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À propos des systèmes de particules en interaction sur un réseau,
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ABSTRACT

We are interested in particles systems located on a lattice, with different type of interaction . For short range interaction on $\mathbb{Z}^d$, we study the large deviation properties for the empirical field of a Gibbs measure ; we also cover the case of random interaction , and derive some applications .

Next we study Glauber dynamics of a local mean field model on the torus , in the asymptotics of a large number of particles . The fluctuation process has to be rescaled in space and time at the critical temperature . We analyse the dynamics of a change of attractor using large deviations techniques : at low temperature , we recover a description for nucleation .

We then need to study the stationary points in such a local mean field model ; this is tackled in the frame of bifurcation theory .

KEY WORDS : Gibbs measure , large deviation , spin-flip process , spin glass model , maximum of entropy , critical renormalization , nucleation , bifurcation theory .

A.M.S. CLASSIFICATION : 60F10 , 60G60 , 60K35 , 65F05 , 45G10 .

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TABLE DES MATIERES

- INTRODUCTION -

Ce travail regroupe les cinq articles [12-16] référencés ci-dessous. Nous y considérons une famille de variables aléatoires dépendantes $X = (X_i)_{i \in S}$ indexées par un réseau S, pour laquelle la difficulté d'une réalisation x est mesurée par une fonction d'énergie H(x).

Ce modèle s'est d'abord développé en mécanique statistique à partir de 1920 autour du célèbre modèle d'Ising, qui décrit un cristal magnétique. Dans ce cas, X_i représente l'orientation du moment magnétique de la particule au site i du réseau, et sa distribution dépend des orientations des autres sites. L'énergie correspondante s'écrit, du moins formellement,

$$H(x) = - \sum_{i,j \in S} J_{ij}\, x_i\, x_j$$

où J_{ij} mesure l'intensité de l'interaction entre les particules situées aux points i et j. D'après les première et seconde lois de la thermodynamique, le système évolue vers une distribution d'équilibre, appelée *mesure de Gibbs*, définie par sa dérivée $\frac{1}{Z} e^{-\beta H}$ par rapport à une distribution de référence (la loi du système sans interaction), $\beta>0$ désignant l'inverse de la température, et la constante de normalisation Z donnant à cette mesure (positive) une masse égale à un. Dans le cas d'un réseau S infini, les deux formules précédentes permettent de comparer les configurations x qui coïncident à l'extérieur d'une partie finie de S, sous réserve d'hypothèses (acceptables) sur les $J_{i,j}$. Nous considérons des formes d'interaction plus générales, comme l'interaction à k corps (k>2), mais aussi celles qui sont elles-mêmes aléatoires (modèles de verres de spin).

Les mesures de Gibbs interviennent dans d'autres domaines, par exemple en neurophysiologie -pour décrire l'activité de certains neurones dans le cerveau-, en biologie -pour la contamination de cellules dans l'étude des tumeurs-, ou encore en épidémiologie.

On peut également considérer une dynamique de ces modèles, suivant un processus de Markov stationnaire qui laisse invariantes les mesures de Gibbs.

L'étude mathématique des mesures de Gibbs, et celle de ces processus, a nourri une vaste littérature (voir la bibliographie de [1] [2]), en particulier autour des phénomènes de transition de phase (coexistence de plusieurs mesures de Gibbs), brisure de symétrie, renormalisation critique, nucléation. Elle s'est avérée difficile, et elle a été menée à l'aide de nombreuses techniques dont certaines propres à ce domaine.

L'ingrédient essentiel de notre approche est la théorie des *grandes déviations*, dans la mesure où la fonction d'énergie H possède une propriété d'additivité, et pour des réseaux de grande taille comme dans les exemples précédents (ce qui écarte pratiquement l'approche combinatoire). Nous avons adopté le point de vue introduit récemment par Donsker et Varadhan dans [3] pour étudier les propriétés de grandes déviations du champ empirique (niveau 3) de différentes mesures de Gibbs sur $S = \mathbb{Z}^d$ au chapitre I. Dans le modèle plus simple du chapitre II, le processus apparaît comme une petite perturbation aléatoire d'un système dynamique : l'étude de la nucléation (II.B) est menée dans l'esprit des travaux de Wentsell et Freidlin [4] ; quant à l'étude fluctuations (II.A), elle ne revet toute sa signification qu'avec celle du comportement de la mesure de Gibbs au sens des grandes déviations traitée dans [5] au niveau 1 du théorème de Chernov.

Pour en revenir aux différents modèles de particules, ceux qui présentent des interactions locales (à courte portée) sont les plus réalistes ; mais leur étude, et l'interprétation des résultats, sont alors difficiles. A l'inverse, le modèle de Curie-Weiss (champ moyen), dans lequel l'intensité de l'interaction ne dépend pas de l'éloignement des sites, est simpliste dans l'asymptotique d'un nombre infini de particules. Le modèle simplifié (champ moyen local) du chapitre II est intermédiaire : $S = S_n$ est un réseau régulier sur le tore à d dimensions, de cardinal n^d ; l'interaction entre les particules situées aux points i et j vaut

$$J_{ij} = n^{-d} J(i-j) \quad ,$$

où J est une fonction régulière, et dépend de la distance séparant les particules. En particulier, il confère au système une géométrie suffisante lorsque $n \longrightarrow \infty$ pour qu'il exhibe des *phénomènes coopératifs locaux* tels que des phases antiferromagnétiques, des fluctuations critiques riches, et la nucléation.

Nous détaillons à présent nos résultats.

Le chapitre I est consacré à l'étude des états d'équilibre.

Dans la partie A, on établit les estimations de grandes déviations du champ empirique pour une mesure de Gibbs sur $\mathbb{Z}^d$ associée à une interaction sommable et invariante par translation. La fonctionnelle I_1 mesurant le taux de décroissance exponentielle, est définie sur l'ensemble des champs stationnaires $\mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ par

$$I_1(Q) = -\beta\mathbb{E}^Q U + \mathcal{I}(Q) - p$$

où $-U(x)$ désigne l'interaction normalisée du site 0 avec les autres sites, $\mathcal{I}$ l'entropie relative par rapport à la mesure de référence, et $p = \inf\{\beta\mathbb{E}^Q U + \mathcal{I}(Q) ; Q\}$ la pression.

Cette fonctionnelle ne dépend que de l'interaction, et de la mesure de référence ; elle est la même pour toutes les mesures de Gibbs lorsqu'il y a transition de phase, et on ne peut pas discriminer celles-ci à l'ordre de grandeur exponentiel du volume. De plus, ces estimations de grandes déviations sont également valables pour celles qui ne sont pas ergodiques (dans le cas de transition de phase), et celles qui ne sont pas stationnaires (dans celui de brisure de la symétrie). Pour établir ce résultat, on traite d'abord le cas sans interaction $H=0$, en généralisant des techniques introduites dans [3] pour $d=1$; puis le cas avec interaction à l'aide d'un changement exponentiel de probabilité.

A la conclusion de cette partie, l'auteur a eu connaissance des résultats identiques de [6], et de [7] par une méthode différente.

Dans la partie I-B, on considère des modèles sur $\mathbb{Z}^d$ avec interaction aléatoire. Reprenant la stratégie précédente, il s'agit d'abord de montrer un principe de grandes déviations conditionnel au niveau 3 pour un champ bivarié (X_i, Y_i) indépendant identiquement distribué ; ceci nous donne accès à des interactions dépendant de $Y = (Y_i)_i$ et par là même, aux exemples usuels de verres de spin à interaction sommable. Pour presque tout Y, on obtient alors une formule variationnelle relativisée pour la pression (résultat déjà obtenu dans [8]), mais aussi les propriétés de grandes déviations pour les mesures de Gibbs, avec une fonctionnelle I_2 déterministe, définie cette fois sur $\mathcal{P}_s\left((\mathbb{R}^2)^{\mathbb{Z}^d}\right)$. Une conséquence de ceci est que, sous certaines hypothèses de convergence et de stationnarité, une limite de mesure de Gibbs à volume fini peut s'écrire $Q(./Y)$ avec $I_2(Q) = 0$, c'est-à-dire comme une distribution d'entropie minimale conditionnelle à l'interaction.

Enfin, nous montrons que le résultat précédent est vrai sans hypothèses dans les modèles de champ moyen. Sur un exemple particulier de verre de spin, nous montrons que la distribution d'une particule est alors la même que celle d'une particule choisie au hasard dans la phase antiferromagnétique d'un modèle avec interaction (non aléatoire) de champ moyen local.

Nous décrivons maintenant les états d'équilibres asymptotiques d'un modèle de champ moyen local [5], avant de détailler le chapitre II. On peut réduire l'étude du système X à celle d'une *mesure de magnétisation*

$$X^n = n^{-d} \sum_{i \in S_n} X_i \, \delta_i \, .$$

Dans l'asymptotique $n \to \infty$, le système sera représenté par une densité de magnétisation sur le tore, l'analogue d'un profil en hydrodynamique.

Remarquant que l'énergie s'écrit alors $- \frac{\beta}{2} \int J*X^n(s) \, X^n(ds)$ où $*$ représente la convolution, on obtient des inégalités de grandes déviations traduisant que

"n^{-d} Log P $\{X^n$ voisin de $u\}$ est approximativement égal à $I_3(u)$",

avec

$$I_3(u) = - \frac{\beta}{2} \int J*u(s) \, u(s) \, ds + \int \iota[u(s)] \, ds$$

où ι est la transformée de Cramèr de la distribution de référence d'une particule. Les minima de I_3 sont les densités d'équilibre ; ils présentent un éventail varié de comportements, suivant les valeurs de J et β, comme la phase ferromagnétique (les densités sont constantes non nulles), ou la phase antiferromagnétique (elles constituent une famille d'ondes).

Dans le chapitre II, nous menons l'étude de la dynamique de Glauber de ces modèles, et tout particulièrement celle des phénomènes liés à la transition de phase. La fonctionnelle I_3, qui décrit le comportement asymptotique de la mesure invariante du processus "mesure de magnétisation" X_t^n , mesure le temps moyen que passe X_t^n au voisinage d'un état ; en particulier, X_t^n apparait comme une perturbation aléatoire(d'autant plus petite que n est grand) d'un système dynamique qui l'entraîne vers les équilibres (minima de I_3), au voisinage desquels il passe la majeure partie du temps.

Dans la partie A, l'auteur établit en collaboration avec T. Eisele des résultats de fluctuations (théorèmes de limite centrale) hors de l'équilibre et à l'équilibre, après avoir prouvé la loi des grands nombres ci-dessus, et un résultat de propagation du chaos (qui justifie l'exhaustivité de la seule étude du processus mesure). Au voisinage d'un minimum non dégénéré u de I_3, le processus de fluctuation

$$n^{d/2}\,(X_t^n - u)$$

converge, en norme de Sobolev, vers un processus de Ornstein-Uhlenbeck généralisé. Lorsque le minimum est dégénéré, la convergence précédente a lieu, mais le processus limite ne possède plus de distribution invariante. Il convient alors de *renormaliser en espace*, mais aussi *en temps* en raison de cette convergence. Nous traitons le cas du point critique d'une transition de phase ferromagnétique, correspondant à une dégénérescence d'ordre m arbitraire dans une seule direction : alors $u = 0$ et le processus de fluctuation critique

$$n^{d/2(m+1)}\; X^n_{tn^{m/(m+1)}}$$

converge vers un processus stationnaire non gaussien occupant la direction de dégénérescence ; cet espace étant ici celui des constantes, le processus de fluctuation critique est donc homogène dans l'espace. Au point critique d'une transition de phase antiferromagnétique de fréquence q avec dégénérescence d'ordre $m=1$, le processus de fluctuation critique converge vers un processus stationnaire non gaussien, de fréquence q sur le tore - cet espace de fréquence (de dimension 2) constituant alors le noyau de dégénérescence -. Enfin, nous traitons le cas où les deux transitions précédentes se combinent. Dans tous les cas, la mesure invariante du processus limite est la limite des fluctuations de la mesure de Gibbs conditionnée au voisinage de l'équilibre. Certains résultats analogues de [9] sont obtenus dans un modèle de champ moyen ; notre cadre, qui introduit une plus grande richesse de paramètres et de géométrie, met en évidence l'universalité des processus limites.

La partie II.B concerne le comportement du processus sur des échelles de temps beaucoup plus longues. L'auteur de cette thèse y étudie les changements d'attracteurs selon l'approche par les grandes déviations de [4], [10], proposée dans ce cadre par G. Ruget [11]. L'idée originelle est d'expliquer le phénomène de *nucléation*, à savoir l'apparition d'un nombre déterminé de noyaux dont la composition approche celle du nouvel attracteur, ces noyaux se propageant par la suite jusqu'à remplir tout l'espace. Il s'agit dans un premier lieu d'obtenir des estimations en temps fixe T, que l'on peut caricaturer par

$$- \frac{1}{n^d} \text{ Log Pr}\{\text{dist } (X_t^n, u_t) < \gamma \; ; \; t \leqslant T\} \text{ est à peu près } I_4(T;u)$$

pour toute trajectoire (déterministe) u sur [0,T] dans l'espace des densités ; I_4 s'annule sur les trajectoires du système dynamique sous-jacent. En généralisant certaines estimations à des durées plus grandes, nous déterminons alors les points de sortie d'un bassin d'attraction, qui se trouvent être les points du bord les plus bas dans le paysage d'énergie défini par I_3. Ces points col vérifient l'équation $\nabla I_3 = 0$, étudiée ci-dessous ; en particulier, ils sont non homogènes à température assez basse : il y a alors nucléation.

Enfin, le chapitre III qui ponctue ce travail, consiste en l'étude des solutions u de $\nabla I_3 = 0$, ou plutôt de l'équation équivalente

$$u = g\ (\beta\ J*u)$$

avec $g = (\iota')^{-1}$. Il résulte d'une collaboration entre T. Eisele, motivé par l'étude des minima de I_3, M. Schatzman, intéressée plus particulièrement par la modélisation du développement du cortex visuel, et l'auteur, pour les différents motifs déjà évoqués. Il révèle la géométrie du paysage d'énergie I_3, dictée par les coefficients de Fourier de l'interaction J . Nous décrivons de manière plutôt complète les branches de bifurcation primaire et secondaire, pour des noyaux de bifurcation de dimension au plus deux, ainsi que leur stabilité. Au passage, nous obtenons un exemple de transition de phase du premier ordre dans le cadre du champ moyen local, où l'équilibre saute brutalement d'une branche de solutions à une autre pour une certaine valeur du paramètre β .

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CHAPITRE I : DISTRIBUTIONS A L' EQUILIBRE .

Partie A : GRANDES DEVIATIONS POUR LES MESURES DE GIBBS AVEC INTERACTION A COURTE PORTEE .

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PROBABILITÉS. — *Grandes déviations pour des champs de Gibbs sur* $\mathbb{Z}^d$. Note de **Francis Comets**, présentée par Robert Fortet.

Un principe de grandes déviations est d'abord établi pour le processus empirique d'un champ de variables indépendantes équidistribuées indexées par $\mathbb{Z}^d$, pour $d \geqq 1$. Ce résultat est ensuite généralisé aux champs de Gibbs stationnaires associés à une interaction sommable, et mène à la formule variationnelle de Gibbs.

PROBABILITY THEORY. — Large deviations results for Gibbsian random fields on $\mathbb{Z}^d$.

A large deviations principle is first proved for the empirical process of i. i. d. random variables indexed by the integer lattice $\mathbb{Z}^d$, $d \geqq 1$. This result is then extended to stationary Gibbsian fields corresponding to a summable interaction, and we obtain the Gibbs variational formula.

I. Énoncé du résultat principal. — Soit X un espace polonais, et $\Omega = X^{\mathbb{Z}^d}$. On considère une suite Λ_n de parallélépipèdes de $\mathbb{N}^d$, $\Lambda_n = \prod_{i=1}^{d} [0, a_n^i]$ où chaque suite a_n^i $(i = 1, \ldots, d)$ tend vers l'infini. Pour $\omega \in \Omega$, on note $\omega^{(n)}$ l'élément de Ω obtenu en prolongeant par périodicité en dehors de Λ_n la restriction de ω à Λ_n. On définit alors le *processus empirique*

$$R_{n,\omega} = \frac{1}{|\Lambda_n|} \sum_{\lambda \in \Lambda_n} \delta_{\theta^\lambda \omega^{(n)}}$$

où $|\Lambda|$ désigne le cardinal de $\Lambda \subset \mathbb{Z}^d$ et θ^λ, $\lambda \in \mathbb{Z}^d$, les opérateurs de shift sur Ω : pour tout ω, $R_{n,\omega}$ appartient à l'ensemble $\mathscr{P}_S(\Omega)$ des mesures de probabilités stationnaires (sous l'action de $\mathbb{Z}^d$) sur Ω.

Soient α une probabilité sur X et P la probabilité produit sur Ω. Considérons une interaction $J = \{ J_A ;\ A \text{ partie finie de } \mathbb{Z}^d \}$, où les J_A sont des fonctions continues sur Ω, mesurables par rapport à la tribu $\sigma(A)$ engendrée par les applications coordonnées $\omega \mapsto \omega_\lambda$ pour $\lambda \in A$; on suppose J invariante par translation, et $\sum_{A \ni 0} \sup_{\omega \in \Omega} |J_A(\omega)| < \infty$.

Pour toute partie finie Λ de $\mathbb{Z}^d$ et toute condition extérieure $\tilde{\omega} \in X^{\Lambda^c}$, on définit le potentiel hamiltonien $U_\Lambda^{\tilde{\omega}}$ pour $\bar{\omega} \in X^\Lambda$ par

$$U_\Lambda^{\tilde{\omega}}(\bar{\omega}) = - \sum_{A :\, A \cap \Lambda \neq \varnothing} J_A(\omega)$$

avec $\omega = (\tilde{\omega}, \bar{\omega})$, $Z_\Lambda^{\tilde{\omega}} = \mathbb{E}^{P_\Lambda} \{ \exp - U_\Lambda^{\tilde{\omega}}(\bar{\omega}) \}$ où P_Λ désigne la restriction de P à $\sigma(\Lambda)$, et enfin

$$\pi_\Lambda^{\tilde{\omega}}(\bar{\omega}) = (Z_\Lambda^{\tilde{\omega}})^{-1} \exp\{ -U_\Lambda^{\tilde{\omega}}(\bar{\omega}) \}$$

la spécification. Soit G l'ensemble des *états de Gibbs invariants par translation*, i. e. des $Q \in \mathscr{P}_S(\Omega)$ tels que, pour toute partie Λ finie, $\pi_\Lambda^{\tilde{\omega}} dP_\Lambda$ soit une version régulière de Q étant donné $\sigma(\Lambda^c)$ (problème de Dobrushin-Lanford-Ruelle).

Théorème. — (i) $\lim_{n \to \infty} 1/|\Lambda_n| \log Z_{\Lambda_n}^{\tilde{\omega}_n}$ *existe, est indépendante de la suite des conditions* $\tilde{\omega}_n$ *extérieures à* Λ_n, *uniforme par rapport à* $\tilde{\omega}_n$, *et est égale à*

$$p = - \inf_{Q \in \mathscr{P}_S(\Omega)} \{ \mathbb{E}^Q(U) + I(Q, P) \}$$

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avec

$$U(\omega) = -\sum_{A \ni 0} \frac{1}{|A|} J_A(\omega)$$

et

$$I(Q, P) = \sup_{\Lambda \text{ partie finie de } \mathbb{Z}^d} \frac{1}{|\Lambda|} h(Q_\Lambda, P_\Lambda),$$

où $h(\mu, \nu)$ *est l'information de Kullback de* μ *par rapport à* ν *lorsque* μ *et* ν *sont deux probabilités définies sur la même tribu.*

(ii) *Si* $Q \in G$, *on a pour tout borélien* B *de* $\mathscr{P}_S(\Omega)$

$$-\inf_{R \in \mathring{B}} \{ \mathbb{E}^R U + I(R, P) + p \} \leqq \varliminf_{n \to \infty} \frac{1}{|\Lambda_n|} \log Q \{ R_{n,\omega} \in B \}$$

$$\leqq \varlimsup_{n \to \infty} \frac{1}{|\Lambda_n|} \log Q \{ R_{n,\omega} \in B \} \leqq -\inf_{R \in \bar{B}} \{ \mathbb{E}^R U + I(R, P) + p \}.$$

Commentaires. — La formule variationnelle de Gibbs (i) est bien connue; cependant la preuve donnée ici, à l'aide de techniques de grandes déviations, est nouvelle.

Du principe de grandes déviations (ii), on déduit

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log Q \{ R_{n,\omega} \in B \} = 0$$

dès que $\mathring{B}$ contient l'un quelconque des états de Gibbs [il est en effet connu que $p = \mathbb{E}^R U + I(R, P)$ équivaut à $R \in G$] : l'apparition d'un autre état de Gibbs, s'il en existe, n'est pas de probabilité exponentiellement petite sous Q.

On montre aisément que $I(., P)$ est s. c. i. sur $\mathscr{P}_S(\Omega)$ et que ses lignes de niveau sont compactes; on retrouve ainsi que $G \neq \varnothing$ sous nos hypothèses.

II. Indication de preuve et le cas indépendant. — Pour $\Lambda, \Lambda' \subset \mathbb{Z}^d$ et Q une probabilité sur Ω, notons $Q_{\Lambda'}^{\Lambda}$ une version régulière de Q conditionnelle à $\sigma(\Lambda)$, restreinte à $\sigma(\Lambda')$ (on écrira 0 pour $\{0\}$).

Proposition. — (*a*) *Soient* $\leqq$ *un ordre total sur* $\mathbb{Z}^d$, *compatible avec les translations,* $(\leftarrow, 0]$ [*resp.* $(\leftarrow, 0)$] *l'ensemble des minorants* [*resp. des minorants stricts*] *de* 0 (*le « passé »*). *Si* $Q \in \mathscr{P}_S(\Omega)$, *le* sup *définissant* I *peut être calculé sur les parallélépipèdes de* $\mathbb{N}^d$ *et est égal à* $\tilde{I} = \mathbb{E}^Q h(Q_0^{(\leftarrow, 0)}, \alpha)$.

(*b*) *Pour tout borélien* B *de* $\mathscr{P}_S(\Omega)$, *la loi du processus empirique* $R_{n,\omega}$ *sous* P *satisfait un principe de grandes déviations avec constantes* $|\Lambda_n|$ *et fonction de taux* $I(., P)$.

Remarques. — Ce résultat est la clé du théorème; il généralise celui de Donsker-Varadhan en dimension 1 ([1], [2]); (*a*) est dû à Föllmer [3] dans le cas X fini et $\leqq$ un ordre lexicographique.

Pour montrer (*a*) on ordonne les éléments d'une partie finie Λ arbitraire de $\mathbb{Z}^d$, et on obtient comme dans [2] : $\mathbb{E}^Q F - \log \mathbb{E}^P F \leqq |\Lambda| \tilde{I}$ pour toute F, $\sigma(\Lambda)$-mesurable bornée. Puis, pour établir $\sup \{ (1/|\Lambda|) h(Q_\Lambda, P_\Lambda);\ \Lambda \text{ parallélépipède} \} \geqq \tilde{I}$, on décompose

$$h(Q_\Lambda, P_\Lambda) = \sum_{\lambda \in \Lambda} \mathbb{E}^Q h(Q_0^{T^{-\lambda}\Lambda \cap (\leftarrow, 0)}, \alpha)$$

où T^λ est la translation de vecteur λ dans $\mathbb{Z}^d$.

L'argument en dimension 1 de [2] mène à $\varliminf_{n\to\infty} \mathbb{E}^Q h(Q_0^{\Delta_n}, \alpha) \geqq \tilde{I}$ pour toute suite Δ_n croissant vers $(\leftarrow, 0)$: on le complète en remarquant que $\Lambda \mapsto \mathbb{E}^Q h(Q_0^\Lambda, \alpha)$ est croissante en $\Lambda \subset (\leftarrow, 0)$ (inégalité de Jensen), et on peut alors adapter l'argument de Cézaro.

En s'inspirant de [2], on montre que $I(Q, P)$ est une fonction affine de Q et que $Q \in \mathscr{P}_S(\Omega)$ a une représentation intégrale

$$\int_{S \text{ ergodique}} S\,\mu_Q(dS) \quad \text{avec} \quad I(Q, P) = \int I(S, P)\,\mu_Q(dS).$$

La majoration dans (*b*) est semblable au cas $d=1$ [2]. Pour la minoration, on établit d'abord

$$(1) \qquad \varliminf_{n\to\infty} \frac{1}{|\Lambda_n|} \log P\{R_{n,\omega} \in V\} \geqq -I(Q, P),$$

pour tout voisinage V dans $\mathscr{P}_S(\Omega)$ d'une probabilité ergodique Q vérifiant $I(Q, P) < \infty$: le point crucial est que la famille filtrante $\{\log dQ_0^\Lambda/d\alpha;\ \Lambda$ partie finie de $(\leftarrow, 0)\}$ est une $Q-\sigma(\Lambda \cup \{0\})$ sous-martingale, convergeant dans $\mathscr{L}^1(Q)$ vers Y,

$$Y \leqq \log dQ_0^{(\leftarrow, 0)}/d\alpha \quad Q \text{ p.s.}$$

On écrit alors

$$P\{R_{n,\omega} \in V\} \geqq \exp\{-|\Lambda_n|(I(Q, P)+2\varepsilon)\}\, Q(\Omega_n^1 \cap \Omega_n^2 \cap \{R_{n,\omega} \in V\})$$

où

$$\Omega_n^1 = \left\{ \frac{1}{|\Lambda_n|} \sum_{\lambda \in \Lambda_n} \log \frac{dQ_0^{(\leftarrow, 0)}}{d\alpha}(\theta^\lambda \omega) \leqq I(Q, P) + \varepsilon \right\}$$

et

$$\Omega_n^2 = \left\{ \frac{1}{|\Lambda_n|} \sum_{\lambda \in \Lambda_n} \left[Y - \log \frac{dQ_0^{(\leftarrow, 0) \cap T^{-\lambda}\Lambda_n}}{d\alpha} \right](\theta^\lambda \omega) \geqq -\varepsilon \right\};$$

la Q-probabilité de Ω_n^2 tend donc vers 1, mais aussi celles des deux autres ensembles d'après le théorème ergodique multiparamétrique.

D'après la représentation intégrale précédente, il suffit alors de vérifier que l'ensemble des $Q \in \mathscr{P}_S(\Omega)$ vérifiant (1) est convexe, en utilisant un argument classique.

La preuve du théorème consiste alors à combiner les résultats de [4], § 3, et les inégalités (*b*) qui sont bien sûr indépendantes de $\tilde{\omega}^n$.

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PREUVES DE LA NOTE:

"GRANDES DEVIATIONS POUR DES CHAMPS DE GIBBS".

Nous reprenons ici les notations définies dans la note. Une bibliographie complémentaire, référencée par le nom des auteurs, figure à la fin des preuves.

Si $\Lambda \subset \mathbb{Z}^d$, on note

$$D_\Lambda = \{ F:\Omega \longrightarrow \mathbb{R} \ ; \ F \ \sigma(\Lambda)\text{-mesurable, bornée, } \mathbb{E}^P e^F \leq 1 \}.$$

Alors, $h(Q_\Lambda, P_\Lambda) = \sup \{\mathbb{E}^Q F \ ; \ F \in D_\Lambda\}$.

□ Preuve de la proposition:

. Montrons d'abord que $I \leq \tilde{I}$. Soient Λ une partie finie de $\mathbb{Z}^d$, dont on note $\lambda_1, \ldots \lambda_{|\Lambda|}$ ses éléments classés par ordre croissant, et $F \in D_\Lambda$. Soient $F_{|\Lambda|} = F$ et

$$F_k(\omega_{\lambda_1}, \ldots, \omega_{\lambda_k}) = \log \int e^{F(\omega)} \alpha(d\omega_{\lambda_{k+1}}) \ldots \alpha(d\omega_{\lambda_{|\Lambda|}})$$

pour $k = 0, \ldots |\Lambda|$. Comme $F_0 \leq 0$ et $\int_X e^{F_{k+1}-F_k} \alpha(d\omega_{\lambda_{k+1}}) = 1$, on a Q-p.s.

$$\mathbb{E}^{Q^{(\leftarrow,\lambda_{k+1})}_{\{\lambda_{k+1}\}}} \{ F_{k+1} - F_k \} \leq h\left(Q^{(\leftarrow,\lambda_{k+1})}_{\{\lambda_{k+1}\}}, \alpha \right) = h(Q_0^{(\leftarrow,0)}, \alpha) \circ \theta^{\lambda_{k+1}}$$

d'après la stationarité de Q; on intègre ces inégalités par rapport à Q, on somme sur k, et il vient

$$\mathbb{E}^Q F \leq |\Lambda| \ \tilde{I} \ .$$

L'inégalité inverse nécessite une étape de plus qu'en dimension d=1. Maintenant, Λ est un parallélépipède; la formule de décomposition de l'entropie (lemme 2.3 de [1]) s'écrit

$$h(Q_\Lambda, P_\Lambda) = h(Q_{\Lambda-\{\lambda_{|\Lambda|}\}}, P_{\Lambda-\{\lambda_{|\Lambda|}\}}) + \mathbb{E}^Q h(Q^{\Lambda-\{\lambda_{|\Lambda|}\}}_{\{\lambda_{|\Lambda|}\}}, \alpha)$$

où le dernier terme est égal à $\mathbb{E}^Q h\left(Q_0^{(\leftarrow,0)\cap T^{-\lambda_{|\Lambda|}}\Lambda}, \alpha\right)$ par stationnarité.

D'où l'on obtient

$$h(Q_\Lambda,P_\Lambda) = \sum_{\lambda\in\Lambda} \mathbb{E}^Q h(Q_0^{(\leftarrow,0)\cap T^{-\lambda}\Lambda},\alpha) \qquad (1).$$

L'argument suivant, suffisant pour d=1, devra ici être complété : si Δ_n est une suite croissante de parties de $(\leftarrow,0)$ avec $\bigcup_n \Delta_n = (\leftarrow,0)$, le théorème usuel de convergence des martingales montre que $Q_0^{\Delta_n} \Rightarrow Q_0^{(\leftarrow,0)}$, Q-p.s. ($\Rightarrow$ désigne la convergence étroite des probabilités). Puisque $\mu \mapsto h(\mu,\nu)$ est s.c.i. , le lemme de Fatou montre que

$$\liminf_{n\to\infty} \mathbb{E}^Q h(Q_0^{\Delta_n},\alpha) \geq \mathbb{E}^Q h(Q_0^{(\leftarrow,0)},\alpha) \qquad (2).$$

D'autre part, on a Q-p.s. $Q_{\Delta-\Delta'}^{\Delta'}\ Q_0^{\Delta} = Q_0^{\Delta'}$ pour $\Delta'\subset\Delta\subset(\leftarrow,0)$, et donc

$$\mathbb{E}^Q h(Q_0^{\Delta},\alpha) = \mathbb{E}^Q\, \mathbb{E}^{Q_{\Delta-\Delta'}^{\Delta'}} \sup_{F\in D_0} \mathbb{E}^{Q_0^{\Delta}} F \geq \mathbb{E}^Q \sup_{F\in D_0} \mathbb{E}^{Q_0^{\Delta'}} F = \mathbb{E}^Q h(Q_0^{\Delta'},\alpha) \ .$$

En combinant cette propriété de croissance avec (1),(2) et un argument de Cezaro, on obtient aisément

$$I \geq \sup\{ \frac{1}{|\Lambda|} h(Q_\Lambda,P_\Lambda) \ :\ \Lambda \text{ parallélépipède de } \mathbb{Z}^d \} \geq \tilde{I} \quad ,$$

ce qui prouve a).

.Nous établissons maintenant la majoration de la probabilité de grande déviation, en suivant l'esprit de la preuve de [2]. Soient Λ un parallélépipède de $\mathbb{N}^d$ contenant l'origine, et $F\in D_\Lambda$: on peut recouvrir Λ_n par des translatés $T^\gamma\Lambda$ de Λ deux à deux disjoints et on majore

$$\exp \frac{1}{|\Lambda|} \sum_{\lambda\in\Lambda_n} F(\theta^\lambda\omega) = \exp \frac{1}{|\Lambda|} \sum_{\lambda\in\Lambda} \sum_{\gamma:\gamma+\lambda\in\Lambda_n} F(\theta^{\gamma+\lambda}\omega)$$

$$\leq \frac{1}{|\Lambda|} \sum_{\lambda\in\Lambda} \exp \sum_{\gamma} F(\theta^{\gamma+\lambda}\omega)$$

par convexité de l'exponentielle. Puisque $F\in D_\Lambda$ et P est une mesure produit, on obtient en intégrant par P

$$\mathbb{E}^P \exp \frac{1}{|\Lambda|} \sum_{\lambda\in\Lambda_n} F(\theta^\lambda\omega) \leq 1$$

Comme $\varepsilon_F(n) = \sup_\omega \left| \frac{1}{|\Lambda_n|} \sum_{\lambda\in\Lambda_n} F(\theta^\lambda\omega) - \int_\Omega F\, dR_{n,\omega} \right|$ est un

$\Theta_d(\|F\|\,|\Lambda|/\min_{j\leq d}\{a_n^j\})$, on en déduit pour tout borélien B de $\mathcal{P}_s(\Omega)$

$$\exp\left\{ \frac{|\Lambda_n|}{|\Lambda|} \varepsilon_F(n) \right\} \geq \mathbb{E}^P \exp\left\{ \frac{|\Lambda_n|}{|\Lambda|} \int_\Omega F\, dR_{n,\omega} \right\}$$

$$\geq P\{R_{n,\omega}\in B\} \exp\left\{ \frac{|\Lambda_n|}{|\Lambda|} \inf_{Q\in B} \int_\Omega F\, dQ \right\}$$

d'après l'inégalité de Chebichev; soit

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log P\{R_{n,\omega}\in B\} \leq -\sup\{\frac{1}{|\Lambda|} \sup_{F\in D_\Lambda} \inf_{Q\in B} \mathbb{E}^Q F \;; \Lambda \text{ parallélépipède}\}.$$

La fin de la preuve est alors identique à la référence ci-dessus.

. Montrons maintenant que I(Q,P) est une fonction affine de Q; notre preuve reprend et explicite celle de [1],[2]. Comme Ω est polonais, on peut trouver une famille dénombrable $\mathcal{U}$ d'éléments de l'espace $\mathcal{C}_b(\Omega)$ -des fonctions continues bornées sur Ω à valeurs réelles- ne dépendant que d'un nombre fini de coordonnées, qui soit déterminante pour la convergence étroite des probabilités sur Ω. Soit Λ_n' une suite de cubes croissante vers $(\mathbb{Z}_-^*)^d$; pour $\omega\in\Omega$ et $f\in\mathcal{U}$, on définit

$$\begin{aligned} \Pi_\omega f &= \lim_{n\to\infty} \sum_{\lambda\in\Lambda_n'} f\circ\theta^\lambda(\omega) && \text{lorsque cette limite existe} \\ &= \infty && \text{sinon .} \end{aligned}$$

Alors, $\Omega_f = \{\omega\in\Omega :\ \Pi_\omega f \in \mathbb{R}\}$ et $\Omega_o = \bigcap_{f\in\mathcal{U}} \Omega_f$ sont des boréliens de Ω, éléments de $\sigma\{(\mathbb{Z}_-^*)^d\}$. D'après le théorème ergodique, on constate que

$$\forall Q\in\mathcal{P}_e(\Omega),\quad Q(\Omega_o) = 1 \quad \text{et} \quad \Pi_\omega = Q \quad Q\text{-p.s.},$$

où on a noté $\mathcal{P}_e(\Omega)$ l'ensemble des probabilités stationnaires ergodiques sur Ω.

On choisit à présent $\leq$ tel que $(\mathbb{Z}_-^*)^d \subset (\leftarrow,0)$ - un ordre lexicographique par exemple -, on note $R(Q,\omega)$ une version régulière de la probabilité $Q\in\mathcal{P}_s(\Omega)$ conditionnelle à $(\leftarrow,0)$ qui soit conjointement mesurable en Q et ω ; soit $R^\omega = R(\Pi_\omega,\omega)$. Pour $Q\in\mathcal{P}_e(\Omega)$, on a

$$Q(d\omega') = \int_\Omega R^\omega(d\omega')\, Q(d\omega) \qquad (3).$$

Maintenant, supposons simplement Q stationnaire. Toujours en vertu du théorème ergodique, $Q(\Omega_o) = 1$ et Q-p.s. $\Pi_\omega = Q^{\mathcal{I}}$, où $\mathcal{I}$ est la tribu des invariants. Mais $Q^{\mathcal{I}}$ est presque sûrement ergodique et

$$Q = \int_\Omega Q^{\mathcal{I}} \, dQ_{\mathcal{I}} = \int_{\mathcal{P}_e(\Omega)} S \quad \mu_Q(dS) \tag{4}$$

où μ_Q est l'image de $Q_{\mathcal{I}}$ par l'application $\mathcal{I}$-mesurable $\omega \mapsto Q^{\mathcal{I}}$: donc (3) reste vrai pour $Q \in \mathcal{P}_s(\Omega)$ et $Q^{(\leftarrow,0)} = R^\omega$ Q-p.s. .

Donc I(Q,P) est une fonction affine de Q et vérifie:

$$\begin{aligned} I(Q,P) &= \int h(R_0^\omega,\alpha) \, Q(d\omega) \\ &= \int \mu_Q(dS) \int h(R_0^\omega,\alpha) \, S(d\omega) \\ &= \int \mu_Q(dS) \, I(S,P) \end{aligned} \tag{5}$$

d'après le théorème de Fubini.

.Nous montrons maintenant la minoration b). Il suffit de prouver que pour tout voisinage V de Q dans $\mathcal{P}_s(\Omega)$ avec $I(Q,P) < \infty$,

$$\liminf_{n\to\infty} \frac{1}{|\Lambda_n|} \log P\{R_{n,\omega} \in V\} \geq -I(Q,P) \tag{6}.$$

Supposons d'abord Q ergodique. La preuve, plus délicate que pour d = 1, nécessite le

<u>Lemme :</u> *Si* $I(Q,P) < \infty$ *, la famille filtrante* $\{ \log \frac{dQ_0^\Lambda}{d\alpha} \; ; \; \Lambda \subset (\leftarrow,0),$ Λ *finie* $\}$ *converge dans* $\mathbb{L}^1(Q)$ *vers une variable* Y *vérifiant*

$$Y \leq \log \frac{dQ_0^{(\leftarrow,0)}}{d\alpha} \quad \text{Q-p.s. .}$$

□ Preuve du lemme: soit $\overline{Q} = Q_{(\leftarrow,0)} \otimes \alpha_0 \in \mathcal{P}(X^{(\leftarrow,0]})$; $\frac{dQ_0^\Lambda}{d\alpha}$ est une $\overline{Q}$-$\sigma(\Lambda\cup\{0\})$ martingale. Puisque $\frac{dQ_{\Lambda\cup\{0\}}}{d\overline{Q}_{\Lambda\cup\{0\}}} = \frac{dQ_0^\Lambda}{d\alpha}$, pour $\Lambda' \subset \Lambda$ la probabilité $Q_{\Lambda-\Lambda'}^{\Lambda'\cup\{0\}}$ est $\overline{Q}$-p.s. absolument continue par rapport à $\overline{Q}_{\Lambda-\Lambda'}^{\Lambda'\cup\{0\}}$ avec dérivée $\frac{dQ_0^\Lambda}{d\alpha} \Big/ \frac{dQ_0^{\Lambda'}}{d\alpha}$. En posant $\Phi(x) = x \log x$ pour $x \geq 0$, on a Q-p.s.

$$\mathbb{E}^{Q^{\Lambda'\cup\{0\}}} \log \frac{dQ_0^{\Lambda}}{d\alpha} = \mathbb{E}^{\bar{Q}^{\Lambda'\cup\{0\}}} \left\{ \Phi\left(\frac{dQ_0^{\Lambda}}{d\alpha}\right) \Big/ \frac{dQ_0^{\Lambda'}}{d\alpha} \right\}$$

$$= \left\{ 1 \Big/ \frac{dQ_0^{\Lambda'}}{d\alpha} \right\} \mathbb{E}^{\bar{Q}^{\Lambda'\cup\{0\}}} \Phi\left(\frac{dQ_0^{\Lambda}}{d\alpha}\right)$$

$$\geq \left\{ 1 \Big/ \frac{dQ_0^{\Lambda'}}{d\alpha} \right\} \Phi\left(\mathbb{E}^{\bar{Q}^{\Lambda'\cup\{0\}}} \frac{dQ_0^{\Lambda}}{d\alpha}\right) = \log \frac{dQ_0^{\Lambda'}}{d\alpha}$$

en utilisant l'inégalité de Jensen pour la fonction convexe Φ et la propriété de martingale; comme $I(Q,P) < \infty$,

$\mathbb{E}^{Q}\left(\log \frac{dQ_0^{\Lambda}}{d\alpha}\right)^+ \leq \mathbb{E}^{Q}\left(\log \frac{dQ_0^{(\leftarrow,0)}}{d\alpha}\right)^+ < \infty$, $\log \frac{dQ_0^{\Lambda}}{d\alpha}$ est une Q sous-martingale filtrante qui converge dans $\mathbb{L}^1(Q)$ vers une limite Y lorsque Λ croît vers $(\leftarrow,0)$ [NEVEU]. Pour $\Lambda = (\leftarrow,0)$, on a $\log \frac{dQ_0^{\Lambda'}}{d\alpha} \leq \mathbb{E}^{Q^{\Lambda'\cup\{0\}}} \log \frac{dQ_0^{(\leftarrow,0)}}{d\alpha}$ Q-p.s.; en passant à la limite sur $\Lambda' \nearrow (\leftarrow,0)$, on obtient la majoration de Y.□

Choisissons à présent $\leq$ tel que $(\leftarrow,0) \cap \mathbb{N}^d \neq \emptyset$. Puisque Q est stationnaire,

$$\frac{dQ_{\Lambda_n}}{dP_{\Lambda_n}}(\omega) = \prod_{\lambda\in\Lambda_n} \frac{dQ_{\{\lambda\}}^{(\leftarrow,\lambda)\cap\Lambda_n}}{d\alpha}(\omega) = \exp \sum_{\lambda\in\Lambda_n} \log \frac{dQ_0^{(\leftarrow,0)\cap T^{-\lambda}\Lambda_n}}{d\alpha}(\theta^\lambda\omega) .$$

On a donc:

$$P\{ R_{n,\omega}\in V \} \geq \mathbb{E}^{Q}\left\{ \left(\frac{dQ_{\Lambda_n}}{dP_{\Lambda_n}}\right)^{-1} \mathbb{1}_{\{R_{n,\omega}\in V\}} \right\}$$

$$= \mathbb{E}^{Q}\left\{ \exp\left(- \sum_{\lambda\in\Lambda_n} \log \frac{dQ_0^{(\leftarrow,0)}}{d\alpha}(\theta^\lambda\omega) + \sum_{\lambda\in\Lambda_n} \left[\log \frac{dQ_0^{(\leftarrow,0)}}{d\alpha} - Y \right](\theta^\lambda\omega) \right.\right.$$

$$\left.\left. + \sum_{\lambda\in\Lambda_n} \left[Y - \log \frac{dQ_0^{(\leftarrow,0)\cap T^{-\lambda}\Lambda_n}}{d\alpha} \right](\theta^\lambda\omega) \right) \mathbb{1}_{\{ R_{n,\omega}\in V \}} \right\} \quad (7).$$

D'après le lemme, le premier crochet est positif, et la moyenne de

Cezaro

$$Z_n = \frac{1}{|\Lambda_n|} \sum_{\lambda\in\Lambda_n} \left[Y - \log \frac{dQ_O^{(\leftarrow,0)\cap T^{-\lambda}\Lambda_n}}{d\alpha} \right] (\theta^\lambda\omega)$$

converge vers 0 dans $\mathbb{L}^1(Q)$: si $\varepsilon>0$, $\Omega_n^2 = \{\omega\in\Omega;\ Z_n \geq -\varepsilon\}$ est de Q-probabilité tendant vers un. D'autre part, comme $\mathbb{E}^Q \log \frac{dQ_O^{(\leftarrow,0)}}{d\alpha} = I(Q,P)$,

celle de $\Omega_n^1 = \{\omega\in\Omega;\ \frac{1}{|\Lambda_n|} \sum_{\lambda\in\Lambda_n} \log \frac{dQ_O^{(\leftarrow,0)}}{d\alpha} (\theta^\lambda\omega) \leq I(Q,P) + \varepsilon\}$ tend aussi vers un d'après le théorème ergodique multiparamétrique [KRENGEL]; il en est de même pour $\{R_{n,\omega}\in V\}$. En écrivant (7) comme

$$P\{ R_{n,\omega}\in V \} \geq \exp(- |\Lambda_n| [I(Q,P) + 2\varepsilon])\ Q(\Omega_n^1 \cap \Omega_n^2 \cap \{R_{n,\omega}\in V\})$$

on obtint (6) pour Q ergodique.

Montrons à présent (6) pour $Q \in \mathcal{P}_s(\Omega)$. Soit $S_1,\ldots S_m,\ldots$ un échantillon de la loi μ_Q sur $\mathcal{P}_e(\Omega)$: d'après (4),(5) et $I(Q,P)<\infty$, la loi des grands nombres entraîne que

$$\lim_{m\to\infty} (1/m) \sum_{i=1}^m S_i = Q \quad \text{et} \quad \lim_{m\to\infty} (1/m) \sum_{i=1}^m I(S_i,P) = I(Q,P) \ , \ \mu\text{-p.s.} \ .$$

$I(.,P)$ étant affine, il suffit donc d'établir (6) lorsque Q est une combinaison linéaire finie d'éléments de $\mathcal{P}_e(\Omega)$, ou, plus simplement, que l'ensemble des $Q\in\mathcal{P}_s(\Omega)$ vérifiant (6) est convexe.

Soit donc $Q = t\,\tilde{Q} + (1-t)\,\overline{Q}$, $0<t<1$, $\tilde{Q}$ et $\overline{Q}$ vérifiant (6), et V un voisinage de Q. Notons ici $R_{n,\omega} = R_{\Lambda_n,\omega}$. Soient b_n un entier tel que $b_n = t\,a_n^1 + o(1)$, $\lambda_n = (b_n,0,\ldots 0)$, $\overline{\omega}^n = \theta^{\lambda_n}\omega$, $\tilde{\Lambda}_n = [0,b_n]\times\prod_{i=2}^d [0,a_n^i]$ et $\overline{\Lambda}_n = [0,a_n^1-b_n]\times\prod_{i=2}^d [0,a_n^i]$: alors $R_{n,\omega}$ est voisin de $t\,R_{\tilde{\Lambda}_n,\omega} + (1-t)\,R_{\overline{\Lambda}_n,\overline{\omega}^n}$, la différence résultant des effets de périodisation au bord des bandes. Puisque la convergence des processus est essentiellement celle de leur marginales de dimension finie, cet effet devient négligeable pour toute portée donnée lorsque $n \longrightarrow \infty$: on peut trouver des voisinages $\tilde{V}$ de $\tilde{Q}$ et $\overline{V}$ de $\overline{Q}$ tels que

$$R_{\tilde{\Lambda}_{n,\omega}} \in \tilde{V} \quad \text{et} \quad R_{\overline{\Lambda}_{n,\omega}^{-n}} \in \overline{V} \implies R_{n,\omega} \in V \qquad \text{pour n assez grand.}$$

Comme P est une mesure produit, on a pour n assez grand

$$P\{R_{n,\omega}\in V\} \geq P\{R_{\tilde{\Lambda}_{n,\omega}} \in \tilde{V}\} \; P\{R_{\overline{\Lambda}_{n,\omega}^{-n}} \in \overline{V}\}$$

$$\geq \exp(- t|\Lambda_n| \; [I(\tilde{Q},P)+\varepsilon]) \quad \exp(- (1-t)|\Lambda_n| \; [I(\overline{Q},P)+\varepsilon])$$

$$= \exp(- |\Lambda_n|[I(Q,P)+\varepsilon]) \; .$$

□

La <u>preuve du théorème</u> est mot pour mot la même que celle des théorèmes IV.1 et IV.2 de l'article "large deviation estimates for a conditional probability distribution ..." figurant dans cette thèse, en prenant pour ν une masse de Dirac (on conditionne par une fonction constante); elle ne sera donc pas répétée ici.

<u>Sur les commentaires :</u>

1) La preuve de la formule variationnelle de Gibbs à l'aide de techniques de grande déviation est nouvelle pour d≥2 [ELLIS,p.161]. Elle est, du reste, analogue à celle donnée dans cette référence pour d=1. Pour une autre preuve, voir [PRESTON], qui contient également beaucoup de résultats sur les états de Gibbs.

Notre résultat montre que la fonctionnelle de grandes déviations est la même pour tous les états de Gibbs correspondant à une interaction donnée; elle ne dépend que des caractéristiques locales. Elle induit une fonctionnelle de grandes déviations pour la mesure empirique $\frac{1}{n}\sum_{i=1}^{n}\delta_{\omega_i}$ qui n'est pas strictement convexe s'il y a transition de phase. Une autre conséquence est que l'on ne peut discriminer les mesures de Gibbs à l'ordre de grandeur exponentiel du volume.

2) $I(Q,P)$ est s.c.i. : en effet, $I(Q,P) = \sup_{\Lambda} \frac{1}{|\Lambda|} h(Q_\Lambda, P_\Lambda)$, où $h(.,\nu)$ est s.c.i. (car X est polonais), et $Q \mapsto Q_\Lambda$ est continue.

3) Les lignes de niveaux de I sont compactes dans $\mathcal{P}_s(\Omega)$: si $l \in \mathbb{R}^+$, la ligne de niveau $\{ Q \in \mathcal{P}_s(\Omega) ; I(Q,P) \leq l \}$ est fermé en vertu du point 2). Pour montrer qu'elle est relativement compacte, il suffit de montrer qu'elle est tendue, ou encore que chacune de ses projections finies-dimensionnelles le sont; mais ceci résulte du fait que sa projection unidimensionnelle est incluse dans le compact $\{ q \in \mathcal{P}(X) ; h(q,\alpha) \leq l \}$.

4) $G \neq \emptyset$: comme U est continue sur Ω, $Q \mapsto \mathbb{E}^Q U + I(Q,P)$ est s.c.i. et atteint son minimum $-p$ sur le compact non vide $\{Q ; \mathbb{E}^Q U + I(Q,P) \leq -p+1\}$. Donc $G \neq \emptyset$.

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CHAPITRE I .

Partie B : MESURES DE GIBBS AVEC INTERACTION ALEATOIRE .

LARGE DEVIATION ESTIMATES

FOR A CONDITIONAL PROBABILITY DISTRIBUTION.

APPLICATIONS TO RANDOM INTERACTION GIBBS MEASURES.

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ABSTRACT

Let $(x_i, y_i), i \in \mathbb{Z}^d$, be independent identically distributed random variables with arbitrary distribution. We show that, for almost every $(y_i)_i$, the conditional law of the empirical field given $(y_i)_i$ satisfies to large deviations inequalities. This applies to the study of Gibbs measures with random interaction, in the case of some mean-field models as well as of short range summable interaction. We show that the pressure is non random, and is given by a variational formula. These random Gibbs measures have the same large deviation rate, which does not depend on the particular realization of the interaction; their local behaviour is described in terms of conditional probabilities given the interaction of solutions to the variational formula.

KEY WORDS

Large deviation , Gibbs measure , random field , spin glass, neural networks , maximum entropy , conditional probability.

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I. INTRODUCTION:

In [7], DONSKER and VARADHAN have recently initiated the large deviation theory for stationary random processes on the "level 3" of the empirical process; their methodology was shown to be fruitful, and has been since applied to various domains. Among these, Gibbs random fields were proved to have large deviation properties depending only on the local characteristic ([3],[14],[21]): a Gibbs field is a random field on $\mathbb{Z}^d$ such that the conditional distribution of a finite set of coordonates given the other ones has exponential density with respect to some independent identically distributed (i.i.d.) field; this density involves a translation invariant interaction, which describes the dependence between the variables. One strategy is to establish first a level 3 large deviation principle for the i.i.d. field, and then to transfer it to the Gibbs field via Laplace's method.

In recent years, Gibbs random fields with random interaction have been extensively used to describe disordered systems; this time, it is assumed that the law of the interaction is translation invariant. In this paper, we study such a field for allmost every realization of the interaction, using large deviation techniques.

We will adopt the same strategy as above, so we will first establish that a conditional large deviation principle for i.i.d. random fields holds with probability one (w.p.1). Let $(W_i)_i$ be an i.i.d. field, with index $i\in\mathbb{Z}^d$ for some integer $d\geq 1$, and values in a Polish space $\mathbb{W}$; let π be continuous on $\mathbb{W}$ to another Polish space, and $Y_i = \pi W_i$: we estimate large deviation probabilities for a regular version of the conditional law of the empirical field $R_{n,W}$ given the Y-field, with $Y=(Y_i)_i$, under typical conditioning (i.e. on a set of Y's with full probability). The rate function in the conditional case coincides with that of the unconditional case on the set of stationary fields with the typical margin, and is infinite elsewhere.

The proof does not reduce to a mere consequence of Bayes formula, as when conditioning consists in an event $\{ R_{n,\cdot} \in B \}$ with non-zero probability as in [18]; by the way, our result implies the latter for typical B's. Our techniques are not either related to the expansions of probability densities of ZABELL [29] for exact conditioning.We will essentially use the non conditional estimates and Borel-Cantelli lemma. The lower bound will be proved by means of an exponential change of probability, which is a central idea in large deviation theory: here, the new probability will be the law of an i.i.d. field indexed by a bigger lattice with some rectangle Λ as unit cell, each variable having cardinal(Λ) components.

In section IV and V, we derive applications to Gibbs measures with random interaction, which randomness will be given by the Y variable. We will consider separately short range summable interaction (§IV) and some mean-field interaction (§V); one can refer to detailed references of such models in *spin-glasses* and *neural networks*.

We show that the pressure exists in the thermodynamic limit w.p.1, is independent on the experiment and is given by a Gibbs variational formula; in particular, we recover results of LEDRAPPIER for Ising spins [19].We obtain large deviation probability estimates, for almost every realization of the interaction: the rate function, which give the rate of exponential decay, does not depend on the particular realization. The problems are tackled with a particular emphasis on uniformity with respect to boundary conditions; our results also apply for Gibbs measures which are obtained in the thermodynamic limit with boundary conditions depending on the interaction itself.

But the thermodynamical limits of finite volume Gibbs measures depend on the interaction, and they are random measures. Therefore we localize the previous results on space averages, and show that these limits are related to the maximum entropy distribution - which are by definition the solutions to the variational problem -, more precisely they are conditional versions of these distributions given the interaction.

In section II, we recall generalities on large deviations and empirical fields, and state some known results used in this paper. Section III is devoted to the conditional large deviation principle in the i.i.d. case. We give now a simple explicit computation showing why it holds at level 1 (of course, the level 3 proof is not trivial like this one): let X_i and Y_i , i=1,2,... be two sequences of two sequences of bounded real i.i.d. random variables; we prove that the conditional distribution of

$$Z_n = (1/n)\sum_{i=1}^{n} X_i Y_i \tag{1.1}$$

given $Y=(Y_i)_i$ obeys a large deviation principle for almost every realization of Y . Indeed, using the independence assumption, one can compute the logarithm of the Laplace transform of nZ_n given Y

$$L^Y_{nZ_n}(t) = \log \mathbb{E}_X \exp(tnZ_n) = \log \prod_{i=1}^{n} \mathbb{E}_X \exp(tY_i X) = \sum_{i=1}^{n} L_X(tY_i)$$

with $L_X(s) = \log \mathbb{E}_X \exp(sX)$ and $\mathbb{E}_X$ the expectation in the X variable; the law of large numbers implies that

$$(1/n)\ L^Y_{nZ_n}(t) \xrightarrow{w.p.1} L(t) = \mathbb{E}_Y\ L_X(t) \qquad (1.2).$$

Denote by ¥ the Borel set where the limit (1.2) holds; it is shown in [6] that (1.2) implies that we have on ¥

$$\begin{aligned} -\inf\{L^*(z); z\in\mathring{B}\} &\leq \liminf_{n\to\infty} (1/n) \log \Pr\{ Z_n\in B / Y\} \\ &\leq \limsup_{n\to\infty} (1/n) \log \Pr\{ Z_n\in B / Y\} \leq -\inf\{L^*(z); z\in\bar{B}\} \end{aligned} \qquad (1.3)$$

with L^* the Legendre transform of L given by

$$L^*(z) = \sup\{tz-L(t); t\in\mathbb{R}\} \qquad (1.4).$$

Notice the set ¥ of conditioning under consideration is typical in the sense it has probability one.

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II. DEFINITIONS and GENERALITIES :

Let E be a Polish space, i.e. a metrizable complete separable topological space, and denote by $\mathcal{P}(E)$ the set of probability measures on E. Consider a sequence of positive numbers a_n going to infinity and a function $I : E \mapsto [0,+\infty]$. A sequence $P_n \in \mathcal{P}(E)$ obeys a *large deviation principle* on E with rate function I and sequence a_n if

i) I is lower semi-continuous on E, and the level set $\{ x \in E \ ; \ I(x) \leq a \}$ is compact for all $a \in \mathbb{R}^+$;

ii) for all Borel subset B of E,

$$- I(\mathring{B}) \leq \liminf_{n \to \infty} a_n^{-1} \log P_n(B) \leq \liminf_{n \to \infty} a_n^{-1} \log P_n(B) \leq - I(\bar{B}) \qquad (2.1)$$

with $I(B)=\inf\{I(x);x \in B\}$.

When concerned with a partial summary of the information weighted by P_n , we will use the *contraction principle* (theorem 2.4 in [27]): let Φ be continuous on E to another Polish space, and assume the above principle holds, then $P_n o \Phi^{-1}$ obeys a large deviation principle too, with same sequence a_n and rate function $\tilde{I}$ given by

$$\tilde{I}(y) = \inf \{ I(x) \ ; \ \Phi(x) = y \}$$

NOTATIONS : Let $\mathbb{W}$ be a Polish space, $d \geq 1$ be integer and $\Omega = \mathbb{W}^{\mathbb{Z}^d}$. If $i=(i^1,\ldots,i^d) \in \mathbb{N}^d$ and $j \in \mathbb{N}^d$, we denote by $\Lambda(i,j)$ the rectangle $\prod_{i=1}^{d} [-i^k,j^k]$ in $\mathbb{Z}^d$. Through all this paper, w consider sequences i_n and j_n with $j_n+i_n \longrightarrow \infty$, in the sense $j_n^k+ i_n^k \longrightarrow \infty$ for each $k \leq d$, and we set $\Lambda_n=\Lambda(i_n,j_n)$. For $\omega \in \Omega$, let $\omega^{(n)}$ be the element of Ω obtained in making periodic the restriction of ω to Λ_n: $\omega_i^{(n)}= \omega_j^{(n)}$ if $i-j= m.(i_n+j_n+\mathbb{1})$ for some $m \in \mathbb{Z}^d$ and $\mathbb{1}=(1,\ldots 1)$. Let $\theta^i, i \in \mathbb{Z}^d$, be the shift operators on Ω given by $\theta^i\omega_. = \omega_{i+.}$, and define the *empirical field*

$$R_{\Lambda_n,\omega} = |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \delta_{\theta^i\omega(n)} \tag{2.2}$$

with $|\Lambda|$ the cardinal of a finite set Λ in $\mathbb{Z}^d$. Then $R_{\Lambda_n,\omega} \in \mathcal{P}_s(\Omega)$ the set of all stationary (shifts invariant) measures on Ω. Except in the proof of theorem III.1, we will write $R_{n,\omega}$ instead of $R_{\Lambda_n,\omega}$.

Notice that space averages may be evaluated asymptotically in terms of the empirical field, since

$$\int f \, dR_{n,\omega} - |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} f(\theta^i\omega) \longrightarrow 0 \qquad \text{as} \qquad n\to\infty$$

for any bounded continuous function f on Ω.

Let α,β in $\mathcal{P}(\mathbb{W})$: *Kullback information* $h(\beta;\alpha)$ of β with respect to α on the Borel field of $\mathbb{W}$ is

$$h(\beta;\alpha) = \begin{cases} \int \frac{d\beta}{d\alpha} \log \frac{d\beta}{d\alpha} \, d\alpha & \text{if } \beta \ll \alpha \text{ and } \frac{d\beta}{d\alpha} \log \frac{d\beta}{d\alpha} \in \mathbb{L}^1(\alpha) \\ \infty & \text{otherwise.} \end{cases}$$

We will denote by P_α the product measure $\alpha^{\otimes\mathbb{Z}^d}$. The law of the empirical field $R_{n,\omega}$ under P_α is known to obey a large deviation principle on $\mathcal{P}_s(\Omega)$ (refer to [3],[14] or [21]) with sequence $|\Lambda_n|$ and rate function $H(\,.\,;P_\alpha)$

$$\begin{aligned} H(Q;P_\alpha) &= \sup\{ \, |\Lambda|^{-1} h(Q_\Lambda;P_{\alpha,\Lambda}) \; : \; \Lambda \subset \mathbb{Z}^d \text{ finite } \} \\ &= \lim_{n\to\infty} |\Lambda_n|^{-1} h(Q_{\Lambda_n};P_{\alpha,\Lambda_n}) \end{aligned} \tag{2.3}$$

with Q_Λ the restriction of Q to the σ-algebra $\sigma(\Lambda)$ generated by $\{ \, \omega_i, i\in\Lambda \, \}$. In fact H is a linear functional of Q.

This result implies Sanov theorem on large deviations of the empirical measure $|\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \delta_{\omega_i}$ as well as Cramèr-Chernov theorem, via the contraction principle; but it also applies to empirical correlations which are space averages too, and, by the way, to Markov random fields which involve spatial dependence between variables.

III. CONDITIONAL PROBABILITY FOR LARGE DEVIATION OF I.I.D. RANDOM FIELDS :

Let $\mathcal{Y}$ be another Polish space, and $\pi : \mathcal{W} \rightarrow \mathcal{Y}$ be continuous. π induces a continuous map Π on $\Omega = \mathcal{W}^{\mathbb{Z}^d}$ to $\mathcal{Y}^{\mathbb{Z}^d}$, $\Pi\omega = y$ with $y_i = \pi\omega_i$, and Π itself induces a continuous $\Pi^*: \mathcal{P}_s(\Omega) \rightarrow \mathcal{P}_s(\mathcal{Y}^{\mathbb{Z}^d})$, $\Pi^*Q = Q\circ\Pi^{-1}$. Notice that Π^*P_α is the product measure based on $\alpha\circ\pi^{-1}$, and $\Pi^*R_{n,\omega} = R_{n,\Pi\omega}$ the *empirical field* based and $\Pi\omega$. Since Ω and $\mathcal{Y}$ are Polish spaces, we can define a *regular version* $P_\alpha\{ . / y \}$ of P_α *condionally on* $\Pi\omega = y$ [22], i.e. a map $y \mapsto P_\alpha\{ . / y \}$ on $\mathcal{Y}^{\mathbb{Z}^d}$ such that i) $\forall y$, $P_\alpha\{./y\} \in \mathcal{P}(\Omega)$, ii) for all Borel subset B in Ω, $y \mapsto P_\alpha\{B/y\}$ is a version of the conditional expectation of $\mathbb{1}_B$ given $\Pi\omega = y$.

THEOREM III.1 : With P_α-probability one, the sequence of conditional distribution of the empirical process under P_α given $\Pi\omega = y$

$$P_\alpha \{ R_{n,\omega} \in . / y \}$$

obeys a large deviation principle on $\mathcal{P}_s(\Omega)$ with sequence $|\Lambda_n|$ and rate function I given by

$$I(Q) = \begin{cases} H(Q;P_\alpha) & \text{if } \Pi^*Q = \Pi^*P_\alpha \\ \infty & \text{otherwise .} \end{cases}$$

REMARKS: .1) In the applications we give in this paper, we restrict to a product space $\mathcal{W} = \mathcal{X} \times \mathcal{Y}$ with projection π on $\mathcal{Y}$ and to a product measure $\alpha = \mu \otimes \nu$, as we did for the computation in the introduction. We then have P_ν a.s.

$$P_\alpha \{ R_{n,\omega} \in B / y \} = \int \mathbb{1}_B(R_{n,\omega}) \, d\mu^{\otimes\Lambda_n} \tag{3.1}$$

for all Borel subsets B of $\mathcal{P}_s(\Omega)$. The estimates (1.3) follow from the contraction principle and the above theorem.

.2) The usual non-conditional case is a consequence of the theorem with π a constant function.

.3) Since $H(.;P_\alpha)$ and Π^* are linear , so is I.

We prove the theorem. Since H is a rate function and since Π^* is continuous, the level sets $\{Q;I(Q)\leq a\} = (\Pi^*)^{-1}\{\Pi^*P_\alpha\} \cap \{Q;H(Q;P_\alpha)\}\leq a\}$ are compact sets in $\mathcal{P}_s(\Omega)$ for all $a\in\mathbb{R}$. Then we only need to prove that (2.1) holds for $P^n = P_\alpha\{ R_{n,\omega}\in./y\}$, P_α a.s.. We begin with the upper bound :

<u>PROPOSITION III.2: There exists a Borel subset $\mathcal{Y}_1$ of $\mathcal{Y}^{\mathbb{Z}^d}$ with P_ν probability one such that, for all $y \in \mathcal{Y}_1$, we have</u>

$$\limsup_{n\to\infty} |\Lambda_n|^{-1}\log P_\alpha\{ R_{n,\omega} \in C / y \} \leq - I(C) \qquad (3.2)$$

<u>for all closed set C in</u> $\mathcal{P}_s(\Omega)$.

□ 1) We first prove that (3.2) holds on a Borel set $\Omega_1(C)$ of full probability, for any closed set C. Let C'_m , $m\in\mathbb{N}$, be a sequence of closed neighbourhoods of Π^*P_α in $\mathcal{P}_s(\mathcal{Y}^{\mathbb{Z}^d})$, decreasing to $\{\Pi^*P_\alpha\}$, and $C_m = C \cap (\Pi^*)^{-1}(C'_m)$. Since H is lower semi-continuous in its first argument, $H(C_m;P_\alpha)$ is non-decreasing to $H(C \cap (\Pi^*)^{-1}\{\Pi^*P_\alpha\};P_\alpha) = I(C)$ by definition of I. For positive ε, fix m such that

$$H(C_m;P_\alpha) \geq I(C) - \varepsilon \qquad (3.3).$$

The ergodic theorem implies that

$$\mathcal{Y}(C,\varepsilon) = \{ y \in \mathcal{Y}^{\mathbb{Z}^d} / \exists\, n_o(y) : \forall\, n\geq n_o(y),\ R_{n,y} \in C'_m \}$$

has P_ν-probability one; on this set, $P_\alpha\{R_{n,\omega}\in C_m / y \} = P_\alpha\{R_{n,\omega}\in C / y \}$ for $n\geq n_o(y)$. From the upper bound (2.3) for P_α , we have for large n

$$P_\alpha\{ R_{n,\omega}\in C_m \} \leq \exp -|\Lambda_n|\{ H(C_m;P_\alpha) - \varepsilon \} \qquad (3.4).$$

Let $¥(C,\varepsilon,n) = \{ y\in\mathscr{Y}^{\mathbb{Z}^d} ;\ P_\alpha\{ R_{n,\omega}\in C_m / y \} \geq \exp -|\Lambda_n|[I(C) - 3\varepsilon] \}$.
Since $P_\alpha\{ R_{n,\omega}\in . \} = \int P_\alpha\{ R_{n,\omega}\in ./y\} dP_\alpha$,Chebichev inequality yields

$$\exp(-|\Lambda_n|[I(C)-3\varepsilon])\ \ P_\alpha\{ ¥(C,\varepsilon,n)\} \leq P_\alpha[\{R_{n,\omega}\in C_m\} \cap ¥(C,e,n)\}$$
$$\leq \exp -|\Lambda_n|[I(C) - 2\varepsilon]$$

combining (3.3,4). Then $P_\alpha\{ ¥(C,\varepsilon,n)\} \leq \exp -|\Lambda_n|\varepsilon$ for large n : Borel-Cantelli lemma implies that $¥_1(C,\varepsilon) = ¥(C,\varepsilon) \cap \{\limsup_{n\to\infty} ¥(C,\varepsilon,n)\}^c$ has P_α-probability one too; let $\omega\in¥_1(C,\varepsilon)$, we have

$$P_\alpha\{ R_{n,\omega}\in C / y \} \leq \exp -|\Lambda_n|[I(C) - 3\varepsilon]$$

for large n. So $¥_1(C) = \bigcap_\varepsilon ¥_1(C,\varepsilon)$ with arbitrary sequence $\varepsilon\to 0$ is such that (4.1) holds .

2) $\mathscr{P}_s(\Omega)$ being separable, we can find a countable basis of open sets $\mathcal{O}_m$, $m\in\mathbb{N}$. Define $¥_1 = \bigcap_{\mathcal{F}} ¥_1(\bigcap_{m\in\mathcal{F}} \mathcal{O}_m^c)$ where $\mathcal{F}$ ranges over the (countable) set of finite subsets of $\mathbb{N}$; then $P_\nu(¥_1) = 1$. If $C \subset \mathscr{P}_s(\Omega)$ is closed, then $C = \bigcap_{m\in\mathcal{F}'} \mathcal{O}_m^c$ for some $\mathcal{F}'\subset \mathbb{N}$; but we can find a finite $\mathcal{F} \subset \mathcal{F}'$ with $I(\bigcap_{m\in\mathcal{F}} \mathcal{O}_m^c) > I(C) - \varepsilon$. Hence,

$$\limsup_{n\to\infty} |\Lambda_n|^{-1}\log P_\alpha\{ R_{n,\omega}\in C /y\} \leq \limsup_{n\to\infty} |\Lambda_n|^{-1}\log P_\alpha\{ R_{n,\omega}\in \bigcap_{m\in\mathcal{F}}\mathcal{O}_m^c /y\}$$
$$\leq - I(\bigcap_{m\in\mathcal{F}} \mathcal{O}_m^c)$$

for $y\in¥_1$, which is less than $-I(C)+\varepsilon$; since ε is arbitrary, $¥_1$ is as in proposition III.2. □

<u>PROPOSITION III.3 :</u> <u>There exists a Borel set</u> $¥_2$ <u>in</u> $\mathscr{Y}^{\mathbb{Z}^d}$ <u>with</u> P_ν <u>probability one such that we have for all</u> $y \in ¥_2$:

$$\liminf_{n\to\infty} |\Lambda_n|^{-1} \log P_\alpha\{ R_{n,\omega}\in \mathcal{O} / y \} \geq - I(\mathcal{O}) \tag{3.5}$$

<u>for all open set $\mathcal{O}$ in</u> $\mathscr{P}_s(\Omega)$.

□ Again we begin to prove that (3.5) holds on some $\mathcal{V}_2(\mathcal{O})$. We assume that the right hand side of (3.5) is finite, since there is nothing to prove otherwise. Let $\varepsilon>0$, and $R\in\mathcal{O}$ such that

$$I(R) < I(\mathcal{O}) + \varepsilon \qquad (3.6).$$

Since $\mathbb{W}$ is Polish, we can pick a finite number p_o of bounded continuous functions f_p on Ω, depending on finitely many ω_i , such that

$$\{ Q \in \mathcal{P}_s(\Omega) \ ; \ |\mathbb{E}^Q f_p - \mathbb{E}^R f_p| < 2 \ , \ \forall p \leq p_o \} \subset \mathcal{O} \ .$$

Let $\mathcal{O}_1 = \{ Q\in\mathcal{P}_s(\Omega) ; |\mathbb{E}^Q f_p - \mathbb{E}^R f_p| < 1, \ \forall p \leq p_o \}$.

