^{\frac{1}{3}}(1 - (3\sqrt{3}\alpha/8))\,) \,. \end{aligned}$$

Since δ_n tends to 0 as n tends to ∞ , there exists $n_0 \geq n'$ such that for all $n \geq n_0$, and all $p \in B_n$,

$$|N_pX - (1:\alpha:0)|_e > (3\sqrt{3}\alpha/16)\delta_n^{\frac{1}{3}} \,.$$

Thus we obtain $(*)$.

Note 3.7 : We have in fact proved slightly more by the above example. Namely that a transverse foliation, with C^1 leaves, which is C^1 with a Lipschitz derivative in the direction transverse to the leaves, cannot detect this (a)-fault. If $(\mathcal{F}^{1,p})$ denotes the condition similar to $(\mathcal{F}^1)$ but restricting to foliations defined by a C^1 diffeomorphism C^1 along the leaves and C^p transverse to the leaves, then clearly $(\mathcal{F}^{1,p})$ implies $(\mathcal{F}^{1,q})$ if $p < q$ (and $(\mathcal{F}^{1,p})$ implies (t) for all $p \leq \infty$). Also it is (now) easy to construct examples showing $(\mathcal{F}^{1,q})$ does not imply $(\mathcal{F}^{1,p})$ when $p < q$. Simply set

$$\theta_n = \sin^{-1}(3\sqrt{3}(\delta_n^{p-\frac{2}{3}} + \delta_n^{2p-\frac{4}{3}})/8) ,$$
$$r_n = (3\sqrt{3}\,\delta_n^{2p-\frac{4}{3}})/8 ,$$

and repeat the argument of 3.6 .

4. Detecting Thom faults in stratified mappings.

Since the regularity condition imposed on a stratified morphism is formally very similar to (a)-regularity we note here the analogues of the results we have proved about (a)-regularity in §§1-3.

Following [6] , let $f : N \longrightarrow P$ be a C^1 map, between C^1 manifolds N and P , and let X and Y be C^1 submanifolds of N such that $f|_X$ and $f|_Y$ have constant rank, and let $0 \in Y \subset \overline{X} - X$. We say that X <u>is</u> <u>(a_f)-regular over</u> Y <u>at</u> 0 (in the terminology of Gibson [6], X <u>is Thom regular over</u> Y <u>at</u> 0 <u>relative to</u> f) if,

(a_f) Given a sequence $\{x_i\}$ in X , such that x_i tends to 0 as i tends to ∞, and $\ker d_{x_i}(f|_X)$ converges to a plane τ , then $\ker d_0(f|_Y) \subseteq \tau$.

Since $f|_X$ is of constant rank, the fibres of $f|_X$ form the leaves of a foliation $\mathcal{F}_X^f$ of X , and similarly for Y . Thus (a_f) may be stated,

(a_f) Given a sequence $\{x_i\}$ in X , such that x_i tends to 0 as i tends to ∞, and $T_{x_i}(\mathcal{F}_X^f)$ converges to a plane τ , then $T_0(\mathcal{F}_Y^f) \subseteq \tau$.

Here $T_0(\mathcal{F}_Y^f)$ denotes the tangent space at 0 to the leaf of $\mathcal{F}_Y^f$ passing through 0 .

The natural analogue of (t)-regularity is,

(t_f) Given a C^1 submanifold S such that S is transverse to $\mathcal{F}_Y^f$ at 0, there is a neighbourhood of 0 in which S is transverse to $\mathcal{F}_X^f$.

Similarly the analogue of $(\mathcal{F}^k)$ is,

$(\mathcal{F}_f^k)$ Given a C^k foliation $\mathcal{F}$ of N transverse to $\mathcal{F}_Y^f$ at 0, there is a neighbourhood of 0 in which $\mathcal{F}$ is transverse to $\mathcal{F}_X^f$.

Note 4.1 : (i) Another way to say that S is transverse to $\mathcal{F}_Y^f$ at 0 is to say that the rank of $f|_{S \cap Y}$ at 0 equals the rank of $f|_Y$.

(ii) If f has rank zero on X and Y then (a_f), (t_f), $(\mathcal{F}_f^k)$ become (a), (t), $(\mathcal{F}^k)$ respectively.

With these definitions all of the results proved in §2 and §3 have corresponding versions, with just some nuances.

Thus, $(a_f) \Longleftrightarrow (\mathcal{F}_f^1) \Longrightarrow (t_f)$ by merely mimicking the proofs that $(a) \Longleftrightarrow (\mathcal{F}^1) \Longrightarrow (t)$.

Example 4.2 : Take Example 2.1 and define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $(x, y, z) \longmapsto z$. (a_f) fails since the tangent to $\mathcal{F}_X^f$ at x_n will be the vector $(2, 1, 0)$ for all n. (t_f) holds since no submanifold transverse to Y intersects the horn containing the sequences on which (a_f) fails. ($\mathcal{F}_Y^f$ is the trivial foliation with one leaf.) Thus (t_f) does not imply (a_f).

Example 4.3 : If we define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $(x, y, z) \longmapsto z$ and examine Example 2.4 we find that X is not (t_f)-regular over Y at 0 : it is easy to find a C^1 submanifold, with tangent plane at the origin spanned by the lines $\{z = y , x = 0\}$ and $\{z = 0 , y = x\}$, which is not transverse to $\mathcal{F}_X^f$ on a sequence of points in X tending to 0.

To obtain an example with (t_f) and not (a_f) we can either take f to be the constant map (see Note 4.1 (ii)), or add a fourth variable w, and consider $X_1 = X \times \mathbb{R}$, $Y_1 = Y \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$ and let $f : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ be projection $(x, y, z, w) \longmapsto w$. Then X_1 is (t_f)-regular over Y_1 at 0, but not (a_f)-regular.

<u>Example 4.4</u> : As in Example 4.3 we take Example 3.6, let $X_1 = X \times \mathbb{R}$, $Y_1 = Y \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$, and take $f : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ to be projection $(x, y, z, w) \longmapsto w$. Then X_1 is $(\mathcal{F}^2_f)$-regular over Y_1 at 0, but not (a_f)-regular. (X is neither $(\mathcal{F}^2_f)$-regular nor (a_f)-regular over Y at 0.) Thus $(\mathcal{F}^2_f)$-regularity does not imply (a_f)-regularity.

The next result is an analogue of Theorem 2.5 .

<u>Theorem 4.5</u> : <u>Let</u> X, Y <u>be</u> C^1 <u>submanifolds of</u> $\mathbb{R}^n$, <u>and let</u> $0 \in Y \subset \overline{X} - X$, <u>and let</u> X <u>be a subanalytic set. Let</u> $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ <u>be a subanalytic map (i.e. the graph of</u> f <u>is subanalytic in</u> $\mathbb{R}^n \times \mathbb{R}^p$), <u>such that</u> $f|_X$ <u>and</u> $f|_Y$ <u>are of constant rank. Then</u> X <u>is</u> (a_f)-<u>regular over</u> Y <u>at</u> 0 <u>if and only if for every semianalytic</u> C^1 <u>submanifold</u> S <u>transverse to</u> $\mathcal{F}^f_Y$ <u>at</u> 0, <u>there is some neighbourhood of</u> 0 <u>in which</u> S <u>is transverse to</u> $\mathcal{F}^f_X$.

<u>Proof</u> : The proof is similar to that of Theorem 2.5 , save that instead of proving that $\{(x,T_xX) : x \in X\}$ is subanalytic, we must prove that $\{(x,T_x(\mathcal{F}^f_X) : x \in X\}$ is subanalytic. But this reduces to proving that $\{(x,T_xX) : x \in X\}$ is subanalytic. For, $T_x(\mathcal{F}^f_X) = \ker d_x(f|_X) = \ker d_xf \cap T_xX$, and $\ker d_xf$ is a fixed subspace of $\mathbb{R}^n$ if we suppose (as we can) that f is a linear projection, since f is the composition of an embedding onto its graph followed by a linear projection (cf. page 30 of [6]). Theorem 4.5 follows.

Finally we consider a possible analogue of Theorem 1.1 . Let $g : M \longrightarrow N$

and $f : N \longrightarrow P$ be C^1 maps between C^1 manifolds, and X a submanifold of N . Then,

$$g \pitchfork \ker d_x(f|_X) \text{ for all } x \in X \iff g \pitchfork \mathcal{F}^f_X$$
$$\iff g \pitchfork \text{fibres of } f|_X$$
$$\iff f|_X \circ g : M \longrightarrow f(X) \text{ is a submersion.}$$

Then the analogue of Theorem 1.1 is as follows, writing " $g \pitchfork \mathcal{F}^f_\Sigma$ " for " $g \pitchfork \mathcal{F}^f_X$ for all X in Σ " .

Hypothesis 4.6 : *Let* Σ *be a locally finite stratification of a closed subset* V *of a* C^1 *manifold* M , *and let* $f : M \longrightarrow P$ *be a* C^1 *map,* P *a* C^1 *manifold, such that for each stratum* X *of* Σ , $f|_X$ *has constant rank. Then the following conditions are equivalent* :

(1) Σ *is* (a_f)*-regular,*

(2) *for every* C^1 *manifold* N , $\{z \in J^1(N,M) : z \pitchfork \mathcal{F}^f_\Sigma\}$ *is open in* $J^1(N,M)$,

(3) *for every* C^1 *manifold* N , $\{g \in C^1(N,M) : g \pitchfork \mathcal{F}^f_\Sigma\}$ *is open in* $C^1(N,M)$ *with the strong* C^1 *topology,*

(4) *there is some integer* r , $1 \leq r \leq \max(1, \min_{X \in \Sigma} (\operatorname{rank} f|_X))$, *and some* C^1 *manifold* N *with* $\dim N = \dim M - r$, *for which* $\{g \in C^1(N,M) : g \pitchfork \mathcal{F}^f_\Sigma\}$ *is open in* $C^1(N,M)$ *with the strong* C^1 *topology.*

One can prove $(1) \iff (2) \Longrightarrow (3) \Longrightarrow (4)$ without much difficulty, by copying the proof of Theorem 1.1 . To make Hypothesis 4.6 into a theorem we must prove (4) implies (1) . If we try to copy the proof that (4) implies (1) in Theorem 1,1 we arrive at,

Question 4.7 : If X is a C^1 submanifold of $\mathbb{R}^m$, $0 \in \overline{X} - X$, and $f : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a C^1 map such that $f|_X$ has constant rank, then given a plane H and a C^1 manifold N with $\dim N = \dim H$, and $n \in N$, is $\{g \in C^1(N,\mathbb{R}^m) : g \pitchfork \mathcal{F}^f_X , g(n) = 0 , d_n g(T_n N) = H\}$ nonempty ? *

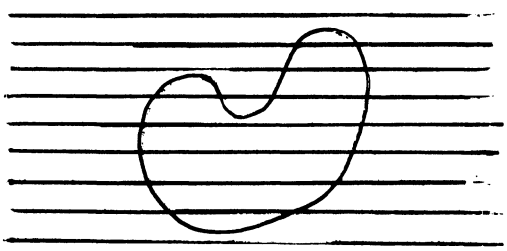
* See over.

A positive answer to Question 4.7 would suffice to prove Hypothesis 4.6. To prove that (3) implies (1) it suffices to answer Question 4.8 , which is *a priori* weaker than 4.7 .

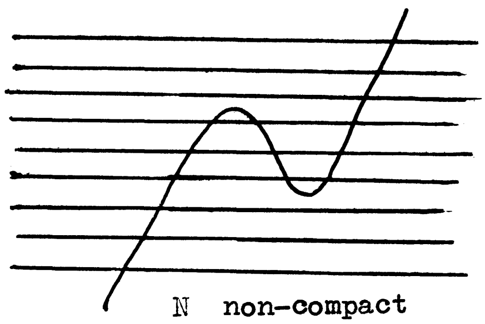
Question 4.8 : Is there some C^1 manifold N for which Question 4.7 has a positive response ? ✱

Note 4.9 : The proof of Lemma 1.3 made use of the local transversality lemma : the set of C^1 maps transverse to a submanifold on a compact coordinate disc is open and dense. The corresponding statement that C^1 maps transverse to the leaves of a foliation on a compact coordinate disc be dense is clearly false (although openness is easy). (Cf. page 193 of [42] .)

Consider



or



So another method of proof is required to attack (3) implies (1) of Hypothesis 4.6 .

Observe also that the figures above show that the set of C^1 maps transverse to $\mathcal{F}^f_\Sigma$ is not dense (cf. Note 1.2 (iv)).

Finally we remark that the results of §§1-3 could also be extended to the "generalised condition (a) for O-bundles" of M.-H. Schwartz in [27] .

✱ An example of David Epstein shows that the answer to Questions 4.7 and 4.8 is no. However Hypothesis 4.6 is still undecided : a finer study is needed.

CHAPTER 2. WHITNEY (b)-REGULARITY

In this chapter we consider various natural ways of detecting (b)-faults.

The most striking property of (b)-regularity in the theory of smooth stratified objects is that a (b)-regular stratification is locally topologically trivial, as proved by Mather in [21]. The proof shows <u>en route</u> that (b)-regularity implies a condition we have called (b_s) in [38], namely that for any C^1 tubular neighbourhood of the base stratum, associated to which are a retraction π and a distance function ρ, the fibres of $(\pi \times \rho)$ (which are embedded spheres) are transverse to the attaching stratum. This has an exact counterpart in the implication (a) implies (a_s) (see §3). In [43] C. T. C. Wall conjectured that (a_s) implied (a) and that (b_s) implied (b); we proved these implications in the semianalytic case in [37] and [38]. In Chapter 1 (Theorem 3.3) we have shown that (a_s) implies (a) in general, by perturbing a transverse foliation with an infinite sequence of ripples so as to detect a given (a)-fault. The same idea will be used in §5 to prove that (b_s) implies (b); this time we use the ripples (of 3.4) to perturb a foliation by spheres (the fibres of $\pi \times \rho$) of the complement of the base stratum, so as to detect a given (b)-fault.

In §6 we study how (b)-regularity behaves with respect to generic sections. We show that, if Y is linear, and if, for a generic set of linear spaces H containing Y, $(X \cap H, Y)_0$ is (b)-regular, then any (b)-fault of (X,Y) at 0 cannot be too "deep". Conversely, we show that if (X,Y) is (b)-regular at 0, then for generic such H, $(X \cap H, Y)$ is (b)-regular at 0.

Knowing that (b)-regularity is generic for subanalytic sets — see the introduction — it is natural to ask what are the strongest generic regularity conditions. In [40] J.-L. Verdier introduced (w)-regularity, proved that it implied (b)-regularity, and showed that it was generic (and also that it gave local trivialisations by integrating <u>continuous</u> vector fields tangent to the strata, whereas the vector fields resulting from (b)-regularity may theoretically be discontinuous). (w)-regularity is easily seen to imply Kuo's ratio test (r), and hence (r) too is generic. In §7 we give examples which show that even for semialgebraic strata, (b) , (r) and (w) are distinct, and that (r) and (w) are not invariant under C^1 diffeomorphisms, although they are preserved by C^2 diffeomorphisms.

5. (b)-regularity and tubular neighbourhoods.

Following Mather in [22] , we first define what is meant by a C^1 tubular neighbourhood.

<u>Definition 5.1</u> : Let X be a C^1 submanifold of a C^1 manifold M . A C^1 <u>tubular neighbourhood</u> T of X in M is a quadruple (p,E,ε,ϕ) where $p : E \longrightarrow X$ is an inner product bundle of class C^1 , $\varepsilon : X \longrightarrow \mathbb{R}^+$ is a positive C^1 function on X , and ϕ is a C^1 diffeomorphism of $B_\varepsilon = \{e \in E : \|e\| < \varepsilon(\pi(e))\}$ onto an open subset of M which commutes with the zero section ζ of E :

$$\begin{array}{ccc} B_\varepsilon & & \\ \uparrow \zeta & \searrow \phi & \\ X & \hookrightarrow & M \end{array}$$

We set $|T| = \phi(B_\varepsilon)$. The map $\pi_T = p \circ \phi^{-1} : |T| \longrightarrow X$ will be called the C^1 <u>retraction</u> π_T <u>associated to</u> T , and the non-negative function $\rho_T = \rho_E \circ \phi^{-1} : |T| \longrightarrow \mathbb{R}$, where $\rho_E(e) = \|e\|^2$ for $e \in E$, will be called the C^1 <u>distance function</u> ρ_T <u>associated to</u> T .

(We have, similarly, C^r tubular neighbourhoods.)

It is clear that the map $(\pi_T, \rho_T) : |T| - X \longrightarrow X \times \mathbb{R}$ is a submersion .

As what follows will be entirely local, we can restrict to the situation of adjacent strata in $\mathbb{R}^n$.

Let X , Y be C^1 submanifolds of $\mathbb{R}^n$ and let $0 \in Y \subset \overline{X} - X$. We say that X is (b_s)-<u>regular over</u> Y if for all C^1 tubular neighbourhoods T of Y , there is a neighbourhood N of Y in $|T|$ such that $(\pi_T, \rho_T)|_{X \cap N}$ is a submersion.

Given a C^1 chart for Y at 0 ,

$$\phi : (U , U \cap Y , 0) \longrightarrow (\mathbb{R}^n , \mathbb{R}^m \times 0^{n-m} , 0) ,$$

the standard tubular neighbourhood of $\mathbb{R}^m \times 0^{n-m}$ in $\mathbb{R}^n$ provides a retraction $\pi_\phi = \phi^{-1} \circ \pi_m \circ \phi : U \longrightarrow Y \cap U$, where $\pi_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m \times 0^{n-m}$ is linear projection taking $(x_1 , \dots , x_n)$ to $(x_1 , \dots , x_m , 0 , \dots , 0)$, and a distance function $\rho_\phi = \rho \circ \phi : U \longrightarrow \mathbb{R}^+$, where $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is the function $\rho(x_1 , \dots , x_n) = \sum_{i=m+1}^{n} x_i^2$. We refer to the tubular neighbourhood T_ϕ of $U \cap Y$ in U .

We say X is (b_s)-<u>regular over</u> Y <u>at</u> 0 when,

(b_s) Given a C^1 chart (U , ϕ) at 0 for Y as a C^1 submanifold of $\mathbb{R}^n$, there is a neighbourhood U' of 0 , $U' \subset U$, such that $(\pi_\phi , \rho_\phi)|_{X \cap U'}$ is a submersion.

The following lemma justifies our use of the term (b_s)-regularity in the local and global cases.

<u>Lemma 5.2</u> : X <u>is</u> (b_s)-<u>regular over</u> Y <u>if and only if</u> X <u>is</u> (b_s)-<u>regular over</u> Y <u>at</u> y , <u>for all</u> $y \in Y$.

<u>Proof</u> : " If " : Given a sequence of points on X tending to Y , at which $(\pi_T , \rho_T)|_X$ is not submersive, there must be some convergent subsequence with a limit y_0 in Y . The implication follows.

" Only if " : Given a point y_0 of Y and a C^1 tubular neighbourhood T_ϕ of a neighbourhood $U \cap Y$ of y_0 in Y defined by a C^1 chart (U , ϕ)

for Y at y_0, it will suffice to find a C^1 tubular neighbourhood T of Y and a neighbourhood U' of y_0, $U' \subset U$, such that $T|_{U' \cap Y} = T_\phi|_{U' \cap Y}$. This follows from the Tubular Neighbourhood Theorem of [22], which is proved in [21].

For a simpler proof, let ψ be a C^1 diffeomorphism of $\mathbb{R}^n$ which is the identity outside some neighbourhood of y_0, and such that there is a smaller neighbourhood W of y_0, $W \subset U$, such that the fibres of the retraction $\psi \circ \pi_\phi \circ \psi^{-1}$ intersect $\psi(W)$ in a C^1 field of planes transverse to $\psi(Y)$, and such that $\rho_\phi \circ \psi^{-1}$ is the square of the function measuring distance from $\psi(Y)$ in R^n. Extend this local C^1 field to a globally defined (over $\psi(Y)$) C^1 field of planes (whose dimension is the codimension of Y) transverse to $\psi(Y)$. In Theorem 4.5.1 of [13] Hirsch shows how to obtain a tubular neighbourhood of $\psi(Y)$, so that the transverse planes contain the fibres of the associated retraction. There is also a very careful proof of this fact by Munkres on page 51 of [54]. Pulling back by ψ^{-1} we have a tubular neighbourhood T of Y with the required properties. This completes the proof of Lemma 5.2.

In [43] C. T. C. Wall conjectured that (b_s)-regularity is a necessary and sufficient condition for (b)-regularity. Applying Lemma 5.2, together with the convention that X is (b)-regular over Y when X is (b)-regular over Y at y for all y in Y, we see that the local and global versions of the conjecture are equivalent. We now prove the local version.

<u>Theorem 5.3</u> : <u>Let</u> X , Y <u>be disjoint</u> C^1 <u>submanifolds of</u> $\mathbb{R}^n$, <u>and let</u> $0 \in Y$. <u>Then</u> X <u>is</u> (b)-<u>regular over</u> Y <u>at</u> 0 <u>if and only if</u> X <u>is</u> (b_s)-<u>regular over</u> Y <u>at</u> 0 .

<u>Proof</u> : " Only if " was proved by Mather as Lemma 7.3 in [21], and in fact in 1964 by Thom on page 10 of [35]. For another published proof see Lemma 2.3 of [48].

It is left to prove " if ".

Suppose X is (b_s)-regular over Y at 0. It follows at once that X is (a_s)-regular over Y at 0 (see §3), so that we can apply Theorem 3.3 to show that (a) holds. Suppose (b) fails : we shall derive a contradiction.

By (0.4), (b') must fail for every C^1 retraction onto Y.

Let π_1 (resp. π_2) be the local linear retraction defined near 0 of $\mathbb{R}^n$ onto Y (resp. T_0Y) orthogonal to T_0Y. Then (b') fails for π_1, and there is a sequence $\{x_i\}$ in X tending to 0 such that $\lambda_i = \frac{x_i\pi_1(x_i)}{|x_i\pi_1(x_i)|}$ tends to a limit λ, and $T_{x_i}X$ tends to a limit τ, and $\lambda \not\subset \tau$.