The following construction is that of FOLLMER and OREY in [14] , lemma 3.2: let $i,j\in\mathbb{N}^d$ and $\Lambda = \Lambda(i,j)$ as given in §II. If $\gamma \in \mathcal{P}(\mathbb{W}^\Lambda)$, we will denote by $\bar{\gamma} \in \mathcal{P}(\Omega)$ the probability measure which coincides with $\gamma \circ \theta^{m.(i+j+\mathbb{1})}$ (where $\mathbb{1} = (1,\ldots 1)$) on the σ-fields $\sigma\{ \Lambda + m.(i+j+\mathbb{1})\}$, $m\in\mathbb{Z}^d$, and making these fields independent; we next randomize the origin in defining $\varphi^\Lambda(\gamma) \in \mathcal{P}_s(\Omega)$ by

$$\varphi^\Lambda(\gamma) = |\Lambda|^{-1} \sum_{k\in\Lambda} \bar{\gamma}\circ\theta^k \ .$$

From (2.3) and the definition of $\mathcal{O}_1$, we can fix i and j such that

$$\big|\ |\Lambda|^{-1} h(R_\Lambda ; P_{\alpha,\Lambda}) - H(R;P_\alpha) \ \big| < \varepsilon \qquad (3.7)$$

and

$$\varphi^\Lambda(R_\Lambda) \in \mathcal{O}_1 \qquad (3.8)$$

for $\Lambda = \Lambda(i,j)$. Clearly φ^Λ is continuous on $\mathcal{P}(\mathbb{W}^\Lambda)$, and $A = (\varphi^\Lambda)^{-1}(\mathcal{O}_1)$ is a neighbourhood of R^Λ . We now need a lemma :

LEMMA III.4 : Let E,F be Polish spaces, ξ : E⟶F continuous, $\beta,\gamma\in\mathcal{P}(E)$ with $\beta\circ\xi^{-1} = \gamma\circ\xi^{-1}$ and $h(\gamma;\beta)<\infty$. For any weak neighbourhood A of γ and any positive ε , there exists $\rho \in A$ such that $\rho\circ\xi^{-1} = \beta\circ\xi^{-1}$, $|h(\rho;\beta) - h(\gamma;\beta)|<\varepsilon$ and $\log \frac{d\rho}{d\beta}$ is β-almost surely equal to a bounded continuous function en E .

The lemma will be proved later; it applies to $E=\mathbb{W}^{\Lambda}$, $F=\mathcal{Y}^{\Lambda}$, $\gamma=R_{\Lambda}$, $\beta=\alpha^{\otimes\Lambda}$, and $\xi((\omega_i)_{i\in\Lambda}) = (\pi\omega_i)_{i\in\Lambda}$. Since $\rho \in A$, $\varphi^{\Lambda}(\rho) \in \Theta_1$.

Let Δ_n be the part of the lattice with unit cell Λ, which is contained in Λ_n : $\Delta_n = \bigcup_{l\in L_n} (\Lambda + l.(j-i+\mathbb{1}))$ where $L_n = \{ l\in\mathbb{N}^d;\ \Lambda + l.(j-i+\mathbb{1}) \subset \Lambda_n \}$. We identify $\mathbb{W}^{\Delta_n}$ to $(\mathbb{W}^{\Lambda})^{L_n}$ as well as $\mathbb{W}^{\mathbb{Z}^d}$ to $(\mathbb{W}^{\Lambda})^{\mathbb{Z}^d}$, with identification

$$\omega_i = \omega_{m,l} \quad \text{for } i\in\Delta_n\ ,\ m\in\Delta,\ l\in L_n\ ,\ \text{and} \quad i = l.(j-i+\mathbb{1}) + m \qquad (3.9)\ .$$

Recalling the definition (2.2), we see that the empirical field $R_{\Delta_n,\omega}$ is the image of $R_{L_n,(\omega_{.,l})_l}$ - that we shall denote by $R^{\Lambda}_{L_n,\omega}$ - built on the lattice with bigger cell Λ, through the map ψ^{Λ},

ψ^{Λ}: $\mathcal{P}_s((\mathbb{W}^{\Lambda})^{\mathbb{Z}^d}) \longrightarrow \mathcal{P}_s(\Omega)$, $Q^{\Lambda} \mapsto \psi^{\Lambda}(Q^{\Lambda}) = Q$ such that

$$\int f\ dQ = \int |\Lambda|^{-1} \sum_{i\in\Lambda} f\circ\theta^i\ dQ^{\Lambda} \qquad (3.10)$$

for all bounded continuous f on Ω. In particular, we have $\psi^{\Lambda}(P_{\gamma}) = \varphi^{\Lambda}(\gamma)$ for $\gamma\in\mathcal{P}(\mathbb{W}^{\Lambda})$. ψ^{Λ} is continuous, hence

$$\Theta(\varepsilon) = (\psi^{\Lambda})^{-1}(\Theta_1) \cap \{ Q^{\Lambda}\in\mathcal{P}_s((\mathbb{W}^{\Lambda})^{\mathbb{Z}^d});\ |\ \mathbb{E}^{Q^{\Lambda}} \log \frac{d\rho}{d\beta} - \mathbb{E}^{P_\rho} \log \frac{d\rho}{d\beta}\ | < \varepsilon \} \qquad (3.11)$$

is open; in the last formula, the expectations are defined acording to the identification (4.8), and we recall that $\beta = \alpha^{\otimes\Lambda}$. But the law of $R^{\Lambda}_{L_n,\omega}$ under P_ρ obeys a large deviation principle with sequence $|L_n| = |\Delta_n|/|\Delta|$ and rate function $H(.;P_\rho)$: since $\Theta(\varepsilon)$ is open and contains P_ρ , this implies that there exists $\eta>0$ such that

$$P_\rho\{ R^{\Lambda}_{L_n,\omega} \in \Theta(\varepsilon) \} \geq 1 - \exp - \eta\ |\Delta_n| \qquad (3.12)$$

for large n . Furthermore we have

$$P_\rho\{ R^\Lambda_{L_n,\omega} \in \Theta(\varepsilon) \} = \mathbb{E}^{P_\beta}\left[1\!\!1_{\Theta(\varepsilon)}[R^\Lambda_{L_n,\omega}] \frac{d\rho^{\otimes L_n}}{d\beta^{\otimes L_n}} \right]$$

$$= \mathbb{E}^{P_\alpha}\left\{ \mathbb{E}^{P_\alpha}\left[1\!\!1_{\Theta(\varepsilon)}[R^\Lambda_{L_n,\omega}] \frac{d\rho^{\otimes L_n}}{d\beta^{\otimes L_n}} \Big/ y \right]\right\}$$

$$\leq \exp(-|\Lambda_n|\varepsilon) + \mathbb{E}^{P_\alpha}\left\{ 1\!\!1_{Y_2(n,\varepsilon)}(\omega) \ \mathbb{E}^{P_\alpha}\left[1\!\!1_{\Theta(\varepsilon)}[R^\Lambda_{L_n,\omega}] \frac{d\rho^{\otimes L_n}}{d\beta^{\otimes L_n}} \Big/ y \right]\right\} \quad (3.13),$$

with

$$Y_2(n,\varepsilon) = \{y\in Y^{\mathbb{Z}^d} ; \ \mathbb{E}^{P_\alpha}\left[1\!\!1_{\Theta(\varepsilon)}[R^\Lambda_{L_n,\omega}] \frac{d\rho^{\otimes L_n}}{d\beta^{\otimes L_n}} \Big/ y \right] \geq \exp(-|\Lambda_n|\varepsilon) \}.$$

Since $\rho\circ\xi^{-1} = \beta\circ\pi^{-1} = (\alpha\circ\pi^{-1})^{\otimes\Lambda}$, $\mathbb{E}^{P_\alpha}\{ \frac{d\rho}{d\beta} / y \} = 1$ P_ν a.s. and the conditional expectation in (3.13) in not more than one : together with (3.12), this implies $P_\nu\{ Y_2(n,\varepsilon) \} \geq 1 - 2\exp(-|\Lambda_n|\inf\{\eta,\varepsilon\})$. From Borel-Cantelli lemma, $Y_2(\varepsilon) = \liminf_{n\to\infty} Y_2(n,\varepsilon)$ has P_ν-probability one.

From the definition of $Y_2(n,\varepsilon)$ and (3.11), we see that for $y\in Y_2(\varepsilon)$

$$\exp(-|\Delta_n|\varepsilon) \leq \mathbb{E}^{P_\alpha}\left[1\!\!1_{\Theta(\varepsilon)}[R^\Lambda_{L_n,\omega}] \exp\left\{ |L_n| \int \log \frac{d\rho}{d\beta} dR^\Lambda_{L_n,\omega} \right\} \Big/ y \right]$$

$$\leq \exp |L_n|\left\{ \mathbb{E}^{P_\rho} \log \frac{d\rho}{d\beta} + \varepsilon \right\} \quad P_\alpha\{ R^\Lambda_{L_n,\omega} \in \Theta(\varepsilon) \ / \ y \}$$

$$\leq \exp|L_n|\{ h(\rho;\beta) + \varepsilon \} \ P_\alpha\{ R_{\Delta_n,\omega} \in \Theta^1 \ / \ y \} \quad (3.14)$$

since $\psi^\Lambda(R^\Lambda_{L_n,\omega}) = R_{\Delta_n,\omega}$ and (3.11) again . But it follows from the lemma and (3.6) that $h(\rho;\beta) \leq |\Lambda|[I(R)+\varepsilon] + \varepsilon$. Moreover, f_p with $p<p_o$ depends on a finite number of coordonates ω_i , hence $\sup_{p<p_o} \left| \int f_p \, dR_{\Delta_n,\omega} - \int f_p \, dR_{\Lambda_n,\omega} \right|$ converges uniformly with respect to ω and $\{ R_{\Delta_n,\omega} \in \Theta^1\} \subset \{ R_{\Lambda_n,\omega} \in \Theta \}$ for large n. So (3.14) yields

$$P_\alpha\{ R_{\Lambda_n,\omega} \in \Theta \ / \ y \} \geq \exp - |\Delta_n| \{ I(R) + 4\varepsilon \} .$$

At last, (3.6) and $\lim_{n\to\infty} |\Delta_n|/|\Lambda_n| = 1$ imply that

$$\liminf_{n\to\infty} |\Lambda_n|^{-1} \log P_\alpha\{ R_{n,\omega} \in \mathcal{O} / y \} \geq - I(\mathcal{O}) - 5\varepsilon$$

holds for $y \in \mathcal{Y}_2(\varepsilon)$. We set $\mathcal{Y}_2(\mathcal{O}) = \bigcap_\varepsilon \mathcal{Y}_2(\varepsilon)$ with any sequence ε going to 0 , and this $\mathcal{Y}_2(\mathcal{O})$ is a desired set.

2) We keep the notations in point 2) of the proof of proposition III.2 . Let $\mathcal{Y}_2 = \bigcap_m \mathcal{Y}_2(\mathcal{O}_m)$. Any open set $\mathcal{O}$ in $\mathcal{P}_s(\Omega)$ is of the form $\mathcal{O} = \bigcup_{m\in\mathcal{F}'} \mathcal{O}_m$ with $\mathcal{F}'\subset \mathbb{N}$; but there exists $m\in\mathcal{F}'$ with $I(\mathcal{O}_m) \leq I(\mathcal{O}) + \varepsilon$. since $\mathcal{O}_m \subset \mathcal{O}$, the left hand side of (3.5) is bounded from point 1) with $-[I(\mathcal{O}) + \varepsilon]$ on $\mathcal{Y}_2$; since ε is arbitrary, (3.5) holds on $\mathcal{Y}_2$.□

□ We now prove lemma III.4 : since $h(\gamma;\beta) < \infty$, γ is absolutely continuous with respect to β; denoting by g the derivative, $\mathbb{E}^\beta(g/\xi) = 1$ β a.s. since γ and β have the same image by ξ ($\mathbb{E}^\beta(g/\xi)$ denoting here the conditional expectation of $g:E\to\mathbb{R}$ given $\xi e = .$).

First we show that we may assume that log g is bounded. Let $m>1$, $D_m = \mathbb{E}^\beta\{ g - \inf(g,m) / \xi \}$ and $g_m = (1-m^{-1})[\inf(g,m) + D_m] + m^{-1}$. Then, $\gamma_m = g_m \beta \in \mathcal{P}(E)$ satisfies to $\gamma_m\circ\xi^{-1}= \beta\circ\xi^{-1}$. According to Lebesgue theorem, g_m converges almost everywhere to g, and, since $g_m \leq g+1$, γ_m goes to γ in the topology of probability measures on E.

Using the convexity inequality $\log(g+1) \leq (\log g)^+ + \log 2$, the sequence $g_m \log g_m$ is bounded from above with $(g+1)[(\log g)^+ + \log 2]$ which is integrable since Kullback information $h(\gamma;\beta)$ is finite : using again Lebesgue theorem, we see that $\lim_{m\to\infty} h(\gamma_m;\beta) = h(\gamma;\beta)$.

As $m^{-1} \le g_m \le m$, β a.s., it is enough to prove the lemma under this extra assumption.

Applying Lusin theorem [13], we find a sequence $\tilde{r}_k \in \mathcal{C}(E)$ such that $\lim \|\tilde{r}_k - g\|_p = 0$ for $1<p<\infty$ and $m^{-1} \le \tilde{r}_k \le m$. Now, we define

$$\rho_k = r_k \beta \qquad \text{with} \qquad r_k = \tilde{r}_k \,/\, \mathbb{E}^\beta\{ \tilde{r}_k / \xi \} .$$

Then, log r_k is bounded and continuous and ρ_k has same image as β by ξ.

Moreover,

$$\|r_k - g\|_p \le \| (\tilde{r}_k - g) / \mathbb{E}^\beta\{\tilde{r}_k/\xi\} \|_p + \| (1-\mathbb{E}^\beta\{\tilde{r}_k/\xi\}) g / \mathbb{E}^\beta\{\tilde{r}_k/\xi\} \|_p$$

$$\le m \| \tilde{r}_k - g \|_p + m^2 \| 1 - \mathbb{E}^\beta\{\tilde{r}_k/\xi\} \|_p .$$

Since $\tilde{r}_k$ converges to g in $\mathbb{L}^p$, $\mathbb{E}^\beta\{\tilde{r}_k/\xi\}$ converges to $\mathbb{E}^\beta\{g/\xi\} = 1$ in $\mathbb{L}^p$ and the above computation shows that r_k goes to g in $\mathbb{L}^p$; in particular, γ_k converges weakly to γ , and the continuity of $f \longmapsto h(f\beta;\beta)$ in the $\|.\|_p$ norm implies $\lim_{k\to\infty} h(\gamma_k;\beta) = h(\gamma;\beta)$, which ends the proof.

□

IV. GIBBS MEASURES WITH SHORT RANGE RANDOM INTERACTION :

From now now on, we assume $\mathbb{W} = \mathcal{X}\times\mathcal{Y}$ with Polish spaces $\mathcal{X},\mathcal{Y}$ as in remark 1 in the last section, with π the second projection. We write $\omega_i = (x_i, y_i)$, where x_i is the spin at site $i\in\mathbb{Z}^d$ and where y, $y = (y_i)_i$ contains the randomness of the interaction; let $\alpha = \mu\otimes\nu$ with $\mu\in\mathcal{P}(\mathcal{X})$ the a priori single spin distribution and $\nu\in\mathcal{P}(\mathcal{Y})$. Notice that we can define the conditional law of ω under P_α given y for all y by $P_\alpha\{B/y\}=\int 1\!\!1_B(\omega)dP_\mu$, which we will denote by $P\{B/y\}$ for simplicity.

For any finite set A in $\mathbb{Z}^d$, let J_A be a continuous function on Ω, $\sigma(A)$-measurable ; for arbitrary fixed y, $J_A(\omega)$ represents the interaction between the spins located in A in the experiment $y = (y_i)_i$. We set $\mathbb{J} = \{J_A : A \text{ finite subset of } \mathbb{Z}^d\}$, and we assume that

$\mathbb{J}$ is translation invariant : $J_A\circ\theta^i = J_{i+A}$

$\mathbb{J}$ is summable : $\|\mathbb{J}\| = \sum_{A\ni 0} \sup_{\omega\in\Omega} |J_A(\omega)| < \infty$ (4.1).

Fix y for a moment. Let $\Lambda^d\subset\mathbb{Z}$ is finite ; a *boundary condition* (b.c.) is a configuration $\chi\in\mathcal{X}^{\Lambda^c}$ of the particle system outside Λ ; we define the *Hamiltonian* $U_\Lambda^{\chi,y}(\bar{x})$, which represents the energy of a configuration $\bar{x}\in\mathcal{X}^\Lambda$ inside Λ, given the configuration χ outside, by

$$U_\Lambda^{\chi,y}(\bar{x}) = - \sum_{A;\,A\cap\Lambda\neq\emptyset} J_A(\omega) \qquad (4.2)$$

with $\omega=(x,y)$ and $x=(\bar{x},\chi)$ the configuration equal to $\bar{x}$ on Λ and to χ outside. We can view the b.c. as governed by a *boundary condition distribution* (b.c.d.) $\Xi \in \mathcal{P}(\mathcal{X}^{\Lambda^c})$.

The *finite volume Gibbs measure* on Λ with b.c.d. Ξ is the the probability measure $G_\Lambda^{\Xi,y}$ on $\mathcal{X}$ given by

$$G_\Lambda^{\Xi,y}(d\bar{x}) = (Z_\Lambda^{\Xi,y})^{-1} \int_{\mathcal{X}^{\Lambda^c}} \exp\{-U_\Lambda^{\chi,y}(\bar{x})\}\ \Xi(d\chi) \prod_{i\in\Lambda} \mu(d\bar{x}_i) \qquad (4.3)$$

with normalizing $Z_\Lambda^{\Xi,y}$. We will write $Z_\Lambda^{\chi,y}$ and $G_\Lambda^{\chi,y}$ when $\Xi=\delta_\chi$ and $Z_n^{\cdot,y}$, $G_n^{\cdot,y}$ when $\Lambda=\Lambda_n$.

An *infinite volume Gibbs measure* in the experiment y is, by definition, a solution to Dobrushin-Lanford-Ruelle (D.L.R.) problem , that is any probability measure G^y on $\mathcal{X}^{\mathbb{Z}^d}$ such that, for all finite $\Lambda \subset \mathbb{Z}^d$,

$$G^y(d\bar{x}/\chi) = G_\Lambda^{\chi,y}(d\bar{x}) \qquad \text{for } G^y\text{-a.e. } \chi\in\mathcal{X}^{\Lambda^c} \qquad (4.4).$$

Our first result is the thermodynamic limit of the pressure:

<u>THEOREM IV.1 :</u> <u>With</u> P_ν <u>probability one,</u> $\lim_{n\to\infty} |\Lambda_n|^{-1} \log Z_n^{\Xi_n,y}$ <u>exists, does not depend on the sequence of boundary condition distribution</u> Ξ_n <u>and is equal to the deterministic number</u>

$$p = -\inf\{\mathbb{E}^Q U + I(Q) ; Q\in\mathcal{P}_s(\Omega)\}$$

<u>with</u>

$$U(\omega) = -\sum_{A\ni 0} |A|^{-1} J_A(\omega) \quad .$$

<u>Moreover, this limit is uniform with respect to</u> $\Xi_n \in \mathcal{P}(\mathcal{X}^{\Lambda_n^c})$:

$$P_\nu\{\lim_{n\to\infty} \sup_{\Xi_n} \left| |\Lambda_n|^{-1} \log Z_n^{\Xi_n,y} - p\right| = 0\} = 1 \quad .$$

<u>REMARK 1 :</u> .1) It is well known in the litterature in physics that the limit exists w.p.1 and is constant [28]. The Gibbs variational formula for the *pressure* p was established by LEDRAPPIER [19], with ergodic theory techniques, in the particular case of nearest neighbour Ising model with free b.c., but for more general conditioning including non typical y's.

.2) Because of the uniformity in the limit with respect to b.c.d. one is allowed to consider <u>b.c.d. depending on the interaction</u> . We put emphasis on this, since some infinite volume Gibbs measures may not be limit of finite volume ones with b.c.d. independent on the interaction. Before proving the theorem, we give

<u>EXAMPLES OF PAIR INTERACTION :</u>

Let $\mathcal{Y} = \mathbb{B}^{\mathbb{Z}^d}$ with $\mathbb{B}$ and $\mathcal{X}$ bounded subsets of $\mathbb{R}$; a generic element y_i of $\mathcal{Y}$ will be written $(y_i^k)_{k\in\mathbb{Z}^d}$. Let J be an even real function on $\mathbb{Z}^d$ such that $\sum_{k\in\mathbb{Z}^d} |J(k)| < \infty$.

.1) Define $\mathbb{J}$ by

$$J_A(\omega) = \begin{cases} - y_i^o \, x_i & \text{if } A=\{i\} \\ - J(i-j) \, y_i^{j-i} \, x_i \, x_j & \text{if } A=\{i,j\} \text{ with } i<j \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

with lexicographic order $<$ on $\mathbb{Z}^d$. The Hamiltonian (4.2) is of the form

$$\sum_{i<j} J(i-j) \, z_{i,j} \, x_i \, x_j \;+\; \sum_i t_i \, x_i \tag{4.6}$$

with $z_{i,j}=y_i^{j-i}$ for $i<j$ and external field $t_i=y_i^o$. Choosing ν as the appropriate product measure, we cover following situations :

i) the $z_{i,j}$'s are i.i.d. with arbitrary distribution and t_i is equal to some constant t : this a usual framework in *spin glass* models (see [0],[1],[11],[12])

ii) the $z_{i,j}$'s are equal to 1 and the t_i's are i.i.d. : this is the *random external field* model [15].

.2) We may also consider Hamiltonian of the form (4.6) with dependent $z_{i,j}$'s. Let's illustrate this in modifying $\mathbb{J}$ in (4.5) by

$$J_A(\omega) = - J(i-j)\ \varphi(y_i^1, y_j^1)\ x_i\ x_j \qquad \text{if } A=\{i,j\} \text{ with } i<j$$

for a symmetric continuous function on $\mathbb{R}^2$. When y_i^1 has values in the finite set $\{1,\dots,l_o\}$, this describes a crystal of a mixture of l_o different kind of particles or isotopes which are randomly distributed in the crystal; $\varphi(l,l')$ represents the energy interaction between particles of type l and l', and is modulated by the intensity $J(i-j)$ which takes into account the distance between the particles (refer to [20],examples in ch.XIV). For $l_o=2$ and $\varphi(l,l')=(l-1)(l'-1)$, this is the *site disorder* model [2] for a non magnetic crystal (particles of type 1) with randomly located magnetic impureties (type 2).

.3) With slight modifications in (4.5), we also cover *XY spin glass* model [11], which correspond to the formal Hamiltonian

$$\sum_{i,j} J(i-j)\ z_{i,j} \cos(x_i - x_j)$$

with i.i.d. $z_{i,j}$'s , $x_i \in \mathcal{X}=[0,2\pi]$ and $\mu(dx_1) = dx_1$.

REMARK 2 : Our method shows its limit in example 1-i); indeed there still exists a limit p when $J(k) = |k|^{-ad}$ with $a>1/2$ and $\mathbb{E} z_{i,j}=0$ [17], which is non summable, but only square summable.

REMARK 3 : With little extra work, we can also consider in the above examples unbounded spins x with some control on the tail of distribution μ; for instance, the results in this section remain valid for Gaussian spins under boundedness assumption on the function J .

□ Proof : Let χ_n an arbitrary b.c. outside Λ_n and $y \in \mathcal{Y}^{\mathbb{Z}^d}$, $\bar{x} \in \mathcal{X}^{\Lambda_n}$, $x=(\chi_n, \bar{x})$, $\omega=(x,y)$. Then,

$$U_{\Lambda_n}^{\chi_n, y}(\bar{x}) = - \{ \sum_{A \subset \Lambda_n} J_A(\omega) + u_1 \} \tag{4.7}$$

with $u_1 = \sum J_A(\omega)$ where the summation is over the sets A which intersect Λ_n but are not contained in it. In other respects, translation invariance of $\mathbb{J}$ implies

$$\int_{\Omega} U\, dR_{n,\omega} = - |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \sum_{A\ni 0} |A|^{-1} J_{i+A}(\omega^{(n)})$$

$$= - |\Lambda_n|^{-1} \{ \sum_{A\subset\Lambda_n} J_A(\omega) + u_2 \} \qquad (4.8)$$

with $u_2 = \sum_{i\in\Lambda_n} \sum J_{i+A}(\omega^{(n)})$, the last summation ranging over A containing O such that i+A is not contained in Λ_n . From (4.1), there exists for any positive ε an integer $m(\varepsilon)$ such that $\sum \sup|J_A(\omega)| \leq \varepsilon$ with summation over all A containing O, with diameter more than $m(\varepsilon)$. Hence we have uniformly in χ_n

$$|\Lambda_n|^{-1}|u_k| \leq \varepsilon + m(\varepsilon)\ \|J\|\ \mathcal{O}(1/\min\{i_n^l + j_n^l ; l\leq d\}) \qquad (4.9)$$

for k=1,2. Then, for n larger than some n_o independent on χ_n :

$$\exp(-4|\Lambda_n|\varepsilon)\ Z_n^{\chi_n,y} \leq \mathbb{E}\{\exp(-|\Lambda_n|\int_{\Omega} U\, dR_{n,\omega})/y\} \leq \exp(4|\Lambda_n|\varepsilon)\ Z_n^{\chi_n,y} \quad (4.10)$$

where conditional expectation means integration with respect to $P\{./y\}$.

According to theorem III.1, there exists a Borel set $\mathcal{Y}$ in $\mathcal{Y}^{\mathbb{Z}^d}$ of P_ν-probability one such that a large deviation principle holds on $\mathcal{Y}$ for the law of $R_{n,\omega}$ under $P\{./y\}$. Notice that U is bounded and continuous on Ω, and so is $Q \longmapsto \mathbb{E}^Q U$; then, for all $y\in\mathcal{Y}$, theorem II.2 in [27] applies and show that the quantity

$$a_n(y) = |\Lambda_n|^{-1}\log \mathbb{E}\{ \exp(-|\Lambda_n|\int_{\Omega} U\, dR_{n,\omega}) /y\} - p$$

(independent on χ_n) converges to O on $\mathcal{Y}$. From (4.10), we derive that

$$\sup_{\chi_n} \left| |\Lambda_n|^{-1}\log Z_n^{\chi_n,y} - p\right| \leq 4\varepsilon + |a_n(y)| \qquad (4.11)$$

holds for $n\geq n_o$, which implies

$$\limsup_{n\to\infty} \sup_{\Xi_n} \left| |\Lambda_n|^{-1}\log Z_n^{\Xi_n,y} - p\right| \leq 4\varepsilon$$

for $y\in\mathcal{Y}$; this ends the proof.□

We now give large deviation estimates for the different empirical fields of Gibbs distributions. Define

$$\mathcal{G} = \{ Q \in \mathcal{P}_s(\Omega) \ ; \ \tilde{I}(Q) = 0 \} \tag{4.12}$$

where

$$\tilde{I}(Q) = \mathbb{E}^Q U + I(Q) + p \tag{4.13}.$$

Let $\Pi' : \Omega = \mathcal{X}^{\mathbb{Z}^d} \times \mathcal{Y}^{\mathbb{Z}^d} \longrightarrow \mathcal{X}^{\mathbb{Z}^d}$ be the first projection $\Pi'\omega = (x_i)_i = x$ and $(\Pi')^* : \mathcal{P}_s(\Omega) \longrightarrow \mathcal{P}_s(\mathcal{X}^{\mathbb{Z}^d})$ the first margin map . Then, $(\Pi')^* R_{n,\omega} = R_{n,x}$ the empirical field for the spin variables only.

<u>THEOREM IV.2 :</u> <u>There exists a Borel</u> $\mathbb{Y}$ <u>set in</u> $\mathcal{Y}^{\mathbb{Z}^d}$ <u>with</u> P_ν<u>-probability one , such that</u>

<u>i) the sequence of the laws of the empirical field</u> $R_{n,\omega}$ [resp. $R_{n,x}$] <u>under</u> $G_n^{\Xi_n, y}$ <u>obeys a large deviation principle on</u> $\mathcal{P}_s(\Omega)$ [resp $\mathcal{P}_s(\mathcal{X}^{\mathbb{Z}^d})$] <u>with rate function</u> $\tilde{I}$ [resp. $\hat{I}(.) = \min\{\tilde{I}(Q) ; (\Pi')^* Q = .\}$] <u>and sequence</u> $|\Lambda_n|$ <u>, for any sequence of boundary condition distribution</u> $\Xi_n \in \mathcal{P}(\mathcal{X}^{\Lambda_n^c})$.

<u>ii) this sequence is tight, and any limit point is concentrated on</u> $\mathcal{G}$ [resp $(\Pi')^*\mathcal{G}$].

<u>iii) the previous points hold for any infinite volume Gibbs measure</u> G^y.

Clearly, $\tilde{I}$ given in (4.13) is lower semi-continuous and linear with compact level sets, $\mathcal{G}$ and $(\Pi')^*\mathcal{G}$ are non-empty convex compact sets, independent on y; but the limit points in ii) may depend on y as well as on Ξ_n.

We emphasize on point iii) : <u>all</u> the solution to D.L.R. equations have the same large deviation properties, and these do not depend on the particular experiment. As in the non random case, the rate function depends only on the structure of the local characteristics, and the Gibbs measures are not discriminated in the order of exponential magnitude of the volume. On the other hand, the rate is non random because the empirical field is only sensitive to the ergodic behaviour of the interaction.

□ Proof: Fix a b.c. χ_n. By definition (4.3), we have

$$G_n^{\chi_n,y}\{R_{n,\omega}\in B\} = \left(Z_n^{\chi_n,y}\right)^{-1} \mathbb{E}\{\exp[-U_{\Lambda_n}^{\chi_n,y}(\bar{x})]\ \mathbb{1}_B[R_{n,\omega}]\ /y\} \qquad (4.14)$$

for any Borel set B in $\mathcal{P}_s(\Omega)$. Combining the estimates (4.7,8,9,11), we obtain for any χ_n and $n\geq n_o$

$$\begin{aligned}\exp\{-|\Lambda_n|(p+|a_n(y)|+8\varepsilon)\}\ &\mathbb{E}\{\exp(-|\Lambda_n|\int_\Omega U\, dR_{n,\omega})\ \mathbb{1}_B[R_{n,\omega}]\ /y\}\\ &\leq G_n^{\chi_n,y}\{R_{n,\omega}\in B\} \qquad (4.15)\\ &\leq \exp\{-|\Lambda_n|(p-|a_n(y)|-8\varepsilon)\}\ \mathbb{E}\{\exp(-|\Lambda_n|\int_\Omega U\, dR_{n,\omega})\ \mathbb{1}_B[R_{n,\omega}]\ /y\}.\end{aligned}$$

Since the bounds do not dependent on χ_n , as well as n_o , we can integrate (4.15) with respect to any b.c.d. Ξ_n ;hence (4.15) holds for $G_n^{\Xi_n,y}$. On the other hand, the techniques of [26], §3, show that for all $y\in\mathbb{Y}$

$$\begin{aligned}-\inf_{Q\in\mathring{B}} \{\mathbb{E}^Q U+I(Q)\} &\leq \liminf_{n\to\infty} |\Lambda_n|^{-1}\log \mathbb{E}\{\exp(-|\Lambda_n|\int_\Omega U\, dR_{n,\omega})\ \mathbb{1}_B[R_{n,\omega}]\ /y\}\\ &\leq \limsup_{n\to\infty} |\Lambda_n|^{-1}\log \mathbb{E}\{\exp(-|\Lambda_n|\int_\Omega U\, dR_{n,\omega})\ \mathbb{1}_B[R_{n,\omega}]\ /y\}\\ &\leq -\inf_{Q\in\overline{B}} \{\mathbb{E}^Q U+I(Q)\} \qquad (4.16).\end{aligned}$$

Together with (4.15), this yields the inequalities (2.1) for the law of $R_{n,\omega}$ under $G_n^{\Xi_n,y}$. But, if G^y satisfies to (4.4), we have

$$G^{y}_{\Lambda_n} = G_n^{\Xi_n, y} \quad , \quad \text{with } \Xi_n = G^{y}_{\Lambda_n^c} \text{ the restriction of } G^{y} \text{ to } \sigma(\Lambda_n^c);$$

hence the results for the infinite volume Gibbs measures are contained in those for finite volume Gibbs measures with arbitrary b.c.d. Ξ_n. Since $R_{n,x} = (\Pi')^* R_{n,\omega}$ with continuous $(\Pi')^*$, the corresponding results for $R_{n,x}$ are a staightforward consequence of the contraction principle. We have proved i).

Since $\mathcal{G}$ is compact and $\tilde{I}$ lower semi-continuous, $\tilde{I}(\mathcal{N})>0$ for all neighbourhood $\mathcal{N}$ of $\mathcal{G}$ in $\mathcal{P}_s(\Omega)$. According to the point i),we have

$$\lim_{n\to\infty} G_n^{\Xi_n, y}\{R_{n,\omega}\in\mathcal{N}\} = 0 \tag{4.17}$$

for all $y\in\mathcal{Y}$ and all $\mathcal{N}$. But (4.17) and compactness of $\mathcal{G}$ implies that the sequence $G_n^{\Xi_n, y}\{R_{n,\omega}\in .\}$ is tight ([22],p.49) for such y's, and that the limit points are concentrated on $\mathcal{G}$. Then ii) follows easily. As above, the same hold for G^{y}. □

The previous results concern space averages; in the next one, we localize our results to study the Gibbs distribution itself, and we express it in terms of solutions to the variational problem. We consider b.c.d. Ξ , depending on y as a <u>measurable</u> function, that is

$$\Xi \ : \ \mathcal{Y}^{\mathbb{Z}^d} \longrightarrow \mathcal{P}(\mathcal{X}^{\Lambda^c}) \quad \text{is a transition probability kernel .}$$

We extend $G_{\Lambda}^{\Xi, y}$ given in (4.3) to any probability measure on $\mathcal{X}^{\mathbb{Z}^d}$, say

$$\mathbb{G}_{\Lambda}^{\Xi, y} = G_{\Lambda}^{\Xi, y} \otimes P_{\mu, \Lambda^c} \quad .$$

<u>THEOREM IV.3 :</u> <u>Assume that , with P_ν probability one , $\mathbb{G}_n^{\Xi_n,y}$ converges weakly to some $\mathbb{G}^y \in \mathcal{P}(\mathcal{X}^{\mathbb{Z}^d})$. If $G = \mathbb{G}^y P_\nu$ is stationary , then</u>

$$G \in \mathcal{G} .$$

<u>In other words, there exists a $G \in \mathcal{G}$ such that</u>

$$\mathbb{G}^y(dx) = G(dx/y) \qquad \underline{\text{w.p.1 .}}$$

<u>COMMENTS :</u> .1) The stationarity assumption on G is needed, since the result does not hold for arbitrary b.c.d. in the (usual) case of deterministic interaction : a finite volume Gibbs measure may converge to a non-stationary solution to D.L.R. equation; this is *symmetry breaking* ([24],p.77). On the other hand, a stationary G is characterized by the sequence of its empirical fields, which asymptotics are known from theorem IV.2 .

.2) We give two examples where the assumptions are fulfilled:

a) random ferromagnetic Ising model with free boundary condition: $\mu = \frac{1}{2}(\delta_1+\delta_{-1})$, $\mathbb{J}$ given in (4.5) with $\mathbb{B} \subset \mathbb{R}^+$, $\Xi_n = \otimes^{\Lambda_n^c} \delta_o$ (free b.c. meaning that the particles in Λ_n are isolated from the outside). Then G.K.S.-2 inequality ([10],p.142) implies that $\mathbb{G}_n^{\Xi_n,y}$ converges, monotonically in some sense, for all y; denoting by $\mathbb{G}^y$ the limit, $\mathbb{G}^y P_\nu$ is stationary because the b.c. are free.

b) let μ as above (Ising spins), d=1 (one-dimensional) , $\mathbb{J}$ be given by (4.5) with $\mathbb{E}^\nu y_i = 0$, and Ξ_n be independent on y . it is shown in [0] that the limit G^y exists w.p.1, is independent on such a sequence Ξ_n. Furthermore, $G^y P_\nu$ is stationary, since free b.c. are among those Ξ_n .

□ Proof: let f(x) and g(y) be bounded real continuous functions on Ω, $\sigma(\Lambda)$-measurable for some finite $\Lambda \subset \mathbb{Z}^d$. The convergence assumption implies

$$\lim_{n\to\infty} \mathbb{E}^{P_\nu}\left(g\, \mathbb{E}^{\mathbb{G}_n^{\Xi_n, y}} f \right) = \mathbb{E}^{P_\nu}\left(g\, \mathbb{E}^{\mathbb{G}^y} f \right) = \mathbb{E}^{G}(\, fg\,)$$
$$= \mathbb{E}^{G}\left(\int_\Omega fg\, dR_{m,\omega} \right) + \varepsilon(m)$$

with $\lim_{m\to\infty} \varepsilon(m) = 0$, because G is stationary and f,g depend on finitely many coordonates. But this last expectation itself is equal to the limit as $n \to \infty$ of

$$\mathbb{E}^{P_\nu}\, \mathbb{E}^{\mathbb{G}_n^{\Xi_n, y}} \int_\Omega fg\, dR_{m,\omega} = \mathbb{E}^{P_\nu}\, \mathbb{E}^{\mathbb{G}_m^{\Xi_m^n, y}} \int_\Omega fg\, dR_{m,\omega}$$

when $n \geq m$ with Ξ_m^n some measurable b.c.d. outside Λ_m; hence we have

$$\mathbb{E}^{G}(\, fg\,) = \mathbb{E}^{P_\nu}\, \mathbb{E}^{\mathbb{G}_m^{\Xi_m^n, y}} \int_\Omega fg\, dR_{m,\omega} + \varepsilon(m) + \varepsilon_m(n) \quad .$$

We choose $n = n(m)$ such that $|\varepsilon_m(n)| \leq 1/m$. Combining (4.17) and Lebesgue theorem, we derive

$$\lim_{m\to\infty} \mathbb{E}^{P_\nu}\, \mathbb{G}_m^{\Xi_m^n, y}\{\, R_{n,\omega} \notin \mathcal{N}\, \} = 0$$

for any neighbourhood $\mathcal{N}$ of $\mathcal{G}$ in $\mathcal{P}_s(\Omega)$: then, the laws of $R_{m,\omega}$ under $P_\nu\, \mathbb{G}_m^{\Xi_m^n, y}$ are a tight sequence with limit points concentrated on $\mathcal{G}$.
Since f and g are arbitrary, $G \in \mathcal{G}$. □

V. MEAN FIELD MODELS WITH RANDOM INTERACTION :

In the case of non random interaction, each pair of particles in a mean field model interacts with constant intensity, so that the Hamiltonian depends only on the empirical measure of the spins. Here, we consider Hamiltonian depending on the empirical measure

$$r_{n,\omega} = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, y_i)} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \tag{5.1}$$

with x_i the spin of the i^{th} particle and y the randomness of the interaction as in the above section; we take $d = 1$ since no geometry is involved in such models. We could also treat a (more general) local mean field as in [8] - where the intensity of the interaction depends also on the distance between particles in a suitable way -, via the adequate (known) techniques.

We do not cover the SHERRINGTON-KIRCKPATRICK model [25], which has a weaker normalization in n and independent couplings. Nevertheless there are some similarities in the two models, such as frustration, strongly oscillating couplings at long distance, and the dependence between couplings decreases in the asymptotics: refer to the discussion of Van HEMMEN, Van ENTER and CANISIUS [16], §2. Examples will be given later on; we first define the Gibbsian set-up.

Denote by l^2 the Hilbert space of square summable real sequences, with scalar product $t.t'$ for t, t' in l^2.

Let $\quad M : \mathcal{X}\times\mathcal{Y} \longrightarrow l^2$ bounded continuous (5.2)

and $\quad v : l^2 \longrightarrow \mathbb{R}$ twice continuously differentiable with bounded derivatives on bounded sets in l^2. (5.3).

Define the Hamiltonian $V_n^y(x) = n\, V(r_{n,\omega})$ with

$$V(r) = v\Big(\int M\, dr \Big) \qquad \forall r \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \qquad (5.4).$$

We are interested in the asymptotics of the Gibbs measure $\mathbb{Q}_n^y \in \mathcal{P}(\mathcal{X}^{\mathbb{N}^*})$

$$\mathbb{Q}_n^y(dx) = Z_n^y \exp\{ - V_n^y(x) \} \overset{\mathbb{N}^*}{\otimes} \mu \quad (dx) \qquad (5.5)$$

with normalizing Z_n^y. The *order parameter* $\int M\, dr$, and its empirical version $m_n(\omega) = \int M\, dr_{n,\omega}$ are a priori infinite dimensional; it is a quantity of interest for it characterizes the equilibria of the system.

For $t \in l^2$, we define $L(t) = \int L^{y_1}(t)\, \nu(dy_1)$ with $L^{y_1}(t) = \log \int \exp\{ t.M(x_1,y_1) \}\, \mu(dx_1)$, and its Legendre transform L^*

$$L^*(m) = \sup\{ t.m - L(t) \; ; \; t \in l^2\} \qquad \text{for } m \in l^2 \qquad (5.6).$$

We assume that μ is not a Dirac mass, which is unrestrictive.

<u>THEOREM V :</u> <u>For</u> P_ν <u>almost every y, we have :</u>

<u>i)</u> $\lim_{n\to\infty} n^{-1} \log Z_n^y = - \inf \{ v(m) + L^*(m) \; ; \; m \in l^2\}$

<u>which we will denote by</u> p <u>.</u>

<u>ii) The law of the empirical order parameter</u> $m_n(\omega)$ <u>under</u> $\mathbb{Q}_n^y$ <u>obeys a large deviation principle with rate function</u> $v + L^* + p$ <u>and sequence n .</u>

<u>iii) The sequence</u> $\mathbb{Q}_n^y$ <u>is tight on</u> $\mathcal{P}(\mathcal{X}^{\mathbb{N}})$ <u>, and any limit point is a mixture</u> $\int_{\mathcal{S}} \overset{\infty}{\otimes} r_m^{y_i}\, \tau(dm)$ <u>for some</u> $\tau \in \mathcal{P}(\mathcal{S})$ <u>depending on y ,</u>

$$\mathcal{S} = \{ m \in l^2 ;\ v(m) + L^*(m) + p = 0 \} \quad , \quad \underline{\text{and}}$$

$$r_m^{y_i}(dx_i) = \exp\{ t.M(x_i,y_i) - L^{y_i}(t) \} \mu(dx_i)$$

<u>where $t = t(m)$ satisfies to $L'(t) = m$.</u>

<u>REMARKS :</u> 1) The local asymptotics in iii) is analogous to theorem IV.3: here, φ consists in the mixtures $\int_{\mathcal{S}} P_{r_m^{\cdot}\otimes\nu}\ \tau(dm)$ with $\tau \in \mathcal{P}(\mathcal{S})$ not depending on y; a version of the conditional probability given y is $\int \otimes_i r_m^{y_i}\ \tau(dm)$.

2) If $|\mathcal{S}| = 1$, the Gibbs measure converges w.p.1, the spins are independent in the limit, and the margin of x_i is the "maximum (conditional) entropy" distribution $r_m^{\cdot}\otimes\nu$ given y_i .

<u>EXAMPLES :</u>

Assume $\mathcal{X}$ be bounded in $\mathbb{R}$, μ symmetric , and $\mathbb{L}^2(\nu)$ separable with complete orthonormal system $(\psi_k)_{k\in\mathcal{K}}$ ($\mathcal{K}$ at most countable) such that $\psi_o = 1$ and ψ_k is continuous and $\sup_{k\in\mathcal{K}} \|\psi_k\|_\infty < \infty$. Let $(a_k)_{k\in\mathcal{K}}$ be an element of $l^2 = l^2(\mathcal{K})$ with $a_k \neq 0$ for all k. We consider

$$M(\omega) = [x_1 a_k \psi_k(y_1)]_{k\in\mathcal{K}} \qquad (5.7).$$

Then we can compute explicitely $L^*(m)$

$$L^*(m) = \begin{cases} \int \lambda^*[\sum_{k\in\mathcal{K}} (m_k/a_k)\ \psi_k]\ d\nu & \text{if } (m_k/a_k)_k \in l^2, \\ +\infty & \text{otherwise} \end{cases} \qquad (5.8),$$

where λ^* is the Cramèr transform of μ given by $\lambda^*(u) = \sup_{s\in\mathbb{R}}\{su-\lambda(s)\}$ and $\lambda(s) = \log \int e^{sx_1}\ \mu(dx_1)$. We will prove this later.

We consider a quadratic v , with diagonal form

$$v(m) = \sum_{k\in\mathcal{K}} v_k\ m_k^2 \qquad \text{with } \sup_{k\in\mathcal{K}} |v_k| < \infty \qquad (5.9).$$

Hence the Hamiltonian is of the form $V_n^y(x) = \frac{1}{n} \sum_{i,j} J_{i,j} x_i x_j$ with

$$J_{i,j} = \sum_{k\in\mathcal{K}} v_k a_k^2 \psi_k(y_i) \psi_k(y_j) \qquad (5.10).$$

Then the couplings $J_{i,j}$ are non correlated, but J_{i,j_1} and J_{i,j_2} are dependent; we cannot obtain independent coupling from (5.7). Using a non diagonal quadratic form v, we can obtain correlated J_{i,j_1} and J_{i,j_2}.

From the definition of $\mathcal{S}$, any $m\in\mathcal{S}$ satisfies to the mean field equation

$$m_l = a_l \int \lambda'(-2 \sum_{k\in\mathcal{K}} v_k a_k m_k \psi_k) \psi_l \, d\nu \qquad \forall l\in\mathcal{K} \qquad (5.11)$$

which is obtained in differentiating $v+L^*$ in the form (5.8,9), using $(\lambda^*)' = (\lambda')^{-1}$ and the oddness of λ'. But (5.11) implies that

$$t = (-2v_k m_k)_k \quad \text{satisfies to} \quad L'(t) = m \qquad (5.12).$$

Notice that -m lies in $\mathcal{S}$ too, with conjugate variable $t = (2v_k m_k)_k$. We are now in position to study some simple equilibrium situations :

a) <u>The ferromagnetic phase :</u>

Assume $\mathcal{S} = \{ m^+, -m^+\}$ with $(m^+)_k = 0$ for $k \neq 0$ and positive $(m^+)_o$ - which will still denoted by m^+ -.

Then, from (5.12), $r^{y_1}_{\pm m^+}(dx_1) = e^{\mp 2v_o a_o m^+ x_1 - \lambda(2v_o a_o m^+)} \mu(dx_1)$ do not depend on y (we then drop superscript y_1 in this notation) , and has mean $\pm m$. Furthermore, the symmetry of x_1 under $\mathbb{Q}_n^y$ shows that $\mathbb{E}^{\mathbb{Q}_n^y} x_1 = 0$; hence, any limit point in iii), theorem V , is such that $m^+\tau(m^+) - m^+\tau(-m^+) = 0$, and then

$$\text{w.p.1,} \qquad \mathbb{Q}_n^y \underset{n\to\infty}{\Longrightarrow} \frac{1}{2} \left(\underset{i}{\otimes} r_{m^+} + \underset{i}{\otimes} r_{-m^+} \right) \qquad (5.13).$$

The thermodynamic limit completely forgets the randomness of the interaction, and is the same as that of the usual mean field model where we merely suppress the random part in $J_{i,j}$.

Notice that the paramagnetic phase $s = \{0\}$ leads to $\mathbb{Q}_n^y \Longrightarrow \overset{\mathbb{N}^*}{\otimes} \mu$ and the same conclusions.

b) A spin glass phase :

Assume $s = \{ m_\ell , -m_\ell \}$ with positive $(m_\ell)_\ell$ and $(m_\ell)_k = 0$ if $k \neq \ell$.

We have $r^{y_1}_{\pm m_\ell}(dx_1) = e^{\mp 2v_\ell a_\ell m_\ell \psi_\ell(y_1) x_1 - \lambda[2v_\ell a_\ell m_\ell \psi_\ell(y_1)]} \mu(dx_1)$, which has mean $\pm\lambda'[2v_\ell a_\ell m_\ell \psi_\ell(y_1)]$; again, $\psi_\ell(y_1)\, \mathbb{E}^{\mathbb{Q}_n^y} x_1 = 0$, and then $\tau(m_\ell) = \tau(-m_\ell) = \frac{1}{2}$ if $\psi_\ell(y_1) \neq 0$ for τ as in theorem V-iii). But $r^{y_1}_{m_\ell} = r^{y_1}_{-m_\ell}$ if $\psi_\ell(y_1)=0$; so the theorem states here

$$\text{w.p.1,} \qquad \mathbb{Q}_n^y \underset{n\to\infty}{\Longrightarrow} \frac{1}{2}\left(\otimes_i r^{y_i}_{m_\ell} + \otimes_i r^{y_i}_{-m_\ell} \right) \qquad (5.14).$$

This means that, in allmost every experiment, the Gibbs measure converges to the average of two inhomogeneous product which are symmetric from each other; in each one of the two components, the law of x_i depends on the randomness y_i of the interaction at site i, and the mean magnetization per site $\frac{1}{n}\sum_{i\leq n} \mathbb{E}x_i = \pm \frac{1}{n}\sum_{i\leq n} \lambda'[2v_\ell a_\ell m_\ell \psi_\ell(y_1)]$ is non zero in general, but goes to zero in the limit with celerity $\sqrt{n}$ if $\mathbb{E}^\nu \lambda'[c\psi_\ell(y_1)]=0$, $\forall c \in \mathbb{R}$.

Example 1: Let ν be Lebesgue measure on the D-dimensional torus $[0,1]^D$, $\mathcal{K} = \mathbb{Z}^D$, $\psi_k = \sqrt{2}\cos 2\pi k.y_1$ if k>0 in the lexicographic order , $\psi_k = \sqrt{2}\sin 2\pi k.y_1$ if k<0 and $\psi_o=1$. Assume $v_k a_k^2 = v_{-k} a_{-k}^2$ for all k. Then,

$$v(m) + L^*(m) = - \int J*u\, u\, d\nu + \int \lambda^*(u)\, d\nu \qquad (5.15),$$

with $u = \sum_k (m_k/a_k)\ \psi_k \in \mathbb{L}^2(\nu)$, $J = -(\ v_o a_o^2 + \sum_{k>0} 2v_k a_k^2 \cos 2\pi k.y_1\ )$ and $*$ the convolution. This is the rate function of a local mean field, where a large number of particles located in $[0,1[^D$ interact with couplin function J [8]; our model consists in picking n of these particles (those located at $y_1,\dots y_n$) independently with uniform distribution on $[0,1]^D$.

The ferromagnetic phase a) occurs for instance if J>0 and $-2v_o a_o^2 > 1/\lambda''(0)$, and b) if $v_k = 0$, $k \neq \pm \ell$, and $-2v_\ell a_\ell^2 > 1/\lambda''(0)$. The set Δ may be studied in general with bifurcation techniques [4].

Example 2 : a classical spin glass model [16].

Let $\mathcal{X} = \{+1,-1\}$ [resp. $\mathcal{Y} = \{+1,-1\}^2$], and μ [resp. ν] the uniform distribution; we put $y_i = (\psi_1,\psi_2)(y_i)$. Then, $\psi_o = 1, \psi_1, \psi_2$, $\psi_3 = \psi_1\psi_2$ is an orthonormal basis of $\mathbb{L}^2(\nu) \simeq \mathbb{R}^4$. In [16], the $J_{i,j}$'s are given by $J_{i,j} = v_o + v_1\ \psi_1(y_i)\ \psi_2(y_j)$ instead of (5.10). This correspond to $v(m) = v_o m_o^2 + v_1 m_1 m_2$. One can rotate the axis in the plane ψ_1,ψ_2 in order to obtain a diagonal form for v; since (ψ_k) is transformed into an orthonormal basis, we cover this situation. In the reference, point i) is proved, and the existence of different phases is studied with v_o,v_1 as parameters; notice that we generalize in (5.8) the trick used in [16] to compute L^*.

Example 3 : HOPFIELD model for neural networks [23] .

This describes the equilibrium distribution of a large number n (in the order of magnitude 10^{10}) of neurons of a certain type. The firing activity $x_i = \pm 1$ characterizes the neuron located at i; they are connected in a complex way, with intensity depending on learned patterns

$\varphi_k \in \{+1,-1\}^n$, $k=1,2,\ldots K$. In this classical model, the neural networks is governed by an Hamiltonian $V_n(x) = -n \sum_{i,j} J_{i,j}\, x_i x_j$ with

$$J_{i,j} = \sum_{k\leq K} (\varphi_k)_i \; (\varphi_k)_j$$

according to HEBB's rule. Let $y_i = [(\varphi_k)_i]_{k\leq K} \in \{+1,-1\}^K$. When the y_i's are independent, the model is of the form (5.7,9 and 10).

☐ We prove (5.8). Denote by $C>0$ the supremum of the support of distribution μ. Let m with $L^*(m)<\infty$; we first show $(m_k/a_k)_k \in l^2$. Since $\lim_{s\to+\infty} \lambda(s)/s = C < \infty$ and since λ is convex symmetric with $\lambda(0)=0$, there exists $C'<\infty$ such that $\lambda(s) \leq C's^2$;then, for all t in l^2,

$$L^*(m) \geq t.m - \int \lambda\Big(\sum_k a_k t_k \psi_k \Big)\, d\nu \geq t.m - C'\int \Big(\sum_k a_k t_k \psi_k\Big)^2 d\nu$$

$$= t.m - C' \sum_k (a_k t_k)^2$$

Then, for any finite sequence s_k, $k\leq K$, $\sum_k s_k\, (m_k/a_k) \leq L^*(m) + C' \sum_k s_k^2$; a classical argument shows this implies $(m_k/a_k)_k \in l^2$. We now prove that

$$\Big| \sum_k (m_k/a_k)\, \psi_k \Big| \leq C \;,\; \nu\text{-p.s.} \;.$$

Let $b\in\mathbb{R}$ and $t\in l^2$ such that $\sum_k t_k a_k \psi_k \geq 0$ ν-p.s. . Since λ is convex symmetric with $\lambda(0)=0$ and $\lim_{s\to+\infty} \lambda(s)/s = C$, we have $\lambda(s) \leq C|s|$; then

$$L^*(m) \geq b \sum_k t_k m_k - \int \lambda\Big(b \sum_k t_k a_k \psi_k \Big)\, d\nu$$

$$\geq |b| \Big[\mathrm{sign}(b) \sum_k t_k m_k - C < \sum_k t_k a_k \psi_k \;,\; \psi_o >\Big] \;,$$

where $<.,.>$ denote the scalar product in $\mathbb{L}^2(\nu)$. Letting $b\to\infty$, we obtain

$$\text{for } \eta=1,-1 \qquad < \sum_k t_k a_k \psi_k \;,\; \eta \sum_k (m_k/a_k)\psi_k - C\,\psi_o > \leq 0 \qquad (5.16).$$

Now let t' in l^2 with $\sum_k t'_k\psi_k \geq 0$; approximating t' in l^2 by a finite sum $\sum_k t_k a_k \psi_k \geq 0$, we see that (5.16) holds for $\sum_k t'_k\psi_k$, and then $\eta \sum_k (m_k/a_k)\psi_k - C\,\psi_o \leq 0$ ν-p.s., which is the desired result.

If $\| \sum_k (m_k/a_k)\psi_k \|_\infty \geq C$,the integrant in (5.8) is infinite on a set of positive ν-measure, and the integral is infinite. So it is enough to prove (5.8) when the converse inequality is satisfied.We have

$$L^*(m) = \sup_t \int \{ (\sum_k t_k a_k \psi_k)\times(\sum_k (m_k/a_k)\psi_k) - \lambda(\sum_k t_k a_k\psi_k) \} \, d\nu .$$

The quantity between {.} is maximized with $\sum_k t_k a_k\psi_k = (\lambda')^{-1}[\sum_k (m_k/a_k)\psi_k] = f(y_1)$, therefore $L^*(m)$ is not more than the integral in (5.8). We begin to prove the equality when $\| \sum_k (m_k/a_k)\psi_k \|_\infty < C$: then, $f \in \mathbb{L}^\infty(\nu) \subset \mathbb{L}^2(\nu)$. We may approximate f with a finite sum $\bar{f} = \sum_k t_k a_k\psi_k$ in $\mathbb{L}^2(\nu)$, and because λ is lipschitz continuous with constant C, $\int \{ \bar{f}\times(\sum_k (m_k/a_k)\psi_k) - \lambda(\bar{f}) \} \, d\nu$ is close to $\int \{ f\times(\sum_k (m_k/a_k)\psi_k) - \lambda(f) \} \, d\nu$, which is the second term in (5.8). If the strict inequality is not satisfied, we truncate f in $f_b = (-b)\vee f\wedge b$ for $b>0$. Since f_b lies between 0 and f, $f_b\times\{\Sigma (m_k/a_k)\psi_k\} - \lambda(f_b)$ is non negative; from Fatou lemma, we derive

$$\liminf_{b\to\infty} \int\{f_b\times(\sum_k (m_k/a_k)\psi_k) - \lambda(f_b)\}\, d\nu \geq \int\{f\times(\sum_k (m_k/a_k)\psi_k) - \lambda(f)\}\, d\nu .$$

Since the previous point apply to f_b , this ends the proof of (5.8).□

We now prove the theorem, making use - for convenience - of theorem III.1; nevertheless, there exist shorcuts using large deviation estimates on a lower level.

□ Proof of theorem V :

Let Φ be the bounded continuous map, $\Phi : \mathcal{P}_s(\Omega) \longrightarrow l^2$, $\Phi(Q) = \int M(x_1,y_1)\, dQ$. We use the contraction principle in § II for Φ and the large deviation principle in theorem III.1) with $\Lambda_n = [1,n]$; we obtain another principle for the conditional law of $m_n(\omega)$ under $P_{\mu\otimes\nu}$ given y with sequence n and rate function I_1 on l^2,
$I_1(m) = \inf \{ I(Q;P_{\mu\otimes\nu}) \; ; \; Q\in\mathcal{P}_s(\Omega) \; , \; \Phi(Q) = m \}$, for allmost every y. The previous condition on Q concerns only with the one dimensional margin : by definition of I in theorem III.1, we have:
$I_1(m) = \inf \{ h(q;\mu\otimes\nu) \; ; \; q\in\mathcal{P}(\mathcal{X}\times\mathcal{Y}) \; , \; \pi^*q = \nu \; , \; \int M\, dq = m \}$ with π^*q the margin of q on $\mathcal{Y}$.

Because of theorem 3.1 in [5], $I_1(m)$ is achieved with $q(dx_1,dy_1) = r_m^{y_1}(dx_1)\, \nu(dy_1)$, with $r_m^{\cdot}$ as in iii) , whenever $I_1(m)$ is finite.

Since $Z_n^y = \mathbb{E}^{P_\mu} \exp -nv\{m_n(\omega)\}$, the conditional large deviation principle for $m_n(\omega)$ shows that $\lim n^{-1} \log Z_n^y = p$ holds with P_ν-probability one: this is point i). To prove ii) , one can easily adapt the proof of theorem IV.2 .

We now prove iii). We first establish for suitable y's, that any l-dimensional margin $\mathbb{Q}_{l,n}^y$ of $\mathbb{Q}_n^y$ is tight, with limit points $\int \bigotimes_{i=1}^{l} r_m^{y_i}\, \tau_l(dm)$ for some $\tau_l\in\mathcal{P}(\mathcal{S})$. Fix some $l\geq 1$; from (5.4,5), $\mathbb{Q}_{l,n}^y$ is given by

$$\frac{d\mathbb{Q}_{l,n}^y}{d\mu^{\otimes l}}(x_1,\ldots x_l) = (Z_n^y)^{-1} \int \exp[-nv\{m_n(\omega)\}]\, \mu(dx_{l+1})\ldots\mu(dx_n) \qquad (5.17).$$

We intend to omit ω in our notations from now on. Define

$$m_{l,n} = \frac{1}{n-l} \sum_{i=l+1}^{n} M(x_i, y_i) \quad \text{and} \quad m_l = \frac{1}{l} \sum_{i=1}^{l} M(x_i, y_i) \qquad (5.18).$$

Then, $m_n = (l/n)\, m_l + (n-l/n)\, m_{l,n}$. From Taylor formula and (5.3),

$$v(m) = v(m_{p,n}) + (l/n)\, v'(m_{l,n}).(m_l - m_{l,n}) + \sigma(1/n) \qquad (5.19).$$

Combining this with (5.17), we obtain

$$\frac{d\mathbb{Q}^y_{l,n}}{d\mu^{\otimes l}} = (\, Z^y_n)^{-1} \;\; Z^y_{l,n} \;\; A^y_{l,n}(m_l) \qquad (5.20)$$

with $A^y_{l,n}(m) = (Z^y_{l,n})^{-1} \int \exp\{-nv(m_{l,n}) - lv'(m_{l,n}).(m-m_{l,n}) + \sigma(1)\}$ $\mu(dx_{l+1})\ldots\mu(dx_n)$

and $Z^y_{l,n}$ such that

$(Z^y_{l,n})^{-1} \exp\{-nv(m_{l,n}) + lv'(m_{l,n}).m_{l,n})\}\, \mu(dx_{l+1})\ldots\mu(dx_n)$ is a probability measure. In the same way as in ii), we can show that the laws of $m_{l,n}$, $n \geq l$, under this measure satisfy a large deviation principle with rate $v+L^*+p$ for $y \in ¥_l$ for some $¥_l$ in $\mathcal{Y}^{\mathbb{Z}^d}$ with full P_ν-probability. We fix y in $¥_l$, and $\varphi_o : \mathbb{N}^* \rightarrow \mathbb{N}^*$ an increasing function; because of tightness, we can find an increasing φ_l such that the subsequence of the laws of $m_{l,.}$ with index $. = \varphi_o \circ \varphi_l(n)$ converges to some $\rho_l \in \mathcal{P}(\Delta)$. Of course, ρ_l and φ_l do not depend on $x_1, \ldots x_l$. Then, for any $\tilde{m}$ in l^2, we have

$$\lim_{n\to\infty} A^y_{l,\varphi_o\circ\varphi_l(n)}(\tilde{m}) = \int_\Delta \exp\{-lv'(m).\tilde{m}\}\, \rho_l(dm) \qquad (5.21).$$

Integrating both sides of (5.20) with respect to $x_1, \ldots x_l$, we derive from this and Lebesgue theorem that

$Z^y_{l,\varphi} = \lim_{n\to\infty} Z^y_{\varphi_o\circ\varphi_l(n)} \left(Z^y_{l,\varphi_o\circ\varphi_l(n)} \right)^{-1}$ exists , and is equal to

$\int \prod_{i=1}^{l} \exp\{ L^{y_i}[-v'(m)]\}\, \rho_l(dm)$ because of (5.18) and Fubini theorem.

Combining this with (5.21) and Lebesgue theorem, we see that $Q^y_{l,\varphi_o\circ\varphi_l(n)}$ converges in the sense of probability measures to

$$\left(Z^y_{l,\varphi} \right)^{-1} \int_{\Delta} \exp\{-l v'(m).m_l\} \; \rho_l(dm) \quad \mu(dx_1)\ldots\mu(dx_l)$$

$$= \left(Z^y_{l,\varphi} \right)^{-1} \int_{\Delta} \prod_{i=1}^{l} \exp\{-v'(m).M(x_i,y_i)\} \; \mu(dx_i) \; \rho_l(dm)$$

because of definition (5.18) of m_l .

We now prove that $L'[-v'(m)] = m$ for all $m \in \Delta$. By definition of Δ, $v+L^*$ achieves its minimum at $m \in \Delta$, so $-v'(m)$ belongs to the subdifferential $\partial L^*(m)$ of the convex function L^* . Since L is convex (as an integral of convex functions L^y) and differentiable, this implies that ([9], cor. 5.2) $m \in \partial L(-v'(m)) = \{ L'(-v'(m)) \}$. Then, $-v'(m) = t(m)$, with $t(m)$ as in iii), theorem V.1.

Let $\tau_l(dm) = \left(Z^y_{l,\varphi} \right)^{-1} \exp\{ \sum_{i=1}^{l} L^{y_i}(t(m)) \} \; \rho_l(dm)$;
from the expression of $Z^y_{l,\varphi}$, we know that $\tau_l \in \mathcal{P}(\Delta)$, which yields

$$\forall y \in Y_l , \qquad Q^y_{l,\varphi_o\circ\varphi_l(n)} \underset{n\to\infty}{\Longrightarrow} \int \bigotimes_{i=1}^{l} r_m^{y_i} \; \tau_l(dm) \quad .$$

The Borel set $Y_\infty = \cap Y_l$ has P_ν-probability one. Using a diagonal procedure, we can find a subsequence of $Q^y_{\varphi_o(n)}$ converging in $\mathcal{P}(\mathcal{X}^l)$ to $\int \bigotimes_{i=1}^{l} r_m^{y_i} \; \tau_l(dm)$ for all $l \geq 1$ and all y in Y_∞ .

Since Δ is compact, $\mathcal{P}(\Delta)$ is compact too; let τ be a limit point of $(\tau_l)_{l\geq 1}$ for such a y. Then the previous measure on $\mathcal{P}(\mathcal{X}^l)$ is the restriction of $\int \bigotimes_{i=1}^{\infty} r_m^{y_i} \; \tau(dm)$. This ends the proof. □

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CHAPITRE II : EVOLUTION TEMPORELLE DE SYSTEMES AVEC INTERACTION DE CHAMP MOYEN LOCAL .

Partie A : FLUCTUATIONS AUTOUR DE LA LOI DES GRANDS NOMBRES , RALENTISSEMENT CRITIQUE .

ASYMPTOTIC DYNAMICS, NON-CRITICAL AND CRITICAL FLUCTUATIONS FOR A GEOMETRIC LONG-RANGE INTERACTING MODEL

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RUNNING TITLE : GEOMETRIC LONG-RANGE INTERACTING MODEL

Abstract.

We study the dynamics of a geometric spin system on the torus with long-range interaction. As the number of particles goes to infinity, the process converges to a deterministic, dynamical magnetization field that satisfies an Euler equation (law of large numbers). Its stable steady states are related to the limits of the equilibrium measures (Gibbs states) of the finite particle system. A related equation holds for the magnetization densities, for which the property of propagation of chaos also is established. We prove a dynamical central limit theorem with an infinite-dimensional Ornstein-Uhlenbeck process as limiting fluctuation process. At the critical temperature of a ferromagnetic phase transition, both a tighter quantity scaling and a time scaling is required to obtain convergence to a one-dimensional critical fluctuation process with constant magnetization fields, which has a non-Gaussian invariant distribution. Similarly, at the phase transition to an antiferromagnetic state with frequency p_0, the fluctuation process with critical scaling converges to a two-dimensional critical fluctuation process, which consists of fields with frequency p_0 and has a non-Gaussian invariant distribution on these fields. Finally, we compute the critical fluctuation process in the infinite particle limit at a triple point, where a ferromagnetic and an antiferromagnetic phase transition coincide.

1. INTRODUCTION.

In this paper, we study the nonequilibrium behaviour of a geometric spin model with weak interaction in the infinite particle limit. For finite $n \in \mathbb{N}$, the n-particle model consists of particles located at the sites $0, 1/n, 2/n, \ldots, n-1/n$ of the unit circle $\mathbb{T} = \mathbb{R} \bmod \mathbb{Z}$. A one-dimensional spin value $\sigma(i/n)$ is associated to each particle, and the spins interact via a mean-field potential depending on the distance between the particles.

In the equilibrium theory, the thermodynamic limit of these geometric models has been studied recently [8,2], and has shown a variety of interesting phase transitions. Depending on the parameters, there exist phase transitions to ferromagnetic states with constant magnetization or transitions to antiferromagnetic states with wave-like magnetization functions of any frequency p. Moreover, secondary phase transitions of first-order occur too (see e.g. the phase diagram in [7]). We find metastable states near these secondary phase transitions. The nucleation behaviour of the system can be described, as it switches from one (meta-) stable state to another stable one ([1]).

Here, however, we are interested in the dynamical laws of these models. We start with a Glauber-type dynamics ([13]) for the n-particle system, where the spins flip from time to time to another value with a jump intensity depending on the gradient of the Hamiltonian felt by the particle. Next, we establish the asymptotic dynamics of the magnetization field in the infinite particle limit (Euler equation). We obtain a similar equation for the density field of the magnetization and show that a propagation of chaos result holds. Our main results are the infinite particle limits of the non-critical fluctuation process and at the critical fluctuations, which -besides an appropriate scaling of the spin values- require a rescaling of the time in order to keep track with the stiffness and long time fluctuations of the critical structure ('critical slowing down'). As a result, only the critical structure survives the critical scaling, and in the limit, the critical fluctuation process is a low dimensional process (of the dimension of the eigenspace of zero of the infinitesimal operator at the critical point), in contrast to the infinite dimensional non-critical fluctuation process. In fact, the critical fluctuations are of dimension 1 at the critical point of a ferromagnetic phase transition, while they are of dimension 2 at an antiferromagnetic phase transition, and of dimension 3 at a ferro~-antiferromagnetic triple point.

Asymptotic dynamics, propagation of chaos results and non-critical fluctuation processes for weakly interacting systems have been extensively studied (see e.g. [19, 23, 24, 25, 27, 28], to mention just a few). Dawson [4] also obtained a critical fluctuation process of dimension 1. All these models have a space-independent weak interaction, and therefore lack a rich structure of phase transitions. In a recent paper, Fritz obtained the Euler equation for a continuous spin model on a lattice with nearest neighbour interaction [12].

We are now going to describe our model and the results of the different sections in more detail. For simplicity, we restrict ourselves here to the case of one space dimension (d=1), though all the results in the later sections are formulated for arbitrary dimension d.

For the system consisting of n particles, located at the points of the lattice $\mathbb{T}_n = \{i/n,\ i=0,\dots,n-1\}$, a spin configuration $\sigma^n = n^{-1} \sum_{x\in\mathbb{T}_n} \sigma(x)\delta_x$ has the internal energy

$$H^n(\sigma^n) = -1/2n \sum_{i,j=1}^{n} \mathfrak{J}(i-j/n)\sigma(i/n)\sigma(j/n)$$

$$= -n/2\iint_{\mathbb{T}_n^2} \mathfrak{J}(x-y)\ \sigma^n(dx)\sigma^n(dy)$$

$$= -n/2 \langle \sigma^n, \mathfrak{J}*\sigma^n \rangle = n\ H(\sigma^n). \tag{1.1}$$

Here δ_x is the Dirac mass at x and * denotes convolution. The single spin distribution, denoted by ρ, is a probability measure on R with compact support. (Only in the last sections of the paper, when we deal with the specific situation at the critical point of a phase transition, do we impose further conditions on ρ). The dynamical process of the n-particle system is a spin-flip process where the intensity of flipping the spin $\sigma(x)$ at $x\in\mathbb{T}_n$ to the new spin value m is equal to

$$-\beta m\ \partial/\partial\sigma(x)\ H(\sigma^n) = \beta m.\mathfrak{J}*\sigma^n(x), \tag{1.2}$$

with $\beta>0$ as the inverse temperature. Therefore, the infinitesimal generator L^n of the system is

$$L^n f(\sigma) = \sum_{x\in\mathbb{T}_n} \int [f(\sigma|_x^n m) - f(\sigma)]\ \exp\{\beta m\ \mathfrak{J}*\sigma^n(x)\}\ \rho(dm), \tag{1.3}$$

where f is a continuous function on the spin configuration space and $\sigma|_x^n m$ is the flipped configuration which is equal to σ except at x where its value is m. It is easy to check, that the unique invariant distribution for the infinitesimal generator L^n is the n-particle Gibbs measure Q^n with the Hamiltonian H^n, given by

$$Q^n(d\sigma^n) = \exp(-\beta H^n(\sigma^n)) \prod_{x\in\mathbb{T}_n} \rho(d\sigma(x))/Z^n, \tag{1.4}$$

with normalizing constant Z^n. Q^n lives on the n-particle configuration space, which is a closed subset of the set $\mathcal{M}$ of bounded (w.r.t. the total variation norm) Radon measures endowed with the weak-* topology. The cumulant generating function of the single spin distribution ρ is defined by

$$\gamma(r) = \log \int_R \exp(rm)\rho(dm). \tag{1.5}$$

Now, we can state the asymptotic dynamics of the spin-flip processes σ^n_t, generated by L^n, in the infinite particle limit.

THEOREM 1′.

The processes σ^n_t *converge in law on the Skorokhod space* $\mathcal{D}([0,T],\mathcal{M})$ *to the magnetization process* $u_t\lambda$, *where* λ *is the Lebesgue measure on* $\mathbb{T}$ *and the density* $u_t\in L^\infty(\mathbb{T})$ *satisfies the deterministic evolution equation*

$$d/dt\ u_t(x) = \exp(\gamma(\beta\mathfrak{J}*u_t))[\gamma'(\beta\mathfrak{J}*u_t) - u_t]. \tag{1.6}$$

As is to be expected, there is a close connection between (1.6) and the Gibbs states Q^n. Indeed, it has been shown in [8] that the Q^n satisfy a large deviation principle on $\mathcal{M}$ with a rate function

$$V(\mu) = I(\mu) + \beta H(\mu) \tag{1.7}$$

with

$$I(\mu) = \begin{cases} \int_{\mathbb{T}} i(d\mu/d\lambda(x))\ \lambda(dx) & \text{if } \mu \ll \lambda, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

where $$i(q) = \sup_{r\in\mathbb{R}}(q.r - \gamma(r)) \tag{1.9}$$

is the Cramèr transform of ρ. The large deviation means heuristically that for a small weak-* neighborhood $U(\mu)$ of $\mu\in\mathcal{M}$

"$Q^n(U(\mu))$ behaves asymptotically like $\exp(-n[V(\mu)-\inf_\nu V(\nu)])$". (1.10)

But the Frechet derivative of $u \mapsto V(u\lambda)$ in the $\|.\|_\infty$-norm is by (1.8) and (1.2)

$$\nabla V(u\lambda)(x) = i'(u(x)) - \beta\mathcal{J}*u(x) \qquad (1.11)$$
$$= (\gamma')^{-1}(u(x)) - \beta\mathcal{J}*u(x)$$

but if $\gamma'(0) = 0$, i.e. ρ has mean zero,

$$\text{sign}(\gamma'(\beta\mathcal{J}*u(x)) - u(x)) = \text{sign}(-\nabla V(u\lambda)(x)), \qquad (1.12)$$

since i' is the inverse of γ' by (1.9), and since $\gamma'(0) = 0$ implies $\text{sign}\,\gamma'(r) = \text{sign}(r)$. This means that the right-hand side of the evolution equation has the same sign as $-\nabla V(u\lambda)$. In particular, its paths go downhill with respect to the potential V, and the stable steady state solutions of (1.6) are exactly the local minima of V.

In Section 4, we study the asymptotic dynamics of the density process

$$\pi^n_t = n^{-1} \sum_{x\in\mathbb{T}_n} \delta_{(\sigma^n_t(x),x)}, \qquad (1.13)$$

which is a probability measure on $\mathbb{R}\times\mathbb{T}$. Again we give the space $\mathcal{P}(\mathbb{R}\times\mathbb{T})$ of all such probability measures the weak-* topology. Notice that since for each $x\in\mathbb{T}_n$, $\pi^n_t(dm,\{x\}) = n^{-1}\delta_{(\sigma^n_t(x))}$ is a one-point measure on $\mathbb{R}$, π^n and σ^n contain mathematically the same information. This is however no longer true in the infinite particle limit.

THEOREM 2'.

π^n_t converges in law to the magnetization density process $h_t(m,x)\rho(dm)\lambda(dx)$, where h_t satisfies the deterministic density evolution equation :

$$d/dt\, h_t(m,x) = \exp\{m\beta\mathcal{J}*u_t(x)\} - h_t(m,x)\exp\{\gamma(\beta\mathcal{J}*u_t(x))\}. \qquad (1.14)$$

(1.14) is a desintegrated version of (1.6). In fact, multiplying both sides of (1.14) with m and integrating with respect to $\rho(dm)$ gives exactly (1.6). In a similar way, we define the higher order correlation densities for different sites of $\mathbb{T}$. It is then easy to show that in the infinite particle limit, these correlations densities satisfy a propagation of chaos property. (See Theorem 3 of Section 4 for details). This result itself implies the usual result of propagation of chaos (see theorem 3 bis).

Next, we look for a first order approximation to u_t ; we define the (non-critical) fluctuation process

$$\zeta^n_t = n^{1/2} (\sigma^n_t - u_t \lambda). \tag{1.15}$$

In order to establish a central limit theorem for these fluctuation processes, we have not only to work in the space $\mathcal{S}'$ of distributions on $\mathbb{T}$, or at least in a Sobolev space H_{r_o} with sufficiently low negative index (see section 5 for technical details), but we also need first a law of large number result for the second moment magnetization fields

$$(\sigma^n_t)^2 = n^{-1} \sum_{x \in \mathbb{T}_n} \sigma^2_t(x)\, \delta_x . \tag{1.16}$$

In fact, like σ^n_t, also $(\sigma^n_t)^2$ converge in law on $\mathcal{D}([0,T],\mathbb{R})$ to the second moment magnetization process $v_t \lambda$, where v_t satisfies the deterministic equation

$$d/dt\, v_t = \exp\{\gamma(\beta\mathcal{J} * u_t)\}[\gamma''(\beta\mathcal{J} * u_t) + (\gamma')^2 (\beta\mathcal{J} * u_t) - v_t]. \tag{1.17}$$

with u_t from (1.6). Now, we can state the central limit theorem for the fluctuation process :

<u>THEOREM 4'.</u>

<u>If</u> $\mathcal{J}$ <u>is sufficiently smooth and</u> ζ^n_o <u>converges in a Sobolev-sense to some</u> $\zeta_o \in \mathcal{S}'$, <u>then the processes</u> ζ^n_t <u>converge in law to a</u> $\mathcal{S}'$-<u>valued diffusion process</u> ζ_t, <u>given by</u>

$$d\zeta_t = -\gamma''(\beta\mathcal{J} * u_t) \exp \gamma(\beta\mathcal{J} * u_t) d^2 V(u_t)\zeta_t \, dt$$

$$+ [\exp \gamma(\beta\mathcal{J} * u_t)(\gamma''(\beta\mathcal{J} * u_t) + (\gamma')^2 (\beta\mathcal{J} * u_t) - 2u_t \gamma'(\beta\mathcal{J} * u_t) + v_t]^{1/2} dW_t . \tag{1.18}$$

Here, d^2V is the second Frechet derivative of V from (1.7), i.e.

$$d^2V(u)\zeta = i''(u)\zeta - \beta\mathcal{J}*\zeta, \tag{1.19}$$

and W_t is the $\mathcal{S}'$-valued Brownian motion with covariance

$$E \langle \varphi, W_s \rangle \langle \psi, W_t \rangle = (s \wedge t) \langle \varphi, \psi \rangle \tag{1.20}$$

for $\varphi, \psi \in \mathcal{C}^\infty(\mathbb{T})$.

To get a better understanding of (1.18), let us suppose that u_e is a stable steady state solution of (1.6) and that we are not in a critical situation of a phase transition. This means that u_e is a local minimum of V and that the second derivative d^2V is a non-degenerate, positive definite operator. Then also v_t converges to its stable solution $v_e = \gamma''(\beta\mathcal{J}*u_e) + (\gamma')^2(\beta\mathcal{J}*u_e)$ such that with $u_e = \gamma'(\beta\mathcal{J}*u_e)$, (1.18) reduces to

$$d\zeta_t = -\gamma''(\beta\mathcal{J}*u_e)\exp\gamma(\beta\mathcal{J}*u_e)d^2V(u_e)\zeta_t\,dt$$
$$+ [2\exp\gamma(\beta\mathcal{J}*u_e)\ \gamma''(\beta\mathcal{J}*u_e)]^{1/2}dW_t. \tag{1.21}$$

Thus, ζ_t is a generalized Ornstein-Uhlenbeck process and its unique stationary distribution is the Gaussian field with mean zero and covariance

$$E\langle \varphi, \zeta_t \rangle \langle \psi, \zeta_t \rangle = \langle \varphi, (d^2V(u_e))^{-1}\psi \rangle. \tag{1.22}$$

On the other hand, it is a consequence of (1.10), that the conditional fluctuation fields of Q^n, restricted to a neighborhood $U(u_e)$ of u_e, $Q^n(d(\sigma^n - u_e/n^{1/2})|U(u_e))$ converges to the mean zero Gaussian field with covariance (1.22) (see also [10]).

In order to investigate the situation at a critical point of a phase transition, we must specify our assumptions in order to make sure that a phase transition indeed occurs.

First, we assume ρ to be an even probability measure on $\mathbb{R}$ with compact support and that the GHS-inequality hold (cf. [9]), a consequence of which is that for some $K_o \geq 2$

$$0=\gamma(0)=\gamma'(0),\ \gamma''(0)>0,\ 0=\gamma^{(3)}(0)=\ldots=\gamma^{(2K_o-1)}(0),\ \gamma^{(2K_o)}(0)<0. \quad (1.23)$$

Again, $\mathcal{J}$ should be sufficiently smooth and symmetric. For a ferromagnetic phase transition, we want its Fourier coefficients to satisfy

$$\hat{\mathcal{J}}(0) - \hat{\mathcal{J}}(p) \geq \delta_o > 0 \quad \text{for all} \quad p\in\mathbb{Z}-\{0\}. \quad (1.24)$$

From [8] or [2], we know that a phase transition indeed occurs at the critical inverse temperature

$$\beta_o = (\gamma''(0)\ \hat{\mathcal{J}}(0))^{-1}. \quad (1.25)$$

Here, the potential $V = V_{\beta_o}$ has a unique minimum at $u_o = 0$, but $d^2V(u_o)$ has a one-dimensional kernel spanned by the constant function $\mathbb{1}$. (1.24) requires that the remainder of the spectrum is positive, bounded away from zero by $\beta_o\delta_o$. We define the critical fluctuation process by

$$\xi^n_t = n^{1/2K_o}\sigma^n_{tn^{1-1/K_o}}, \quad (1.26)$$

where the new time scale tn^{1-1/K_o} compensates the effect of critical slowing down, mentioned above. We decompose ξ^n_t into its ferromagnetic component $\theta^n_t = \hat{\xi}^n_t(0)\lambda^n$, where $\lambda^n = n^{-1}\sum_{x\in\mathbb{T}_n}\delta_x$ is the discrete Haar measure on $n^{-1}\mathbb{Z}/\mathbb{Z}$, and its complement η^n_t,

$$\xi^n_t = \theta^n_t + \eta^n_t. \quad (1.27)$$

Since $d^2V(0)$ is not degenerate in the direction of η^n_t, the stronger scaling $n^{1/2K_o}$, instead of $n^{1/2}$ at the non-critical fluctuation, has the effect that the processes η_t^n collapses to the zero process, and the dynamics of θ_t^n, in which direction $d^2V(0)$ is degenerate, has to be expanded to higher order terms of θ_t^n. For the following result on critical fluctuations, we need in addition some more complicated assumptions on the starting configurations ξ_o^n, for which we refer to Section 6, mainly to insure that η_o^n already collapses sufficiently fast.

THEOREM 5'.

The critical fluctuation process $\xi_t^n = \theta_t^n + \eta_t^n$ converges in law to the one-dimensional process $\xi_t = \hat{\theta}_t(0)\lambda$ with

$$d\hat{\theta}_t(0) = \gamma^{(2K_o)}(0)\left[(2K_o-1)!(\gamma''(0))^{2K_o-1}\right]^{-1}\hat{\theta}_t^{2K_o-1}(0)dt+(2\gamma''(0))^{1/2}dw_t, \quad (1.28)$$

and with w_t as the standard Brownian motion.

The stationary distribution of the process $\hat{\theta}_t(0)$ is given by the non-Gaussian distribution

$$\exp\{\gamma^{(2K_o)}(0)\left[2(2K_o)!(\gamma''(0))^{2K_o}\right]^{-1}\theta^{2K_o}\}\, d\theta/Z_1, \quad (1.29)$$

with normalizing constant Z_1. Notice that the surviving process $\hat{\theta}_t(0)$ in (1.28) depends only on quantities coming from the cumulant generating function γ of the single spin distribution ρ. It is invariant from the specific interaction function $\mathfrak{J}$, except for the implicit assumption that we are indeed at the critical point of ferromagnetic second-order phase transition. This phenomenon is called universality. This kind of result on critical fluctuation processes was first obtained by Dawson [4] for a non-geometric model with mean-field interaction, with a one-dimensional kernel of the second derivative of the large deviation potential V at the critical point, this proof is based on a semi-group perturbation theory. Our proofs use martingale decompositions and martingale inequalities, which allows us in Section to treat also critical fluctuations at an antiferromagnetic phase transition, where the kernel of $d^2V(0)$ has dimension 2. However, we have to strengthen the assumption (1.23) by requiring

$$\gamma^{(4)}(0) < 0 \text{ ; i.e. } K_o = 2, \qquad (1.30)$$

and instead of (1.24-25), we now have for $\hat{\jmath}(p_o) = \hat{\jmath}(-p_o)$

$$\hat{\jmath}(p_o) - \hat{\jmath}(q) \geq \delta_o > 0 \quad \text{for all} \quad q \in \mathbb{Z}\setminus\{\pm p_o\}, \qquad (1.31)$$
$$\beta_{p_o} = (\gamma''(0)\ \hat{\jmath}(p_o))^{-1}.$$

These conditions assure that we are at the critical point of a second-order phase transition to an antiferromagnetic state with frequency p_o (cf. [2]). This time, we split the critical fluctuation process $\xi^n_t = n^{1/4}\sigma^n_{tn^{1/2}}$ into the two-dimensional p_o-antiferromagnetic components

$$\varphi^n_t = [2\ \Re e(\hat{\xi}^n_t(p_o))\cos(2\pi\ p_o x) + 2\ \operatorname{Im}(\hat{\xi}^n_t(p_o))\sin(2\pi\ p_o x)]\lambda^n(dx) \qquad (1,32)$$

and its complement ψ^n_t : $\xi^n_t = \varphi^n_t + \psi^n_t$.

Here, we again omit the assumptions on the initial configurations.

<u>THEOREM 6´.</u>

<u>At the critical point of an antiferromagnetic phase transition of frequency</u> p_o, <u>the critical fluctuation process</u> ξ^n_t <u>converges in law to the two-dimensional antiferromagnetic process of frequency</u> p_o

$$\varphi_t(dx) = 2[\Re e(\hat{\varphi}_t(p_o))\cos(2\pi\ p_o x) + \operatorname{Im}(\hat{\varphi}_t(p_o))\sin(2\pi\ p_o x)]\lambda(dx), \qquad (1.33)$$

<u>where</u> $\hat{\varphi}_t(p_o) \in \mathbb{C}$ <u>is given by</u>

$$d\hat{\varphi}_t(p_o) = \gamma^{(4)}(0)\left[2\ \gamma''(0)^3\right]^{-1} |\hat{\varphi}_t(p_o)|^2\ \hat{\varphi}_t(p_o)dt + (2\gamma''(0))^{1/2} dw^{\mathbb{C}}_t, \qquad (1.34)$$

<u>with</u> $w^{\mathbb{C}}_t$ <u>the complex Brownian motion.</u>

Again, the stationary distribution of $\hat{\varphi}_t(p_o) \in \mathbb{C}$ is non-Gaussian :

$$\exp(\gamma^{(4)}(0)\left[16(\gamma''(0))^4\right]^{-1}|Z|^4)\, dZ/Z_2 \qquad (1.35)$$

with normalizing Z_2.

Finally, we calculate in Section 8 the limit of the critical fluctuation process at a triple point, where a ferromagnetic and an anti-ferromagnetic phase transition fall together. This means that for some $p_o \neq 0$

$$\hat{J}(0) = \hat{J}(p_o) = \hat{J}(-p_o) \quad \text{and} \quad \hat{J}(0)-\hat{J}(q) \geq \delta_o > 0 \qquad (1.36)$$

for all $q \in \mathbb{Z}\setminus\{0,\pm p_o\}$ and

$$\beta_o = (\gamma''(0)\,\hat{J}(0))^{-1}. \qquad (1.37)$$

Now in the infinite particle limit, the critical fluctuation process has the form

$$\mu_t(dx) = [\hat{\mu}_t(0) + 2\,\mathrm{Re}(\hat{\mu}_t(p_o))\cos(2\pi p_o x) + 2\,\mathrm{Im}(\hat{\mu}_t(p_o))\sin(2\pi p_o x)]\lambda(dx), \qquad (1.38)$$

and $(\hat{\mu}_t(0), \hat{\mu}_t(p_o)) \in \mathbb{R} \times \mathbb{C}$ is driven by the coupled stochastic differential equation

$$d\hat{\mu}_t(0) = \gamma^{(4)}(0)\left[3!(\gamma''(0))^3\right]^{-1}(\hat{\mu}_t^2(0) + 6|\hat{\mu}_t(p_o)|^2)\,\hat{\mu}_t(0)dt + [2\gamma''(0)]^{1/2}dw_t$$

$$d\hat{\mu}_t(p_o) = \gamma^{(4)}(0)\left[2(\gamma''(0))^3\right]^{-1}(\hat{\mu}_t^2(0) + |\hat{\mu}_t(p_o)|^2)\,\hat{\mu}_t(p_o)dt + [2\gamma''(0)]^{1/2}dw_t^{\mathbb{C}} \qquad (1.39)$$

with w_t and $w_t^{\mathbb{C}}$ independent real, resp. complex Brownian motions.

In the appendix, we add a useful proposition on collapsing processes, which is of interest in its own right.

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2. NOTATIONS AND MAIN EXAMPLE.

Let $\mathbb{T}$ be the d-dimensional torus $(\mathbb{R}/\mathbb{Z})^d$ For any natural number $n \in \mathbb{N}$ we consider the lattice torus $\mathbb{T}_n = (n^{-1}\mathbb{Z}/\mathbb{Z})^d$ with spacing n^{-1}, consisting of the $N = n^d$ sites $x = (k_1/n, \ldots, k_d/n)$ where $k_j = 0, \ldots, n-1$ for $j = 1, \ldots, d$.

To each lattice site $x \in \mathbb{T}_n$, we associate a real-valued spin $\sigma(x)$, whose ensemble defines the magnetization field

$$\sigma^n = N^{-1} \sum_{x \in \mathbb{T}_n} \sigma(x)\, \delta_x \in \mathcal{M} \tag{2.1}$$

with δ_x the Dirac mass at x and $\mathcal{M} = \mathcal{M}(\mathbb{T})$ the set of Radon measures on $\mathbb{T}$. We endow $\mathcal{M}$ with the weak-*-topology, which makes $\mathcal{M}$ a metrizable space. Let $\mathcal{M}^n$ be the set of all measure of the form (2.1) $\mathcal{M}^n_b = \{\sigma^n \in \mathcal{M}^n,\ |\sigma(x)| \leq b$ for all $x \in \mathbb{T}_n\}$, and $\mathcal{M}_b = \{\mu \in \mathcal{M},\ \|\mu\| \leq b\}$, where $\|\mu\|$ means the total variation of μ. $\mathcal{M}^n$ and $\mathcal{M}^n_b$ are closed subsets of $\mathcal{M}$, resp. $\mathcal{M}_b$, and $\mathcal{M}_b$ is compact in the weak-*-topology.

We assume the single spin distribution ρ to be a probability measure on $\mathbb{R}$ with compact support, say contained in $B = [-b, +b]$. (In Section 5, we shall impose further restrictions on ρ). Let

$$\gamma(u) = \log \int \exp\{mu\}\, \rho(dm) \tag{2.2}$$

be the logarithm of the moment generating function. γ is a convex function with $\gamma(0) = 0$. Note that

$$\int m \exp\{mu\}\, \rho(dm) = \gamma'(u) \exp \gamma(u), \tag{2.3}$$

$$\int m^2 \exp\{mu\}\, \rho(dm) = [\gamma''(u) + (\gamma'(u))^2] \exp \gamma(u). \tag{2.4}$$

Let λ be the Lebesgue measure on $\mathbb{T}$ and

$$\lambda^n = N^{-1} \sum_{x \in \mathbb{T}_n} \delta_x \tag{2.5}$$

its discrete analogue on $\mathbb{T}_n$. Finally for $\sigma \in \mathcal{M}$, $m \in \mathbb{R}$, we define

$$\sigma|_x^n m(.) = \sigma(.\backslash C_n(x)) + m/N\ \delta_x(.), \tag{2.6}$$

where $C_n(x) = (x_1 - 1/2n,\ x_1 + 1/2n] \times \ldots \times (x_d - 1/2n,\ x_d + 1/2n] \subseteq \mathbb{T}$ is the cube in $\mathbb{T}$ with centre x and edge length $1/n$.

Now we define the operators L^n on $\mathcal{C}(\mathcal{M})$ by

$$L^n f(\sigma) = \int_{B\times\mathbb{T}} [f(\sigma|_x^n m) - f(\sigma)]\ N\ A^n(m,x,\sigma)\ \rho(dm)\ \lambda^n(dx) \tag{2.7}$$

with

$$A^n(m,x,\delta) = \exp\{G_o(x,\sigma) + m\ G_1(x,\delta) + G_2^n(m,x,\sigma)\}, \tag{2.8}$$

$$G_o,\ G_1 \in \mathcal{C}(\mathbb{T} \times \mathcal{M}), \tag{2.9}$$

and

$$G_2^n \xrightarrow[n\to\infty]{} 0 \tag{2.10}$$

in a sense to be made precise in the following sections.

We set

$$A(m,x,\sigma) = \exp\{G_o(x,\sigma) + mG_1(x,\sigma)\}. \tag{2.11}$$

Clearly, there exists a unique Markov process P^n on the Skorokhod space $\Omega = \mathcal{D}(\mathbb{R}^+, \mathcal{M})$, the space of right-continuous, $\mathcal{M}$-valued functions with left-hand limits, with L^n as its infinitesimal generator, i.e.

$$f(\sigma_t) - f(\sigma_o) - \int_{(0,t]} L^n f(\sigma_s) ds = M_t^n(f) \quad \text{is a } P^n\text{-martingale} \tag{2.12}$$

for all $f \in \mathcal{C}(\mathcal{M})$. This martingale can be written in the integral form

$$M_t^n(f) = \int_{(0,t]}\int_{B\times\mathbb{T}} [f(\sigma_{s-}|_x^n m) - f(\sigma_{s-})]\tilde{\Lambda}^n(dm,dx,ds), \tag{2.13}$$

where for $\sigma \in \Omega$

$$\tilde{\Lambda}^n(dm,dx,ds)(\sigma) = \Lambda^n(dm,dx,ds)(\sigma) - N\ A^n(m,x,\sigma_s)\rho(dm)\ \lambda^n(dx)ds$$

with a pure point process $\Lambda^n(dm,dx,ds)(\sigma)$. The corresponding increasing process (see [16], II.3.9) is

$$\langle M^n(f), M^n(f)\rangle_t = \int_0^t \int_{B\times\mathbb{T}} [f(\sigma_s|_x^n m) - f(\sigma_s)]^2\ N\ A^n(m,x,\sigma_s)\rho(dm)\lambda^n(dx)ds. \tag{2.14}$$

Example.

The general q-body long-range interaction between the spins of a magnetic field has the internal energy

$$H(\sigma)=-\sum_{j=1}^{q} 1/j! \int_{\mathbb{T}^j} \mathfrak{d}_j(x_1,\dots,x_j)\sigma(dx_1),\dots,\sigma(dx_j)=-\sum_{j=1}^{q}\langle \mathfrak{d}_j,\sigma^{\otimes j}\rangle, \quad (2.15)$$

where $\mathfrak{d}_j \in \mathscr{C}(\mathbb{T}^j)$. Its Frechet derivative is

$$\nabla H(\sigma)(x) = -\sum_{j=1}^{q} 1/j! \sum_{i=1}^{j} \langle \mathfrak{d}_j, \sigma^{\otimes i-1} \otimes \delta_x \otimes \sigma^{j-i} \rangle \in \mathscr{C}(\mathbb{T}). \quad (2.16)$$

Now, let G_o be any continuous function on $\mathbb{T} \times \mathcal{M}$, with

$$\sup_{x,\sigma\in\mathcal{M}_b^n} \{G_o(x,\sigma) - G_o(x,\sigma|_x^n o) = \mathcal{O}(N^{-1}), \quad (2.17)$$

and

$$G_1(x,\sigma) = -\beta\ \nabla H(\sigma)(x), \quad (2.18)$$

where $\beta>0$ is the inverse temperature. We set

$$\begin{aligned} G_2^n(m,x,\sigma) = {} & G_o(x,\sigma|_x^n o) - G_o(x,\sigma) + \beta\{NH(\sigma) - NH(\sigma|_x^n o) \\ & - \sigma(x)\ \nabla H(\sigma|_x^n o) + m(\nabla H(\sigma) - \nabla H(\sigma|_x^n o))\}. \end{aligned} \quad (2.19)$$

By (2.15-17), it is easy to check that

$$\sup_{\substack{m\in B_n, x\in\mathbb{T}_n \\ \sigma\in\mathcal{M}_b}} |G_2^n(m,x,\sigma)| = \mathcal{O}(N^{-1}). \quad (2.20)$$

The detailed balanced condition (see [29]) shows that the unique invariant probability distribution for the process P^n with infinitesimal generator L^n, given by (2.7-8), is the Gibbs state

$$Q^n(d\sigma^n) = \exp\{-\beta NH(\sigma^n)\} \prod_{x\in\mathbb{T}_n} \rho(d\sigma(x))/Z^n \quad (2.21)$$

with σ^n from (2.1) and Z^n as normalizing constant. The thermodynamic limit of (2.21) has been investigated in [8].

3. ASYMPTOTIC DYNAMICS OF THE MAGNETIZATION.

Besides (2.9-10), we assume that

G_o and G_1 are Lipschitz-continuous in $\sigma \in \mathcal{M}_b$ in the total variation norm, and that (3.1)

$$\sup_{\substack{m\in B, x\in \mathbb{T}_n \\ \sigma\in \mathcal{M}^n_b}} |G^n_2(m,x,\sigma)| = o(1). \tag{3.2}$$

Set $$(\sigma^n)^2(dx) = N^{-1} \sum_{y\in \mathbb{T}_n} \sigma^2(y)\, \delta_y(dx). \tag{3.3}$$

THEOREM 1.

(i) Let $\sigma^n_o \in \mathcal{M}^n_b$ converge in law to $u_o\lambda$, i.e.

$$u_o \in L^\infty_b = \{u\in L^\infty,\ \|u\|_\infty \leqslant b\}.$$

Then the process $(\sigma^n_t)_{t\leqslant T}$ converges in law to $(u_t\lambda)_{t\leqslant T}$, where $u_t \in L^\infty_b$ is the unique solution of the mean-field evolution equation (1)

$$d/dt\ u_t = G(u_t), \tag{3.4}$$

starting at u_o, and

$$G(u)(x) = \exp\{G_o(x,u) + \gamma(G_1(x,u))\}[\gamma'(G_1(x,u))-u(x)]. \tag{3.5}$$

(ii) Moreover, let $(\sigma^n_o)^2$ converge in law to some $v_o\lambda$,

$$v_o \in L^\infty_{[0,b^2]} = \{\alpha\in L^\infty;\ 0 \leqslant \text{ess inf}\ \alpha \leqslant \text{ess sup}\ \alpha \leqslant b^2\}.$$

Then, $(\sigma^n_t)^2$ converges in law to $v_t\lambda$, where $v_t \in L^\infty_{[0,b^2]}$ is the unique solution of

$$d/dt\ v_t = F(u_t, v_t), \tag{3.6}$$

starting at v_o, with

$$F(u,v)(x)=\exp\{G_o(x,u)+\gamma(G_1(x,u))\}[\gamma''(G_1(x,u))+(\gamma'(G_1(x,u)))^2-v(x)]. \tag{3.7}$$

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(1) in fact, this convergence holds (in probability) with an exponential rate (see [1] in the case of Ising spins).

<u>Proof</u>. The Lipschitz properties of G_o and G_1 imply that (3.4) and (3.6) have unique solutions. Since $\gamma'(y)\in(-b,+b)$ and $\gamma''(y)+(\gamma')^2(y)\in(0,b^2)$ for all $b\in\mathbb{R}$ the solutions u_t, v_t, satisfy $-b\leq u_t\leq b$, $0\leq v_t\leq b^2$. In order to show the tightness of $(\sigma^n_t)_{t\leq T}$, notice that $\sigma^n_o\in\mathcal{M}^n_b$ implies $\sigma^n_t\in\mathcal{M}^n_b\subset\mathcal{M}_b$ for all $t\geq 0$, P^n a.e., and that $\mathcal{M}_b$ is compact in its weak-*-topology. It suffices therefore to show uniform continuity in the following form :

For any $g\in\mathcal{C}(\mathbb{T})$, η, $\varepsilon>0$, there exists $n_o\in\mathbb{N}$ and $\delta>0$ such that

$$\sup_{n\geq n_o}\ \sup_{0\leq\tau_1\leq\tau_2\leq(\tau_1+\delta)\wedge T} P^n\{|\langle g,\sigma^n_{\tau_2}\rangle-\langle g,\sigma^n_{\tau_1}\rangle| > \eta\} \leq \varepsilon, \qquad (3.8)$$

where τ_1, τ_2 are stopping times (cf. [18], I.3.4). From (2.7-13), we get

$$\langle g,\sigma^n_{\tau_2}\rangle-\langle g,\sigma^n_{\tau_1}\rangle = N^{-1}\int_{(\tau_1,\tau_2]}\int_{B\times\mathbb{T}} g(x)(m-\sigma^n_{s-}(x))\tilde{\Lambda}^n(dm,dx,ds)$$
$$+\int_{\tau_1}^{\tau_2}\int_{B\times\mathbb{T}} g(x)(m-\sigma^n_s(x))A^n(m,x,\sigma^n_s)\rho(dm)\lambda^n(dx)ds \qquad (3.9)$$

with the last term being in absolute value less than $2b|g|_\infty|A+1|_\infty.\delta$ for n sufficiently large, using (3.2). Therefore, by (2.14)

$$P^n\{|\langle g,\sigma^n_{\tau_2}\rangle-\langle g,\sigma^n_{\tau_1}\rangle| > \eta\} \leq \eta^{-2}E^n((\langle g,\sigma^n_{\tau_2}\rangle-\langle g,\sigma^n_{\tau_2}\rangle)^2)$$
$$\leq 8b^2|g|^2_\infty|A+1|^2_\infty.\delta^2/\eta^2 + N^{-1}8b^2|g|^2_\infty|A+1|_\infty.\delta/\eta^2, \qquad (3.10)$$

which is less than ε for all $n\in\mathbb{N}$, if δ is sufficiently small. Furthermore, the jump sizes go to zero uniformly, so any limit law is concentrated on continuous paths.

The tightness of the processes $(\sigma^{n\,2}_t)_{t\leq T}$ is shown similarly.

As in (3.10), we get by Doob's inequality

$$P^n(\sup_{t\leq T}|N^{-1}\int_0^t\int_{B\times\mathbb{T}} g(x)(m-\sigma^n_{s-}(x))\tilde{\Lambda}^n(dm,dx,ds)|>\eta N^{-1/3}) = O(N^{-1/3}). \quad (3.11)$$

Hence, outside of a set of very small P^n-probability, we have

$$\langle g,\sigma_t^n\rangle = \langle g,\sigma_o^n\rangle + \int_0^t\int_{B\times\mathbb{T}} g(x)(m-\sigma_s^n(x))A^n(m,x,\sigma_s^n)\rho(m)\lambda^n(dx)ds+o(1)$$

$$\underset{\substack{(2.3)\\(3.2)}}{=} \langle g,\sigma_o^n\rangle + \int_0^t\int_{\mathbb{T}} g(x)\exp\{G_o(x,\sigma_s^n)+\gamma(G_1(x,\sigma_s^n))\}$$

$$[\gamma'(G_1(x,\sigma_s^n))\lambda^n(dx)-\sigma_s^n(dx)]ds + o(1).$$

But the maps $\mu \mapsto G_i(.,\mu)\in\mathcal{C}(\mathbb{T})$, i=1,2, are continuous on the (compact) set $\mathcal{M}_b$; then, from Ascoli's theorem, their range is a uniformly equicontinuous family of $\mathcal{C}(\mathbb{T})$, and the Riemann sum λ^n in the last term converges uniformly to the λ-integral. Combining this with (3.5), we derive that with large probability

$$\langle g,\sigma_t^n\rangle = \langle g,\sigma_o^n\rangle + \int_0^t\langle g,G(\sigma_s^n)\rangle ds + o(1), \qquad (3.12)$$

and so any limit process of $(\sigma_t^n)_{t\leq T}$ must be concentrated on the solution of (3.4), which is unique however. In the case of $(\sigma^n)^2$, we obtain

$$\langle g,(\sigma_t^n)^2\rangle=\langle g,(\sigma_o^n)^2\rangle+\int_0^t\int_{B\times\mathbb{T}} g(x)(m^2-(\sigma_s^n(x))^2)A^n(m,x,\sigma_s^n)\rho(dm)\lambda^n(dx)ds+o(1)$$

$$\underset{(2.4)}{=} \langle g,(\sigma_o^n)^2\rangle+\int_0^t\int_{\mathbb{T}} g(x)\exp\{G_o(x,\sigma_s^n)+\gamma(G_1(x,\sigma_s^n))\}$$

$$[(\gamma''(G_1(x,\sigma_s^n))+(\gamma')^2(G_1(x,\sigma_s^n)))\lambda(dx) - (\sigma_s^n)^2(dx)]ds + o(1)$$

$$= \langle g,(\sigma_o^n)^2\rangle + \int_0^t \langle g,F(\sigma_s^n,(\sigma_s^n)^2)\rangle ds + o(1). \qquad (3.13)$$

This completes the proof of the Theorem.

4. ASYMPTOTIC DYNAMICS OF THE DENSITIES AND PROPAGATION OF CHAOS.

To a magnetization field $\sigma^n \in \mathcal{M}_b^n$, we associate the empirical magnetization density

$$\pi^n = N^{-1} \sum_{x\in\mathbb{T}_n} \delta_{(\sigma^n(x),x)} \in \mathcal{P}(B \times \mathbb{T}) \tag{4.1}$$

where $\mathcal{P}(B\times\mathbb{T})$ denotes the set of all probability measures on $B\times\mathbb{T}$. $\mathcal{P}(B\times\mathbb{T})$ is compact in the weak-* topology.

We first show that the density process π^n converges to a deterministic density, governed by the asymptotic magnetization process :

THEOREM 2.

Assume (3.1-2) and that π_o^n converge in law to $h_o(m,x)\,\rho(dm)\,\lambda(dx) \in \mathcal{P}(B \times \mathbb{T})$, $h_o \in L^\infty(B \times \mathbb{T})$. Then the empirical density process π_t^n converges in law to $h_t(m,x)\,\rho(dm)\,\lambda(dx)$, where the density $h_t \in L^\infty(B \times \mathbb{T})$ is the solution

$$\begin{aligned} d/dt\ h_t(m,x) = {} & \exp\{G_o(x,u_t)+m\,G_1(x,u_t)\} \\ & -h_t(m,x)\exp\{G_o(x,u_t)+\gamma(G_1(x,u_t))\} \end{aligned} \tag{4.2}$$

starting at h_o; and where $u_t(x) = \int_B m\, h_t(m,x)\rho(dm)$ is the solution of (3.4) with $u_o(x) = \int_B m\, h_o(m,x)\,\rho(dm)$.

Since by (2.9-10)

$$\sup_{\substack{m\in B,\,x\in\mathbb{T}\\ \sigma\in\mathcal{M}_b}} \exp\{m\,G_1(x,\sigma) - \gamma(G_1(x,\sigma))\} = C < \infty, \tag{4.3}$$

$0 \leqslant h_t(m,x) \leqslant C$, if this property holds for h_o. Therefore $h_t \in L^\infty(B\times\mathbb{T})$ for all t. Moreover

$$\int h_t(m,x)\,\rho(dm) = 1. \tag{4.4}$$

Proof : Since (4.2) is linear in h, it has a unique solution in $L^{\infty}(B \times \mathbb{T})$ satisfying (4.4). Let $g \in \mathcal{C}(B \times \mathbb{T})$. Then

$$\langle g, \pi_t^n \rangle = \langle g, \pi_o^n \rangle + \int_0^t \int_{B\times\mathbb{T}} \left[g(m,x) - g(\sigma_s^n(x),x) \right] A^n(m,x,\sigma_s^n) \rho(dm) \lambda^n(dx) ds + \int_0^t \int_{B\times\mathbb{T}} N^{-1} \left[g(m,x) - g(\sigma_s^n(x),x) \right] \tilde{\Lambda}^n(dm,dx,ds). \quad (4.5)$$

The uniform continuity can now be shown in the same way as is (3.8-10). By the compactness of $\mathcal{P}(B \times \mathbb{T})$, the sequence of processes π_t^n is therefore tight.

Doob's inequality implies again

$$P^n\left\{ \sup_{t\leq T} \left| \int_0^t \int_{B\times\mathbb{T}} N^{-1} [g(m,x) - g(\sigma_s^n(x),x)] \tilde{\Lambda}^n(dm,dx,ds) \right| > \eta N^{-1/3} \right\} = \mathcal{O}(N^{-1/3}), \quad (4.6)$$

which gives outside a set of uniformly small probability

$$\begin{aligned}\langle g, \pi_t^n \rangle &= \langle g, \pi_o^n \rangle + \int_0^t \Big[\int_{B\times\mathbb{T}} \exp\{G_o(x,\sigma_s^n) + mG_1(x,\sigma_s^n)\} g(m,x) \rho(dm) \lambda^n(dx) \\ &\quad - \int_{B\times\mathbb{T}} \exp\{G_o(x,\sigma_s^n) + \gamma(G_1(x,\sigma_s^n))\} g(m,x) \pi_s^n(dm,dx) \Big] ds + o(1) \\ &= \langle g, \pi_o^n \rangle + \int_0^t \Big[\int_{B\times\mathbb{T}} \exp\{G_o(x,u_s) + mG_1(x,u_s)\} g(m,x) \rho(dm) \lambda(dx) \\ &\quad - \int_{B\times\mathbb{T}} \exp\{G_o(x,u_s) + \gamma(G_1(x,u_s))\} g(m,x) \pi_s^n(dm,dx) \Big] ds + o(1) \\ &\quad + \mathcal{O}\big(\sup_{s\leq T} \sup_{x\in\mathbb{T}} (|G_o(x,u_s) - G_o(x,\sigma_s^n)| + |G_1(x,u_s) - G_1(x,\sigma_s^n)|)\big). \quad (4.7)\end{aligned}$$

Since, by Theorem 1, the last term converges to zero uniformly in probability, we find that any limit π_s of the processes π_s^n satisfies the following equation, which is deterministic except for π_o :

$$\langle g,\pi_t\rangle=\langle g,\pi_o\rangle+\int_0^t\Big[\int_{B\times\mathbb{T}}\exp\{G_o(x,u_s)+mG_1(x,u_s)\}g(m,x)\rho(dm)\lambda(dx)$$

$$-\int_{B\times\mathbb{T}}\exp\{G_o(x,u_s)+\gamma(G_1(x,u_s))\}g(m,x)\pi_s(dm,dx)\Big]ds. \quad (4.8)$$

But the solution of (4.8) is unique, and if $\pi_o = h_o\, d\rho d\lambda$ then also π_t has a density $h_t(m,x)$ with respect to $d\rho d\lambda$, and h_t is the solution of (4.2). This completes the proof.

Notice, that if $h_o \in \mathscr{C}(B \times \mathbb{T})$, then $h_t \in \mathscr{C}(B \times \mathbb{T})$ for all $t>0$. Since the right-hand side of (4.2) depends only on the single site x, it is obvious that results of the type of 'propagation of chaos' should hold. In fact, we shall derive two versions of propagations of chaos. The first one will be at the level of empirical measures. In analogy to the weak-* topology on $\mathscr{P}(B \times \mathbb{T})$, used in Theorem 2, we shall obtain only a weak version at this level.

The second result is the usual 'propagation of chaos' for the random spin variables $\sigma_t^n(x)$. It says that, if the spins at distinct sites are independently distributed at $t=0$, then in the limit $n \to \infty$, they continue to behave independently at any time $t>0$ according to a distribution which satisfies (4.2), i.e. they constitute a sample of the empirical density. Of course, this is not true for finite n, where the spins are dependent. We shall see, that this is a consequence of the first version, yielding here a new proof of the standard result.

Let $x_1,\ldots,x_K$ be distinct sites in $\mathbb{T}$.

Let ε_n be a sequence of positive numbers with

$$\varepsilon_n \searrow 0 \quad \text{and} \quad N^{-1}\varepsilon_n^{-2d} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty . \qquad (4.9)$$

We define

$$\bar{\pi}_t^n = \prod_{i=1}^K \bar{\pi}_t^n(x_i) = \prod_{i=1}^K \Big[(N\varepsilon_n^d)^{-1} \sum_{y_i\in C_{\varepsilon_n}(x_i)\cap\mathbb{T}_n} \delta_{\sigma_t^n(y_i)}\Big] \qquad (4.10)$$

as a positive measure on B^K, where $C_{\varepsilon_n}(x_i)$ are the cubes with center x_i and edge length ε_n.

THEOREM 3.

Assume (3.1-2), that σ^n_o converges in law to $u_o\lambda$, and that $\bar{\pi}^n_o$ converges in law to $\prod_{i=1}^{K} h_o(m_i,x_i)\,\rho(dm_i)$. Then, for $t>0$, $\bar{\pi}^n_t$ converges in law to $\prod_{i=1}^{K} h_t(m_i,x_i)\,\rho(dm_i)$ with h_t satisfying (4.2).

Proof. First, notice that it is enough to prove the Theorem for $K=1$, since $\bar{\pi}^n_t = \prod_i \bar{\pi}^n_t(x_i)$ and $h(.,x_i)\,d\rho$ is deterministic.

Now, for $g\in\mathcal{C}(B)$,

$$\langle g,\bar{\pi}^n_t(x)\rangle=\langle g,\bar{\pi}^n_o(x)\rangle + \int_0^t\int_{B\times\mathbb{T}} \mathbf{1}_{C_{\varepsilon_n}(x)}(y)\varepsilon_n^{-d}(g(m)-g(\sigma^n_s(y)))A^n(m,y,\sigma^n_s)\rho(dm)\lambda^n(dy)ds + M^n_t, \tag{4.11}$$

where

$$M^n_t = \int_0^t\int_{B\times\mathbb{T}} \mathbf{1}_{C_{\varepsilon_n}(x)}(y)(N\varepsilon_n^d)^{-1}(g(m)-g(\sigma^n_{s-}(y))\tilde{\Lambda}^n(dm,dy,ds), \tag{4.12}$$

implies $E^n((M^n_t)^2) = \mathcal{O}(N^{-1}\varepsilon_n^{-2d}) \xrightarrow[n\to\infty]{} 0$ by (4.9).

By the same argument as in the proof of the last theorem, we see that $\bar{\pi}^n_t(x)$ converges in law to a positive measure $\bar{\pi}_t$ on B, which satisfies

$$\langle g,\bar{\pi}_t(x)\rangle = \langle g,\bar{\pi}_o(x)\rangle + \int_0^t(\int_B g(m)\exp(G_o+mG_1)(x,u_s)\rho(dm)$$
$$\langle g,\bar{\pi}_s(x)\rangle\exp\{G_o+\gamma(G_1)\}(x,u_s))ds. \tag{4.13}$$

(4.13) is linear in $\bar{\pi}$, and therefore, has a unique solution, which is $\bar{\pi}_t(x) = h_t(.,x)d\rho$ by (4.2) and the initial condition $\bar{\pi}_o(x) = h_o(.,x)d\rho$. This proves the Theorem.

As a consequence of the last result, we get the propagation of chaos for the random variables $\sigma_t^n(x)$: corresponding to the distinct $x_1,\dots,x_K \in \mathbb{T}$, let x_i^n be sequences with

$$x_i^n \in \mathbb{T}_n \quad \text{and} \quad \lim_n x_i^n = x_i \quad \text{for} \quad i = 1,\dots,K. \tag{4.14}$$

THEOREM 3 bis.

Besides (3.1-2), (4.14), *assume that* σ_o^n *converges in law to* $u_o\lambda$ *and that the distribution of* $(\sigma_o^n(x_1^n),\dots,\sigma_o^n(x_K^n))$ *converges to* $\prod_{i=1}^K h_o(.,x_i)d\rho$ *as* $n \to \infty$. *Then, for* $t>0$, *the distribution of* $(\sigma_t^n(x_1^n),\dots,\sigma_t^n(x_K^n))$ *converges to* $\prod_{i=1}^K h_t(.,x_i)d\rho$ *with* $h_t(.,x_i)$ *from* (4.2).

Proof. Without loss of generality, we may add the assumption that $\prod_i \bar{\pi}_o^n(x_i)$ converges in law to $\prod_{i=1}^K h_o(.,x_i)d\rho$: indeed, this assumption may be achieved via the change in the initial distribution of particles in proportion $\Theta(\varepsilon_n^d)$, then being without any influence on the asymptotic distribution of $(\sigma_t^n(x_1^n),\dots,\sigma_t^n(x_K^n))$.

First, we regard the case $K=1$. Let $g \in \mathscr{C}(B)$.

Using (4.11) and a similar expression for $g(\sigma_t^n(x_1^n))$, it is easy to get the following inequality :

$$\begin{aligned}|E^n g(\sigma_t^n(x_1^n))-E^n\langle g,\bar{\pi}_t^n(x_1)\rangle| &\leq |E^n g(\sigma_o^n(x_1^n))-E^n\langle g,\bar{\pi}_o^n(x_1)\rangle| \\ &+ \int_0^t (2\|g\|_\infty \sup_{y\in C_{\varepsilon_n}(x_1)} \|A^n(.,x_1^n,\dots)-A^n(.,y,\dots)\|_\infty \\ &+ \|A^n\|_\infty |E^n g(\sigma_s^n(x_1^n))-E^n\langle g,\bar{\pi}_s^n(x_1)\rangle|)ds. \end{aligned} \tag{4.15}$$

Hence, Gronwall's lemma together with the assumptions on the initial distributions and Theorem 3 implies

$$\lim_n E^n g(\sigma_t^n(x_1^n)) = \lim_n E^n \langle g, \bar{\pi}_t^n(x_1) \rangle = \int g(m) h_t(m, x_1) \rho(dm). \tag{4.16}$$

For the general case, we take $g_1, \ldots, g_K \in \mathscr{C}(B)$ and n so large that $C_{\varepsilon_n}(x_1), \ldots, C_{\varepsilon_n}(x_K)$ are all disjoint. Similar to (4.15), we get

$$\begin{aligned}
&|E^n \prod_{j=1}^K g_j(\sigma_t^n(x_j^n)) - E^n \prod_{j=1}^K \langle g_j, \bar{\pi}_t^n(x_j) \rangle| \\
&\leq |E^n \prod_{j=1}^K g_j(\sigma_o^n(x_j^n)) - E^n \prod_{j=1}^K \langle g_j, \bar{\pi}_o^n(x_j) \rangle| \\
&\quad + \sum_{j=1}^K \int_0^t \Big[2 \prod_{i=1}^K |g_i|_\infty \sup_{y \in C_{\varepsilon_n}(x_j)} |A^n(.,x_j,\ldots) - A^n(.,y,\ldots)|_\infty \\
&\qquad + |g_j|_\infty |A^n|_\infty |E^n \prod_{i \neq j} g_i(\sigma_s^n(x_i^n)) - E^n \prod_{i \neq j} \langle g_i, \bar{\pi}_s^n(x_i) \rangle| \\
&\qquad + |A^n|_\infty |E^n \prod_i g_i(\sigma_s^n(x_i^n)) - E^n \prod_i \langle g_i, \bar{\pi}_s^n(x_i) \rangle| \Big] ds.
\end{aligned} \tag{4.17}$$

By an induction hypothesis, the second integrand goes to zero uniformly in s, and we conclude by the same argument with Gronwall's lemma as above, that

$$\begin{aligned}
\lim_n E^n \prod_{j=1}^K g_j(\sigma_t^n(x_j^n)) &= \lim_n E^n \prod_{j=1}^K \langle g_j, \bar{\pi}_t^n(x_j) \rangle \\
&= \prod_{j=1}^K \int g_j(m_j) h_t(m_j, x_j) \rho(dm_j),
\end{aligned} \tag{4.18}$$

which proves the Theorem.

5. NON-CRITICAL FLUCTUATIONS.

For $r\geq 0$, we introduce the Sobolev space

$$H_r = \{g\in L^2(\lambda)\ ;\ \|g\|_r < +\infty\}, \tag{5.1}$$

where
$$\|g\|_r^2 = \sum_{p\in\mathbb{Z}^d} (1+|p|^2)^r\ |\hat{g}(p)|^2 \tag{5.2}$$

with the Fourier coefficients

$$\hat{g}(p) = \langle\ \exp(2\pi ip.),\ g.\lambda\ \rangle\ ,\ p\in\mathbb{Z}^d. \tag{5.3}$$

Let

$$H_{-r} = H_r' \tag{5.4}$$

be the dual space of H_r under the duality product $\langle.\rangle$, with the norm

$$\|\mu\|_{-r}^2 = \sum (1+|p|^2)^{-r}\ |\hat{\mu}(p)|^2. \tag{5.5}$$

For each $r\in\mathbb{R}$, we have the scalar product on $\underline{H_r}$, given by

$$\langle\ \mu,\nu\ \rangle_r = \sum_{p\in\mathbb{Z}} (1+|p|^2)^r \hat{\mu}(p)\ \hat{\nu}(p) \in \mathbb{R}, \tag{5.6}$$

which makes $(H_r, \langle,\rangle_r)$ a Hilbert space.

Obviously, for $r_2\geq r_1\geq 0$,

$$\mathcal{C}^\infty(\mathbb{T})=H_\infty = \cap_{r'} H_{r'} \subseteq H_{r_2} \subseteq H_{r_1} \subseteq H_o = L^2 \subseteq H_{-r_1} \subseteq H_{-r_2} \subseteq H_{-\infty} = \mathcal{C}^\infty(\mathbb{T})', \tag{5.7}$$

and the embedding $H_r \subseteq H_s$ for any $r\geq s$ is Hilbert-Schmidt, whenever $r-s > d/2$, due to the fact that

$$C_{-r} = \sum_{p\in\mathbb{Z}^d} (1+|p|^2)^{-r} < \infty \quad \text{if and only if} \quad r>d/2. \tag{5.8}$$

In particular,

$$\|\delta_x\|_{-r}^2 = \sum_p (1+|p|^2)^{-r} = C_{-r},$$

so that
$$\mathcal{M}_b^n \subseteq \{\mu\in H_{-r},\ \|\mu\|_{-r} \leq C_{-r}^{1/2}\, b\} \quad \text{for} \quad r>d/2, \tag{5.9}$$

and

$$\|\lambda-\lambda^n\|_{-r}^2 = \sum_{p\in\mathbb{Z}^d}(1+|p|^2)^{-r}\ |N^{-1} \sum_{x\in\mathbb{T}_n} \exp(2\pi ipx)-\delta_o(p)|^2$$

$$= \sum_{p\in(n\mathbb{Z})^d\setminus\{0\}} (1+|p|^2)^{-r}$$

$$\leq C'_{-r}\ n^{-2r}, \tag{5.10}$$

for some constant C'_{-r}.

Let $\Omega_{-r} = \mathcal{D}([0,\infty), H_{-r})$. For $r>d/2$, the H_{-r}-valued Brownian motion W_t with covariance

$$E(\langle g_1, W_{t_1}\rangle . \langle g_2, W_{t_2}\rangle) = (t_1 \wedge t_2).\langle g_1, g_2\rangle, \qquad (5.11)$$

$g_1, g_2 \in H_r$, is well-defined on Ω_{-r} (cf. [14] ch. 3, th. 3.1).

We shall use the following <u>tightness criterion</u> on Ω_{-r}, $r>d/2$: a sequence of processes ζ^n_t with laws P^n on $\mathcal{D}([0,T], H_{-r})$ is tight, if

(i) for each $\varepsilon>0$ we find $K>0$ such that

$$\sup_n P^n\{\sup_{s\leq T} |\zeta^n_s|^2_{-r} \geq K\} \leq \varepsilon, \qquad (5.12)$$

and

(ii) for all $g\in H_r$, $\varepsilon>0$, $\eta>0$ there exists $\delta>0$ such that

$$\sup_n \sup_{0\leq\tau_1\leq\tau_2\leq(\tau_1+\delta)\wedge T} P^n\{|\langle g,\xi^n_{\tau_2}\rangle - \langle g,\xi^n_{\tau_1}\rangle| > \eta\} \leq \varepsilon. \qquad (5.13)$$

These conditions are an immediate consequence of Mitoma's result (see [22] Theorem 4.1 and Remark 1, and notice that (5.13) implies the uniform r-continuity of P^n).

Finally, we strengthen the assumptions made in (2.9-10) :

(i) There exists $r_o>d/2$ and a map dG from $L^\infty_b \cap H_{r_o}$ into the space of continuous linear operators on H_{-r_o} such that for $g\in H_{r_o}$, $u\in L^\infty_b \cap H_{r_o}$, $\mu\in H_{-r_o}$

$$|\langle g,G(\mu)\rangle - \langle g,G(u\lambda)\rangle - \langle g,dG(u)(\mu-u\lambda)\rangle| = |g|_{r_o}.o(|\mu-u.\lambda|_{-r_o}) \quad (5.14)$$

and

$$\sup_{u\in L^\infty_b \cap H_{r_o}} \sup_{\mu\in H_{-r_o}} |dG(u)\mu|_{-r_o} / |\mu|_{-r_o} < +\infty . \qquad (5.15)$$

Here, $G(\mu)$ is the natural extension of G from (3.5) to a H_{-r_o}-valued function by

$$G(\mu)(dx) = \exp(G_o(x,\mu)+\gamma(G_1(x,\mu)))[\gamma'(G_1(x,\mu))\lambda(dx)-\mu(dx)]. \quad (5.16)$$

(ii) G_0 and G_1 are bounded continuous functions from

$$\mathcal{M}_b \cap \{\mu\in H_{-r_0} : |\mu|_{-r_0} \leq bC_{-r_0}\} \text{ into } H_{r_0},$$

and G_2^n satisfies

$$\sup_{m\in B,\mu\in\mathcal{M}_b^n} |G_2^n(m,.,\mu)|_{r_0} = o(N^{-1/2}). \tag{5.17}$$

Notice that by the interpolation theorem and Sobolev's Theorem for $r_0>d/2$

$$|g.h|_{r_0} \leq C'_{r_0}(|g|_\infty|h|_{r_0}+|g|_{r_0}|h|_\infty) \leq C''_{r_0}|g|_{r_0}|h|_{r_0} \tag{5.18}$$

so that H_{r_0} is an algebra (cf. [26] II, 2.1).

In (5.18), we used that for $r_0>d/2$

$$|g|_\infty \leq \sum_p |\hat{g}(p)| \leq C_{r_0}^{1/2} |g|_{r_0}, \tag{5.19}$$

which also shows that $H_{r_0} \subseteq \mathcal{C}(\mathbb{T})$.

Moreover, for any $\mathcal{C}^\infty$-function on φ on R, $\varphi(g)\in H_{r_0}$ for $g\in H_{r_0}$ with

$$|\varphi(g)|_{r_0} \leq \psi(|g|_{r_0}) \tag{5.20}$$

for some continuous function ψ. In particular,

$$|\varphi(g_1)-\varphi(g_2)|_{r_0} \leq C_M|g_1-g_2|_{r_0} \tag{5.21}$$

for all g_1, g_2 with $|g_1|_{r_0}$, $|g_2|_{r_0} \leq M$.

As a consequence of (5.17-21), we get for our functions G and F from (3.4) and (3.6) that for $u\in L_b^\infty\cap H_{r_0}$, $v\in L^\infty_{[0,b^2]}\cap H_{r_0}$,

$$|G(u)|_{r_0} \leq C(1 + |u|_{r_0}), \tag{5.22}$$

$$|F(u,v)|_{r_0} \leq C(1 + |v|_{r_0}). \tag{5.23}$$

Therefore, if $u_o \in L_b^\infty \cap H_{r_o}$, $v_o \in L^\infty_{[0,b^2]} \cap H_{r_o}$ then the solutions u_t of (3.3) and v_t of (3.5) satisfy

$$u_t \in L_b^\infty \cap H_{r_o}, \quad v_t \in L^\infty_{[0,b^2]} \cap H_{r_o} \tag{5.24}$$

for all $t \in \mathbb{R}^+$.

Now, we are ready to study the asymptotics of the non-critical fluctuation processes

$$\zeta_t^n = N^{1/2}(\sigma_t^n - u_t) \in H_{-r_o}. \tag{5.25}$$

THEOREM 4.

<u>We assume</u> (5.14-17), $\sigma_o^n \in \mathcal{M}_b$, $u_o \in L_b^\infty \cap H_{r_o}$, $v_o \in L^\infty_{[0,b^2]} \cap H_{r_o}$ <u>and that</u> $\zeta_o^n = N^{1/2}(\sigma_o^n - u_o)$ <u>converge in law to</u> $\zeta_o \in H_{-r_o}$ <u>with</u>

$$\sup_n E^n \|\zeta_o^n\|^2_{-r_o} < \infty. \tag{5.26}$$

<u>Then on</u> $\mathcal{D}([0,T], H_{-r_o})$, <u>the fluctuation processes</u> ζ_t^n (5.25) <u>converge in law to the process</u> ζ_t <u>satisfying</u>

$$d\zeta_t = dG(u_t)\zeta_t\,dt + (B(u_t,v_t))^{1/2}.dW_t \tag{5.27}$$

<u>with the</u> H_{-r_o}<u>-valued Brownian motion</u> W_t <u>from</u> (5.11) <u>and</u>

$$B(u,v)(x) = \exp\{G_o(x,u) + \gamma(G_1(x,u))\}$$
$$[\gamma''(G_1(x,u)) + (\gamma')^2(G_1(x,u)) - 2u(x).\gamma'(G_1(x,u)) + v(x)]. \tag{5.28}$$

<u>Proof</u> : We first notice that (5.27) implies

$$d\langle g,\zeta_t\rangle = \langle (dG(u_t))^* g,\zeta_t\rangle dt + \langle g.B(u_t,v_t)^{1/2}, dW_t\rangle. \tag{5.29}$$

Therefore, if $T^*_{s,t}$ is the semigroup on H_{r_o} with generator $dG(u_t)^*$, the adjoint of $dG(u_t)$, we have for $0 \leqslant s \leqslant t \leqslant T$:

$$P(\langle g,\zeta_t\rangle \in dy \mid \zeta_s) = p\left(\int_s^t \langle T^*_{\tau,t} g, B(u_\tau,v_\tau).T^*_{\tau,t} g\rangle d\tau,\ y - \langle T^*_{s,t} g,\zeta_s\rangle\right)dy, \tag{5.30}$$

where $p(t,y) = (2\pi t)^{-1/2} \exp(-y^2/2t)$ is the heat kernel.

This shows that the process ζ_s is uniquely determined by the following martingale problem : for $f \in \mathcal{C}_b^2(\mathbb{R}^k)$ and $g_i \in H_{r_0}$ for $i = 1,\dots,k$, $k \in \mathbb{N}$, we have with $\tilde{f}(\zeta) = f(\langle g_1,\zeta\rangle,\dots,\langle g_k,\zeta\rangle)$ that

$$\tilde{f}(\zeta_t) - \int_0^t L(u_s,v_s)\tilde{f}(\zeta_s)ds \quad \text{is a P-martingale,} \tag{5.31}$$

where

$$L(u,v)\tilde{f}(\zeta) = \sum_{i=1}^k \widetilde{\partial_i f(\zeta)} \langle g_i, dG(u)\zeta\rangle + 1/2 \sum_{i,j}^k \widetilde{\partial_i \partial_j f(\zeta)} \langle g_i, B(u,v) g_j\rangle \tag{5.32}$$

(cf. [15] Theorem 1.4).

From (2.12-13) and (3.4), we obtain the martingale decomposition

$$\langle g,\zeta_t\rangle = \langle g,\zeta_o\rangle + \int_0^t N^{1/2} \int_{\mathbb{T}} g(x) \left[\int_B A^n(m,x,\sigma_s^n)(m-\sigma_s^n(x))\rho(dm)\lambda^n(dx) - G(u_s)(dx)\right]ds$$

$$+ N^{-1/2} \int_0^t \int_{B\times\mathbb{T}} g(x)(m-\sigma_{s^-}^n(x))\bar{\Lambda}^n(dm,dx,ds). \tag{5.33}$$

Set

$$\tau_M^n = \inf(t\ ;\ |\zeta_t^n|_{-r_o}^2 \geq M^2). \tag{5.34}$$

Then, for $t_1 < t_2 \leq \tau_M^n$, Ito's formula yields

$$|\zeta_{t_2}^n|_{-r_o}^2 = |\zeta_{t_1}^n|_{-r_o}^2$$

$$+ 2N^{1/2} \int_{t_1}^{t_2} \left\langle \zeta_s^n, \int_B [A^n(m,.,\sigma_s^n)(m\lambda^n(.) - \sigma_s^n(.))\rho(dm) - G(u_s)\right\rangle_{-r_o} ds$$

$$+ \int_{t_1}^{t_2} \int_{B\times\mathbb{T}} \|(m-\sigma_s^n(x))\delta_x\|_{-r_o}^2 A^n(m,x,\sigma_s^n)\rho(dm)\lambda^n(dx)ds$$

$$+ \int_{t_1}^{t_2} \int_{B\times\mathbb{T}} \left(|\zeta_{s^-}^n + N^{-1/2}(m-\sigma_{s^-}^n(x))\delta_x|_{-r_o}^2 - |\zeta_{s^-}^n|_{-r_o}^2\right)\bar{\Lambda}^n(dm,dx,ds). \tag{5.35}$$

The second term of the right-hand side of (5.35) gives

$$|\text{second term}| = \Big| 2N^{1/2}\int_{t_1}^{t_2} \langle \zeta^n_s, G(\sigma^n_s)-G(u_s)\rangle_{-r_o}\, ds$$

$$+ 2N^{1/2}\int_{t_1}^{t_2} \langle\langle \zeta^n_s, \int_B \big[A^n(m,.,\sigma^n_s)(m\lambda^n(.)-\sigma^n(.))-A(m,.,\sigma^n_s)(m\lambda(.)-\sigma^n(\cdot))\big]d\rho\rangle_{-r_o}\, ds\Big|$$

$$\underset{(5.14-17-21)}{\leq} 2\int_{t_1}^{t_2} |\langle \zeta^n_s, dG(u_s)\zeta^n_s\rangle_{-r_o}| + o(|\zeta^n_s|^2_{-r_o}+1)ds$$

$$+ N^{1/2}|\lambda^n-\lambda|_{-r_o}\, 2\int_{t_1}^{t_2} |\zeta^n_s|_{-r_o}\, |\int_B A^n(m,.,\sigma^n_s)m\ \rho(dm)|_{r_o}\, ds$$

$$\underset{(5.10-15-17-21)}{\leq} C_1/2 \int_{t_1}^{t_2} (|\zeta^n_s|^2_{-r_o} + 1)ds. \qquad (5.36)$$

Also the integrand of the third term of (5.35) is bounded, by $C_1/2$ say. Therefore

$$|\zeta^n_{t\wedge\tau^n_M}|^2_{-r_o} \pm C_1\int_0^{t\wedge\tau^n_M} (|\zeta^n_s|^2_{-r_o} + 1)ds$$

is a submartingale, respectively a supermartingale. Taking expectations of this supermartingale and using Gromwall's lemma together with (5.26), we get

$$E^n(|\zeta^n_t|^2_{-r_o}) = \lim_{M\to\infty} E^n(|\zeta^n_{t\wedge\tau^n_M}|^2_{-r_o}) \leq C_2 e^{C_1 t} \qquad (5.37)$$

for all $t>T$ and $n>n_o$, where n_o depends only on the values in (5.9) and (5.13-20). By Doob's submartingale inequality, we get

$$P^n(\sup_{t\leq T}|\zeta^n_t|^2_{-r_o} > K) \leq P^n(\sup_{t\leq T}(|\zeta^n_t|^2_{-r_o} + C_1\int_0^t(|\zeta^n_s|^2_{-r_o}+1)ds) > K)$$

$$\leq K^{-1}[E^n(|\zeta^n_T|^2_{-r_o}) + C_1\int_0^T(E(|\zeta^n_s|^2_{-r_o})+1)ds]$$

$$\leq K^{-1}[2C_2 e^{C_1 T} + C_1 T] < \varepsilon \qquad (5.38)$$

for K large enough. This shows (5.12).

Next, let $\tau_1 \leq \tau_2 \leq (\tau_1+\delta)\wedge T$ be stopping times and $g\in H_{r_o}$.

Applying similar inequalities like in (5.36) to (5.33), we find

$$\langle g,\zeta^n_{\tau_2}-\zeta^n_{\tau_1}\rangle = \int_{\tau_1}^{\tau_2}(\langle g,dG(u_s)\zeta^n_s\rangle + |g|_{r_0}\, o(|\zeta^n_s|_{-r_0}+1)+o(1))ds$$
$$+ N^{-1/2}\int_{\tau_1}^{\tau_2} g(x)(m-\sigma^n_{s-}(x))\bar{\Lambda}^n(dm,dx,ds). \qquad (5.39)$$

$$E^n(\langle g,\zeta^n_{\tau_2}-\zeta^n_{\tau_1}\rangle^2)\leq 6E^n[(\tau_2-\tau_1)\int_{\tau_1}^{\tau_2}(C_3|g|^2_{r_0}|\zeta^n_s|^2_{-r_0}+|g|^2_{r_0}\, o(|\zeta^n_s|^2_{-r_0}+1)+o(1))ds]$$
$$+2E^n[\int_{\tau_1}^{\tau_2} g^2(x)(m-\sigma^n_s(x))^2 A^n(m,x,\sigma^n_s)\rho(dm)\lambda^n(dx)ds]$$
$$\leq C_4|g|^2_{r_0}(e^{C_1T}+1)\,\delta^2 + C_5|g|^2_{L^\infty} b^2\delta, \qquad (5.40)$$

which implies for δ sufficiently small

$$P^n(|\langle g,\zeta^n_{\tau_2}-\zeta^n_{\tau_1}\rangle| > \eta) \leq \eta^{-2}E^n(\langle g,\zeta^n_{\tau_2}-\zeta^n_{\tau_1}\rangle^2) \leq \varepsilon. \qquad (5.41)$$

This shows the tightness of the fluctuation processes $\zeta^{n\cdot}_t$.

In order to characterize the limit process of ζ^n_t by the martingale problem (5.30), we apply Ito's formula to

$\tilde{f}(\zeta^n_t)=f(\langle g_1,\zeta^n_t\rangle,\ldots,\langle g_k,\zeta^n_t\rangle)$, $f\in\mathcal{C}^2_b(R^k)$, $g_i\in H_{r_0}$ for $i = 1,\ldots,k$.

We write M^n_t for $M^n_t(f(\ldots,N^{1/2}\langle g_i,\sigma-u\rangle,\ldots))$ form (2.13) and use estimates similar to (5.36).

$$\bar f(\zeta^n_t) = \bar f(\zeta^n_0)$$

$$+\int_0^t \sum_{i=1}^k \Big\{ \widetilde{\partial_i f}(\zeta^n_s) \int_{\mathbb{T}} g_i(x) N^{1/2} \Big[\int_B A^n(m,x,\sigma^n_s)(m-\sigma^n_s(x))\rho(dm)\lambda^n(dx) - G(u_s)\lambda(dx) \Big]$$

$$+ \int_{B\times\mathbb{T}} [f(.,\langle g_i,\zeta^n_s\rangle + N^{-1/2} g_i(x)(m-\sigma^n_s(x)),.) - \bar f(\zeta^n_s) - \widetilde{\partial_i f}(\zeta^n_s) N^{-1/2} g_i(x)(m-\sigma^n_s(x))$$

$$N A^n(m,x,\sigma^n_s)\rho(dm)\lambda^n(dx)\Big\} ds + M^n_t$$

$$= \bar f(\zeta^n_0) + \int_0^t \sum_{i=1}^k \widetilde{\partial_i f}(\zeta^n_s) \langle g_i, dG(u_s)\, \zeta^n_s \rangle ds$$

$$+ 1/2 \int_0^t \sum_{ij} \widetilde{\partial_i \partial_j f}(\zeta^n_s) \int_{B\times\mathbb{T}} g_i(x) g_j(x)(m-\sigma^n_s(x))^2 A^n(m,x,\sigma^n_s)\rho(dm)\lambda^n(dx)ds + M^n_t$$

$$+ \int_0^t [o(|\zeta^n_s|_{-r_0} + 1) + O(N^{1/2}|\lambda^n - \lambda|_{-r_0})] ds$$

$$= \bar f(\zeta^n_0) + \int_0^t \sum_{i=1}^k \widetilde{\partial_i f}(\zeta^n_s) \langle g_i, dG(u_s)\zeta^n_s \rangle ds$$

$$+ 1/2 \int_0^t \sum_{ij=1}^k \widetilde{\partial_i \partial_j f}(\zeta^n_s) \int_{\mathbb{T}} g_i(x) g_j(x) B(\sigma^n_s, (\sigma^n_s)^2)\lambda^n(dx)ds + M^n_t$$

$$+ \int_0^t [o(N^{-1/2} + |\zeta^n_s|_{-r_0} + 1) + O(N^{-r/d+1/2})] ds$$

$$= \bar f(\zeta^n_0) + \int_0^t L(u_s, v_s)\bar f(\zeta^n_s) ds + M^n_t$$

$$+ \int_0^t [o(N^{1/2} + |\zeta^n_s|_{-r_0} + 1) + O(N^{-r/d+1/2}) + C(\sum_{i=0,1} |G_i(\sigma^n_s) - G_i(u_s)|_{r_0}$$

$$+ \sum_{i,j=1}^k |\langle g_i g_j \exp(G_0(u_s) + \gamma(G_1(u_s))), v_s \lambda - (\sigma^n_s)^2 \rangle + |\sigma^n_s - u_s\lambda|_{-r_0})] ds. \quad (5.42)$$

The last integral vanishes in the limit $n\to\infty$, and so any limit process ζ_t of ζ^n_t satisfies the martingale problem (5.31), which has the unique solution (5.27). This completes the proof of Theorem 4. ∎

We review the example of Section 2 in the light of the last theorem. Let

$$G_0(x,\sigma) = f_0(g_1 * \sigma(x), \ldots, g_k * \sigma(x)) \quad (5.43)$$

with $f_o \in \mathcal{C}^{[r_o]+1}(\mathbb{R}^k)$, $g_i \in H_{r_o}$. We also assume that in (2.16) $\vartheta_j \in H_{r_o}(\mathbb{T}^j)$, $j = 1,\dots,q$. Then both G_o and G_1 are continuous bounded functions from $\mathcal{M}_b \cap \{\mu \in H_{-r_o}, |\mu|_{-r_o} \leq bC_{-r_o}\}$ into H_{r_o}.

Then G_2^n, given in (2.19), satisfies

$$\sup_{m \in B, \mu \in \mathcal{M}_b^n} |G_2^n(m,\cdot,\mu)|_{r_o} = \mathcal{O}(N^{-1}). \qquad (5.44)$$

Thus Theorem 4 applies to our example.

6. CRITICAL FLUCTUATIONS AT THE FERROMAGNETIC PHASE TRANSITION.

Here, we consider the special case of a translation invariant, two-body interaction without external field. In the context of our example of Section 2, this means

$$q = 2, \qquad \mathfrak{J}_1 = 0, \quad \mathfrak{J}_2(x,y) = \mathfrak{J}'(x-y), \tag{6.1}$$
$$\nabla H(\sigma)(x) = -\mathfrak{J}*\sigma(x) = -\int_{\mathbb{T}} \mathfrak{J}(x-y)\,\sigma(dy)$$

with the symmetrization $\mathfrak{J}(x) = (\mathfrak{J}'(x) + \mathfrak{J}'(-x))/2$.

We know that if ρ is symmetric and satisfies the GHS-inequality (see below), if (5.1) holds, and

$$\hat{\mathfrak{J}}(0) - \hat{\mathfrak{J}}(p) \geq \delta_0 > 0 \quad \text{for all} \quad p \in \mathbb{Z}^d \setminus \{0\}, \tag{6.2}$$

then the Gibbs states to the Hamiltonian (2.15) have a second order phase transition at the critical inverse temperature

$$\beta_0 = (\gamma''(0)\,\hat{\mathfrak{J}}(0))^{-1}. \tag{6.3}$$

This is the first phase transition as the temperature decreases from the high-temperature region. The new phase, which appears immediately below the critical temperature β_0^{-1}, is ferromagnetic, i.e. it has constant non-zero magnetization. In order to study critical fluctuations of the dynamical model, we make the following assumptions :

(A1) Let ρ be a symmetric measure on $\mathbb{R}$ with support contained in $[-b,b]$, $b>0$, and let ρ satisfy the GHS-condition :

$$\gamma^{(3)}(x) \leq 0 \quad \text{for} \quad x \in [0,\infty). \tag{6.4}$$

Since γ is convex and symmetric with $\gamma''(0) > 0$, (6.4) implies that there exists $K_0 \geq 2$, such that

$$\gamma(0)=\gamma'(0)=0,\ \gamma''(0)>0,\ \gamma^{(3)}(0)=\ldots=\gamma^{(2K_0-1)}(0)=0,\ \gamma^{(2K_0)}(0)<0. \tag{6.5}$$

(A2) Assume

$$G_0 = 0, \quad G_1(x,\sigma) = -\beta_0 \nabla H(\sigma)(x) = \beta_0\, \mathfrak{J}*\sigma(x) \tag{6.6}$$

with β_0 from (6.3) and $\mathfrak{J}$ satisfying (6.2).