The C^1 diffeomorphism defined near 0,

$$\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$p \longmapsto p + (\pi_2(p) - \pi_1(p))$$

preserves $\{\lambda_i\}$, λ and τ, and sends Y onto T_0Y, hence we may identify Y with $\mathbb{R}^m \times 0^{n-m}$ in $\mathbb{R}^n$. Write $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m \times 0^{n-m}$ for the projection mapping $(x_1,\dots,x_n)$ to $(x_1,\dots,x_m,0,\dots,0)$. Then, continuing to write $\{x_i\}$ and X for their images by α, we have that $\lambda_i = \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|}$ tends to λ, which is not contained in $\tau = \lim T_{x_i}X$.

Now let A be a linear automorphism of $0^m \times \mathbb{R}^{n-m}$ such that $A(\lambda)$ and $A(\tau \cap \mathbb{R}^{n-m})$ are orthogonal. By applying the linear change of coördinates $(I_m, A) : \mathbb{R}^m \times \mathbb{R}^{n-m} \circlearrowleft$ we may suppose that λ and τ are orthogonal. The function measuring distance from Y is $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, taking $(x_1,\dots,x_n)$ to $\sum_{i=m+1}^{n} x_i^2$. We shall construct a C^1 diffeomorphism ϕ of $\mathbb{R}^n$ with $\phi|_{\mathbb{R}^m \times 0^{n-m}} =$ identity, such that the tangent space to X is contained in the tangent space to the fibre of $\rho_\phi = \rho\circ\phi$ on an infinite subsequence of the sequence $\{x_i\}$, so that (b_s) fails for (X, Y) at 0.

As in the proof of Theorem 3.3, pick an infinite sequence of pairwise disjoint balls $B_{r_i}(x_i) = B_i$ with centre x_i and of radius r_i. Then $0 \notin B_i$ for all i. We shall obtain ϕ by perturbing the foliation of $\mathbb{R}^n - (\mathbb{R}^m \times 0^{n-m})$ by the level hypersurfaces of ρ, within each B_i.

Let $H = \lambda^{\perp} \in G^{n}_{n-1}(\mathbb{R})$, and note that $H = \tau \oplus (\tau \oplus \lambda)^{\perp}$ because τ and λ have been assumed orthogonal. Since $T_{x_i}X$ tends to τ, and λ_i tends to λ, as i tends to ∞, there is some i_o such that $i \geq i_o$ implies $\lambda_i \not\subset T_{x_i}X$. Then for all $i \geq i_o$ we define a hyperplane

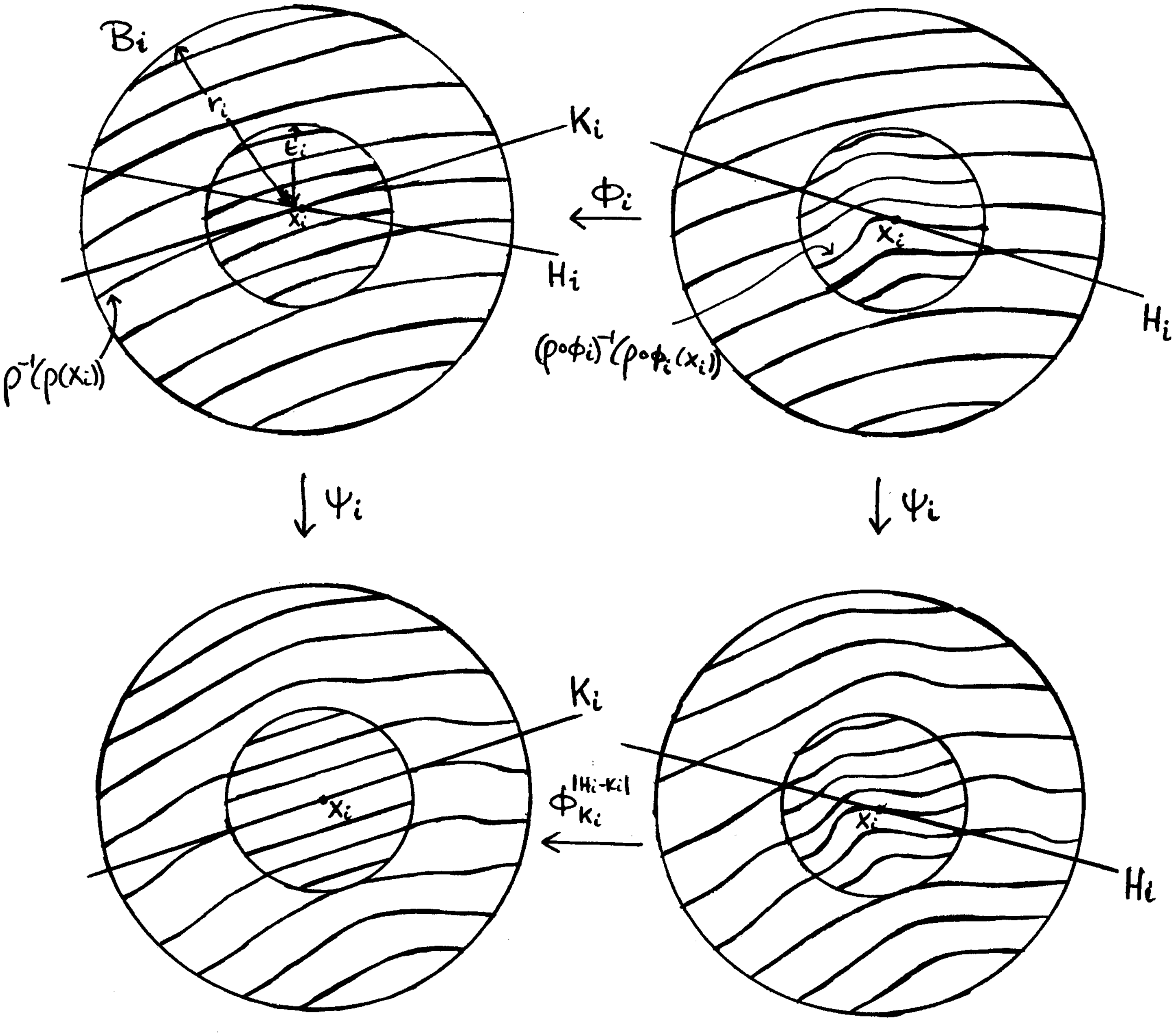
$$H_i = T_{x_i}X \oplus (T_{x_i}X \oplus \lambda_i)^{\perp} \subset T_{x_i}\mathbb{R}^n .$$

H_i tends to H as i tends to ∞. Pick $i_1 \geq i_o$ such that $|H_i - H| < 1/4$ for $i \geq i_1$.

Let $\delta_i > 0$. Then it is clear that we can find a C^1 diffeomorphism $\psi_i : (B_i , x_i) \circlearrowleft$, equal to the identity near ∂B_i, such that $d\psi_i(x_i) = I_n$ (the identity matrix), $|j^1(\psi_i)(p) - j^1(id_{\mathbb{R}^n})(p)| < \delta_i$ and $|j^1(\psi_i^{-1})(p) - j^1(id_{\mathbb{R}^n})(p)| < \delta_i$ for all $p \in B_i$, and such that for some t_i, $0 < t_i < r_i$, the image by ψ_i of the foliation of $B_{t_i}(x_i)$ by the level hypersurfaces of ρ is the trivial foliation by hyperplanes parallel with $K_i = T_{x_i}(\rho^{-1}(\rho(x_i)))$. Now $K_i = \lambda_i^{\perp}$, by definition of λ_i, and so K_i tends to $H = \lambda^{\perp} = (\lim \lambda_i)^{\perp}$ as i tends to ∞. Pick $i_2 \geq i_1$ such that $| K_i - H | < 1/4$ for all $i \geq i_2$. Then $|K_i - H_i| \leq 1/2$ for $i \geq i_2$, by our choice of i_1 and i_2.

For all $i \geq i_2$ we now perturb the trivial foliation of $B_{t_i}(x_i)$ by planes parallel with K_i by placing inside $B_{t_i}(x_i)$ a "ripple" : a foliated ball $B_{\frac{1}{2}t_i}(y_i)$ of radius $\frac{1}{2}t_i$, centre y_i, with the foliation $\mathcal{F}^{|H_i-K_i|}_{K_i}$ given by Construction 3.4, such that $x_i = x_{H_i}$ (the tangent at x_i to the leaf of the foliation passing through x_i is H_i). In the notation of 3.4, $\phi^{|H_i-K_i|}_{K_i}$ is the C^1 diffeomorphism defining the resulting foliation of $B_{t_i}(x_i)$, and we may extend $\phi^{|H_i-K_i|}_{K_i}$ by the identity to the rest of B_i.

Set $\phi_i = \psi_i \circ \phi^{|H_i-K_i|}_{K_i} \circ \psi_i^{-1} : B_i \circlearrowleft$. ϕ_i is a C^1 diffeomorphism, and the tangent space at x_i to $(\rho \circ \phi_i)^{-1}(\rho(\phi_i(x_i)))$ is H_i which contains $T_{x_i}X$ by definition (we have used here for the second time that $d\psi_i(x_i) = I_n$). Compare the figure overleaf.



<u>Figure : Construction of</u> Φ_i .

We have yet to fix δ_i . It is easy to verify that $\sup_{p\in B_i} |d\phi_i(p) - I_n|$ may be set as near as we please to $\sup_{p\in B_i} |d\phi_{K_i}^{|H_i-K_i|}(p) - I_n|$, by choosing δ_i small.

Let δ_i be chosen such that,

$$\sup_{p\in B_i} |d\phi_i(p) - I_n| \leq 2 \sup_{p\in B_i} |d\phi_{K_i}^{|H_i-K_i|}(p) - I_n| . \qquad (*)$$

Define $\phi : \mathbb{R}^n \circlearrowleft$ by setting $\phi|_{\mathbb{R}^n-(\bigcup_{i\geq i_2} B_i)}$ = identity , and $\phi|_{B_i} = \phi_i$ for $i \geq i_2$. To verify that ϕ is a C^1 diffeomorphism it is enough to check that $d\phi(p)$ is continuous at 0 , and that $d\phi(0) = I_n$.

Given $\varepsilon > 0$, (4) of Construction 3.4 gives an $s_{\frac{1}{2}\varepsilon} > 0$. Pick $i_3 \geq i_2$ such that $|H_i - H|$ and $|K_i - H|$ are each less than $\frac{1}{2}s_{\frac{1}{2}\varepsilon}$ for all $i \geq i_3$. Then $|H_i - K_i| < s_{\frac{1}{2}\varepsilon}$ for all $i \geq i_3$. Let $\delta = \min_{\substack{p\in B_i \\ i_2\leq i<i_3}} \{|p|\}$.

Then δ is well-defined and nonzero since $0 \notin \bigcup_{i=i_2}^{i_3-1} B_i$.

Let $p \in \mathbb{R}^n$ be such that $|p| < \delta$. Then $p \notin \bigcup_{i=i_2}^{i_3-1} B_i$, and thus

$$\begin{aligned}
|d\phi(p) - I_n| &\leq \max_{\substack{p'\in B_i \\ i\geq i_3}} \{|d\phi_i(p') - I_n|\} \\
&\leq 2 \max_{\substack{p'\in B_i \\ i\geq i_3}} \{|d\phi_{K_i}^{|H_i-K_i|}(p') - I_n|\} && \text{(by (*))} \\
&\leq 2. \tfrac{1}{2}\varepsilon && \text{(by choice of } i_3 \text{ and } s_{\frac{1}{2}\varepsilon} \text{ - see 3.4)} \\
&= \varepsilon .
\end{aligned}$$

Hence $d\phi(p)$ is continuous at 0 , and $d\phi(0)$ is the identity matrix.

By construction, the fibre of $\rho_\phi = \rho\circ\phi$ is not transverse to X at x_i, and hence neither is the fibre of $(\pi_\phi, \rho_\phi) = (\pi\circ\phi, \rho\circ\phi)$, so that $(\pi_\phi, \rho_\phi)|_X$ is not a submersion near x_i. Hence we have shown that X fails to be (b_s)-regular over Y at 0, using the hypothesis that X is not (b)-regular over Y at 0.

This completes the proof of Theorem 5.3.

Corollary 5.4 : (b)-regularity is a C^1 invariant.

Example 5.5 : Theorem 5.3 is sharp : C^2 tubular neighbourhoods do not detect all (b)-faults. Consider once again Example 3.6. There we have a (b)-fault, since it is an (a)-fault. However for all C^1 distance functions ρ (associated to a C^1 tubular neighbourhood), the fibres of ρ are transverse to X near 0. For, all limiting tangent planes to X at 0 contain the z-axis, and near 0 all points (x, y, z) on X have x/z small, and at such points the normal to the fibre of ρ will be close to $(0 : 0 : 1)$. (To see that near 0, if (x, y, z) is on X, then x/z is small, notice that the x-coordinate of the points in each barrow B_n is bounded above by $m_n r_n$, while the z-coordinate is bounded below by m_n, and r_n tends to 0 as n tends to ∞ and we approach 0.)

Since we have shown in 3.6 that all C^2 retractions have their fibres transverse to X near 0, it follows that for all C^2 tubular neighbourhoods T of Y, the fibres of (π_T, ρ_T) are transverse to X near 0.

Note 5.6 : A semianalytic version of 5.3.

We refer to [38] for a proof that (b_s) implies (b) when X and Y

are semianalytic. A careful reading of the proof in [38] shows that semianalytic (b)-faults can be detected by C^1 semianalytic tubular neighbourhoods, i.e. we can suppose the maps in the definition of tubular neighbourhood to have semianalytic graphs.

Note 5.7 : On μ-constant implies topological triviality.

In [17] Lê Dũng Tràng and Ramanujam prove that for a family of complex hypersurfaces (with isolated singularity) defined by

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^k , 0 \times \mathbb{C}^k) \longrightarrow (\mathbb{C} , 0)$$

with $F(z, t) = F_t(z)$, that $\mu(F_t)$ constant implies that the topological type of $F_t^{-1}(0)$ is constant, provided $n \neq 2$. Timourian has proved further that the family is topologically trivial (see [33]).

If one could prove that $\mu(F_t)$ constant implied the existence of a C^1 tubular neighbourhood T of $0 \times \mathbb{C}^k$ with the fibres of (π_T, ρ_T) transverse to $F^{-1}(0)$ near 0 , one could then apply the proof of Mather in [21] to give topological triviality, so removing the restriction $n \neq 2$. Applying Theorem 5.3 , we know from the counterexamples of Briançon and Speder in [2] that $\mu(F_t)$ constant does not imply that $(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k)$ is (b)-regular, and hence does not imply (b_s) , and indeed following the proof in [38] that (b_s) implies (b) it is easy to construct explicit semianalytic tubular neighbourhoods T with the fibres of (π_T, ρ_T) nontransverse to $F^{-1}(0)$ along the curve through 0 for which (b) fails. There are though <u>some</u> tubular neighbourhoods T for which the fibres of (π_T, ρ_T) are transverse to $F^{-1}(0)$ in their examples, since in each case $F(z, t)$ is weighted homogeneous in z , and so the <u>standard</u> spheres cut $F^{-1}(0)$ transversally. Thus, even though $n = 2$, we can derive topological triviality from [21] .

A more promising way of removing the restriction that $n \neq 2$ looks to be a new theorem of Kuo (Theorem 2 in [15]) which may give topological triviality directly from the hypothesis that $\mu(F_t)$ be constant. This depends on whether $\mu(F_t)$ constant implies that there is some constant $C < 1$ and a neighbourhood U of 0 such that $(\partial F/\partial t\ (z,t))/|\text{grad } F| < C\ |z|/|t|$ whenever $(z,t) \in U \cap F^{-1}(0)$. We shall leave this question for the present.

6. (b)-regularity and generic sections

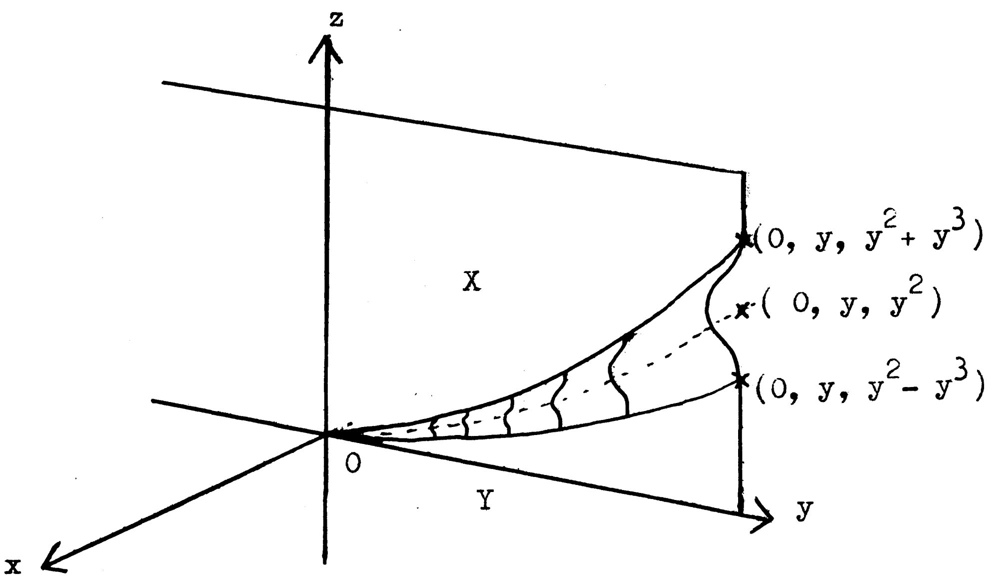
Part I . Detecting (b)-faults with generic sections.

The work in this section was motivated by the result of Teissier in [30] that " μ^*-constant " implies (b)-regularity for a family of complex hypersurfaces. Using the converse result (proved by Briançon and Speder in [3]) we find that if we have topological triviality, and (b) for generic hyperplane sections, then (b) follows. That this result does not generalise to real semialgebraic strata is shown by the next example.

Example 6.1 : In the open subset of $\mathbb{R}^3$ (with (x, y, z) as coordinates) where $y^2 < 1$, let Y be the y-axis, and let X be

$$\{x = 0\ ,\ (z - y^2)^2 \geq y^6,\ z > 0\} \cup \{y^9 x = ((z - y^2)^2 - y^6)^2,\ (z - y^2)^2 \leq y^6,\ z > 0\}.$$

X is a C^1 manifold, and a semialgebraic set.



Then X is topologically trivial along Y and , since the non-linear part of X is contained in a horn tangent to Y , X is (a)-regular over Y .

But X is not (b)-regular over Y at 0 : on the curve

$$\gamma(t) = (9t^3/16 , t , t^2 + \tfrac{1}{2}t^3)$$

which lies in X , the normal tends to $(1, 0, 3/2)$, so that the limiting tangent space does not comtain Oz , which is the limit of $\dfrac{x_i \pi(x_i)}{|x_i \pi(x_i)|}$ for all sequences $\{x_i\}$ on X tending to 0 , since the radius (y^3) of the horn tends to 0 faster than the height (y^2) above Y of the centre of the horn.

Also if $x = \alpha z$ defines the plane H_α , which contains Y , then H_α intersects X near 0 only if $\alpha = 0$. Thus $(X \cap H_\alpha , Y)$ is not a (b)-fault (by default) for generic sections H_α containing Y .

<u>Notation</u> : Let (X,Y) be a pair of adjacent strata, and let $0 \in Y \subset \overline{X} - X$. Suppose Y is a linear space, and that π is orthogonal projection onto Y . We let $\mathcal{K}_0(X,Y)$ (resp. $\Lambda_0(X,Y)$) denote the set of limit vectors for which (b) (resp. (b')) fails.

$$\mathcal{K}_0(X,Y) = \left\{ \lambda : \exists \{x_i\} \in X , \{y_i\} \in Y , \lambda = \lim \frac{x_i y_i}{|x_i y_i|} \not\subset \tau = \lim T_{x_i} X \right\}$$

$$\Lambda_0(X,Y) = \left\{ \lambda : \exists \{x_i\} \in X , \lambda = \lim \frac{x_i \pi(x_i)}{|x_i \pi(x_i)|} \not\subset \tau = \lim T_{x_i} X \right\}$$

In Example 6.1 , $\Lambda_0(X,Y) = \{(0: 0: 1)\}$, $\mathcal{K}_0(X,Y) = \{(0: a: b) : b \neq 0\}$.

It is easy to see that (when $\dim \Lambda_0(X,Y)$ is defined),

$$\dim \Lambda_0(X,Y) + \dim Y \leq \dim \mathcal{K}_0(X,Y)$$

if $\mathcal{K}_0(X,Y) \neq \phi$.

If X is (a)-regular over Y at 0 , then by the proof of (0.4) that (b) is equivalent to (a) + (b') ,

$$\mathcal{K}_0(X,Y) \subset \Lambda_0(X,Y) \oplus T_0 Y ,$$

and hence,

$$\dim \mathcal{K}_0(X,Y) \leq \dim \Lambda_0(X,Y) + \dim Y .$$

Thus if (a)-regularity holds and $\Lambda_0(X,Y) \neq \emptyset$ (or, equivalently, $\mathcal{K}_0(X,Y) \neq \emptyset$),

$$\dim \mathcal{K}_0(X,Y) = \dim \Lambda_0(X,Y) + \dim Y .$$

That is, the dimension of $\Lambda_0(X,Y)$ determines the dimension of $\mathcal{K}_0(X,Y)$, so that we can restrict our attention to $\Lambda_0(X,Y)$.

We say that X <u>is</u> $(b_{\text{cod } k})$-<u>regular over</u> Y <u>at</u> 0 for $0 \leq k \leq \text{cod } Y - 1$, when Y is linear (as it will be throughout this first part of §6), if

$(b_{\text{cod } k})$ There is an open dense subset $\mathcal{L}$ of the set of linear subspaces of codimension k containing Y , such that if $L \in \mathcal{L}$, $L \pitchfork X$ near 0 , and $X \cap L$ is (b)-regular over Y at 0 in L .