Moreover, we require

$$\partial \in H_{2r_o} \quad \text{for some} \quad r_o > d(1-1/K_o) \geq d/2, \tag{6.7}$$

which yields by (5.19)

$$|\partial * \sigma|_\infty \leq C_{r_o} |\partial * \sigma|_{r_o} \leq C_{r_o} [\sum_p (1+|p|^2)^{2r_o} |\hat{\partial}(p)|^2 . (1+|p|^2)^{-r_o} |\hat{\sigma}(p)|^2]^{1/2}$$

$$\leq C_{r_o} |\partial|_{2r_o} |\sigma|_{-r_o} \quad , \tag{6.8}$$

such that G_1 is a continuous bounded function from

$$\mathcal{M}_b \cap \{\mu \in H_{-r_o}, |\mu|_{-r_o} \leq bC_{-r_o}\} \quad \text{into} \quad H_{r_o},$$

and

$$\sup_{m \in B, \mu \in \mathcal{M}_b^n} |G_2^n(m,.,\mu)|_{r_o} = o(N^{-(1-1/K_o)}), \tag{6.9}$$

which is satisfied if we define G_2^n by (2.19) (see 5.41)).

The critical fluctuation process is defined by

$$\xi_t^n = N^{1/2K_o} \sigma^n_{tN^{1-1/K_o}} \in H_{-r_o} \cap \mathcal{M}^n. \tag{6.10}$$

We split ξ_t^n into its ferromagnetic and non-ferromagnetic components :

$$\theta_t^n = \hat{\xi}_t^n(0)\, \lambda, \quad \eta_t^n = \xi_t^n - \hat{\xi}_t^n(0)\, \lambda^n. \tag{6.11}$$

θ_t^n and η_t^n are orthogonal in $(H_{-r_o}, \langle . \rangle_{-r_o})$. Notice that $\sigma^n \in \mathcal{M}^n$ implies

$$\hat{\sigma}^n(p) = \hat{\sigma}^n(p+nq) \quad \text{for all} \quad p, q \in \mathbb{Z}^d, \tag{6.12}$$

and

$$\widehat{\partial * \sigma^n \lambda^n}(p) = \widehat{\partial \lambda}^n(p)\, \hat{\sigma}^n(p), \quad p \in \mathbb{Z}^d . \tag{6.13}$$

THEOREM 5.

Besides assumptions (A1) and (A2), we suppose for the starting configurations ξ_o^n that

(i) $\qquad \sigma_o^n \in \mathcal{M}_b^n$, θ_o^n converges in law to some $\theta_o \lambda$; $\qquad$ (6.14)

(ii) $\qquad E^n |\eta_o^n|_{-r_o}^{2\kappa} \leq C_2\, \alpha_n^{-\kappa}$ for all large n, $\qquad$ (6.15)

where $\kappa > \kappa' > K_o - 1$ and α_n an increasing (to infinity) sequence with

$$N^{(1-2/K_o)+(1-1/K_o)/\kappa'}\alpha_n^{-1} \longrightarrow 0, \text{ and } \alpha_n N^{-(1-1/K_o)} \longrightarrow 0\ ; \tag{6.16}$$

(iii) $$E^n(|(\sigma_o^n(.))^2 - \gamma''(0))\lambda^n|_{-r_o}^{2\bar{\kappa}}) \leq C_3 \bar{\alpha}_n^{\bar{\kappa}} \tag{6.17}$$

<u>where</u> $\bar{\kappa} > 1$ <u>and</u> $\bar{\alpha}_n$ <u>a sequence with</u>

$$N^{(1-1/K_o)/\bar{\kappa}}\bar{\alpha}_n^{-1} \longrightarrow 0, \quad \bar{\alpha}_n N^{1-1/K_o} \longrightarrow 0. \tag{6.18}$$

<u>Then the critical fluctuation process</u> ξ_t^n <u>converges in law on</u> $\mathcal{D}([0,T], H_{-2r_o})$ <u>to the ferromagnetic process</u> $\xi_t = \hat{\theta}_t(0)\lambda$, <u>where</u> $\hat{\theta}_t(0) \in R$ <u>is given by</u>

$$d\hat{\theta}_t(0)=\gamma^{(2K_o)}(0)/[(2K_o-1)!\gamma''(0)^{2K_o-1}]\,\hat{\theta}_t^{2K_o-1}(0)dt+(2\gamma''(0))^{1/2}dw_t, \tag{6.19}$$

<u>starting at</u> θ_o, <u>and where</u> w_t <u>is the standard Brownian motion</u>.

<u>Proof</u> : We start with the semimartingale decomposition of $\langle g,\xi_t^n\rangle$ with $g\in H_{r_o}$:

$$\langle g,\xi_t^n\rangle = \langle g,\theta_t^n\rangle+\langle g,\eta_t^n\rangle$$

$$= \langle g,\xi_o^n\rangle+\int_0^t N^{1-1/2K_o}\int_{B\times\mathbb{T}} g(x)A^n(m,x,\sigma^n_{sN^{1-1/K_o}})(m-\sigma^n_{sN^{1-1/K_o}}(x))\rho(dm)\lambda^n(dx)ds$$

$$+N^{-(1-1/2K_o)}\int_0^{tN^{1-1/K_o}}\int_{B\times\mathbb{T}} g(x)(m-\sigma^n_{s-}(x))\tilde{\Lambda}^n(dm,dx,ds) \tag{6.20}$$

By Ito's formula, we get for $t_1<t_2<\tau_M^n=\inf\{t,\ |\xi_t^n|_{-r_o}>M\}$

$$
\begin{aligned}
|\eta^n_{t_2}|^2_{-r_0} &= |\eta^n_{t_1}|^2_{-r_0} \\
&+2\int_{t_1}^{t_2} [\langle \eta^n_s, N^{1-1/2K_0} \exp\gamma(\beta_0 \partial*\sigma^n_{sN^{1-1/K_0}}(.))(\gamma'(\beta_0 \partial*\sigma^n_{sN^{1-1/K_0}}(.))\lambda^n - \sigma^n_{sN^{1-1/K_0}})\rangle_{-r} \\
&\qquad + o(1)|\eta^n_s|_{-r_0} |\xi^n_s|_{-r_0}]ds \\
&+\int_{t_1}^{t_2}\int_{B\times\mathbb{T}} |(m-\sigma^n_{sN^{1-1/K_0}}(x))(\delta_x-\lambda^n)|^2_{-r_0} A^n(m,x,\sigma^n_{sN^{1-1/K_0}})\rho(dm)\lambda^n(dx)ds \\
&+ M^n_{t_1,t_2} \qquad (6.21)
\end{aligned}
$$

where

$$
\begin{aligned}
M^n_{t_1,t_2} &= \int_{t_1 N^{1-1/K_0}}^{t_2 N^{1-1/K_0}} \int_{B\times\mathbb{T}} \Big[|\eta^n_{sN_-^{-(1-1/K_0)}} + N^{-(1-1/2K_0)} (m-\sigma^n_{s-}(x))(\delta_x-\lambda^n)|^2_{-r_0} \\
&\qquad - |\eta^n_{sN_-^{-(1-1/K_0)}}|^2_{-r_0}\Big] \tilde{\Lambda}^n(dm,dx,ds). \qquad (6.22)
\end{aligned}
$$

We estimate the first integral of the right-hand side of (6.21), using (6.8) and

$$\gamma'(z) = \gamma''(0)z + \mathcal{O}(z^{2K_0-1}),$$

first integral of (6.21)

$$
\begin{aligned}
&\underset{(5.18\text{-}20)(6.8)}{=} 2\int_{t_1}^{t_2}\Big[N^{1-1/K_0}(\langle\eta^n_s, \gamma''(0)\beta_0\partial*\eta^n_s\lambda^n - \eta^n_s\rangle_{-r_0} + \mathcal{O}(|\eta^n_s|^2_{-r_0}|\xi^n_s|^2_{-r_0} N^{-1/K_0})) \\
&\qquad + \mathcal{O}(|\eta^n_s|_{-r_0}|\xi^n_s|^{2K_0-1}_{-r_0}) + o(1)|\eta^n_s|_{-r_0}|\xi^n_s|_{-r_0}\Big]ds \\
&= 2\int_{t_1}^{t_2}\Big[N^{1-1/K_0}\Big(\sum_{p\notin(n\mathbb{Z})^d}(1+|p|^2)^{-r_0}|\hat{\eta}^n_s(p)|^2(\gamma''(0)\beta_0\widehat{\partial\lambda^n}(p)-1) \\
&\qquad + \mathcal{O}(N^{-1/K_0}|\eta^n_s|^2_{-r_0}|\xi^n_s|^2_{-r_0})\Big) \\
&\qquad + \mathcal{O}(|\eta^n_s|_{-r_0}|\xi^n_s|^{2K_0-1}_{-r_0}) + o(1)|\eta^n_s|_{-r_0}|\xi^n_s|_{-r_0}\Big]ds \qquad (6.23)
\end{aligned}
$$

since $\widehat{\eta^n_s}(p) = 0$ for all $p \in (n\mathbb{Z})^d$.

By $\widehat{\partial\lambda^n}(p) = \sum_{q\in\mathbb{Z}^d} \hat{\partial}(p+nq)$, we get by (5.19) and (6.2)

$$\widehat{\partial\lambda^n}(p) \leq \hat{\partial}(0) - \delta_o/2, \text{ for all } p \notin n\mathbb{Z}^d \text{ and } n \geq n_o,$$

which implies by (6.3)

$$\gamma''(0)\beta_o \widehat{\partial\lambda^n}(p) - 1 \leq - 2/3\ \gamma''(0)\beta_o\delta_o,\ p \notin n\mathbb{Z}^d,\ n \geq n_o.$$

Therefore, assuming $\Theta(N^{-1/K_o} M^2) \leq 1/3\ \gamma''(0)\beta_o\delta_o$ for n large, first integral of (6.21)

$$\leq \int_{t_1}^{t_2} (-N^{1-1/K_o} \gamma''(0)\beta_o\delta_o |\eta^n_s|^2_{-r_o} + C_1(M)\ |\eta^n_s|_{-r_o})ds \tag{6.24}$$

for $n \geq n_o(M)$. The second integral of (6.21) is bounded by $C_2(t_2-t_1)$. With $C_3(M) = C_1(M).M+C_2$, we have for $t_1<t_2\leq\tau^n_M$, $n \geq n_o(M)$,

$$|\eta^n_{t_2}|^2_{-r_o} \leq |\eta^n_{t_1}|^2_{-r_o} - \int_{t_1}^{t_2} (N^{1-1/K_o}\delta_o|\eta^n_s|^2_{-r_o} - C_3(M))ds + M^n_{t_1,t_2}. \tag{6.25}$$

The drift term in the last member is strongly attractive to zero. To (6.25), we apply the proposition on collapsing processes, given in the appendix, with $m = N^{1-1/K_o}$. (6.16) and (6.15) imply (A.2) and (A.3) and since here $Y = B \times \mathbb{T}$ and

$$f^n_t(m,x) = 2N^{-(1-1/2K_o)} \langle \eta^n_t, (m-\sigma^n_{tN^{(1-1/K_o)}}(x))(\delta_x-\lambda^n)\rangle_{-r_o}$$
$$+ N^{-(2-1/K_o)} |(m-\sigma^n_{tN^{(1-1/K_o)}}(x))(\delta_x-\lambda^n)|^2_{-r_o}, \tag{6.26}$$

$$g^n_t(dm,dx) = N^{2-1/K_o)} A^n(m,x,\sigma^n_{tN^{(1-1/K_o)}})\rho(dm)\lambda^n(dx), \tag{6.27}$$

it is easy to check that

$$\sup|f_t^n(m,x)| \leq 4bN^{-(1-1/2K_o)} C_{-r_o}^{1/2} |\eta_t^n|_{-r_o} + b^2 C_{-r_o} N^{-(2-1/K_o)}$$

$$\leq C_4(M)\, N^{-(1-1/2K_o)}, \qquad (6.28)$$

$$\int_{B\times\mathbb{T}} |f_t^n(m,x)|^2 g_t(dm,dx) \leq C_5(|\eta_t^n|^2_{-r_o} + N^{-(2-1/K_o)}), \qquad (6.29)$$

which are both sharper than required by (A.5) and (A.8). Therefore

$$P^n\left\{ \sup_{t\leq T\wedge\tau_M^n} |\eta_t^n|^2_{-r_o} > C_6(M) N^{(1-1/K_o)/\kappa} \alpha_n^{-1} \right\} \leq \varepsilon \qquad (6.30)$$

for all large n. Since $C_6(M) \leq N^{(1-1/K_o)(1/\kappa'-1/\kappa)}$ for large n, we find that the sets

$$A_n = \left\{ \sup_{t\leq T\wedge\tau_M^n} |\eta_t^n|^2_{-r_o} \leq N^{(1-1/K_o)/\kappa'} \alpha_n^{-1} \leq 1 \right\} \qquad (6.31)$$

have P^n-probabilities greater than $1-\varepsilon$ for $n > n_o(M,\varepsilon)$.

Similarly to (6.21-23), we investigate the ferromagnetic component θ_t^n in $t_1 \leq t_2 \leq T\wedge\tau_M^n$, using (5.20), (6.8) and the expansion

$$\gamma'(Z) = \gamma''(0)Z + \gamma^{(2K_o)}(0)Z^{2K_o-1}/(2K_o-1)! + \Theta(Z^{2K_o}).$$

$$\begin{aligned}|\theta_{t_2}^n|^2_{-r_o} = |\theta_{t_1}^n|^2_{-r_o} + 2\int_{t_1}^{t_2} \Big[&\Theta(N^{1-1/K_o} |\theta_s^n|_{-r_o} |\gamma''(0)\beta_o \mathfrak{z}*\theta_s^n \lambda^n - \theta_s^n|_{-r_o}) \\ &+ \langle \theta_s^n, \gamma^{(2K_o)}(0)/(2K_o-1)!\, (\beta_o \mathfrak{z}*\xi_s^n)^{2K_o-1} \lambda^n \rangle_{-r_o} + o(1)|\theta_s^n|_{-r_o} |\xi_s^n|_{-r_o} \\ &+ N^{1-1/K_o} \langle \theta_s^n, \{\exp\gamma(\beta_o \mathfrak{z}*\sigma^n_{sN^{1-1/K_o}}(.))-1\}(\gamma''(0)\beta_o \mathfrak{z}*\eta_s^n \lambda^n - \eta_s^\kappa)\rangle_{-r_o} \\ &+ \Theta(N^{-1/K_o} |\theta_s^n|_{-r_o} |\xi_s^n|^{2K_o+1}) + \Theta(N^{-1/2K_o} |\theta_s^n|_{-r_o} |\xi_s^n|^{2K_o}_{-r_o}) \Big] ds\end{aligned}$$

$$+\int_{t_1}^{t_2} \int_{B\times\mathbb{T}} |(m-\sigma^n_{sN^{1-1/K_o}}(x))\lambda^n|^2_{-r_o} A^n(m,x,\sigma^n_{sN^{1-1/K_o}})\rho(dm)\lambda^n(dx)ds + \tilde{M}^n_{t_1,t_2}, \qquad (6.32)$$

where

$$\tilde{M}^n_{t_1,t_2} = \int_{t_1 N^{1-1/K_0}}^{t_2 N^{1-1/K_0}} \int_{B\times\mathbb{T}} \left(|\theta^n_{sN^{-(1-1/K_0)}} + N^{-(1-1/2K_0)}(m-\sigma^n_{s-}(x))\lambda^n|^2_{-r_0} - |\theta^n_{sN^{-(1-1/K_0)}}|^2_{-r_0} \right) \tilde{\Lambda}^n(dm,dx,ds). \qquad (6.33)$$

By (6.3) and (5.10), we have the estimates

$$N^{1-1/K_0} |\gamma''(0)\beta_0 \widehat{\partial * \theta^n_s \lambda^n}(0)\lambda^n - \hat{\theta}^n_s(0)\lambda^n|_{-r_0}$$

$$= N^{1-1/K_0} |\gamma''(0)\beta_0 (\widehat{\partial\lambda^n}(0) - \hat{\partial}(0))| |\hat{\theta}^n_s(0)\lambda^n|_{-r_0}$$

$$\leq N^{1-1/K_0} \gamma''(0)\beta_0 |\partial|_{r_0} |\lambda^n - \lambda|_{-r_0} |\theta^n_s|_{-r_0} = \mathcal{O}(N^{-r_0+1-1/K_0} |\theta^n_s|_{-r_0}), \qquad (6.34)$$

$$N^{1-1/K_0} \langle \theta^n_s, (\exp \gamma(\beta_0 \partial * \sigma^n_{sN^{1-1/K_0}}(.)) - 1)(\gamma''(0)\beta_0 \partial * \eta^n_s \lambda^n - \eta^n_s) \rangle_{-r_0}$$

$$= \mathcal{O}(N^{1-2/K_0} |\theta^n_s|_{-r_0} |\xi^n_s|_{-r_0} |\eta^n_s|^2_{-r_0}), \qquad (6.35)$$

and

$$\langle \theta^n_s, \gamma^{(2K_0)}(0)/(2K_0-1)!\, (\beta_0 \partial * \xi^n_s)^{2K_0-1} \lambda^n \rangle_{-r_0}$$

$$= \gamma^{(2K_0)}(0)/(2K_0-1)!\, \beta_0^{2K_0-1} \langle \theta^n_s, (\partial * \theta^n_s + \partial * \eta^n_s)^{2K_0-1} \lambda^n \rangle_{-r_0}$$

$$\underset{(6.8)}{=} \gamma^{(2K_0)}(0)/(2K_0-1)!\, \beta_0^{2K_0-1} \widehat{\partial\lambda^n}(0)^{2K_0-1} \hat{\theta}^n_s(0)^{2K_0} |\lambda^n|^2_{-r_0}$$

$$+ \mathcal{O}(|\theta^n_s|^{2K_0-2}_{-r_0} |\eta^n_s|^2_{-r_0} + |\theta^n_s|_{-r_0} |\eta^n_s|^{2K_0-1}_{-r_0}) \qquad (6.36)$$

since $\widehat{\partial * \eta^n_s \lambda^n}(0) = \widehat{\partial\lambda^n}(0)\, \hat{\eta}^n_s(0) = 0$ implies $\langle \theta^n_s, \partial * \eta^n_s \lambda^n \rangle_{-r_0} = 0$.

The second integral in (6.32) is again bounded by $(t_2 - t_1) C_7$ with C_7 independent of M. Hence

$$|\theta^n_{t_2}|^2_{-r_0} \leqslant |\theta^n_{t_1}|^2_{-r_0} + 2\int_{t_1}^{t_2} \gamma^{(2K_0)}(0)/(2K_0-1)!\,\beta_0^{2K_0-1}\,\widehat{\partial\lambda^n}(0)^{2K_0-1}\,\hat{\theta}^n_s(0)^{2K_0}|\lambda^n|^2_{-r_0}\,ds$$
$$+\int_{t_1}^{t_2}\Big[\Theta_M\big(N^{-r_0+1-1/K_0}+(N^{1-2/K_0}+1)|\eta^n_s|^2_{-r_0}+|\eta^n_s|^{2K_0-1}_{-r_0}+N^{-1/K_0}+N^{-1/2K_0}\big)+o(1)M^2\Big]d$$
$$+ C_7(t_2-t_1) + \bar{M}^n_{t_1,t_2}\,, \tag{6.37}$$

where the constants in the term Θ_M depend on M. The first condition of (6.16) shows that we can find $n_1(\varepsilon,M) \geqslant n_0(\varepsilon,M)$ such that on the sets A_n from (6.31), the second integrand in (6.37) is less than 1 for all $n > n_1(\varepsilon,M)$, and the first integrand is non-positive, thanks to $\gamma^{(2K_0)}(0) < 0$. (6.14) implies $P^n\{|\theta^n_0|^2_{-r_0} \geqslant C_8\} < \varepsilon$ for C_8 large enough and for all n, by which, together with (6.37), we obtain for $n \geqslant n_1(\varepsilon,M)$

$$\{|\theta^n_0|^2_{-r_0} \leqslant C_8\} \cap A_n \cap \{\sup_{t\leqslant T\wedge\tau^n_M} |\theta^n_t|^2_{-r_0} \geqslant T(C_7+1)+C_8+C_9\} \subseteq \{\sup_{t\leqslant T\wedge\tau^n_M} \bar{M}^n_t \geqslant C_9\}. \tag{6.38}$$

But

$$P^n\{\sup_{t\leqslant T\wedge\tau^n_M} \bar{M}^n_t \geqslant C_9\} \leqslant C_9^{-2} E(\bar{M}^n_{T\wedge\tau^n_M})^2 \leqslant C_9^{-2} C_{10} \leqslant \varepsilon, \tag{6.39}$$

where C_{10} is independent of n and M and $C_9 \geqslant (C_{10}/\varepsilon)^{1/2}$. By (6.31) and (6.38-39), we finally get for $M > 1 + T(C_7+1) + C_8 + C_9$

$$\{\tau^n_M < T\} = \{\sup_{t\leqslant T\wedge\tau^n_M} |\xi^n_t|^2_{-r_0} \geqslant M\} \subseteq \{\sup_{t\leqslant T\wedge\tau^n_M} |\eta^n_t|^2_{-r_0} > 1\} \cup \Omega\backslash A_n \cup \{|\theta^n_0|^2_{-r_0} > C_8\}$$
$$\cup \{|\theta^n_0|^2_{-r_0} \leqslant C_8\} \cap A_n \cap \{\sup_{t\leqslant T\wedge\tau^n_M} |\theta^n_t|^2_{-r_0} \geqslant C_8 + T(C_7+1)\},$$

which show $P^n(\tau^n_M < T) \leqslant 4\varepsilon$. (6.40)

The condition (5.12) is satisfied. In order to establish (5.13) for ξ^n_t, it is enough to show it for θ^n_t, since (6.30) and (6.40) show that the sequence of processes η^n_t is tight and converges in law to

$\eta_t = 0$. Thus, let $0 \le \tau_1 \le \tau_2 \le (\tau_1 + \delta) \wedge T$. We have

$$\langle g, \theta^n_{\tau_2} - \theta^n_{\tau_1} \rangle$$

$$= \widehat{g\lambda^n}(0) \int_{\tau_1}^{\tau_2} N^{1-1/2K_0} (m - \sigma^n_{sN^{1-1/K_0}}(x)) A^n(m, x, \sigma^n_{sN^{1-1/K_0}}) \rho(dm) \lambda^n(dx) ds$$

$$+ \bar{M}^n_{\tau_1, \tau_2} \tag{6.41}$$

with

$$\bar{M}^n_{\tau_1, \tau_2} = \widehat{g\lambda^n}(0) \int_{\tau_1 N^{1-1/K_0}}^{\tau_2 N^{1-1/K_0}} N^{-(1-1/2K_0)} (m - \sigma^n_{s-}(x)) \bar{A}^n(dm, dx, ds). \tag{6.42}$$

After the same expansion of the first integral as in (6.32-37), we have for large n

$$1_{\{\tau^n_M > T\}} (\langle g, \theta^n_{\tau_2} - \theta^n_{\tau_1} \rangle)^2 \le C_9(M)(\tau_2 - \tau_1)^2 + 2(\bar{M}^n_{\tau_1, \tau_2})^2, \tag{6.43}$$

such that by (6.40)

$$P^n\{|\langle g, \theta^n_{\tau_2} - \theta^n_{\tau_1} \rangle| > \eta\} \le 4\epsilon + P^n\{|1_{\{\tau^n_M > T\}} \langle g, \theta^n_{\tau_2} - \theta^n_{\tau_1} \rangle| > \eta\}$$

$$\le 4\epsilon + C_9(n)\delta^2/\eta^2 + C_{10}\delta/\eta^2 \le 5\epsilon \tag{6.44}$$

for $n > n_2(\epsilon, M)$ and δ sufficiently small. This completes the proof of the tightness of the critical fluctuation processes ξ^n_t.

Before we can characterize the limit process of ξ^n_t, we need the following result :

Let
$$X^n_t = ((\sigma^n_{tN^{1-1/K_0}}(.))^2 - \gamma''(0))\lambda^n \in H_{-r_0}. \tag{6.45}$$

We claim that for some $n(\epsilon)$

$$\sup_{n > n(\epsilon)} P^n(\sup_{t < T \wedge \tau^n_M} |X^n_t|^2_{-r_0} > N^{(1-1/K_0)/2\bar{\kappa}} \bar{\alpha}_n^{-1/2}) < \epsilon. \tag{6.46}$$

Ito's formula shows for $t_1 \leq t_2 \leq \tau_M^n$

$$|X_{t_2}^n|_{-r_0}^2 = |X_{t_1}^n|_{-r_0}^2 + 2N^{1-1/K_0}\int_{t_1}^{t_2}$$

$$\langle X_s^n, \int_{B\times\mathbb{T}} \delta_x(m^2-(\sigma^n_{sN^{1-1/K_0}}(x))^2)A^n(m,x,\sigma^n_{sN^{1-1/K_0}})\rho(dm)\lambda^n(dx)\rangle_{-r_0} ds$$

$$+\int_{t_1}^{t_2} N^{-1/K_0}\int_{B\times\mathbb{T}}|(m^2-(\sigma^n_{sN^{1-1/K_0}}(x))^2)\delta_x|_{-r_0}^2 A^n(m,x,\sigma^n_{sN^{1-1/K_0}})\rho(dm)\lambda^n(dx)ds$$

$$+M'^n_{t_1,t_2} \tag{6.47}$$

where

$$M'^n_{t_1,t_2} = \int_{t_1N^{1-1/K_0}}^{t_2N^{1-1/K_0}}\left[|X^n_{sN_-^{-(1-1/K_0)}}+N^{-1}(m^2-\sigma^n_{s-}(x)^2)\delta_x|_{-r_0}^2-|X^n_{sN_-^{-(1-1/Ko)}}|_{-r_0}^2\right]$$

$$\bar{\Lambda}^n(dm,dx,ds). \tag{6.48}$$

Using Sobolev-norm estimates as above, we get

$$|X_{t_2}^n|_{-r_0}^2 = |X_{t_1}^n|_{-r_0}^2-\int_{t_1}^{t_2} 2N^{1-1/K_0}|X_s^n|_{-r_0}^2\left[1+o(N^{-(1-1/K_0)})+\Theta(N^{-1/K_0}|\xi_s^n|_{-r_0}^2)\right]ds$$

$$+\Theta(N^{-1/K_0})(t_2-t_1)+M'^n_{t_1,t_2}$$

$$\leq |X_{t_1}^n|_{-r_0}^2 - \int_{t_1}^{t_2}(N^{1-1/K_0}|X_s^n|_{-r_0}^2+1)ds + M'^n_{t_1,t_2} \tag{6.49}$$

for $t_2 < \tau_M^n$ and $n > n(M)$. Applying the proposition on collapsing processes to $|X^n_{t\wedge\tau_M^n}|_{-r_0}^2$ with $\bar{\alpha}_n$ from (6.18), we see that (6.17) corresponds to (A.3) and (6.49) to (A.4). Here

$$f_t^n(m,x)=2N^{-1}\langle X_t^n,(m^2-(\sigma^n_{tN^{1-1/K_0}}(x))^2)\delta_x\rangle_{-r_0}$$

$$+N^{-2}|(m^2-(\sigma^n_{tN^{1-1/K_0}}(x))^2)\delta_x|_{-r_0}^2 = \Theta(N^{-1}),$$

and

$$g_t^n(dm,dx) = N^{2-1/K_o} A^n(m,x,\sigma^n_{tN^{1-1/K_o}})\rho(dm)\lambda^n(dx),$$

such that (A.5) and (A.8) are also satisfied. Therefore, the proposition on collapsing processes implies (6.46).

Finally, we compute the limit of

$$\tilde{f}(\xi_t^n)=\tilde{f}(\theta_t^n+\eta_t^n)=f(g_1\widehat{\lambda^n}(0)\widehat{\theta_t^n}(0)$$

$$+\langle g_1-g_1\widehat{\lambda^n}(0),\eta_t^n\rangle,\dots,g_\ell\widehat{\lambda^n}(0)\widehat{\theta_t^n}(0)+\langle g_\ell-g_\ell\widehat{\lambda^n}(0),\eta_t^n\rangle) \qquad (6.50)$$

with $f\in\mathcal{C}_b^2(R^\ell)$, $g_j\in H_{r_o}$ for $j=1,\dots,\ell$, and for $t\leq T\wedge\hat{\tau}$, where

$$\hat{\tau} = \tau_M^n \wedge \inf(t;|\eta_t^n|^2_{-r_o} > N^{(1-1/K_o)/\kappa'}\alpha_n^{-1})$$

$$\wedge \inf(t;|X_t^n|^2_{-r_o} > N^{(1-1/K_o)/2\bar{\kappa}}\alpha_n^{-1/2}) \qquad (6.51)$$

with $P^n\{\hat{\tau}\leq T\} < 6\epsilon$ by (6.30-31, 40, 46).

Setting

$$M^{f,n}_{t_1,t_2}=\int_{t_1N^{1-1/K_o}}^{t_2N^{1-1/K_o}}[f(.,\langle\xi^n_{sN_-^{-(1-1/K_o)}},g_j\rangle+N^{-(1-1/2K_o)}(m-\sigma^n_{s_-}(x))g_j(x),.)$$

$$-\tilde{f}(\xi^n_{sN_-^{-(1-1/K_o)}})]\tilde{\Lambda}^n(dm,dx,ds) \qquad (6.52)$$

we get for $t_1<t_2<\hat{\tau}$, using the same estimates as above,

$$\tilde f(\xi^n_{t_2}) = \tilde f(\xi^n_{t_1}) + \int_{t_1}^{t_2} \sum_j \widetilde{\partial_j f(\xi^n_s)} \int_{\mathbb{T}\times B} g_j(x)(m-\sigma^n_{sN^{1-1/K_o}}(x))$$

$$N^{1-1/2K_o} A^n(m,x,\sigma^n_{sN^{1-1/K_o}})\rho(dm)\lambda^n(dx)ds$$

$$+ \int_{t_1}^{t_2} \sum_j \int_{\mathbb{T}\times B} \Big[f(.,\langle \xi^n_s, g_j\rangle + N^{-(1-1/2K_o)}(m-\sigma^n_{sN^{1-1/K_o}}(x))g_j(x),\ldots)$$

$$\tilde f(\xi^n_s) - \sum_j \widetilde{\partial_j f(\xi^n_s)} g_j(x)(m-\sigma^n_{sN^{1-1/K_o}}(x))N^{-(1-1/2K_o)}\Big]$$

$$N^{2-1/K_o} A^n(m,x,\sigma^n_{sN^{1-1/K_o}})\rho(dm)\lambda^n(dx)ds + M^{f,n}_{t_1,t_2}$$

$$= \tilde f(\xi^n_{t_1}) + \int_{t_1}^{t_2} \sum_j \widetilde{\partial_j f(\xi^n_s)} \Big[N^{1-1/K_o} \int_{\mathbb{T}} (g_j(x) - \widehat{g_j\lambda^n}(0))(\gamma''(0)\beta_o \partial^* \eta^n_s(x)\lambda^n(dx) - \eta^n_s(dx)$$

$$(1+\Theta(N^{-1/K_o}(\widehat{\theta^n_s}(0))^2)) + \gamma^{(2K_o)}(0)/(2K_o-1)!\ \ \widehat{g_j\lambda^n}(0)(\beta_o \widehat{\partial\lambda^n}(0)\widehat{\theta^n_s}(0))^{2K_o-1}$$

$$+\Theta(N^{1-2/K_o}|\eta^n_s|^2 M) + \Theta(N^{-r_o+1-1/K_o} M) + \Theta(|\eta^n_s| M^{2K_o-2}) + \Theta(N^{-1/2K_o} M^{2K_o})$$

$$+\Theta(N^{-1/K_o} M^{2K_o+1}) + o(1).M\Big] ds$$

$$+1/2 \int_{t_1}^{t_2} \sum_{ij} \widetilde{\partial_{ij} f(\xi^n_s)}(1+o(1))2\gamma''(0)\Big[\widehat{g_i\lambda^n}(0)\widehat{g_j\lambda^n}(0) + \int_{\mathbb{T}} (g_i(x) - \widehat{g_i\lambda^n}(0))$$

$$(g_j(x) - \widehat{g_j\lambda^n}(0))\lambda^n(dx)\Big] ds$$

$$+o(N^{-(1-1/K_o)}) + \Theta(N^{-1/K_o} M^2) + \Theta(\sup_{t_1 \le s \le t_2} |X^n_s|)_{-r_o} + M^{f,n}_{t_1,t_2}. \qquad (6.53)$$

Here, we can even replace λ^n by λ everywhere making additional errors of the order $|\lambda^n - \lambda|_{-r_o} = \Theta(N^{-r_o})$, which however become arbitrarily small for large n, since $r_o > 1-1/K_o$. By (6.16), (6.18), (6.51) and the fact that

$$N^{1-1/K_o}\int_{\mathbb{T}}(g_j(x)-g_j\widehat{\lambda^n}(0))(\gamma''(0)\beta_o\partial*\eta^n_s\lambda^n(dx)-\eta^n_s(dx))(1+\Theta(N^{-1/K_o}M^2))$$

$$=N^{1-1/K_o}\sum_{p\in(\mathbb{Z}/n\mathbb{Z})^d\setminus\{0\}} g_j\widehat{\lambda^n}(p)\ \overline{\widehat{\eta^n_s}}(p)(\gamma''(0)\beta_o\widehat{\partial\lambda^n}(p)-1)(1+\Theta(N^{-1/K_o}M^2)) \quad (6.54)$$

has coefficients $(\gamma''(0)\beta_o\widehat{\partial\lambda^n}(p)-1)(1+\Theta(N^{-1/K_o}M^2)) \leq -\gamma''(0)\beta_o\delta_o/2$ for large n, uniformly in $p\in\mathbb{Z}^d\setminus(n\mathbb{Z})^d$, we see that any limit process $\xi_t = \theta_t + \eta_t$ of ξ^n_t satisfies

$$\eta_t = 0, \quad (6.55)$$

and $\theta_t = \widehat{\theta_t(0)\lambda}$ solves the martingale problem :

$$\tilde{f}(\theta_t)-\tilde{f}(\theta_o)-\int_0^t \sum_j \widetilde{\partial_j f}(\theta_s)\gamma^{(2K_o)}(0)/(2K_o-1)!\ \hat{g}_j(0)(\beta_o\hat{\partial}(0)\hat{\theta}_s(0))^{2K_o-1}ds$$

$$-\int_0^t \sum_{i,j}\widetilde{\partial_{ij}f}(\theta_s)2\gamma''(0)\hat{g}_i(0)\hat{g}_j(0)ds \text{ is a martingale,} \quad (6.56)$$

with $\tilde{f}$ from (6.50). But (6.56) is equivalent to (6.19).

This completes the proof of Theorem 5. §

The unique invariant probability measure of the process $\hat{\theta}_t(0)\in\mathbb{R}$ from (6.19) is

$$\nu_1(dx)=\exp(\gamma^{(2K_o)}(0)/[2(2K_o)!(\gamma''(0))^{2K_o}]\ X^{2K_o})dX/Z_1 \quad (6.57)$$

where Z_1 is the normalization constant.

7. CRITICAL FLUCTUATIONS AT AN ANTIFERROMAGNETIC PHASE TRANSITION.

Instead of the critical fluctuations at the ferromagnetic phase transition, we now study critical fluctuations at the point of an antiferromagnetic transition with frequency $p_o \neq 0$. This means that instead of (6.2) and (6.3), we now have

$$\hat{\partial}(p_o)=\hat{\partial}(-p_o)>0 \quad \text{and} \quad \hat{\partial}(p_o)-\hat{\partial}(q)\geq\delta_o>0 \quad \text{for all} \quad q\in\mathbb{Z}^d\backslash\{\pm p_o\}, \tag{7.1}$$

and

$$\beta_{p_o} = (\gamma''(0)\hat{\partial}(p_o))^{-1} . \tag{7.2}$$

In addition, we strengthen assumption (A1) of the last section by requiring

$$\gamma^{(4)}(0) < 0 \tag{7.3}$$

i.e. $K_o = 2$ in (6.5). For example, this is true for Ising spins with $\rho = (\delta_1+\delta_{-1})/2$, where $\gamma''(0) = 1$ and $\gamma^{(4)}(0) = -2$.

We keep the assumption (A2) of the last section with $K_o = 2$.

We now split the critical fluctuation process

$$\xi^n_t = N^{1/4}\sigma^n_{tN^{1/2}} \tag{7.4}$$

into the p_o-antiferromagnetic component and its complement

$$\varphi^n_t(dx) = 2[\text{Re}(\widehat{\xi^n_t}(p_o))\cos(2\pi p_o x)+\text{Im}(\widehat{\xi^n_t}(p_o))\sin(2\pi p_o x)]\lambda^n(dx) \tag{7.5}$$

$$\psi^n_t(dx) = \xi^n_t(dx) - \varphi^n_t(dx). \tag{7.6}$$

THEOREM 6.

<u>Let</u> (7.1-2), (A1) <u>with</u> (7.3) <u>and</u> (A.2) <u>hold. For the starting configurations, we assume</u>

(i) $\sigma^n_o\in\mathcal{M}^n_b$, <u>and</u> $\widehat{\varphi^n_o}(p_o) = N^{1/4}\widehat{\sigma^n_o}(p_o)$ <u>converges in law to some</u> $\hat{\varphi}_o(p_o)$; (7.7)

(ii) *for some* $\kappa>1$ *and an increasing sequence* α_n *with*

$$N^{1/2\kappa}\alpha_n^{-1} \longrightarrow 0, \quad \text{and} \quad N^{-1/2}\alpha_n \longrightarrow 0, \tag{7.8}$$

we have

$$E^n \|\psi_o^n\|_{-r_o}^{2\kappa} \leq C_1 \alpha_n^{-\kappa} ; \tag{7.9}$$

and

$$E^n(\|((\sigma_o^n(.))^2 - \gamma''(0))\lambda^n\|_{-r_o}^{2\kappa}) \leq C_2 \alpha_n^{-\kappa} \tag{7.10}$$

for all large n. *Then the critical fluctuation process converges in law to the* p_o*-antiferromagnetic process*

$$\varphi_t(dx) = 2[\mathrm{Re}(\hat{\varphi}_t(p_o))\cos(2\pi p_o x)+\mathrm{Im}(\hat{\varphi}_t(p_o))\sin(2\pi p_o x)]\lambda(dx), \tag{7.11}$$

where $\hat{\varphi}_t(p_o)\in\mathbb{C}$ *satisfies the complex diffusion equation*

$$d\hat{\varphi}_t(p_o)=\gamma^{(4)}(0)/[2\gamma''(0)^3]\,|\hat{\varphi}_t(p_o)|^2\hat{\varphi}_t(p_o)dt + (2\gamma''(0))^{1/2}dw_t^c, \tag{7.12}$$

starting at $\hat{\varphi}_o(p_o)$. *Here,* w_t^c *denotes a complex-valued Brownian motion.*

Proof. Since the proof follows the same lines as that of the last section, we will give only the main estimates. Like in (6.21-25), we obtain for

$$t_1<t_2<\tau_M^n = \inf\{t \;;\; \|\xi_t^n\|_{-r_o} \geq M\}$$

$$\|\psi_{t_2}^n\|_{-r_o}^2 = \|\psi_{t_1}^n\|_{-r_o}^2 + 2\int_{t_1}^{t_2}\Big[N^{1/2}\langle\psi_s^n,\gamma''(0)\beta_{p_o}\mathfrak{F}*\psi_s^n\lambda^n - \psi_s^n\rangle_{-r_o} + O(\|\psi_s^n\|_{-r_o}M^3)$$
$$+ o(1)\|\psi_s^n\|_{-r_o}\|\xi_s^n\|_{-r_o}\Big]ds$$
$$+ O(1)(t_2-t_1) + Q_{t_1t_2}^n, \tag{7.13}$$

with the martingale

$$Q_{t_1,t_2}^n = \int_{t_1N^{1/2}}^{t_2N^{1/2}}\int_{B\times\mathbb{T}}\Big[\|\psi_{sN_-^{-1/2}}^n + N^{-3/4}(m-\sigma_{s_-}^n(x))(\delta_x - 2\cos(2\pi p_o(x-.))\lambda^n\|_{-r_o}^2$$
$$-\|\psi_{sN_-^{1/2}}^n\|_{-r_o}^2\Big]\,\tilde{\Lambda}^n(dm,dx,ds). \tag{7.14}$$

Using (7.1-2), we have

$$\gamma''(0)\beta_{p_o}\,\partial\widehat{\lambda^n}(q)-1 \le -1/2\,\gamma''(0)\beta_{p_o}\delta_o \quad \text{for all } q\in(\mathbb{Z}/n\mathbb{Z})^d\setminus\{\pm p_o\}. \tag{7.15}$$

and

$$|\psi^n_{t_2}|^2_{-r_o} \le |\psi^n_{t_1}|^2_{-r_o} + \int_{t_1}^{t_2}(-N^{1/2}\gamma''(0)\beta_{p_o}\delta_o|\psi^n_s|^2_{-r_o} + C_3(M))ds + Q^n_{t_1,t_2}. \tag{7.16}$$

By (7.8-9) and estimates, similar to (6.28-29), we see that the assumptions of the proposition of the Appendix with $m = N^{1/2}$ are satisfied, so that

$$P^n\Big(\sup_{t\le T\wedge\tau^n_M}|\psi^n_t|^2_{-r_o} > N^{1/4\kappa-1/2}\alpha_n\Big) \le P^n\Big(\sup_{t\le T\wedge\tau^n_M}|\psi^n_t|^2_{-r_o} > C_4(M)N^{1/2\kappa-1}\alpha_n\Big) < \varepsilon \tag{7.17}$$

for all $n \ge n_o(M,\varepsilon)$. For the p_o-antiferromagnetic component and $t_1 < t_2 < \tau^n_M$, we get the estimate

$$|\varphi^n_{t_2}|^2_{-r_o} = |\varphi^n_{t_1}|^2_{-r_o}$$

$$+2\int_{t_1}^{t_2}\Big[\Theta(N^{-r_o+1/2}|\varphi^n_s|^2_{-r_o}) + \langle\varphi^n_s,\ \gamma^{(4)}(0)/3!(\beta_{p_o}\partial*\varphi^n_s)^3\lambda^n\rangle_{-r_o}$$

$$+\Theta(|\varphi^n_s|_{-r_o}|\psi^n_s|_{-r_o}|\xi^n_s|^2_{-r_o}) + \Theta(N^{-1/2}|\varphi^n_s|_{-r_o}|\xi^n_s|^5_{-r_o}) + o(1)|\varphi^n_s|_{-r_o}|\xi^n_s|_{-r_o}\Big]ds$$

$$+\ \Theta(1)(t_2-t_1) + \bar{Q}^n_{t_1,t_2}, \tag{7.18}$$

$$\bar{Q}^n_{t_1,t_2} = \int_{t_1N^{1/2}}^{t_2N^{1/2}}\int_{B\times\mathbb{T}}\Big[|\varphi^n_{sN^{-1/2}} + N^{-3/4}(m-\sigma^n_s(x))2\cos(2\pi p_o(x-.))\lambda^n|^2_{-r_o}$$

$$-|\varphi^n_{sN^{1/2}}|^2_{-r_o}\Big]\bar{\Lambda}^n(dm,dx,ds). \tag{7.19}$$

We calculate

$$\gamma^{(4)}(0)/3!\ \beta^3_{p_o}\langle\varphi^n_s,(\partial*\varphi^n_s)^3\lambda^n\rangle$$

$$= \gamma^{(4)}(0)/3!\ \beta^3_{p_o}\,\partial\widehat{\lambda^n}(p_o)^3\,|\widehat{\varphi^n_s}(p_o)|^4\sum_{q\in(n\mathbb{Z})^d}(1+|p_o+q|^2)^{-r_o}$$

$$< 0 \tag{7.20}$$

Since $\gamma^{(4)}(0)<0$. Therefore, using (7.17), we find that for $n \geq n_o(\varepsilon, M)$

$$|\varphi^n_t|^2_{-r_o} \leq |\varphi^n_0|^2_{-r_o} + (C_4+1)t + \bar{Q}^n_{0,t} \qquad (7.21)$$

with C_4 independent of M and $t < \tau^n_M$. Reasoning in the same way as in (6.38-40), we conclude from (7.17) and (7.21) that

$$P^n\{\tau^n_M \leq T\} \leq 4\varepsilon \qquad (7.22)$$

for M large enough and $n \geq n_1(\varepsilon, M)$. The modulus of continuity of φ^n_t is shown to be uniform in probability in the same way as in (6.41-44). Thus, by (5.12-13), the sequence of processes ξ^n_t is tight. Of course, (6.46) also holds here. Thus, it only remains to identify the limit process of the critical fluctuations ξ^n_t. Thus, let

$$g^{p_o,n}(x) = 2[\mathrm{Re}\ \widehat{g\lambda^n}(p_o)\cos(2\pi p_o x) + \mathrm{Im}\ \widehat{g\lambda^n}(p_o)\sin(2\pi p_o x)] \qquad (7.23)$$

with $g \in H_{r_o}$, and for $f \in \mathcal{C}^2_b(\mathbb{R}^\ell)$, $g_j \in H_{r_o}$ for $j=1,\ldots,\ell$ set

$$\bar{f}(\xi^n_t) = f(\langle g_1^{p_o,n}, \varphi^n_t\rangle + \langle g_1 - g_1^{p_o,n}, \psi^n_t\rangle, \ldots, \langle g_\ell^{p_o,n}, \varphi^n_t\rangle + \langle g_\ell - g_\ell^{p_o,n}, \psi^n_t\rangle). \qquad (7.24)$$

Again, we may restrict ourselves to $t_1 < t_2 < \hat{\tau}$ with

$$\hat{\tau} = \tau^n_M \wedge \inf\{t, |\psi^n_t|^2_{-r} \vee |(\sigma^n_{tN^{1/2}}(.)^2 - \gamma''(0))\lambda^n|^2_{-r_o} > N^{1/4\kappa}\alpha_n^{-1/2}\}, \qquad (7.25)$$

and

$$P^n\{\hat{\tau} \leq T\} \leq 6\varepsilon \quad \text{for} \quad n \geq n_1(\varepsilon, M). \qquad (7.26)$$

Now, with $M^{f,n}_{t_1,t_2}$ from (6.52), we get

$$\tilde{f}(\xi^n_{t_2})=\tilde{f}(\xi^n_{t_1})+\int_{t_1}^{t_2}\sum_j \widetilde{\partial_j f}(\xi^n_s)\Big[N^{1/2}\langle g_j-g_j^{p_0,n},\gamma''(0)\beta_{p_0}\partial^*\psi^n_s\lambda^n-\psi^n_s\rangle$$

$$+\gamma^{(4)}(0)/3!\;\beta^3_{p_0}\langle g_j,(\partial^*\varphi^n_s)^3\lambda^n\rangle + O(|\psi^n_t|_{-r_0}M^2)$$

$$+O(N^{-r_0+1/2}M)+O(N^{-1/2}M^5)+o(1)M\Big]ds$$

$$+1/2\int_{t_1}^{t_2}\sum_{i,j}\widetilde{\partial_{ij}f}(\xi^n_s)(1+o(1))2\gamma''(0)\Big[\langle g_i^{p_0,n},g_j^{p_0,n}\rangle+\langle g_i-g_i^{p_0,n},g_j-g_j^{p_0,n}\rangle$$

$$+o(N^{-1/2})+O(N^{-1/2}M^2)+O(N^{1/4\kappa-1/2}\alpha_n)\Big]ds+M^{f,n}_{t_1,t_2}\,. \qquad (7.27)$$

Now

$$\langle g,(\partial^*\varphi^n)^3\lambda^n\rangle=\sum_{q\in(\mathbb{Z}/n\mathbb{Z})^d}\widehat{g\lambda^n}(q)\overline{\widehat{(\partial^*\varphi^n)^3\lambda^n}(q)}$$

$$=\sum_{q_1,q_2,q_3\in\{\pm p_0\}}\widehat{g\lambda^n}(q_1+q_2+q_3)\prod_{i=1}^{3}\widehat{\partial\lambda^n}(q_i)\,\overline{\widehat{\varphi^n}(q_i)}$$

$$=\widehat{\partial\lambda^n}^3(p_0)\Big[3|\widehat{\varphi^n}(p_0)|^2\,2\mathrm{Re}(\widehat{g\lambda^n}(p_0)\overline{\widehat{\varphi^n}(p_0)})+2\mathrm{Re}(\widehat{g\lambda^n}(3p_0)\overline{\widehat{\varphi^n}^3(p_0)})\Big]$$

$$=\widehat{\partial\lambda^n}^3(p_0)\Big[3|\widehat{\varphi^n}(p_0)|^2\langle g^{p_0,n},\varphi^n\rangle+\langle g-g^{p_0,n},2[\mathrm{Re}(\widehat{\varphi^n}^3(p_0))\cos(2\pi 3p_0\cdot)$$

$$+\mathrm{Im}(\widehat{\varphi^n}^3(p_0))\,\sin(2\pi 3p_0\cdot)]\lambda^n\rangle\Big], \qquad (7.28)$$

and therefore

$$\tilde{f}(\varphi^n_{t_2}+\psi^n_{t_2})=\tilde{f}(\varphi^n_{t_1}+\psi^n_{t_1})$$

$$+\int_{t_1}^{t_2}\Big[\sum_j\widetilde{\partial_j f}(\varphi^n_s+\psi^n_s)N^{1/2}\langle g_j-g_j^{p_0,n},\gamma''(0)\beta_{p_0}\partial^*\psi^n_s\lambda^n-\psi^n_s+O(N^{-1/2}M^3)\lambda^n$$

$$+1/2\sum_{i,j}\widetilde{\partial_{ij}f}(\varphi^n_s+\psi^n_s)2\gamma''(0)\langle g_i-g_i^{p_0,n},g_j-g_j^{p_0,n}\rangle\Big]ds$$

$$+\int_{t_1}^{t_2}\Big[\sum_j\widetilde{\partial_j f}(\varphi^n_s+\psi^n_s)\gamma^{(4)}(0)/2(\beta_{p_0}\widehat{\partial\lambda^n}(p_0))^3|\widehat{\varphi^n_s}(p_0)|^2\langle g_j^{p_0,n},\varphi^n_s\rangle$$

$$+1/2\sum_{i,j}\widetilde{\partial_{ij}f}(\varphi^n_s+\psi^n_s)2\gamma''(0)\,\langle g_i^{p_0,n},g_j^{p_0,n}\rangle\Big]ds$$

$$+M^{f,n}_{t_1,t_2}$$

$$+O(\sup_t|\psi^n_t|_{-r_0}M^2+N^{-r_0+1/2}M+N^{-1/2}M^5+N^{1/4\kappa}\alpha_n^{-1/2})+o(M+N^{-1/2}), \qquad (7.29)$$

where we know that for large n

$$N^{1/2}\langle g_j - g_j^{p_o,n}, \gamma''(0)\beta_{p_o}\partial^*\psi_s^n\lambda^n - \psi_s^n + O(N^{-1/2}M^3)\lambda^n\rangle$$

$$\leq - N^{1/2}\gamma''(0)\beta_{p_o}\delta_o/2\,\langle g_j - g_j^{p_o,n}, \varphi_s^n\rangle + C\,\| g_M - g_j^{p_o,n}\|_{r_o}.$$

Hence, by (7.17), any limit process $\xi_t = \varphi_t + \psi_t$ of ξ_t^n has $\psi_t = 0$, and φ_t satisfies the martingale problem

$$\tilde{f}(\varphi_{t_2}) - \tilde{f}(\varphi_{t_1}) - \int_{t_1}^{t_2}\Big[\sum_j \widetilde{\partial_j f(\varphi_s)}\ \gamma^{(4)}(0)/2\ (\gamma''(0))^{-3}\,|\hat{\varphi}_s(p_o)|\langle g_j^{p_o}, \varphi_s\rangle$$

$$+ 1/2\sum_{i,j}\widetilde{\partial_{ij} f(\varphi_s)}\ 2\gamma''(0)\ \langle g_i^{p_o}, g_j^{p_o}\rangle\Big] ds, \qquad (7.30)$$

where g^{p_o} is defined as in (7.23) with λ^n replaced by λ. (7.30) is equivalent to (7.12). This completes the proof.

The unique invariant probability measure of the process $\hat{\varphi}_t(p_o)\in\mathbb{C}$ is

$$\nu_2(dz) = \exp\{\gamma^{(4)}(0)/[16\gamma''(0)^4]\ |z|^4\}dz/Z_2 \qquad (7.31)$$

with normalization constant Z_2.

8. CRITICAL FLUCTUATIONS AT A TRIPLE POINT.

Let us suppose that we are at a triple point where a ferromagnetic second-order phase transition and an antiferromagnetic one of frequency p_o occur simultaneously. This means that

$$\hat{\partial}(0) = \hat{\partial}(p_o) = \hat{\partial}(-p_o)>0 \quad \text{and} \quad \hat{\partial}(0)-\hat{\partial}(q) \geq \delta_o >0 \tag{8.1}$$

for all $q\in\mathbb{Z}^d\backslash\{0,\pm p_o\}$, and

$$\beta_o = (\gamma''(0)\hat{\partial}(0))^{-1} = (\gamma''(0)\hat{\partial}(p_o))^{-1}. \tag{8.2}$$

We continue to let assumption (A1) and (A2) from Section 6 hold, with (7.3), i.e. $K_o = 2$, like in the last section. The surviving component of the critical fluctuation process ξ^n_t from (7.4) is now

$$\mu^n_t(dx)=\widehat{\xi^n_t}(0)\lambda^n(dx)+2[Re(\widehat{\xi^n_t}(p_o))\cos(2\pi p_o x)$$
$$+Im(\widehat{\xi^n_t}(p_o))\sin(2\pi p_o x)]\lambda^n(dx) \tag{8.3}$$

$$\nu^n_t(dx) = \xi^n_t(dx) - \mu^n_t(dx). \tag{8.4}$$

THEOREM 7.

Let (8.1-2), (A1) *with* (7.3), *and* (A2) *from section 6 hold. Assume*

(i) $\sigma^n_o\in\mathcal{M}^n_b$, *and* μ^n_o *converges in law to* μ_o; (8.5)

(ii) (7.8) *and* (7.9) *hold, together with*

$$E^n|\nu^n_t|^{2\kappa}_{-r_o} \leq C_1\alpha_n^{-\kappa}. \tag{8.6}$$

Then ξ^n_t *converges in law to the mixed-phase process*

$$\mu_t(dx)=\hat{\mu}_t(0)\lambda(dx)+2[Re(\hat{\mu}_t(p_o))\cos(2\pi p_o x)$$
$$+Im(\hat{\mu}_t(p_o))\sin(2\pi p_o x)]\lambda(dx), \tag{8.7}$$

where $(\hat{\mu}_t(0), \hat{\mu}_t(p_o))$ *satisfies the coupled stochastic equation*

$$d\hat{\mu}_t(0)=\gamma^{(4)}(0)/[3!(\gamma''(0))^3]\,(\hat{\mu}_t(0)^2+6|\hat{\mu}_t(p_o)|^2)\hat{\mu}_t(0)dt$$
$$+[2\gamma''(0)]^{1/2}dw_t, \qquad (8.8)$$

$$d\hat{\mu}_t(p_o)=\gamma^{(4)}(0)/[2(\gamma''(0))^3]\,(\hat{\mu}_t(0)^2+|\hat{\mu}_t(p_o)|^2)\hat{\mu}_t(p_o)dt$$
$$+[2\gamma''(0)]^{1/2}dw_t^{\mathbb{C}}, \qquad (8.9)$$

<u>starting at</u> $(\hat{\mu}_o(0),\ \hat{\mu}_o(p_o))$, <u>where</u> w_t <u>and</u> $w_t^{\mathbb{C}}$ <u>are independent real, resp. complex-valued Brownian motions.</u>

<u>Proof</u> : Again, we give only the main estimates and formulas, the arguments being the same as in the proof of Section 6. For $t_1<t_2\leq\tau^n_M$, we have

$$|\nu^n_{t_2}|^2_{-r_o}=|\nu^n_{t_1}|^2_{-r_o}+2\int_{t_1}^{t_2}\Big[N^{1/2}\langle\nu^n_s,\gamma''(0)\beta_o\partial*\nu^n_s\lambda^n-\nu^n_s\rangle_{-r_o}+O(|\nu^n_s|^3_{-r_o}M)$$
$$+o(1)|\nu^n_s|_{-r_o}M\Big]ds+O(1)(t_2-t_1)+R^n_{t_1,t_2} \qquad (8.10)$$

with

$$R^n_{t_1,t_2}=\int_{t_1N^{1/2}}^{t_2N^{1/2}}\int_{B\times\mathbb{T}}\Big[|\nu^n_{sN^{-1/2}}+N^{-3/4}(m-\sigma^n_s(x))(\delta_x-\lambda^n-2\cos(2\pi p_o(x-.))\lambda^n|^2_{-r_o}$$
$$-|\nu^n_{sN^{-1/2}}|^2_{-r_o}\Big]\bar{\Lambda}^n(dm,dx,ds). \qquad (8.11)$$

Since
$$\gamma''(0)\beta_o\widehat{\partial\lambda^n}(q)-1\leq-1/2\,\gamma''(0)\beta_o\delta_o \qquad (8.12)$$

for all $q\in(\mathbb{Z}/n\mathbb{Z})^d\setminus\{0,\pm p_o\}$, we get

$$|\nu^n_{t_2}|^2_{-r_o}=|\nu^n_{t_1}|^2_{-r_o}-\int_{t_1}^{t_2}(N^{1/2}\gamma''(0)\beta_o\delta_o|\nu^n_s|_{-r_o}+C_3(M))ds+R^n_{t_1,t_2}, \qquad (8.13)$$

for which the proposition of the appendix yields

$$P^n(\sup_{t\leq T\wedge\tau^n_M}|\nu^n_t|^2_{-r_o}>N^{1/4\kappa}\alpha_n^{-1/2})<\varepsilon \qquad (8.14)$$

for all $n>n_o(M,\varepsilon)$. Similarly,

$$|\mu^n_{t_2}|^2_{-r_0}=|\mu^n_{t_1}|^2_{-r_0}+2\int_{t_1}^{t_2}\Big[O(N^{-r_0+1/2}|\mu^n_s|^2_{-r_0})+\langle\mu^n_s,\gamma^{(4)}(0)/3!\,(\beta_0\partial*\mu^n_s)^3\lambda^n\rangle_{-r_0}$$

$$+O(N^{1/2}|\mu^n_s|_{-r_0}|\xi^n_s|^5_{-r_0})+O(|\mu^n_s|_{-r_0}|\nu^n_s|_{-r_0}M^2)+o(1)|\mu^n_s|_{-r_0}|\xi^n_s|_{-r_0}\Big]ds$$

$$+O(1)(t_2-t_1)+\tilde{R}^n_{t_1,t_2}, \tag{8.15}$$

with

$$\tilde{R}^n_{t_1,t_2}=\int_{t_1N^{1/2}}^{t_2N^{1/2}}\int_{B\times\mathbb{T}}\Big[|\mu^n_{sN_-^{-1/2}}+N^{3/4}(m-\sigma^n_{s_-}(x))(\lambda^n+2\cos(2\pi p_0(x-.))\lambda^n|^2_{-r_0}$$

$$-|\mu^n_{sN_-^{-1/2}}|^2_{-r_0}\Big]\tilde{\Lambda}^n(dm,dx,ds), \tag{8.16}$$

and

$$\gamma^{(4)}(0)/3!\;\beta_0^3\langle\mu^n_s,(\partial*\mu^n_s)^3\lambda^n\rangle_{-r_0}$$

$$=\gamma^{(4)}(0)/3!\;\beta_0^3\sum_{q_1,q_2,q_3\in\{0,\pm p_0\}}\hat{\mu}^n_s(q_1+q_2+q_3)\prod_{i=1}^3\widehat{\partial\lambda}^n(q_i)\overline{\hat{\mu}^n_s(q_i)}$$

$$\sum_{q\in(n\mathbb{Z})^d}(1+|q_1+q_2+q_3+q|^2)^{-r_0}$$

$$=\gamma^{(4)}(0)/3!\;\beta_0^3\Big[(\widehat{\partial\lambda}^n{}^3(0)\hat{\mu}^n_s{}^4(0)+6\,\widehat{\partial\lambda}^n(0)\widehat{\partial\lambda}^n{}^2(p_0)\hat{\mu}^n_s{}^2(0)|\hat{\mu}^n_s(p_0)|^2)$$

$$\cdot\sum_{q\in(n\mathbb{Z})^d}(1+|q|^2)^{-r_0}+(6\,\widehat{\partial\lambda}^n{}^2(0)\widehat{\partial\lambda}^n(p_0)\hat{\mu}^n_s{}^2(0)|\hat{\mu}^n_s(p_0)|^2$$

$$+6\,\widehat{\partial\lambda}^n{}^3(p_0)|\hat{\mu}^n_s(p_0)|^4)\cdot\sum_{q\in(n\mathbb{Z})^d}(1+|p_0+q|^2)^{-r_0}$$

$$<0. \tag{8.17}$$

Hence $|\mu^n_t|^2_{-r_0}\le|\mu^n_0|^2_{-r_0}+(C_4+1)t+\tilde{R}^n_{0,t}$, which implies, like in (6.38-40), $P^n\{\tau^n_M<T\}\le 4\varepsilon$ for large M and $n\ge n_1(\varepsilon,M)$. μ^n_t is shown to have a modulus of continuity uniform in probability, such that by (5.12-13) ξ^n_t is tight. Since (6.46) still holds, we only need to calculate the limit of ξ^n_t. For $g\in H_{r_0}$, define

$$\bar{g}^n(x) = g\widehat{\lambda^n}(0) + g^{p_o,n}(x) \quad \text{with} \quad g^{p_o,n} \quad \text{from (7.23).} \qquad (8.18)$$

Thus, $f\in\mathcal{C}_b^2(R^\ell)$, $g_j\in H_{r_o}$ for $j=1,\dots,\ell$, have the decomposition

$$\tilde{f}(\xi_t^n)=f(\langle\bar{g}_1^n,\mu_t^n\rangle+\langle g_1-\bar{g}_1^n,\nu_t^n\rangle,\dots,\langle\bar{g}_\ell^n,\mu_t^n\rangle+\langle g_\ell-\bar{g}_\ell^n,\nu_t^n\rangle). \qquad (8.19)$$

We define $\hat{\tau}$ as in (7.25) with ψ_t^n replaced by ν_t^n, such that (7.26) still holds, because of (8.14), and we get for $t_1<t_2<\hat{\tau}$

$$\begin{aligned}\tilde{f}(\xi_{t_2}^n) = \tilde{f}(\xi_{t_1}^n)&+\int_{t_1}^{t}\sum_j \widetilde{\partial_j f(\xi_s^n)}\Big[N^{1/2}\langle g_j-\bar{g}_j^n,\gamma''(0)\beta_o\partial^*\nu_s^n\lambda^n-\nu_s^n\rangle\\ &+\langle g_j,\gamma^{(4)}(0)/3!\,\beta_o(\partial^*\mu_s^n)^3\lambda^n\rangle+\mathcal{O}(N^{-r_o+1/2}M)+\mathcal{O}(|\nu_s^n|_{-r_o}^2M)\\ &+\mathcal{O}(N^{-1/2}M^5)+o(1)M\Big]ds\\ &+1/2\int_{t_1}^{t_2}\sum_{i,j}\widetilde{\partial_{ij}f(\xi_s^n)}(1+o(1))2\gamma''(0)\Big[\langle\bar{g}_i^n,\bar{g}_j^n\rangle+\langle g_i-\bar{g}_i^n,g_j-\bar{g}_j^n\rangle\\ &+o(N^{1/2})+\mathcal{O}(N^{-1/2}M^2)+\mathcal{O}(N^{1/4\kappa}\alpha_n^{-1/2})\Big]ds+M_{t_1,t_2}^{f,n}\end{aligned} \qquad (8.20)$$

with $M_{t_1,t_2}^{f,n}$ from (6.52) with $K_o=2$. We compute $\langle g,(\partial^*\mu^n)^3\lambda^n\rangle$, similar to (8.17), and obtain

$$\begin{aligned}\langle g,(\partial^*\mu^n)^3\lambda^n\rangle&=g\widehat{\lambda^n}(0)(\partial\widehat{\lambda^n}(0))^3\,\widehat{\mu^n}(0)^3+6\,\partial\widehat{\lambda^n}(0)\partial\widehat{\lambda^n}(p_o)^2\,|\widehat{\mu^n}(p_o)|^2\,\widehat{\mu^n}(0))\\ &+6\mathrm{Re}\Big[g\widehat{\lambda^n}(p_o)(\partial\widehat{\lambda^n}(0)^2\,\partial\widehat{\lambda^n}(p_o)\widehat{\mu^n}(0)^2\,\overline{\widehat{\mu^n}}(p_o)+\partial\widehat{\lambda^n}(p_o)|\widehat{\mu^n}(p_o)|^2\,\overline{\widehat{\mu^n}}(p_o))\Big]\\ &+6\mathrm{Re}\Big[g\widehat{\lambda^n}(2p_o)\partial\widehat{\lambda^n}(0)\partial\widehat{\lambda^n}(p_o)^2\,\mu^n(0)\overline{\widehat{\mu^n}}(p_o)^2\Big]\\ &+2\mathrm{Re}\Big[g\widehat{\lambda^n}(3p_o)\partial\widehat{\lambda^n}(p_o)^3\,\overline{\widehat{\mu^n}}(p_o)^3\Big].\end{aligned} \qquad (8.21)$$

This shows

$$\bar f(\mu^n_{t_2}+\nu^n_{t_2})$$

$$=\bar f(\mu^n_{t_1}+\nu^n_{t_1})+\int_{t_1}^{t_2}\Big[\sum_j \widetilde{\partial_j f(\mu^n_s+\nu^n_s)}N^{1/2}\langle g_j-\bar g^n_j,\gamma''(0)\beta_o\partial^*\nu^n_s\lambda^n-\nu^n_s+\Theta(N^{-1/2}M^3)\lambda^n\rangle$$

$$+1/2\sum_{i,j}\widetilde{\partial_{ij} f(\mu^n_s+\nu^n_s)}2\gamma''(0)\langle g_i-\bar g^n_i,\ g_j-\bar g^n_j\rangle\Big]ds$$

$$+\int_{t_1}^{t_2}\Big[\sum_j \widetilde{\partial_j f(\mu^n_s+\nu^n_s)}\gamma^{(4)}(0)/3!\ (\beta_o\hat\partial(0)(1+\Theta(N^{-r_o}M^3)))^3.$$

$$\Big\{(\widehat{\mu^n_s}^2(0)+6|\widehat{\mu^n_s}(p_o)|^2)g_j\widehat{\lambda^n}(0)\widehat{\mu^n_s}(0)+3(\widehat{\mu^n_s}(0)^2+|\widehat{\mu^n_s}(p_o)|^2)\langle g_j^{p_o,n},\mu^n_s\rangle\Big\}$$

$$+1/2\sum_{i,j}\widetilde{\partial_{ij} f(\mu^n_s+\nu^n_s)}2\gamma''(0)\langle\bar g^n_i,\bar g^n_j\rangle\Big]ds\ +\ M^{f,n}_{t_1,t_2}$$

$$+\Theta(\sup_t|\nu^n_{t}|_{t-r_o}M^2+N^{-r_o+1/2}M+N^{-1/2}M^5+N^{1/4\kappa}\alpha_n^{-1/2})+o(M+N^{-1/2}). \qquad (8.22)$$

Taking the limit $n\longrightarrow\infty$, we get a limit process $\xi_t=\mu_t+\nu_t$ with $\nu_t=0$ by (8.14) and μ_t given by (8.7-9), since this is the only solution to the martingale problem

$$\bar f(\mu_{t_2})-\bar f(\mu_{t_1})$$

$$=\int_{t_1}^{t_2}\Big[\sum_j\widetilde{\partial_j f(\mu_s)}\gamma^{(4)}(0)/[3!(\gamma''(0))^3]\ \Big\{(\hat\mu_s(0)^2+6|\hat\mu_s(p_o)|^2)\hat g_j(0)\hat\mu_s(0)$$

$$+3(\hat\mu_s(0)^2+|\hat\mu_s(p_o)|^2)\langle g_j^{p_o},\mu_s\rangle\Big\}$$

$$+1/2\sum_{i,j}\widetilde{\partial_{ij}f(\mu_s)}2\gamma''(0)\Big\{\hat g_i(0)\hat g_j(0)+2\mathrm{Re}(\hat g_i(p_o)\overline{\hat g_j(p_o)})\Big\}\Big]ds. \qquad (8.23)$$

The proof of Theorem 7 is complete.

APPENDIX

A PROPOSITION ON COLLAPSING PROCESSES

PROPOSITION.

(i) Let $X_t^m \geq 0$ be a sequence of positive semimartingales with

$$dX_t^m = S_t^m dt + \int f_{t-}^m(y)\,[\Lambda^m(dt,dy) - g_t^m(dy)dt]. \qquad (A.1)$$

Here S_t^m and f_t^m are adapted processes, Λ^m is a point process on some measurable space Y with compensator $g_t^m(dy)dt$. Let $\kappa > 1$ and let α_m be an increasing sequence with

$$m^{1/\kappa}\alpha_m^{-1} \longrightarrow 0, \quad \alpha_m . m^{-1} \longrightarrow 0, \qquad (A.2)$$

$$E^m(X_o^m)^\kappa \leqslant C_1 \alpha_m^{-\kappa} \quad \text{for all } m. \qquad (A.3)$$

Furthermore, τ^m are stopping times such that for $t \in [0,\tau^m]$, $m \geq 1$,

$$S_t^m \leqslant -m\delta X_t^m + C_2, \quad \delta > 0 \qquad (A.4)$$

$$\sup_{\omega\in\Omega, y\in Y, t\leqslant\tau^m} |f_t^m| \leqslant C_4 \alpha_m^{-1}, \qquad (A.5)$$

$$\int_Y (f_t^m(y))^2 g_t(dy) \leqslant C_5. \qquad (A.6)$$

(Here, and in the sequel, C_i are constants independent of m and X_t^m).

Then for any $\varepsilon > 0$, there exist $C_6 > 0$ and m_o such that

$$\sup_{m\geq m_o} P^m\{\sup_{0\leqslant t\leqslant T\wedge\tau^m} X_t^m > C_6(m^{1/\kappa}\alpha_m^{-1} \vee \alpha_m m^{-1})\} \leqslant \varepsilon. \qquad (A.7)$$

(ii) If instead of (A.6), we have even

$$\int_Y (f_t^m(y))^2 g_t(dy) \leqslant C_5(X_t^m + m^{-1}), \qquad (A.8)$$

then we get instead of (A.7)

$$\sup_{m\geq m_0} P^m\left(\sup_{0\leq t\leq T\wedge\tau^m} X_t^m > C_6\, m^{1/\kappa}\, \alpha_m^{-1}\right) \leq \varepsilon. \qquad (A.9)$$

Proof : (We drop the superscript m everywhere). Let h be a smooth, positive, increasing, convex function on $\mathbb{R}$ with

$$(y^+)^\kappa \leq h(y) \leq a + (y^+)^\kappa \,. \qquad (A.10)$$

and

$$\sup_{y_1\in\mathbb{R}} \sup_{|y_2|\leq C_4} h''(y_1+y_2)/h'(y_1) = C_7 < \infty \,. \qquad (A.11)$$

(A.11) implies $h(y_1+y_2)-h(y_1)-h'(y_1)y_2 \leq 1/2\, h'(y_1)C_7 y_2^2$ for all

$|y_2| \leq C_4$ and all $y_1\in\mathbb{R}$. | (A.12)

Now for $\ell = 1,\ldots,[Tm]+1$ and $t \leq (\ell/m)\wedge T\wedge\tau$, let

$$Z_t^\ell = \tilde{h}(X_t)$$
$$:= h\left(\alpha_m e^{\delta(mt-\ell)}(X_t - C_2/\delta m) - C_7\int_0^t \alpha_m^2 e^{2\delta(ms-\ell)}\int_Y |f_s(y)|^2 g_s(dy)ds\right). \qquad (A.13)$$

Ito's formula gives

$$dZ_t^\ell = \tilde{h}'(X_t)\alpha_m e^{\delta(mt-\ell)}(m\delta X_t - C_2 + S_t)$$
$$+\int_Y\left[\tilde{h}(X_t+f_t)-\tilde{h}(X_t)-\tilde{h}'(X_t)(\alpha_m e^{\delta(mt-\ell)}f_t + C_7\alpha_m^2 e^{2\delta(ms-\ell)}(f_t(y))^2)\right]g_t(dy)$$
$$+\int_Y[\tilde{h}(X_t+f_t)-\tilde{h}(X_t)](\Lambda(dy,dt)-g_t(dy)dt). \qquad (A.14)$$

Using (A.5), $mt-\ell\leq 0$ and (A.12), the first two terms in (A.14) are non-positive, such that Z_t^ℓ are positive supermartingales on $t \leq (\ell/m)\wedge T\wedge\tau$. Doob's inequality and (A.3) yield

$$P\left(\bigcup_{\ell=1}^{[mT]+1}\left\{\sup_{t\leq(\ell/m)\wedge T\wedge\tau} Z_t^\ell > m\eta^{-1}\right\}\right) \leq m^{-1}\eta\sum_{\ell=1}^{[Tm]+1} E(Z_0) \leq \eta(T+1)(a+C_1) \leq \varepsilon \quad (A.15)$$

for η sufficiently small. But $\sup_{t\leq(\ell/m)\wedge T\wedge\tau} Z_t^\ell \leq m\eta^{-1}$ is equivalent to

$$\alpha_m e^{\delta(mt-\ell)}(X_t - C_2/\delta m) \leq h^{-1}(m\eta^{-1}) + C_7\int_0^t \alpha_m^2 e^{2\delta(ms-\ell)}\int_Y |f_s(y)|^2 g_s(dy)ds \quad (A.16)$$

for all $t \leq (\ell/m)\wedge T\wedge\tau$. If we restrict t to the interval $[(\ell-1)/m, (\ell/m)\wedge T\wedge\tau]$, we see that by (A.10) and (A.6), resp. (A.8), (A.16) implies

$$X_t \leq C_2\delta^{-1}m^{-1} + e^{\delta}h^{-1}(m\eta^{-1})\alpha_m^{-1}$$
$$+ e^{\delta}C_7\alpha_m \sup_{s\leq t}\int_Y |f_s(y)|^2 g_s(dy)[e^{2\delta(mt-\ell)} - e^{-2\delta\ell}]/4\delta m$$
$$\leq C_2\delta^{-1}m^{-1} + C_8 m^{1/\kappa}\alpha_m^{-1} + C_9\alpha_m m^{-1}\begin{bmatrix} 1 \\ \sup_{s\leq t} X_s + m^{-1}\end{bmatrix}, \tag{A.17}$$

where the first component in the last bracket refers to the condition (A.6) and the second to (A.8). Thus by (A.2)

$$\bigcap_{\ell=1}^{[mT]+1}\left\{\sup_{t\leq(\ell/m)\wedge T\wedge\tau} Z_t^{\ell} \leq m\eta^{-1}\right\}$$
$$\subseteq\left\{\sup_{s\leq t} X_s \leq C_2\delta^{-1}m^{-1} + C_8 m^{1/\kappa}\alpha_m^{-1} + C_9\alpha_m m^{-1}\begin{bmatrix} 1 \\ \sup_{s\leq t} X_s + m^{-1}\end{bmatrix}\right\}$$
$$\subseteq\left\{\sup_{t\leq T\wedge\tau} X_s \leq C_{10}\begin{bmatrix} m^{1/\kappa}\alpha_m^{-1} \vee \alpha_m m^{-1} \\ m^{1/\kappa}\alpha_m^{-1}\end{bmatrix}\right\} \tag{A.18}$$

for m sufficiently large. (A.15) and (A.18) prove (A.7), resp. (A.9).

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CHAPITRE II .

Partie B : GRANDS ECARTS A LA LOI DES GRANDS NOMBRES . NUCLEATION .

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Nucleation for a long range magnetic model

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ABSTRACT. — We are interested in a local mean-field Ising model on the torus which exhibits two stable equilibria at low temperature and in the limit of infinite number of particles. Using large deviations techniques, we analyse the behaviour of the system during dynamical transitions from one equilibrium to the other: it is shown to be crucially dependent on the temperature and the interaction structure; symmetry breaking may occur, as in the asymptotic behaviour of the Gibbs measure.

Key-words: Mean-field, ising model, large deviations, nucleation.

RÉSUMÉ. — On considère un modèle d'Ising de champ moyen local sur le tore, qui présente deux états d'équilibre stable, dans l'asymptotique d'un nombre infini d'aimants et à température suffisamment basse. A l'aide de techniques de grandes déviations, on décrit le comportement du systeme lors des transitions dynamiques d'un de ces équilibres à l'autre : il dépend crucialement de la température ainsi que de la structure fine des interactions, et peut présenter une brisure de symétrie analogue à celle de la mesure de Gibbs.

I. INTRODUCTION

We are interested in long-time behaviour for a magnetic system, consisting in a large number N of Ising spins with fixed sites, and weak pair interaction (depending on distance between particles).

In the case of a ferromagnetic mean-field model without external influence, the Gibbs measure is concentrated on the neighbourhood of two stable steady states u^+, $-u^+$, at low temperature [*11*]. We consider a dynamic process, whose invariant probability is the Gibbs measure; on finite time intervals, it behaves—in first approximation—like the solution of an ordinary differential equation (the bigger N the better approximation) with u^+, $-u^+$ as stable equilibria. Because of ergodicity, the process starting near u^+ leaves the domain of attraction of u^+ in a finite time. Through this paper we study this type of dynamical phase transition and establish results conjectured by G. Ruget [*24*]. Such transitions can be studied using the theory of large deviations: one can refer to [*2*] [*15*] for finite dimensional processes. A quite recent reference to large deviations for distribution-valued processes is [*8*], with an application to the empirical distribution of a system of N weakly coupled diffusions; however, their model is quite different from the one studied in this paper.

Using large deviations estimates, we show under some conditions that the transition occurs at the neighbourhood of one of the « lowest saddle points » separating the two domains of attraction. We then give an example, where these saddle points can be found explicitely, and show how these results yield an explanation to nucleation [*23*]: at low temperature, the decisive step during a transition is the constitution of nuclei (of macroscopic size) in which local magnetization approaches that of the new equilibrium; these nuclei will later agregate as the whole system tends to the new equilibrium. The structure of the nuclei depends on the interaction function.

To make this more precise, we first define the *static model.*

For every integer n, we consider on $\mathbf{T} = (\mathbb{R}_{/\mathbb{Z}})^d$, the d-dimensional torus, $N = n^d$ magnets located at each point x of a square lattice with mesh $\frac{1}{n}$; the magnetization at each point is represented by a spin $\eta^n(x) \in \{-1, +1\}$.

Let $\mathcal{S}^n = \left\{ x \in \mathbf{T};\ x = \left(\frac{r_1}{n_n}, \ldots, \frac{r_d}{n}\right),\ r_1, \ldots, r_d \in \{0, 1, \ldots, n-1\} \right\}$ be the set of N sites, and $\mathcal{E}^n = \{-1, +1\}^{\mathcal{S}^n}$ the set of configurations η^n, $\eta^n = (\eta^n(x))_{x \in \mathcal{S}^n}$.

These magnets undergo an external field, represented by an element h of $C(\mathbf{T})$, the space of real continuous functions on $\mathbf{T}$, and interact according to a symmetric translation-invariant coupling represented by a symmetric function $J \in C(\mathbf{T})$. In statistical mechanics (cf. [*25*] [*26*]), one defines the internal energy of a configuration η^n as:

$$H^n(\eta^n) = -\sum_{x \in \mathcal{S}^n} h(x)\eta^n(x) - \frac{1}{2N} \sum_{x, y \in \mathcal{S}^n} J(x - y)\eta^n(x)\eta^n(y) \qquad (1.1)$$

and the Gibbs measure on $\mathscr{E}^n$ as:

$$G^n(\eta^n) = \frac{1}{2^N Z_h^n} \exp - \beta H^n(\eta^n) \tag{1.2}$$

where β is proportional to the inverse of the temperature, and where the constant Z_h^n makes G^n a probability. The multiplicative coefficient $\frac{1}{N}$ in the interaction term in (1.1) ensures the existence of asymptotics when n goes to infinity. Notice that the interaction is long range, wherefore this model is qualitatively different from nearest neighbour ones (for example see [*26*]); but interaction intensity depends on the distance between particles, thus being more general than the *Curie-Weiss model*, in which h and J are constant [*12*] [*13*]. This is a *local mean field* model (or long range model).

Let us describe now the *dynamics*.

For each N-particles system, the configuration will evolve with time, according to a stationary and reversible Markov process, whose invariant measure is the Gibbs measure G^n; spins are allowed to flip, at most one at a time (Glauber's dynamics, see [*17*]).

For $x \in \mathscr{S}^n$, let $\tau_x : \mathscr{E}^n = \{-1, +1\}^{\mathscr{S}^n} \to \mathscr{E}^n$ the operator of flip at site x:

$$\tau_x \eta^n(y) = \begin{cases} \eta^n(y) & \text{if} \quad y \neq x \\ -\eta^n(x) & \text{if} \quad y = x \end{cases}$$

and Δ_x operating on functions $f : \mathscr{E}^n \to \mathbb{R}$,

$$\Delta_x f = f \circ \tau_x - f.$$

The configuration being η^n at time t, we imagine for each site x a clock delivering a random time t_x with exponential law with intensity parameter $c^n(x, \eta^n)$.

All these variables are supposed to be independent of one another, and of the past. Let x_0 be the site with shortest time t_{x_0}; at time $t + t_{x_0}$, one flips the spin in x_0, and the previous mechanism is restarted. The resulting random process of configurations is denoted by η_t^n; its infinitesimal generator is

$$L^n f(\eta^n) = \sum_{x \in \mathscr{S}^n} c^n(x, \eta^n) \Delta_x f(\eta^n) \tag{1.3}$$

In order to obtain the previous properties together with asymptotics as n goes to infinity, we will restrict to jump parameters c^n of a suitable form given below in (1.9 to 1.11). Our purpose is to establish large deviation results for the configuration process: these being closely related to the large deviations results for Gibbs measure, we recall now the latter ones.

As the set $\mathcal{S}^n$ of configurations depends on n, we will represent the state of the system by a measure σ^n

$$\sigma^n = \frac{1}{N} \sum_{x \in \mathcal{S}^n} \eta^n(x)\delta_x = \eta^n \lambda^n \tag{1.4}$$

where δ_x is the Dirac mass at point x, and $\lambda^n = \frac{1}{N} \sum_{x \in \mathcal{S}^n} \delta_x$. As in [11], we could as well consider the density of magnetization

$$\xi^n = \sum_{x \in \mathcal{S}^n} \eta^n(x) \mathbf{1}_{x+\left[0,\frac{1}{n}\right[^d} \tag{1.5}$$

which is constant on the cubes $x + \left[0, \frac{1}{n}\right[^d$, $x \in \mathcal{S}^n$.

It's easy to transfer properties obtained for one of the representations to the other. We will use (1.4) for calculations, which can be written formally in a simpler way: for instance, $H^n(\eta^n)$ is equal to $-N \left\langle h + \frac{1}{2} J * \sigma^n, \sigma^n \right\rangle$ where $*$ denotes the convolution and $\langle , \rangle$ duality brackets. Nevertheless, in § 8, 9, we will consider ξ^n which is more suggestive.

Then σ^n belongs to the set $M_1(\mathbb{T})$ of all bounded measures μ on the Borel field of $\mathbb{T}$ with total variation norm $\|\mu\| \leq 1$. $M_1(\mathbb{T})$ will be furnished with the weak-* topology τ^* (weakened by $C(\mathbb{T})$); since $\lambda^n \xrightarrow[n\to\infty]{\tau^*} \lambda$ the Haar probability measure on $\mathbb{T}$, the states of the system will be represented in the limit $n \to \infty$ by measures $u\lambda$, with density $u \in B$ the closed unit ball of $\mathbb{L}^\infty(\mathbb{T}) = \mathbb{L}^\infty(\mathbb{T}; \lambda)$.

The following results are due to Eisele and Ellis [11], for general spin distribution; see [5] for the lower bound; the techniques of [16] also extend to this situation.

THEOREM 1.1.

1)
$$\lim_{n\to\infty} \frac{-1}{N\beta} \log Z_h^n = F_h$$

where the specific free energy F_h *is given by the variational problem*

$$F_h = \inf \{ V_h(u) ; u \in B \} \tag{1.6}$$

The potential V_h *is the* τ^**-lower-semi-continuous (l. s. c.) functional*

$$V_h(\mu) = -\left\langle u, h + \frac{1}{2} J * u \right\rangle + \frac{1}{\beta}\int_T \phi(u(x))dx \quad \text{if } \mu = u\lambda, \ \text{for some } u \in B \qquad (1.7)$$

$V_h(\mu) = \infty$ *otherwise*

and ϕ *denotes the Cramer transform of the single spin distribution* $\frac{1}{2}(\delta_1 + \delta_{-1})$:

$$\phi(W) = \frac{1 + W}{2}\log(1 + W) + \frac{1 - W}{2}\log(1 - W), \qquad W \in [-1, 1].$$

2) *For all* $A \subset M_1(T)$,

$$F_h - \inf_{\mu \in \mathring{A}} V_h(\mu) \leq \varliminf_{n \to \infty} \frac{1}{N\beta}\log G^n(A) \leq \varlimsup_{n \to \infty} \frac{1}{N\beta}\log G^n(A) \leq -\inf_{\mu \in \bar{A}} V_h(\mu) + F_h$$

(here, and in the following, we identify G^n *and its image by the application* $\eta^n \to \sigma^n$*).*

Therefore the support of any accumulation point of the sequence of probabilities G^n (on $M_1(T)$) is contained in the set of all the solutions of the variational problem (1.6). We will call stable equilibrium (or phase) any global minimum of V_h, metastable equilibrium any τ^*-local minimum of V_h, and more generally equilibrium any zero for the gradient ([1])

$$-dV_h(u) = -h - J * u + \frac{1}{\beta}\tanh^{-1} u \qquad (1.8)$$

Notice that an equilibrium is λ-equivalent to some element of $\mathscr{C}(T;]-1, 1[)$.

If $h = 0$ and $J \geq 0$, the model shows a phase transition (see previous references); for β greater the critical value $\beta_c = (\langle 1, J \rangle)^{-1}$, there are two stable equilibria, with constant densities u^+, and $-u^+$, where u^+ is the unique positive solution of the real equation associated to (1.7):

$$\tanh \frac{\beta}{\beta_c} u^+ = u^+.$$

Now we define the jump parameters

$$c^n(x, \eta^n) = c(x, \sigma^n)\exp\{-\eta^n(x)\beta(h + J * \sigma^n)(x)\} \qquad (1.9)$$

([1]) V_h is differentiable on $\{u; \|u\|_\infty < 1\}$ with respect to uniform norm with differential $v \to \langle dV_h(u), v \rangle$. In (1.8), $\tanh^{-1}$ denotes the inverse function of tanh.

with c a *continuous* function on $\mathbb{T} \times M(\mathbb{T})$ (set of all bounded measures on $\mathbb{T}$, furnished with topology τ^*) to $]0, +\infty[$. We furthermore assume that

$$\forall x \in \mathbb{T}, \quad \forall \mu \in M_1(\mathbb{T}), \qquad c(x, \mu - \mu\{x\}\delta_x) = c(x, \mu) \tag{1.10}$$

and that there exists some C_0 (capital C will denote constants) such that

$$\| c(u_1) - c(u_2) \|_1 \leqq C_0 \| u_1 - u_2 \|_1 \qquad \forall u_1, u_2 \in L^1(\mathbb{T}) \tag{1.11}$$

Relations (1.9, 1.10) imply that « detailed balanced conditions » are fulfilled with respect to G^n (see [25]); the form of the multiplicative factor c of the exponential in (1.9) ensures us with the existence of asymptotics and (1.11) with the uniqueness of the limit process.

The simplest case is $c(x, \mu) = 1$, which is the situation considered in [5]. Other examples are given by $c(x, \mu) = f(\theta_1 * \mu(x), \ldots, \theta_K * \mu(x))$ with $\theta_k \in \mathscr{C}(\mathbb{T})$ and $\theta_k(0) = 0$ for $k = 1, \ldots, K$, and f a Lipschitz continuous function on $\mathbb{R}^K$.

For any sign $\eta \in \{-1, +1\}$ let

$$c_\eta(x, \mu) = c(x, \mu) \exp\{-\eta\beta(h + J * \mu)(x)\} \tag{1.11 b}$$

Then
$$c^n(x, \eta^n) = \sum_{\eta \in \{-1, +1\}} \frac{1 + \eta\eta^n(x)}{2} c_\eta(x, \sigma^n) \tag{1.12}$$

Let g be a bounded measurable function on $\mathbb{T}$, $F_g : \mu \to \langle g, \mu \rangle$; applying (1.3) to $f(\eta^n) = \frac{1}{N} \sum_{x \in \mathscr{S}^n} g(x)\eta^n(x)$, we derive the infinitesimal generator ([2]) of the measure-value process σ_t^n, restricted to such linear functional F_g:

$$L^n F_g(\mu) = - \sum_{\eta \in \{-1, +1\}} \langle \mu + \eta\lambda^n, g c_\eta(\mu) \rangle \tag{1.13}$$

Because the particles are weakly interacting, it turns out that this process converges uniformly on finite time intervals to the solution $u_t \in B$ of the ordinary differential equation

$$\frac{d}{dt} u_t = - \sum_{\eta \in \{-1, +1\}} (u_t + \eta) c_\eta(u_t) \tag{1.14}$$

$$= -2c(u_t)\sqrt{1 - u_t^2} \sinh \beta d V_h(u_t) \tag{M. E.}$$

([2]) Still denoted by L^n.

the mean evolution equation ([3]): the right hand side of (1.14) is obtained in taking the limit $n \to \infty$ in (1.13). In the simpler case of Curie-Weiss model, this law of large numbers may be found in physical literature (see [*18*]), and in [*22*] for a global mean field on $\mathbb{Z}^d$.

Notice that the equilibrium are the stationary points for equation (1.14). Furthermore, one can show that V_h is a Lyapunov function ([4]) for the dynamical system (1.14), in the sense that V_h is decreasing along its trajectories.

Hence, the transitions from the neighbourhood of a stable equilibrium to another are large deviations from the law of large numbers: we need estimates for the probability of such an event. We will obtain the following result:

let $T > 0$, $u_0 \in \mathscr{C}(\mathbb{T};]-1, 1[)$, σ_0^n a sequence of initial magnetization measures such that $\tau^* - \lim_{n\to\infty} \sigma_0^n = u_0$, and $A \subset \mathscr{D}\{[0,T]; M_1(\mathbb{T})\}$ the space of all right-continuous left-limited functions on $[0,T]$, with values in $(M_1(\mathbb{T}); \tau^*)$.

Let (A) be the set of interior points of A with respect to the uniform convergence topology, [A] its closure.

THEOREM 1.2. — *There exists a functional* I_{0T} *such that the inequalities*

$$- \inf \{ I_{0T}(\varphi); \varphi \in (A), \varphi_0 = u_0 \} \leq \liminf_{n\to\infty} \frac{1}{N} \operatorname{Log} P_{\sigma_0^n}^n \{ \sigma^n \in A \} \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{N} \operatorname{Log} P_{\sigma_0^n}^n \{ \sigma^n \in A \} \leq - \inf \{ I_{0T}(\varphi); \varphi \in [A], \varphi_0 = u_0 \}$$

hold whenever $\{ \sigma^n \in A \}$ *is measurable for all* n($P_{\sigma_0^n}^n$ *denotes the law of the magnetisation process starting at* σ_0^n).

The action functional I_{0T}, or « energy », will be defined in section 3. It is such that $I_{0T}(\varphi) \geq 0$, with equality if and only if φ satisfies (1.14); furthermore, the least energy trajectories which leave a potential wells are time-reversed solutions of M. E., this least energy being related to the potential V_h.

In section 3, we also give some properties of I_{0T}, which are proved in appendix. We establish the Vent'sel-Freidlin estimates for large deviations in § 4.5. Technical difficulties essentially arise from the lack of regularity

([3]) From (1.11), (1.14) has a unique solution in $L^1(\mathbb{T})$; a precise study on the of B shows that $\| u_t \|_\infty < 1$ for all $t > 0$.

([4]) Use inequality $z \sinh z \geq z^2$, z real (notice that the vector field is not a gradient field).

of various functionals at the boundary (local magnetization equal to $+1$ or -1), this boundary not being rare enough (in the sense of large deviations probability) to be negligible. The lower bound for the large deviations probability is obtained in a manner slightly different from [27] in the finite dimensional case (another problem being the structure of neighbourhood of 0 in the weak topology); as for the upper bound, we first show a local estimate, then extend it similarly to the proof of Sanov's theorem [3]. The law of large numbers is a by-product of theorem V.1: it justifies intuitively some further choices, but will not be used in the proofs: therefore we do not give a more precise statement of it. Theorem I.2 is a straight consequence of theorems IV.1 and V.1 (see 7.6 in [2] for the proof). In § 7, we solve the problem of exit points from a basin of attraction; the result extends the well known one in [15]. The quasi-potential $W(u_e, u)$, which represents the minimal energy to go from an equilibrium u_e to u, is a lower semi-continuous function of u; but this doesn't change anything compared to the classical situation, as we can guess from the result of [14]. As an application, we study nucleation in a simple model.

II. BASIC PROPERTIES AND PRELIMINARIES

Since $\mathcal{S}^n$ is finite, there exist a probability space $(\Omega^n, \mathbf{F}, \mathrm{P}^n)$ and a process η^n on Ω^n with generator L^n given by (1.3). For $\eta_0^n \in \mathcal{E}^n$, $\mathrm{P}^n_{\eta_0^n}$ will denote the law of the configuration process $(\eta_t^n)_{t \in \mathbb{R}^+}$ starting at η_0^n, or, equivalently, of the measure value process $(\sigma_t^n)_{t \in \mathbb{R}^+}$. Let $\mathbf{F}_t$ be the σ-field generated by the variables η_s^n, $s \leq t$.

Let $g(t, x)$ be a bounded measurable function on $\mathbb{R}^+ \times \mathbb{T}$, such that the set $\{ t \in \mathbb{R}^+ ; \exists x \in \mathbb{T}, s \to g(s, x)$ is discontinuous at point $t \}$ is discrete. The process η_t^n is of bounded variation on every finite interval of $\mathbb{R}^+$ with probability 1, so we can define as Stieltjes integrals the quantities

$$\int_0^t \langle g_s, d\sigma_s^n \rangle = \frac{1}{N} \sum_{x \in \mathcal{S}^n} \int_0^t g(s^-, x) d\eta_s^n(x).$$

In the following, we shall use the following probabilistic results (see [19] or [20]), and use (1.12):

$i)$ $$\mathrm{M}_t^n(g) = \int_0^t \langle g_s, d\sigma_s^n \rangle - \int_0^t \mathrm{L}^n \mathrm{F}_{g_s}(\sigma_s^n) ds$$

is a $(P^n, \mathbb{F}_t)$-martingale with increasing process

$$\langle M^n(g) \rangle_t = \frac{2}{N} \int_0^t \sum_{\eta \in \{-1,+1\}} \langle \lambda^n + \eta \sigma_s^n, g_s^2 c_\eta(\sigma_s^n) \rangle \, ds. \qquad (2.1)$$

ii) For $\mu \in M_1(\mathbb{T})$ and h' bounded measurable function on $\mathbb{T}$ (so-called because it is formally an external field) let's define

$$\Gamma_n^*(\mu, h') = \sum_{\eta \in \{-1,+1\}} \left\langle \frac{\lambda^n + \eta\mu}{2}, (e^{-\eta\beta h'} - 1) c_\eta(\mu) \right\rangle \qquad (2.2)$$

Then $$R_t^n(g) = \exp\left\{ N \int_0^t \frac{\beta}{2} \langle g_s, d\sigma_s^n \rangle - \int_0^t \Gamma_n^*(\sigma_s^n, g_s) ds \right\} \qquad (2.3)$$

is a $(P^n, \mathbb{F}_t)$-martingale.

Let's define the probability $\tilde{P}^n$, by its restriction to $\mathbb{F}_T$

$$d\tilde{P}^n / \mathbb{F}_T = R_T^n(g) \, dP^n / \mathbb{F}_T .$$

Denoting by $\tilde{c}_t^n$, $\tilde{L}_t^n$ for $t \leq T$ the analogues to (1.3, 1.12) with

$$\tilde{c}_{\eta,t} = c_\eta . \exp - \eta\beta g_t \qquad (2.4)$$

instead of c_η, $\tilde{L}_t^n$ is the infinitesimal generator of the process $\tilde{P}^n$. In particular, the analogue of property *i)* is valid for this last process.

Because of (1.12, 2.4), $\tilde{P}^n$ is the law of the magnetization process evolving under external field $h + g_t$. This fact is the counterpart of the duality relationship (1.6), in which F_h is written like the Legendre transform of V_0 (i. e. V_h for $h = 0$): magnetization u and external field h are conjugate variables. We will prove that the law of large numbers remains valid —with the coefficients $\tilde{c}$— for a large class of such (non stationary) processes $\tilde{P}^n$ (see (4.7)).

We need some topological properties of the space $M_1(\mathbb{T})$, that we state here for convenience:

PROPOSITION II.1. — $(M_1(\mathbb{T}), \tau^*)$ *is a metrizable compact space.*

$M_1(\mathbb{T})$ is the closed unit ball of $M(\mathbb{T})$, so it is compact for weak-* topology. $\mathscr{C}(\mathbb{T})$ is a separable space according to Stone-Weierstrass theorem, and $M_1(\mathbb{T})$ is strongly bounded; so [*21*] τ^* is metrizable on $M_1(\mathbb{T})$, and defined by the metric ρ

$$\rho(\mu, \nu) = \sup_{m \in \mathbb{Z}^d} \{ (1 + |m|)^{-1} | \langle \mu - \nu, e^{2i\pi m.x} \rangle | \}.$$

Notice that $\mu \in B$ is equivalent to $0 \leq \frac{\mu + \lambda}{2} \leq \lambda$, and therefore B is τ^*-compact too.

Let $\rho_{0T}(\mu, \nu) = \sup \{ \rho(\mu_t, \nu_t) : t \in [0, T] \}$ be the uniform metric on the finite time interval $[0, T]$. By computations similar to those of the end of § 4, we can show that $u_0 \to u$ the solution of (1.4) starting at u_0, is continuous on (B, τ^*) to $\mathscr{C}([0, T]; B)$.

Through this paper, $\mathscr{A} = \{ A_k ; k = 1, 2 \dots K \}$ will denote a partition of $\mathbf{T}$ in rectangles (i. e.: product of connected sets of $\mathbb{R}/\mathbb{Z}$) with non-empty interior.

Let $\pi^{\mathscr{A}}$ the projection operator associating to a measure μ the Radon-Nikodym derivative of its restriction $\mu/\mathscr{A}$ to the algebra generated by $\mathscr{A}$ with respect to $\lambda/\mathscr{A}$:

$$\pi^{\mathscr{A}}\mu = \frac{d\mu/\mathscr{A}}{d\lambda/\mathscr{A}} = \sum_{k=1}^{K} \frac{\mu(A_k)}{\lambda(A_k)} 1_{A_k}. \tag{2.5}$$

For $\mathscr{A}_n$ the algebra generated by the cubes $x + \left[0, \frac{1}{n}\right[^d$, $x \in \mathscr{S}^n$, one sees that $\xi^n = \pi^{\mathscr{A}_n}\sigma^n$. In § IV, V, we will use operator $\pi^{\mathscr{A}}$ to define sets that are approximately neighbourhoods of 0:

PROPOSITION II.2. — i) *Given such a partition $\mathscr{A}_0$, and a τ^*-neighbourhood $\mathscr{V}$ of 0 in $M(\mathbf{T})$, there exist a finer partition $\mathscr{A}$ and $\varepsilon > 0$ such that*

$$\forall \mu, \nu \in M_1(\mathbf{T}), \qquad \| \pi^{\mathscr{A}}(\mu - \nu) \|_1 < \varepsilon \Rightarrow \mu - \nu \in \mathscr{V}.$$

ii) *Given $\mathscr{A}$ and $\varepsilon > 0$, there exist and integer n_0 and a weak neighbourhood $\mathscr{V}$ of 0 in $M(\mathbf{T})$ such that for all $u \in B$, $n \geq n_0$ and $\sigma^n \in \mathscr{E}^n$.*

$$\sigma^n - u \in \mathscr{V} \Rightarrow \| \pi^{\mathscr{A}}(\sigma^n - u) \|_1 < \varepsilon.$$

To prove *i)* use uniform approximation of continuous functions by step functions on $\mathscr{A}$, then recall the inequality $\| \mu \| \leq 1$; for *ii)* notice that a strip of width α on the torus contains at most $\left(\alpha + \frac{1}{n}\right)$N points of $\mathscr{S}^n$ lattice.

III. THE ACTION FUNCTIONAL I_{0T}

In this section we state some standard properties of the action functional I_{0T}. The proofs of the results III.3, 4 and 6, somewhat technical, are carried out in the appendix.

First of all, we anticipate the demonstration of theorem IV.1 in order to introduce the action functional in a heuristic manner. Let's fix some time T, and consider a smooth enough trajectory φ defined on $[0, T]$ with values in B; let's try and estimate the probability for the process σ^n to be uniformly close to φ on $[0, T]$, following the idea of [27].

We look for some exponential change of probability making φ the central path; since magnetization and external field are conjugate variables (see § 2), it will consist in an adequate choice of some extra external field $\tilde{h}_t$, under which φ *satisfies the mean evolution equation* M.E.: let $\tilde{P}^n$ be the probability law on $(\Omega^n, \mathbb{F}_T)$ of the magnetization process with external field $h + \tilde{h}_t$

$$\frac{d\tilde{P}^n}{dP^n} = R_T^n = \exp N\left\{\int_0^T \frac{\beta}{2}\langle \tilde{h}_t, d\sigma_t^n \rangle - \int_0^T \Gamma_n^*(\sigma_t^n, \tilde{h}_t)dt\right\} \qquad (3.1)$$

We then require the analogue of (1.14) for $\tilde{P}^n$

$$\dot{\varphi}_t = -\sum_{\eta\{-1,1\}} (\varphi_t + \eta)\tilde{c}_{\eta,t}(\varphi_t) \qquad (3.2)$$

with $\dot{\varphi}_t$ the time derivative of φ_t and $\tilde{c}_{\eta,t}$ given by relation (2.4). Using (1.11 *b*), we derive the following expression for $\tilde{h}_t$:

$$\tilde{h}_t = -h - J * \varphi_t + \beta^{-1}\tanh^{-1}\varphi_t + \beta^{-1}\sinh^{-1}\frac{\dot{\varphi}_t}{2c(\varphi_t)\sqrt{1-\varphi_t^2}} \qquad (3.3)$$

where $\tanh^{-1}$, $\sinh^{-1}$ denote the inverse functions of $\tanh$, $\sinh$.

Formally, the computation will consist in writing $P^n(\sigma^n \sim \varphi)$ as $\tilde{\mathbb{E}}^n\{\mathbb{1}_{\{\sigma^n \sim \varphi\}}(R_T^n)^{-1}\}$, with $\tilde{\mathbb{E}}^n$ the expectation for $\tilde{P}^n$. For trajectories σ^n, close to φ, we replace approximately $\Gamma_n^*(\sigma_t^n, \tilde{h}_t)$ with $\Gamma_n^*(\varphi_t, \tilde{h}_t)$ and $\int_0^T \langle \tilde{h}_t, d\sigma_t^n \rangle$ with $\int_0^T \langle \tilde{h}_t, \dot{\varphi}_t \rangle dt$ using the law of large numbers for $\tilde{P}^n$. We now recall that φ is the central path for the process $\tilde{P}^n$, and obtain the estimate $\exp -N\int_0^T \{\langle \tilde{h}_t, \dot{\varphi}_t \rangle - \Gamma_n^*(\varphi_t, \tilde{h}_t)\}dt$ for the previous probability. This justifies the

III.1. Definition of the action functional I_{0T}.

Because of (1.11 *b*, 2.2), we define for $u \in B$, $a \in \mathbb{R}$ and $h' \in \mathbb{L}^\infty(\mathbb{T})$

$$\Gamma(u, a, x) = c(\varphi)\sum_{\eta\in\{-1,+1\}} \frac{1+\eta u}{2} e^{-\eta\beta(h+J*u)}(e^{-\eta\beta a} - 1)(x)$$

and

$$\Gamma^*(u, h') = \int_{\mathbb{T}} \Gamma(u, h'(x), x)dx. \tag{3.4}$$

For an evolution speed $v \in \mathbb{L}^1(\mathbb{T})$ of the magnetization, its Legendre transform is

$$\mathcal{H}^*(u, v) = \sup_{h' \in \mathbb{L}^\infty(\mathbb{T})} \left\{ \frac{\beta}{2} \langle v, h' \rangle - \Gamma^*(u, h') \right\} \tag{3.5}$$

$\Gamma^*(u, .)$ is a convex differentiable function on $\mathbb{L}^\infty(\mathbb{T})$. If $\| u \|_\infty < 1$, the supremum (3.5) is achieved for h' given by the right-hand side of formula (3.3) with u instead of φ_t, and is equal to

$$\mathcal{H}^*(u, v) = \int_{\mathbb{T}} \left[\frac{v}{2} \operatorname{Log} \frac{\dfrac{v}{2c(u)} + \sqrt{1 - u^2 + (v/2c(u))^2}}{1 - u} \right.$$
$$- \beta \frac{v}{2}(h + J * u) + c(u) \left\{ -\sqrt{1 - u^2 + (v/2c(u))^2} \right.$$
$$\left. \left. + \cosh \beta(h + J * u) - u \sinh \beta(h + J * u) \right\} \right](x)dx \tag{3.6}$$

Troughout this paper, we furnish $\mathscr{C}([0, T]; B)$ with metric ρ_{0T} defined in § 2; for an element φ of this space, we denote by (D) the following differentiability condition:

$\exists \dot{\varphi} \in \mathbb{L}^1([0, T] \times \mathbb{T})$ *such that for all* $t \leq T$,

$$\varphi_t(x) - \varphi_0(x) = \int_0^t \dot{\varphi}(s, x)ds \quad \lambda\text{-a. s.}$$

We will then denote $\dot{\varphi}(s, x) = \dot{\varphi}_s(x)$.

DEFINITION III.1. — *The action functional* I_{0T} *is*

$$I_{0T}(\varphi) = \begin{cases} \displaystyle\int_{[0,T]} \mathcal{H}^*(\varphi_t, \dot{\varphi}_t)dt & \textit{if } \varphi \textit{ satisfies to property (D)} \\ \infty & \textit{otherwise} \end{cases}$$

We shall say that an element φ of $\mathscr{C}([0, T]; B)$ is absolutely continuous if for all $\varepsilon > 0$, there exists some $\Delta > 0$ such that for all integer i_0 and all rectangles $A_1, \ldots, A_{i_0}$ of $\mathbb{T}$, and all real numbers $s_1, t_1, \ldots, s_{i_0}, t_{i_0}$ satisfying to $0 \leq s_i < t_i \leq T$, the inequality $\sum_{i \leq i_0} | t_i - s_i | \lambda(A_i) < \Delta$ implies $\sum_{i \leq i_0} | \langle \varphi_{t_i} - \varphi_{s_i}, \mathbf{1}_{A_i} \rangle | < \varepsilon$.

PROPOSITION III.2. — *$\varphi \in \mathscr{C}([0, T]; B)$ satisfies* (D) *if and only if φ is absolutely continuous.*

The proof of the proposition is standard (see [*10*]), and is not carried out here.

III.2. Some properties of the action functional.

We first notice that if φ satisfies to (D), we can find a modification of $\dot{\varphi}$ such that $\varphi\dot{\varphi} \leq 0$ at all points (t, x) such that $| \varphi | = 1$. We will then suppose this condition fulfilled by functions u, v in the following of this section. We need some technical results for obtaining usual properties of I_{0T}:

PROPERTIES III.3.

a) $$I_{0T}(\varphi) = \sup_{f \in L^\infty([0,T] \times T)} \left\{ \int_0^T \left[\frac{\beta}{2} \langle f(t, .), \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, f(t, .)) \right] dt \right\}$$
$$= \int_{[0,T] \times T} \mathscr{H}(\varphi_t, \dot{\varphi}_t(x), x) dt dx$$

with

$$\mathscr{H}(u, v(x), x) = \sup_{a \in \mathbb{R}} \left\{ \frac{\beta}{2} v(x) a - \Gamma(u, a, x) \right\} \qquad (3.7)$$

b) $I_{0T}(\varphi) < \infty$ if and only if $\dot{\varphi} \operatorname{Log} | \dot{\varphi} |$, $\varphi \operatorname{Log} \frac{1}{1 - \varphi} \mathbf{1}_{(\dot{\varphi} > 0)}$ and $\dot{\varphi} \operatorname{Log} \frac{1}{1 + \varphi} \mathbf{1}_{(\dot{\varphi} < 0)}$ are elements of $L^1([0, T] \times \mathbb{T})$.

c) There exists some constant K such that

$$\mathscr{H}(u, v, x) \leq \frac{| v |}{2} \left[\operatorname{Log} | v | + \mathbf{1}_{v > 0} \operatorname{Log} \frac{1}{1 - u} + \mathbf{1}_{v < 0} \operatorname{Log} \frac{1}{1 + u} + K \right] (x) + K$$

(here, and up to property *e*) we write v for $v(x)$, no confusion being possible).

d) There exists some constant $K > 0$ such that

$$\mathscr{H}(u, v, x) \geq \frac{1}{2} | v | [\operatorname{Log} | v | - K] - K$$

e) For $\gamma > 0$ we have

$$| \mathcal{H}(u, v, x) - \mathcal{H}(u_1, v, y) |$$
$$= (1 + |v|)\{ C_\gamma[\, | u(x) - u_1(y) |\,] + c_\gamma[\, | x - y | + \rho(u, u_1)\,] \}$$

for all u, u_1 such that $\| u \|_\infty, \| u_1 \|_\infty \leq 1 - \gamma$, all $x, y \in \mathbb{T}$ and $v \in \mathbb{R}$.

The property *a*) shows that one can reverse the order of the supremum and the integrals; *b*) is a characterisation of finite energy trajectories. With upper bound *c*) one can limit to consider magnetization densities avoiding the boundary points -1, $+1$. The continuity property *e*) is somewhat similar to condition (C) in [*27*]; « outer » speeds being forbidden at these boundary points, it only holds for non-zero γ. The regularity in the x variable is a (new) property that enables us to replace magnetization u with a smooth function on $\mathbb{T}$ in proposition III.6 *d*) shows how $\mathcal{H}$ increases at infinity; it is an usual property for Cramer transforms.

Furthermore one can notice that the condition required in [*2*] is not satisfied here, because the set of possible speeds is discontinuous at the boundary points -1, $+1$.

THEOREM III.4. — 1) $D_{I_0} = \{ \varphi ; I_{0T}(\varphi) \leq I_0 \}$ *is compact in* $\mathcal{C}([0, T]; B)$ *for all non negative* I_0.

2) *The functional* I_{0T} *is lower semi-continuous on* $\mathcal{C}([0, T]; B)$.

This result ensures us with existence of solution to variational problem $\min \{ I_{0T}(\varphi) ; \varphi \in A \}$ for closed subset A of $\mathcal{C}([0, T]; B)$.

Remark. — Whenever φ satisfies to (D), φ is continuous on $[0, T]$ with values in B furnished with $\| \cdot \|_1$ norm; but this topology is too fine to make D_{I_0} compact.

In the proof of theorem IV.1, we will need a large enough class of smooth functions: piecewise $\mathcal{C}^{1,0}$ functions.

DEFINITIONS III.5. — *We define* $\mathcal{C}P_T^{1,0}$ *as the class of all* φ *of* $\mathcal{C}([0, T] \times \mathbb{T} ;\ ]-1, 1[)$ *such that there exists a subdivision* $S = (t_k)_{k \leq k_0}$ *of* $[0, T]$ *with*:

$$\forall k \leq k_0 - 1, \frac{\partial \varphi}{\partial t} \textit{ exists on } [t_k, t_{k+1}] \times \mathbb{T} \textit{ and is continuous.}$$

Then φ satisfies to (D), and $\dot{\varphi} = \dfrac{\partial \varphi}{\partial t}$.

PROPOSITION III.6. — *Let* φ *with* $I_{0T}(\varphi) < \infty$, $\varphi_0 \in \mathcal{C}(\mathbb{T};\ ]-1, 1[)$, γ, $\delta > 0$. *Then, there exists* $\tilde{\varphi} \in \mathcal{C}P^{1,0}$ *such that*

$$\varphi_0 = \tilde{\varphi}_0, \qquad \rho_{0T}(\varphi, \tilde{\varphi}) < \delta \qquad \textit{and} \qquad | I_{0T}(\varphi) - I_{0T}(\tilde{\varphi}) | < \gamma \quad (3.8)$$

IV. LARGE DEVIATIONS: LOWER BOUND FOR THE PROBABILITY OF PASSAGE IN A TUBELET

For $\varphi \in \mathscr{C}([0,T]; B)$ and $\delta > 0$, we define the $[0,T]$-tubelet with axis φ and radius δ as the set of all $\mu: [0,T] \to M_1(\mathbb{T})$ such that $\rho_{0T}(\mu, \varphi) < \delta$. We shall denote it shortly by $\{ \varphi \}^\delta$.

THEOREM IV.1. — *Let $\delta > 0$ and $\varphi \in \mathscr{C}([0,T]; B)$ with $\varphi_0 \in \mathscr{C}(\mathbb{T};]-1,1[)$. For all $\gamma > 0$, there exist an integer n_0 and $\delta_1 > 0$ such that $n \geq n_0$ implies*

$$P^n_{\sigma_0^n} \{ \rho_{0T}(\sigma^n, \varphi) < \delta \} \geq \exp - N \{ I_{0T}(\varphi) + \gamma \}$$

on the set $\{ \rho(\sigma_0^n, \varphi_0) < \delta_1 \}$.

□ *Proof.* — Suppose first $\varphi \in \mathscr{C}P_T^{1,0}$ (see def. III.5).

We can define the extra external field $\tilde{h}_t$ by (3.3) and the probability $\tilde{P}^n$ by (3.1); as written in the beginning of § III,

$$P^n_{\sigma_0^n}(\{ \varphi \}^\delta) = \tilde{E}^n_{\sigma_0^n} \{ (R^n_T)^{-1} \mathbb{1}_{\{\varphi\}^\delta} \}. \tag{4.1}$$

φ being a smooth function, there exists a finite subset S of $[0,T]$ such that the family $\{ \tilde{h}_t ; t \notin S \}$ be equicontinuous on $\mathbb{T}$; so is $\{ c_\eta(\mu); \eta \in \{ -1, +1 \}, \mu \in M_1(\mathbb{T}) \}$. Then, the Riemann sums in $\Gamma^*_n(\mu, \tilde{h}_t)$ converges to the λ-integral, uniformly for $t \notin S$ and $\mu \in M_1(\mathbb{T})$, and this last quantity converges uniformly to

$$\Gamma^*(\mu, \tilde{h}_t) = \sum_{\eta \in \{-1,+1\}} \left\langle \frac{\lambda + \eta\mu}{2}, (e^{-\eta\beta\tilde{h}_t} - 1) c_\eta(\mu) \right\rangle \tag{4.2}$$

which is an extension of (3.4). Recall that the generator $\tilde{L}^n_t$ of the process $\tilde{P}^n$ is given by (1.13) with $\tilde{c}_{\eta,t} = c_\eta . \exp - \eta\beta\tilde{h}_t$ instead of c_η: in particular, $M^n(\tilde{h}) = \int_0^T \langle \tilde{h}_t, d\sigma^n_t \rangle - \int_0^T \tilde{L}^n_t(F_{\tilde{h}_t})(\sigma^n_t)dt$ (notation F_g being defined just before (1.13) is a random variable with mean 0 for $\tilde{P}^n$ and variance less than $C_1 N^{-1}$, relation (2.1) showing that the constant C_1 depends only on φ. Using Chebicheff's inequality, we choose some integer n_1 such that

$$\forall n \geq n_1, \quad \forall \sigma_0^n \in \mathscr{E}^n, \quad \tilde{P}^n_{\sigma_0^n} \{ |M^n_T(\tilde{h})| \leq \gamma/6 \} \geq 3/4 \tag{4.3}$$

As above, we notice that $\tilde{L}^n_t(F_{\tilde{h}_t})(\mu)$ converges informly to $\tilde{L}_t(F_{\tilde{h}_t})(\mu)$, with

$$\tilde{L}_t(F_g)(\mu) = - \sum_{\eta \in \{-1,+1\}} \langle \mu + \eta\lambda, g\,\tilde{c}_{\eta,t}(\mu) \rangle \tag{4.4}$$

we then choose n_2 such that, for $n \geq n_2$, we can replace on $\{ M_T^n(\tilde{h}) \leq \gamma/6 \}$, up to an error of magnitude $\gamma/6$ for each operation, $\int_0^T \langle \tilde{h}_t, d\sigma_t^n \rangle$ with $\int_0^T \tilde{L}_t^n(F_{\tilde{h}_t})(\sigma_t^n)dt$, this last term with $\int_0^T \tilde{L}_t(F_{\tilde{h}_t})(\sigma_t^n)dt$, and Γ_n^* with Γ^*; we obtain:

$$\mathbf{1}_{\{M_T^n(\tilde{h}) \leq \frac{\gamma}{6}\}} R_T^n \leq \exp N \left\{ \int_0^T \left[\frac{\beta}{2} \tilde{L}_t(F_{\tilde{h}_t})(\sigma_t^n) - \Gamma^*(\sigma_t^n, \tilde{h}_t) \right] dt + \frac{\gamma}{2} \right\}. \quad (4.5)$$

We need the following result, where $L^{h'}$ denotes the operator given by (4.4) with h' instead of $\tilde{h}_t$ (i.e.: $c_\eta \exp - \eta\beta h'$ instead of $\tilde{c}_{\eta,t}$):

LEMMA IV.2. — *Let Q be a compact subset of $\mathscr{C}(\mathbf{T})$. The family* $\{ \mu \to \Gamma^*(\mu, h'), \mu \to L^{h'}(F_{h'})(\mu) ; h' \in Q \}$ *is equicontinuous on* $(M_1(\mathbf{T}), \rho)$.

We go on the proof of the theorem: since φ is smooth, Ascoli's theorem shows that $\{ \tilde{h}_t ; t \notin S \}$ is relatively compact; the lemma yields some $\delta' < \delta$ such that:

$$\left\{ \begin{array}{l} \mu_1, \mu_2 \in M_1(\mathbf{T}) \\ \rho(\mu_1, \mu_2) < \delta' \end{array} \right. \Rightarrow \left\{ \begin{array}{l} | \tilde{L}_t(F_{\tilde{h}_t})(\mu_1) - \tilde{L}_t(F_{\tilde{h}_t})(\mu_2) | < (3\beta T)^{-1}\gamma \\ | \Gamma^*(\mu_1, \tilde{h}_t) - \Gamma^*(\mu_2, \tilde{h}_t) | \leq (6T)^{-1}\gamma \end{array} \right. \forall t \in [0, T] - S \quad (4.6)$$

Now, we claim it's enough to find $n_3 \in \mathbb{N}$, $\delta_1 > 0$ with:

$$\rho(\sigma_0^n, \varphi_0) < \delta_1, \qquad n \geq n_3 \Rightarrow \tilde{P}_{\sigma_0^n}^n(\{ \varphi \}^{\delta}) \geq \frac{3}{4}. \quad (4.7)$$

Indeed, for $n \geq n_1 \vee n_2 \vee n_3$, relations (4.4, 8, 9) imply:

$$P_{\sigma_0^n}^n(\{ \varphi \}^{\delta}) \geq \tilde{E}_{\sigma_0^n}^n \{ (R_T^n)^{-1} \mathbf{1}_{\{\varphi\}^{\delta'}} \mathbf{1}_{\{|M_T^n(\tilde{h})| \leq \frac{\gamma}{6}\}} \}$$

$$\geq P_{\sigma_0^n}^n\left(\{ \varphi \}^{\delta'} \cap \left\{ | M_T^n(\tilde{h}) | \leq \frac{\gamma}{6} \right\} \right)$$

$$\exp - N\left(\int_0^T \left[\frac{\beta}{2} \tilde{L}_t(F_{\tilde{h}_t})(\varphi_t) - \Gamma^*(\varphi_t, \tilde{h}_t) \right] dt + 5\frac{\gamma}{6} \right).$$

Combining (4.3 and 7), we see that the last probability is not less than 0.5. φ being the central path for $\tilde{P}^n$, $\tilde{L}_t(F_{\tilde{h}_t})(\varphi_t) = \langle \tilde{h}_t, \dot{\varphi}_t \rangle$ holds for all $t \notin S$ (one can compute it from (3.2) and (4.4)): recalling then that $\tilde{h}_t$ is the solution to variational problem (3.5), we see that the term between brackets in the last exponential is equal to $\mathscr{H}^*(\varphi_t, \dot{\varphi}_t)$, this yields the desired result.

We now prove (4.7). From proposition I.4.*i*) we first fix some partition $\mathscr{A}$ of $\mathbb{T}$ in rectangles with non empty interior and positive ε such that

$$\forall \mu, \quad \nu \in M_1(\mathbb{T}), \quad \| \pi^{\mathscr{A}}(\mu - \nu) \|_1 \leq \varepsilon \Rightarrow \rho(\mu, \nu) < \delta' \qquad (4.8)$$

Let's consider a finer partition $\mathscr{A}_0 = \{ A_k ; k \leq K_0 \}$: for $\eta \in \{ -1, +1 \}^{K_0}$, we set $h'_{\eta} = \sum_{k=1}^{K_0} \eta_k \mathbb{1}_{A_k}$. We recall property *i*) in section II:

$$M_t^n(\underline{\eta}) = \langle \sigma_t^n - \sigma_0^n, h'_{\eta} \rangle - \int_0^t \tilde{L}_s^n(F_{h'_{\eta}})(\sigma_s^n) ds$$

are $(\tilde{P}^n - F_t)$ martingales, which increasing process is uniformly bounded over $[0, T]$ with C_2/N for some constant C_2 depending on φ. Since equality $\tilde{L}_s(F_g)(\varphi_s) = \langle g, \dot{\varphi}_s \rangle$ holds for all bounded measurable function g on $\mathbb{T}$ and all $s \notin S$, and since $\langle \mu, \pi^{\mathscr{A}_0} f \rangle = \langle \pi^{\mathscr{A}_0} \mu, \pi^{\mathscr{A}_0} f \rangle$ for all $f \in \mathscr{C}(\mathbb{T})$, we derive:

$$M_t^n(\underline{\eta}) = \langle \pi^{\mathscr{A}_0}(\sigma_t^n - \varphi_t), h'_{\eta} \rangle - \langle \pi^{\mathscr{A}_0}(\sigma_0^n - \varphi_0), h'_{\eta} \rangle$$
$$- \int_0^t X_s ds - \int_0^t [\tilde{L}_s^n(F_{h'_{\eta}})(\sigma_s^n) - \tilde{L}_s(F_{h'_{\eta}})(\sigma_s^n)] ds \qquad (4.9)$$

with $X_s = \tilde{L}_s(F_{h'_{\eta}})(\sigma_s^n) - \tilde{L}_s(F_{h'_{\eta}})(\varphi_s)$.

We state it's enough to show:

$$\forall s \notin S \qquad | X_s | \leq C_3 \| \pi^{\mathscr{A}_0}(\sigma_s^n - \varphi_s) \|_1 + \varepsilon_0 (\operatorname{diam} \mathscr{A}_0) \qquad (4.10)$$

where diam $\mathscr{A}_0$ denotes the diameter $\sup \{ | x - y | ; x, y \in A_k, k = 1, \ldots, K_0 \}$ of partition $\mathscr{A}_0$, ε_0 a function with limit zero, and C_3 some positive constant.

Indeed, we then fix partition $\mathscr{A}_0$ finer than $\mathscr{A}$ such that last term in (4.10) be less than $(\varepsilon/4T) \exp - C_3 T$. As above, we can suppose the last integral in (4.9) to be bounded with $(\varepsilon/4) \exp - C_3 T$ for all n superior to some n_4: this time, the functions to be integrated with λ^n are equicontinuous on the rectangles A_k. At last, using property (2.1, *ii*) we can choose $\delta_1 > 0$ and n_5 such that $n \geq n_5$, $\rho(\sigma_0^n, \varphi_0) < \delta_1$ imply $\| \pi^{\mathscr{A}_0}(\sigma_0^n - \varphi_0) \|_1 < (\varepsilon/4) \exp - C_3 T$. Then, (4.9) yields

$$| \langle \pi^{\mathscr{A}_0}(\sigma_t^n - \varphi_t), h'_{\eta} \rangle | \leq | M_t^n(\underline{\eta}) | + (3\varepsilon/4) \exp - C_3 T$$
$$+ C_3 \int_0^t \| \pi^{\mathscr{A}_0}(\sigma_s^n - \varphi_s) \|_1 ds \qquad (4.11)$$

Using Doob's inequality for each martingale $M_t^n(\underline{\eta})$, $\underline{\eta} \in \{ -1, +1 \}^{K_0}$, we can control the probability of

$$\mathscr{X}^n = \{ \max_{\eta} \max_{t \leq T} | M_t^n(\underline{\eta}) | \leq (\varepsilon/4) \exp - C_3 T \}$$

with

$$\tilde{P}^n_{\sigma^n_0}(\mathscr{X}^n) \geq 1 - 2^{K_\alpha}(25C_2/\varepsilon^2 N)\exp 2C_3T, \geq 3/4 \text{ whenever } n \text{ is more than some } n_6.$$

Notice that $\| \pi^{\mathscr{A}}\circ(\sigma^n_t - \varphi_t)\|_1 = \max_\eta \langle \pi^{\mathscr{A}}\circ(\sigma^n_t - \varphi_t), h'_\eta \rangle$: for $n \geq n_3 = n_4 \vee n_5 \vee n_6$, relation (4.11) shows that

$$\| \pi^{\mathscr{A}}\circ(\sigma^n_t - \varphi_t)\|_1 \leq C_3 \int_0^T \| \pi^{\mathscr{A}}\circ(\sigma^n_s - \varphi_s)\|_1 ds + \varepsilon \exp - C_3 T$$

holds on the set $\mathscr{X}^n \cap \{ \rho(\sigma^n_0, \varphi_0) < \delta_1 \}$. Using Gromwall's lemma, we derive $\sup_{t \leq T} \| \pi^{\mathscr{A}}\circ(\sigma^n_t - \varphi_t)\|_1 \leq \varepsilon$; since $\mathscr{A}_0$ is finer than $\mathscr{A}$, Jensen's inequality and (4.8) imply (4.7).

Now, let's prove (4.10): denoting the random function

$$2c(\sigma^n_s)\cosh \beta(h + \tilde{h}_s + J^*\sigma^n_s) \quad \text{by} \quad \psi_s,$$

we have:

$$\begin{aligned} |X_s| \leq\ & |\langle \varphi_s - \sigma^n_s, h'_\eta \pi^{\mathscr{A}_0}\psi_s \rangle| + |\langle \varphi_s - \sigma^n_s, h'_\eta(\psi_s - \pi^{\mathscr{A}_0}\psi_s)\rangle| \\ & + \sum_{\eta \in \{-1,+1\}} \langle |\varphi_s + \eta| \, e^{-\eta\beta(h+\tilde{h}_s+J^*\sigma^n_s)}, c(\varphi_s) \, | e^{-\eta\beta J^*(\varphi_s - \sigma^n_s)} - 1| \\ & + |c(\varphi_s) - c(\pi^{\mathscr{A}_0}\sigma^n_s)| + |c(\pi^{\mathscr{A}_0}\sigma^n_s) - c(\sigma^n_s)| \rangle . \end{aligned}$$

The first bound is not more than $\| \psi_s \|_\infty \| \pi^{\mathscr{A}}\circ(\sigma^n_s - \varphi_s)\|_1$; the second one can be controled with the continuity modulus of the (equicontinuous) family $\{ c(\mu);\ \mu \in M_1(T) \} \cup \{ \tilde{h}_t;\ t \notin S \} \cup \{ J, h \}$. For the last one, we use mean-value theorem for the derivative and inequality

$$|J^*\mu|(x) \leq \| J \|_\infty \| \pi^{\mathscr{A}_0}\mu \|_1 + \| \mu \| \, \| J_x - \pi^{\mathscr{A}_0} J_x \|_\infty$$

(denoting by $J_x : y \to J(x-y)$): $|e^{-\eta\beta J^*(\varphi_s - \sigma^n_s)} - 1|$ is bounded with $C_4 \| \pi^{\mathscr{A}}\circ(\sigma^n_s - \varphi_s)\|_1 + \varepsilon_1(\operatorname{diam} \mathscr{A}_0)$ for some function ε_1 with limit zero.

Next we use relation (1.11) to get

$$\| c(\varphi_s) - c(\pi^{\mathscr{A}_0}\sigma^n_s)\|_1 \leq C_0 \{ \| \varphi_s - \pi^{\mathscr{A}_0}\varphi_s \|_1 + \| \pi^{\mathscr{A}}\circ(\sigma^n_s - \varphi_s)\|_1 \}.$$

At last, $\| c(\mu) - c(\nu)\|_\infty$ goes to zero with $\rho(\mu, \nu)$; but for all continuous function f on T and all measure $\mu \in M_1(T)$,

$$\begin{aligned} |\langle \mu - \pi^{\mathscr{A}_0}\mu, f \rangle| &= |\langle \mu, f - \pi^{\mathscr{A}_0} f \rangle| \\ &\leq \langle |\mu|, |f - \pi^{\mathscr{A}_0} f| \rangle \\ &\leq \| f - \pi^{\mathscr{A}_0} f \|_\infty \end{aligned}$$

then $\sup \{ \rho(\mu, \pi^{\mathscr{A}_0}\mu);\ \mu \in M_1(T) \}$ goes to zero with $\operatorname{diam} \mathscr{A}_0$. All the considered functions being bounded, these estimates prove the statement (4.10).

In the general case, we suppose $I_{0T}(\varphi) < \infty$, the opposite case being trivial. Then proposition III.6 for φ, γ, $\delta/2$ yields a smooth trajectory $\tilde{\varphi}$, for which the previous computation apply. □

□ To end, we prove lemma IV.2: the functionals Γ^* and $L^{h'}$ are composed of two kinds of terms,

$$\mathcal{E}_1(\mu) = \langle \lambda, \theta \exp \eta\beta J^*\mu \rangle \qquad \text{and} \qquad \mathcal{E}_2(\mu) = \langle \mu, \theta c(\mu) \exp \eta\beta J^*\mu \rangle$$

with

$$\theta \exp - \eta\beta h = 1, \exp \eta\beta h', \qquad \text{or} \qquad h' \exp \eta\beta h'.$$

Since $\mu \in M_1(\mathbb{T}) \to c(\mu) \in \mathcal{C}(\mathbb{T})$ is equicontinuous and bounded, equicontinuity for the first kind term will result from that of $\mu \to J^*\mu$. According to Stone-Weierstrass theorem one can uniformly approximate J with some trigonometric polynomial $f(x) = \sum_{\substack{q \in \mathbb{Z}^d \\ |q| \leq m}} a_q \exp 2i\pi q.x$. Furthermore

$$\| J^*(\mu - \nu) \|_\infty \leq 2 \| J - f \|_\infty + \sum_{|q| \leq m} |a_q| \, | \langle \mu - \nu, \exp 2i\pi q.x \rangle |$$

for $\mu, \nu \in M_1(\mathbb{T})$, where the last duality brackets are linear continuous forms: one easily derive that $| \mathcal{E}_1(\mu) - \mathcal{E}_1(\nu) | = \varepsilon(\rho(\mu, \nu))$ for some function ε independent of μ, ν, with limit zero.

Using this to control $| \mathcal{E}_2(\mu) - \mathcal{E}_2(\nu) |$, one sees that the only extra work necessary is to bound $| \langle \mu - \nu, \theta c(\mu) \exp \eta\beta J^*\mu \rangle |$. Because of Ascoli's theorem, the family $\theta c(\mu) \exp \eta\beta J^*\mu$ is equicontinuous and bounded on $\mathbb{T}$, then totally bounded: taking a finite covering of this set with $\| - \|_\infty$-balls centered at points $g_k \in \mathcal{C}(\mathbb{T})$, $k \leq K$, and radius $\delta > 0$, one can see that the previous quantity is some $\mathcal{O}\left(\delta + \sum_{k \leq K} | \langle \mu - \nu, g_k \rangle | \right)$, which ends the proof. □

V. UPPER BOUND FOR LARGE DEVIATIONS PROBABILITIES

Recall that $D_{I_0} = \{ \varphi \in \mathcal{C}([0, T]; B); I_{0T}(\varphi) \leq I_0 \}$.

THEOREM V.1. — *Let $\gamma > 0$, $\varepsilon > 0$, $I_0 > 0$. There exists an integer n_0 such that for all $n \geq n_0$ and all σ_0^n,*

$$P^n_{\sigma_0^n} \{ \rho_{0T}(\sigma^n, D_{I_0}) \geq \varepsilon \} \leq \exp \{ - N(I_0 - \gamma) \}.$$

In order to prove theorem, we need the two following results: the first one is a local upper bound, and will be obtained from Markov exponential inequality; the second one is a (very) rough global estimate.

LEMMA V.2. — *For all* $\varphi \in \mathscr{C}([0,T]; B)$, *and all* $I < I_{0T}(\varphi)$ *there exist* $\delta' > 0$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\forall n \geq n_1, \qquad P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} \leq \exp(-NI).$$

LEMMA V.3. — *For all* $a \geq 0$, *there exist a compact subset* Λ *of* $\mathscr{C}([0,T]; B)$ *with following property*: $\forall \delta > 0$, $\exists n_2$ *such that* $\forall n \geq n_2$,

$$P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \Lambda) \geq \delta \} \leq \exp(-Na).$$

□ We first prove theorem V.1:

Choose a compact set Λ from lemma V.3 with $a = I_0$. Then $\Lambda \cap \{ \varphi; \rho_{0T}(\varphi, D_{I_0}) \geq \varepsilon/2 \}$ is compact; for each element φ of this set, apply Lemma V 2 with $I = I_0$, and obtain some integer $n_1(\varphi)$ and some $\delta'(\varphi)$, that can be supposed less than ε without loss of generality. Then make a covering of the previous compact with a finite number K of open neighbourhoods $\left\{ \varphi; \rho_{0T}(\varphi, \varphi_k) < \frac{1}{2}\delta'(\varphi_k) \right\}$ where the φ_k belong to this compact.

Let $\delta = \frac{1}{2}\min\{ \delta'(\varphi_k); k \leq K \}$; $\rho_{0T}(\sigma^n, \Lambda) \leq \delta$ and $\rho_{0T}(\sigma^n, D_{I_0}) \geq \varepsilon$ imply $\rho_{0T}(\sigma^n, \varphi_k) < \delta'(\varphi_k)$ for some $k \leq K$; hence

$$P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, D_{I_0}) \geq \varepsilon \} \leq P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \Lambda) > \delta \} + \sum_{k \leq K} P^n_{\sigma_0^n}\{ \rho_{0T}(\sigma^n, \varphi_k) < \delta'(\varphi_k) \},$$

which is less than $(K+1)e^{-NI_0}$, when $n \geq n_2 \vee \max_{k \leq K} n_1(\varphi_k)$; finally, for large n the last bound is less than $e^{-N(I_0-\gamma)}$. □

□ We now prove lemma V.2:

a) If φ is absolutely continuous, let $I < I_{0T}(\varphi)$ and $\gamma > 0$ with $I + 3\gamma < I_{0T}(\varphi)$; according to property III.3 *a*, there exists some $f \in L^\infty([0,T] \times \mathbb{T})$ such that

$$\int_0^T \left[\frac{\beta}{2} \langle \dot{\varphi}_t, f_t \rangle - \Gamma^*(\varphi_t, f_t) \right] dt \geq I + 3\gamma.$$

The functions h, J and f being bounded, Lusin's theorem shows that we can suppose f to be continuous with respect to (t, x), and even $f \in \mathscr{C}^{1,0}$, using a density argument.

Let $\tilde{P}^n$ be the probability on $(\Omega^n, \mathbb{F}_T)$ defined by its Radon-Nikodym derivative with respect to the restriction of P^n to $\mathbb{F}_T$:

$$\frac{d\tilde{P}^n}{dP^n/\mathbb{F}_T} = R_T^n = \exp N\left\{ \int_0^T \frac{\beta}{2} \langle f_t, d\sigma_t^n \rangle - \int_0^T \Gamma_n^*(\sigma_t^n, f_t)dt \right\} \tag{5.1}$$

Then, we have

$$P_{\sigma_0}^n \{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} = \tilde{\mathbb{E}}_{\sigma_0}^n([R_T^n]^{-1} . \mathbf{1}_{\{\rho_{0T}(\sigma^n, \varphi) < \delta'\}}) . \tag{5.2}$$

As $\eta_t^n(x)$ is $\tilde{P}^n$-almost surely of bounded variation on $[0, T]$ for every $x \in \mathscr{S}^n$, we integrate by parts:

$$\int_0^T \langle f_t, d\sigma_t^n \rangle = \int_0^T \langle f_t, \dot{\varphi}_t \rangle dt - \int_0^T \langle \sigma_t^n - \varphi_t, \dot{f}_t \rangle dt + [\langle \sigma_t^n - \varphi_t, f_t \rangle]_0^T \tag{5.3}$$

Like in the proof of theorem IV.1, we have for large n:

$$\left| \int_0^T [\Gamma_n^*(\sigma_t^n, f_t) - \Gamma^*(\sigma_t^n, f_t)]dt \right| \leq \gamma \quad \text{for all path } \sigma^n .$$

According to lemma IV.2, the family $\{ \mu \to \Gamma^*(\mu, f_t) ; t \leq T \}$ is equicontinuous on the compact $M_1(T)$; furthermore, $\{ \dot{f}_t, t \leq T \}$ is totaly bounded in $\mathscr{C}(T)$. Therefore we can choose $\delta' > 0$ such that $\rho_{0T}(\sigma^n, \varphi) < \delta'$ implies the inequalities

$$\left| \int_0^T [\Gamma^*(\sigma_t^n, f_t) - \Gamma^*(\varphi_t, f_t)]dt \right| \leq \gamma$$

and

$$\left| \int_0^T \langle \sigma_t^n - \varphi_t, \dot{f}_t \rangle dt + [\langle \sigma_t^n - \varphi_t, f_t \rangle]_0^T \right| \leq \gamma .$$

Then, the last three inequalities, together with relations (5.1 to 3) yield for large n:

$$P_{\sigma_0}^n \{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} \leq \exp\left\{ -N \int_0^T \left[\frac{\beta}{2} \langle f_t, \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, f_t) \right] dt + 3\gamma N \right\}$$
$$\leq \exp(-NI) .$$

b) In the case of a non absolutely continuous function φ, let's fix $\gamma > 0$ such that for all $\Delta > 0$ there exist $s_i, t_i \in [0, T]$, $i = 1, \ldots, i_0$, $s_i < t_i$, and i_0

rectangles A_i of $\mathbb{T}$ with positive λ-measure, satisfying both inequalities $\sum_{i=1}^{i_0}(t_i - s_i)\lambda(A_i) < \Delta$ and $\sum_{i=1}^{i_0} | \langle \varphi_{t_i} - \varphi_{s_i}, \mathbf{1}_{A_i} \rangle | \geq \gamma$.

By parting some A_i', s, then modifying them at the boundary, and increasing i_0, we can suppose without loss of generality that $\{ A_i ; i \leq i_0 \}$ is included in some partition $\mathscr{A}$ of $\mathbb{T}$ in rectangles with non-empty interior. Let b a positive real number, η_i the sign of

$$\langle \varphi_{t_i} - \varphi_{s_i}, \mathbf{1}_{A_i} \rangle \qquad \text{and} \qquad f = b \sum_{i=1}^{i_0} \eta_i \mathbf{1}_{]s_i,t_i] \times A_i}.$$

Let's define probability $\tilde{P}^n$ by (5.1) and this function f; we have:

$$-\frac{1}{N} \operatorname{Log} R_T^n = -\frac{\beta}{2} b \sum_{i=1}^{i_0} \eta_i \langle \mathbf{1}_{A_i}, \varphi_{t_i} - \varphi_{s_i} \rangle$$

$$- \frac{\beta b}{2} \sum_{i=1}^{i_0} \eta_i \langle \mathbf{1}_{A_i}, (\sigma_{t_i}^n - \varphi_{t_i}) - (\sigma_{s_i}^n - \varphi_{s_i}) \rangle + \int_0^T \Gamma_n^*(\sigma_t^n, f_t) dt \quad (5.3b)$$

In the right-hand side member of this equality, the first term is not more than $-\frac{\beta b}{2}\gamma$, the second one not more than $\frac{\beta b}{2} i_0 \sup_{t \leq T} \| \pi^{\mathscr{A}}(\sigma_t^n - \varphi_t) \|_1$.
For the measure $\lambda^n + \eta\sigma_t^n$ is positive

$$\Gamma_n^*(\sigma_t^n, b\mathbf{1}_A) \leq C_1 \sum_{\eta \in \{-1,+1\}} \left\langle \frac{\lambda^n + \eta\sigma_t^n}{2}, e^{\beta b \mathbf{1}_A} - 1 \right\rangle = C_1 \lambda^n(A)(e^{\beta b} - 1)$$

with constant $C_1 = \max \{ c_\eta(x, \mu) ; \eta \in \{ -1, +1 \}, x \in \mathbb{T}, \mu \in M_1(\mathbb{T}) \}$; so last term of (5.3 b) is less than $C_1(e^{\beta b} - 1)(\Delta + i_0 T \| \pi^{\mathscr{A}}\lambda^n - 1 \|_1)$.

We now choose $b = 8I(\beta\gamma)^{-1}$ and $\Delta = I[C_1(e^{\beta b} - 1)]^{-1}$ .Proposition I.4.ii) for partition $\mathscr{A}$ and $\varepsilon = 2I(i_0\beta b)^{-1}$ yields $\delta' > 0$ such that, for large n, $\rho_{0T}(\sigma^n, \varphi) < \delta'$ implies $\| \pi^{\mathscr{A}}(\sigma_t^n - \varphi_t) \|_1 < \varepsilon$ for all $t \leq T$. $\| \pi^{\mathscr{A}}\lambda^n - 1 \|_1 \leq \Delta(i_0 T)^{-1}$ for large enough n, so that (5.2) leads to

$$P_{\sigma_0^n}^n \{ \rho_{0T}(\sigma^n, \varphi) < \delta' \} \leq \sup \{ [R_T^n]^{-1} ; \rho_{0T}(\sigma^n, \varphi) < \delta' \}$$
$$\leq \exp - NI \qquad \square$$

□ We next prove lemma V.3: suppose $a > 0$ (the case $a = 0$ is trivial). First of all, fix a sequence Δ_j, $j = 1, 2, \ldots$ such that $T/\Delta_1 \in \mathbb{N}$,

$$\Delta_j \leq (a+1)/C_1 (\exp [2\beta(a+1)/j] - 1), \qquad \Delta_j/\Delta_{j+1} \in \mathbb{N} - \{ 0, 1 \},$$

for $j = 1, 2, \ldots$; these conditions imply in particular that $T/\Delta_j \in \mathbb{N}$ and $j\Delta_j$ is decreasing.