We must suppose $L \pitchfork X$ to be able to talk of (b)-regularity of $X \cap L$ over Y . In the case where X is the nonsingular part of a family of complex analytic hypersurfaces with singular locus Y , there is a Zariski open dense subset of the set of linear subspaces of (complex) codimension k containing Y, consisting of subspaces transverse to X (moreover the topological type of their intersection with X is well-defined : see Chapter 1 , §1 of [30]). It was this situation which motivated the work in this section : see Note 6.9 .

The following theorem says that $(b_{\text{cod } k})$ implies that $\dim \Lambda_0(X,Y) < k$. Here $\dim \Lambda_0(X,Y)$ is the maximal integer r , $-1 \leq r \leq \text{cod } Y - 1$, for which $\Lambda_0(X,Y)$ has a point near which it is a differentiable submanifold of $G_1^{\text{cod } Y}(\mathbb{R})$ of dimension r . This is the same as the usual dimension of $\Lambda_0(X,Y)$ when X is subanalytic, for then $\Lambda_0(X,Y)$ is the union of countably many compact manifolds-with-boundary of varying dimensions, the largest of which being the dimension of $\Lambda_0(X,Y)$; this will follow from the proof of the theorem.

We point out that a section of a pair (X,Y) , as in the title of §6 , is a linear subspace of $\mathbb{R}^n$ containing Y , which is assumed to be linear. Thus Theorem 6.2 describes the extent to which generic sections detect (b)-faults.

Theorem 6.2 : Let Y be a linear subspace of $\mathbb{R}^n$ containing 0 , and let X be a C^2 submanifold (resp. and a subanalytic subset) of $\mathbb{R}^n$ such that $Y \subset \overline{X} - X$. Suppose there is an open dense (resp. dense) subset $\mathcal{L}_k'$ of the set $\mathcal{L}_k$ (of linear subspaces of codimension k in $\mathbb{R}^n$ which contain Y) such that $L \in \mathcal{L}_k'$ implies $L \pitchfork X$ near 0 and $X \cap L$ is (b)-regular over Y at 0 .

Then $\dim \Lambda_0(X,Y) < k$.

Proof : We first state two assertions which we shall prove once we have shown how they give the theorem.

Assertion 6.3 : Let $Y \subset \mathbb{R}^n$ be linear, $0 \in Y$, and X a C^2 submanifold of $\mathbb{R}^n$, $Y \subset \overline{X} - X$, such that $\dim \Lambda_0(X,Y) = i \geq k$.

Then there is a dense subset $\mathcal{L}_k^d$ of a nonempty open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$, such that if $L \in \mathcal{L}_k^d$, there is a sequence $\{x_i\}$ in $X \cap L$ such that x_i tends to 0 as i tends to ∞ , and $\lim \frac{x_i\pi(x_i)}{x_i\pi(x_i)} \not\subset \lim T_{x_i}X$.

Assertion 6.4 : In Assertion 6.3 , if X is also a subanalytic subset of $\mathbb{R}^n$, we may take $\mathcal{L}_k^d = \mathcal{L}_k^o$.

(The conclusion of Assertion 6.4 is that there is a nonempty subset of $\mathcal{L}_k$ consisting of linear sections containing "bad" sequences, and that this subset may be taken to be open, not merely dense in some open set.)

Suppose that Theorem 6.2 is false.

Take Y and X which satisfy the hypotheses of Theorem 6.2 , and yet

$$\dim \Lambda_0(X,Y) = i \geq k .$$

Assume for the moment that X is not subanalytic, and apply Assertion 6.3 .

Assertion 6.3 gives $\mathcal{L}_k^d$, whioh is dense in the nonempty open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$, and hence meets the open dense subset $\mathcal{L}_k'$ of $\mathcal{L}_k$ described in the hypotheses of Theorem 6.2 .

Take $L \in \mathcal{L}_k' \cap \mathcal{L}_k^d$. Then $L \pitchfork X$ near 0 and $(X \cap L, Y)_0$ is (b)-regular. Hence for <u>all</u> sequences $\{x_i\}$ in $X \cap L$ tending to 0 ,

$$\lim \frac{x_i \pi(x_i)}{x_i \pi(x_i)} \subset \lim T_{x_i}(X \cap L) .$$

But $T_{x_i}(X \cap L) \subset T_{x_i}X$ for all i , and so $\lim T_{x_i}(X \cap L) \subset \lim T_{x_i}X$.

Thus,

$$\lim \frac{x_i \pi(x_i)}{x_i \pi(x_i)} \subset \lim T_{x_i} X .$$

However this is not true for <u>all</u> $\{x_i\}$ in $X \cap L$ since $L \in \mathcal{L}_k^d$, by Assertion 6.3 . Thus we find a contradiction, showing that Theorem 6.2 is valid when X is not subanalytic so long as Assertion 6.3 is true.

The argument for subanalytic X is similar : the dense subset $\mathcal{L}_k'$ of $\mathcal{L}_k$ must meet the open subset $\mathcal{L}_k^o$ of $\mathcal{L}_k$ given by Assertion 6.4 .

We shall have to prove Assertions 6.3 and 6.4 separately, but we first set up the situation which is common to both.

Rotate the coordinate axes so that $Y = \mathbb{R}^{n-m} \times 0^m$. Let $\dim X = d$.

Define $\gamma : X \longrightarrow G_1^m \times G_d^n$ and let G denote the closure of the graph of

$$x \longmapsto \left(\frac{x\pi(x)}{|x\pi(x)|}, T_x X \right)$$

γ in $\mathbb{R}^n \times G_1^m \times G_d^n$ (we write G_1^m for $G_1^m(\mathbb{R})$,etc.). Since X is C^2, γ is a C^1 map. Let p and q denote the projections from $\mathbb{R}^n \times G_1^m \times G_d^n$ onto $\mathbb{R}^n$ and G_1^m respectively. $p|_{\gamma(X)}$ is a C^1 diffeomorphism.

If ℓ is a line through 0 in $\mathbb{R}^m$, let $\hat{\ell}$ denote the line in $\mathbb{R}^n$ given by the inclusion $0^{n-m} \times \mathbb{R}^m \hookrightarrow \mathbb{R}^n$. Then $B = \{(\ell,\tau) \in G_1^m \times G_d^n : \hat{\ell} \not\subset \tau\}$ is an open subset of $G_1^m \times G_d^n$.

From now on we write Λ for $\Lambda_0(X,Y)$. Observe that

$$\Lambda = q(G \cap p^{-1}(0) \cap (\mathbb{R}^n \times B)) .$$

Given a subspace L in $\mathcal{L}_k$ we can write $L = Y \times \tilde{L}$ where $\tilde{L} \in G_{m-k}^m$. Given $A \in G_{m-k}^m$, write $A^* = \{\ell \in G_1^m : \ell \subset A\} \subset G_1^m$.

Let D_0 be a compact coordinate disc (of dimension $m-1$) for Λ as a C^1 submanifold of G_1^m of dimension i. D_0 exists by hypothesis on dim Λ.

<u>Proof of Assertion 6.3</u>:

<u>Lemma 6.5</u> : <u>There is a dense subset $\mathcal{L}_k^d$ of the open set</u>

$$\mathcal{L}_k^o = \{L \in \mathcal{L}_k : (\tilde{L})^* \pitchfork \Lambda \text{ on } \Lambda \cap D_0\}$$

<u>such that for all</u> $L \in \mathcal{L}_k^d$, $L \pitchfork X$ <u>near</u> 0 <u>and there is an open ball</u> B_L <u>such that</u> (i) $\bar{B}_L \subset \mathbb{R}^n \times B$ <u>and</u> $q(B_L) \subset D_0$,

(ii) <u>if</u> $F_L = q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(X) \cap \{z \in G : G \pitchfork q^{-1}(\tilde{L})^* \text{ at } z\}$, <u>then</u> $q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(0)$ <u>has nonempty intersection with</u> $\bar{F}_L$.

Assuming Lemma 6.5, let $L \in \mathcal{L}_k^d$, and let $\{z_i\}$ be a sequence of points in F_L tending to a limit z_0 in $q^{-1}(\tilde{L})^* \cap G \cap B_L \cap p^{-1}(0)$. Let $x_i = p(z_i)$ for all i. Then $\{x_i\}$ is a sequence of points in X tending to $p(z_0) = 0$. Also for all i, $x_i \in L$ since $q(\gamma(x_i)) \in (\tilde{L})^*$. Finally

$\lim \frac{x_i \pi(x_i)}{|x_i \pi(x_i)|} = \hat{\ell} \not\subset \tau = \lim T_{x_i} X$ since $(0, \ell, \tau) \in \bar{F}_L \subset \bar{B}_L \subset \mathbb{R}^n \times B$, by (i) and (ii) of Lemma 6.5, and so $(\ell, \tau) \in B$, i.e. $\hat{\ell} \not\subset \tau$. This completes the proof of Assertion 6.3.

Proof of Lemma 6.5 :

Sublemma 6.6 : Given a C^1 retraction $r : D_0 \longrightarrow \Lambda \cap D_0$, there is a dense subset W of $\Lambda \cap D_0$ such that if $\ell \in W$, $(r \circ q)^{-1}(\ell)$ contains a sequence $\{a_i\}$ in $p^{-1}(X) \cap G \cap (\mathbb{R}^n \times B)$ tending to a point in $p^{-1}(0) \cap q^{-1}(D_0)$ such that $(r \circ q)^{-1}(\ell)$ is transverse to G at a_i for all i.

Proof (after L. Siebenmann) :

Let $W_N = \{ \ell \in \Lambda \cap D_0 : \exists\ a_\ell^N \in G \cap (r \circ q)^{-1}(\ell)$ with $(r \circ q)^{-1}(\ell) \pitchfork G$ at a_ℓ^N and $0 < |\pi(p(a_\ell^N))| < 1/N \}$, for N a positive integer. a_ℓ^N is inside a region R_N of radius $1/N$ around $p^{-1}(0)$. W_N is open since transversality is an open condition. W_N is dense (and hence nonempty) by Sard's theorem applied to the C^1 map $(r \circ q)|_{G \cap R_N \cap q^{-1}(D_0)}$. Note that $G \cap R_N \cap q^{-1}(D_0)$ is nonempty since,

$$(\Lambda \cap D_0) \subset q(p^{-1}(0) \cap \overline{(G \cap R_N \cap q^{-1}(D_0))}\ ) .$$

Because $\Lambda \cap D_0$ is a C^1 manifold, it is locally compact and Hausdorff, and hence is a Baire space. Thus $W = \bigcap_{N=1}^{\infty} W_N$ is dense in $\Lambda \cap D_0$. Given $\ell \in W$, there is a limit point of $\{a_\ell^N\}$ in $p^{-1}(0)$ since $p^{-1}(0)$ is compact ($p^{-1}(0) \cong G_1^m \times G_d^n$). This limit point will be in $q^{-1}(D_0)$ since D_0 is closed. Then W satisfies the properties required for Sublemma 6.6 .

Now we can prove Lemma 6.5 .

Given L in $\mathcal{L}_k^o$ with $(\tilde{L})^* \pitchfork \Lambda$ at ℓ in $\Lambda \cap D_0 \cap (\tilde{L})^*$, there is a neighbourhood U of L in the k-dimensional family in $\mathcal{L}_k^o$ which is defined by the (k+1)-dimensional linear subspace orthogonal to L and containing the line ℓ, such that if $L' \in U$, $(\tilde{L'})^* \pitchfork \Lambda$ in $\Lambda \cap D_0$. $\{(\tilde{L'})^* : L' \in U\}$

defines a foliation of codimension k transverse to Λ near ℓ .

Choose a C^1 retraction $r : D_0 \longrightarrow \Lambda \cap D_0$ such that $r^{-1}(\ell) \subset (\tilde{L})^*$ and for all ℓ' in some neighbourhood of ℓ in $\Lambda \cap D_0$, $r^{-1}(\ell') \subset (\tilde{L}')^*$, where L' is the element of U such that $\ell' \in (\tilde{L}')^*$. By Sublemma 6.6 , arbitrarily near ℓ there is some $\ell' \in W$. Hence arbitrarily near L in $\mathcal{L}_k^o$ there is some L' (in U) with $(\tilde{L}')^* \pitchfork \Lambda$ in D_0 and such that $q^{-1}((\tilde{L}')^*)$ contains a sequence of points $\{a_i\}$ in $G \cap (\mathbb{R}^n \times B)$ tending to a limit a_0 in $G \cap p^{-1}(0) \cap q^{-1}((\tilde{L}')^*)$ such that for all i , $q^{-1}((\tilde{L}')^*)$ is transverse to G at a_i . Choose an open ball $B_{L'}$ around a_0 such that $q(B_{L'}) \subset D_0$ and $\overline{B}_{L'} \subset \mathbb{R}^n \times B$. Then (i) and (ii) of Lemma 6.5 are satisfied since $a_0 \in \overline{F}_{L'} \cap q^{-1}((\tilde{L}')^*) \cap G \cap B_{L'} \cap p^{-1}(0)$. This completes the proof of Lemma 6.5 .

Proof of Assertion 6.4 :

Lemma 6.7 : There is a compact coordinate disc D for Λ as a submanifold of dimension i in G_1^m , with $D \subset \operatorname{Int} D_0$, such that if T is a C^1 submanifold of G_1^m of dimension $(m-k-1)$ which is transverse to Λ on $D \cap \Lambda$, then there is an open ball $B_T \subset \mathbb{R}^n \times B$ such that,

(i) $\overline{B}_T \subset \mathbb{R}^n \times B$ and $q(B_T) \subset D$,

(ii) $F_T = q^{-1}(T) \cap G \cap p^{-1}(X) \cap B_T$ is a C^1 submanifold of G of codimension k .

(iii) $\emptyset \neq q^{-1}(T) \cap G \cap p^{-1}(0) \cap B_T \subset \overline{F}_T$.

We leave the proof of Lemma 6.7 for the moment.

Let $\mathcal{L}_k^o = \{L \in \mathcal{L}_k : (\tilde{L})^* \pitchfork \Lambda$ on $D\}$. Let $M_L = p(F_{(\tilde{L})^*})$ for L in $\mathcal{L}_k^o$. By Lemma 6.7(ii) and the fact that $p|_{\gamma(X)}$ is a C^1 diffeomorphism , M_L is a C^1 submanifold of X of codimension k , and $0 \in \overline{M}_L$ by (iii) . Also if $x \in M_L$ then $q(\gamma(x)) \in (\tilde{L})^*$ by definition of M_L , and hence

$q(\gamma(x)) \subset \tilde{L}$ by definition of $(\ )^*$, so that $x \in \pi(x) \times \tilde{L} \subset Y \times \tilde{L} = L$. Thus $M_L \subset L$ for $L \in \mathcal{L}_k^o$.

Let $\{x_i\}$ be a sequence in M_L tending to 0 as i tends to ∞. To complete the proof of Assertion 6.4 we must show that

$$\hat{\ell} = \lim \frac{x_i \pi(x_i)}{|x_i \pi(x_i)|} \not\subset \lim T_{x_i} X = \tau .$$

Now for all i $(x_i, \gamma(x_i)) \in F_{(\tilde{L})*}$, by (ii) of Lemma 6.7 and the definition of M_L. Hence $(0, \lim \gamma(x_i)) = (0, \ell, \tau) \in \overline{F}_{(\tilde{L})*} \subset \overline{B}_{(\tilde{L})*} \subset \mathbb{R}^n \times B$ using (i) and (ii) of Lemma 6.7. Thus $(\ell, \tau) \in B$, i.e. $\hat{\ell} \not\subset \tau$, by the definition of B. This completes the proof of Assertion 6.4.

<u>Proof of Lemma 6.7</u> : First, G is subanalytic in $\mathbb{R}^n \times G_1^m \times G_d^n$. For we can partition X into a locally finite set of real analytic submanifolds by [12] (See also [10] and [40]), then complexify each real analytic part, apply the argument of §17 in [46], take real parts, and finally take closures, using that the closure of a subanalytic set is subanalytic [12]. The closures match up since X is C^2.

Then apply Lemma 4.8.3 of [12] to G to give a (b)-regular stratification $\mathcal{G}$ of G such that $G \cap p^{-1}(Y)$ and $G \cap p^{-1}(0)$ are each the union of strata of $\mathcal{G}$. Since $\Lambda = q(G \cap p^{-1}(0) \cap (\mathbb{R}^n \times B))$ has dimension i there is some stratum S of $\mathcal{G}$ contained in $G \cap p^{-1}(0)$ such that $\dim(q(S) \cap D_0) = i$. By the implicit function theorem there is an open subset V of S contained in $\mathbb{R}^n \times B$ such that $q(V) \subset \Lambda \cap D_0$ is a C^1 submanifold of dimension i, and $q|_V$ has rank i. Let D be a compact coordinate disc for $D_0 \cap q(V)$ as a submanifold of dimension i.

Suppose T is a C^1 submanifold of dimension $(m-k-1)$ in G_1^m, transverse to Λ on $D \cap \Lambda$. Then $q^{-1}(T)$ is transverse to S on V since $q|_V$ has constant rank. Let $z \in V \cap q^{-1}(T)$. By (a)-regularity of $\mathcal{G}$ there is an open ball B_T in $\mathbb{R}^n \times B$ such that $z \in (B_T \cap S) \subset V$ and such that $q^{-1}(T)$ is transverse to every stratum of $\mathcal{G}$ within B_T. We may further suppose that $q(B_T) \subset D$, proving (i) of Lemma 6.7.

By definition of G, there is a stratum S_1 of $\mathcal{G}$, not meeting $p^{-1}(Y)$, such that $z \in \overline{S}_1$, i.e. $S \cap \overline{S}_1 \neq \emptyset$. Then by 10.4 of [21], $q^{-1}(T) \cap S \cap B_T \subset \overline{q^{-1}(T) \cap S_1 \cap B_T}$.

Repeating the argument given above for S for each stratum of $\mathcal{G}$ in $p^{-1}(0)$ adjacent to S we find that $q^{-1}(T) \cap G \cap p^{-1}(0) \cap B_T$ is nonempty and contained in $\overline{F}_T$, where $F_T = q^{-1}(T) \cap G \cap p^{-1}(X) \cap B_T$, and that F_T is a C^1 submanifold of G of codimension k. This proves (ii) and (iii) and completes the proof of Lemma 6.7.

We have now completed the proof of Theorem 6.2.

Note 6.8 : (1) In the proof of Lemma 6.7 we cited the result of Mather (10.4 of [21]) that if X is (b)-regular over Y in $\mathbb{R}^n$ and S is a submanifold of $\mathbb{R}^n$ transverse to Y then $S \cap Y \subset \overline{S \cap X}$. It is amusing that for complex analytic X, Y, and S, this follows from (a)-regularity : see the appendix of [25].

(2) If X, Y are complex analytic in $\mathbb{C}^n$ we obtain the same theorem, but involving complex linear subspaces of complex codimension k, and with the conclusion that $\dim_{\mathbb{C}} \Lambda_0(X,Y) < k$.

Note 6.9 : In the context of a family of complex hypersurfaces with isolated singularity, if one could prove that $\mu(F_t)$ constant implies that

$$\dim_{\mathbb{C}} \Lambda_0(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k) \neq 0,$$

then using Theorem 6.2 we would obtain an inductive proof of the result of Teissier that μ^*-constant implies (b)-regularity for the pair $(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k)$ ([30]).

In the only known examples of a μ-constant family which is not (b)-regular (due to Briançon and Speder), $\dim_{\mathbb{C}} \Lambda_0(F^{-1}(0) - (0 \times \mathbb{C}^k), 0 \times \mathbb{C}^k) > 0$.

For example, consider $F(x, y, z, t) = x^3 + txy^3 + y^4z + z^9$ (due to Speder. Cf. [2]). Analogous to the calculation in [2] we find that (b) fails on a curve $\gamma(u) = (\beta u^5, \alpha u^3, h(\alpha u^3)\alpha u^3, u)$ where $h : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies $h(0) = 1$ and $h(y)y^5 + (h(y)y)^9 = y^5$ (h exists by the implicit function theorem), and α , β are complex numbers defined by the equations $\beta^3 + \alpha^3\beta + \alpha^5 = 0$, $3\beta^2 + \alpha^3 = 0$. The limit of orthogonal secant vectors λ is $(0 : 1 : 1)$ and the limit of normal vectors ν is $(0 : 3\beta + 4\alpha^2 : \alpha^2)$. λ is not contained in the limiting tangent space orthogonal to ν since $3\beta + 5\alpha^2 \neq 0$.

Now consider the curve $\gamma_\theta(u) = (\beta_\theta u^5, \alpha_\theta u^5, h_\theta(\alpha_\theta u^3)\alpha_\theta u^3(1+\theta), u)$ where $\theta \in \mathbb{C}$, $|\theta| < \varepsilon$ for some positive $\varepsilon < 1$, and $h_\theta : \mathbb{C} \longrightarrow \mathbb{C}$ satisfies $h_\theta(0) = 1$ and $h_\theta(y)y^5(1+\theta) + (h_\theta(y)y(1+\theta))^9 = y^5(1+\theta)$, and α_θ , β_θ are complex numbers defined by the equations $\beta_\theta{}^3 + \alpha_\theta{}^3\beta_\theta + (1+\theta)\alpha_\theta{}^5 = 0$, $3\beta_\theta{}^2 + \alpha_\theta{}^3 = 0$. Then $\lambda_\theta = (0 : 1 : 1+\theta)$, and $\nu_\theta = (0 : 3\beta_\theta + 4(1+\theta)\alpha_\theta{}^2 : \alpha_\theta{}^2)$. λ_θ is not contained in the limiting tangent space orthogonal to ν_θ since $3\beta_\theta + 5(1+\theta)\alpha_\theta{}^2 \neq 0$ for small θ , i.e. for ε sufficiently small. As θ varies we obtain a complex 1-dimensional subset of $\Lambda_0(X,Y)$ and thus $\dim_{\mathbb{C}} \Lambda_0(X,Y) \geq 1$. In fact $\dim_{\mathbb{C}} \Lambda_0(X,Y) = 1$ here since the family is equimultiple (with multiplicity 3), which is the same as saying that $(X \cap L, Y)$ is (b)-regular for generic complex linear subspaces L of codimension 2 containing Y , or again that $X \cap L = \emptyset$ for generic L . (Recall $X = F^{-1}(0) - (0 \times \mathbb{C})$, and $Y = 0 \times \mathbb{C}$, the t-axis)

Note 6.10 (added 1980) : When X is subanalytic and Y is 1-dimensional Vicente Navarro Aznar and the author have obtained a converse to Theorem 6.2, i.e. a proof that $\dim \Lambda < k$ implies $(b_{cod\ k})$. If either of these two conditions is not satisfied we have examples showing that the converse to 6.2 does not hold in general. For details see [57] .

Part II . Preservation of (b)-regularity under generic sections.

Let X , Y be C^1 submanifolds of $\mathbb{R}^n$, and $0 \in Y \subset \overline{X} - X$. We call a C^1 submanifold of dimension $(n-k)$ containing Y a section of codimension k (cod $Y < k \leqslant 0$).(This term was reserved for linear subspaces in Part I.) Denote the set of germs at 0 of sections of codimension k by $\mathcal{S}_k$. In the notation of Whitney [46] [47], the set of limits of tangent planes to X given by sequences on X tending to 0 is $\mathcal{T}(X,0) \subset G^n_{\dim X}(\mathbb{R})$. Let $\mathcal{S}^*_k$ denote the subset of $\mathcal{S}_k$ consisting of germs at 0 of sections S of codimension k such that T_0S is transverse to every element of $\mathcal{T}(X,0)$ in $T_0\mathbb{R}^n$. We give $\mathcal{S}_k$ the topology induced from the topology on $G^n_{n-k}(\mathbb{R})$ by the map $\sigma \longmapsto T_0\sigma$.

Theorem 6.11 : Let X be (b)-regular over Y at 0 , and let S be a representative of $\sigma \in \mathcal{S}^*_k$. Then $S \pitchfork X$ near 0 and $X \cap S$ is (b)-regular over Y at 0 .

Proof : It suffices to prove the result for $k = 1$, since we may consider a section of codimension k as the intersection of k sections of codimension 1 . Let $\sigma \in \mathcal{S}^*_1$, and let S be a representative of σ . It is clear that $S \pitchfork X$ near 0 , so that it makes sense to test for (b)-regularity.

Let $\{x_i\}$ and $\{y_i\}$ be sequences in $X \cap S$ and Y tending to 0 so that $\frac{x_iy_i}{|x_iy_i|}$ tends to λ , $T_{x_i}(X \cap S)$ tends to $\mathcal{T}_S$ and $T_{x_i}X$ tends to $\mathcal{T}$.

$T_0S \pitchfork \mathcal{T}$ since $S \in \mathcal{S}^*_1$, and clearly $\mathcal{T}_S \subset \mathcal{T} \cap T_0S$. Thus $\mathcal{T}_S = \mathcal{T} \cap T_0S$. Since X is (b)-regular over Y at 0 , $\lambda \subset \mathcal{T}$. But S is a C^1 submanifold, and thus $\lambda \subset T_0S$, and $\lambda \subset \mathcal{T}_S$, showing that $X \cap S$ is (b)-regular over Y at 0 , and completing the proof of the theorem.

If $\mathcal{S}^*_k$ were open and dense in $\mathcal{S}_k$ (in the topology given by the tangents at $0)$, we would have proved that (b) implies $(b_{\text{cod }k})$. Our next result will give sufficient conditions for this to be so.

Theorem 6.12 : Let $\Gamma \subset G^n_{p+q}$ be a subset of Hausdorff dimension at most $p-k$. Let $Y = \mathbb{R}^q \times 0^{n-q} \subset \mathbb{R}^n$. Then $\Omega = \{H \in G^n_{n-k} \mid Y \subset H$ and $H \pitchfork T, \forall T \in \Gamma\}$ is a dense subset of $\{H \in G^n_{n-k} \mid Y \subset H\}$.

Proof : Write $Y^{\perp} = 0^q \times \mathbb{R}^{n-q}$, and let $\Gamma_i = \{T \in \Gamma \mid \dim_{\mathbb{R}}(T \cap Y^{\perp}) = p+i\}$. Then $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \dots \cup \Gamma_r$, where $r = \inf(q, n-p-q)$, and so $\Omega = \bigcap_{i=0}^{r} \Omega_i$, where $\Omega_i = \{H \in G^n_{n-k} \mid Y \subset H$ and $H \pitchfork T, \forall T \in \Gamma_i\}$. Note that $\dim \Gamma_i \leqslant \dim \Gamma < p-k+1$, so that we can assume $\Gamma = \Gamma_i$. Also if $H \cap Y^{\perp}$ and $T \cap Y^{\perp}$ are transverse in $Y^{\perp}$ it follows that H and T are transverse in $\mathbb{R}^n$, and thus it will suffice to prove the theorem in the case of $q = 0$. Also because $\dim \Gamma_i < p-k+1 < (p+i)-k+1$ $(i \geqslant 1)$, if we prove the result when $\Gamma = \Gamma_0$, this will include the case when $\Gamma = \Gamma_i$ $(i \geqslant 1)$. We are left with the following lemma to prove.

Lemma 6.13 : Let $\Gamma_0 \subset G^n_p$ have Hausdorff dimension at most $p-k$. Then $\Omega_0 = \{H \in G^n_{n-k} \mid H \pitchfork T, \forall T \in \Gamma_0\}$ is dense in G^n_{n-k}.

Proof of Lemma 6.13 : Let

$$A_j = \{(H,T) \in G^n_{n-k} \times G^n_p \mid \dim_{\mathbb{R}}(H \cap T) = p-k+j, \text{ and } T \in \Gamma_0\} \quad (1 \leq j \leq n-p).$$

Then if $\pi_1 : G^n_{n-k} \times G^n_p \longrightarrow G^n_{n-k}$ denotes projection onto the first factor, $\Omega_0 = G^n_{n-k} - (\pi_1(\bigcup_{j=1}^{n-p} A_j))$. It will suffice to show that $\dim_h A_j$ is less than $k(n-k) = \dim G^n_{n-k}$, where $\dim_h$ denotes Hausdorff dimension, for each j, $1 \leqslant j \leqslant n-p$. For then $\dim_h \pi_1(\bigcup_{j=1}^{n-p} A_j) < k(n-k)$, so that $\pi_1(\bigcup_{j=1}^{n-p} A_j)$ has nonempty interior in G^n_{n-k}, and hence its complement Ω_0 will be dense.

Now A_j fibres over Γ_0 by projection $(H,T) \longmapsto T$, with fibre $A_{j,T} = \{H \in G^n_{n-k} \mid \dim(H \cap T) = p-k+j\}$. Hence $\dim_h A_j = \dim_h A_{j,T} + \dim_h \Gamma_0$. Also $A_{j,T}$ fibres over G^p_{p-k+j} by intersection $H \longmapsto H \cap T$ with fibre isomorphic with $G^{n-p+k-j}_{n-p-j}$.

Thus $\dim_h A_{j,T} = \dim G^{p}_{p-k+j} + \dim G^{n-p+k-j}_{n-p-j}$

$$= (k-j)(p-k+j) + k(n-p-j)$$

$$= k(n-k) - j(p-k+j) .$$

Since $\dim_h \Gamma_0 < p-k+1$, we obtain

$$\dim_h A_j < k(n-k) - (j-1)(p-k+j) ,$$

hence $\dim_h A_j < k(n-k)$ as required.

This completes the proof of the lemma and hence of the theorem.

Corollary 6.14 : Let X , Y be C^1 submanifolds of $\mathbb{R}^n$, and $0 \in Y \cap (\overline{X} - X)$. If $\tau(X,0)$ in $G^n_{\dim X}$ has Hausdorff dimension at most $\dim X - \dim Y - k$, then $\mathfrak{J}^*_k$ is open and dense in $\mathfrak{J}_k$.

Proof : Apply the theorem to show density. Because $\tau(X,0)$ is closed it is also compact and the required openness follows easily.

Corollary 6.15 : If Y is linear, X is (b)-regular over Y at 0 and $\tau(X,0)$ has Hausdorff dimension at most $\dim X - \dim Y - k$, then X is $(b_{\mathrm{cod}\ k})$-regular over Y at 0 .

Proof : Apply Theorem 6.11 and Corollary 6.14 .

Note 6.16 : If X is subanalytic, $\tau(X,0)$ is also subanalytic and its Hausdorff dimension coincides with the maximal dimension of a stratum of an analytic stratification. See Lemma 2.7 .

Note 6.17 : An example described by Mark Goresky at I. H. E. S. in June 1979 shows that the dimension hypothesis in Corollary 6.15 is required in general. For details see [57] . This example is not subanalytic ; however semialgebraic examples have recently been found of (b)-regular pairs (X,Y) for which $\tau(X,0) = \dim X - \dim Y$ (again see [57]) , although Teissier has shown that these cannot occur in the complex analytic case [58] .

7. Stronger generic regularity.

Let X be a C^1 submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset \overline{X} - X$.

According to Verdier [40], X is (w)-regular over Y at 0 if,

(w) There is a constant $C > 0$ and a neighbourhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$, $y \in U \cap Y$, then $d(T_x X, T_y Y) \leq C\|x - y\|$. *

Verdier proves that (w) implies (b). Here we give an example showing that (b) does not imply (w), even for algebraic strata.

Example 7.1 : In $\mathbb{R}^3$ with (x, y, t) as coordinates, let V be $\{y^4 = t^4 x + x^3\}$. Let Y be the t-axis, and X be $(V - Y)$.

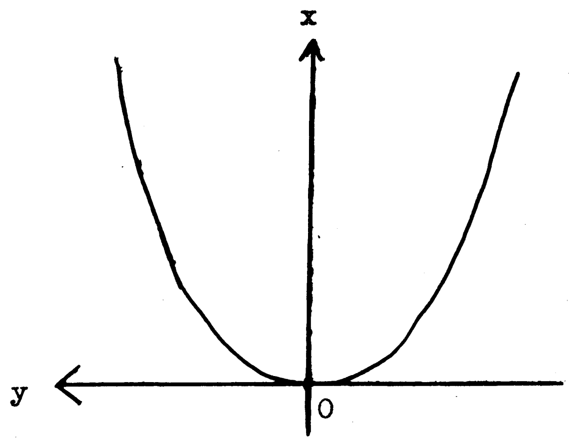


Figure : $t = 0$.

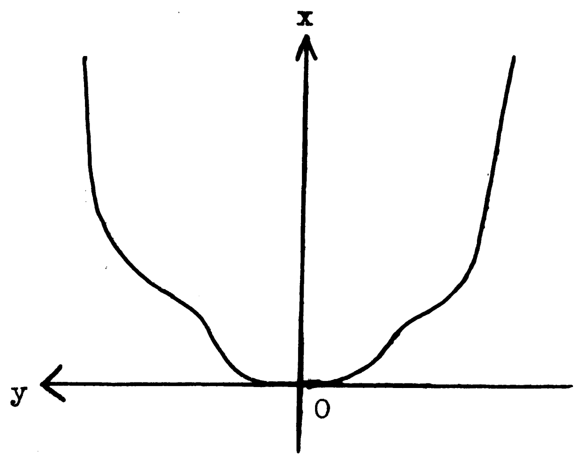


Figure : $t \neq 0$.

From the figures it is clear that V is a topological manifold near 0, and in particular that X is topologically trivial along Y. It will follow from the calculations of §8 that X is (b)-regular over Y at 0. In fact in this example X is C^1 trivial along Y : V is a C^1 submanifold. We show that at 0 there is a unique limiting tangent plane, with normal $(1 : 0 : 0)$ — a chart for V at 0 follows easily.

* See Addendum 7.13 for the definition of $d(,)$.

The normal to X at $(x, y, t) = (x, (t^4x + x^3)^{1/4}, t)$ is

$$(3x^2 + t^4 : -4(t^4x + x^3)^{3/4} : 4t^3x) \qquad (7.2)$$

Since X is algebraic it suffices to consider curves on X through 0 defined by an analytic arc $\gamma(s) = (x(s), t(s))$, $s \in [0,1]$. If $|t(s)/x(s)|$ is bounded as s tends to 0, the normal is

$$(3 + t^2(t/x)^2 : -4(t^{4/3}(t/x)^{8/3} + x^{1/3})^{3/4} : 4t^2(t/x))$$

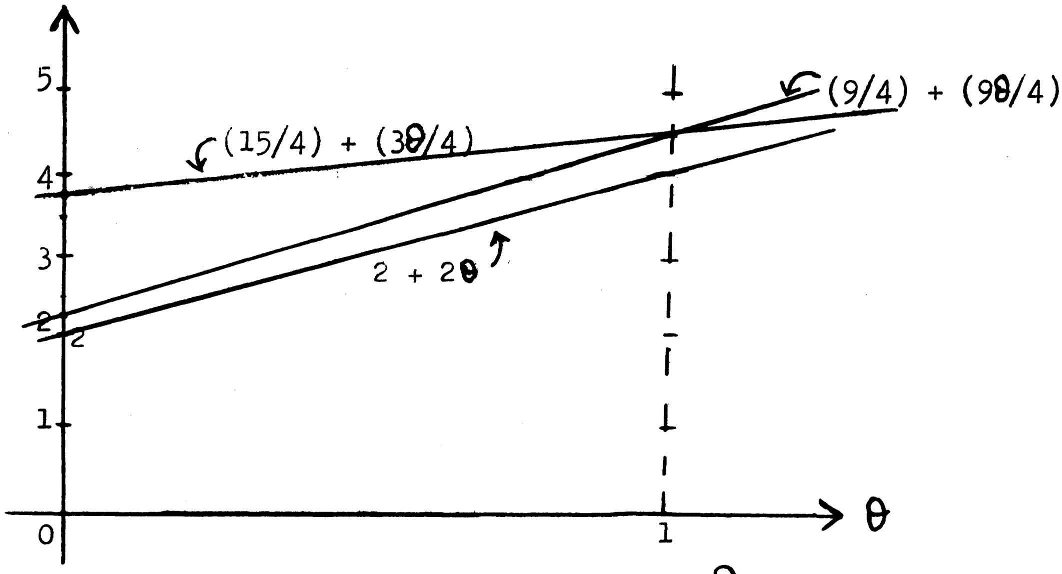
and tends to $(1 : 0 : 0)$. If $|t(s)/x(s)|$ is not bounded as s tends to 0 we set $x = ct^{1+\theta}$ + (higher terms in t), $\theta > 0$. The normal becomes

$$(3c^2t^{2+2\theta} + t^4 : -4(ct^{5+\theta} + c^3t^{3+3\theta})^{3/4} : 4ct^{4+\theta})$$

disregarding higher terms.

<u>$\theta \geq 1$</u> : $4 < 18/4 = \min((15/4) + (3\theta/4), (9/4) + (9\theta/4)) < 5 \leq 4 + \theta$, hence the normal tends to $(1 : 0 : 0)$.

<u>$\theta < 1$</u> : $2 + 2\theta < (9/4) + (9\theta/4) < (15/4) + (3\theta/4)$, and so once again we find $(1 : 0 : 0)$.



<u>Figure</u> : <u>Justification of the inequalities when</u> $\theta < 1$.

<u>(w) fails</u> : Consider the curve $\gamma(s) = (s^2, (2s^6)^{1/4}, s)$ on X. From (7.2) we find that the normal to X at $\gamma(s)$ is $(4s^4 : -4(2s^6)^{3/4} : 4s^5)$ and hence that $d(T_{\gamma(s)}X, T_0Y) = 4s^5/((4s^4)^2 + \ldots)^{\frac{1}{2}} \sim s$. Now $\|\gamma(s) - \pi_Y(\gamma(s))\| = \|(s^2, (2s^6)^{1/4}, 0)\| \sim s^{3/2}$. Hence X fails to be (w)-regular over Y at 0.

As a consequence (w)-regularity is not a C^1 diffeomorphism invariant. However it is clear from the definition of (w) that it is a C^2 diffeomorphism invariant, or more precisely that it is invariant under a C^1 diffeomorphism with a Lipschitz derivative.

Note 7.3 : No example has been found so far of complex analytic strata for which (b) holds and (w) fails. In the special case of a family of complex hypersurfaces with isolated singularity parametrised by Y it is known that (b) and (w) are equivalent. This is because (w) is a trivial consequence of (c)-cosecance as defined by Teissier in [32] . It follows from [3] and [31] that (b) implies (c)-cosecance.

Now we suppose that Y is linear (apply a local analytic isomorphism at 0 to $\mathbb{R}^n$). Let π denote orthogonal projection onto Y .

We can reformulate (w) by saying that for x , y near 0 , $\dfrac{d(T_xX,T_yY)}{\|x-y\|}$ is bounded, and so in particular $\dfrac{d(T_xX,T_0Y)}{\|x-\pi(x)\|}$ is bounded for x near 0 .

Then it is clear that if X is (w)-regular over Y at 0 then $(X,Y)_0$ satisfies the ratio test (r) of Kuo (defined in [14]) :

(r) Given any vector $v \in T_0Y$, $\displaystyle\lim_{\substack{x\to 0\\ x\in X}} \frac{|\pi_x(v)|.\|x\|}{\|x-\pi(x)\|} = 0$.

Here π_x denotes orthogonal projection onto the normal space to X at x , so that $|\pi_x(v)| = d(T_xX, v)$.

Kuo proved in [14] ,

Theorem 7.4 (Kuo) : (1) (r) *implies* (b) ,

(2) (b) *implies* (r) *if* Y *is of dimension one.*

Proof : In each case the proof in [14] uses the curve selection lemma with the assumption that X be a semianalytic set. Using Lemma 2.6 we can use the same proof when X is a subanalytic set.

Corollary 7.5 : (w) implies (b) .

Example 7.6 : For an example showing that (r) does not imply (w) apply Theorem 7.4 (2) to Example 7.1 .

Actually we can make more precise what was proved in [14] . It is shown there that (b) is equivalent to the conjunction of (a) and

(r') If $\gamma(t)$, $t \in [0,1]$, is an analytic arc on X with $\gamma(0) = 0$, then $\lim_{t \to 0} \frac{|\pi_t(v)| \, \|\gamma(t)\|}{\|\gamma(t) - \pi(\gamma(t))\|} = 0$, where v is the tangent at 0 to the arc in Y defined by $\pi \circ \gamma(t)$ (when nonzero) and π_t is projection onto the normal space to X at $\gamma(t)$.

It is obvious that (r) implies (a) + (r') , and that (a) + (r') implies (r) when Y is of dimension one. With this in mind we now give an example of a pair of semialgebraic strata, with Y of dimension two, X (b)-regular over Y , and where (r) fails to hold for a curve $\gamma(t)$ and a vector v spanning the orthogonal complement in T_0Y to the subspace spanned by the tangent at 0 to the curve in Y defined by $\pi \circ \gamma(t)$.

This example, discovered at Oslo in August 1976 (see [39]), gives the first (b)-regular pair of subanalytic strata which fail the ratio test (r) (introduced in 1970) . It is an open question whether real algebraic or complex analytic examples exist, although from the argument for (w) in Note 7.3 we see that (b) is equivalent to (r) when X is the nonsingular part of a complex hypersurface.

Example 7.7 : Let (x, y, z, w) be coordinates in $\mathbb{R}^4$, and let Y be the plane $\{z = w = 0\}$. Define the semialgebraic set,

$$X = \{w = 0 , 2(x^2 + (z - y^p)^2) \geqslant y^{2p} , z > 0\}$$
$$\cup \{y^q w = (x^2 + (z - y^p)^2 - y^{2p}/2)^2 , 2(x^2 + (z - y^p)^2) \leqslant y^{2p} , z > 0\}$$

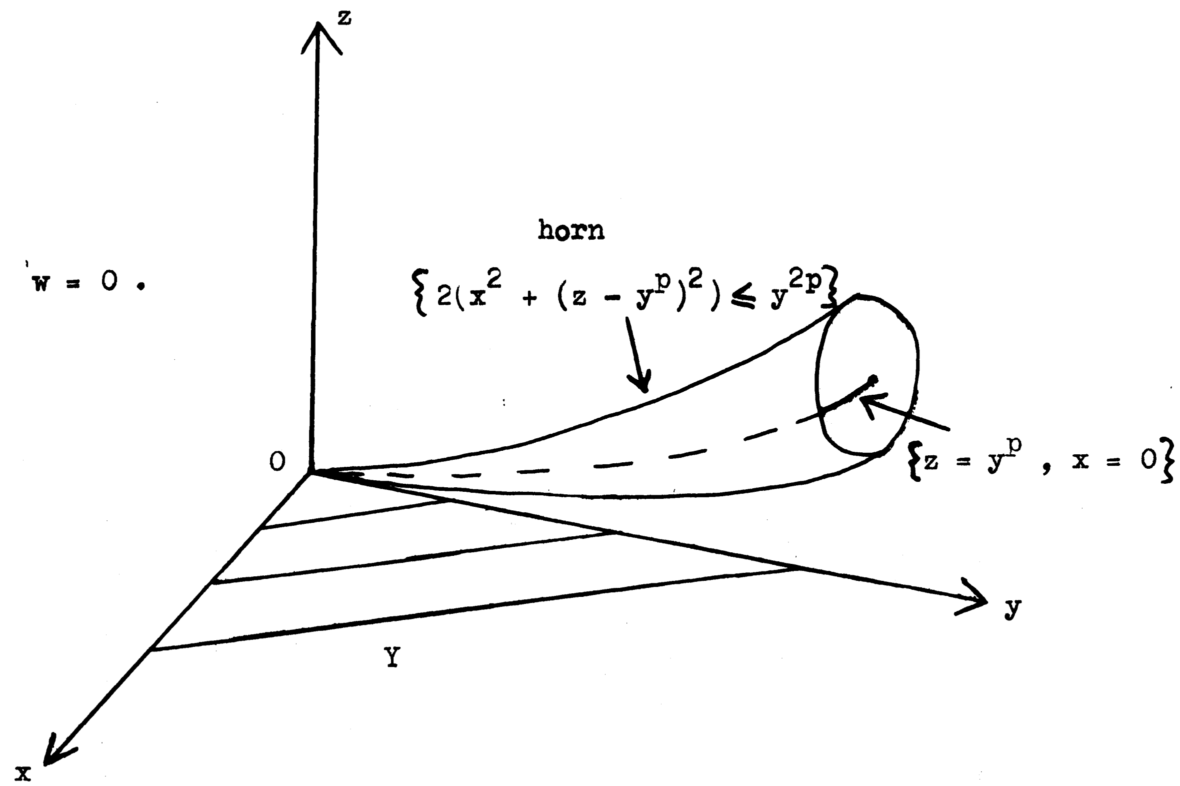
where p and q are positive integers satisfying

$$2p < q < 3p . \qquad (7.8)$$

(For example let $p = 2$, $q = 5$.)