For φ: $[0, T] \rightarrow B$, let $\Delta^{\varphi}(j) = \sup \left\{ \Delta \geq 0; \sup_{\substack{|t-t'| \leq \Delta \\ t, t' \in [0,T]}} \| \varphi_t - \varphi_{t'} \|_1 < \frac{1}{j} \right\}$ be its modulus of uniform continuity for $\| . \|_1$ norm. Let's define $\Lambda' = \{ \varphi \in \mathscr{C}([0, T]; B); \Delta^{\varphi}(j) \geq \Delta_j, \forall j \geq 1 \}$; because $\| . \|_1$ norm dominates metric ρ Ascoli's theorem proves that Λ' is relatively compact in space $(\mathscr{C}([0, T]; B), \rho_{0T})$; so its closure Λ is compact.

Given $\delta > 0$, from proposition I.4.*i*) we can choose a partition $\mathscr{A} = \{ A_k; k \leq K \}$ of $\mathbf{T}$ in rectangles with non-empty interior and $\varepsilon > 0$ such that for every $\mu \in M_1(\mathbf{T})$, the δ-neighbourhood of μ in metric ρ contains all $\nu \in M_1(\mathbf{T})$ satisfying to $\| \pi^{\mathscr{A}}(\mu - \nu) \|_1 < \varepsilon$. Let's fix now $j_0 = [2/\varepsilon] + 1$, $t_m = m\Delta_{j_0}$, $m = 1, 2 \ldots m_0 = T/\Delta_{j_0}$. For $b > 0$ and $\underline{\eta} \in \{ -1, 1 \}^K$ we define $h'_{\eta} = \sum_{k=1}^{K} \eta_k \mathbf{1}_{A_k}$. Then,

$$M_t^{n,b}(m, \underline{\eta}) = \exp N \left\{ \langle bh'_{\underline{\eta}}, \sigma_t^n - \sigma_{t_m}^n \rangle - \int_{t_m}^{t} \Gamma_n^*(\sigma_s^n, bh'_{\underline{\eta}}) ds \right\} \qquad (5.4)$$

is a P^n-martingale for $t \geq t_m$. Recall inequality $\Gamma_n^*(\sigma_s^n, bh'_{\underline{\eta}}) \leq C_1(e^{\beta b} - 1)$ from the proof of lemma V.2 (part *b*); since $\langle h'_{\underline{\eta}}, \sigma_t^n - \sigma_{t_m}^n \rangle = \langle h'_{\underline{\eta}}, \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \rangle$ is equal to $\| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1$ for at least one choice of $\underline{\eta}$,

$$\left\{ \sup_{t_m \leq t \leq t_{m+1}} \| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1 \geq \frac{1}{j_0} \right\}$$
$$\subset \bigcup_{\underline{\eta}} \{ \sup_{t_m \leq t \leq t_{m+1}} M_t^{n,b}(m, \underline{\eta}) \geq \exp N \{ (b/j_0) - \Delta_{j_0} C_1(e^{\beta b} - 1) \} \}.$$

For each $\underline{\eta}$, we bound from above the conditional probability of this event using Kolmogorov's maximal inequality [7] by

$$\exp - N \{ (b/j_0) - \Delta_{j_0} C_1(e^{\beta b} - 1) \}.$$

Take $b = 2(a + 1)j_0$; this term is not more than $\exp - N(a + 1)$, because of the properties of the sequence Δ_j.

As a conclusion,

$$P_{\sigma_0^n}^n \{ R(j_0)^c \} \leq 2^K m_0 \exp - N(a + 1) \qquad (5.5)$$

where $R(j_0)$ denotes the event $\{ \sup_{m < m_0} \sup_{t_m \leq t \leq t_{m+1}} \| \pi^{\mathscr{A}}(\sigma_t^n - \sigma_{t_m}^n) \|_1 < 1/j \}$.

Let l^n be the random polygon on $[0, T]$, $\mathscr{A}$-measurable valued, with vertices at the points $(t_m, \pi^{\mathscr{A}} \sigma^n_{t_m})$. Of course, on $R(j_0)$, $\rho_{0T}(\sigma^n, l^n) < \delta$, so

$$P^n_{\sigma_3} \{ \rho_{0T}(\sigma^n, \Lambda) \geq \delta \} \leq P^n_{\sigma_3}(R(j_0)^c) + P^n_{\sigma_3}(R(j_0) \cap \{ l^n \notin \Lambda' \}) \quad (5.6)$$

In the following, we bound the last term of (5.6).

On $R(j_0)$, the slope of l^n satisfies

$$\| l^n_t - l^n_{t'} \|_1 / | t - t' | < (j_0 \Delta_{j_0})^{-1} \leq (j \Delta_j)^{-1} \quad \text{if} \quad j \geq j_0$$

(see remark at the beginning of the proof).

We derive from this $\Delta^{l^n}(j) \geq \Delta_j$ for $j \geq j_0$. For $j < j_0$, we show first that:

$$\max_{|t - t'| \leq \Delta_j} \| l^n_t - l^n_{t'} \|_1 = \max_{\substack{m, r \\ |t_m - t_r| \leq \Delta_j}} \| l^n_{t_m} - l^n_{t_r} \|_1 . \quad (5.7)$$

t [resp. t'] belongs to some interval $[t_m, t_{m+1}]$ [resp. $[t_r, t_{r+1}]$].

For $s \in [t_m \vee (t + t_r - t'), t_{m+1} \wedge (t + t_{r+1} - t')]$, $s \to l^n_s - l^n_{s+t'-t}$ is an affine function; $u \to \| u \|_1$ being a convex function, so is the product function: this one achieves its maximum value on the boundary of the interval. Thus, it's enough to show that $\| l^n_t - l^n_{t_m} \|_1$ is not more than the right-hand side of (5.7) when $t \in]t_r, t_{r+1}[$ and $| t - t_m | \leq \Delta_j$. In this situation, Δ_j / Δ_{j_0} being integer implies that $| t_r - t_m | \vee | t_{r+1} - t_m | \leq \Delta_j$. Combining this with the convexity of $s \to \| l^n_s - l^n_{t_m} \|_1$ yields the desired result.

From (5.7), we derive the inclusion

$$\{ \Delta^{l^n}(j) < \Delta_j \} \subset \bigcup_{(m, r)} \{ \| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r}) \|_1 > 1/j \},$$

where the union extends to all couples (m, r) such that

$$0 \leq m < r \leq m_0 \wedge (m + \Delta_{j_0} / \Delta_j) .$$

For such a couple and $b_j > 0$, $\| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r} \|_1 \geq 1/j$ implies for at least one $\underline{\eta}$:

$$M^{n, b_j}_{t_r}(m, \underline{\eta}) \geq \exp N \{ (b_j / j) - \Delta_j C_1 (e^{\beta b_j} - 1) \} .$$

We now choose $b_j = 2(a + 1)j$, we apply Bienaymé inequality to the positive variables $M^{n, b_j}_{t_r}(m, \underline{\eta})$ with expected value 1:

$$P^n_{\sigma_3} \{ \| \pi^{\mathscr{A}}(\sigma^n_{t_m} - \sigma^n_{t_r}) \|_1 \geq 1/j \} \leq 2^K \exp - N(a + 1) .$$

Next, using the rough upper bound $m_0\Delta_j/\Delta_{j_0}$ of the number of couples (m, r), we get

$$P^n_{\sigma_0}\Big(\bigcup_{j<j_0}\{\Delta'^n(j)<\Delta_j\}\Big)\leq m_0\Big(\sum_{j=1}^{j_0-1}\Delta_j/\Delta_{j_0}\Big)2^K\exp-N(a+1).$$

Combining this with (5.5 and 6), we find

$$P^n_{\sigma_0}\{\rho_{0T}(\sigma^n,\Lambda)\geq\delta\}\leq 2^K m_0\Big(\sum_{j=1}^{j_0}\Delta_j/\Delta_{j_0}\Big)\exp-N(a+1),$$

so we can choose n_2 (depending on δ) such that the last bound is less than $\exp - Na$ for all $n \geq n_2$. □

VI. PROPERTIES OF THE QUASIPOTENTIAL $W(u_e, u)$

The quasipotential $W(u_e, u)$ is the least energy necessary to join an equilibrium u_e to some point $u \in B$:

$$W(u_e,u)=\inf\{I_{0T}(\varphi);\ \varphi\in\mathscr{C}([0,T];B),\ \varphi_0=u_e,\ \varphi_T=u,\ T\in\mathbb{R}^+\}\tag{6.1}$$

Before studying the exit points of an attracting domain, we show some properties of the quasipotential. We say that u is attracted by u_e (or u is in the basin of attraction of u_e) if the solution u_t of (M. E.) starting at u goes to u_e as t tends to ∞ (τ^*-convergence implying here convergence in norm $\|\cdot\|_\infty$, see [5]).

Hamilton-Jacobi equation corresponding to the free-time variational problem (6.1) is $\Gamma^*\Big(u, \frac{2}{\beta}dW\Big) = 0$ where dW denotes the gradient of $W(u_e, u)$ with respect to u. Combining (1.6), (3.4) one computes that $\Gamma^*(u, 2dV_h(u)) = 0$; this shows the relation between large deviations results for the magnetization process, and the ones for the Gibbs measure we recalled in § 1.

PROPOSITION VI.1. — *a)* $\forall u \in B,\ W(u_e, u) \geq \beta\{V_h(u) - V_h(u_e)\}$.

b) *If u is attracted by u_e, the equality holds in* a).

c) *If φ is the line segment $[u, u']$ covered in the time $T = \|u - u'\|_2$ with constant speed, $I_{0T}(\varphi) = \mathcal{O}(\|u - u'\|_2^{1/2-\varepsilon})$ for all $\varepsilon > 0$ and $u, u' \in B$.*

We just give the sketch of the proof: refer to [5] for more details. Using the above remark, we have

$$\mathcal{H}^*(\varphi_t, \dot\varphi_t) \geq \frac{\beta}{2}\langle \dot\varphi_t, 2\, d\mathrm{V}_h(\varphi_t)\rangle - \Gamma^*(\varphi_t, 2\, d\mathrm{V}_h(\varphi_t)) \geq \beta \langle \dot\varphi_t, d\mathrm{V}_h(\varphi_t)\rangle .$$

Integrating over $[0, \mathrm{T}]$, we obtain the first inequality.

The path φ described in *c*) is $\varphi_t = u + \frac{t}{\mathrm{T}}(u' - u)$. Let's fix $x \in \mathbb{T}$, and suppose that $u'(x) > u(x)$, the other case being treated similarly; we shortly denote $u(x)$ by u, $u'(x)$ by u'. From property III.3.*c*), we derive

$$\int_0^t \mathcal{H}(\varphi_t, \dot\varphi_t, s)dx \leq \frac{u' - u}{2}[\log(u' - u) - \log \mathrm{T}] + [\theta(\log\theta - 1)]_{1-u}^{1-u'} + \frac{\mathrm{K}}{2}(u' - u) + \mathrm{KT} \qquad (6.2)$$

$f(\theta) = \theta \log \theta$ being a convex function on $[0, 2]$, $0 \leq \theta' \leq \theta \leq 2$ implies $f(\theta') - f(\theta) \leq f(0) - f(\theta - \theta') = -f(\theta - \theta')$; the second term of (6.2) is then bounded from above by $-(u' - u)\log(u' - u) + (u' - u)$. Using Hölder's inequality together with the boundedness of $\theta^\varepsilon \log \theta$ on $[0, 2]$, we can easily prove *c*).

In order to show *b*) let's notice that there exists a unique function on $]-\infty, 0]$ in B such that $\varphi_0 = u$ and the field h' maximizing (3.5) along the trajectory be equal to $2\, d\mathrm{V}_h(\varphi_t)$: because of (3.2), it is the solution starting at $\varphi_0 = u$ of the mean evolution equation time being reversed

$$\frac{d}{dt}\varphi_t = 2c(\varphi_t)\sqrt{1 - \varphi_t^2}\, \mathrm{sh}\, \beta d\mathrm{V}_h(\varphi_t) . \qquad (6.3)$$

Such a trajectory φ will be called and extremal. Since φ_t converges to u_e in $\mathbb{L}^2(\mathbb{T})$ as t goes to $-\infty$, *b*) is a consequence of *c*).

The previous results are valid in general finite dimensional situations [15]. But, in our case, the potential V_h (and $d\mathrm{V}_h$ too) is not continuous in the weak topology. We then need some extra results:

PROPOSITION VI.2. — *There exist positive constants* K, K′ *such that for all trajectories* φ *on* $[0, \infty[$ *with values in* B, *and all* $\mathrm{T} > 0$

$$\mathrm{I}_{0\mathrm{T}}(\varphi) \geq -\mathrm{K} + \mathrm{K}' \int_0^{\mathrm{T}} \| 1 \wedge | d\mathrm{V}_h(\varphi_t) | \|_2^2 dt .$$

In particular, whenever $\mathrm{I}_{0\infty}(\varphi) < \infty$, *there exists a sequence* $t_m \to \infty$ *such that* φ_{t_m} *converges to an equilibrium in* $\mathbb{L}^2(\mathbb{T})$.

By an easy calculation, one sees that

$$\Gamma(u, d\mathrm{V}_h(u), x) = c(u)\left[- \sum_{\eta \in \{+1, -1\}} \frac{1 + u}{2} e^{-\eta\beta(h + \mathrm{J}*u)} + \sqrt{1 - u^2} \right],$$

the last quantity being evaluated at point x.

On one hand we see that

$$\Gamma(u, d\mathrm{V}_h(u), x) \leq c(u)[- \mathrm{K}'' + \sqrt{1 - u^2}] \tag{6.4}$$

with constant $\mathrm{K}'' = \exp - \beta(\| h \|_\infty + \| \mathrm{J} \|_\infty) < 1$. On the other hand,

$$\begin{aligned} \Gamma(u, d\mathrm{V}_h(u), x) &= - 2c(u)\sqrt{1 - u^2}\, \mathrm{sh}^2 \frac{\beta}{2} d\mathrm{V}_h(u) \\ &\leq - 2c(u)\sqrt{1 - u^2}\left[\frac{\beta}{2} d\mathrm{V}_h(u)\right]^2 \end{aligned} \tag{6.5}$$

because of the inequality $|\mathrm{sh}\, z| \geq |z|$ for $z \in \mathbb{R}$.

Combining (6.4 and 5), we deduce that $\Gamma^*(u, d\mathrm{V}_h(u)) \leq -\mathrm{K}' \| 1 \wedge |d\mathrm{V}_h(u)| \|_2^2$ for some positive constant K' depending on K'' and $\min_{x,u} c(x, u)$. We have

$$\begin{aligned} \mathrm{I}_{0\mathrm{T}}(\varphi) &\geq \int_0^{\mathrm{T}} \left[\frac{\beta}{2} \langle d\mathrm{V}_h(\varphi_t), \dot{\varphi}_t \rangle - \Gamma^*(\varphi_t, d\mathrm{V}_h(\varphi_t))\right] dt \\ &\geq \frac{\beta}{2} [\mathrm{V}_h(\varphi_\mathrm{T}) - \mathrm{V}_h(\varphi_0)] + \mathrm{C}' \int_0^{\mathrm{T}} \| 1 \wedge |d\mathrm{V}_h(\varphi_t)| \|_2^2 dt ; \end{aligned}$$

since V_h is bounded on B, this yields the desired inequality.

Suppose now that $\mathrm{I}_{0\infty}(\varphi) < \infty$. We can find some sequence $t_m \to \infty$ such that $\| 1 \wedge |d\mathrm{V}_h(\varphi_{t_m})| \|_2^2 = \lambda(\mathrm{A}_m^c) + \| d\mathrm{V}_h(\varphi_{t_m}) . \mathbb{1}_{\mathrm{A}_m} \|_2^2$ goes to 0, A_m denoting the subset $\{ |d\mathrm{V}_h(\varphi_{t_m})| < 1 \}$ of $\mathbb{T}$.

Then, write φ_{t_m} as $\tanh \{ \beta(h + \mathrm{J}*\varphi_{t_m}) + d\mathrm{V}_h(\varphi_{t_m}) . \mathbb{1}_{\mathrm{A}_m} \} . \mathbb{1}_{\mathrm{A}_m} + \varphi_{t_m} . \mathbb{1}_{\mathrm{A}_m^c}$; $\| \varphi_{t_m} \|_\infty \leq 1$ implies that the second term tends to 0 in $\mathbb{L}^2(\mathbb{T})$; considering a subsequence of $(t_m)_m$, we may suppose that φ_{t_m} converges in the weak topology to some $u \in \mathrm{B}$; then, $\mathrm{J} * \varphi_{t_m}$ goes to $\mathrm{J} * u$ uniformly, and we easily deduce that the first term converge in $\mathbb{L}^2(\mathbb{T})$ to $\tanh \beta(h + \mathrm{J} * u)$. To see that u is an equilibrium, write

$$\tanh \beta(h + \mathrm{J} * u) = \| \cdot \|_2 - \lim_{m \to \infty} \varphi_{t_m} = \tau^* - \lim_{m \to \infty} \varphi_{t_m} = u .$$

VII. THE EXIT POINTS FROM THE ATTRACTING DOMAIN OF A METASTABLE STATE

In the next two sections, we will consider the magnetization density ζ^n given by (1.5). Obviously there exists some function ε with $\lim_{n\to\infty} \varepsilon(n) = 0$ and: for all configuration

$$\rho(\zeta^n, \sigma^n) \le \varepsilon(n). \tag{7.1}$$

Hence theorems IV.1 and V.1 are still valid for the process ζ^n_t. For a subset Z of B, we will denote by $\partial Z, \overline{Z}, \dots$ [resp. $\partial^2 Z, \overline{Z}^2, \dots$] the boundary, the closure,... of Z in τ^* topology [resp. in the $\| \cdot \|_2$ norm topology on B]; for positive δ, $\mathcal{V}_\delta(Z)$ will denote the (closed) δ-neighbourhood of Z in metric ρ: $\mathcal{V}_\delta(Z) = \{ u \in B; \rho(u, Z) \le \delta \}$.

In this section, we make somewhat general assumptions, which are satisfied in the example of § VIII: we consider an equilibrium u_e, τ^*-asymptotically stable in the Lyapunov sense for the mean evolution equation (M. E.). Because of the continuity of $u_0 \to u$ (see § 2), its basin of attraction $\mathcal{B}_e$ is a weakly open subset of B. We are interested in the situation where there exist at least two locally stable equilibria, so we suppose $\overline{\mathcal{B}}_e \neq B$. Let Ex be the set of the « lowest saddle points » on the boundary:

$$Ex = \{ u \in \partial\mathcal{B}_e ; V_h(u) = \min \{ V_h(w); w \in \partial\mathcal{B}_e \} \} \tag{7.2}$$

V_h being l. s. c., Ex is weakly compact; since $u_0 \to u$ is continuous, and since V_h is a Lyapunov function for M. E., its elements are equilibria.

Throughout this section, $V_h(Ex)$ will denote the value of V_h on Ex, and $\Delta = \beta \{ V_h(Ex) - V_h(u_e) \}$ the height of the potential barrier.

We require the following hypothesis (H):

i) $Ex \cap \partial^2\mathcal{B}_e \neq \varnothing$; let u_{Ex} be one of its elements.

ii) $Ex \subset \partial^2 [(\overline{\mathcal{B}}_e)^c]$.

There exists positive δ_0 such that

iii) $\mathcal{V}_{\delta_0}(Ex) \cap \{ u \in \partial\mathcal{B}_e - Ex ; dV_h(u) = 0 \} = \varnothing$

iv) $\mathcal{V}_{\delta_0}(\partial\mathcal{B}_e) \cap \{ u \in B ; dV_h(u) = 0 \} \subset \partial\mathcal{B}_e$.

THEOREM VII.1. — *Let τ be the exit time for ζ^n_t from the basin of* *attraction $\mathcal{B}_\epsilon$. Under assumptions* (H), *we have for all weakly closed subset* F *of $\mathcal{B}_\epsilon$ and all $\delta > 0$*

$$\lim_{n\to\infty} \inf_{\xi_0^n \in \mathcal{F}} P^n_{\xi_0^n}\{\, \xi^n_\tau \in \mathcal{V}_\delta(Ex) \,\} = 1$$

where $P^n_{\xi_0^n}$ denotes the law of process ξ^n_t starting at ξ^n_0.

The theorem states that, for large enough n, the process leaves the basin of attraction of such an equilibrium at the neighbourhood of one of the « lowest saddle points » on the boundary. It extends Vent'sel-Freidlin result ([15] [1]) about the exit point of a compact set strictly contained in an attracting domain, under the assumption that the vector field at the boundary be transverse and pointed inwards.

The technical assumptions (H) *i*) *ii*) cannot reduce to only local conditions holding at exit points; they are satisfied whenever the frontier $\partial\mathcal{B}_\epsilon$ is smooth, for example a one dimensional Banach $\mathcal{C}^1$ manifold. The hypothesis (H) *iii*) *iv*) concern the accumulation points of equilibria at the neighbourhood of $\partial\mathcal{B}_\epsilon$.

As in the previous references, we will study long time behaviour using finite time estimates of theorems IV.1 and V.1 together with the fact that the magnetization process restarts afresh from Markov stopping times. The structure of the stopping times we use is quite different from the one of [15] [1], because the quasipotential W is not continuous in the weak topology, but only in a strong one; we must furthermore take into account the equilibria located in $\partial\mathcal{B}_\epsilon$.

We will outline the proof after relation (7.5); we first reduce the analysis of the random path to its final part.

☐ It's enough to prove the theorem for $\delta < \delta_0 \wedge \rho(u_\epsilon, Ex)$. Recall definition (7.2); since V_h is lower semicontinuous, and $\partial\mathcal{B}_\epsilon - \mathcal{V}_\delta(Ex)$ is τ^*-compact, one can find positive numbers α and $\delta_1 < \delta/2$ such that [5]

$$\forall u \in \mathcal{V}_{2\delta_1}(NEx), \qquad \beta V_h(u) \geq \beta V_h(Ex) + \alpha \tag{7.3}$$

where NEx stands for $\partial\mathcal{B}_\epsilon - \mathcal{V}_\delta(Ex)$.

We consider (small) neighbourhoods $\mathcal{V}_{\gamma_\epsilon}(u_\epsilon)$, $\mathcal{V}_{\gamma_3}(Ex)$, and (large) time T. We first carry out the proof with initial condition ξ^n_0 in $\mathcal{V}_{\gamma_\epsilon}(u_\epsilon)$ instead of $\mathcal{F}$.

(5) Subscripts 1 will be used for NEx, subscripts 3 for Ex, ϵ for u_ϵ and 2 for points outside of $\mathcal{B}_\epsilon$. (NEx is defined after next relation (7.3)).

Let's define the stopping times:

$\tau_e^0 = 0$

$\tau_3^0 =$ the entrance time of ξ_t^n in $\mathcal{V}_{\gamma_3}(Ex)$, and for $k = 0, 1, \ldots$

$\tau_e^{k+1} = \min \{ t \geq \tau_3^k \wedge (\tau_e^k + T) ; \xi_t^n \in \mathcal{V}_{\gamma_e}(u_e) \}$

$\tau_3^{k+1} = \min \{ t \geq \tau_e^{k+1} ; \xi_t^n \in \mathcal{V}_{\gamma_3}(Ex) \}$

$\tau_1 =$ the entrance time in $\mathcal{V}_{\delta_1}(NEx)$.

Let ν_e be the last integer k such that $\tau_e^k < \tau$, and $R_k = \{ \nu_e = k \}$.

It is enough to show that, for $\xi_0^n \in \mathcal{V}_{\gamma_e}(u_e)$ and sufficiently large n:

$$\forall k, \quad P^n_{\xi_0^n} \{ \xi_\tau^n \in \mathcal{V}_\delta(Ex) ; R_k \} \geq q^2 P^n_{\xi_0^n} \{ R_k \} \tag{7.4}$$

with $q = 1 - 2 \exp - N\alpha/6$: indeed, summing this relation over all k provides us with the theorem.

Using strong Markov property on the set $\{ \tau_e^k < \tau \}$, we obtain:

$$\begin{aligned} P^n_{\xi_0^n} \{ \xi_\tau^n \in \mathcal{V}_\delta(Ex) ; R_k \} &= E^n_{\xi_0^n} \{ 1_{\tau_e^k < \tau} . P^n_{\xi_0^n} \{ \xi_\tau^n \in \mathcal{V}_\delta(Ex) ; \tau \leq \tau_e^{k+1} / F_{\tau_e^k} \} \} \\ &= E^n_{\xi_0^n} \{ 1_{\tau_e^k < \tau} . P^n_{\xi^n_{\tau_e^k}} \{ \xi_\tau^n \in \mathcal{V}_\delta(Ex) ; \tau \leq \tau_e^1 \} \} . \end{aligned}$$

The same computation for the right hand side probability in (7.4) shows we only need to prove this inequality for $k = 0$. We now decompose R_0 according to

$$\begin{aligned} \Omega^n = \{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2T \} \cup \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T \} \\ \cup \{ \tau \leq \tau_3^0 \wedge 2T, \tau < \tau_1 \} \cup \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T \} \end{aligned} \tag{7.5}$$

The main contribution in decomposition (7.5) to the probability of R_0 is given by the two last terms. The contribution of first term will be negligible, because the process cannot spend too much time far away from the equilibria (lemma 4). The trajectories close to the second set hit

$$\mathcal{V}_{2\delta_1}(\partial \mathcal{B}_e - \mathcal{V}_\delta(Ex)) ,$$

so they have a large action functional value (lemma 3); this set will be negligible too. On the third set, we have $\xi_\tau^n \in \mathcal{V}_\delta(Ex)$ for large enough n. To bound from below the contribution to $P^n_{\xi_0^n}(R_0)$ of the last set in (7.5), we shall look for some tubelet in it; but we also need to study the random paths starting close to Ex which leave $\mathcal{B}_e$ before returning near u_e:

LEMMA VII.2. — *There exists* γ_3 *such that for all* $\gamma_e < \gamma_e' = \rho(u_e, \mathcal{V}_\delta(Ex))$, *the inequality* $P^n_{\xi_0^n} \{ \xi_\tau^n \in \mathcal{V}_\delta(Ex), \tau_e > \tau \} \geq q P^n_{\xi_0^n} \{ \tau_e > \tau \}$ *holds on the set* $\{ \xi_0^n \in \mathcal{V}_{\gamma_3}(Ex) \}$ *for all sufficiently large* n, *where* τ_e *denotes the entrance time in* $\mathcal{V}_{\gamma_e}(u_e)$, *and* q *is the same as in (7.4).*

From now on, the radius γ_3 is fixed as above. As for γ_e, we use it for controling the energy value of some trajectories.

LEMMA VII.3. — *There exists* $\gamma''_e > 0$ *such that for all* $T > 0$, $\varphi \in \mathscr{C}([0,T]; B)$, *the conditions*

$$\varphi_0 \in \mathcal{V}_{2\gamma''_e}(u_e), \quad \varphi_T \in \mathcal{V}_{2\delta_1}(NEx), \quad \varphi[0,T] \subset \mathcal{V}_{\delta_0}(\mathscr{B}_e) - \mathcal{V}_{\gamma_3/2}(Ex)$$

imply $I_{0T}(\varphi) \geq 3\alpha/4$.

These two lemmas will be proved further. In order to fix γ_e, we now look for a tubelet included in the last set of (7.5). According to the assumption (H) *i*), and to the $\|.\|_2$-continuity (6) of V_h, we can pick some $u_3 \in \mathscr{B}_e \cap \mathcal{V}_{\gamma_3/2}(u_{Ex})$ with $V_h(u_3) \leq V_h(Ex) + \alpha/5$; using proposition VI.1.*b*) we can find some trajectory $\overline{\varphi}$ on $[0, T_3]$ joining u_e to u_3 with energy $I_{0,T_3}(\overline{\varphi}) \leq \Delta + \alpha/4$; in particular we derive from (7.3) that $\overline{\varphi}$ does not enter $\mathcal{V}_{2\delta_1}(NEx)$. Furthermore, we can assume that $\overline{\varphi}$ does not return to u_e on $]0, T_3]$. We can therefore choose $\gamma < \delta_1 \wedge (\gamma_3/2)$ such that the random paths ξ^n with $\rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma$ don't return in $\mathcal{V}_\gamma(u_e)$ after reaching $\mathcal{V}_{\gamma_3}(Ex)$, and enter $\mathcal{V}_{\gamma_3}(Ex)$ before time T_3 and before hitting $\mathcal{V}_{\delta_1}(NEx)$. Applying theorem IV.1, we obtain (7), for some $\gamma_e < \gamma$,

$$P^n_{\xi^n_0} \{ \rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma \} \geq \exp - N(\Delta + \alpha/3) \qquad (7.6)$$

for sufficiently large n and $\xi^n_0 \in \mathcal{V}_{\gamma_e}(u_e)$. Of course, we can impose the condition $\gamma_e < \gamma'_e \wedge \gamma''_e$.

At last, we need the following

LEMMA VII.4. — *Let* $\mathscr{F}_1, \mathscr{F}'_1$ *be weakly closed subsets of* B *such that* $\mathscr{F}_1 \overset{c}{\subset} \mathscr{F}'_1$, *and no equilibrium lies in* $\mathscr{F}'_1$. *Then for all positive* I *there exists* T_0 *with* $P^n_{\xi^n_0} \{ \xi^n_t \in \mathscr{F}_1 : \forall t \leq T_0 \} \leq \exp - NI$ *for all sufficiently large* n, *and all* ξ^n_0.

Because of (H) *iii*) and *iv*), we can apply this result to

$$\mathscr{F}'_1 = \mathcal{V}_{\delta_0}(\mathscr{B}_e) - \mathring{\mathcal{V}}_{\gamma_e/2}(u_e) - \mathring{\mathcal{V}}_{\gamma_3/2}(Ex) - \mathring{\mathcal{V}}_{\delta_1/2}(NEx),$$
$$\mathscr{F}_1 = \mathscr{B}_e - \mathring{\mathcal{V}}_{\gamma_e}(u_e) - \mathring{\mathcal{V}}_{\gamma_3}(Ex) - \mathring{\mathcal{V}}_{\delta_1}(NEx),$$

and $I = \Delta + \alpha/2$. We now fix $T = T_0 \vee T_3$, and come back to decomposition (7.5).

(6) See [5] 1.2 lemme 1. Or, for this particular point, use proposition VI.1 together with τ^*-lower semicontinuity of V_h.

(7) Recall that every equilibrium belongs to $\mathscr{C}(T;]-1, 1[)$.

Since $\{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2T \} \cap R_0$ is contained in $\{ \xi_t^n \in \mathscr{F}_1 ; \forall t \in [T, 2T] \}$, we bound its probability using the Markov property and the previous lemma.

Because of lemma VII.3, trajectories φ such that

$$\rho_{0,2T}(\varphi, \xi^n) \leq (\delta_0/2) \wedge \delta_1 \wedge \gamma_e \wedge (\gamma_3/2)$$

for some

$$\xi^n \in \{\tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T \} \cap \{ \xi_0^n \in \mathscr{V}_{\gamma_e}(u_e) \}$$

satisfy $I_{0,2T}(\varphi) \geq \Delta + 3\gamma/4$; theorem V.1 provides:

$$P_{\xi_0}^n \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T ; R_0 \} \leq \exp - N(\Delta + \alpha/2) \quad \text{on} \quad \xi_0^n \in \mathscr{V}_{\gamma_e}(u_e).$$

At last, $\{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T, R_0 \}$ contains the tubelet $\{ \rho_{0,T_3}(\xi^n, \overline{\varphi}) < \gamma \}$. Combining (7.6) and the two last inequalities, we obtain for large enough n

$$P_{\xi_0}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T ; R_0 \} \geq 2^{-1} \exp(N\alpha/6)$$
$$P_{\xi_0}^n([\{ \tau \wedge \tau_1 \wedge \tau_3^0 > 2T \} \cup \{ \tau_1 \leq \tau \wedge \tau_3^0 \wedge 2T \}] \cap R_0).$$

Recalling that $\{ \tau \leq \tau_3^0, \tau < \tau_1 \} \subset \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex) \}$, we then derive from (7.5)

$$q^{-1} P_{\xi_0}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T ; R_0 \} + P_{\xi_0}^n \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex), \tau \leq \tau_3^0 ; R_0 \} \geq P_{\xi_0}^n(R_0). \quad (7.7)$$

Furthermore,

$$P_{\xi_0}^n \{ \tau_3^0 < \tau \wedge \tau_1 \wedge 2T ; R_0 \} = \mathbb{E}_{\xi_0}^n \{ \mathbb{1}_{\tau_3^0 \leq \tau \wedge \tau_1 \wedge 2T} \cdot P_{\xi_0}^n(\tau_e^1 \geq \tau / \mathbb{F}_{\tau_3^0}) \quad (7.8)$$

Applying the strong Markov property on the set $\{ \tau_3^0 \leq \tau \}$, we see that $P_{\xi_0}^n(\tau_e^1 \geq \tau / \mathbb{F}_{\tau_3^0}) = P_{\xi^n_{\tau_3^0}}^n(\tau_e \geq \tau)$ with τ_e as in lemma VII.2; from this lemma we deduce that (7.8) is not more than

$$q^{-1} \mathbb{E}_{\xi_0}^n \{ \mathbb{1}_{\tau_3^0 \leq \tau \wedge \tau_1 \wedge 2T} \cdot P_{\xi^n_{\tau_3^0}}(\tau_e^1 \geq \tau, \xi_\tau^n \in \mathscr{V}_\delta(Ex)) \} \leq q^{-1} P_{\xi_0}^n \{ \tau_3^0 \leq \tau, \ \xi_\tau^n \in \mathscr{V}_\delta(Ex) ; R_0 \}.$$

Then (7.7) yields

$$q^{-2} P_{\xi_0}^n \{ \xi_\tau^n \in \mathscr{V}_\delta(Ex) ; R_0 \} \geq P_{\xi_0}^n \{ R_0 \}$$

which is the desired result.

We end with the case $\xi_0^n \in \mathscr{F}$: denoting by τ_e^0 the entrance time in $\mathscr{V}_{\gamma_e}(u_e)$, we must show $\lim_{n \to \infty} \inf_{\xi_0 \in \mathscr{F}} P_{\xi_0}^n(\tau_e^0 < \tau) = 1$. This can be carried out in the same way as in [15]. (Lyapunov stability implies that $\rho(\mathscr{B}_e^c, \hat{\mathscr{F}}) > 0$, where $\hat{\mathscr{F}}$ denotes the set of all points visited by the solutions of (M. E.) starting from $\mathscr{F}$) □

□ *Proof of lemma* VII.2. — Let γ_3, $T' > 0$ and $\gamma_e < \gamma'_e$. We define stopping times τ_1 as above, and

$$\tau_3^0 = 0, \qquad \tau_3^{k+1} = \min\{ t > \tau_3^k + T' : \xi_t^n \in \mathscr{V}_{\gamma_3}(Ex) \}.$$

Let $\nu_3 = \max\{ k > 0 ; \tau_3^k < \tau \}$, and $R'_k = \{ \tau < \tau_e, \nu_3 = k \}$.

Using the same argument as above, we see it is enough to show that

$$P^n_{\xi_0^n}\{ \xi_\tau^n \in \mathscr{V}_\delta(Ex), R'_0 \} \geq q P^n_{\xi_0^n}(R'_0) \tag{7.9}$$

holds for $\xi_0^n \in \mathscr{V}_{\gamma_3}(Ex)$ and sufficient large n.

This time, we will decompose R'_0 according to

$$\Omega^n = \{ \tau \wedge \tau_1 > 2T' \} \cup \{ \tau_1 \leq 2T' \wedge \tau \} \cup \{ \tau \leq 2T', \tau < \tau_1 \} \tag{7.10}$$

To show that the most important contribution to $P^n_{\xi_0^n}(R'_0)$ comes from the last set in (7.10), we look for a family of tubelets included in it. Using hypothesis (H) *ii*) and proposition VI.1.*c*, we find for each $u \in Ex$ some $u_2[u] \in (\overline{\mathscr{B}}_e)^c$ with $\rho(u_2[u], u) < \delta/4$ and some line segment $\varphi[u]$ with end-points u, $u_2[u]$ on the time interval $[0, T_2[u]]$ such that $I_{0,T_2[u]}(\varphi[u]) \leq \alpha/6$. Let $\delta_2[u] \leq \delta_1/2$ with $\mathscr{V}_{\delta_2[u]}(u_2[u]) \subset (\overline{\mathscr{B}}_e)^c$; then,

$$\mathscr{V}_{\delta_2[u]}(\varphi[u][0, T_2[u]]) \subset \mathscr{V}_{\delta/2}(Ex). \tag{7.11}$$

Since u is an equilibrium, we apply theorem IV.1 and find some $\delta_3[u] < \delta_2[u]$ such that for large enough n

$$P^n_{\xi_0^n}\{ \rho_{0,T_2[u]}(\xi^n, \varphi[u]) < \delta_2[u] \} \geq \exp - N\alpha/3$$

on $$\{ \rho(\xi_0^n, u) < \delta_3[u] \}. \tag{7.12}$$

Ex being compact, there exist $\delta_3 > 0$ and a finite number K of elements u_k of Ex with $\mathscr{V}_{\delta_3}(Ex) \subset \bigcup_{k \leq K} \mathring{\mathscr{V}}_{\delta_3[u_k]}(u_k)$. We now claim the analogue to lemma VII.3:

LEMMA VII.5. — *There exists* $\gamma_3 > 0$ *such that for all* $T \geq 0$ *and* $\varphi \in \mathscr{C}([0, T]; B)$ *the conditions* $\varphi_0 \in \mathscr{V}_{2\gamma_3}(Ex)$,

$$\varphi_T \in \mathscr{V}_{2\delta_1}(NEx) \qquad \textit{and} \qquad \varphi[0, T] \subset \mathscr{V}_{\delta_0}(\mathscr{B}_e)$$

imply $I_{0T}(\varphi) \geq 3\alpha/4$.

We fix γ_3; of course we may suppose $\gamma_3 \leq \delta_3$. Recall time T_0 we obtained from lemma VII.4: from now, we set $T' = T_0 \vee \max_{k \leq K} T_2[u_k]$.

We now come back to the decomposition (7.10). The set $\{ \tau \wedge \tau_1 > 2T' \} \cap R'_0$ being contained in $\{ \xi_t^n \in \mathscr{F}_1, \forall t \in [T, 2T] \}$, we derive an uniform upper bound for its probability from lemma VII.4.

The trajectories φ on $[0, 2T']$ uniformly close to the set $\{ \tau_1 \leq \tau \wedge 2T', \xi_0^n \in \mathcal{V}_{\gamma_3}(Ex) \}$ up to a distance $(\delta_0/2) \wedge \delta_1 \wedge \gamma_3$ have action functional value not less than $3\alpha/4$, according to lemma VII.5: thus, theorem V.1 for $D_{2\alpha/3}$ yields for large enough n

$$P_{\xi_0^n}^n \{ \tau_1 \geq 2T' \wedge \tau, R'_0 \} \leq \exp - N\alpha/2 \quad \text{on} \quad \{ \xi_0^n \in \mathcal{V}_{\gamma_3}(Ex) \}.$$

At last, whenever $\xi_0^n \in \mathcal{V}_{\gamma_3}(Ex)$, ξ_0^n lies in some $\mathcal{V}_{\delta_3[u_k]}(u_k)$; but (7.11), and the conditions on $\delta_1, \delta_2[u_k], T'$ show that the tubelet with axis $\varphi[u_k]$ and radius $\delta_2[u_k]$ is included in $\{ \tau \leq 2T', \tau < \tau_1 \} \cap R'_0$: thus, the $P_{\xi_0^n}^n$-probability of this set is not less than $\exp - N\alpha/3$ because of (7.12), and combining the three last estimates:

$$P_{\xi_0^n}^n \{ \tau \leq 2T', \tau < \tau_1 ; R'_0 \} \geq 2^{-1} \exp - N\alpha/6$$
$$. [P_{\xi_0^n}^n \{ \tau \wedge \tau_1 > 2T'; R'_0 \} + P_{\xi_0^n}^n \{ \tau_1 \leq 2T' \wedge \tau; R'_0 \}].$$

Because of (7.10), the term between brackets is equal to

$$P_{\xi_0^n}^n(R'_0) - P_{\xi_0^n}^n \{ \tau \leq 2T', \tau < \tau_1 ; R'_0 \} ;$$

then, relation (7.9) easily follows from $\{ \tau < \tau_1 \} \subset \{ \xi_\tau^n \in \overline{\mathcal{V}_\delta(Ex)} \}$. □

□ *Proof of lemma* VIII.3. — Suppose the results is false: then, there exist time T^k, trajectories φ^k with $\tau^* - \lim_{k \to \infty} \varphi_0^k = u_e$, $\varphi_{T_k}^k \in \mathcal{V}_{2\delta_1}(NEx)$,

$$\varphi^k[0, T^k] \subset \mathcal{V}_{\delta_0}(\mathcal{B}_e) - \mathring{\mathcal{V}}_{\gamma_3}(Ex), \qquad I_{0T^k}(\varphi^k) \leq \Delta + 3\alpha/4 \quad (7.11)$$

We may suppose—shortening T^k if necessary—that T^k is the entrance time of φ^k in $\mathcal{V}_{2\delta_1}(NEx)$. If $(T^k)_k$ was bounded, say with $T^\infty \in \mathbb{R}^+$, we would extend φ^k on $[T^k, T^\infty]$ as being the solution of (M. E.) starting at $\varphi_{T^k}^k$ (without changing action value); according to the theorem III.4, there should exist some accumulation point φ^∞ with $\varphi_0^\infty = u_e$, $I_{0T^\infty}(\varphi^\infty) \leq \Delta + 3\alpha/4$ and $\varphi_t^\infty \in \mathcal{V}_{2\delta_1}(NEx)$ for some accumulation point t of $(T^k)_k$, which would contradict (7.3).

So we may suppose that the times T^k increase to infinity. Let's shift φ^k in ψ^k: $\psi_t^k = \varphi_{t+T^k}^k$, $t \in [-T^k, 0]$. Using the same argument as before for all $K \in \mathbb{N}^*$, one can find a subsequence—still denoted by ψ^k—uniformly converging on $[-K, 0]$ to some $\tilde{\psi}^k$ such that

$$I_{-K,0}(\tilde{\psi}^k) \leq \Delta + 3\alpha/4, \qquad \tilde{\psi}_0^K \in \mathcal{V}_{2\delta_1}(NEx),$$
$$\tilde{\psi}^K[-K, 0] \subset \mathcal{V}_{\delta_0}(\mathcal{B}_e) - \mathring{\mathcal{V}}_{\gamma_3}(Ex) - \mathring{\mathcal{V}}_{\delta_1}(NEx) \qquad (7.12)$$

By a classical argument, we then find a subsequence—still denoted by ψ^k—uniformly converging on $[-K, 0]$ to $\tilde{\psi}^K$ for all K: of course, the $\tilde{\psi}^K$ are the restrictions of some $\tilde{\psi}$ defined on $]-\infty, 0]$ with $I_{-\infty,0}(\tilde{\psi}) \leq \Delta + 3\alpha/4$; thus, using proposition VII.2, there exist times $t' < 0$ such that $\tilde{\psi}_{t'}$ converges in $\|.\|_2$ norm to an equilibrium which must be u_e because of (7.12). For large enough l, $\tilde{\psi}_{t'} \in \mathscr{B}_e$: hence we derive from proposition VI.1 that $W(u_e, \tilde{\psi}_{t'}) = \beta\{V_h(\tilde{\psi}_{t'}) - V_h(u_e)\}$ goes to zero; for large l, we can find a function $\tilde{\varphi}$ on $[0, s]$ with $\tilde{\varphi}_0 = u_e$, $\tilde{\varphi}_s = \tilde{\psi}_{t'}$, $I_{0s}(\tilde{\varphi}) < \alpha/4$; making a trajectory φ from pieces $\tilde{\varphi}$ on $[0, s]$ and $\tilde{\psi}$ on $[s, s - t']$, we would obtain $\varphi_{s-t'} \in \mathscr{V}_{2\delta_1}(\text{NEx})$ and

$$\Delta + \alpha > I_{0,s-t'}(\varphi) > \beta\{V_h(\varphi_{s-t'}) - V_h(u_e)\},$$

which contradicts (7.3). □

□ The proof of lemma VII.5 is carried out in the same way as the previous one: if the result was false, we could find an accumulation point $\tilde{\psi}$ of some sequence satisfying to $I_{-\infty,0}(\tilde{\psi}) \leq 3\alpha/4$, $\tilde{\psi}_0 \in \mathscr{V}_{2\delta_1}(\text{NEx})$; this time, there would exist a sequence of times t' such that $\tilde{\psi}_{t'}$ converges in the $\|.\|_2$ norm to u_e or some element of Ex. In both cases, we are lead to a contradiction. □

The proof of lemma VII.4 is much simpler here than in general frameworks; ω-limit sets [8] consist in equilibria. First, notice that $\min\{\|1 \wedge |dV_h(u)|\|_2^2 ; u \in \mathscr{F}_1'\} > 0$: otherwise, an argument we used in the end of the proof of proposition IV.2 would conclude to the existence of some equilibrium in $\mathscr{F}_1'$.

This proposition therefore shows there exist constants C, C′ such that $I_{0T}(\varphi) \geq CT - C'$ for all $T > 0$ and trajectory φ on $[0, T]$ with values in $\mathscr{F}_1'$. So the lemma is an easy consequence of the theorem V.I. □

VIII. AN EXAMPLE. NUCLEATION PHENOMENON

Studying the equilibrium equation $dV_h(u) = 0$ is difficult in the general situation; it requires techniques of bifurcation (parameter β varying in $\mathbb{R}^+$). In the case $h \neq 0$ one can hardly derive a few quantitative results [5]. If $h = 0$ the energy landscape defined by V_0 only depends on β and the Fourier structure of interaction J: somewhat general results about bifurcation branches in the set of equilibria are shown in [6].

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[8] For the mean equation (M. E.).

We consider here the simplest example exhibiting nucleation phenomenon, which is the (ferromagnetic) case

$$J(x) = 1 + 2b \cos 2\pi p.x, \quad h = 0 \tag{8.1}$$

with $p \in \mathbb{Z}^d - \{0\}$, $0 < b \leq 1/2$.

Then, all the equilibria are (see [6]) $u = 0$, constants u^+, $-u^+$ if $\beta > \beta_c = 1$ (given in the end of section I.1), and u_{p,x_0}, $x_0 \in \mathbb{T}$ if $\beta > \beta_p = [\hat{J}(p)]^{-1}(b^{-1}$, where u_{p,x_0} is given by

$$u_{p,x_0}(x) = \tanh \{ 2\beta ba \cos 2\pi p.(x - x_0) \} \tag{8.2}$$

and a is the unique positive root of $a = \langle \tanh \{ 2\beta ba \cos 2\pi p.x \}, \cos 2\pi p.x \rangle$; a depends on β, and is equal to $\hat{u}_{p,0}(p)$.

At critical value β_c, the branch of constant solutions $\pm u^+$ bifurcates from the branch of null solution, with some stability transfer: $\pm u^+$ are stable equilibria for $\beta > \beta_c$, while 0, being stable up to β_c, becomes a saddle point and $Ex = \{0\}$ for $\beta \in]\beta_c, \beta_p]$; this is symmetry breaking. At value β_p, the branch $\{ u_{p,x_0} ; x_0 \in \mathbb{T} \}$ bifurcates from zero solution branch with stability transfer: as in [5] we can compute that the relation

$$V_0(u) = (2\beta)^{-1} \langle \theta(u), 1 \rangle,$$

θ being the concave function $u \tanh^{-1} u + \log(1 - u^2)$, holds for all equilibrium u, and therefore $Ex = \{ u_{p,x_0} ; x_0 \in \mathbb{T} \}$ as soon as $\beta > \beta_p$. It's easy to see that the assumptions of section VII are satisfied with the stable equilibria $\pm u^+$.

For the sake of simplicity let's assume $d = 1$. If $\beta > \beta_p$, theorem VIII.1 shows that the magnetization process when leaving the attracting domain of u^+ must pass the potential barrier close to one of lowest saddle points u_{p,x_0}; these states exhibit p areas on the torus—« nuclei »—where local magnetization approaches the new phase $-u^+$.

In this simple example, it seems to be difficult to study the extremal trajectories from u^+ to u_{p,x_0} (recall these are the solutions to (6.2) with $\lim_{t \to -\infty} \varphi_t = u^+$, $\lim_{t \to +\infty} \varphi_t = u_{p,x_0}$), which are the exit paths from the attracting domain of u^+ for the process (see [1] [15]). Nevertheless one may conjecture, with a slight act of faith, that, during such a dynamic phase transition and for $\beta > \beta_p$, small clusters initially appear, among which some, very small, are due to stochastic fluctuations; they next order in p main nuclei, and grow untill they attain approximately the structure of some u_{p,x_0}. At last, the process is attracted by $-u^+$, the nuclei go on spreading till they occupy the entire space.

For $\beta \in]\beta_c, \beta_p]$ the only one exit point is 0, so nucleation phenomenon does not occur. In the Curie-Weiss model (J = 1), every equilibrium is a constant function, and so nucleation never occurs. For more general ferromagnetic interaction function J, a bifurcation temperature is given by $\beta_p = [\hat{J}(p)]^{-1}$ with $\hat{J}(p) = \max_{q \in \mathbb{Z}^d} \{ \hat{J}(q); q \neq 0, \pm p \}$, and, under additional assumptions, (8.2) still defines saddle points when $\beta > \beta_p$ (see [5]).

IX. APPENDIX: PROOFS OF III.3, 4 AND 6

□ We begin with properties III.3:

a) We show the different formulas for I_{0T}. Let's denote by I_1, I_2 the second and last expression in the desired equality. We have clearly $I_1 \leq I_{0T}(\varphi) \leq I_2$. Let's define for all t, x, $a_0(t, x) \in \overline{\mathbb{R}}$ maximizing (3.7): a_0 is the (measurable) function given by (3.3) if $|\varphi_t(x)| < 1$, and, if $\varphi_t(x) = \eta \in \{-1, +1\}$, by $-h(x) - J * \varphi_t(x) - \eta\beta^{-1} \log \dfrac{-\eta\dot{\varphi}_t(x)}{2c(\varphi_t, x)}$ with the convention $\log y = -\infty$ for $y \leq 0$. Then, for fixed (t, x), $a_m = \operatorname{sign}(a_0) \times [|a_0| \wedge m]$ converges to a_0 in $\overline{\mathbb{R}}$ as $m \to \infty$, and $b_m(t, x) = \dfrac{\beta}{2}\dot{\varphi}_t(x) . a_m(t, x) - \Gamma(\varphi_t, a_m(t, x), x)$ converges to $\mathcal{H}(\varphi_t, \dot{\varphi}_t(x), x)$ in $\overline{\mathbb{R}}^+$. As $a \to \Gamma(\varphi_t, a, x)$ is convex and $a_m(t, x)$ is between 0 and $a_0(t, x)$, b_m is non negative. Fatou's lemma then shows that $I_2 \leq \underline{\lim}_{m \to \infty} \int_{[0,T] \times \mathbf{T}} a_m$; this last term being less than I_1, we obtain property *a*).

Proof of the lower bound *d*) of $\mathcal{H}$: using the inequality $e^{\beta a} - 1 \leq e^{\beta|a|} - 1$, we obtain for $\Gamma(u, a, x)$ an uniform upper bound $A(e^{\beta|a|} - 1)$ with constant A, whose Legendre transform in the sense of (3.7) is

$$\max\left\{\frac{|v|}{2}\left[\log\frac{|v|}{2A} - 1\right] + A, 0\right\}.$$

To show the upper bound *c*) for $\mathcal{H}$, we notice that

$$\theta_c(w, v) = \frac{v}{2}\log\frac{\dfrac{v}{2c} + \sqrt{1 - w^2 + (v/2c)^2}}{1 - w}$$

(with parameter $c \in \mathbb{R}^+$) is an even function on $\mathbb{R}^2$; we restrain to $v > 0$, and see that $|\theta_c| \leq \frac{v}{2}\left\{ 1.5 \log 2 + e^{-1} + \left(\log \frac{v}{c}\right)^+ + \log \frac{1}{1-w} \right\}$. Then, the result can be easily derived.

The conditions mentioned in *b*) are sufficient because of *c*); the first one is necessary because of *d*). Inequalities

$$\theta_c(w, v) - \frac{v}{2}\log\frac{v}{2c} > \frac{v}{2}\log\frac{1}{1-u} \geq -\frac{v}{2}\log 2$$

hold for positive v, so the last two ones are necessary too.

In the proof of the regularity property *e*) of $\mathscr{H}$, the most difficult bound to get is for

$$|\theta_{c_1}(w_1, v) - \theta_c(w, v)| \leq |v| \left\{ \left|\log \frac{1-w}{1-w_1}\right| + \left|\log \frac{c}{c_1}\right| + \left|\log \frac{v + \sqrt{4c_1(1-w_1^2)+v^2}}{v+\sqrt{4c(1-w^2)+v^2}}\right| \right\};$$

we use inequality $\log 1 + z \leq z$ to bound the first two terms, and control the derivative of $a \to \log v + \sqrt{a+v^2}$ for the last one.

□ *Proof of the theorem III.4.* — We show first that D_{l_0} is relatively compact. Let φ^m be a sequence in D_{l_0}; because of property III.3.*d*) $(\dot\varphi^m)_{m\in\mathbb{N}}$ is uniformly integrable on $[0,T] \times \mathbb{T}$. According to Dunford-Pettis' theorem [9], there exists a subsequence that we still denote by φ^m, such that $\dot\varphi^m$ converges to some $\dot\varphi^\infty \in \mathbb{L}^1([0,T] \times \mathbb{T})$ in the weak topology $\sigma(\mathbb{L}^1([0,T] \times \mathbb{T}); \mathbb{L}^\infty([0,T] \times \mathbb{T}))$.

Since $\|\varphi^m_{t'} - \varphi^m_t\|_1 \leq \int_{[t,t'] \times \mathbb{T}} |\dot\varphi^m_s(x)| \, ds dx$, uniform integrability shows that (φ^m) is equicontinuous on $[0,T]$ in the $\|\,.\,\|_1$ norm, and then so it is in the metric ρ; B being τ^*-compact, Ascoli's theorem in the space $(\mathscr{C}([0,T]; B), \rho_{0T})$ yields the relative compactness of D_{l_0}.

Let φ be an accumulation point of (φ^m): we now show $\varphi \in D_{l_0}$. Without loss of generality, we may suppose that φ^m goes to φ in metric ρ_{0T}, and that $\dot\varphi^m$ goes to some $\dot\varphi^\infty$ in weak topology $\sigma(\mathbb{L}^1; \mathbb{L}^\infty)$.

For $t \leq T$ and $g \in \mathscr{C}(\mathbb{T})$, we have

$$\begin{aligned} \langle g, \varphi_t - \varphi_0 \rangle &= \lim_{m\to\infty} \langle g, \varphi^m_t - \varphi^m_0 \rangle \\ &= \lim_{m\to\infty} \int_{[0,t]\times \mathbb{T}} \dot\varphi^m_s(x) g(x) ds dx \\ &= \int_{[0,t]\times\mathbb{T}} \dot\varphi^\infty_s(x) g(x) ds dx, \end{aligned}$$

since $\mathbb{1}_{[0,t]} \times g \in \mathbb{L}^{\infty}([0, T] \times \mathbb{T})$. So φ satisfies to the differentiability condition (D) with $\dot{\varphi} = \dot{\varphi}^{\infty}$.

Let's suppose $I_{0T}(\varphi) = I < \infty$. For $\varepsilon > 0$, property III.3.a) yields some $f \in \mathbb{L}^{\infty}([0, T] \times \mathbb{T})$ such that

$$\int_{[0,T]\times\mathbb{T}} \left[\frac{\beta}{2}\dot{\varphi} f - \Gamma(\varphi, f, x)\right] dt dx \geq I - \varepsilon. \tag{9.1}$$

Because of the convergence of $\dot{\varphi}^m$ to $\dot{\varphi} = \dot{\varphi}^{\infty}$, we have

$$\lim_{m\to\infty} \int_{[0,T]\times\mathbb{T}} \dot{\varphi}^m f = \int_{[0,T]\times\mathbb{T}} \dot{\varphi} f .$$

We then study the convergence of $\int_{[0,T]\times\mathbb{T}} \Gamma(\varphi^m, f, x) dt dx$. The difficult point concerns terms of the type

$$a(\varphi^m) - a(\varphi) \quad \text{with} \quad a(\psi) = c(\psi)\psi \exp \beta(h + f + J * \psi);$$

using Lebesgue's theorem, it's enough to show that $\int_{[0,T]\times\mathbb{T}} b^m \underset{m\to\infty}{\longrightarrow} 0$ where $b^m = (\varphi - \varphi^m)c(\varphi)e^{\beta(h+f+J*\varphi)}$. φ_t^m being uniformly bounded, we may suppose $f(t,.)$ to be a continuous function according to Lusin's theorem; then we derive $\int_{\mathbb{T}} b^m \to 0$, and, together with Lebesgue's theorem, $\int_{[0,T]\times\mathbb{T}} b^m \to 0$.

We have showed

$$\underline{\lim}_{m\to\infty} \int_{[0,T]\times\mathbb{T}} \left[\frac{\beta}{2}\dot{\varphi}^m f - \Gamma(\varphi^m, f, x)\right] dt dx \geq I - \varepsilon,$$

so

$$\underline{\lim}_{m\to\infty} I_{0T}(\varphi^m) \geq I - \varepsilon .$$

The case $I_{0T}(\varphi) = \infty$ is impossible, because (9.1) would otherwise be true for all I, and the previous demonstration would conclude to $I_0 \geq I$. So we showed the first part of the result. Since $I_{0T}^{-1}([0, I_0])$ is closed, I_{0T} is a l. s. c. function. □

□ At last, we prove the result III.6 of approximation by smooth trajectories: we first show that (3.8) is satisfied by some trajectory $\tilde{\varphi}$ staying far away from the boundary points -1, $+1$; then by a polygon in t variable, with vertices on the previous trajectory: to end, by a last one which is furthermore continuous on $\mathbb{T}$.

Let $\varphi_t^m = \varphi_0 + \left(1 - \frac{1}{m}\right)(\varphi_t - \varphi_0)$. For $\delta = 1 - \| \varphi_0 \|_\infty$, we have $\| \varphi_t^m \|_\infty \leq 1 - \frac{\delta}{m}$; furthermore $\varphi^m \to \varphi$ and $\dot\varphi^m = \left(1 - \frac{1}{m}\right)\dot\varphi \to \dot\varphi$ for all (t, x), one can notice that, for all (t, x) such that $\varphi_t(x) = \eta \in \{ -1, +1 \}$,

$$\frac{1}{2}\dot\varphi^m \log \frac{[\dot\varphi^m/2c(\varphi^m)] + \sqrt{1-(\varphi^m)^2 + [\dot\varphi^m/2c(\varphi^m)]^2}}{1-\varphi^m} \underset{m\to\infty}{\longrightarrow} \frac{|\dot\varphi^m|}{2}\log - \eta\frac{\dot\varphi^m}{2c(\varphi^m)}$$

with the previous notation $\log a = -\infty$ for $a \in \mathbb{R}_*^-$, so that

$$\mathscr{H}(\varphi_t^m, \dot\varphi_t^m(x), x) \to \mathscr{H}(\varphi_t, \dot\varphi_t(x), x).$$

In order to apply Lebesgue's theorem, we look for an upper bound of $\mathscr{H}(\varphi_t^m, \dot\varphi_t^m(x), x)$ using property III.3.*c*): remark that, on $\{ \varphi^m \geq 1 - \delta \}$, $\varphi - \varphi_0 \geq 0$ and $1 - \varphi^m \geq 1 - \varphi$ which is $(t - x)$ a. s. non zero on the set $\{ \dot\varphi > 0 \} = \{ \dot\varphi^m > 0 \}$. We then obtain the following bound, independent of m:

$$\mathscr{H}(\varphi_t^m, \dot\varphi_t^m, x)$$

$$\leq K(\delta)\left[|\dot\varphi| \{ \log|\dot\varphi| + 1 \} + 1 + \sum_{\eta\in\{-1,+1\}} \mathbb{1}_{\{\eta\dot\varphi>0 \text{ and } 1-\delta<\eta\varphi<1\}} |\dot\varphi| \log\frac{1}{1-\eta\varphi}\right].$$

Property III.3.*b*) and hypothesis $I_{0T}(\varphi) < \infty$ imply that the bound is integrable; then $\lim_{m\to\infty} I_{0T}(\varphi^m) = I_{0T}(\varphi)$. As φ^m clearly goes to φ in metric ρ_{0T}, we can fix some m such that φ^m satisfies to (3.8).

We will prove further on the following

LEMMA A.1. — *Let ψ satisfying to* (D) *and* $\gamma > 0$ *with* $I_{0T}(\psi) < \infty$, $\sup_{t\leq T} \| \psi_t \|_\infty \leq 1 - \gamma$. *For all subdivision* $S = \{ t_0 = 0 < t_1 < \ldots < t_{k_0} = T \}$, *we define the polygon l^S with vertices at points (t_k, ψ_{t_k}). As S becomes finer, $I_{0T}(l^S)$ goes to $I_{0T}(\psi)$ and l^S goes to ψ uniformly on* $[0, T]$ *in* $\| . \|_1$ *norm.*

Applying this to $\psi = \varphi^m$, we find a polygon l satisfying to (3.8).

To end, we make l smoother in the x variable, using a kernel $\alpha^m \in \mathscr{C}(\mathbb{T}; \mathbb{R}^+)$, with support contained in $\left[-\frac{1}{m}, \frac{1}{m}\right]$ and integral equal to 1:

LEMMA A.2. — *Let ψ satisfying to* (D) *and* $\gamma > 0$ *with* $I_{0T}(\psi) < \infty$, $\sup_{t\leq T} \| \psi_t \|_\infty \leq 1 - \gamma$ *and* $\psi_0 \in \mathscr{C}(\mathbb{T})$. *Denote by ψ^m the function*

$$\psi_t^m(x) = \psi_0(x) + \alpha^m * (\psi_t - \psi_0)(x).$$

Then, as $m \to \infty$, $\psi^m \to \psi$ *uniformly on* $[0, T]$ *in* $\|\cdot\|_1$ *norm*, $I_{0T}(\psi^m) \to I_{0T}(\psi)$ *and* $\overline{\lim}_{m \to \infty} (\sup_{t \le T} \|\psi_t^m\|_\infty) \le 1 - \gamma$.

We apply this lemma to $\psi = l$, and find some m such that (3.8) is satisfied, and $\sup_{t \le T} \| l_t^m \|_\infty \le 1 - \gamma/2$; we show in the proof of the lemma that $\dot{l}_t^m = a^m * l_t$, which is a stepwise function on $[0, T]$, with values in $\mathscr{C}(\mathbb{T})$; so $l^m \in \mathscr{C}P_T^{1,0}$, which ends the proof. □

□ We now prove lemma A.1; we will forget the index S in notation l^S. l satisfies to (D), with

$$\dot{l}_t = \frac{\psi_{t_{k+1}} - \psi_{t_k}}{t_{k+1} - t_k}, \quad \text{for} \quad t \in I_k =]t_k, t_{k+1}[, \qquad \text{and} \qquad \sup_{t \le T} \| l_t \|_\infty \le 1 - \gamma .$$

Applying Jensen's inequality to the convex function $a \to \mathscr{H}(l_{t_k}, a, x)$ and to $\dot{\psi}_t(x)$, we derive

$$(t_{k+1} - t_k)\mathscr{H}(l_{t_k}, \dot{l}_{t_k^+}(x), x) \le \int_{I_k} \mathscr{H}(l_{t_k}, \dot{\psi}_t(x), x)dt - \lambda \quad \text{a. s.}$$

Next, we integrate this relation on $\mathbb{T}$, and use property III.3.e) with $y = x$: we obtain:

$$\int_{I_k \times T} \mathscr{H}(l_t, \dot{l}_t(x), x)dx \le \int_{I_k \times T} \mathscr{H}(\psi_t, \dot{\psi}_t(x), x)dtdx + A_k \qquad (9.2)$$

where

$$A_k = \int_{I_k \times T} \{ (1 + |\dot{\psi}_t(x)|)(\mathcal{O}_\gamma[|\psi_t - l_{t_k}|(x)] + \varepsilon_\gamma[\rho(\psi_t, l_{t_k})]) + (1 + |\dot{l}_{t_k^+}(x)|)(\mathcal{O}_\gamma[|\psi_t - l_{t_k}|(x)] + \varepsilon_\gamma[\rho(\psi_t, l_{t_k})] \} dtdx .$$

Let's denote $r = \sup \{ \| \psi_t - \psi_{t_{k'}} \|_1 ; t \in I_{k'}, k' = 0, \ldots, k_0 - 1 \}$; then $\| l_t - l_{t_k} \|_1 \le r$ if $t \in I_k$. Metric ρ being dominated by $\|\cdot\|_1$ norm and $|\dot{l}_{t_k^+}(x)|$ being bounded from above by $|t_{k+1} - t_k|^{-1} \int_{I_k} |\dot{\psi}_s(x)| dsdx$, we see that

$$A_k \le \varepsilon'_\gamma(r) \int_{I_k \times T} (1 + |\dot{\psi}_t|)dtdx + K_\gamma A'_k \qquad (9.3)$$

for some constant K_γ depending on γ and

$$A'_k = \int_{I_k \times T} |\psi_t - l_{t_k}| |\dot{\psi}_t| dtdx + (t_{k+1} - t_k)^{-1} \int_{I_k \times I_k \times T} | l_t - l_{t_k} | |\dot{\psi}_s| dtdsdx .$$

Recall that $|l_t - l_{t_k}|, |\psi_t - l_{t_k}| \leq 2$. For any $C > 0$ we have

$$A'_k \leq 4\left[Cr(t_{k+1} - t_k) + \int_{I_k \times T} |\dot{\psi}_t| \, 1_{\{|\dot{\psi}_t| > C\}} dt dx\right].$$

Combining this with (9.2 and 3) we finally derive

$$I_{0T}(l) \leq I_{0T}(\psi) + \varepsilon''_{\gamma,C}(r)\left[1 + \int_{[0,T]\times T} |\dot{\psi}| \, dt dx\right] + 4K_\gamma \int_{\{|\dot{\psi}| > C\}} |\dot{\psi}| \, dt dx \quad (9.4)$$

where $\varepsilon''_{\gamma,C}$ depends on γ, C and goes to 0 with r. According to the remark after theorem III.4, $t \to \psi_t$ is continuous in $\| \cdot \|_1$ norm; so l goes to ψ in $\mathscr{C}([0,T], \mathbf{L}^1(\mathbf{T}))$ as the subdivision S becomes finer, and then theorem III.4 implies $\underline{\lim}\, I_{0T}(l) \geq I_{0T}(\psi)$. On the other hand, the integrability of $\dot{\psi}$ and (9.4) shows that $\overline{\lim}\, I_{0T}(l) \leq I_{0T}(\psi)$; so the lemma is proved. □

□ At last we prove lemma A.2:

1) Note that $\| \psi_t^m \|_\infty \leq \| \alpha^m * \psi_t \|_\infty + \| \psi_0 - \alpha^m * \psi_0 \|_\infty$. ψ_0 being continuous, the last term converges to 0; the first one being less than $1-\gamma$, we derive the last part of the result.

In the following, we will suppose m large enough so that

$$\sup_{t \leq T} \| \psi_t^m \|_\infty \leq 1 - \frac{\gamma}{2}.$$

Notice that $\psi_t^m = \psi_0^m + \int_0^t \alpha^m * \dot{\psi}_s ds$. As $\alpha^m * \dot{\psi}_s$ goes to $\dot{\psi}_s$ in $\| \cdot \|_1$ norm for a.e. $s \in [0,T]$, the inequality $\| \psi_t^m - \psi_t \|_1 \leq \int_{[0,T]\times T} |\alpha^m * \dot{\psi}_s - \dot{\psi}_s| ds dx$ shows that $\lim_{m\to\infty} \psi^m = \psi$ in $\mathscr{C}([0,T]; \mathbf{L}^1(\mathbf{T}))$; in particular, $\underline{\lim}_{m\to\infty} I_{0T}(\psi^m) \geq I_{0T}(\psi)$.

2) First apply Jensen's inequality to the probability $\alpha^m(x-y)dy$:

$$\mathscr{H}(\psi_t, \alpha^m * \dot{\psi}_t(x), x) \leq \int_T \alpha^m(y) \mathscr{H}(\psi_t, \dot{\psi}_t(x-y), x) dy \quad \text{for a.e. } (t,x).$$

Combining the relation

$$\int_{T\times T} \alpha^m(y) \mathscr{H}(\psi_t, \dot{\psi}_t(x-y), x-y) dy dx = \int_T \mathscr{H}(\psi_t, \dot{\psi}_t(x), x) dx,$$

and the property III.3.f) we obtain for a.e. t:

$$\begin{aligned} &\mathcal{H}^{\rho*}(\psi_t^m, \dot{\psi}_t^m) - \mathcal{H}^{\rho*}(\psi_t, \dot{\psi}_t) \\ &\leq \int_{\mathbb{T}} \{ \mathcal{O}_{\frac{\gamma}{2}}[\,|\psi_t - \psi_t^m|(x)] + \varepsilon_{\frac{\gamma}{2}}(\|\psi_t - \psi_t^m\|_1) \}(1 + |\dot{\psi}_t^m|(x))dx \\ &+ \int_{\mathbb{T}\times\mathbb{T}} \alpha^m(y)\{ \mathcal{O}_{\frac{\gamma}{2}}[\,|\psi_t(x) - \psi_t(x-y)|\,] + \varepsilon_{\frac{\gamma}{2}}(|y|) \}(1 + |\dot{\psi}_t(x-y)|)dxdy \end{aligned} \tag{9.5}$$

the first bound in (9.5) can be studied as we did in the proof of Lemma A.1: $(\dot{\psi}^m)_m$, being convergent in $\mathbb{L}^1([0,T] \times \mathbb{T})$, is uniformly integrable on $[0,T] \times \mathbb{T}$; as for $|\psi_t - \psi_t^m|$, it is less than 2 and goes to 0 in space $\mathbb{L}^1(\mathbb{T})$.

In order to use the same arguments for the last term of (9.5), we only need to show that $z \to \int_{\mathbb{T}} \alpha^m(x-z)\,|\psi_t(x) - \psi_t(z)|\,dx$ goes to 0 in $\mathbb{L}^1(\mathbb{T})$ (we set $z = x - y$). Denoting by $\mathcal{C}_{-y}\psi_t : x \to \psi_t(x-y)$, we have

$$\int_{\mathbb{T}\times\mathbb{T}} \alpha^m(x-z)\,|\psi_t(x) - \psi_t(z)|\,dxdz = \int_{\mathbb{T}} \alpha^m(y)\,\|\psi_t - \mathcal{C}_{-y}\psi_t\|_1 dy\,;$$

but translation operator is continuous in space $\mathbb{L}^1(\mathbb{T})$, so this last term goes to 0. We then showed $\overline{\lim_{m\to\infty}}\, I_{0T}(\psi^m) \leq I_{0T}^{\circ}(\psi)$, which ends the proof. □

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CHAPITRE III :

ETUDE DES POINTS STATIONNAIRES DANS UN MODELE DE CHAMP MOYEN LOCAL . BIFURCATIONS .

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ON SECONDARY BIFURCATIONS FOR SOME NONLINEAR CONVOLUTION EQUATIONS[1]

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ABSTRACT. On the d-dimensional torus $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$, we study the nonlinear convolution equation

$$u(t) = g\{\lambda \cdot w * u(t)\}, \qquad t \in \mathbf{T}^d, \ \lambda > 0.$$

where $*$ is the convolution on $\mathbf{T}^d$, w is an integrable function which is not assumed to be even, and g is bounded, odd, increasing, and concave on $\mathbf{R}^+$. A typical example is $g = \text{th}$.

For a general function w, we show by the standard theory of local bifurcation that, if the eigenspace of the linearized problem is of dimension 2, a branch of solutions bifurcates at $\lambda = (g'(0)\hat{w}(p))^{-1}$ from the zero solution, and we show that it can be extended to infinity.

For special simple forms of w, we show that the first bifurcating branch has no secondary bifurcation, but the other branches can.