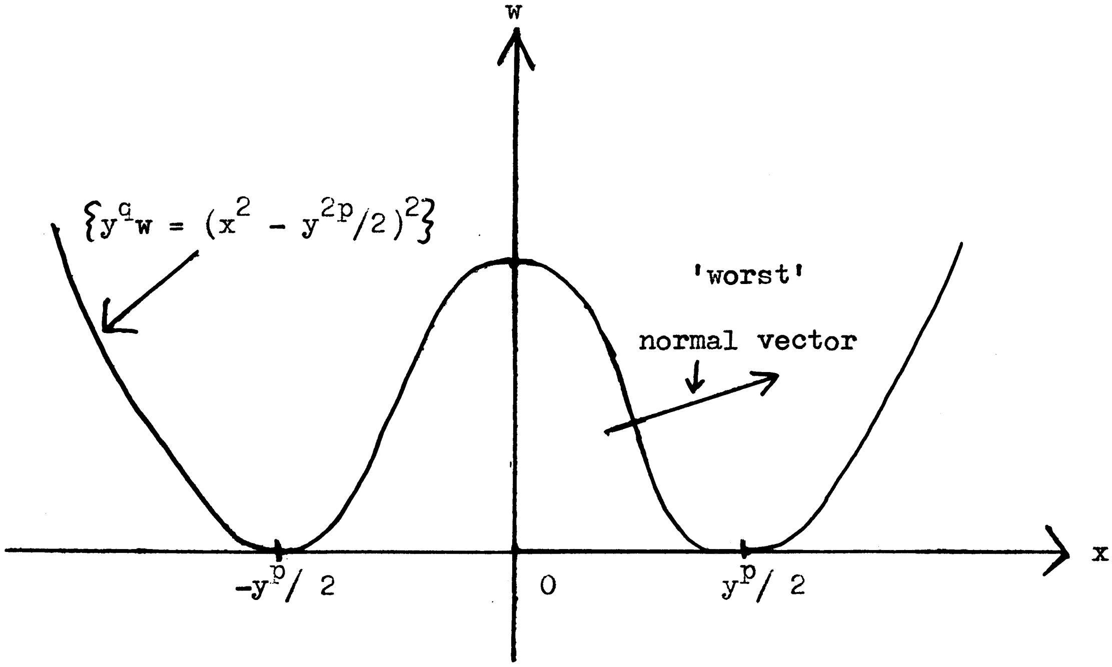
Observe that because the algebraic variety defined by the equality in the second part of the expression for X has $\{w = 0\}$ as tangent space at every point of its intersection with $\{2(x^2 + (z - y^p)^2) = y^{2p}\}$, X is a C^1 submanifold of $\mathbb{R}^4$ (compare Construction 2.2).

Figure : $w = 0$.



Assertion 7.9 : (b) holds.

Proof : We show that there is a single limiting tangent 3-plane for sequences on X tending to 0, namely $\{w = 0\}$. It suffices to consider the points on $\{y^q w = (x^2 - y^{2p}/2)^2\}$ (with y fixed) where $d^2w/dx^2 = 0$, since at these points the normal is furthest from the (w)-direction (cf. 2.2).



<u>Figure</u> : $z = y^p$, y fixed.

$d^2w/dx^2 = 0$ when $6x^2 = y^{2p}$, and the normal vector is $(\pm(4/3\sqrt{6})y^{3p} : -y^q)$ which tends to $(0 : 1)$ as y tends to 0 since $q < 3p$ by (7.8) . Hence $\{w = 0\}$ is the unique limiting tangent plane.

At the points on X where the secant vector defined by orthogonal projection onto Y is furthest from the z-direction , the secant vector is contained in the tangent space to X . Hence Oz is the unique limit of tangent vectors, and (b') holds. (a) holds (since $\{w = 0 , z = 0\} \subset \{w = 0\}$) , so we can apply the result that (a) + (b') is equivalent to (b) (0.4) to show that (b) holds, proving the assertion.

<u>Assertion 7.10</u> : (r) <u>fails to hold</u>.

<u>Proof</u> : Consider the curve $\gamma(t) = (t^p/\sqrt{6} , t , t^p , t^{4p-q}/9)$ which lies on X . The normal vector to X at $\gamma(t)$ is,

$$((4/3\sqrt{6})t^{3p} : ((2p/3) - (q/9))t^{4p-1} : 0 : -t^q) .$$

Let π_t denote projection onto this normal space. Then

$$|\pi_t(0x)| \sim \frac{t^{3p}}{\|(t^{3p} , t^{4p-1} , 0 , t^q)\|} \sim \frac{t^{3p}}{t^q} ,$$

since, by (7.8) , $q < 3p$.

$$\frac{\|\gamma(t)\|}{\|\gamma(t) - \pi(\gamma(t))\|} = \frac{\|(t^p/6\ ,\ t\ ,\ t^p\ ,\ t^{4p-q}/9)\|}{\|(\ 0\ ,\ 0\ ,\ t^p\ ,\ t^{4p-q}/9)\|} \sim \frac{t}{t^p} .$$

Hence the ratio (as in the definition of (r)) becomes t^{2p-q+1} , which does not tend to zero since $2p < q$ by (7.8) . This proves Assertion 7.10 .

Finally we check that (w) fails to hold.

$$d(T_{\gamma(t)}X, T_{\pi(\gamma(t))}Y) \sim t^{3p-q} ,$$

$$d(\gamma(t), \pi(\gamma(t))) \sim t^p ,$$

so that (w) fails exactly when $2p < q$.

<u>Note 7.11</u> : The proof of Assertion 7.9 gives in fact that $\overline{X}$ is a C^1 manifold-with-boundary. Basing the construction on $\{w = (x^{2k} - 1/2)^2\}$, $1 < k < \infty$, instead of $k = 1$ as here, we can build similar examples with X a C^k submanifold and semialgebraic subset of $\mathbb{R}^4$. However $\overline{X}$ will still be a submanifold-with-boundary of class C^1 , not C^2 . (r) , like (w) , is a C^2 diffeomorphism invariant, but not a C^1 diffeomorphism invariant. In this context note that there is no C^2 version of the lemma showing that wings are generically submanifolds-with-boundary of class C^1 (see [43]). Hence the proof in [43] that (b) is generic does not apply directly to (r) or (w) .

(As a counterexample to a C^2 version it suffices to take the product of $\mathbb{R}$ and a semi-cubical cusp in $\mathbb{R}^3$.)

<u>Note 7.12</u> : In [14] there is an example of Kuo showing that (r) does not imply (b) if X is merely smooth. Kuo has also an example where Y is 1-dimensional, (b) holds, and (r) fails, and of course X merely smooth (private communication). This is why we assumed subanalyticity of X from the beginning of §7 .

<u>Addendum 7.13.</u> If A, B are vector subspaces of $\mathbb{R}^n$, let

$$d(A, B) = \sup_{\substack{b \in B \\ |b|=1}} |b - \pi_A(b)|$$

where π_A is orthogonal projection onto A. This is not symmetric in A and B. Clearly $d(A, B) = 0$ if and only if $A \supseteq B$.

(Compare [14], [40], [46], [47] in all of which the order is the reverse of the above.)

CHAPTER 3. COMPUTATIONS

During a talk delivered at the Göttingen Catastrophe Theory Conference in October 1973, C. T. C. Wall suggested that it would be useful to determine Whitney regularity in the following case : $X \equiv \{y^a = t^b x^c + x^d\} - \{t\text{-axis}\}$ in $\mathbb{R}^3$ or $\mathbb{C}^3$, $Y \equiv \{t\text{-axis}\}$, with a , b , c , d positive integers.

We determine (a)- and (b)-regularity completely in the complex case and record this together wih the calculations for the real case that have been made. These calculations have proved useful in providing Example 7.1 (showing (b) to be strictly weaker than (w) even for algebraic strata), and in answering several questions posed by J.-J. Risler concerning algebraic stratifications not regular over $\mathbb{C}$, yet regular over $\mathbb{R}$.

8. $y^a = t^b x^c + x^d$.

The tables below collect the results which are obtained.

Key : ✓ - regularity holds ; ✗ - there is a fault at 0 ; ? - undecided .

Table 8.1 : (a)-regularity over $\mathbb{C}$.

$a = 1$ ✓ (8.6)

$$a > 1 \begin{cases} d \leq c & \checkmark \ (8.7) \\ c < d < b + c & \begin{cases} a \leq b & \checkmark \ (8.12) \\ a > b & \begin{cases} d < ac/(a-b) & \checkmark \ (8.12) \\ d \geq ac/(a-b) & \times \ (8.12) \end{cases} \end{cases} \\ b + c \leq d & \times \ (8.8) \end{cases}$$

Table 8.2 : (a)-regularity over $\mathbb{R}$.

- $a = 1$ ✓ (8.6)
- $a > 1$
  - $d \leqslant c$ ✓ (8.7)
  - $c < d < b+c$
    - $a \leqslant b$ ✓ (8.11, 8.12)
    - $a > b$
      - $d < ac/(a-b)$ ✓ (8.12)
      - $d \geqslant ac/(a-b)$
        - $b \equiv 0 \ (2)$
          - $d \equiv c \ (2)$ ✓ (8.14)
          - $d \equiv c+1 \ (2)$ ✗ (8.13)
        - $b \equiv 1 \ (2)$ ✗ (8.13)
  - $b+c \leqslant d$
    - $b \equiv 0 \ (2)$
      - $d \equiv c+1 \ (2)$ ✗ (8.9)
      - $d \equiv c \ (2)$
        - $a \leqslant b$ ✓ (8.11)
        - $b < a < b+c$
          - $d = b+c$ ✓ (8.15)
          - $b+c < d$?
        - $b+c \leqslant a$ ✗ (8.10)
    - $b \equiv 1 \ (2)$ ✗ (8.9)

Table 8.3 : (b)-regularity over $\mathbb{C}$.

- $a = 1$ ✓ (8.16)
- $a > 1$
  - $d \leqslant c$ ✓ (8.17)
  - $c < d$ ✗ (8.18)

Table 8.4 : (b')-regularity over $\mathbb{R}$. (Not (b))

- $a = 1$ ✓ (8.16)
- $a > 1$
  - $d \leqslant c$ ✓ (8.17)
  - $c < d$
    - $b \equiv 0 \ (2)$
      - $d \equiv c \ (2)$
        - $d < a$ ✓ (8.20)
        - $a \leqslant d$?
      - $d \equiv c+1 \ (2)$ ✗ (8.19)
    - $b \equiv 1 \ (2)$ ✗ (8.19)

Note 8.5 : It is easy to show that if (a) (resp (b'), resp. (b)) holds over $\mathbb{C}$, then (a) (resp. (b') , resp. (b)) holds over $\mathbb{R}$.

Write $f(x, y, t) = -y^a + t^b x^c + x^d$. Then (a) holds at 0 if and only if $\dfrac{\partial f/\partial t(x, y, t)}{|\mathrm{grad} f(x, y, t)|}$ tends to 0 as (x, y, t) tends to 0 on X, i.e. if and only if at least one of $\dfrac{\partial f/\partial t}{\partial f/\partial x}$ and $\dfrac{\partial f/\partial t}{\partial f/\partial y}$ tend to 0. We have that

$$\begin{aligned}\mathrm{grad}\, f &= (\partial f/\partial x, \partial f/\partial y, \partial f/\partial t) \\ &= (dx^{d-1} + cx^{c-1}t^b, -ay^{a-1}, bt^{b-1}x^c).\end{aligned}$$

<u>8.6</u>: (a) <u>holds if</u> $a = 1$.

$$\frac{\partial f/\partial t}{\partial f/\partial y} = \frac{bt^{b-1}x^c}{-1} \longrightarrow 0 \text{ as } x \longrightarrow 0.$$

<u>8.7</u> : (a) <u>holds if</u> $d \leq c$.

We may suppose $\partial f/\partial x \not\equiv 0$, for $\partial f/\partial x$ is identically zero only on $\{dx^{d-c} + ct^b\} = 0$, and since $d \leq c$, this surface intersects X in an isolated point at 0. Then $\left|\dfrac{\partial f/\partial t}{\partial f/\partial x}\right| \sim \dfrac{t^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} = \dfrac{t^{b-1}x^{c-d+1}}{d + cx^{c-d}t^b} \longrightarrow 0$ as x tends to 0 if $d \leq c$.

<u>8.8</u> : (a) <u>fails over</u> $\mathbb{C}$ <u>if</u> $d \geq b + c$ <u>and</u> $a > 1$.

Consider the curve on which $\partial f/\partial y \equiv 0$, i.e. $y = 0 = x^c(t^b + x^{d-c})$. Let $t^b = -x^{d-c}$. Then

$$\begin{aligned}\frac{\partial f/\partial t}{\partial f/\partial x} = \frac{bt^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} &\sim \frac{x^{c+(d-c)(b-1)/b}}{dx^{d-1} - cx^{d-1}} \\ &\sim x^{c-d+1+(d-c)(b-1)/b} \\ &= x^{(b+c-d)/b} \not\longrightarrow 0\end{aligned}$$

if $d \geq b + c$. Hence if $d \geq b + c$, (a) fails on $\{y = 0 = t^b + x^{d-c}\}$.

<u>8.9</u> : (a) <u>fails over</u> $\mathbb{R}$ <u>if</u> $d \geq b+c$, $a > 1$ <u>and either</u> $b \equiv 1 \pmod 2$ <u>or</u> $(d-c) \equiv 1 \pmod 2$, <u>or both</u>.

As in 8.8 $\{t^b = -x^{d-c}\} \cap X$ has a branch through 0.

<u>8.10</u> : (a) <u>fails over</u> R <u>if</u> $b+c \leq a$, $b+c \leq d$, $b \equiv 0 \pmod 2$ <u>and</u> $d \equiv c \pmod 2$.

$\partial f/\partial y \not\equiv 0$ since $\{t^b = -x^{d-c}\} \cap X$ has no branches near 0 . Let $x = \lambda t$, $\lambda \neq 0$. Then $\dfrac{\partial f/\partial t}{\partial f/\partial y} \sim \dfrac{x^{b+c-1}}{(x^{b+c} + x^d)^{(a-1)/a}} \sim x^{b+c-1-(b+c)(a-1)/a} = x^{(b+c-a)/a}$.

Thus $\dfrac{\partial f/\partial t}{\partial f/\partial y} \to 0$ along $\{x = \lambda t\}$ if $a > b+c$.

Also $\dfrac{\partial f/\partial t}{\partial f/\partial x} \sim \dfrac{x^{b+c-1}}{dx^{d-1} + cx^{b+c-1}}$ on $\{x = \lambda t\}$

$$\sim \frac{x^{b+c-1}}{cx^{b+c-1}} \quad \text{since } d \geq b+c$$

$$\not\to 0 .$$

Hence (a) fails along $\{x = \lambda t\}$.

<u>8.11</u> : (a) <u>holds over</u> R <u>if</u> $a \leq b$, $b \equiv 0 \pmod 2$, <u>and</u> $d \equiv c \pmod 2$.

$\partial f/\partial y \not\equiv 0$ since $\{t^b x^c + x^d\} \neq 0$ except at 0 if b and (d-c) are even. $\dfrac{\partial f/\partial t}{\partial f/\partial y} \sim \dfrac{t^{b-1}x^c}{(t^b x^c + x^d)^{1-1/a}} \leq \dfrac{t^{b-1}x^c}{(t^b x^c)^{1-1/a}} = t^{b-1-b(a-1)/a} x^{c/a} = t^{(b-a)/a} x^{c/a}$

$$\to 0 \quad \text{if } a \leq b .$$

<u>8.12</u> : <u>Let</u> $c < d < b+c$. <u>Then</u> (a) <u>holds over</u> $\mathbb{C}$ <u>if and only if either</u> $a \leq b$ <u>or</u>, $a > b$ <u>and</u> $d < ac/(a-b)$.

After curve selection (2.6) we can reduce to the case of curves along which $|t/x|$ is bounded or unbounded as x and t tend to 0 .

(i) $|t/x|$ is bounded. Then $\partial f/\partial x \not\equiv 0$ and

$$\frac{\partial f/\partial t}{\partial f/\partial x} \sim \frac{t^{b+c-d}(t/x)^{d-c-1}}{d + ct^{b+c-d-1}(t/x)^{d-c}} \to 0 \text{, if } c < d < b+c .$$

(ii) $|x/t|$ tends to 0 .

$$\frac{\partial f/\partial t}{\partial f/\partial x} \sim \frac{(x/t)}{c + dx^{d-c}/t^b} \to 0 \text{ if } dx^{d-c}/ct^b \not\to -1 .$$

Let $dx^{d-c}/ct^b \to -1$.

Then $\dfrac{\partial f/\partial t}{\partial f/\partial y} \sim \dfrac{t^{b-1}x^c}{(t^b x^c + x^d)^{1-1/a}} \sim \dfrac{x^{c+(d-c)(b-1)/b}}{((1-d/c)x^d)^{1-1/a}}$

$\sim x^{c+(d-c)(b-1)/b - d(a-1)/a}$ since $d > c$

$\sim x^{(ac-d(a-b))/ab}$ which

$\to 0$ if $d(a-b) < ac$

$\not\to 0$ if $d(a-b) \geq ac$, when (a) fails

along $dx^{d-c} + ct^b = 0$.

<u>8.13</u> : (a) <u>fails over</u> $\mathbb{R}$ <u>if</u> $c < d < b+c$, $a > b$, $d \geq ac/(a-b)$ <u>and either</u> $b \equiv 1$ (<u>mod</u> 2) <u>or</u> $d \equiv c+1$ (<u>mod</u> 2) , <u>or both</u>.

As in 8.12 , (a) fails along $dx^{d-c} + ct^b = 0$.

<u>8.14</u> : (a) <u>holds over</u> $\mathbb{R}$ <u>if</u> $c < d < b+c$, $b \equiv 0$ (<u>mod</u> 2) , $d \equiv c$ (<u>mod</u> 2) .

8.12 shows that (a) fails only for curves on which $dx^{d-c}/ct^b \to -1$, and these curves have no points on X near 0 if b and d-c are even.

<u>8.15</u> : (a) <u>holds over</u> $\mathbb{R}$ <u>if</u> $b \leq a < b+c = d$, $b \equiv 0$ (<u>mod</u> 2), $d \equiv c$ (<u>mod</u> 2) .

(i) $|x/t|$ bounded near 0 .

$\dfrac{\partial f/\partial t}{\partial f/\partial y} \sim t^{b-1-b(a-1)/a}\, x^{c-c(a-1)/a} = t^{(b-a)/a}\, x^{c/a}$

$= x/t^{\,(a-b)/a}\, x^{(b+c-a)/a}$

$\to 0$ if $b \leq a < b+c$.

(ii) $|t/x|$ tends to 0 .

Suppose t tends to x^θ , $\theta > 1$.

$\dfrac{\partial f/\partial t}{\partial f/\partial x} = \dfrac{bt^{b-1}x^c}{dx^{d-1} + cx^{c-1}t^b} \sim x^{c+b\theta-\theta-d+1} = x^{(b-1)(\theta-1)}$ if $d = b+c$.

$\to 0$.

This completes our calculations of (a)-regularity — the inquisitive reader can work out for himself the remaining cases of (a)-regularity over $\mathbb{R}$: when $b < a < b+c < d$ and $b \equiv 0$ (mod 2) , $d \equiv c$ (mod 2) .

(b') holds at 0 if and only if $\dfrac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y, \partial f/\partial t)|}$ tends to 0 as (x, y, t) tends to $(0, 0, 0)$.

<u>8.16</u> : (b) <u>holds if</u> $a = 1$.

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y,\partial f/\partial t)|} = \frac{(d-1)x^d + (c-1)t^b x^c}{|(x,y)|.|(\partial f/\partial x,\ 1\ ,\partial f/\partial t)|}$$

$$= \frac{(d-1)x^{d-1} + (c-1)t^b x^{c-1}}{|(1,y/x)|.|(\partial f/\partial x, 1, \partial f/\partial t)|}$$

$$\longrightarrow 0 .$$

Now use (8.6) and (0.4).

<u>8.17</u> : (b) <u>holds if</u> $d \leq c$.

Since by (8.7) (a) holds, by (0.4) it is enough to show that $\dfrac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y)|}$ tends to 0, i.e. $\dfrac{(c-a)t^b x^c + (d-a)x^d}{|\cdot\cdot|\,|\ \cdots\cdots\ |}$ tends to 0.

Since $d \leq c$, it is enough to show that $\dfrac{x^d}{|\cdot|.|\cdot\cdot|}$ tends to 0 when $d \neq a$, and $\dfrac{t^b x^c}{|\cdot|.|\cdots|}$ tends to 0 when $d = a$.

(i) <u>$d > a$</u>. $$\frac{x^d}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y)|} = \frac{x^{d-1}}{|(1,y/x)|.|(cx^{c-1}t^b+dx^{d-1},-a(t^b x^c+x^d)^{1-1/a})|}$$

$$\sim \frac{x^{(d/a)-1}}{|(1,y/x)|.|(\ldots, -a(t^b x^{c-d} + 1)^{1-1/a})|}$$

$\longrightarrow$ 0 as $d > a$, unless $t^b x^{c-d} + 1$ tends to 0, but there are no such points near 0 as $d \leq c$.

(ii) <u>$d < a$</u>. $$\frac{x^d}{|(x,y)|.|(\partial f/\partial x,\partial f/\partial y)|} = \frac{x}{|(x,-(t^b x^c+x^d)^{1/a})|.|(d+cx^{c-d}t^b,\ldots)|}$$

$$= \frac{x^{1-d/a}}{|(x,-(t^b x^{c-d}+1)^{1/a})|.|(d+cx^{c-d}t^b,\ldots)|}$$

$\longrightarrow$ 0 since $d < a$, and $d \leq c$.

(iii) <u>$d = a$</u>. $$\frac{t^b x^c}{|(x,y)|.|(cx^{c-1}t^b+dx^{d-1},\partial f/\partial y)|} = \frac{t^b\, x^{c-d}}{|(1,y/x)|.|(ct^b + d,\ldots)|}$$

$\longrightarrow$ 0 since $d \leq c$.

8.18 : (b') fails over $\mathbb{C}$ if $c < d$ and $a > 1$, and (a) holds.

$y = 0$ and $\partial f/\partial y \equiv 0$ on $t^b x^c + x^d = 0$. Then

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y)|} = \frac{(d-c)x^d}{|(x,0)|.|(ct^b x^{c-1} + dx^{d-1}, 0)|}$$

$$= \frac{(d-c)x^{d-1}}{|(1,0)|.|((d-c)x^{d-1}, 0)|}$$

$\longrightarrow 0$, so (b') fails, and hence (b) fails.

8.19 : (b') fails over $\mathbb{R}$ if $a > 1$, $c < d$ and either $b \equiv 1 \pmod 2$ or $d \equiv c+1 \pmod 2$ or both.

$X \cap \{t^b x^c + x^d = 0\}$ has real branches through 0 if b or $(d-c)$ is odd.

8.20 : (b') holds over $\mathbb{R}$ if $d < a$, $b \equiv 0 \pmod 2$ and $d \equiv c \pmod 2$.

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|.|(\partial f/\partial x, \partial f/\partial y, \partial f/\partial t)|} = \frac{(d-a)x^d + (c-a)t^b x^c}{|(x,(t^b x^c + x^d)^{1/a})|.|(ct^b x^{c-1} + dx^{d-1}, \dots, \dots)|}$$

$$= \frac{(d-a)x^{1-d/a}}{|(x^{1-d/a},(t^b x^{c-d} + 1)^{1/a})|.|(ct^b x^{c-d} + d, \dots, \dots)|}$$

$$+ \frac{(c-a)x^{1-d/a}}{|(x^{1-d/a},(t^b x^{c-d} + 1)^{1/a})|.|(c + dx^{d-c} t^{-b}, \dots, \dots)|}$$

$\longrightarrow 0$ if $d < a$.

This completes our calculations of (b')- and (b)-regularity save for the case $1 < a \leq d$, $c < d$, $b \equiv 0 \pmod 2$, $d \equiv c \pmod 2$, over $\mathbb{R}$.

Example 8.21 : J. J. Risler asked for an example which was (a)-regular over $\mathbb{R}$, but not over $\mathbb{C}$. By 8.11 and 8.8 it suffices that $a \leq b \leq d-c$, $b \equiv 0 \pmod 2$ and $d \equiv c \pmod 2$. For example $\{y^2 = t^2 x + x^3\}$.

Example 8.22 : For an example which is (b)-regular over $\mathbb{R}$ but not over $\mathbb{C}$, 8.12, 8.18, and 8.20 give $c < d < a \leq b$ (or $c < d < a$, $b < a$, $d < ac/(a-b)$) $b \equiv 0 \pmod 2$, $d \equiv c \pmod 2$. For example $\{y^4 = t^4 x + x^3\}$ or $\{y^5 = t^4 x + x^3\}$.

Example 8.23 : If an equimultiple example is demanded, satisfying the requirements of 8.22 , consider $\{y^2 = t^2x^2 + x^4\}$. By 8.8 (a) fails over $\mathbb{C}$, and by 8.11 (a) holds over $\mathbb{R}$. It remains to check that (b') holds over $\mathbb{R}$, using (0.4) .

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)|\,.\,|(\partial f/\partial x, \partial f/\partial y)|} = \frac{2x^4}{|(x,(t^2x^2+x^4)^{\frac{1}{2}})|\,.\,|(4x^3+2t^2x,\ -2(x^4+t^2x^2)^{\frac{1}{2}})|}$$

$$= \frac{2\,x}{|(1,(t^2+x^2)^{\frac{1}{2}})|\,.\,|(4x+2t^2/x,\ -2(1+(t/x)^2)^{\frac{1}{2}})|}$$

$\longrightarrow 0$ as (x,t) tends to 0 since $X \cap \{t^2 + x^2 = 0\}$ has no branches passing through 0 . Hence (b) holds over $\mathbb{R}$.

Note 8.24 : Table 8.3 corresponds with the known fact that for families of plane curves, " μ -constant " is equivalent to (b)-regularity ([30]) .

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Geometric versions of Whitney regularity

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Let X^m and Y^n be C^1 manifolds embedded in $\mathbb{R}^p$, $m < n < p$, and let $x \in X \subset \bar{Y} - Y$. In (4) C. T. C. Wall considered the following conditions:

(a_s) For any local C^1 retraction at x, π: $\mathbb{R}^p \to X$, x has a neighbourhood U such that $\pi|_{Y \cap U}$ is a submersion.

(b_s) For any local C^1 tubular neighbourhood of X at x, given by π: $\mathbb{R}^p \to X$ and ρ: $\mathbb{R}^p \to \mathbb{R}_+ \cup \{0\}$, where $\rho^{-1}(0) = X$, x has a neighbourhood U such that $(\pi, \rho)|_{Y \cap U}$ is a submersion.

Wall conjectured that (a_s) and (b_s) are respectively equivalent to Whitney's conditions (a) and (b):

(a) Given $y_i \in Y$ so that, as $i \to \infty$, $y_i \to x$ and $T_{y_i} Y \to \tau$, then $T_x X \subset \tau$.

(b) Given $y_i \in Y$ and $x_i \in X$ so that, as $i \to \infty$, $y_i \to x$, $x_i \to x$, $T_{y_i} Y \to \tau$ and $y_i - x_i/|y_i - x_i| = \lambda_i \to \lambda$, then $\lambda \subset \tau$.

It is not difficult to show that (a) implies (a_s). See (2), p. 35, for a proof that (b) implies (b_s); this enabled Mather to show that if X is a stratum of a (b)-regular stratification Σ, then Σ is locally topologically trivial over X. In (3), § 3, it is proved that (a_s) implies (a) if Y is semianalytic. Here we prove the following,

THEOREM. (b_s) *implies* (b) *if* X *and* Y *are semianalytic*. (C. G. Gibson has also obtained this result.)

Note. The conjectured equivalences have been verified in exactly the cases where the curve selection lemma is applicable. It would be interesting to know if they are true in the general, i.e. non-semianalytic, case, so as to have geometric versions of the regularity conditions available, avoiding sequences.

Proof of the theorem. Suppose (b) fails; we shall show that (b_s) fails.

We have sequences $x_i \in X$, $y_i \in Y$ tending to x, $T_{y_i} Y \to \tau$, and $y_i - x_i/|y_i - x_i| = \lambda_i \to \lambda$. Since $\lambda \not\subset \tau$ we may suppose that $d(\lambda, \tau) > \epsilon > 0$ for some ϵ, with distance $d(\ ,\ )$ defined appropriately. Then, for some i_0, $d(\lambda_i, T_{y_i} Y) > \epsilon$ when $i \geq i_0$.

Let G^r_s denote the Grassmannian of s-planes in $\mathbb{R}^r$, a compact analytic manifold. Set

$$V_1 = \{(v, P) \in G^p_1 \times G^p_n : d(v, P) > \epsilon\}$$

and

$$V_2 = \{(x, y, y - x/|y - x|, T_y Y) : x \in X, y \in Y\}.$$

Then V_1 is semialgebraic, and V_2 is semianalytic since both X and Y are semianalytic by hypothesis. Hence

$$V = (X \times Y \times V_1) \cap V_2$$

is a semianalytic subset of $\mathbb{R}^p \times \mathbb{R}^p \times G_1^p \times G_n^p$, and $(x, x, \lambda, \tau) \in \bar{V}$ satisfies the hypotheses of the curve selection lemma. See (1), p. 103. This provides an analytic curve

$$\alpha\colon [0,1] \to X \times \bar{Y} \times G_1^p \times G_n^p,$$
$$t \mapsto (x_t, y_t, \lambda_t, T_{y_t} Y),$$

where $\lambda_t = y_t - x_t / |y_t - x_t|$, $y_t \in Y$ if $t \neq 0$, and $d(\lambda_t, T_{y_t} Y) > \epsilon$.

Write η for the C^1 manifold-with-boundary $\bigcup_t y_t$, and ξ for $\bigcup_t x_t$, contracting the domain of α if necessary.

Since we are trying to show that (b_s) fails, and (b_s) implies (a_s), we may assume that (a_s) holds. Then by (3), § 3, since Y is semianalytic, (a) holds. This implies that

$$T_x \eta = T_x \xi. \tag{*}$$

For, suppose not. Then

$$\begin{aligned} \lambda &\subset T_x \xi \oplus T_x \eta \\ &\subset T_x X + T_x \eta \\ &\subset \tau \end{aligned}$$

using (a). But $\lambda \not\subset \tau$ by hypothesis, giving (*).

Notation. Given distinct lines λ, λ' in the plane meeting at a point q, and a point q' on λ' at unit distance from q, consider the circles with tangent λ at q which contain q' in their interior. If $\epsilon = d(\lambda, \lambda')$ let r_ϵ denote the lower limit of the radii of these circles.

Lemma. *There exists a local C^1 retraction defined on a neighbourhood U of x in $\mathbb{R}^p$, $\pi\colon U \to X$, such that for each t, $\pi^{-1}(x_t)$*

(i) *is the intersection with U of a $(p-m)$-plane containing λ_t,*

(ii) *is transverse to Y in U,*

(iii) *contains a $(p-m)$-disc D_t of radius $r_\epsilon |y_t - x_t|$ with $y_t \in \partial D_t$, $x_t \in \operatorname{Int} D_t$, and*

$$T_{y_t}(Y \cap \pi^{-1}(x_t)) \subset T_{y_t}(\partial D_t),$$

(iv) *intersects η only at y_t.*

Proof. Because (b) fails and (a) holds, $\lambda \not\subset T_x X$. Thus there exists a $(p-m)$-plane transverse to X at x, and containing λ. Using (*) and the analytic dependence of y_t, λ_t, and $T_{y_t} Y$ upon t, we can find an analytic, and hence a C^1, fibre bundle over ξ, restricting α if necessary, so that the fibre over x_t is a $(p-m)$-plane containing λ_t. Choose a C^1 diffeomorphism ϕ of an open neighbourhood U of x in $\mathbb{R}^p$, so that $\phi(X \cap U)$ is affine and $\phi(\xi \cap U)$ is a line. Extend the fibration over $\phi(\xi)$ to the rest of $\phi(X \cap U)$ by parallel translation, and pull back by ϕ^{-1} to give a C^1 retraction $\pi\colon U \to X$ with each fibre C^1 diffeomorphic to $\mathbb{R}^{p-m}$, and which satisfies (i).

For (ii) use (a_s), shrinking U if necessary, and observe that $\pi|_Y$ is a submersion at y if and only if $\pi^{-1}(\pi(y))$ is transverse to Y at y. (ii) tells us that $Y \cap \pi^{-1}(x_t)$ is a C^1 $(n-m)$-manifold.

Let D_t be a disc of radius $r_\epsilon |y_t - x_t|$ in the $(p-m)$-plane of (i), with y_t on its boundary and so that

$$T_{y_t}(Y \cap \pi^{-1}(x_t)) \subset T_{y_t}(\partial D_t).$$

Because $d(\lambda_t, T_{y_t} Y) > \epsilon$ and r_ϵ is a decreasing function of ϵ, x_t belongs to the interior of D_t. For sufficiently small t, $D_t \subset \pi^{-1}(x_t)$, giving (iii).

Finally use (*), restricting α if necessary, to ensure that $\eta \cap \pi^{-1}(x_t) = y_t$. This proves (iv) and completes the proof of the lemma.

Project λ_t onto $N_{y_t}(\partial D_t)$ to give $\mu_t \in G_1^p$. By (iii) each μ_t is non-zero and

$$\mu_t \subset N_{y_t}(Y \cap \pi^{-1}(x_t)).$$

Now we construct a tubular function ρ so that $\rho(y_t) = t$ and

$$\mu_t \subset N_{y_t}((\pi, \rho)^{-1}(x_t, t)).$$

This will show that Y is not transverse to the fibre of (π, ρ) at y_t, for each t, which is the same as saying that $(\pi, \rho)|_Y$ is not a submersion at y_t, for each t, so that (b_s) fails.

It suffices then to find ρ so that

$$\partial D_t = (\pi, \rho)^{-1}(x_t, t)$$

for each t. Let ϕ be as in the proof of the lemma, and for each $t > 0$ let P_t be obtained by first translating $\phi(\partial D_t)$ along $\phi(\xi)$, using (iv), and then over $\phi(X \cap U)$ orthogonal to $\phi(\xi)$. Shrink U so that

$$\bigcup_{t>0} \phi^{-1}(P_t) = U \backslash (X \cap U).$$

Then we have a C^1 fibration

$$\rho \colon U \backslash (X \cap U) \to (0, 1],$$

with $\rho^{-1}(t) = \phi^{-1}(P_t)$ a C^1 manifold C^1 diffeomorphic to $S^{p-m-1} \times \mathbb{R}^m$. Setting $\rho|_{X \cap U} \equiv 0$ extends ρ to be C^1 on U, and ρ is the required tubular function. This completes the proof of the theorem.

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WHITNEY (b)-REGULARITY IS WEAKER THAN KUO'S RATIO TEST FOR REAL ALGEBRAIC STRATIFICATIONS

HANS BRODERSEN and DAVID TROTMAN[1]

We give examples of real algebraic hypersurfaces such that the full partition by dimension gives a stratification which is Whitney (b)-regular, but which fails to satisfy Kuo's ratio test (r), and hence also fails to satisfy the (w)-regularity of Verdier. Such a hypersurface can be a C^1 submanifold, so that the stratification is C^1 trivial, showing that (r) and (w) are not invariant under C^1 changes of coordinates, although they are C^2 invariant. We show that (w)-regularity is characterised by the possibility of extending rugose vector fields defined on some strata to rugose vector fields tangent to the remaining strata.

1. On regularity.

Let X be a C^1 submanifold of $\mathbf{R}^n$, and a subanalytic set (defined in [2]). Let Y be an analytic submanifold of $\mathbf{R}^n$ such that $0 \in Y \subset \bar{X} \setminus X$. Verdier [8] defines X to be (w)-*regular* over Y at 0 if,

(w) There is a constant $C>0$ and a neighborhood U of 0 in $\mathbf{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqq C|x-y|$.

Here $d(.,.)$ is defined as follows.

Definition. Let A, B, be vector subspaces of $\mathbf{R}^n$.

$$d(A,B) = \sup_{\substack{a \in A \\ |a|=1}} |a-\pi_B(a)| ,$$

where π_B is orthogonal projection onto B.

This is not symmetric in A and B. Clearly $d(A,B)=0$ if and only if $A \subseteqq B$. It is clear from the definition of (w) that it is a C^2 invariant, or more precisely

[1] The work on this paper was done while both authors were at the Matematisk Institut in Aarhus, Denmark.

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that it is invariant under a C^1 diffeomorphism with Lipschitz derivative. We shall see below that it is not a C^1 invariant.

Kuo's ratio test.

We suppose that Y is linear (apply a local analytic isomorphism at 0 to $\mathbf{R}^n$). Let π_Y denote orthogonal projection onto Y.

Reformulate (w) by the condition that $d(T_yY, T_xX)/|x-y|$ is bounded near 0. Then in particular $d(T_0Y, T_xX)/|x-\pi_Y(x)|$ is bounded for x near 0 (recall Y is linear). Then it is clear that if X is (w)-regular over Y at 0, then $(X, Y)_0$ satisfies the ratio test of Kuo [3]:

(r) Given any vector $v \in T_0Y$,

$$\lim_{\substack{x \to 0 \\ x \in X}} \frac{|\pi_x(v)| \cdot |x|}{|x - \pi_Y(x)|} = 0 .$$

Here π_x denotes orthogonal projection onto the normal space to X at x, so that for unit vectors v, $|\pi_x(v)| = d(\langle v \rangle, T_xX)$. In [3] Kuo proved that (r) implies Whitney (b)-regularity (defined in [9]) and that (b) implies (r) when Y is 1-dimensional. In [6] a fairly complicated semialgebraic example was given with Y 2-dimensional showing that (b) is weaker than (r). We give a simple algebraic example below.

First observe that if (b) (respectively (w)) holds for a pair of strata (X, Y) at 0 in $\mathbf{R}^n$, then (b) (respectively (w)) holds for $(X \times \mathbf{R}, Y \times \mathbf{R})$ along $0 \times \mathbf{R}$ in $\mathbf{R}^n \times \mathbf{R}$. However (r) does not have this property.

Proposition 1. *Let (X, Y) be a pair of strata in $\mathbf{R}^n$ not* (w)*-regular at* 0 (*but possibly satisfying* (r)) *and let Y be linear. Then $(X \times \mathbf{R}, Y \times \mathbf{R})$ fails to satisfy* (r) *at any point of $0 \times \mathbf{R}$ in $\mathbf{R}^n \times \mathbf{R}$.*

Proof. Let X, Y have dimensions m, p respectively and identify the set of one dimensional subspaces of T_0Y with the Grassmannian G_1^p.

Define three subsets of $\mathbf{R}^n \times \mathbf{R}^n \times G_m^n \times G_1^p \times \mathbf{R}$:

$$V_1 = \{(x, \pi_Y(x), T_xX) : x \in X\} \times G_1^p \times \mathbf{R}$$

$$V_2 = \{(x, y, T, \langle v \rangle, \varepsilon) : |x-y| < \varepsilon d(\langle v \rangle, T)\}$$

$$V_3 = \mathbf{R}^n \times \mathbf{R}^n \times \{(T, \langle v \rangle) : d(\langle v \rangle, T) = d(T_0Y, T)\} \times \mathbf{R}$$

V_1 is subanalytic using Verdier [8, Lemma 1.6] (by restricting to a compact neighbourhood of 0 in $\mathbf{R}^n$ if necessary), V_2 is semialgebraic, and V_3 is algebraic, Hence $V = V_1 \cap V_2 \cap V_3$ is a subanalytic set.

We have that (w) fails for the pair (X, Y) at 0, which is equivalent to the existence of $\tau \in G_m^n$ and $v \in T_0 Y$ with $\|v\| = 1$ such that

$$(0, 0, \tau, \langle v \rangle, 0) \in \bar{V} \subset \mathbf{R}^n \times \mathbf{R}^n \times G_m^n \times G_1^p \times \mathbf{R} .$$

By curve selection [2] we can find an analytic arc

$$\alpha\colon [0, 1] \to \mathbf{R}^n \times \mathbf{R}^n \times G_m^n \times G_1^p \times \mathbf{R} ,$$

such that $\alpha(0) = (0, 0, \tau, \langle v \rangle, 0)$ and such that $\alpha(t) \in V$ if $t \neq 0$. Write

$$\alpha(t) = (x_t, \pi_y(x_t), T_{x_t}X, \langle v_t \rangle, \varepsilon_t)$$

where $v_t \in T_0 Y$, $\|v_t\| = 1$ and $v_t \to v$ as $t \to 0$. Then

$$\frac{d(\langle v_t \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|}$$

is unbounded as t tends to 0. We assert that

$$d(\langle v \rangle, T_{x_t}X) \geqq \tfrac{1}{2} d(\langle v_t \rangle, T_{x_t}X)$$

for t sufficiently small. This is a consequence of the definition of V_3, as follows:

Let $v = v_t \cos\varphi_t + u_t \sin\varphi_t$ where $\|u_t\| = 1$, $v_t \perp u_t$ and φ_t is the positive angle between v and v_t, we can assume $0 \leqq \varphi_t < \pi/2$. Let π_t denote the orthogonal projection onto $T_{x_t}X$. Then

$$\begin{aligned} d(\langle v \rangle, T_{x_t}X) = |v - \pi_t(v)| &= |(v_t - \pi_t(v_t)) \cos\varphi_t + (u_t - \pi_t(u_t)) \sin\varphi_t| \\ &\geqq |v_t - \pi_t(v_t)| \cos\varphi_t - |u_t - \pi_t(u_t)| \sin\varphi_t \\ &\quad \text{(using the triangle inequality)} \\ &\geqq |v_t - \pi_t(v_t)| (\cos\varphi_t - \sin\varphi_t) \\ &\quad \text{(By definition of } V_3, |v_t - \pi_t(v_t)| \geqq |u_t - \pi_t(u_t)|) \\ &= d(\langle v_t \rangle, T_{x_t}X)(\cos\varphi_t - \sin\varphi_t) \end{aligned}$$

Since φ_t tends to 0 as t tends to 0, it follows that, for t sufficiently small,

$$d(\langle v \rangle, T_{x_t}X) \geqq \tfrac{1}{2} d(\langle v_t \rangle, T_{x_t}X) .$$

We deduce that $d(\langle v \rangle, T_{x_t}X)/|x_t - \pi_y(x_t)|$ is also unbounded as t tends to 0. After reparametrisation we can suppose that

$$\frac{d(\langle v \rangle, T_{x_t}X)}{|x_t - \pi_y(x_t)|} \sim t^{-k} \quad \text{for some } k \geqq 1$$

In $\mathbf{R}^n \times \mathbf{R}$ consider the curve $q(t) = (x_t, t_0 + t)$. Using the canonical inclusion $T_0 Y \subset T_{(0, t_0)}(Y \times \mathbf{R})$, we can consider v as a unit vector of $T_{(0, t_0)}(Y \times \mathbf{R})$. Then

$$\frac{d(\langle v\rangle, T_{q(t)}(X\times \mathbf{R}))\cdot|q(t)-(0,t_0)|}{|q(t)-\pi_{Y\times\mathbf{R}}(q(t))|}$$

$$= \frac{d(\langle v\rangle, T_{x_t}X)\cdot|(x_t,t)|}{|x_t-\pi_y(x_t)|}$$

$$\geq \frac{d(v, T_{x_t}X)\cdot t}{|x_t-\pi_Y(x_t)|} \sim t^{-(k-1)},$$

which does not tend to zero as t approaches zero since $k\geq 1$. Hence the ratio test (r) fails for the pair $(X\times\mathbf{R}, Y\times\mathbf{R})$ at every point $(0,t_0)$ of $0\times\mathbf{R}$ in $\mathbf{R}^n\times\mathbf{R}$, completing the proof of Proposition 1.