These results are related to the theory of spin models on $\mathbf{T}^d$ in statistical mechanics, where they allow one to show the existence of a secondary phase transition of first order, and to some models of periodic structures in the brain in neurophysiology.

1. Introduction. The aim of this paper is to analyse the branches of solutions of a nonlinear convolution equation on the d-dimensional torus $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$. The equations are of the general form

$$u(t) = g\{\lambda w * u(t)\}, \tag{1.1}$$

where $t \in \mathbf{T}^d$, $\lambda \in \mathbf{R}_+$, $*$ is the convolution operator, w a given integrable function on $\mathbf{T}^d$, which is not assumed to be even, and g is a bounded, odd, increasing function, which is concave on $\mathbf{R}^+$. The positivity of λ does not reduce the generality.

There is a large number of models where equations of the above kind appear, in particular within the theory of statistical mechanics and some mathematical models of biology. In statistical mechanics, (1.1) corresponds to the mean field equation of an interacting spin system (see [**20**, **2**]). In the thermodynamical limit, the free energy $\psi(\beta)$ of the system is given by a variational principle

$$-\beta\psi(\beta) = \sup_{u \in L_2(\mathbf{T}^d)} \left[\beta \iint_{(\mathbf{T}^d)^2} w(t-s)u(s)u(t)\,ds\,dt - \int_{\mathbf{T}^d} i_\rho(u(t))\,dt \right], \tag{1.2}$$

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where $w(t-s)$ represents the interaction potential between a spin at site t and a spin at site s, w is assumed to be even, but not necessarily positive, and i_ρ is the entropy function of the single spin distribution ρ:

$$i_\rho(x) = \sup_y \left\{ xy - \ln \int \exp(yz)\rho(dz) \right\} \tag{1.3}$$

(see also §2). If u_0 is a (local) maximum of the variational problem (1.2), then the first Fréchet derivative of $\beta F - I$ must vanish; i.e.,

$$\beta F'(u_0) - I'(u_0) = 0, \tag{1.4}$$

or equivalently,

$$\beta w * u_0(t) - i'_\rho(u_0(t)) = 0 \tag{1.5}$$

almost everywhere. This mean field equation is equivalent to (1.1) if we set $i'_\rho = g^{-1}$ and replace the inverse temperature β by the parameter λ. The global maxima of (1.2) correspond to equilibrium states, while local maxima represent metastable states. Both are stable solutions of (1.5) or (1.1) (see §6). Moreover, in the theory of nucleation (see [**15**, **23**]), one is interested in solutions of (1.5) which are saddle points of the potential $\beta F - I$. They are unstable, or more precisely hyperbolic, solutions of (1.5) in the sense of dynamical systems.

Phase transitions of the spin system are nonanalytic changes of the global maximum u_0 of the variational principle. They are in general linked with a bifurcation of the solutions of the mean field equation (1.5) and simultaneously with a change of the stability of the solutions of (1.5). In [**2**] it has been shown that there are primary stable bifurcations of the solutions of (1.1) not only for the nonzero constant solutions (Curie-Weiss model), but also for periodic solutions of (1.1), which appear in the antiferromagnetic case.

Beside these models, in which equation (1.1) appears literally, there are a number of models where one gets equations of a similar type. We like to refer especially to the spin-glass model of van Hemmen et al. [**10**], since in particular one studies there secondary phase transitions—corresponding to secondary bifurcations of (1.1) here—which establish the existence of so-called mixed phases. The similarity of the equations mentioned in this reference and (1.1) will become even more obvious after we have transformed (1.1) into the corresponding equations for the Fourier transforms in §2. The methods developed in this paper, and in particular those of the associated dynamical system (§8) allow one to understand better the results of [**10**].

Nonlinear evolution equations involving a convolution term appear also in some mathematical models of biological systems. We shall mention [**0**, **1**, **5**, **6**, and **16**], where further references are quoted, but let us give more details about the problems addressed in [**16**] because they are the closest to the ones we consider here.

The adult brain of higher organisms such as mammals displays a remarkable mixture of highly specific connectivity patterns with large amounts of randomness. The cortex is the external part of the brain; it is an envelope about 2 mm thick, with many folds. The visual cortex, which has been extensively studied, is located in the occipital region, and it receives indirect projections from the two retinae. The existence of ocular dominance stripes is among the striking organization patterns uncovered in the sixties: in the brain of adult animals, the cells are segregated into stripes which are sensitive either to left eye or to right eye stimuli; but this is not true in newborn animals.

It is a major problem to understand the rules which guide the formation of this circuitry during pre- and postnatal development. A theoretical explanation should show how microscopic mechanisms governing the growth and decay of synapses—the individual contacts between neurones—yield the observed macroscopic behavior. Models of development of ocular dominance stripes stipulate that growth of contacts at points x depends on the density of fibres or contacts not only at x, but in a neighborhood of x.

Two alternative types of mechanisms may be invoked. In the first, afferent fibres carry chemical markers which diffuse laterally within the cortical tissue; at point x in cortex, the rate of growth of synapses of a certain type—i.e., coming from either the left eye of the right eye—is governed by the similarity between the marker carried by the fibre, and the concentration of this marker at x [**13**]. In the second type of model, synaptic growth depends solely on short-term temporal correlations between pre- and postsynaptic activities: this is an application of the Hebb principle of synaptic modification [**8**]. According to this principle, the strength of connections between two cells grows proportionally to the correlation between the activities of the two cells. Activities in fibres of different origins—right and left eye—are assumed to be uncorrelated, and correlations or anticorrelations are carried through the cortex via a pre-existing circuitry [**22**].

It has been pointed out [**18**] that, in spite of different mechanisms, the two models are theoretically equivalent; both are conveniently summarized by an evolution equation with a spatial convolution term of a particular type: the central part of the convolution kernel is positive, the outer part negative. If the variable u designates the difference between the density of left-eye and right-eye contacts, the evolution of u is described by the following equation [**18**], where w is a given convolution kernel depending only on space and $*$ is the spatial convolution:

$$\partial u/\partial t = (w * u) \cdot f(u). \tag{1.6}$$

The nonlinearity f serves to express a saturation or constraint; a modification of this equation, which has the advantage of exhibiting better the effect of the constraints, if for instance there is a physically maximal density of contacts, is

$$\partial u/\partial t = w * u - h(u), \tag{1.7}$$

where h is an increasing function of u, which can be taken multivalued if sharp constraints are desired. We would like to study the behavior of (1.6) and (1.7), as time increases infinitely.

It is shown in [**16**] that the nontrivial stable solutions of (1.6) when $f(u) = 1-u^2$ satisfy, under a suitable functional hypothesis on w,

$$u = \operatorname{sgn}(w * u), \tag{1.8}$$

and that the nontrivial stable stationary solutions of (1.7) satisfy

$$h(u) = w * u. \tag{1.9}$$

Clearly, if g is the reciprocal of the signum function, which means that, in (1.7), u is constrained to stay between -1 and $+1$, problems (1.8) and (1.9) are identical.

If we write (1.9) as

$$(1.10) \qquad u = h^{-1}(w * u),$$

it is natural to imbed (1.10) in a family of similar problems depending on a parameter $u = h^{-1}(\lambda w * u)$, which is precisely problem (1.1) considered above in a statistical physics setting. If, in particular, we take $g = h^{-1} = \mathrm{th}$ as in (2.10), we obtain

$$(1.11) \qquad u = \mathrm{th}(\lambda w * u).$$

Observe that as λ goes to infinity, problem (1.11) resembles more and more problem (1.8). We expect to gain some understanding of problems (1.6) and (1.7) through a careful study of the set of their stable stationary solutions, which are the main candidates to be asymptotic states of (1.6) and (1.7) as time grows infinitely. Thus we are interested in a rather complete description of *all* solutions of (1.1), at least for some natural choices of the function w.

This paper contains

(a) The proof that if $\lambda \in (0, (g'(0)\widehat{|w|}(0))^{-1})$, the only solution of (1.1) is zero. Here and below, $\hat{f}(p) = \int_{\mathbf{T}^d} f(t)\exp\{-2\pi ipt\}\,dt$ denotes the Fourier coefficient of the function f on $\mathbf{T}^d$, $p \in \mathbf{Z}^d$.

(b) A description of the primary bifurcation picture. Assuming $\hat{w}(p)$ real and $\hat{w}(q) \neq \hat{w}(p)$ for all $q \neq \pm p$, we obtain in some cases a branch starting at $\lambda_p = (g'(0)\hat{w}(p))^{-1}$ and extending to infinity. We do not presently cover the cases when w has symmetries in $\mathbf{T}^d, d > 1$, i.e., $\hat{w}(p) = \hat{w}(q)$ for some $q \neq \pm p$, because this would lead to bifurcation kernels of dimension larger than 2.

(c) A description of secondary bifurcations for some special choices of w. More precisely, if we assume that

$$(1.12) \qquad w(t) = \alpha\cos(2\pi pt) + \beta\cos(2\pi qt) + w_0(t)$$

with $\alpha, \beta > 0$

$$\hat{w}_0(r) = 0 \quad \text{for } r \in [(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q],$$

and p, q satisfying either the noncollinearity condition

$$(1.13) \qquad p = 0 \neq q \quad \text{or} \quad (pp)(qq) - (pq)^2 > 0,$$

or in the collinear case, the arithmetic condition

$$(1.14) \qquad q \notin (2\mathbf{Z}+1)p \quad \text{and} \quad p \notin (2\mathbf{Z}+1)q,$$

then we are able to give a rather complete picture of the secondary bifurcations in Theorems 5 and 7. In particular, no second bifurcation from the first-appearing branch occurs, but some may occur from the second branch. This secondary bifurcation is connected with an exchange in the stability of the primary branch. In the noncollinear case, this branch is unstable, or more precisely hyperbolic (see §8 for the definition), until the appearance of the secondary bifurcation, but it is stable after the occurrence of the secondary bifurcation. The solutions on the secondary branch are, in general, hyperbolic.

In an example we show that the mentioned exchange in stability on the second of the primary branches, which goes together with the secondary bifurcation, is of physical relevance in some models of statistical mechanics. It gives rise to a secondary phase transition of first order, where the equilibrium state jumps from the first primary branch to another stable solution.

The stability analysis of the different branches is done by reducing the problem to a finite-dimensional one on the Fourier coefficients $\hat{u}(\pm p)$ and $\hat{u}(\pm q)$ and by studying the geometric properties of an associated mapping. When the noncollinearity condition (1.13) holds, the set of solutions can be completely described. Moreover, in this case we characterize the fixed points as stable, hyperbolic, or totally unstable.

Of course, these results depend heavily on the oddness of g. Small perturbations from this condition would lead to nonconnected manifolds of solutions, which show turning points and so-called two-sided bifurcations. They appear, for example, in the spin model of the beginning of this section if there exists an additional external magnetic field $h = h(s)$.

Also higher-dimensional spin variables may be treated similarly: for example, X-Y spins or Heisenberg spins with values in S^2. Their mean field equations have the form of systems of nonlinear convolution equations. However, these generalizations will not be discussed in this paper.

2. General assumptions and preliminary results. Let $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$ be the d-dimensional torus. By dt we denote the Lebesgue measure on $\mathbf{T}^d$. We are concerned with the nonlinear convolution equation

$$u(t) = g\{\lambda \cdot w * u(t)\}, \qquad t \in \mathbf{T}^d, \tag{2.1}$$

where $\lambda \in \mathbf{R}_+ = [0, +\infty)$. Here, for a given Lebesgue-integrable function w on $\mathbf{T}^d$, we define the convolution operator

$$w * u(t) = \int_{\mathbf{T}^d} w(t-s)u(s)\,ds, \tag{2.2}$$

and we assume $g\colon \mathbf{R} \to \mathbf{R}$ to be an odd, increasing, bounded function which is

$$\text{concave on } [0, \infty). \tag{2.3}$$

Here and in the rest of the paper, we understand increasing, decreasing, etc., in the weak sense of nondecreasing, nonincreasing, etc., respectively, and similarly for concave. Otherwise, we say strictly increasing, strictly decreasing, strictly concave, etc. Of course, we exclude the trivial cases $g \equiv 0$ or $w \equiv 0$.

REMARK. In principle, there is no restriction in having $\lambda \geq 0$ instead of $\lambda \in \mathbf{R}$, since the pair $(-\lambda, w)$ gives the same equation (2.1) as the pair $(\lambda, -w)$. However, the formulation of the theorems is much simplified by considering only $\lambda \in \mathbf{R}_+$.

In the examples from statistical mechanics, the interaction potential is given by the function w, and a thermodynamical state u on $\mathbf{T}^d$ has internal energy

$$E = \frac{1}{2}\langle u, w * u\rangle = \frac{1}{2}\int_{\mathbf{T}^d} u(t) \cdot w * u(t)\,dt. \tag{2.4}$$

On the other hand, the nonlinear function g reflects in some sense the entropy of the system. To be more precise, let us recall (see [**12**, **2**], e.g.) the definition of the ϕ-function for a measure ρ:

$$\phi_\rho(x) = \ln \int \exp(xy)\rho(dy). \tag{2.5}$$

The entropy function i_ρ of ρ can then be calculated as the Legendre transformation of ϕ_ρ:

$$i_\rho(y) = \sup_x \{xy - \phi_\rho(x)\}. \tag{2.6}$$

Now, g is the derivative of the function ϕ_ρ or, equivalently, by (2.6), the inverse function of the derivative of the entropy i_ρ:

$$g(x) = \phi'_\rho(x) = (i'_\rho)^{-1}(x). \tag{2.7}$$

In examples with Ising spins, we have

$$\rho_0 = (\delta_{+1} + \delta_{-1})/2 \tag{2.8}$$

such that

$$\phi_{\rho_0}(x) = \ln \cosh(x), \tag{2.9}$$

and

$$g_0(x) = \phi'_{\rho_0}(x) = \text{th}(x). \tag{2.10}$$

Obviously, g_0 satisfies the desired properties (2.3). It is even real analytic on $\mathbf{R}$ and strictly concave on $(0, \infty)$. In general, the concavity condition for g is tantamount to the GHS-inequality for the measure ρ (see [**4**, **2**]).

For a one-dimensional problem and in connection with a quadratic internal energy, this inequality guarantees that there is exactly one higher-order phase transition for the equilibrium state (see also [**3**], in particular the remark at the end of §5).

We note some simple consequences from our assumptions on g: g being odd, we have

$$g(0) = 0. \tag{2.11}$$

Because g does not vanish identically and is concave on $\mathbf{R}^+$, we find for $x \neq 0$ that

(2.12) $g(x)/x$ is strictly positive and decreasing with respect to $|x|$.

Therefore,

$$g'(0) := \lim_{|x|\to 0} \frac{g(x)}{x} > 0 \tag{2.13}$$

exists and is positive. Until §4 inclusively, we allow $g'(0) = +\infty$. Set

$$\gamma = \lim_{x\to+\infty} g(x) \in (0, \infty). \tag{2.14}$$

By the concavity condition, g is necessarily continuous on $\mathbf{R}\backslash\{0\}$, possibly with two symmetric jumps at zero. Finally, we set

$$\bar{g}'(x) = \limsup_{|\varepsilon|\to 0} \frac{g(x+\varepsilon) - g(x)}{\varepsilon} \geq 0, \tag{2.15}$$

and

$$\underline{g}'(x) = \liminf_{|\varepsilon|\to 0} \frac{g(x+\varepsilon) - g(x)}{\varepsilon} \geq 0. \tag{2.16}$$

$\bar{g}'$ and $\underline{g}'$ are symmetric, decreasing in $|x|$, and

$$g'(0) = \bar{g}'(0) = \underline{g}'(0). \tag{2.17}$$

We mention in particular, that if g is strictly concave on $(0, \infty)$ then

$$\bar{g}'(x) - \bar{g}'(x') < 0 \quad \text{and} \quad \underline{g}'(x) - \underline{g}'(x') < 0 \tag{2.18}$$

for all $x, x' \in \mathbf{R}$ with $|x'| < x$.

We study naturally our equation (2.1) by considering the Fourier coefficients of u: For $p \in \mathbf{Z}^d$ let

$$\hat{u}(p) = \int u(t) \exp(-2\pi i p \cdot t)\, dt. \tag{2.19}$$

The inverse transformation is given by Parseval's formula

$$u(t) = \sum_{p \in \mathbf{Z}^d} \hat{u}(p) \exp(2\pi i p \cdot t). \tag{2.20}$$

Here and in the sequel the equality is understood in the sense of $L_2(\mathbf{T}^d)$. Since u and w are real functions on $\mathbf{T}^d$, we have

$$\hat{u}(-p) = \overline{\hat{u}(p)} \quad \text{and} \quad \hat{w}(-p) = \overline{\hat{w}(p)}; \tag{2.21}$$

in particular,

$$\hat{u}(0) \in \mathbf{R} \quad \text{and} \quad \hat{w}(0) \in \mathbf{R}. \tag{2.22}$$

By the convolution rule $\widehat{w * u}(p) = \hat{w}(p)\hat{u}(p)$, (2.1) can now be rewritten as

$$u(t) = g\left\{\lambda \sum_{q \in \mathbf{Z}^d} \hat{w}(q)\hat{u}(q) \exp(2\pi i q t)\right\}, \qquad t \in \mathbf{T}^d, \tag{2.23}$$

or

$$\hat{u}(p) = \left[g\left\{\lambda \sum_{q \in \mathbf{Z}^d} \hat{w}(q)\hat{u}(q) \exp(2\pi i q t)\right\}\right]^{\wedge}(p) \tag{2.24}$$

for all $p \in \mathbf{Z}^d$.

DEFINITION. We say that a solution u of (2.1) *is p-stable*, $p \in \mathbf{Z}^d$, if

$$\lambda |\hat{w}(p)| \int \bar{g}'\{\lambda w * u(t)\}\, dt < 1. \tag{2.25}$$

It is called *p-unstable or critical* if

$$\lambda |\hat{w}(p)| \int \underline{g}'\{\lambda w * u(t)\}\, dt \geq 1. \tag{2.26}$$

We conclude this section with some simple results about solutions of (2.1):

(i) Set

$$G(\lambda, u) = u - g\{\lambda w * u\}. \tag{2.27}$$

Then

$$G(\lambda, 0) = 0 \quad \text{for all } \lambda, \tag{2.28}$$

since (2.1) has always the trivial solution $u \equiv 0$. If $g'(0) < +\infty$, the linearized operator at $u \equiv 0$ is given by

$$D_u G(\lambda, 0) \cdot v = v - g'(0)\lambda w * v \tag{2.29}$$

with $v \in L_2(\mathbf{T}^d)$. The operator $v \longrightarrow w * v$ is compact in $L_2(\mathbf{T}^d)$, so that the spectrum of $D_u G(\lambda, 0)$ is

$$\mathrm{sp} D_u G(\lambda, 0) = \{1\} \cup \bigcup_{p \in \mathbf{Z}^d} \{1 - g'(0)\lambda \hat{w}(p)\}. \tag{2.30}$$

Therefore by the implicit function theorem, there is no bifurcation for λ not in

$$\mathbf{R} \cap \{(g'(0)\hat{w}(p))^{-1},\ p \in \mathbf{Z}^d \text{ with } \hat{w}(p) \neq 0\}. \tag{2.31}$$

(ii) There are two kinds of invariance for the set of solutions of (2.1): First, (2.1) is translation invariant; i.e., if u is a solution of (2.1), then so is

$$u_s(t) = u(t + s) \tag{2.32}$$

for all $s \in \mathbf{T}^d$, since

$$G(\lambda, u_s)(t) = G(u, \lambda)(t + s). \tag{2.33}$$

Recall that

$$\widehat{u_s}(p) = \hat{u}(p) \exp\{2\pi i p s\}. \tag{2.34}$$

Second, if u is a solution of (2.1), so is $-u$, since

$$G(\lambda, -u) = -G(\lambda, u). \tag{2.35}$$

THEOREM 1. (i) *Let* $g'(0) < \infty$ *and* $\lambda \in (0, (g'(0) \cdot \|w\|_{L^1})^{-1})$. *Then* (2.1) *has only the trivial solution* $u \equiv 0$.

(ii) *Let* $\hat{w}(0) > 0$ *and*

$$\lambda_0 = \begin{cases} 0 & \text{if } g'(0) = +\infty, \\ (g'(0)\hat{w}(0))^{-1} & \text{otherwise.} \end{cases} \tag{2.36}$$

At λ_0 *a branch of constant nontrivial solutions* $u_\lambda \equiv \pm\hat{u}_\lambda(0)$ *bifurcates from the trivial solution, where* $\hat{u}_\lambda(0) = \hat{u}(0) > 0$ *is the unique positive solution of*

$$\hat{u}(0) = g\{\lambda \hat{u}(0)\hat{w}(0)\}, \qquad \lambda \in (\lambda_0, +\infty). \tag{2.37}$$

If, moreover, $w \geq 0$, *then this branch does not have secondary bifurcations.*

PROOF. (i) g being odd, we have for any solution u,

$$\sup_t |u(t)| = \sup_t g\{\lambda |w * u(t)|\} \leq g\left\{\lambda \|w\|_{L^1} \sup_t |u(t)|\right\},$$

which for $\lambda < (g'(0)\|w\|_{L^1})^{-1}$ implies $\sup_t |u(t)| = 0$.

(ii) The first assertion of (ii) is well known (see, e.g., [**2**, Appendix B],). If $w \geq 0$, $w \not\equiv 0$, then $\hat{w}(p) < \hat{w}(0)$ for all $p \in \mathbf{Z}^d - \{0\}$. If g is differentiable on $(0, +\infty)$, the spectrum of the linearization at $u_\lambda \equiv \hat{u}(0)$,

$$D_u G(\lambda, \hat{u}(0))v = v - \lambda g'\{\lambda \hat{w}(0)\hat{u}(0)\} w * v,$$

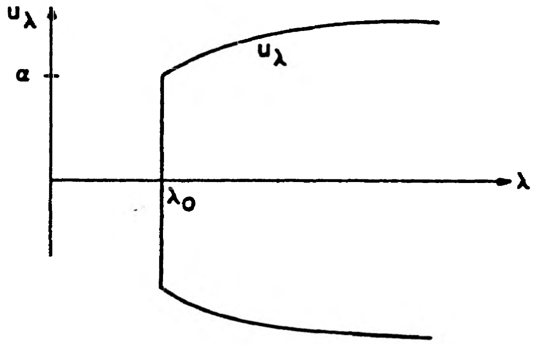

FIGURE 1

consists of the values 1 and $1 - \lambda\hat{w}(p)g'\{\lambda\hat{w}(0)\hat{u}(0)\}$, $p \in \mathbf{Z}^d$. But

$$\lambda\hat{w}(p)g'\{\lambda\hat{w}(0)\hat{u}(0)\} < \lambda\hat{w}(0)g'\{\lambda\hat{w}(0)\hat{u}(0)\} = \frac{d}{dx}g\{\lambda x\hat{w}(0)\}_{|x=\hat{u}(0)} < 1 \tag{2.38}$$

and the branch $u \equiv \pm\hat{u}(0)$ cannot have a secondary bifurcation. For a general function g, a simple approximation by a smooth function $\tilde{g}$ shows that there are no secondary bifurcations on $\pm u(0)$ for g either. □

REMARK. Let again $\hat{w}(0) > 0$. We are interested in the behavior of $\hat{u}_\lambda(0)$ for $\lambda \searrow \lambda_0$. Assume first that g is linear in some interval $[-\alpha, +\alpha]$, $0 < \alpha < +\infty$; i.e.

$$g(x) = g'(0)x, \qquad 0 < g'(0) < +\infty, \tag{2.39}$$

for $x \in [-\alpha, +\alpha]$. Of course, we suppose α to be maximal with this property. Then at $\lambda = \lambda_0 = (g'(0)\hat{w}(0))^{-1}$ we have in addition to (2.37) the constant solutions (see Figure 1)

$$u_{\lambda_0} \equiv x \quad \text{with } x \in [-\alpha, +\alpha]. \tag{2.40}$$

Conversely, if g is not linear in a neighborhood of 0, then the concavity of g implies that either

$$\lambda_0 = 0 \text{ and then } g\{\lambda_0\hat{w}(0)x\} = 0 \text{ for all } x,$$

or

$$\lambda_0 > 0, \text{ and then } 0 < g'(0) < +\infty$$

and

$$g\{\lambda_0\hat{w}(0)x\} < g'(0)\lambda_0\hat{w}(0)x = x \quad \text{for all } x \in (0, \infty). \tag{2.41}$$

In both cases there are no nontrivial constant solutions at λ_0. This shows that we have, in addition to Theorem 1(ii), nontrivial constant solutions of (2.1) if and only if g is linear in a neighborhood of 0. (2.37) and (2.40) are the only nontrivial constant solutions of (2.1).

We set the maximal α from (2.39) equal to 0 if g is not linear in a neighborhood of zero. It is then easy to see that

$$\lim_{\lambda \searrow \lambda_0} \hat{u}_\lambda(0) = \lim_{x \searrow \alpha} g(x). \tag{2.42}$$

In particular, if g is continuous at 0, but not linear in a neighborhood of 0, then

$$\lim_{\lambda \searrow \lambda_0} \hat{u}_\lambda(0) = 0, \tag{2.43}$$

and graphically the nontrivial constant solutions branch indeed from the trivial solution.

Finally, we note some simple properties of the function $\lambda \to \hat{u}_\lambda(0)$ if $\hat{w}(0) > 0$. For $\lambda \in (\lambda_0, +\infty)$ we set

$$\phi_0(\lambda, x) = g\{\lambda \hat{w}(0) x\} \tag{2.44}$$

and

$$\bar{\partial}\phi_0(\lambda, x) = \lambda \hat{w}(0) \bar{g}'\{\lambda \hat{w}(0) x\}. \tag{2.45}$$

On $(\lambda_0, +\infty)$ the function $\lambda \to \hat{u}_\lambda(0) = \phi_0(\lambda, \hat{u}_\lambda(0))$ is continuous and increasing with

$$\lim_{\lambda \to +\infty} \hat{u}_\lambda(0) = \gamma, \tag{2.46}$$

where γ stems from (2.14). The solution $u_\lambda \equiv \pm \hat{u}_\lambda(0)$ is 0-stable since

$$0 \leq \bar{\partial}\phi_0(\lambda, \hat{u}_\lambda(0)) < 1. \tag{2.47}$$

Moreover,

$$\lim_{\lambda \to +\infty} \bar{\partial}\phi_0(\lambda, \hat{u}_\lambda(0)) = 0. \tag{2.48}$$

To see the last equality, we fix $0 < \hat{x} < \gamma$ and $\bar{\lambda} > \lambda_0$ with $u_0(\bar{\lambda}) > \bar{x}$ by (2.46). The concavity of g on $\mathbf{R}^+$ shows for $\lambda > \bar{\lambda}$ that

$$0 \leq \bar{\partial}\phi_0(\lambda, \hat{u}_\lambda(0)) \leq (g\{\lambda \hat{w}(0) \hat{u}_\lambda(0)\} - g\{\lambda \hat{w}(0) \bar{x}\})/(\hat{u}_\lambda(0) - \bar{x}). \tag{2.49}$$

By (2.14) the right side of (2.47) goes to zero as $\lambda \to +\infty$.

3. Some invariance results for the Fourier coefficients. In this section, we show that the conditions on g imply the existence of classes of functions u, characterized by their Fourier coefficients, which are invariant under the operation $u \to g\{\lambda w * u\}$. Therefore, solutions of (2.1) can be studied independently in each of these classes.

Let us define, for $p \in \mathbf{Z}^d$ and $A \subseteq \mathbf{Z}$,

$$Ap = \{kp,\ k \in A\}. \tag{3.1}$$

PROPOSITION 1. *Let $p, q \in \mathbf{Z}^d$ and m be an integrable function on $\mathbf{T}^d$. Then*

(i)

$$\hat{m}(r) = 0 \quad \textit{for all } r \notin \mathbf{Z}p \tag{3.2}$$

if and only if

$$m(\cdot + s) = m(\cdot) \quad \textit{for all } s \in \mathbf{T}^d \textit{ with } p \cdot s \equiv 0 \bmod 1. \tag{3.3}$$

(ii)

$$\hat{m}(r) = 0 \quad \textit{for all } r \notin (2\mathbf{Z} + 1)p \tag{3.4}$$

if and only if

$$m(\cdot + s) = -m(\cdot) \quad \textit{for all } s \in \mathbf{T}^d \textit{ with } p \cdot s \equiv \tfrac{1}{2} \bmod 1. \tag{3.5}$$

(iii) *Assume that $p \neq 0$ and $q \neq 0$ are not collinear, i.e.,*

$$(pp)(qq) - (pq)^2 > 0. \tag{3.6}$$

Then

(3.7) $\hat{m}(r) = 0$ *for all* $r \notin [(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q]$

if and only if

(3.8) $m(\cdot + s) = m(\cdot)$ *for all* $s \in \mathbf{T}^d$ *with* $ps \bmod 1 \equiv qs \bmod 1 \equiv \frac{1}{2}$.

(iv) *Assume that* $p, q \in \mathbf{Z}^d \backslash \{0\}$ *are collinear, i.e.,*

(3.9) $$n_1 p = n_2 q \neq 0$$

for some $n_1, n_2 \in \mathbf{Z}\backslash\{0\}$ *with* $gcd(n_1, n_2) = 1$. *Here, gcd denotes the greatest common divisor. Set* $r_0 = p/n_2 = q/n_1$. *Then* $r_0 \in \mathbf{Z}^d$, *and*

(3.10) $$[(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q] = \begin{cases} \mathbf{Z}r_0 & \text{if } n_1 \cdot n_2 \text{ even,} \\ (2\mathbf{Z}+1)r_0 & \text{if } n_1 \cdot n_2 \text{ odd.} \end{cases}$$

Now, if $n_1 \cdot n_2$ *is odd, then*

(3.7) *is equivalent to* (3.8).

But if $n_1 \cdot n_2$ *is even, then* (3.7) *is equivalent to*

(3.11) $m(\cdot + s) = m(\cdot)$ *for all* $s \in \mathbf{T}^d$ *with* $ps \bmod 1 \equiv qs \bmod 1 \equiv 0$.

PROOF. (i) (3.2) and (2.20) imply (3.3) immediately. Conversely, by (3.3) and (2.34), we get for all $s \in \mathbf{T}^d$ with $ps \equiv 0 \bmod 1$ and $r \in \mathbf{Z}^d$ that

(3.12) $$\hat{m}(r)[\exp\{2\pi i r s\} - 1] = 0.$$

Let $r = (r_1, \ldots, r_d)$ with $\hat{m}(r) \neq 0$. Then

(3.13) $r \cdot s \equiv 0 \bmod 1$ for all $s \in \mathbf{T}^d$ with $ps \equiv 0 \bmod 1$.

Considering, in particular, $s = (0, \ldots, s_k, 0, \ldots)$, we have $r_k s_k \in \mathbf{Z}$ for all s_k with $p_k s_k \in \mathbf{Z}$, which can only hold if $r_k = n_k p_k$ for some $n_k \in \mathbf{Z}$. Moreover, if $n_k \neq n_l$ for $k \neq l$ and $p_k, p_l \neq 0$, we take $s = (0, \ldots, s_k, \ldots, s_l, 0, \ldots)$ with $s_k = \frac{1}{2}(n_k - n_l)p_k$ and $s_l = -\frac{1}{2}(n_k - n_l)p_l$, which satisfies $ps = 0$ but $rs = \frac{1}{2}$. We have a contradiction to (3.13). Therefore, $n_k = n_l$ and $r \in \mathbf{Z}p$.

(ii) Obviously, if (3.4) is satisfied, then so is (3.5).

Let (3.5) hold. Then (3.3) holds also, and we get $\hat{m}(r) = 0$ for all $r \notin \mathbf{Z}p$. But if $r = 2np \neq 0$ with $n \in \mathbf{Z}$, we take $s = (0, \ldots, 1/2p_k, 0, \ldots)$ for some k with $p_k \neq 0$, such that $ps = \frac{1}{2}$. By (3.5)

$$0 = \hat{m}(r)[\exp\{2\pi i r s\} + 1] = \hat{m}(r)2,$$

which shows $\hat{m}(r) = 0$. Thus (3.4) holds.

(iii) Evidently, (3.8) follows from (3.7). Conversely, assume (3.8). First, one checks that the set of all s satisfying $ps \equiv qs \equiv \frac{1}{2} \bmod 1$ is given by

(3.14) $$s = \tfrac{1}{2}((pp)(qq) - (pq)^2)^{-1} \times \{p[(2k+1)(qq) - (2l+1)(pq)] + q[(2l+1)(pp) - (2k+1)(pq)]\} + \tilde{s}$$

with $k, l \in \mathbf{Z}$ and $(\tilde{s}p) = (\tilde{s}q) = 0$. (3.8) says that if $\hat{m}(r) \neq 0$ then $r \cdot s \equiv \frac{1}{2} \bmod 1$ for all s from (3.14). Setting $r = \alpha p + \beta q + \tilde{r}$ with $\tilde{r}p = \tilde{r}q = 0$, we find for all such s that

$$rs = \tfrac{1}{2}[\alpha(2k+1) + \beta(2l+1)] + \tilde{r}\tilde{s} \equiv \tfrac{1}{2} \bmod 1$$

for all $k, l \in \mathbf{Z}$ and all $\tilde{s}$. Hence, $\tilde{r} = 0$ and

$$(\alpha, \beta) \in [(2\mathbf{Z}+1) \times 2\mathbf{Z}] \cup [2\mathbf{Z} \times (2\mathbf{Z}+1)], \tag{3.15}$$

which is $r \in [(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q]$.

(iv) Since $\gcd(n_1, n_2) = 1$, there exists $k_1, k_2 \in \mathbf{Z}$ with

$$k_1 n_1 + k_2 n_2 = 1. \tag{3.16}$$

Therefore

$$k_1 q + k_2 p = r_0 \in \mathbf{Z}^d. \tag{3.17}$$

Now if n_1 and n_2 are odd, then

$$l_1 n_1 + l_2 n_2 \in (2\mathbf{Z}+1) \quad \text{iff} \quad (l_1, l_2) \in [(2\mathbf{Z}+1) \times 2\mathbf{Z}] \cup [2\mathbf{Z} \times (2\mathbf{Z}+1)], \tag{3.18}$$

which by $l_1 q + l_2 p = (l_1 n_1 + l_2 n_2) r_0$ shows (3.10) for $n_1 \cdot n_2$ odd. On the other hand, if $n_1 \cdot n_2$ is even—i.e., $(n_1, n_2) \in [(2\mathbf{Z}+1) \times 2\mathbf{Z}] \cup [2\mathbf{Z} \times (2\mathbf{Z}+1)]$—then

$$l_1 q + l_2 p = (l_1 + n_2) q + (l_2 - n_1) p = (l_1 n_1 + l_2 n_2) r_0$$

shows

$$[(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q] = \mathbf{Z}p + \mathbf{Z}q = \mathbf{Z}r_0. \tag{3.19}$$

Now, let $n_1 \cdot n_2$ be odd. Then, since $(k_1, k_2) \in [(2\mathbf{Z}+1) \times 2\mathbf{Z}] \cup [2\mathbf{Z} \times (2\mathbf{Z}+1)]$ by (3.18),

$$ps \equiv qs \equiv \tfrac{1}{2} \bmod 1 \quad \text{iff} \quad r_0 s \equiv \tfrac{1}{2} \bmod 1,$$

and (ii) shows the equivalence of (3.7) and (3.8). It is clear that

$$ps \equiv qs \equiv 0 \bmod 1 \quad \text{iff} \quad r_0 s \equiv 0 \bmod 1.$$

If $n_1 \cdot n_2$ is even, then (3.19) and (i) show that (3.7) and (3.11) are equivalent. □

DEFINITION. We denote by $\mathcal{F}_p$, $\mathcal{F}_p'$ and $\mathcal{F}_{pq}$ the sets of integrable functions on $\mathbf{T}^d$ which satisfy (3.2), (3.4), and (3.7) respectively.

We note some immediate consequences of the proposition:

COROLLARY. *If $w \in \mathcal{F}_p, w \in \mathcal{F}_p'$, or $w \in \mathcal{F}_{pq}$ for noncollinear p and q, then all solutions of (2.1) are in $\mathcal{F}_p, \mathcal{F}_p'$, and $\mathcal{F}_{pq}$, respectively.*

REMARK. For $p \in \mathbf{Z}^d \backslash \{0\}$ set

$$w_p(t) = \sum_{r \notin (2\mathbf{Z}+1)p \backslash \{\pm p\}} \hat{w}(r) \exp\{2\pi i r t\}. \tag{3.20}$$

If $w = w_p$, i.e., if

$$\hat{w}(r) = 0 \quad \text{for all } r \in (2\mathbf{Z}+1)p \backslash \{+p, -p\}, \tag{3.21}$$

then any function $u \in \mathcal{F}_p'$ is a solution of (2.1) if and only if

$$u(t) = g\left\{\lambda \sum_{r=\pm p} \hat{w}(r)\hat{u}(r) \exp(2\pi i r t)\right\}. \tag{3.22}$$

Of course, the last statement also holds trivially for $p = 0$. $\mathcal{F}_0 = \mathcal{F}_0'$ consists only of constant functions, and $u \equiv \hat{u}(0)$ is a solution of (2.1) if and only if $\hat{u}(0)$ satisfies

$$\hat{u}(0) = g\{\lambda \hat{w}(0)\hat{u}(0)\}. \tag{3.23}$$

Similarly, for $p, q \in \mathbf{Z}^d$, $p \neq q \neq 0$, we set

$$(3.24) \qquad w_{pq} = \sum_{r=\pm p, \pm q} \hat{w}(r) \exp\{2\pi irt\} + \sum_{r \notin [(2\mathbf{Z}+1)p+2\mathbf{Z}q] \cup [2\mathbf{Z}p+(2\mathbf{Z}+1)q]} \hat{w}(r) \exp\{2\pi irt\}.$$

If $w = w_{pq}$, i.e., if

$$(3.25) \quad \hat{w}(r) = 0 \quad \text{for all } r \in ([(2\mathbf{Z}+1)p + 2\mathbf{Z}q] \cup [2\mathbf{Z}p + (2\mathbf{Z}+1)q]) \backslash \{\pm p, \pm q\},$$

then any function $u \in \mathcal{F}_{pq}$ is a solution of (2.1) if and only if

$$(3.26) \qquad u(t) = g\left\{\lambda \sum_{r=\pm p, \pm q} \hat{w}(r)\hat{u}(r) \exp(2\pi irt)\right\}.$$

The simple forms of (3.22), (3.23), and (3.26) lead to the following definitions.

DEFINITION. A function $u \in \mathcal{F}'_p$ is called a *p-primary* solution, $p \in \mathbf{Z}^d$, if

$$(3.27) \qquad \hat{u}(p) \neq 0$$

and

$$(3.28) \qquad u(t) = g\{\lambda w_p * u(t)\}$$

with w_p from (3.20). $u \in \mathcal{F}_{pq}$ is called a (p,q)-*secondary* solution if

$$(3.29) \qquad \hat{u}(p) \neq 0, \qquad \hat{u}(q) \neq 0,$$

and

$$u(t) = g\{\lambda w_{pq} * u(t)\}$$

with w_{pq} from (3.24).

Note in particular that p-primary solutions and (p,q)-secondary solutions are, in general, *not* solutions of (2.1) unless $w = w_p, w = w_{pq}$, respectively. p-primary solutions and (p,q)-secondary solutions are nontrivial by definition. Only for $p = 0$, the 0-primary solutions are always the nontrivial constant solutions of (2.1), which are treated in Theorem 1.

For $p \neq 0$ we investigate p-primary solutions in the next section. (p,q)-secondary solutions are studied in §§5 and 7.

4. Primary solutions. For $p \in \mathbf{Z}^d \backslash \{0\}$ we shall study the existence of (nontrivial) p-primary solutions in $\mathcal{F}'_p$, i.e., solutions of (3.28). This means implicitly that we assume $w = w_p$ with w_p from (3.20) or that (3.21) holds. Let us define (assuming for a moment that g is a function on $\mathbf{C}$)

$$(4.1) \qquad \Phi_p(\lambda, z_p, z_{-p}) = \int g\left\{\lambda \sum_{q=\pm p} \hat{w}(q) z_q \exp\{2\pi iqt\}\right\} \exp\{-2\pi ipt\}\, dt,$$

$$(4.2) \qquad \begin{aligned} \phi_p(\lambda, z) &= \mathrm{Re}\Phi_p(\lambda, z, \bar{z}) \\ &= \int g\{\lambda 2\, \mathrm{Re}(\hat{w}(p) z \exp(2\pi ipt))\} \cos(2\pi pt)\, dt, \end{aligned}$$

and its 'formal' symmetric derivative

$$(4.3)\qquad \begin{aligned}\bar{\partial}\phi_p(\lambda, z) &= \tfrac{1}{2}(\bar{\partial}_{z_p}\Phi_p + \bar{\partial}_{z_{-p}}\Phi_{-p})(\lambda, z, \bar{z}) \\ &= \lambda \operatorname{Re}\hat{w}(p) \int \bar{g}'\{\lambda 2 \operatorname{Re}(\hat{w}(p) z \exp(2\pi i p t))\}\, dt.\end{aligned}$$

Note that even if $\hat{w}(p) = \hat{w}(-p) \in \mathbf{R}$ and $\bar{z} = z \in \mathbf{R}$,

$$(4.4)\qquad \begin{aligned}\bar{\partial}\phi_p(\lambda, z) &\neq \frac{\bar{\partial}}{\partial z}\phi_p(\lambda, z) \\ &= \lambda\hat{w}(p) \int \bar{g}'\{\lambda\hat{w}(p) z 2\cos(2\pi p t)\} 2\cos^2(2\pi p t)\, dt.\end{aligned}$$

The following result is generalized in §5.

THEOREM 2. *If* $\operatorname{Im}\hat{w}(p) \neq 0$, *then there do not exist p-primary solutions in* $\mathcal{F}'_p$.

REMARK. We know already from (2.31) that the condition of Theorem 2 is necessary for local bifurcations from zero. The theorem and its generalization in §5 is more interesting as a gobal result. It has nothing to do with our restriction to $\lambda \in \mathbf{R}_+$: If $\operatorname{Im}\hat{w}(p) \neq 0$, then there are no p-primary solutions even for all $\lambda \in \mathbf{R}$. However, the restriction to $\lambda \in \mathbf{R}_+$ makes it necessary to have $\hat{w}(p) > 0$, since we must have $\lambda\hat{w}(p) > 0$ for the existence of p-primary solutions, as can be seen from (3.22).

In this context we want to mention that in [2] the assumptions of Theorems 1.2 and 2.3 have been formulated somewhat sloppily. Instead of simply supposing $\nu \neq 0$, we must demand $\nu > 0$, as in the proof there (see also [2, p. 336]). Thus, we get the following assumptions for the existence of p-primary solutions in $\mathcal{F}'_p$ with $\lambda \in \mathbf{R}_+$.

THEOREM 3. *For* $p \in \mathbf{Z}^d \backslash \{0\}$ *let*

$$(4.5)\qquad 0 < \hat{w}(p) \in \mathbf{R} \text{ and } \hat{w}(r) = 0 \quad \text{for all } r \in (2\mathbf{Z}+1)p\backslash\{+p, -p\}.$$

Define

$$(4.6)\qquad \lambda_p = \begin{cases} 0 & \text{if } g'(0) = +\infty, \\ (g'(0)\hat{w}(p))^{-1} & \text{if } 0 < g'(0) < +\infty. \end{cases}$$

With exceptions for $\lambda = \lambda_p$, *the functions*

$$(4.7)\qquad u_p^s(\lambda, t) = g\{\lambda\hat{w}(p)|\hat{u}_\lambda(p)| 2\cos(2\pi p(t+s))\},$$

$\lambda \in (\lambda_p, +\infty)$, $s \in \mathbf{T}^d$, *are the only p-primary solutions of* (2.1), *where* $|\hat{u}(p)| = |\hat{u}_\lambda(p)| > 0$ *is the unique positive solution of*

$$(4.8)\qquad |\hat{u}(p)| = \int g\{\lambda\hat{w}(p)|\hat{u}(p)| 2\cos(2\pi p t)\}\cos(2\pi p t)\, dt.$$

$\lambda \to |\hat{u}(p)|$ *is continuous and increasing on* $(\lambda_p, +\infty)$ *with*

$$(4.9)\qquad \lim_{\lambda\to\infty} |\hat{u}(p)| = \frac{2\gamma}{\pi}.$$

The p-primary solutions (4.7) *are p-stable*; *we even have*

$$(4.10)\qquad \bar{\partial}\phi_p(\lambda, |\hat{u}(p)|) \in (\tfrac{1}{2}, 1), \qquad \lambda \in (\lambda_p, +\infty),$$

and

$$\lim_{\lambda\to\infty} \bar\partial\phi_p(\lambda, |\hat u(p)|) = \tfrac{1}{2}. \tag{4.11}$$

REMARK. At $\lambda = \lambda_p$ there are (nontrivial) p-primary solutions if and only if g is linear on some interval $[-\alpha, +\alpha]$, with α maximal (see (2.39)). If the latter is the case, all p-primary solutions at $\lambda = \lambda_p$ are of the form

$$u_p^s(\lambda_p, t) = y2\cos(2\pi p(t+s)) \tag{4.12}$$

with $y \in (0, \alpha/2]$, $s \in \mathbf{T}^d$.

PROOF. For $z > 0$ the function $\phi_p(\lambda, \cdot)$ is positive and concave. Excluding the case $\lambda = \lambda_p$, treated in the remark, there exists a unique positive fixed point $|\hat u(p)|$ of $\phi_p(\lambda, \cdot)$ if and only if $\lambda g'(0)\hat w(p) > 1$, or, equivalently, if $\lambda > \lambda_p$. $|\hat u(p)|$ is increasing in λ.

(4.9) is evident. Obviously, (4.7) is a p-primary solution. Conversely, if v is any p-primary solution at λ with $\hat v(p) \neq 0$, then $|\hat v(p)|$ is a positive fixed point of $\phi_p(\lambda, \cdot)$. Excluding $\lambda = \lambda_p$, we must have $\lambda > \lambda_p$ and $|\hat v(p)| = |\hat u(p)|$ such that v has the form (4.7). To prove (4.10), we first show

$$\lambda \hat w(p) \int \bar g'\{\lambda \hat w(p)|\hat u(p)|2\cos(2\pi pt)\}\sin^2(2\pi pt)\,dt = \frac{1}{2}. \tag{4.13}$$

For this purpose we approximate g uniformly by a differentiable function $\tilde g$ with the same properties as g. Assuming without loss of generality that $p_1 \neq 0$, we get by partial integration that

$$\begin{aligned} \lambda\hat w(p) \int_{\mathbf{T}^d} &\tilde g'\{\lambda\hat w(p)|\hat u(p)|2\cos(2\pi pt)\}\sin^2(2\pi pt)\,dt \\ &= -\int_{\mathbf{T}^{d-1}} [\tilde g\{\cdots\}\sin(2\pi pt)/2|\hat u(p)|2\pi p_1]_{t_1=0}^{t_1=1}\, d(t_2,\dots,t_d) \\ &\quad + \frac{1}{2|\hat u(p)|}\int_{\mathbf{T}^d} \tilde g\{\cdots\}\cos(2\pi pt)\,dt. \end{aligned} \tag{4.14}$$

The argument of the $\{\cdots\}$ is always the same as in the first line. Since the second term in (4.14) vanishes and the last term tends to $\frac{1}{2}$ as $\tilde g$ tends to g, (4.13) is proved. Now the concavity of $\phi_p(\lambda, z)$ for $z > 0$ yields, with (4.4),

$$\bar\partial\phi_p(\lambda, |\hat u(p)|) = \frac{1}{2}\left(1 + \frac{\bar\partial}{\partial z}\phi_p(\lambda, |\hat u(p)|)\right) \in \left(\frac{1}{2}, 1\right), \tag{4.15}$$

which is (4.10). For (4.11) we have to show by the last equality that

$$\lim_{\lambda\to\infty} \frac{\bar\partial}{\partial z}\phi_p(\lambda, |\hat u(p)|) = 0. \tag{4.16}$$

Fix $z_0 \in (0, 2\gamma/\pi)$ such that, by (4.9), $|\hat u(p)| > z_0$ for all sufficiently large λ. The concavity of $\phi_p(\lambda, z)$ for $z > 0$ implies

$$\begin{aligned} 0 &\le \frac{\bar\partial}{\partial z}\phi_p(\lambda, |\hat u(p)|) \\ &\le \int [g\{\lambda\hat w(p)|\hat u(p)|2\cos(2\pi pt)\} - g\{\lambda\hat w(p)z_0 2\cos(2\pi pt)\}] \\ &\qquad\qquad \times \cos(2\pi pt)\,dt/(|\hat u(p)| - z_0) \to 0 \quad \text{for } \lambda\to\infty. \end{aligned} \tag{4.17}$$

This shows (4.16), and the proof is complete. $\square$

5. Secondary bifurcations for p, q noncollinear. For the rest of the paper we assume that

(5.1) $$0 < g'(0) < +\infty,$$

and

(5.2) $$g \text{ is strictly concave on } (0, +\infty).$$

This does not allow g to be linear or constant on some interval. In particular, g is strictly increasing. For convenience we suppose, moreover, that g is differentiable on $\mathbf{R}$, though this condition is not really necessary and can be overcome by approximating g suitably (see the proof of Theorem 4 for such an approximation). We study the following problem: In which cases do there exist secondary bifurcating branches of solutions from branches of primary solutions? We restrict this problem to the investigation of (p, q)-secondary solutions. Moreover, we assume in this section that $p, q \in \mathbf{Z}^d$ are noncollinear in the sense that

(5.3) $$\textit{either } p = 0 \neq q \quad \textit{or} \quad (pp)(qq) - (pq)^2 > 0.$$

Secondary bifurcations for collinear $p, q \neq 0$ are studied in §7.

In the following theorems the formulas for $(0, q)$-secondary solutions, $q \neq 0$, and for (p, q)-secondary solutions, $p, q \in \mathbf{Z}^d \backslash \{0\}$ noncollinear, are different. The proofs, however, follow the same lines. If necessary, we use square brackets containing two lines, the first of which corresponds to $p = 0$ and the second to $p \neq 0$; for example,

(5.4) $$\sum_{r=\pm p} \hat{w}(r) \exp\{2\pi i r t\} = \begin{bmatrix} \hat{w}(0) \\ 2\,\mathrm{Re}(\hat{w}(p) \exp\{2\pi i p t\}) \end{bmatrix}.$$

We prove the following generalization of Theorem 2.

THEOREM 4. *Let $p, q \in \mathbf{Z}^d$ satisfy* (5.3), *let* (3.25) *hold and $\hat{w}(p) \neq 0$, $\hat{w}(q) \neq 0$. There exist (p, q)-secondary solutions only if*

(5.5) $$\mathrm{Im}\,\hat{w}(p) = 0, \quad \mathrm{Im}\,\hat{w}(q) = 0, \quad \textit{and} \quad \hat{w}(p) > 0, \ \hat{w}(q) > 0.$$

PROOF. Let v be a (p, q)-secondary solution. By assumption (5.3) we can find $s \in \mathbf{T}^d$ with

(5.6) $$\begin{bmatrix} s \text{ arbitrary} \\ 2\pi s p = -\arg(\hat{w}(p)\hat{v}(p)) \end{bmatrix}, \qquad 2\pi s q = -\arg(\hat{w}(q)\hat{v}(q)).$$

After a rotation of u by s, we have

(5.7) $$v^s(t) = g\left\{\lambda \begin{bmatrix} \hat{w}(0)\hat{v}(0) \\ |\hat{w}(p)\hat{v}(p)| 2\cos(2\pi p t) \end{bmatrix} + \lambda |\hat{w}(q)\hat{v}(q)| 2\cos(2\pi q t)\right\},$$

such that v^s is even and therefore $\widehat{v^s}(p) \in \mathbf{R}\backslash\{0\}$ and $\widehat{v^s}(q) \in \mathbf{R}\backslash\{0\}$. Since v^s is a (p, q)-secondary solution, too, and g is invertible as a strictly increasing function, we find for all $t \in \mathbf{T}^d$ that

$$\begin{bmatrix} \hat{w}(0)\hat{v}^s(0) \\ \hat{v}^s(p) 2\,\mathrm{Re}(\hat{w}(p)\exp(2\pi i p t)) \end{bmatrix} + \hat{v}^s(q) 2\,\mathrm{Re}(\hat{w}(q)\exp(2\pi i q t))$$
$$= \begin{bmatrix} \hat{w}(0)\hat{v}(0) \\ |\hat{w}(p)\hat{v}(p)| 2\cos(2\pi p t) \end{bmatrix} + |\hat{w}(q)\hat{v}(q)| 2\cos(2\pi q t),$$

and therefore $\hat{w}(p) \in \mathbf{R}$, $\hat{w}(q) \in \mathbf{R}$.

We know by Proposition 1(ii) that

$$\left[g\left\{\lambda\hat{w}(p)\hat{v}^s(p)\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}\right\}\right]^\wedge(q)=0$$

and

$$[g\{\lambda\hat{w}(q)\hat{v}^s(q)2\cos(2\pi qt)\}]^\wedge(p)=0.$$

Therefore

$$\begin{aligned}0&\neq\hat{v}^s(p)\\&=\int\Bigg(g\left\{\lambda\hat{w}(p)\hat{v}^s(p)\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}+\lambda\hat{w}(q)\hat{v}^s(q)2\cos(2\pi qt)\right\}\\&\qquad-g\left\{\lambda\hat{w}(q)\hat{v}^s(q)2\cos(2\pi qt)\right\}\Bigg)\cos(2\pi pt)\,dt,\end{aligned}\tag{5.8}$$

$$\begin{aligned}0&\neq\hat{v}^s(q)\\&=\int\Bigg(g\left\{\lambda\hat{w}(p)\hat{v}^s(p)\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}+\lambda\hat{w}(q)\hat{v}^s(q)2\cos(2\pi qt)\right\}\\&\qquad-g\left\{\lambda\hat{w}(p)\hat{v}^s(p)\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}\right\}\Bigg)\cos(2\pi qt)\,dt.\end{aligned}\tag{5.9}$$

But both equations can only hold if $\lambda\hat{w}(p)>0$ and $\lambda\hat{w}(q)>0$. By our restriction to $\lambda>0$ we find the second part of assertion (5.5). □

If we now assume (5.5) in addition to (3.18), we know by Theorems 1 and 3 that on $(\max(\lambda_p,\lambda_q),+\infty)$ we have both p-primary and q-primary solutions. On secondary bifurcations we get the following result, which will be proved at the end of §8. By $|\hat{u}(p)|=|\hat{u}_\lambda(p)|$ we denote the unique positive solution of (4.8), and (2.37), respectively, on $(\lambda_p,+\infty)$.

THEOREM 5. *Let $p,q\in\mathbf{Z}^d$ satisfy* (5.3), *and let* (3.25) *hold with* $\hat{w}(p)>0$, $\hat{w}(q)>0$.

(i) *If $\frac{1}{2}>\hat{w}(q)/\hat{w}(p)>0$, then in $\mathcal{F}_{pq}$ there is no secondary bifurcation on the branch of p-primary solutions or on the branch of q-primary solutions.*

(ii) *If $1>\hat{w}(q)/\hat{w}(p)>\frac{1}{2}$, then in $\mathcal{F}_{pq}$ there is no secondary bifurcation on the p-primary branch, but on the q-primary branch there occurs a secondary bifurcation of a branch of (p,q)-secondary solutions at*

$$\lambda_{qp}=\inf\{\lambda>\lambda_q;\partial\phi_q(\lambda'|\hat{u}_{\lambda'}(q)|)<\hat{w}(q)/\hat{w}(p)\ \textit{for all}\ \lambda'>\lambda\},\tag{5.10}$$

with

$$0<\lambda_p<\lambda_q<\lambda_{qp}<+\infty.\tag{5.11}$$

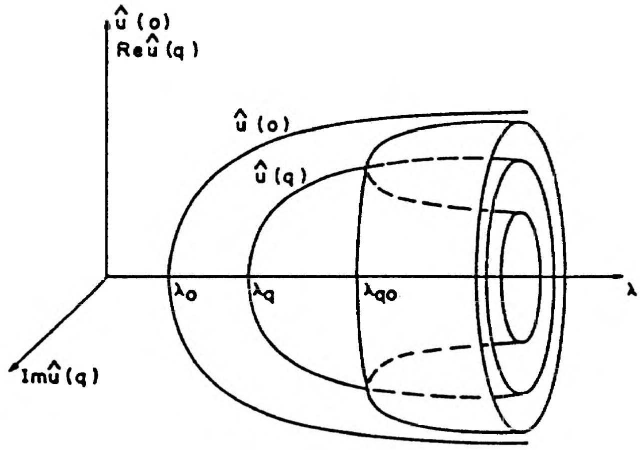

FIGURE 2. Bifurcations for $2\hat{w}(q) > \hat{w}(0) > \hat{w}(q) > 0$. The numerically exact picture is given in Example 3.3 of [**24**]

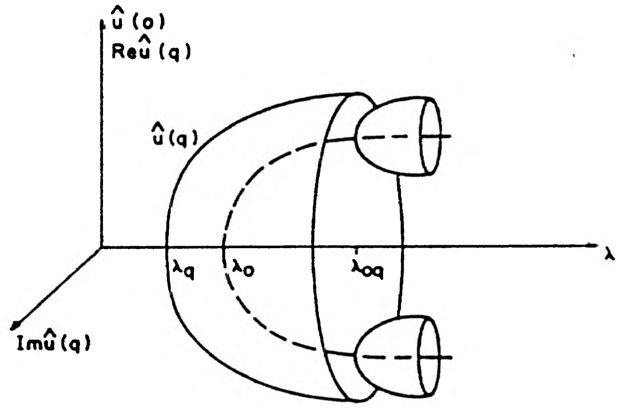

FIGURE 3. Bifurcations for $\hat{w}(q) > \hat{w}(0) > 0$

This branch exists for all $\lambda \in (\lambda_{qp}, +\infty)$ *and consists of* (p, q)*-secondary solutions of the form*

$$u(t) = g\left\{\lambda\hat{w}(p)|\hat{v}(p)|\begin{bmatrix}\pm 1\\ 2\cos(2\pi p(t+s))\end{bmatrix} + \hat{w}(q)|\hat{v}(q)|2\cos(2\pi q(t+s))\right\} \tag{5.12}$$

with $s \in \mathbf{T}^d$ *(recall* $|\hat{v}(p)| \neq 0$ *and* $|\hat{v}(q)| \neq 0$*, by definition).*

(iii) *If* $p = 0$ *and* $1 > \hat{w}(0)/\hat{w}(q) > 0$*, then in* $\mathcal{F}_{0q}$ *there exists no secondary bifurcation on the branch of q-primary solutions, but on the branch of nontrivial constant solutions there occurs a secondary bifurcation of a branch of* $(0, q)$*-secondary solutions at*

$$\lambda_{0q} = \inf\{\lambda, \partial\phi_0(\lambda', \hat{u}_{\lambda'}(0)) < \hat{w}(0)/\hat{w}(q) \text{ for all } \lambda' > \lambda\} \tag{5.13}$$

with

$$(5.14) \qquad 0 < \lambda_q < \lambda_0 < \lambda_{0q} < +\infty.$$

This branch exists for all $\lambda \in (\lambda_{0q}, +\infty)$ *and consists of* $(0, q)$*-secondary solutions of the form* (5.12) *with* $p = 0$.

We want to clarify the bifurcating situation by the following two figures in the case $0 = p \neq q$. In order to take care of the rotational invariance of $\hat{u}(q) \in \mathbf{C}$, we superpose the real axis of $\hat{u}(q)$ on the $\hat{u}(0)$-axis. At the bifurcation points it will be clear from the context in which direction the branch bifurcates. (See Figures 2 and 3.)

REMARKS. (1) Note that in the theorem the case $\hat{w}(q) > \hat{w}(p)$ for $p \neq 0$, i.e., $(pp)(qq) - (pq)^2 > 0$, is covered by (i) and (ii) with p and q exchanged.

(2) By continuity we get from (5.10) and (5.13) the bifurcation conditions

$$(5.15) \qquad \partial\phi_q(\lambda, |\hat{u}(q)|) = \hat{w}(q)/\hat{w}(p) \quad \text{at } \lambda = \lambda_{qp},$$

$$(5.16) \qquad \partial\phi_0(\lambda, |\hat{u}(0)|) = \hat{w}(0)/\hat{w}(q) \quad \text{at } \lambda = \lambda_{0q}.$$

We know from (2.47) that

$$(5.17) \qquad 0 \leq \partial\phi_0(\lambda, |\hat{u}(0)|) < 1 \quad \text{for } \lambda \in (\lambda_0, +\infty),$$

where the upper and lower bounds are approached for $\lambda \searrow \lambda_0$ and $\lambda \to +\infty$, respectively. Similarly, by (4.10),

$$(5.18) \qquad \tfrac{1}{2} < \partial\phi_q(\lambda, |\hat{u}(q)|) < 1$$

for $\lambda \in (\lambda_q, +\infty)$, and again the bounds are approached for $\lambda \searrow \lambda_q$ and $\lambda \to +\infty$. But unfortunately, the functions $\partial\phi_q(\lambda, |\hat{u}(q)|)$ and $\partial\phi_0(\lambda, |\hat{u}(0)|)$ are, in general, not decreasing. Therefore, the sets

$$(5.19) \qquad \Delta_{qp} = \{\lambda > \lambda_q, \partial\phi_q(\lambda, |\hat{u}(q)|) < \hat{w}(q)/\hat{w}(p)\} \supseteq (\lambda_{qp}, +\infty),$$

$$(5.20) \qquad \Delta_{0q} = \{\lambda > \lambda_0, \partial\phi_0(\lambda, |\hat{u}(0)|) < \hat{w}(0)/\hat{w}(q)\} \supseteq (\lambda_{0q}, +\infty)$$

may be composed by several nonconnected intervals. It is now easy to generalize the results of Theorem 5, such that for each $\lambda \in \Delta_{qp}, \lambda \in \Delta_{0q}$, respectively, there are (p, q)-secondary solutions, $(0, q)$-secondary solutions, respectively. Thus, if we have strict inclusions in (5.19) and (5.20), we get secondary bifurcating branches, which again vanish. Schematically, we get the bifurcation picture in Figure 4.

As a common phenomenon (see e.g. [7, Chapter II.11]), the appearance of the secondary bifurcations is followed by an exchange of stability. Here, we note this stability behavior only in terms of definitions (2.25)–(2.26). The results are consequences of more detailed stability investigations in §8.

THEOREM 6. *Let the general assumptions of Theorem* 5 *hold.*

(i) *If* $\frac{1}{2} > \hat{w}(q)/\hat{w}(p) > 0$, *then the p-primary branch is p-stable and q-stable, while the q-primary branch is q-stable but not p-stable.*

(ii) *If* $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$, *then the p-primary branch is again p-stable and q-stable, the q-primary branch is q-stable on* $(\lambda_q, +\infty)$, *but at the bifurcations it*

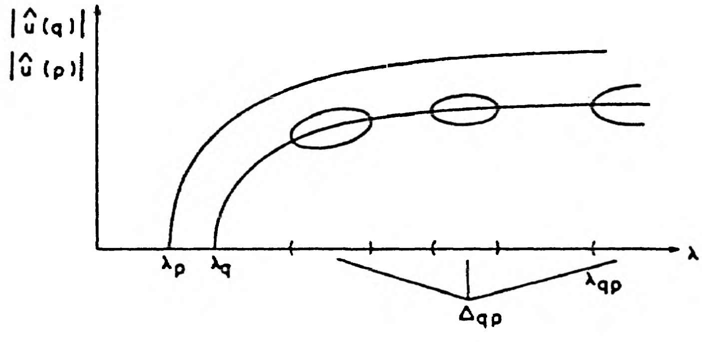

FIGURE 4

changes from a p-unstable or critical solution on $(\lambda_q, +\infty)\backslash\Delta_{qp}$ *to a p-stable solution on* Δ_{qp}.

(iii) *If* $p = 0$ *and* $1 > \hat{w}(0)/\hat{w}(q) > 0$, *then the q-primary branch is q-stable and* 0*-stable, the* 0*-primary branch is* 0*-stable, but it is q-unstable or critical on* $(\lambda_0, +\infty)\backslash\Delta_{0q}$ *and q-stable on* Δ_{0q}.

6. An example: A secondary phase transition of first order. In this section we give an extension of some results from [2]. There, for different interaction potentials J, the equilibrium state in the thermodynamic limit for some mean-field models from statistical mechanics on the circle $\mathbf{T}$ are studied.

It is shown that in the ferromagnetic, but also in the antiferromagnetic case, there exist phase transitions of the equilibrium states. In particular, the phase transition for the antiferromagnetic circle is linked with a breaking of the continuous symmetry group $\mathbf{T}$. In the context of the present paper, the phase transitions of the ferromagnetic and antiferromagnetic circles correspond to the bifurcations of 0-primary, respectively, p-primary, solutions of (2.1) from the trivial solution $u \equiv 0$, which represents the paramagnetic state. The secondary solutions, which we have found in the last section, cannot, however, represent equilibrium states of the corresponding models of statistical mechanics, since the secondary solutions found are not stable. Nevertheless, in an indirect way, the secondary solutions are of physical relevance. Though they do not appear directly, they give rise to a secondary phase transition of first order.

The secondary bifurcation of the secondary solutions is linked with a change of the stability behavior of the primary solutions; in the words of Theorem 6(iii): if $\hat{w}(q) > \hat{w}(0) > 0$, the 0-primary solutions $u_0^{+/-}$ are unstable on $(\lambda_0, +\infty)\backslash\Delta_{0q}$, but are stable for $\lambda \in \Delta_{0q} \supseteq (\lambda_{0q}, +\infty)$, while the p-primary solutions are stable for all $\lambda > \lambda_p$. Therefore, for $\lambda \in \Delta_{0q}$, both primary branches are stable. Now the equilibrium state has to make its choice between these two possible candidates by a variational principle. For λ between λ_q and λ_0 the equilibrium state will certainly be one of the q-primary solutions, since these are the only stable solutions there. By continuity, the equilibrium state will remain a q-primary solution even for values λ, which are little greater than λ_0. But for very large λ it is possible that the newly stable 0-primary solutions win the variational principle. If this is the case, there must be an intermediate value λ^* where the equilibrium state jumps from a

q-primary solution to a 0-primary solution. We have a secondary phase transition of first order.

The following example shows that this phenomenon may really happen. In order to make things as easy as possible and to have a close connection to the representation in [**2**], we restrict ourselves to the case $d = 1$, though the results hold for general dimension d.

At the sites $\alpha/n \in \mathrm{T}$, $\alpha = 1, \dots, n$, there are fixed magnetic spins X_α^n. Without interaction, the X_α^n take independently the values $+1$ and -1 with probability $\frac{1}{2}$; i.e.,

$$\rho_0 = (\delta_{+1} + \delta_{-1})/2. \tag{6.1}$$

We let the interaction potential have the form

$$J(s,t) = w(s-t) = 1 + 2b\cos(2\pi q(s-t)) \tag{6.2}$$

with $q \in \mathbf{N}$, and

$$1 < b < \pi^2/4. \tag{6.3}$$

The Hamiltonian of the interacting system is then given by

$$H_n(X^n) = -\frac{1}{2n} \sum_{\alpha_1,\alpha_2=1}^{n} J\left(\frac{\alpha_1}{n}, \frac{\alpha_2}{n}\right) X_{\alpha_1}^n X_{\alpha_2}^n, \tag{6.4}$$

and the common distribution of $(X_\alpha^n)_{\alpha=1,\dots,n}$ is the Gibbs state to the Hamiltonian H_n:

$$\mathrm{Prob}_{n\beta}(X_\alpha^n \in dx_\alpha,\ \alpha = 1,\dots,n) = \frac{\exp(-\beta H_n(x)) \prod_{\alpha=1}^n \rho(dx_\alpha)}{Z_{n\beta}}, \tag{6.5}$$

where $x = (x_1, \dots, x_n)$ and $Z_{n\beta}$ is the normalizing constant

$$Z_{n\beta} = \int_{\mathbf{R}^n} \exp(-\beta H_n(x)) \prod_{\alpha=1}^n \rho(dx_\alpha). \tag{6.6}$$

In [**2**, Theorems 1.3 and 2.1] it is shown that in the thermodynamic limit the free energy $\psi(\beta)$ is given by the variational principle

$$-\beta\psi(\beta) := \lim_{n\to\infty} n^{-1} \ln Z_{n\beta} = \sup_{f \in \mathcal{H}} [\beta F(f) - I(f)]. \tag{6.7}$$

Here the functionals F and I are defined on $\mathcal{H} = L^2(\mathrm{T})$ by

$$F(f) = \frac{1}{2} \iint_{\mathbf{T}^2} J(s,t) f(s) f(t)\, ds\, dt = \frac{1}{2}\langle f, w * f\rangle, \tag{6.8}$$

and

$$I(f) = \int_{\mathbf{T}} i(f(t))\, dt \tag{6.9}$$

with

$$i(u) = \begin{cases} [(1+u)\ln(1+u) + (1-u)\ln(1-u)]/2 & \text{for } |u| \le 1, \\ +\infty & \text{for } |u| > 1. \end{cases} \tag{6.10}$$

(See formulas (1.16)–(1.22) in [**2**].)

By [2, Theorem 5.1] the supremum in (6.7) is always achieved, and any maximizing function f satisfies the mean field equation

$$i'(f(t)) = \beta(F'f)(t) \quad \text{for almost all } t \in \mathbf{T}. \tag{6.11}$$

In our example we have from (6.8) and (6.10) that

$$\mathrm{th}^{-1}(f(t)) = \beta \cdot w * f(t), \tag{6.12}$$

or, equivalently, (2.1) with $\lambda = \beta$ and $g = (i')^{-1} = \mathrm{th}$ (see also (2.10)).

Next, we make use of Fenchel's duality (see [2, Appendix C]).

$$\sup_{f\in\mathcal{H}}[\beta F(f) - I(f)] = \sup_{f\in\mathcal{H}}[I^*(f) - (\beta F)^*(f)], \tag{6.13}$$

where I^* and $(\beta F)^*$ are the Legendre transforms of I and βF, respectively. In our case we get, by [2, Lemma 3.6 and the remark thereafter],

$$I^*(f) = (\Gamma^*)^*(f) = \Gamma(f) = \int \phi_{\rho_0}(f(t))\,dt, \tag{6.14}$$

where ϕ_{ρ_0} is given in (2.9), and $\phi'_{\rho_0} = g$ (2.10).