EXAMPLE 1. Let $V=\{y^3=z^2x^3+x^5\}\subset\mathbf{R}^3$, and let Y be the z-axis and $X=V-Y$.

$(z^2x^3+x^5)^{1/3}$ is a C^1 function of x and z, and so V, as the graph of a C^1 map, is a C^1 submanifold of $\mathbf{R}^3$. Hence X is (b)-regular over Y. By Theorem 2 of [3] we deduce that (X,Y) satisfies (r) at 0, since dim $Y=1$.

Consider the curve $p(t)=(t^3,\sqrt[3]{2}\cdot t^5,t^3)$ from the origin into X. The normal direction to X at (x,y,z) is $(3x^2z^2+5x^4: -3(z^2x^3+x^5)^{2/3}: 2zx^3)$. At $p(t)$ this becomes

$$(8t^2: -3\cdot 2^{2/3}: 2t^2)\,.$$

So

$$d(T_0Y, T_{p(t)}X) = \frac{2t^2}{(68t^4+18\sqrt[3]{2})^{\frac{1}{2}}}$$

and

$$\frac{d(T_0Y, T_{p(t)}X)}{|p(t)-\pi_Y(p(t))|} \sim \frac{t^2}{t^3} \sim \frac{1}{t},$$

which is unbounded as t approaches zero, so that (w) fails for (X,Y) at 0.

Now let

$$V' = V\times\mathbf{R} = \{y^3=z^2x^3+x^5\} \subset \mathbf{R}^4 = \{(x,y,z,u)\}\,.$$

Let

$$Y' = Y\times\mathbf{R} = \{y=x=0\} \subset \mathbf{R}^4 \quad\text{and}\quad X' = V'-Y'\,.$$

By Proposition 1, (X',Y') fails to satisfy (r) at any point of $0\times\mathbf{R}$ (for example consider the curve $q(t)=(p(t),t)$ from 0 into X'). But since V' is a C^1 submanifold, (X',Y') is (b)-regular.

Example 1 describes the first example of a pair (X,Y) satisfying (b) but not (r) where X is the regular part of an algebraic variety and Y the singular locus. Contrast this with the complex hypersurface case where (b)-regularity, the ratio test, and (w)-regularity are equivalent. This is a consequence of the equivalence of (b)-regularity with Teissier's (c)-cosecance [5] (references for the implications giving this equivalence may be found in [1]); (c)-cosecance trivially implies (w)-regularity, and hence also the ratio test. It remains to be seen whether (b), (r) and (w) are distinct when V is a complex analytic variety of codimension greater than 1.

EXAMPLE 2 (from [7]). $V \equiv \{y^4 = z^4 x + x^3\} \subset \mathbf{R}^3$, $Y = \{z\text{-axis}\}$, $X = V \setminus Y$. Here y is not a C^1 function of x and z, but V is still a C^1 submanifold of $\mathbf{R}^3$, so that (b) holds for (X, Y). (w) fails along the curve $p(t) = (t^4, \sqrt[4]{2} \cdot t^3, t^2)$. As with Example 1 we can apply Proposition 1 to show that $(X \times \mathbf{R}, Y \times \mathbf{R})$ fails to satisfy (r) on $0 \times \mathbf{R}$ in $\mathbf{R}^4$, but (b) clearly holds.

EXAMPLE 3 (due to Kuo [4]). $V \equiv \{y^4 = z^2 x^5 + x^7\} \subset \mathbf{R}^3$, Y the z-axis, $X = V - Y$. V is no longer a C^1 submanifold – for each z, $y^4 = z^2 x^5 + x^7$ defines a plane curve of "cusp type" near 0. However (b) does hold and (w) fails. We can apply Proposition 1 as before.

Examples 1 and 2, and indeed the second discordant horn of [6], show that (r) and (w) are not invariant under C^1 diffeomorphisms. So (b) is more natural in differential topology; it is a C^1 invariant.

Looking closely at the proofs in [3] we see why it is not surprising that (r) is strictly stronger than (b) when $\dim Y \geqq 2$. It is proved in [3] that (b) is equivalent to the conjunction of (a) and (r′) defined as follows.

(r′) If $p(t)$, $t \in [0, 1]$ is an analytic arc in $\mathbf{R}^n$ with $p(0) = 0$ and $p(t) \in X$ for $t \neq 0$, then

$$\lim_{t \to 0} \frac{|\pi_t(v)||p(t)|}{|p(t) - \pi_Y(p(t))|} = 0 ,$$

where v is the tangent at 0 to the arc $\pi_Y \circ p([0, 1])$ on Y, and π_t is projection onto the normal space to X at $p(t)$.

It is obvious that (r) implies (a) + (r′) and that (a) + (r′) implies (r) when Y has dimension one. Being able to choose a vector v in $T_0 Y$ and a curve whose tangent at 0 is orthogonal to v suggested the counterexample in [6], and gives rise to the examples here too.

Rugose vector fields.

Given a (b)-regular stratification, one might hope to be able to find rugose vector fields tangent to the strata. Verdier shows that these exist on (w)-regular stratifications [8] and derives rugose trivialisations. However it can be impossible to extend a constant vector field on a base stratum Y to a rugose vector field on an attaching stratum X when (X, Y) is (b)-regular. This is a consequence of our next proposition and the existence of (b)-regular examples which do not satisfy (w).

We refer to [8] for the definition of rugose vector field. (Note the misprint in the definition of rugose function on page 307 of [8], as described below).

PROPOSITION 2. *Let X be a C^2 submanifold of $\mathbf{R}^n$ and let $Y=\mathbf{R}^m\times 0\subset \mathbf{R}^n$. Suppose that each of the constant vector fields $\{\partial/\partial y_i\}$, $i=1,\ldots,m$, on Y extends to a rugose vector field on $X\cup Y$. Then X is (w)-regular over Y.*

PROOF. Let $\tilde{v}_i$ denote the extension of $\partial/\partial y_i$. For each i there exists a constant C and a neighbourhood U of 0 such that

$$\left|\tilde{v}_i(x)-\frac{\partial}{\partial y_i}\right| \leqq C|x-y|$$

for all $x \in U\cap X$, $y\in U\cap Y$. We can assume that C and U are the same for all i. Let $x\in U$. Then

$$d\left(\frac{\partial}{\partial y_i}, T_xX\right) \leqq \left|\frac{\partial}{\partial y_i}-\tilde{v}_i(x)\right|,$$

hence

$$(*)\qquad d\left(\frac{\partial}{\partial y_i}, T_xX\right) \leqq C|x-y| \quad \text{for all } x\in X\cap U,\ y\in Y\cap U\,.$$

Take $v\in T_yY$ with $|v|=1$.

$$v = \sum_{i=1}^m a_i\frac{\partial}{\partial y_i}, \quad \text{with } \sum_{i=1}^m a_i^2 = 1\,.$$

Let N_xX denote the orthogonal complement of T_xX in $\mathbf{R}^n$ and $\pi_x\colon \mathbf{R}^n\to N_xX$ the orthogonal projection.

$$\begin{aligned} d(v, T_xX) = |\pi_x(v)| &= \left|\sum_{i=1}^m a_i\pi_x\left(\frac{\partial}{\partial y_i}\right)\right|\,. \\ &\leqq \sum_{i=1}^m \left|\pi_x\left(\frac{\partial}{\partial y_i}\right)\right| \\ &= \sum_{i=1}^m d\left(\frac{\partial}{\partial y_i}, T_xX\right). \\ &\leqq mC|x-y| \quad \text{by } (*)\,. \end{aligned}$$

Hence

$$d(T_yY, T_xX) = \sup_{\substack{|v|=1 \\ v\in T_yY}} d(v, T_xX) \leqq mC|x-y| \quad \text{for all } x \in X\cap U,\ y \in Y\cap U\ ,$$

i.e. X is (w)-regular over Y at 0. Repeating the above argument for each $y \in Y$, we obtain that X is (w)-regular over Y, completing the proof of Proposition 2.

COROLLARY. *Let $A = X\cup B$ be a closed subset of $\mathbf{R}^n$, $B\cap X = \emptyset$, X a C^2 submanifold, B a closed subset, and let (B, Σ) be a* (w)*-regular stratification, with each stratum a C^2 submanifold. Then the stratification Σ' of A given by adding X to Σ is* (w)*-regular if and only if every rugose vector field on B tangent to Σ can be extended to a rugose vector field on A tangent to Σ'.*

PROOF. "Only if" is proved by Verdier [8]. "If" follows from Proposition 2 above by making the stratum containing a given point y, affine near y, by a C^2 change of local coordinates.

WARNING. The definition of rugosity in [8] should read "for all $x \in S_\alpha$, there is a constant C and a neighbourhood V of x such that for all $x' \in V\cap S_\alpha$ and all $y \in V\cap A$,

(**) $$|f(x')-f(y)| \leqq C|x'-y|\text{"}$$

and *not*

(***) $$\text{"}|f(x')-f(y)| \leqq C|x-y|\text{"}\ .$$

To see that these are effectively distinct notions in the case of vector fields we can use Example 2. (w) fails, so by Proposition 2 no lift of $\partial/\partial z$ satisfies (**). However the canonical lift of $\partial/\partial z$ (namely the vector field $v(x,y,z)$ on V defined by projecting $\partial/\partial z$ onto the tangent space to X at each point of X) satisfies (***) as follows.

Let $f(x,y,z) = -y^4+z^4x+x^3$. Then

$$v(x,y,z) = (0,0,1)-\frac{(f_x, f_y, f_z)}{|\mathrm{grad}\, f|}\cdot\frac{f_z}{|\mathrm{grad}\, f|}\ .$$

Hence

$$|v(x,y,z)-(0,0,1)| = \frac{|f_z|}{|\mathrm{grad}\, f|}\ .$$

We must check that $|v(x,y,z)-(0,0,1)|/|(x,y,z)|$ is bounded as (x,y,z) tends to 0 on X.

$$\frac{|v(x,y,z)-(0,0,1)|}{|(x,y,z)|} = \frac{|f_z|}{|\text{grad}\, f|\cdot|(x,y,z)|}$$

$$= \frac{|4z^3x|}{|(z^4+3x^2, -4(z^4x+x^3)^{3/4}, 4z^3x)|\cdot|(x,(z^4x+x^3)^{1/4}, z)|}$$

CASE 1. $|x/z^2| \leqq 1$. Dividing through by z^5, gives

$$\frac{|4x/z^2|}{|(1+(3x^2/z^4),\ldots)|\cdot|(x/z,\ldots,1)|}$$

which is at most 4.

CASE 2. $|z^2/x| \leqq 1$. Dividing through by x^2z, gives

$$\frac{|4z^2/x|}{|(z^4/x^2+3,\ldots,4z^3/x)|\cdot|(x/z,\ldots,1)|}$$

which is at most 4/3.

We have shown that (***) is satisfied.

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Partial results on the topological invariance of the multiplicity of a complex hypersurface.

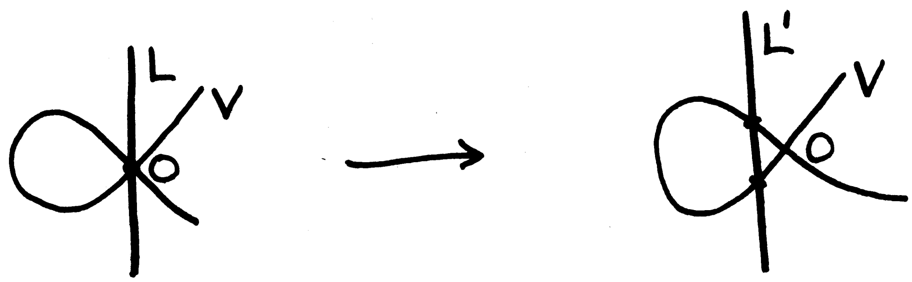
David Trotman

The Problem. In 1971, Zariski posed several problems concerning eqùisingularity [21] ; most of these have been resolved by the French School in Paris or Nice [5],[17] . However the first problem, apparently the simplest, has not yet been decided.

Question : Is the multiplicity of a complex hypersurface a topological invariant ?

That is, given two hypersurfaces V_1 , V_2 in $\mathbb{C}^{n+1}$, with $0 \in V_i$ ($i = 1,2$) and a homeomorphism h of a neighbourhood N_1 of 0 in $\mathbb{C}^{n+1}$ onto a neighbourhood N_2 sending $N_1 \cap V_1$ onto $N_2 \cap V_2$ and fixing 0 , is the multiplicity $m(V_1)$ of V_1 at 0 equal to that of V_2 ?

Definition. The multiplicity $m(V)$ at 0 of a complex hypersurface V is the number obtained as follows : take a generic complex line L passing through 0 , i.e. L not in the tangent cone to V at 0 (so that $L \cap V = \{0\}$ near 0) , then perturb L away from 0 slightly and count the number of points of intersection of V with the new line.



Clearly $m(V)$ is also equal to the degree of the leading form of a function f defining V $(= f^{-1}(0))$.

$$f(z_0,\dots,z_n) = f_m(z) + f_{m+r_1}(z) + \dots .$$

homogeneous polynomials

Concerning this problem, in his survey paper [21] Zariski said,

" I would be disappointed if topologists do not provide an answer in a short time."

Thom has pronounced it a scandal that the answer is not known.

Let me describe what is known.

Curves $(n = 1)$ [15]

In this case the result has been known for more than 40 years (work of Burau, Brauner, Zariski). Two plane curves are topologically equivalent if and only if their branches correspond under the equivalence, and the multiplicity of a curve is just the sum of the multiplicities of the branches (and these are always positive in the complex case). Moreover for a single branch the Puiseux exponents are topological invariants, and there is a formula for the multiplicity in terms of these exponents.

Surfaces $(n = 2)$

Nothing is known.

For hypersurfaces of small Milnor number ($\mu \leq 15$) there is no counterexample - see Arnol'd's lists.

If we ask what happens for small $m(V)$ we only get as far as the following result. Let V_1 be nonsingular and topologically equivalent near 0 with a hypersurface V_2 with an isolated singularity at 0. Then $m(V_1) = 1$, and the link $K_1 = V_1 \cap S_\varepsilon^{2n+1}$ is an unknotted $(2n-1)$-sphere, and so $(S^{2n+1} - K_1)$ has the homotopy type of S^1. The homotopy exact sequence of the Milnor fibration $S_\varepsilon - K_2 \xrightarrow{f/\|f\|} S^1$ becomes

$$\cdots \longrightarrow \pi_{n+1}(S^1) \longrightarrow \pi_n(F) \longrightarrow \pi_n(S_\varepsilon - K_2) \longrightarrow \cdots .$$

Thus $H_n(F_2) = 0$, and $\mu(F_2) = 0$, so that $m(V_2) = 1$.

<u>Question</u> : Are there two surfaces V_1, V_2 in $\mathbb{C}^3$ with $m(V_1) = 2$, $m(V_2) = 3$, and $(V_1,0) \underset{top}{\sim} (V_2,0)$?

(It is evident that we know very little.)

A recent result which ought to be helpful is due to Henry King [9].

<u>Theorem(King)</u>: If $n \neq 2$, and V_1, V_2 are topologically equivalent hypersurfaces, then if $V_i = f_i^{-1}(0)$, the functions f_1, f_2 are topologically equivalent (up to conjugation).

Thus we may assume that the homeomorphism h preserves the level surfaces of the functions defining the hypersurfaces near 0. (This is false for real hypersurfaces if $n \geqslant 6$, see King [8].)

Some restriction on the nature of any counterexample may be derived from the following theorem.

<u>Theorem (A'Campo [1])</u> : Let h be the characteristic homeomorphism of the monodromy of a hypersurface V. Then the Lefschetz number of h^k is zero if $k < m(V)$.

Corollary : $V_1 \underset{top}{\sim} V_2$, $m_1 > m_2$ $\Longrightarrow \Lambda(h_2^{m_2}) = 0$.

Unfortunately it is possible for $(h_2^{m_2})$ to equal zero.

Example: Weighted homogeneous f , $V = f^{-1}(0)$, h periodic of period p , $m(V) = m$. If $(m,p) = 1$, then $\Lambda(h^m) = 0$.

For an explicit case, let $f(x,y,z) = x^2y + y^4 + z^8$.

This has type $(8/3, 4, 8)$, $m = 3$, $p = 8$.

$h(x, y, z) = (u^3x , u^2y , uz)$ where $u = e^{2\pi i/8}$.

$\mathrm{Fix}\,(h^3) = \emptyset \Longrightarrow \chi(\mathrm{Fix}\,(h^3)) = 0$, so $\Lambda(h^3) = 0$.

Intersection number.

Looking at the definition of multiplicity given above we see that $m(V)$ is the intersection number $i(V,L)$ of V with a generic complex line L passing through 0 [12] . Thus if we could replace the given homeomorphism (perhaps by isotopy) by one which mapped some generic complex line (for V_1) onto a complex line generic with respect to V_2 then we would obtain the desired result, since intersection number is invariant under homeomorphism of pairs.

By Lemma 1.4 of [20] it is in fact enough to show that h (can be replaced by some homeomorphism which) maps a generic complex line onto a generic real 2-plane. This gives the following partial answer to the original problem.

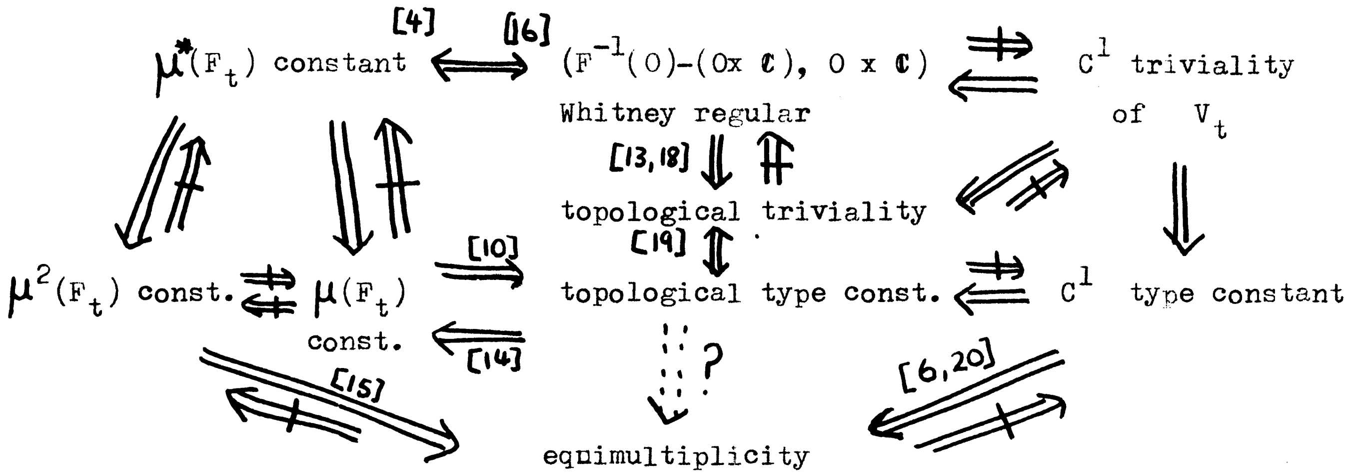
Theorem (Ephraim-Trotman): Multiplicity is a C^1 invariant, i.e. if we suppose V_1 and V_2 are topologically equivalent by a homeomorphism h which is a C^1 diffeomorphism of $\mathbb{R}^{2n+2}$, then $m(V_1) = m(V_2)$.

Proof : (Given in the talk)

See [6] or [20] . One proves that $i(V,P) \leq m(V)$ for a generic real 2-plane P . Then use symmetry.

Families of hypersurfaces.

Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0 \times \mathbb{C}) \longrightarrow (\mathbb{C}, 0)$ be an analytic function with $(0 \times \mathbb{C})$ the critical set of F, and suppose each $V_t = F_t^{-1}(0)$ has an isolated singularity at 0 ($F(z,t) = F_t(z)$). The following diagram describes the results known.



In the only known examples where $\mu(F_t)$ is constant and $\mu^*(F_t)$ is not constant (due to Briancon and Speder [3]), the F_t are weighted homogeneous of fixed type, so that the multiplicity is constant.

Example (Speder): $F(x, y, z, t) = x^3 + txy^3 + y^4z + z^9$.
Then $m(F_t) = 3$, $\mu^3(F_t) = 56$, $\mu^2(F_t) = 7$ ($t \neq 0$), $\mu^2(F_0) = 8$.
F_t has type $(3, 9/2, 9)$ for all t.

We do have a useful characterisation of μ constant families due to Lê Dũng Tráng and Saito [11]:

Theorem (Lê-Saito): $\mu(F_t)$ constant $\Longleftrightarrow$ F has a "bonne stratification"
$\Longleftrightarrow$ normal vectors to the level varieties of F tend to be orthogonal to $0 \times \mathbb{C}$ near 0.