On the other hand, we find by easy calculations that

$$\begin{aligned}(\beta F)^*(f) &:= \sup_{h\in\mathcal{H}}\{\langle f, h\rangle - \beta F(h)\} \\ &= \begin{cases} \beta F(f_0) & \text{if } f = \beta w * f_0 \text{ for some } f_0 \in \mathcal{H}, \\ +\infty & \text{otherwise.}\end{cases}\end{aligned} \tag{6.15}$$

Note that (6.15) is well defined, since $f = \beta w * f_1 = \beta w * f_2$ implies

$$\beta F(f_1) = \tfrac{1}{2}\langle f_1, \beta w * f_2\rangle = \tfrac{1}{2}\langle f_2, \beta w * f_2\rangle = \beta F(f_2).$$

Thus, we can rewrite (6.7) as

$$-\beta\psi(\beta) = \lim_{n\to\infty} n^{-1}\ln Z_n = \sup_{f\in\mathcal{H}}[\Gamma(\beta w * f) - \beta F(f)]. \tag{6.16}$$

Now, if the maximum of (6.16) is achieved at f, then f has to satisfy the mean field equation, which is now written in the form

$$\langle g\{\beta w * f\}, \beta w * h\rangle - \langle f, \beta w * h\rangle = 0 \tag{6.17}$$

for all $h \in \mathcal{H}$. Note that by (6.15) we have reduced the variational principle to the space $w * \mathcal{H}$. But, moreover, f must satisfy the second-order condition

$$\langle g'\{\beta w * f\} \cdot \beta w * h, \beta w * h\rangle - \langle h, \beta w * h\rangle \le 0 \tag{6.18}$$

for all $h \in \mathcal{H}$. By the form (6.2) of w, (6.18) implies, in particular (by calculations analogous to (4.18)),

$$\beta\hat{w}(r)\int g'\{\beta w * f(t)\}\,dt \le 1 \tag{6.19}$$

for $r = 0, q$. This is the stability condition (2.25) with $\le$ instead of $<$. For $0 < \beta \le 1/b = \beta_q$, the trivial solution $u \equiv 0$ is the only solution of (6.17) and

$$-\beta\psi(\beta) = 0, \qquad \beta \in (0, 1/b]. \tag{6.20}$$

By (5.16) the bifurcation point β_{0q} for the $(0,q)$-secondary solutions satisfies

$$b\beta_{0q}(1 - \mathrm{th}^2(\beta_{0q}\hat{u}_{\beta_{0q}}(0)) = 1 \tag{6.21}$$

and $\beta_{0q} > \beta_0 = 1 > \beta_q = 1/b$. For $\beta \in (\beta_q, 1)$ the q-primary solutions u_q^s (4.7) are the only stable solutions, and

$$-\beta\psi(\beta) = \beta b|\hat{u}_\beta(q)|^2/2 - \int i(\mathrm{th}(\beta b|\hat{u}_\beta(q)|2\cos(2\pi qt)))\,dt > 0. \tag{6.22}$$

So we have a first phase transition at $\beta_q = 1/b$. The phase transition is of second order, since $\hat{u}_\beta(q) \to 0$ as $\beta \searrow \beta_q$. But for $\beta > \beta_{0q}$ there are at least two different types of stable solutions: the q-primary and the 0-primary solutions. For $\beta \to +\infty$ we find by (2.46) with $\gamma = 1$,

$$\beta F\ (\equiv \hat{u}_\beta(0)) - I\ (\equiv \hat{u}_\beta(0)) = \beta\hat{u}_\beta(0)^2/2 - i(\hat{u}_\beta(0)) \approx \beta/2, \tag{6.23}$$

since i is bounded by $\ln 2$ for $|\hat{u}_\beta(0)| \le 1$, while

$$\beta F(u_q^s(\beta,\cdot)) - I(u_q^s(\beta,\cdot)) = \beta b(\hat{u}_\beta(q))^2/2 - I(u_q^s(\beta,\cdot)) \approx \beta b 2/\pi^2 \tag{6.24}$$

by (4.9). Now (6.3) implies that (6.23) is greater than (6.24) for β large enough. The maximum of (6.16) is not attained any longer on the q-primary solutions. But by Theorem 5 the q-primary solutions do not have bifurcations. Therefore, there exists a $\beta^* \in (1, +\infty)$, where the maximum point jumps from a q-primary solution to another solution of (2.1). We have found a secondary phase transition of first order, as claimed at the beginning of the section. For β large enough the new maximum is attained by a constant nontrivial solution, which corresponds to a ferromagnetic equilibrium state.

7. Secondary bifurcations for collinear p, q. As one may expect, the behavior of secondary bifurcations is different if p and q are collinear; i.e.,

$$n_1 p = n_2 q \neq 0 \quad \text{for some } n_1, n_2 \in \mathbf{Z} \tag{7.1}$$

with $\gcd(n_1, n_2) = 1$. As in (3.9) we set

$$r_0 = p/n_2 = q/n_1 \in \mathbf{Z}^d. \tag{7.2}$$

Of course, we assume that w again satisfies condition (3.25), which by (3.10) can be rewritten as

$$\hat{w}(r) = 0 \quad \text{for all } r \in \begin{cases} \mathbf{Z}r_0\backslash\{\pm p, \pm q\} & \text{if } n_1 \cdot n_2 \text{ even}, \\ (2\mathbf{Z}+1)r_0\backslash\{\pm p, \pm q\} & \text{if } n_1 \cdot n_2 \text{ odd}, \end{cases} \tag{7.3}$$

and that

$$\hat{w}(p) > \hat{w}(q) > 0. \tag{7.4}$$

In the noncollinear case (3.25) implied condition (3.21) for p and for q (instead of p). Therefore, we could consider the p-primary and the q-primary branches in the last section. To guarantee this also in the collinear case, we must assume

$$p \notin (2\mathbf{Z}+1)q \quad \text{and} \quad q \notin (2\mathbf{Z}+1)p. \tag{7.5}$$

Now, we get the following result about secondary bifurcations which is proved in §9. The assumptions about g from the beginning of §5 are still valid.

THEOREM 7. *Let $p,q \in \mathbf{Z}^d$ satisfy* (7.1) *and* (7.5), *and let* (7.3) *and* (7.4) *hold.*

(i) *There are never in $\mathcal{F}_{pq}$ bifurcations from the p-primary solutions.*

(ii) *If $\frac{1}{2} \geq \hat{w}(q)/\hat{w}(p) > 0$ and $p \notin \mathbf{Z}q$, i.e., $n_1 \neq 1$, then the branch of q-primary solutions does not have a secondary bifurcation in $\mathcal{F}_{pq}$.*

(iii) *If $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$ and $p \notin \mathbf{Z}q$, then*

$$(\lambda_q, +\infty) \supseteq \Delta_{qp} \supseteq (\lambda_{qp}, +\infty) \neq \emptyset \tag{7.6}$$

with Δ_{qp} and λ_{qp} from (5.19) *and* (5.10), *respectively.*

For $\lambda \in \Delta_{qp}$ there are the following branches of (p,q)-secondary solutions, which bifurcate from the q-primary solutions:

$$\begin{aligned} v_1(t) = g\{&\lambda\hat{w}(p)|\hat{v}_1(p)|2\sin(2\pi(pt + (\tau_1 + j_1)/n_1 + l_1/2)) \\ &+ \lambda\hat{w}(q)|\hat{v}_1(q)|2\cos(2\pi(qt + (\tau_1 + k_1)/n_2 + m_1/2))\} \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} v_2(t) = g\{&\lambda\hat{w}(p)|\hat{v}_2(p)|2\cos(2\pi(pt + (\tau_2 + j_2)/n_1 + l_2/2)) \\ &+ \lambda\hat{w}(q)|\hat{v}_2(q)|2\sin(2\pi(qt + (\tau_2 + k_2)/n_2 + m_2/2))\}, \end{aligned} \tag{7.8}$$

$t \in \mathbf{T}^d$, with the parameters $\tau_1, \tau_2 \in \mathbf{T}$; $j_1, j_2 \in \{0, \dots, n_1 - 1\}$; $k_1, k_2 \in \{0, \dots, n_2 - 1\}$; $l_1, l_2, m_1, m_2 \in \{0, 1\}$.

(iv) *If $p \in 2q\mathbf{Z}$, i.e., $n_1 = 1$ and n_2 even, then there exists always a secondary bifurcation in $\mathcal{F}_{pq}$. It takes place at*

$$\begin{aligned} \lambda_{qp}^1 = \inf\Big\{\lambda > \lambda_q, \lambda'\hat{w}(q)\int g'\{\lambda'\hat{w}(q)|\hat{u}(q)|2\cos(2\pi s)\} \\ \times(1 - \cos(4\pi n_2 s))\, ds < \hat{w}(q)/\hat{w}(p) \text{ for all } \lambda' > \lambda\Big\} \end{aligned} \tag{7.9}$$

with

$$\lambda_q \leq \lambda_{qp}^1 < +\infty. \tag{7.10}$$

For $\lambda \in (\lambda_{qp}^1, +\infty)$, we have branches of (p,q)-secondary solutions of the form v_1 from (7.7) *(with $n_1 = 1$).*

REMARKS. (1) Mutans mutandum, remark (2) after Theorem 5 also holds here: On some intervals there may be bifurcating branches of the forms v^1 or v^2 described above, which appear, disappear, and reappear according to the conditions appearing in (5.19) and (7.9), respectively. We define

$$\begin{aligned} \Delta_{qp}^1 = \Big\{\lambda > \lambda_q, \lambda\hat{w}(q)\int g'\{\lambda\hat{w}(q)|\hat{u}(q)|2\cos(2\pi n_1 s)\} \\ \times\,(1 - \cos(4\pi n_2 s))\, ds < \hat{w}(q)/\hat{w}(p)\Big\}, \end{aligned} \tag{7.11}$$

$$\begin{aligned} \Delta_{qp}^2 = \Big\{\lambda > \lambda_q, \lambda\hat{w}(q)\int g'\{\lambda\hat{w}(q)|\hat{u}(q)|2\sin(2\pi n_1 s)\} \\ \times(1 + \cos(4\pi n_2 s))\, ds < \hat{w}(q)/\hat{w}(p)\Big\}. \end{aligned} \tag{7.12}$$

If $n_2 \notin n_1\mathbf{Z}$, then by Proposition 1(i), we can cancel the last cosine term in (7.11) and (7.12) and get

$$\Delta_{qp} = \Delta^1_{qp} = \Delta^2_{qp}. \tag{7.13}$$

So we may have $\Delta^1_{qp} \supseteq (\lambda^1_{qp}, +\infty) \neq \emptyset$ with strict inclusion. In the case $p \in 2q\mathbf{Z}$, i.e., $n_1 = 1$ and n_2 even, Δ^2_{qp} is always a bounded, possibly empty region in $\mathbf{R}^+$. If $\lambda \in \Delta^2_{qp} \neq \emptyset$, we have secondary bifurcating solutions of the form v_2 from (7.8) with $n_1 = 1$. However, these solutions disappear again as $\lambda \to +\infty$.

(2) In (7.7) and (7.8) let us disregard the rotation group $\tau \in \mathbf{T}$ for a moment; i.e., put $\tau = 0$. Then since g is invertible, v^1 and v^2 represent $8 \cdot n_1 \cdot n_2$ different secondary solutions if $n_1 \cdot n_2$ is odd. If $n_1 \cdot n_2$ is even, let n_1 be even, for example; then the parameters $i_1 = n_1/2$, $k_1 = 1$, and $i_1 = k_1 = 0$ give the same solution. Similarly for i_2 and k_2. Therefore, we have only $4 \cdot n_1 \cdot n_2$ different secondary solutions if $n_1 \cdot n_2$ is even. This fact corresponds to result (3.10).

(3) We refer to the end of §9 for some considerations concerning the stability of the solutions in the collinear case.

8. The associated dynamical system for noncollinear p, q. We continue to let g satisfy the additional conditions from the beginning of §5, let p and q be noncollinear in the sense of (5.3), and let (3.25) hold with $\hat{w}(p) > 0$, $\hat{w}(q) > 0$. To prove the results from Theorems 5 and 6, we need a good knowledge of the fixed point problem for $z = (z_1, z_2) \in \mathbf{R}^2$:

$$z = \phi(z) = (\phi_1(z_1, z_2), \phi_2(z_1, z_2)), \tag{8.1}$$

where

$$\begin{aligned} \phi_1(x, y) = \int g \Bigg\{ \lambda \hat{w}(p) x \left[\begin{matrix} 1 \\ 2\cos(2\pi p t) \end{matrix} \right] \\ + \lambda \hat{w}(q) y 2 \cos(2\pi q t) \Bigg\} \cos(2\pi p t)\, dt, \end{aligned} \tag{8.2}$$

$$\phi_2(x, y) = \int g\{\cdots\} \cos(2\pi q t)\, dt, \tag{8.3}$$

where we have in $\{\cdots\}$ the same argument as in (8.2). Of course, all fixed points of ϕ are contained in

$$\Omega = \bigcap_{n \geq 0} \overline{\phi^n(\mathbf{R}^2)}, \tag{8.4}$$

where $\phi^n = \phi \circ \cdots \circ \phi$ (n times). We shall see that here Ω is exactly the set of all fixed points: There are no periodic orbits or more complicated variant limit sets. It turns out that $\overline{\phi(\mathbf{R}^2)}$ is a very nice compact convex set, independent of $\lambda, \hat{w}(p)$ and $\hat{w}(q)$.

THEOREM 8.

$$\overline{\phi(\mathbf{R}^2)} = \begin{cases} \left\{(\gamma x, \gamma y);\ |y| \le \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right),\ |x| \le 1\right\} & \text{for } p = 0 \ne q, \\ \Big\{(\gamma x_1, \gamma x_2);\ |x_i| = \frac{8}{\pi}\int_0^{1/4} \frac{\mu \sin^2(2\pi r)}{\sqrt{1-\mu^2\cos^2(2\pi r)}}\,dr, \\ \qquad |x_{3-i}| \le \frac{8}{\pi}\int_0^{1/4}\sqrt{1-\mu^2\cos^2(2\pi r)}\,dr, \\ \qquad 0 \le \mu \le 1,\ i = 1,2\Big\} \\ & \text{for } (pp)(qq) - (pq)^2 > 0. \end{cases} \tag{8.5}$$

PROOF. First assume $p = 0 \ne q$. For $0 \ne |x| \le 1$ we set $m(x) = 1/\sin(\pi x/2)$ and check that

$$\operatorname{sign}\{1 + m(x)\cos(2\pi r)\} = \operatorname{sign}(x)\cdot(2\cdot 1_{[0,(1+x)/4]}(|r|) - 1) \tag{8.6}$$

for $|r| \le \frac{1}{2}$ and $4|r| \ne 1 + x$, $x \ne 0$. Then

$$\begin{aligned} \lim_{\alpha\to\infty} \phi\left(\frac{\alpha x}{\hat{w}(0)\lambda}, \pm\frac{\alpha x m(x)}{2\lambda\hat{w}(q)}\right) &= \gamma\left(\int_{\mathbf{T}^d} \operatorname{sign}\{x(1 \pm m(x)\cos(2\pi q t))\}\,dt,\right. \\ &\qquad \left.\int_{\mathbf{T}^d} \operatorname{sign}\{x(1 \pm m(x)\cos(2\pi q t))\}\cos(2\pi q t)\,dt\right) \\ &= \gamma\cdot\operatorname{sign}(x)\left(\int_{-1/2}^{+1/2} \operatorname{sign}\{1 + m(x)\cos(2\pi r)\}\,dr,\right. \\ &\qquad \left.\pm\int_{-1/2}^{+1/2} \operatorname{sign}\{1 + m(x)\cos(2\pi r)\}\cos(2\pi r)\,dr\right) \\ &= \gamma\cdot\left(x, \pm\frac{2}{\pi}\sin\left(\frac{2\pi(1+x)}{4}\right)\right) \\ &= \left(\gamma x, \pm\frac{2\gamma}{\pi}\cos\left(\frac{\pi x}{2}\right)\right). \end{aligned} \tag{8.7}$$

Since $\overline{\phi(\mathbf{R}^2)}$ is simply connected, we have

$$\overline{\phi(\mathbf{R}^2)} \supseteq \{(\gamma x, \gamma y),\ |y| \le (2/\pi)\cos(\pi x/2)\}.$$

If we had strict inclusion in the last line, there would exist $(x_0, y_0) \in \mathbf{R}^2$ with

$$\phi(x_0, y_0) = (\gamma\phi_1, \gamma\phi_2)$$

and

$$\phi_2 = (2/\pi)\cos(\pi\phi_1/2).$$

The curve $y = (2/\pi)\cos(\pi x/2)$ has in (ϕ_1, ϕ_2) the outer normal direction $\bar{n} = (\sin(\pi\phi_1/2), 1)$. For $\alpha \in \mathbf{R}^+$ set

$$h(\alpha) = \bar{n}\cdot\phi(x_0 + (\alpha/\lambda\hat{w}(0))\sin(\pi\phi_1/2), y_0 + \alpha/2\lambda\hat{w}(q)).$$

Then, for all $\alpha \in \mathbf{R}^+$,

$$\frac{d}{d\alpha}h(\alpha) = \int g'\{\lambda\hat{w}(0)x_0 + \alpha\sin(\pi\phi_1/2) + (\lambda 2\hat{w}(q)y_0 + \alpha)\cos(2\pi qt)\} \times(\sin(\pi\phi_1/2) + \cos(2\pi qt))^2\, dt > 0.$$

But, as in (8.7),

$$\begin{aligned}\lim_{\alpha\to\infty} h(\alpha) &= \gamma\cdot\bar{n}\cdot\left(\int \operatorname{sign}\{\sin(\pi\phi_1/2) + \cos(2\pi qt)\}\, dt,\right.\\ &\qquad\left.\int \operatorname{sign}\{\sin(\pi\phi_1/2) + \cos(2\pi qt)\}\cos(2\pi qt)\, dt\right)\\ &= \gamma\cdot\bar{n}\cdot(\phi_1, (2/\pi)\cos(\pi\phi_1/2)) = h(0)\end{aligned}$$

gives a contradiction.

For $(pp)(qq) = (pq)^2 > 0$ and $0 \le \mu \le 1$, we calculate

$$\lim_{\alpha\to\infty} \phi(\alpha\mu/2\lambda\hat{w}(p), \pm\alpha/2\lambda\hat{w}(q)) = \gamma\begin{pmatrix}\int \operatorname{sign}\{\mu\cos(2\pi pt) \pm \cos(2\pi qt)\}\cos(2\pi pt)\, dt\\ \int \operatorname{sign}\{\mu\cos(2\pi pt) \pm \cos(2\pi qt)\}\cos(2\pi qt)\, dt\end{pmatrix}. \tag{8.8}$$

Set $B_{qp} = \{(r,s),\ r = tp,\ s = tq,\ t \in \mathbf{T}^d\}$, and the last expression equals

$$\begin{aligned}&2\gamma/|B_{pq}|\begin{pmatrix}\int_{B_{pq}} 1_{\{\mu\cos(2\pi r)\pm\cos(2\pi s)>0\}}\cos(2\pi r)\, dr\, ds\\ \int_{B_{pq}} 1_{\{\mu\cos(2\pi r)\pm\cos(2\pi s)>0\}}\cos(2\pi s)\, dr\, ds\end{pmatrix} \qquad (8.9)\\
&= 2\gamma\begin{pmatrix}\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2} 1_{\{|s|\lessgtr\arccos(\mp\mu\cos(2\pi r))/2\pi\}}\cos(2\pi r)\, dr\, ds\\ \int_{-1/2}^{+1/2}\int_{-1/2}^{-1/2} 1_{\{|s|\lessgtr\arccos(\mp\mu\cos(2\pi r))/2\pi\}}\cos(2\pi s)\, dr\, ds\end{pmatrix}\\
&= \gamma 2/\pi\begin{pmatrix}\int_{-1/2}^{+1/2} \pm\arccos(\mp\mu\cos(2\pi r))\cos(2\pi r)\, dr\\ \int_{-1/2}^{+1/2} \pm\sin\arccos(\mp\mu\cos(2\pi r))\, dr\end{pmatrix}\\
&= \gamma 4/\pi\begin{pmatrix}\int_0^{1/4} (\pi - 2\arccos(\mu\cos(2\pi r))\cos(2\pi r))\, dr\\ \int_0^{1/4} \pm 2\sin\arccos(\mu\cos(2\pi r))\, dr\end{pmatrix}\\
&= \gamma 8/\pi\begin{pmatrix}\int_0^{1/4} \frac{\mu\sin^2(2\pi r)}{\sqrt{1-\mu^2\cos^2(2\pi r)}}\, dr\\ \pm\int_0^{1/4} \sqrt{1-\mu^2\cos^2(2\pi r)}\, dr\end{pmatrix}\end{aligned}$$

by partial integration. For $\alpha \to -\infty$ or $\mu = 1/\mu' \in [1, +\infty)$ we get the other boundary points of the right side of (8.5). Thus $\overline{\phi(\mathrm{R}^2)}$ contains the right side of (8.5). The converse inclusion is shown by a similar argument as in the first part of the proof. □

REMARK. Thanks to Theorem 8, it is sufficient to regard ϕ as acting on the universal set $\overline{\phi(\mathrm{R}^2)}$, which is independent of λ, $\hat{w}(p)$, and $\hat{w}(q)$. Moreover, the geometric form of $\overline{\phi(\mathrm{R}^2)}$ can be used to analyse the behavior of the equilibrium states and their phase transitions. In particular, the ground states can be nicely discussed with the help of the set $\overline{\phi(\mathrm{R}^2)}$. See also [10, §§VII and VIII], where the ground states of a spin-glass model are studied in detail.

The essential step for determining the fixed points of ϕ is to analyse the fixed points of the components ϕ_1 and ϕ_2 separately. We note the following easy facts:

(i) (x, y) is a fixed point of ϕ if and only if x is a fixed point of $\phi_1(\cdot, y)$ and y is a fixed point of $\phi_2(x, \cdot)$.

(ii) 0 is a fixed point for $\phi_1(\cdot, y)$ and $\phi_2(x, \cdot)$ for all $y \in \mathrm{R}$, $x \in \mathrm{R}$, respectively.

(iii) If (x, y) is a stable fixed point of ϕ, then so is x for $\phi_1(\cdot, y)$ and y for $\phi_2(x, \cdot)$.

However, the converse of the last statement is not true, as we shall see later.

In contrast to g, the functions $\phi_1(\cdot, y)$ and $\phi_2(x, \cdot)$ need not be concave on $(0, \infty)$, in general. Instead of concavity, we use the following result.

LEMMA. *Under the conditions from the beginning of this section, $\phi_1(x, y)$ is strictly increasing and odd in x but even in y. For $x \in \mathrm{R}\backslash\{0\}$ fixed, $|\phi_1(x, \cdot)|$ is strictly decreasing in $|y|$ with*

$$\lim_{|y| \to \infty} |\phi_1(x, y)| = 0. \tag{8.10}$$

The same assertions hold for ϕ_2 with x and y exchanged.

PROOF. Similarly to the first equations in (8.8)–(8.9), we get

$$\phi_1(x, y) = \int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} g\left\{\lambda \hat{w}(p) x \begin{bmatrix} 1 \\ 2\cos(2\pi r) \end{bmatrix} + \lambda \hat{w}(q) y 2 \cos(2\pi s)\right\} \times \begin{bmatrix} 1 \\ \cos(2\pi r) \end{bmatrix} dr\, ds, \tag{8.11}$$

$$\phi_2(x, y) = \int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} g\{\cdots\} \cos(2\pi s)\, dr\, ds, \tag{8.12}$$

again repeating the argument in $\{\cdots\}$ from (8.11). The first assertions are then easily verified by the properties of g. For the second assertion we can assume

$x > 0,\ y > 0$. Then

(8.13)

$$\begin{aligned}
\partial_y \phi_1(x,y) &= 2\hat{w}(q)\begin{bmatrix} 1/\hat{w}(0) \\ 1/2\hat{w}(p)\end{bmatrix} \partial_x \phi_2(x,y) \\
&= 2\lambda\hat{w}(q)\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2} g'\left\{\lambda\hat{w}(p)x\begin{bmatrix} 1 \\ 2\cos(2\pi r)\end{bmatrix} + \lambda\hat{w}(q)y2\cos(2\pi s)\right\} \\
&\qquad\qquad \times \begin{bmatrix} 1 \\ \cos(2\pi r)\end{bmatrix}\cos(2\pi s)\,dr\,ds \\
&= \lambda\hat{w}(q)\int_{-1/2}^{+1/2}\int_{-1/2}^{+1/2}\left(g'\left\{\lambda\hat{w}(p)x\begin{bmatrix} 1 \\ 2\cos(2\pi r)\end{bmatrix} + \lambda\hat{w}(q)y2\cos(2\pi s)\right\}\right. \\
&\qquad\qquad \left. - g'\left\{\lambda\hat{w}(p)x\begin{bmatrix} 1 \\ 2\cos(2\pi r)\end{bmatrix} - \lambda\hat{w}(q)y2\cos(2\pi s)\right\}\right) \\
&\qquad\qquad \times \begin{bmatrix} 1 \\ \cos(2\pi r)\end{bmatrix}\cos(2\pi s)\,dr\,ds \\
&< 0,
\end{aligned}$$

since the integrand is a.e. negative, as seen by cases.

Finally, $\phi_1(x,y) \to_{|y|\to\infty} 0$ and $\phi_2(x,y) \to_{|x|\to\infty} 0$ follow from (8.11) and (8.12). □

By the lemma, we can now describe the fixed points of ϕ_1 and ϕ_2 separately. Recall the definition of λ_p, λ_q from (2.36) or (4.6) and of $|\hat{u}(p)|, |\hat{u}(q)|$ from (2.37) or (4.8).

THEOREM 9. *For $\lambda \in (0, \lambda_p]$, $x = 0$ is the only fixed point of $\phi_1(\cdot, y)$ for all y. For $\lambda \in (\lambda_p, +\infty)$ there exists a unique, positive, symmetric, continuously differentiable function*

$$\psi_1 \colon (-|\hat{u}(p)|, +|\hat{u}(p)|) \to (0, \infty)$$

with

$$\phi_1(x, \psi_1(x)) = \phi_1(x, -\psi_1(x)) = x \tag{8.14}$$

for all $x \in (-|\hat{u}(p)|, +|\hat{u}(p)|)$. We have

$$\lim_{|x|\to|\hat{u}(p)|} \psi_1(x) = 0. \tag{8.15}$$

$\psi_1(0) > 0$ is uniquely determined by

$$\lambda\hat{w}(p)\int g'\{\lambda\hat{w}(q)\psi_1(0)2\cos(2\pi qt)\}\,dt = 1. \tag{8.16}$$

Similar assertions hold for $\phi_2(x, \cdot)$ by a function

$$\psi_2 \colon (-|\hat{u}(q)|, +|\hat{u}(q)|) \to (0, \infty),$$

where p and q, x and y are exchanged everywhere. Here, $\psi_2(0)$ is uniquely determined by

$$\lambda\hat{w}(q)\int g'\left\{\lambda\hat{w}(p)\psi_2(0)\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}\right\}dt = 1. \tag{8.17}$$

REMARK. We want to point out that the strict concavity of g is essential for Theorem 9. If g were linear on some intervals, then as in the remarks following Theorems 1 and 3, we would not have uniqueness for the values of ψ_1 satisfying (8.14). The points of a whole interval would then satisfy these equations, and equally the set of fixed points of ϕ in $(0,\infty)^2$ could then have a two-dimensional subset.

PROOF. For $\lambda \in (0,\lambda_p]$ and $x > 0$, we get by the lemma and the definition of λ_p that

$$0 < \phi_1(x,y) \leq \phi_1(x,0) = \int g\left\{\lambda\hat{w}(p)x\begin{bmatrix}1\\2\cos(2\pi pt)\end{bmatrix}\right\}\cos(2\pi pt)\,dt < x \tag{8.18}$$

for all $y \in \mathbf{R}$. ϕ_1 being odd in x, $x = 0$ is thus the only fixed point of $\phi_1(\cdot,y)$ for all y.

Now let $\lambda \in (\lambda_p,+\infty)$. $\phi_1(x,0) = \phi_p(\lambda,x)$ from (2.44) or (4.2) is strictly concave in $x > 0$ with $|\hat{u}_\lambda(p)|$ as the unique positive fixed point. Thus

$$\begin{aligned} |\phi_1(x,0)| &> |x| \quad \text{for } |x| \in (0,|\hat{u}(p)|),\\ |\phi_1(x,0)| &< |x| \quad \text{for } |x| \in (|\hat{u}(p)|,+\infty). \end{aligned} \tag{8.19}$$

Hence, by the second assertion of the lemma, exactly for $|x| \in (0,|\hat{u}(p)|)$, there exists a unique $y = \psi_1(x) > 0$ with

$$\phi_1(x,y) = \phi_1(x,-y) = x. \tag{8.20}$$

Since ϕ_1 is odd in x, ψ_1 is symmetric in x. By the implicit function theorem ψ_1 is continuously differentiable and

$$\frac{d}{dx}\psi_1(x) = \frac{1-\partial_x\phi_1(x,\psi_1(x))}{\partial_y\phi_1(x,\psi_1(x))}, \tag{8.21}$$

where $\partial_x\phi_1$ (resp. $\partial_y\phi_1$) denote the partial derivatives of ϕ_1 with respect to x (resp. y). Next, we show that ψ_1 is bounded on $(0,|\hat{u}(p)|)$. Set

$$\tilde{g}'(v) = \sup\{g'\{v + 2\lambda\hat{w}(p)x\};\ 0 \leq |x| \leq |\hat{u}(p)|\}. \tag{8.22}$$

Since $\int \tilde{g}'\{\lambda\hat{w}(q)y2\cos(2\pi qt)\}\,dt \to_{|y|\to\infty} 0$, we find $y_0 > 0$ with

$$\lambda\hat{w}(p)\int \tilde{g}'\{\lambda\hat{w}(q)y_0 2\cos(2\pi qt)\}\begin{bmatrix}1\\2\cos^2(2\pi pt)\end{bmatrix}dt < 1. \tag{8.23}$$

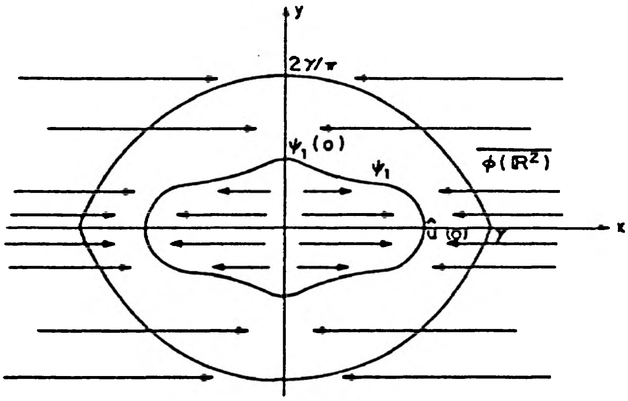

FIGURE 5. The function ψ_1 and the action of $\phi_1(\cdot, y)$, y fixed, $p = 0 \neq q$.

Therefore, by the mean value theorem

$$\begin{aligned}
(8.24)\qquad \phi_1(x, y_0) &= \int g\left\{\lambda\hat{w}(p)x\begin{bmatrix}1\\ 2\cos(2\pi pt)\end{bmatrix}\right.\\
&\qquad\qquad \left. + \lambda\hat{w}(q)y_0 2\cos(2\pi qt)\right\}\cos(2\pi pt)\,dt\\
&\leq \lambda\hat{w}(p)|x|\int \tilde{g}'\{\lambda\hat{w}(q)y_0 2\cos(2\pi qt)\}\begin{bmatrix}1\\ 2\cos^2(2\pi pt)\end{bmatrix}dt\\
&< |x|
\end{aligned}$$

for all $|x| \leq |\hat{u}(p)|$, and

$$(8.25)\qquad \psi_1(x) < y_0 \quad \text{for all } |x| < |\hat{u}(p)|.$$

Since by definition of $|\hat{u}(p)|$, $\phi_1(\pm|\hat{u}(p)|, 0) = \pm|\hat{u}(p)|$, the boundedness of ψ_1 implies (8.15). Since g' is strictly decreasing on $\mathbf{R}^+$, so is $\partial_x\phi_1(0, \cdot)$. If $\partial_x\phi_1(x, y) \lesssim 1$ for $(x, y) = (0, y_0)$, this inequality also holds in a neighborhood U of $(0, y_0)$ in $(0, \infty)^2$. Since $\phi_1(0, y) = 0$, we find $\phi_1(x, y) \lesssim x$ for all $(x, y) \in U$, and $(0, y_0)$ cannot be an accumulation point of $(x, \psi_1(x))$. Hence, $\psi_1(0)$ is uniquely determined by

$$(8.26)\qquad \partial_x\phi_1(0, \psi_1(0)) = 1,$$

which is equivalent to (8.16), since p and q are noncollinear. The analogue of (8.21) for ψ_2 is

$$(8.27)\qquad \frac{d}{dy}\psi_2(y) = \frac{1 - \partial_y\phi_2(\psi_2(y), y)}{\partial_x\phi_2(\psi_2(y), y)}. \quad \square$$

For $p = 0 \neq q$ we get the picture of ψ_1 shown in Figure 5. The arrows indicate the action of $\phi_1(\cdot, y)$ for y fixed.

The proof of Theorem 9 and fact (ii) preceding the lemma give the following complete description of the fixed points of ϕ.

THEOREM 10.

$$F = [\{0\} \times \mathbf{R} \cup \{(x, \pm\psi_1(x)),\ |x| \le |\hat{u}(p)|\}] \cap [\mathbf{R} \times \{0\} \cup \{(\pm\psi_2(y), y),\ |y| \le |\hat{u}(q)|\}] \tag{8.28}$$

is the set of all fixed points of ϕ.

To formulate the following relations between $\psi_1(0)$ and $|\hat{u}(q)|$ and between $\psi_2(0)$ and $|\hat{u}(p)|$, we recall the dependence on λ of ψ_1 and ψ_2, though not made explicit, and the definition of Δ_{qp} and Δ_{0q} in (5.19)–(5.20). For $\lambda \le \lambda_p$ we set, for convenience, $|\hat{u}(p)| = 0$ and $\psi_1 = 0$, and similarly $|\hat{u}(q)| = 0$ and $\psi_2 = 0$ for $\lambda \le \lambda_q$.

THEOREM 11. (i) *If* $\frac{1}{2} > \hat{w}(q)/\hat{w}(p) > 0$, *then*

$$\left.\begin{aligned} |\hat{u}(p)| &> \psi_2(0) \quad (8.29)\\ \psi_1(0) &> |\hat{u}(q)| \quad (8.30) \end{aligned}\right\} \quad \textit{for } \lambda \in (\lambda_p, +\infty).$$

(ii) *If* $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$, *then*

$$|\hat{u}(p)| > \psi_2(0) \quad \textit{for } \lambda \in (\lambda_p, +\infty), \tag{8.31}$$

$$\psi_1(0) > |\hat{u}(q)| \quad \textit{for } \lambda \in (\lambda_p, \lambda_q), \tag{8.32}$$

but

$$\psi_1(0) < |\hat{u}(q)| \ \textit{exactly for } \lambda \in \Delta_{qp} \supseteq (\lambda_{qp}, +\infty). \tag{8.33}$$

(iii) *If* $p = 0$ *and* $1 > \hat{w}(0)/\hat{w}(q) > 0$, *then*

$$\psi_1(0) < |\hat{u}(q)| \quad \textit{for } \lambda \in (\lambda_q, +\infty), \tag{8.34}$$

$$|\hat{u}(0)| < \psi_2(0) \quad \textit{for } \lambda \in (\lambda_q, \lambda_0), \tag{8.35}$$

but

$$|\hat{u}(0)| > \psi_2(0) \ \textit{exactly for } \lambda \in \Delta_{0q} \supseteq (\lambda_{0q}, +\infty). \tag{8.36}$$

PROOF. First, remark that (8.32) holds trivially since $\psi_1(0) > 0 = |\hat{u}(q)|$ for $\lambda \in (\lambda_p, \lambda_q)$. Similarly for (8.35). By (2.47) or (4.10) we find

$$\lambda\hat{w}(q) \int g'\left\{\lambda\hat{w}(p)|\hat{u}(p)| \begin{bmatrix} 1 \\ 2\cos(2\pi pt) \end{bmatrix}\right\} dt \in \hat{w}(q)/\hat{w}(p) \cdot (0, 1) \tag{8.37}$$

for $\lambda > \lambda_p$, and

$$\lambda\hat{w}(p) \int g'\{\lambda\hat{w}(q)|\hat{u}(q)| 2\cos(2\pi qt)\}\, dt \in \hat{w}(p)/\hat{w}(q) \cdot (\tfrac{1}{2}, 1) \tag{8.38}$$

for $\lambda > \lambda_p$. Thus, for $\lambda > \lambda_p$, (8.17) implies $|\hat{u}_\lambda(p)| > \psi_2(0)$ if $\hat{w}(q)/\hat{w}(p) < 1$, i.e., (8.29) and (8.31); and for $\lambda > \lambda_q$, (8.16) implies $\psi_1(0) > |\hat{u}(q)|$ if $\hat{w}(p)/2\hat{w}(q) > 1$, i.e., (8.30), or, for $p = 0$, $\psi_1(0) < |\hat{u}(q)|$ for $\hat{w}(0)/\hat{w}(q) < 1$, i.e., (8.34). But if $1 \in \hat{w}(p)/\hat{w}(q) \cdot (\frac{1}{2}, 1)$, then by definition (5.19),

$$\lambda\hat{w}(p) \int g'\{\lambda\hat{w}(q)|\hat{u}(q)| 2\cos(2\pi qt)\} < 1 \tag{8.39}$$

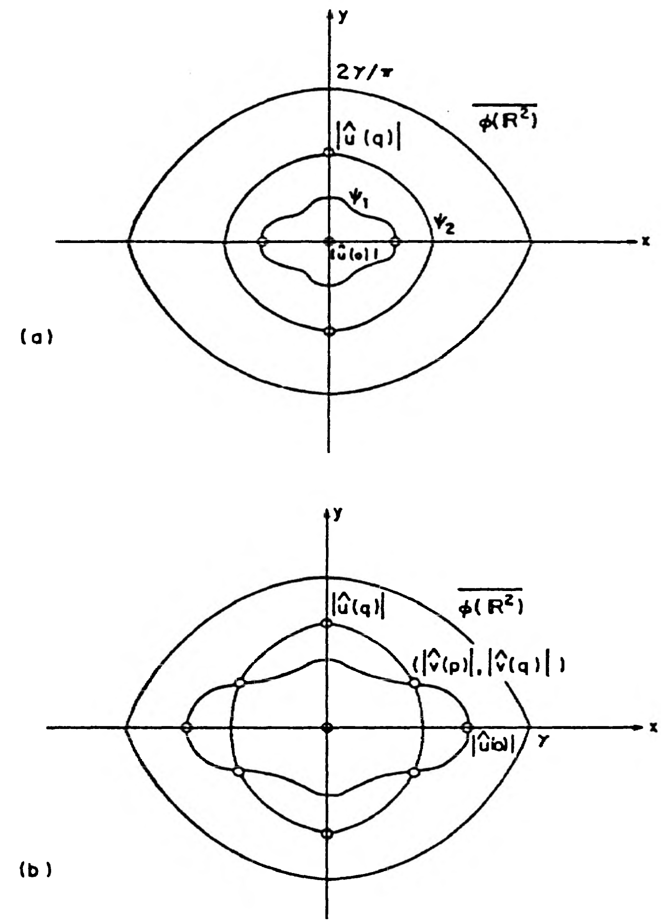

FIGURE 6. The functions $\psi_1(x)$ and $\psi_2(y)$ for $p = 0 \neq q$, $\hat{u}(q) > \hat{w}(q) > \hat{w}(0) > 0$, and $\lambda \in (\lambda_0, \lambda_{0_q}, \lambda_{0_q})$ in (a) (resp. $\lambda > \lambda_{0_q}$ in (b)). The fixed points of ϕ are noted by a small circle.

if and only if $\lambda \in \Delta_{qp} \neq \varnothing$, such that (8.16) implies (8.33). Similarly, for $p = 0$ and $1 \in \hat{w}(q)/\hat{w}(0) \cdot (0, 1)$, we have, by (5.20),

$$\lambda \hat{w}(q) g'\{\lambda |\hat{u}(0)| \hat{w}(0)\} < 1 \tag{8.40}$$

if and only if $\lambda \in \Delta_{0_q}$, i.e., (8.36). □

REMARKS. (i) Remark (i) after Theorem 5 also applies here.

(ii) The proof of (8.29) verifies just the branching condition: there are no bifurcations on the p-primary branch of solutions into the direction of (p, q)-secondary solutions. Conversely, (8.32) and (8.33) prove that indeed a secondary bifurcation occurs on the branch of q-primary solutions, and similarly in case (iii) with $p = 0$ and q exchanged. In case (i) there does not exist a secondary bifurcation on either branch of primary solutions. Nevertheless, to know, in this case, if there are no (p, q)-secondary solutions at all, one has to compute the functions ψ_1 and ψ_2 and to see if their graphs intersect as in Figure 6.

We like to note that Figure 6 is a little optimistic, since for general g one cannot prove without additional assumptions that the graphs of ψ_1 and ψ_2 have no intersection in case $\lambda < \lambda_{0_q}$ (a), and exactly one intersection point in $(0, \infty)^2$ in case $\lambda > \lambda_{0q}$ (b), though this is what we expect in most examples. The results of Theorems 10 and 11 have the following immediate consequence according to fact (i) preceding the lemma.

COROLLARY. *If* (ii) $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$ *and* $\lambda \in \Delta_{qp}$, *or if* (iii) $p = 0 \neq q$, $1 > \hat{w}(0)/\hat{w}(q) > 0$ *and* $\lambda \in \Delta_{0q}$, *then there exists at least one fixed point of* ϕ *in* $(0,\infty)^2 \cap \phi(\mathbf{R}^2)$. *By* $(|\hat{v}(p)|, |\hat{v}(q)|)$ *we denote that fixed point of* ϕ *in* $(0,\infty)^2$, *for which* $|\hat{v}(q)|$ *in case* (ii) (*resp.* $|\hat{v}(0)|$ *in case* (iii)) *is maximal.*

The results of this section enable us to prove Theorem 5 of §5.

PROOF OF THEOREM 5. (i) If $\frac{1}{2} > \hat{w}(q)/\hat{w}(p) > 0$, then the expression in (8.37) is less than $\frac{1}{2}$ for all $\lambda > \lambda_p$, and the expression in (8.38) is greater than 1 for all $\lambda > \lambda_q$. So in $\mathcal{F}_{pq}$ there are no secondary bifurcations either on the p-primary, or on the q-primary branch.

(ii) If $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$, then the expression in (8.37) is still less than 1 for all $\lambda > \lambda_p$, and in $\mathcal{F}_{pq}$ there is no secondary bifurcation on the p-primary branch. But (4.11) for q shows that λ_{qp}, defined in (5.10), is finite and $\Delta_{qp} \supseteq (\lambda_{qp}, +\infty) \neq \emptyset$. $\lambda_q < \lambda_{qp}$ follows from $|\hat{u}(q)| \to_{\lambda \searrow \lambda_q} 0$. By continuity (8.31)–(8.33) and the corollary show that there exists a bifurcation of (p,q)-secondary solutions of the form (5.12), which branches off the q-primary solution and exists for all $\lambda \in \Delta_{qp}$.

(iii) Let $p = 0 \neq q$ and $1 > \hat{w}(0)/\hat{w}(q) > 0$. The expression in (8.38) is less than 1 for all $\lambda > \lambda_q$, and in $\mathcal{F}_{0q}$ there is no secondary bifurcation on the branch of q-primary solutions. Here, (2.48) yields $\lambda_{0q} < +\infty$, and (8.35)–(8.36) show the existence of a secondary bifurcation in $\mathcal{F}_{0q}$ on the branch of nontrivial constant solutions. The $(0,q)$-secondary solutions are of the form (5.12) with $p = 0$ and exist for all $\lambda \in \Delta_{0q}$. By (2.43) we have $\lambda_0 < \lambda_{0q}$. □

We finish this section by describing the stability properties of the fixed points of ϕ. We use the following terminology:

DEFINITION. A fixed point z of ϕ is called *stable* if all eigenvalues μ_i of the linearization $\partial\phi$ of ϕ at z have modulus less than 1: $|\mu_i| < 1$ for all eigenvalues μ_i. z is called a ***hyperbolic*** fixed point if for some eigenvalues μ_{i_0}, μ_{i_1} of $\partial\phi$ at z we have $|\mu_{i_0}| < 1$, $|\mu_{i_1}| > 1$, and $|\mu_i| \neq 1$ for all other eigenvalues. z is called (*totally*) *unstable* if $|\mu_i| > 1$ for all eigenvalues μ_i of $\partial\phi$ at z. z is called *critical* if $|\mu_i| = 1$ for at least one eigenvalue μ_i of $\partial\phi$ at z.

THEOREM 12. *The fixed points of* ϕ *have the following properties*:

(i) *If* $\frac{1}{2} > \hat{w}(q)/\hat{w}(p) > 0$, *then*

$$(0,0) \text{ is } \begin{cases} \text{stable for } \lambda \in (0,\lambda_p), \\ \text{hyperbolic for } \lambda \in (\lambda_p,\lambda_q), \\ \text{unstable for } \lambda \in (\lambda_q,+\infty); \end{cases} \tag{8.41}$$

$$(\pm|\hat{u}(p)|, 0) \text{ is stable for } \lambda \in (\lambda_p, +\infty); \tag{8.42}$$

$$(0, \pm|\hat{u}(q)|) \text{ is hyperbolic for all } \lambda \in (\lambda_q, +\infty). \tag{8.43}$$

(ii) *If* $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$, *then* (8.41) *and* (8.42) *hold again, but*

$$(0, \pm|\hat{u}(q)|) \text{ is } \begin{cases} \text{hyperbolic or critical for } \lambda \in (\lambda_q, +\infty)\setminus\Delta_{qp}, \\ \text{stable for } \lambda \in \Delta_{qp}; \end{cases} \tag{8.44}$$

(8.45) $(\pm|\hat{v}(p)|, \pm|\hat{v}(q)|)$ *is hyperbolic or critical for all* $\lambda \in \Delta_{qp}$.

(iii) *If* $p = 0 \neq q$ *and* $1 > \hat{w}(0)/\hat{w}(q) > 0$, *then*

$$(0,0) \text{ is } \begin{cases} \text{stable for } \lambda \in (0, \lambda_q), \\ \text{hyperbolic for } \lambda \in (\lambda_q, \lambda_0), \\ \text{unstable for } \lambda \in (\lambda_0, +\infty); \end{cases} \tag{8.46}$$

$$(0, \pm|\hat{u}(q)|) \text{ is stable for } \lambda \in (\lambda_q, +\infty); \tag{8.47}$$

$$(\pm|\hat{u}(0)|, 0) \text{ is } \begin{cases} \text{hyperbolic or critical for } \lambda \in (\lambda_0, +\infty)\backslash\Delta_{0q}, \\ \text{stable for } \lambda \in \Delta_{0q}; \end{cases} \tag{8.48}$$

$$(\pm|\hat{v}(0)|, \pm|\hat{v}(q)|) \text{ is hyperbolic or critical for } \lambda \in \Delta_{0q}. \tag{8.49}$$

REMARK. $\pm|\hat{v}(p)|$ is a stable fixed point of $\phi_1(\cdot, \pm|\hat{v}(q)|)$, and $\pm|\hat{v}(q)|$ is a stable fixed point of $\phi_2(\pm|\hat{v}(p)|, \cdot)$, since they lie on the graphs of ψ_1 and ψ_2, respectively, which represent stable fixed points for $\phi_1(\cdot, y)$ and $\phi_2(x, \cdot)$, respectively. This implies that the corresponding solution (5.12) is p-stable and q-stable. However, with regard to ϕ, $(\pm|\hat{v}(p)|, \pm|\hat{v}(q)|)$ is not stable. See also the remark at fact (iii) preceding the lemma.

PROOF. The linearization of ϕ is given by

$$\partial\phi = \begin{pmatrix} \partial_x\phi_1 & \partial_y\phi_1 \\ \partial_x\phi_2 & \partial_y\phi_2 \end{pmatrix}, \tag{8.50}$$

where

$$\partial_x\phi_1(x,y) = \lambda\hat{w}(p)\int g'\left\{\lambda\hat{w}(p)x\begin{bmatrix}1 \\ 2\cos(2\pi pt)\end{bmatrix} + \lambda\hat{w}(q)y2\cos(2\pi pt)\right\} \times \begin{bmatrix}1 \\ 2\cos^2(2\pi pt)\end{bmatrix} dt \tag{8.51}$$

$$\partial_y\phi_2(x,y) = \lambda\hat{w}(q)\int g'\{\cdots\}2\cos^2(2\pi qt)\,dt, \tag{8.52}$$

and

$$\partial_y\phi_1(x,y) = 2\hat{w}(q)\begin{bmatrix}1/\hat{w}(0) \\ 1/2\hat{w}(p)\end{bmatrix}\partial_x\phi_2(x,y)$$

is given in (8.13). By a calculation similar to the first equations in (8.8)–(8.9), we get

$$\partial_y\phi_1(x,y) = \partial_x\phi_2(x,y) = 0 \quad \text{if } x = 0 \text{ or } y = 0. \tag{8.53}$$

(8.41) and (8.46) are obvious from the definition of λ_p, λ_q in (2.36) or (4.6). (2.38) or the concavity of $\phi_q(\lambda, z)$ show

$$\partial_x\phi_1(\pm|\hat{u}(p)|, 0) = \frac{\partial}{\partial z}\phi_p(\lambda, |\hat{u}(p)|) \in (0,1) \quad \text{for } \lambda \in (\lambda_p, +\infty)$$

and

$$\partial_y\phi_2(0, \pm|\hat{u}(q)|) = \frac{\partial}{\partial z}\phi_q(\lambda, |\hat{u}(q)|) \in (0,1) \quad \text{for } \lambda \in (\lambda_q, +\infty).$$

By (8.37), (8.38), and the noncollinearity of p, q, we get

$$\partial_y \phi_2(\pm|\hat{u}(p)|, 0) = \lambda \hat{w}(q) \int g' \left\{ \pm\lambda \hat{w}(p)|\hat{u}(p)| \begin{bmatrix} 1 \\ 2\cos(2\pi pt) \end{bmatrix} \right\} 2\cos^2(2\pi qt)\, dt$$
$$\in \hat{w}(q)/\hat{w}(p) \cdot (0, 1),$$

and

$$\partial_x \phi_1(0, \pm|\hat{u}(q)|) \in \hat{w}(p)/\hat{w}(q) \cdot (\tfrac{1}{2}, 1).$$

Thus, if $\hat{w}(q)/\hat{w}(p) < 1$ in case (i) or (ii), then $(\pm|\hat{u}(p)|, 0)$ is stable for $\lambda \in (\lambda_p, +\infty)$. If $\hat{w}(p)/\hat{w}(q) > 2$ in case (i), then $(0, \pm|\hat{u}(q)|)$ is hyperbolic, and if $\hat{w}(0)/\hat{w}(q) < 1$ in case (iii), then it is stable for $\lambda \in (\lambda_q, +\infty)$. If $\hat{w}(p)/\hat{w}(q) \in (1, 2)$ in case (ii), then by (5.19) and (4.3) we obtain

$$\partial_x \phi_1(0, \pm|\hat{u}(q)|) = \frac{\hat{w}(p)}{\hat{w}(q)} \partial\phi_q(\lambda, |\hat{u}(q)|) < 1 \quad \text{iff} \quad \lambda \in \Delta_{qp},$$

while in case (iii) with $\hat{w}(0)/\hat{w}(q) < 1$ by (5.20),

$$\partial_y \phi_2(\pm|\hat{u}(0)|, 0) = \frac{\hat{w}(q)}{\hat{w}(0)} \partial\phi_0(\lambda, |\hat{u}(0)|) < 1 \quad \text{iff} \quad \lambda \in \Delta_{0q}.$$

This shows (8.44) and (8.48). For assertions (8.45) and (8.49), we have to calculate the eigenvalues

(8.54) $\mu_{1/2} = (\partial_x\phi_1 + \partial_y\phi_2)/2 \pm [(\partial_x\phi_1 + \partial_y\phi_2)^2/4 + \partial_y\phi_1\partial_x\phi_2 - \partial_x\phi_1\partial_y\phi_2]^{1/2}$

at $(\pm|\hat{v}(p)|, \pm|\hat{v}(q)|)$. The maximality condition in the corollary says that in case (ii) the graph $\{(\psi_2(y), y); y \in (|\hat{v}(q)|, |\hat{u}(q)|)\}$ lies above the graph $\{(x, \psi_1(x)); x \in (0, |\hat{v}(p)|)\}$, while in case (iii) the graph $\{(x, \psi_1(x)); x \in (|\hat{v}(p)|, |\hat{u}(p)|)\}$ lies above $\{(\psi_2(y), y); y \in (0, |\hat{v}(q)|)\}$ (see Figure 6). Both cases imply that

(8.55) $$|\psi_1'(|\hat{v}(p)|)| \leq 1/|\psi_2'(|\hat{v}(q)|)|.$$

By (8.21) and (8.27) this shows that at $(|\hat{v}(p)|, |\hat{v}(q)|)$,

(8.56) $$0 \leq |(1 - \partial_x\phi_1)(1 - \partial_y\phi_2)| \leq \partial_y\phi_1\partial_x\phi_2.$$

Assume first that $(1 - \partial_x\phi_1)(1 - \partial_y\phi_2) \geq 0$. Then in (8.54) we have

$$\partial_y\phi_1\partial_x\phi_2 - \partial_x\phi_1\partial_y\phi_2 \geq 1 - (\partial_x\phi_1 + \partial_y\phi_2)$$

such that with $a = (\partial_x\phi_1 + \partial_y\phi_2)/2 > 0$, we get $\mu_1 \geq a + (a^2 + 1 - 2a)^{1/2} = a + |1 - a| \geq 1$, and $\mu_2 \leq a - |1 - a| \leq 1$.

If, on the other hand, $(1 - \partial_x\phi_1)(1 - \partial_y\phi_2) < 0$—i.e., $0 < \partial_x\phi_1 < 1 < \partial_y\phi_2$ or $0 < \partial_y\phi_2 < 1 < \partial_x\phi_1$—then, since

$$\partial_y\phi_1\partial_x\phi_2 = \begin{bmatrix} \hat{w}(0) \\ 2\hat{w}(p) \end{bmatrix} \frac{(\partial_y\phi_1)^2}{2\hat{w}(q)} \geq 0,$$

we get

$$\mu_1 \geq (\partial_x\phi_1 + \partial_y\phi_2)/2 + |\partial_x\phi_1 - \partial_y\phi_2|/2 = \max(\partial_x\phi_1, \partial_y\phi_2) > 1,$$
$$\mu_2 \leq (\partial_x\phi_1 + \partial_y\phi_2)/2 - |\partial_x\phi_1 - \partial_y\phi_2|/2 = \min(\partial_x\phi_1, \partial_y\phi_2) < 1.$$

To finish the proof of (8.45) and (8.49), we need only show that $\mu_2 \geq 0$. But at $(|\hat{v}(p)|, |\hat{v}(q)|)$

$$0 \leq \partial_y\phi_1\partial_x\phi_2 \leq \begin{bmatrix} 2 \\ 4 \end{bmatrix} \lambda^2\hat{w}(p)\hat{w}(q) \times \left(\int g' \left\{ \lambda\hat{w}(p)|\hat{v}(p)| \begin{bmatrix} 1 \\ 2\cos(2\pi pt) \end{bmatrix} + \lambda\hat{w}(q)|\hat{v}(q)| 2\cos(2\pi qt) \right\} \times |\cos(2\pi pt)| \cdot |\cos(2\pi qt)|\, dt \right)^2$$

$$\leq \partial_x\phi_1\partial_y\phi_2$$

such that

$$\mu_2 = (\partial_x\phi_1 + \partial_y\phi_2)/2 - ((\partial_x\phi_1 - \partial_y\phi_2)^2/4 + \partial_y\phi_1\partial_x\phi_2)^{1/2} \geq a - |a| = 0.$$

This completes the proof of Theorem 12. □

The proof of Theorem 6 is now an immediate consequence of Theorem 12. We only note that for primary solutions, the linearization $\partial\phi$ has diagonal form by (8.53). The definition of p- or q-stability is then by (4.14)–(4.15), and the non-collinearity of p, q equivalent to the fact that $\partial_x\phi_1$ (resp. $\partial_y\phi_2$) is less than 1.

9. The dynamical system for collinear p, q. We assume the collinearity conditions (7.1) and (7.5) for p, q, and (7.3) and (7.4) for the function w. In order to get in $\mathcal{F}_{pq}$ a secondary bifurcation from the p-primary solutions, one of the following bifurcation conditions must be satisfied:

$$(9.1)\quad 1 = \lambda\hat{w}(q) \int g'\{\lambda\hat{w}(p) 2\,\mathrm{Re}(\hat{u}(p)\exp(2\pi ipt))\} 2\cos^2(2\pi qt)\, dt = \lambda\hat{w}(q) \int_{-1/2}^{1/2} g'\{\lambda\hat{w}(p)|\hat{u}(p)| 2\cos(2\pi n_2 s + \arg\hat{u}_\lambda(p))\} \times (1 + \cos(4\pi n_1 s))\, ds$$

or

$$(9.2)\quad 1 = \lambda\hat{w}(q) \int_{-1/2}^{1/2} g'\{\lambda\hat{w}(p)|\hat{u}(p)| 2\cos(2\pi n_2 s + \arg\hat{u}_\lambda(p))\} \times (1 - \cos(4\pi n_1 s))\, ds.$$

But we claim that the right expressions of (9.1) or (9.2) are always less than $\hat{w}(q)/\hat{w}(p) < 1$. To verify this, set $\mu = 2\lambda\hat{w}(p)|\hat{u}(p)| > 0$ and $\alpha = \arg\hat{u}(p)$ for $\lambda > \lambda_p$. First consider the case $n_1 \notin n_2\mathbf{Z}$. By Proposition 1(i) we have

$$(9.3)\qquad \int g'\{\mu\cos(2\pi n_2 s + \alpha)\}\cos(4\pi n_1 s)\, ds = 0,$$

and (4.3) and (4.10) yield

$$(9.4)\qquad \lambda\hat{w}(q) \int g'\{\mu\cos(2\pi n_2 s + \alpha)\}\, ds \in \hat{w}(q)/\hat{w}(p)(\tfrac{1}{2}, 1),$$

which proves our claim in this case. On the other hand, assume $n_1 = l \cdot n_2 \in n_2\mathbf{Z}$, $l > 1$. We note first that for all $\beta \in \mathbf{T}$ and $z \in (0, \frac{1}{2})$

$$\begin{aligned}(9.5)\qquad \int_{-z}^{z} [\cos(2\pi s) \pm \cos(2\pi(ls - \beta))]\, ds &= (1/\pi l)[l \sin(2\pi z) \pm \sin(2\pi l z) \cos(2\pi\beta)] \\ &\geq (1/\pi l)[l \sin(2\pi z) - |\sin(2\pi l z)|] > 0.\end{aligned}$$

Since g' is decreasing on $\mathbf{R}^+$, we can define $\int dg'(y)$ as a Lebesgue-Stieltjes integral on $\mathbf{R}^+$ with

$$(9.6)\qquad \int_a^b dg'(y) < 0 \quad \text{for all } 0 \leq a < b,$$

and

$$(9.7)\qquad g'\{\mu|\cos(\pi s)|\} = g'(0) + \int_0^{\mu} 1_{[y/\mu,1]}(|\cos(\pi s)|)\, dg'(y).$$

Then by (9.5)–(9.7),

$$\begin{aligned}(9.8)\qquad &\int_{-1/2}^{+1/2} g'\{\mu \cos(2\pi n_2 s + \alpha)\}[\cos(4\pi n_2 s + 2\alpha) \pm \cos(4\pi n_1 s)]\, ds \\ &= \int_{-1/2}^{+1/2} g'\{\mu|\cos(\pi s)|\}[\cos(2\pi s) \pm \cos(2\pi l s - 2l\alpha)]\, ds \\ &= \int_0^{\mu} dg'(y) \int_{-\arccos(y/\mu)/\pi}^{\arccos(y/\mu)/\pi} [\cos(2\pi s) \pm \cos(2\pi l s - 2l\alpha)]\, ds < 0,\end{aligned}$$

or, by (4.13),

$$\begin{aligned}0 &< \lambda \hat{w}(q) \int_{-1/2}^{+1/2} g'\{\mu \cos(2\pi n_2 s + \alpha)\} \\ &\qquad \times [2\sin^2(2\pi n_2 s + \alpha) - (1 \pm \cos(4\pi n_1 s))]\, ds \\ &= \frac{\hat{w}(q)}{\hat{w}(p)} - \lambda \hat{w}(q) \int_{-1/2}^{+1/2} g'\{\mu \cos(2\pi n_2 s + \alpha)\}(1 \pm \cos(4\pi n_1 s))\, ds,\end{aligned}$$

which proves our claim, following (9.2). Therefore, under the assumptions of Theorem 7, there exists in $\mathcal{F}_{pq}$ no bifurcation from the p-primary branch of solutions.

To prove the existence of secondary bifurcations from the q-primary solutions, we use the same technique as in §8. Here we look for nondegenerate fixed points (x, y), $x \neq 0 \neq y$, of the following pair of operators:

$$(9.9)\qquad \phi^1(x,y) = (\phi_1^1(x,y), \phi_2^1(x,y)) \quad \text{and} \quad \phi^2(x,y) = (\phi_1^2(x,y), \phi_2^2(x,y)),$$

where

$$\begin{aligned}\phi_1^1(x,y) &= \int g\{\lambda \hat{w}(p) x 2 \sin(2\pi p t) + \lambda \hat{w}(q) y 2 \cos(2\pi q t)\} \sin(2\pi p t)\, dt \\ &= \int_{-1/2}^{+1/2} g\{\lambda \hat{w}(p) x 2 \sin(2\pi n_2 s) + \lambda \hat{w}(q) y 2 \cos(2\pi n_1 s)\} \sin(2\pi n_2 s)\, ds,\end{aligned}$$

and similarly

$$\phi_2^1(x,y) = \int_{-1/2}^{+1/2} g\{\lambda\hat{w}(p)x2\sin(2\pi n_2 s) + \lambda\hat{w}(q)y2\cos(2\pi n_1 s)\} \times \cos(2\pi n_1 s)\, ds,$$

$$\phi_1^2(x,y) = \int_{-1/2}^{+1/2} g\{\lambda\hat{w}(p)x2\cos(2\pi n_2 s) + \lambda\hat{w}(q)y2\sin(2\pi n_1 s)\} \times \cos(2\pi n_2 s)\, ds,$$

$$\phi_2^2(x,y) = \int_{-1/2}^{+1/2} g\{\lambda\hat{w}(p)x2\cos(2\pi n_2 s) + \lambda\hat{w}(q)y2\sin(2\pi n_1 s)\} \times \sin(2\pi n_1 s)\, ds.$$

For the pairs (ϕ_1^1, ϕ_2^1) and (ϕ_1^2, ϕ_2^2) we get the same results as in the lemma and Theorem 9 of §8.

THEOREM 13. *The functions ϕ_1^1 and ϕ_1^2 are strictly increasing and odd in x but even in y. For $x \neq 0$ they are strictly decreasing in $|y|$ with*

$$\lim_{|y|\to\infty} |\phi_1^1(x,y)| = \lim_{|y|\to\infty} |\phi_1^2(x,y)| = 0. \tag{9.10}$$

For $\lambda \in (0, \lambda_p]$, $x = 0$ is the only fixed point of $\phi_1^1(\cdot, y)$ and $\phi_1^2(\cdot, y)$ for all y, while for $\lambda \in (\lambda_p, +\infty)$ there exist unique, positive, symmetric, continuously differentiable functions ψ_1^1 and ψ_1^2 on $(-|\hat{u}(p)|, +|\hat{u}(p)|)$ with

$$\phi_1^1(x, \pm\psi_1^1(x)) = \phi_1^2(x, \pm\psi_1^2(x)) = x \tag{9.11}$$

for all $x \in (-|\hat{u}(p)|, +|\hat{u}(p)|)$, and

$$\lim_{|x|\to|\hat{u}_\lambda(p)|} \psi_1^i(x) = 0, \qquad i = 1, 2. \tag{9.12}$$

The same facts hold for ϕ_2^1 and ϕ_2^2 with functions ψ_2^1 and ψ_2^2 on $(-|\hat{u}(q)|, +|\hat{u}(q)|)$ if we exchange x and y and p and q everywhere. At zero ψ_1^1 and ψ_1^2 are uniquely determined by the equations

$$1 = \lambda\hat{w}(p) \int_{-1/2}^{+1/2} g'\{\lambda\hat{w}(q)\psi_1^1(0)2\cos(2\pi n_1 s)\}(1 - \cos(4\pi n_2 s))\, ds, \tag{9.13}$$

$$1 = \lambda\hat{w}(p) \int_{-1/2}^{+1/2} g'\{\lambda\hat{w}(q)\psi_1^2(0)2\sin(2\pi n_1 s)\}(1 + \cos(4\pi n_2 s))\, ds. \tag{9.14}$$

Analogous equations determine $\psi_1^2(0)$ and $\psi_2^2(0)$ uniquely.

The proof of this theorem follows the same lines as those of the lemma and Theorem 9 in §8. (8.13) is now replaced by

$$\begin{aligned} \partial_y\phi_1^1(x,y) &= \hat{w}(q)/\hat{w}(p)\partial_x\phi_2^1(x,y) \\ &= \lambda\hat{w}(q) \int_{-1/2}^{+1/2} [g'\{\lambda\hat{w}(p)x2\sin(2\pi n_2 s) + \lambda\hat{w}(q)y2\cos(2\pi n_1 s)\} \\ &\qquad - g'\{-\lambda\hat{w}(p)x2\sin(2\pi n_2 s) + \lambda\hat{w}(q)y2\cos(2\pi n_1 s)\}] \\ &\qquad \times \sin(2\pi n_2 s)\cos(2\pi n_1 s)\, ds, \end{aligned} \tag{9.15}$$

which is less than 0 if $x > 0$ and $y > 0$. Similarly to (8.24), the boundedness of ψ_1^1 follows from

$$\phi_1^1(x,y) \leq \lambda\hat{w}(p)|x| \int \tilde{g}'\{\lambda\hat{w}(q)y2\cos(2\pi n_1 s)\}(1-\cos(4\pi n_2 s))\,ds, \tag{9.16}$$

which is less than $|x|$ if y is only large enough. Here, $\tilde{g}'$ is taken from (8.22).

For the following result, recall the definitions of Δ_{qp}, Δ_{qp}^1, and Δ_{qp}^2 from (5.19), and (7.11), (7.12), respectively.

THEOREM 14. (i) *Assume $p \notin q\mathbf{Z}$. Then*

$$\Delta_{qp}^1 = \Delta_{qp}^2 = \Delta_{qp} \supseteq (\lambda_{qp}, +\infty) \neq \emptyset \quad \textit{iff} \quad 1 > \hat{w}(q)/\hat{w}(p) > 1/2.$$

(ii) *If $p \in 2q\mathbf{Z}$, then for all $\hat{w}(p) > \hat{w}(q) > 0$,*

$$\Delta_{qp}^1 \supseteq (\lambda_{qp}^1, +\infty) \neq \emptyset,$$

but Δ_{qp}^2 is a bounded (possibly empty) region in $\mathbf{R}^+$.

(iii) *For $i = 1, 2$ we have*

$$|\hat{u}(p)| > \psi_2^i(0) \quad \textit{for all } \lambda \in (\lambda_p, +\infty),$$
$$\psi_1^i(0) < |\hat{u}(q)| \quad \textit{iff } \lambda \in \Delta_{qp}^i.$$

PROOF. (i) follows from (7.13) and (4.10)–(4.11). Now, let $p \in 2q\mathbf{Z}$; i.e., $n_1 = 1$ and n_2 even. We consider the positive measures on $\mathbf{T}$:

$$\mu_1(ds) = \lambda\hat{w}(q)g'\{\lambda\hat{w}(q)|\hat{u}(q)|2\cos(2\pi s)\}\,ds, \tag{9.17}$$

and

$$\mu_2(ds) = \lambda\hat{w}(q)g'\{\lambda\hat{w}(q)|\hat{u}(q)|2\sin(2\pi s)\}\,ds. \tag{9.18}$$

For $\lambda \searrow \lambda_q$ we have $|\hat{u}(q)| \to 0$ and $\lambda_q\hat{w}(q)g'(0) = 1$ such that

$$\mu_i(ds) \to ds \quad \text{in the weak sense,} \qquad i = 1, 2. \tag{9.19}$$

For $\lambda \to \infty$, (4.11) shows $\mu_i(\mathbf{T}) \to \frac{1}{2}$, but $\mu_1(ds) \to 0$ for all $s \neq \pm\frac{1}{4}$ and $\mu_2(ds) \to 0$ for all $s \neq 0, \frac{1}{2}$. By the symmetry of μ_1 on 0 and the symmetry of μ_2 on $\frac{1}{4}$, we get

$$\lim_{\lambda\to\infty} \mu_1 = \tfrac{1}{4}(\delta_{1/4} + \delta_{3/4}), \qquad \lim_{\lambda\to\infty} \mu_2 = \tfrac{1}{4}(\delta_0 + \delta_{1/2}). \tag{9.20}$$

Now the positive functions

$$h_i(\lambda) = \int (1 + (-1)^i \cos(4\pi n_2 s))\mu_i(ds), \tag{9.21}$$

which by the assertion following (9.2) with p, q and n_1, n_2 exchanged are always less than 1, satisfy

$$\lim_{\lambda\searrow\lambda_q} h_i(\lambda) = 1 \quad \text{for } i = 1, 2, \tag{9.22}$$

$$\lim_{\lambda\to\infty} h_1(\lambda) = \frac{1}{4}\left(2 - \cos\left(\frac{4\pi n_2}{4}\right) - \cos\left(\frac{4\pi n_2 3}{4}\right)\right) = 0, \tag{9.23}$$

and

(9.24) $$\lim_{\lambda\to\infty} h_2(\lambda) = \frac{1}{4}\left(2+1+\cos\left(\frac{4\pi n_2}{2}\right)\right) = 1.$$

Therefore

(9.25) $$\Delta^1_{qp} = \{\lambda, h_1(\lambda) < \hat{w}(q)/\hat{w}(p)\} \supseteq (\lambda^1_{qp}, +\infty) \neq \emptyset,$$

while

$$\Delta^2_{qp} = \{\lambda, h_2(\lambda) < \hat{w}(q)/\hat{w}(p) < 1\}$$

is a bounded, possibly empty region in $\mathbf{R}^+$. This proves (ii). Now, the equations uniquely determining $\psi^1_i(0)$ and $\psi^2_i(0)$ in Theorem 14, the assertion after (9.2), and the definition of Δ^i_{qp} yield (iii) immediately. □

PROOF OF THEOREM 7. We have already seen at the beginning of this section that the branch of p-primary solutions does not have a secondary bifurcation in $\mathcal{F}_{pq}$. Similar calculations as in (9.3)–(9.4), with p, q and n_1, n_2 exchanged, show that if $\frac{1}{2} \geq \hat{w}(q)/\hat{w}(p) > 0$ and $p \notin \mathbf{Z}q$, then there are no secondary bifurcation from the q-primary solutions. But if $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$ and $p \notin \mathbf{Z}q$, then Theorem 14 and the symmetry properties of ϕ^1, ϕ^2 give us the existence of eight nondegenerated ($|\hat{v}_i(p)| \neq 0 \neq |\hat{v}_i(q)|$) fixed points

(9.26) $$(\pm|\hat{v}_1(p)|, \pm|\hat{v}_1(q)|) \quad \text{and} \quad (\pm|\hat{v}_2(p)|, \pm|\hat{v}_2(q)|)$$

of ϕ^1 and ϕ^2, respectively, which branch from the fixed points $(0, \pm|\hat{u}(q)|)$. The fixed points (9.26) establish the secondary solutions v_1 and v_2 of (7.7)–(7.8) with $\tau_i = j_i = k_i = 0$, $i = 1, 2$. Rotating these solutions v_i by $r_0 \cdot (\tau + j'_i n_2 + k'_i n_1)/n_1 n_2 (r_0 r_0)$ with

$$j'_i n_2 \equiv j_i \bmod n_1 \quad \text{and} \quad k'_i n_1 \equiv k_i \bmod n_2, \qquad i = 1, 2,$$

it is easily proved by the invariance of the set of solutions of (2.1) under rotations in $\mathbf{T}^d$ that v_1 and v_2 given in (7.7)–(7.8) are indeed secondary solutions by any choice of the parameters.

In the same way, part (iv) of Theorem 7 follows from the results about Δ^1_{qp} in Theorem 14. □

Let us conclude with some remarks about the stability of the solutions. The fact following (9.2) proves that the p-primary solutions are p-stable and stable with respect to all directions $\hat{u}(q) \in \mathbf{C}$. Similarly, the q-primary solutions are q-stable. In the case $p \notin q\mathbf{Z}$ they are also p-stable if and only if $\lambda \in \Delta_{qp} \supseteq (\lambda_{qp}, +\infty)$, i.e., if $1 > \hat{w}(q)/\hat{w}(p) > \frac{1}{2}$ and $h_1(\lambda) = h_2(\lambda) < \hat{w}(q)/\hat{w}(p)$. If, however, $p \in 2\mathbf{Z}q$, then one can show that the q-primary solutions are stable with respect to all directions $\hat{u}(p) \in \mathbf{C}$ only if $\lambda \in \Delta^1_{qp} \cap \Delta^2_{qp}$, which is bounded in $\mathbf{R}^+$. It is hyperbolic or critical otherwise. If, for $\lambda \in \Delta^i_{qp}$, we denote by $(|\hat{v}_i(p)|, |\hat{v}_i(q)|)$ that fixed point of ϕ^i in $(0,\infty)^2$ with $|\hat{v}_i(q)|$ maximal, then we have a hyperbolic or critical fixed point of ϕ^i, which also gives hyperbolic or critical secondary solutions v_i, $i = 1, 2$, by (7.7)–(7.8).

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