Corollary1: If F_0 is semihomogeneous (i.e. F_0 equivalent to its lowest homogeneous part), and $\mu(F_t)$ is constant, then $\mu^*(F_t)$ is constant - in particular one has equimultiplicity.

Proof: See Briançon and Speder [5] or Gabrielov and Koushnirenko [7].

Note: If F_t is semihomogeneous $(t \neq 0)$ and $\mu(F_t)$ is constant, then we have equimultiplicity. For suppose in general that V_0 and V_1 are two topologically equivalent hypersurfaces with $V_i = f_i^{-1}(0)$, then if f_1 is semihomogeneous, $m(V_1) \geqslant m(V_0)$. (If f_1 is semihomogeneous, $h_{V_1}^{m_1}$ is the identity, so $\Lambda(h_{V_1}^{m_1}) = \chi(F_{V_1}) = 1 + (-1)^n \mu(f_1)$ which is nonzero if $\mu(f_1) > 1$, so that $\Lambda(h_{V_0}^{m_1})$ is also nonzero and $m(V_0) \leqslant m(V_1)$. If $\mu(f_1) = 1$, then so is $\mu(f_0)$, and both V_0 and V_1 have multiplicity 2.) Now use semicontinuity of multiplicity.

Corollary 2 : $\mu(F_t)$ constant implies that the leading form in z and t is equimultiple in z as t varies.

Proof : Let $H(z,t)$ be the leading form of $F(z,t)$, i.e. the homgeneous polynomial of lowest degree in the decomposition of F into its homogeneous parts. If the conclusion of the corollary fails to hold, t divides $H(z,t)$, which we can write as $H(z,t) = t\,G(z,t)$. Let $z_i = a_i u$, $t = bu$, then for sufficiently general a_i and b, $G(z,t) \sim u^{g-1}$.

$$\begin{aligned}\text{Now} \quad \operatorname{grad} F &= (\partial F/\partial z_0, \ldots, \partial F/\partial z_n, \partial F/\partial t) \\ &= (\underbrace{t\partial G/\partial z_0}_{\sim u^{g-1}} + \ldots, \ldots, \underbrace{t\partial G/\partial z_n}_{\sim u^{g-1}} + \ldots, \underbrace{G}_{\sim u^{g-1}} + \underbrace{t\partial G/\partial t}_{\sim u^{g-1}} + \ldots)\end{aligned}$$

$$\not\to (c_0 : \ldots : c_n : 0).$$

This contradicts the theorem of Le and Saito which says that grad F tends to $(c_0 : \ldots, c_n : 0)$ with $\sum_i |c_i| \neq 0$.

Thus for families of the form $F(z,t) = F(z) + tG(z)$, or $F(z) + p(t)G(z)$

(this is the same line in the jet space J^k), μ constant implies equimultiplicity.

If all μ constant families could be expressed in the form $F + tG$, we would have the required result.

However Arnol'd has the following example of a μ constant family :

$$x^3 + y^3 + z^3 + w^3 + (ax + by + cz + dw)^3 + exyzw .$$

Finally we give the simplest type of family where our problem is unresolved.

$$F_t(x,y,z) = z^3 + t^2(z + uy)^2 + B(x, y, t) \quad \text{with} \quad B_t(x,y) \in (\mathcal{M}(x,y))^3$$

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MULTIPLICITY IS A C^1 INVARIANT

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0. Introduction

Let V_1, V_2 be complex hypersurfaces in $\mathbb{C}^{n+1}$ each containing the origin 0, and let N_1, N_2 be neighbourhoods of 0 in $\mathbb{C}^{n+1}$ such that there exists a C^1 diffeomorphism $\Phi: (N_1, N_1 \cap V_1, 0) \longrightarrow (N_2, N_2 \cap V_2, 0)$. We prove that the multiplicity of V_1 at 0 is the same as that of V_2 at 0.

By multiplicity we mean the number of points of intersection of the hypersurface with a generic complex line passing close to, but not through, the origin (i.e. the intersection number of the hypersurface with a generic complex line through the origin, as defined by Lefschetz [3]).

The result was motivated by the first of the problems posed by Zariski in [7] : is multiplicity a topological invariant ? Most of the problems of [7] have been decided, but this, the simplest in statement, remains unsolved.

Our result has also been proved by R. Ephraim [2] : the idea of his proof is much the same as ours, but the details differ because he uses a different interpretation of multiplicity. In view of the fact that the classical topological characterisation of plane curves ([1] , [5]) uses intersection number, it seemed worthwhile to record the present proof.

The proof has grown out of fertile conversations with Henry King (sustained by Kiki's strawberries), Bernard Teissier, Norbert A'Campo, and Bob MacPherson (who acted as a catalyst in the process of transforming my intuitive conviction into a proof). I thank Jean-Jacques Risler for suggesting the problem.

In §1 we state the key lemmas and use them to prove that multiplicity is a C^1 invariant; in §2 we prove the lemmas; in §3 we describe an example showing that not all excellent real 2-planes for V have intersection number with V equal to the multiplicity of V .

1. The Result

Let V be a complex hypersurface in $\mathbb{C}^{n+1}$. We write $m(V)$ for the multiplicity of V at 0 (as defined in §0), and $i(V,S)$ for the intersection number at 0 of V with a subset S of $\mathbb{C}^{n+1}$, whenever this is defined (see [3]). $i(V,S)$ is defined when S is a real differentiable submanifold of $\mathbb{C}^{n+1}$ of class C^1 and dimension 2 such that 0 is an isolated point of intersection of V and S. (By [4] V is triangulable, and hence Lefschetz' definition applies to V.)

Theorem 1.1. Let V_1, V_2, N_1, N_2, ϕ be as in §0. Then $m(V_1) = m(V_2)$.

Lemma 1.2. Let S be a real 2-dimensional C^1 submanifold of $\mathbb{C}^{n+1}$, such that $(T_0 S)\cap(C(V,0)) = \{0\}$. Then $i(V,S)$ and $i(V,T_0 S)$ are defined and equal.

Remarks 1.3. (i) $C(V,0)$ is the usual tangent cone to V at 0. ($C(V,0) = C_3(V,0)$ in Whitney's notation of [6])

(ii) We implicitly identify $T_0 S$ with the real 2-plane through 0 tangent to S.

(iii) We shall not distinguish between a 2-plane at 0 in $\mathbb{C}^{n+1}$ and its canonical image in $G_2^{2n+2}(\mathbb{R})$. We regard $G_1^{n+1}(\mathbb{C})$ as a subset of $G_2^{2n+2}(\mathbb{R})$: it is a compact submanifold of dimension (and codimension) $2n$.

(iv) The subset of definition of $i(V,P)$ for P in $G_2^{2n+2}(\mathbb{R})$ is clearly open and dense. We call this the set of good planes for V and denote it by $\mathcal{G}_V$. ($P\in\mathcal{G}_V \Leftrightarrow$ 0 is an isolated point of intersection of V and P, i.e. $i(V,P)$ is defined.) $\mathcal{G}_V \cap G_1^{n+1}(\mathbb{C})$ is precisely the set of good complex lines for V at 0 as defined in [6, p.232], whence the name.

(v) Let $\mathcal{E}_V$ denote the (open dense) set of excellent planes for V in $G_2^{2n+2}(\mathbb{R})$: those planes which intersect $C(V,0)$ only at 0. $\mathcal{E}_V \cap G_1^{n+1}(\mathbb{C})$ is the set of excellent complex lines for V at 0 as defined in [6 , p.234] . By Lemma 1.2 , $\mathcal{E}_V \subset \mathcal{G}_V$ (excellency implies goodness).

(vi) Let $\mathcal{P}_V$ be the (open) subset of $\mathcal{E}_V$ consisting of those planes P for which $i(V,P) = m(V)$, the <u>perfect</u> planes. By [6 , Theorem 7P , p.234] , $\mathcal{P}_V \cap G_1^{n+1}(\mathbb{C}) = \mathcal{E}_V \cap G_1^{n+1}(\mathbb{C})$. It is not true in general that $\mathcal{E}_V = \mathcal{P}_V$: see §3. However the following lemma defines a limit to the possible values of $i(V,P)$ for excellent P .

<u>Lemma</u> 1.4. If P is excellent, then $i(V,P) \leq m(V)$.

<u>Proof of Theorem 1.1.</u> Let $P \in \mathcal{E}_{V_1} \cap G_1^{n+1}(\mathbb{C})$.

By Remark 1.3 (vi) ,

$$m(V_1) = i(V_1,P) \tag{1}$$

Because the intersection number of a pair is an invariant of the topological embedding type of the pair (see [3]),

$$i(V_1,P) = i(V_2,\Phi(P)) \tag{2}$$

Φ is assumed to be a C^1 diffeomorphism, thus, by Remark 1.3 (iii), there is an induced homeomorphism of pairs,

$$\begin{array}{ccc} (G_2^{2n+2}(\mathbb{R}),\ C(V_1,0)) & \xrightarrow{\tilde{\Phi}} & (G_2^{2n+2}(\mathbb{R}),\ C(V_2,0)) \\ P & \longmapsto & T_0(\Phi(P)) \end{array}$$

By Lemma 1.2 ,

$$i(V_2,\Phi(P)) = i(V_2,\tilde{\Phi}(P)) \tag{3}$$

But, since $\tilde{\Phi}(C(V_1,0)) = C(V_2,0)$ and $\tilde{\Phi}$ is a homeomorphism, then by the definition of $\mathcal{E}_V$ (Remark 1.3(v)), $\tilde{\Phi}(\mathcal{E}_{V_1}) = \mathcal{E}_{V_2}$ and $\tilde{\Phi}(P) \in \mathcal{E}_{V_2}$.

Applying Lemma 1.4 gives,

$$i(V_2,\tilde{\Phi}(P)) \leq m(V_2) \tag{4}$$

From (1), (2), (3), (4), and symmetry, we find the required result,

$$m(V_1) = m(V_2).$$

This completes the proof of Theorem 1.1.

2. Proof of the Lemmas

Notation. Let $S_\rho^{2n+1}(0)$ be the $(2n+1)$-sphere in $\mathbb{R}^{2n+2}$, centred at 0, and with radius ρ. For each subset A of $\mathbb{R}^{2n+2}$ we define, following the notation of [6],

$$A_\rho = A \cap S_\rho^{2n+1}(0)$$

$$U_{\varepsilon\rho}(A_\rho) = \{ x \in S_\rho^{2n+1}(0) : \exists\, y \in A_\rho \text{ such that } |x-y| < \varepsilon\rho \},$$

$$U_\varepsilon^\rho(A) = \operatorname{Int}\left(\overline{\bigcup_{0<\rho'<\rho} U_{\varepsilon\rho'}(A_{\rho'})}\right)$$

Proof of Lemma 1.2. A chart for S at 0 yields $\delta_1 > 0$ and a diffeomorphism $\psi : (\mathbb{R}^{2n+2}, \mathbb{R}^2 \times 0, 0) \longrightarrow (B_{\delta_1}, B_{\delta_1} \cap S, 0)$ where B_{δ_1} is the open ball of radius δ_1 centred at 0.

Sublemma 2.1. We can find $\eta > 0$ such that for all positive $\varepsilon \leq \eta$, there exists some positive $\delta(\varepsilon) \leq \delta_1$ such that for all $\rho \leq \delta(\varepsilon)$,

(i) $U_{\varepsilon\rho}(C(V,0)_\rho) \cap U_{\varepsilon\rho}((T_0S)_\rho) = \emptyset$,

(ii) $V_\rho \subset U_{\varepsilon\rho}(C(V,0)_\rho)$,

(iii) $S_\rho \subset U_{\varepsilon\rho}((T_0S)_\rho)$,

(iv) if $\|\psi(x)\| = \rho$, then $d(\psi(x), \psi'(0)(x)) < \varepsilon\rho$.

Proof. Because T_0S and $C(V,0)$ are closed cones, and $T_0S \cap C(V,0) = \{0\}$, there exists $\eta > 0$ such that $U_\eta^\infty(T_0S) \cap U_\eta^\infty(C(V,0)) = \{0\}$, and (i) follows for all ρ.

Now observe that it suffices to find some $\delta(\varepsilon)$ for each of (ii), (iii), (iv) separately, and to then take the smallest of the three.

For (ii) see [6, p.219].

Elementary analysis (using only the continuity and differentiability of ψ and ψ^{-1} at 0) gives (iii) and (iv), completing the proof of 2.1.

Using (i), (ii) and (iii) of Sublemma 2.1 and setting $\delta = \delta(\eta)$ gives

$$B_\delta \cap V \cap S = B_\delta \cap V \cap T_0S = \{0\}.$$

Thus $i(V,S)$ and $i(V,T_0S)$ are defined (see the discussion at the beginning of §1). It remains to show that they are equal.

Let $\alpha : [0,\infty) \longrightarrow [0,\delta)$ be a C^∞ diffeomorphism. Then

$$x \longmapsto \Psi(x) + t\left(\alpha(|\Psi'(0)(x)|).\frac{\Psi'(0)(x)}{|\Psi'(0)(x)|} - \Psi(x)\right)$$

defines a homology between $B_\delta \cap S$ and $B_\delta \cap T_0S$ within

$$U_\varepsilon^\delta(T_0S) \subset B_\delta \setminus U_\varepsilon^\delta(C(V,0)) \subset B_\delta \setminus (V \cap B_\delta)$$

using (ii), (iii), and (iv) of Sublemma 2.1. Hence by the definition of intersection number ([3]) we have that $i(V,S) = i(V,T_0S)$.

(Bob MacPherson pointed out that a homology is sufficient; it becomes messy, although possible, to construct a homeomorphism of pairs.)

This completes the proof of Lemma 1.2.

<u>Proof of Lemma 1.4</u> : Consider the smooth fibration

$$G_2^{2n+2}(\mathbb{R}) \setminus G_1^{n+1}(\mathbb{C}) \xrightarrow{\ \beta\ } G_2^{n+1}(\mathbb{C})$$

$$P \longmapsto \text{complex 2-plane spanned by } P$$

Each fibre of β is isomorphic to the 4-dimensional connected open submanifold of $G_2^4(\mathbb{R})$ which is the complement of $G_1^2(\mathbb{C})$.

Let $\mathcal{H}_V$ be the open dense subset of $G_2^{n+1}(\mathbb{C})$ such that $H \in \mathcal{H}_V$ if and only if the complex 1-dimensional submanifold of $G_1^{n+1}(\mathbb{C})$, $\{P \in G_1^{n+1}(\mathbb{C}) : P \subset H\}$, is transverse to $C(V,0)$ in $G_1^{n+1}(\mathbb{C})$.

Because β is the projection of a smooth fibration, β is both open and continuous, hence $\beta^{-1}(\mathcal{H}_V)$ is both dense and open in $G_2^{2n+2}(\mathbb{R})$. In particular, $\mathcal{E}_V \cap \beta^{-1}(\mathcal{H}_V)$ is dense in $\mathcal{E}_V$.

(Note that in general $\mathcal{E}_V \not\subset \beta^{-1}(\mathcal{H}_V)$, $\beta^{-1}(\mathcal{H}_V) \not\subset \mathcal{E}_V$.)

<u>Assertion 2.2</u>: If $H \in \mathcal{H}_V$, $P \in \mathcal{G}_V$ and $P \subset H$, then $i(V,P) = i(V \cap H,P)$.

(Note that if $P \subset H$ and $P \notin G_1^{n+1}(\mathbb{C})$, then $H = \beta(P)$.)

<u>Assertion 2.3 (Curve case)</u> : If $P \in \mathcal{E}_V$, and $n = 1$, then $i(V,P) \leq m(V)$.

Assume for the moment that we have proved the two assertions.

Let $P \in \mathcal{E}_V \cap \beta^{-1}(\mathcal{H}_V)$. Then in particular $P \in \mathcal{G}_V$ since $\mathcal{E}_V \subset \mathcal{G}_V$ by Lemma 1.2. So, by Assertion 2.2,

$$i(V,P) = i(V \cap \beta(P),P) \tag{1}$$

and by Assertion 2.3,

$$i(V \cap \beta(P),P) \leqslant m(V \cap \beta(P)) \tag{2}$$

Let L be an excellent complex line for V contained in $\beta(P)$. By Remark 1.3 (vi), $m(V) = i(V,L)$. Also $i(V,L) = i(V \cap \beta(P),L)$ by Assertion 2. And clearly, $i(V \cap \beta(P),L) = m(V \cap \beta(P))$, since $L \in \mathcal{E}_{V \cap \beta(P)} \cap G_1^2(\mathbb{C})$. Thus

$$m(V \cap \beta(P)) = m(V) \tag{3}$$

(1), (2), and (3) show that $i(V,P) \leq m(V)$ for all $P \in \mathcal{E}_V \cap \beta^{-1}(\mathcal{H}_V)$. But since $\mathcal{E}_V \cap \beta^{-1}(\mathcal{H}_V)$ was shown above to be dense in $\mathcal{E}_V$, and since $i(V,P)$ is locally constant for P in $\mathcal{E}_V$, we find that for all $P \in \mathcal{E}_V$, $i(V,P) \leqslant m(V)$.

Hence, modulo the assertions, we have proved Lemma 1.4.

<u>Proof of Assertion 2.2</u> : The intersection number $i(V,P)$ (defined since $P \in \mathcal{G}_V$) is the linking number $\ell(V_\delta,P_\delta)$ for sufficiently small positive δ, and $\ell(V_\delta,P_\delta)$ is given by the number of points of intersection (counted with + or − signs, depending on whether orientations match up or not) of V_δ with a disc in $S_\delta^{2n+1}(0)$, spanning P_δ (which is a circle), and transverse to V_δ. (See [3])

If $H \in \mathcal{H}_V$ and $P \subset H$, we can find such a transverse disc lying in the 3-sphere H_δ. Hence $\ell(V_\delta,P_\delta) = \ell(V_\delta \cap H_\delta,P_\delta)$. Since $V_\delta \cap H_\delta = (V \cap H)_\delta$, the assertion follows.

<u>Proof of Assertion 2.3</u> : V is the union of r (distinct) branches $\{V_k\}_{k=1,..,r}$ and $i(V,P) = \sum_{k=1}^{r} i(V_k,P)$ for all $P \in \mathcal{E}_V$, and $m(V) = \sum_{k=1}^{r} m(V_k)$.

Hence it suffices to prove the assertion for $r = 1$, so that we may suppose that $C(V,0)$ is a single point in $G_1^2(\mathbb{C})$.

Let u, v be independent vectors spanning P. Then projection parallel to $C(V,0)$ onto the complex line P_0 orthogonal to $C(V,0)$ sends u, v onto independent vectors u_0, v_0 spanning P_0. The family of pairs of vectors

$$\{(tu + (1-t)u_0 \, , \, tv + (1-t)v_0)\}_{t \in [0,1]}$$

defines a family of planes, each transverse to $C(V,0)$, i.e. in $\mathcal{E}_V$.

Now using Sublemma 2.1, properties (i) and (ii), we can find a positive radius δ such that this family defines a homology in $S^3_\delta(0)$ between P_δ and $(P_0)_\delta$, with support in the complement of V. We deduce (as in the proof of Lemma 1.2) that $i(V,P)$ is $\pm\, i(V,P_0)$, depending upon an initial choice of orientation for P. $i(V,P_0) = m(V)$ since P_0 is an excellent complex line for V (Remark 1.3 (vi)). Thus $i(V,P) = \pm\, m(V)$, and summing over each branch we obtain the result of Assertion 2.3.

3. Possible Values of $i(V,P)$

In the proof of Assertion 2.3 we saw that for V_k a branch of a plane curve V, and $P \in \mathcal{E}_V$, then $i(V_k,P)$ is $+\, m(V_k)$ or $-\, m(V_k)$, depending on our choice of orientation for P. Given a branch V_k, we can obviously assign an orientation to each $P \in \mathcal{E}_V$ so that $i(V_k,P)$ is positive. However, given any other branch $V_{k'}$, there will be some excellent P for which $i(V_{k'},P)$ is negative using this preassigned orientation.

Example 3.1. $C(V,0) \equiv \{z_1 z_2 = 0\} \subset \mathbb{C}^2$. Let C_1 be the z_1-axis, and C_2 be the z_2-axis. We can parametrise the 4-dimensional open submanifold of $G^4_2(\mathbb{R})$ consisting of the planes P such that $P \cap C_1 = \{0\}$, as follows.

Let $P_{abcd} = \{(a\lambda + b\mu, \, c\lambda + d\mu, \, \lambda, \, \mu) : \lambda, \mu \in \mathbb{R}\}$ with $a, b, c, d \in \mathbb{R}$.

Then $P_{0000} = C_2$.

It is easy to check that

$$P_{abcd} \in G^2_1(\mathbb{C}) \iff c = -b \, , \, d = a \, .$$

Thus if P_{abcd} is complex and not equal to C_2, $ad > bc$.

Also, P_{abcd} is excellent for $C_1 \cup C_2 \iff P_{abcd} \cap C_2 = \{0\}$

$$\iff ad \neq bc \, .$$

Hence P_{abcd} is isotopic to a complex line by an isotopy with support in the complement of $C_1 \cup C_2$ if and only if $ad > bc$.

Further, orienting each P_{abcd} such that $i(C_1, P_{abcd}) = +1$, one finds that

$$i(C_2, P_{abcd}) = \begin{cases} +1 & \text{if } ad > bc \\ -1 & \text{if } ad < bc \end{cases}$$

(One way to see this is to consider the projection parallel to C_2 , of the P_{abcd}'s near C_2 , onto C_1 (as in the proof of Assertion 2.3).)

Thus if C_i is the tangent cone of several branches of V whose multiplicities add up to m_i $(i = 1,2)$ and $C(V,0) = C_1 \cup C_2$, then $\mathcal{E}_V = \{P_{abcd} : ad \neq bc\}$ and

$$i(V, P_{abcd}) = \begin{cases} m_1 + m_2 & \text{if } ad > bc \\ m_1 - m_2 & \text{if } ad < bc \end{cases}$$

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