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Quelques problèmes concernant le comportement pour les grands temps des équations d'évolution dissipatives, 1989

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Quelques problèmes concernant le comportement pour les grands temps des équations d'évolution dissipatives

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SOME PROBLEMS RELATED TO THE LONG TIME BEHAVIOUR OF DISSIPATIVE EVOLUTION EQUATIONS.

Abstract : In this work, we consider the long time behaviour of dissipative evolution equations. More precisely we study the existence of attracting sets such as attractors and inertial manifolds.

In the first part, we describe a general method to construct inertial manifolds for a nonlinear parabolic equation. We obtain an existence theorem under the same type of assumptions as the methods that already exist. Our method is based on the resolution of a hyperbolic partial differential equation (the Sacker's equation) such that the graph of its solution is a positively invariant manifold.

The second part is devoted to the existence of approximate inertial manifolds. These are substitute to inertial manifolds when their existence is not known. We prove in two cases (the reaction diffusion equation and the Cahn-Hilliard equation) the existence of an infinite family of approximate inertial manifolds with increasing order of approximation. Our method is general and can be applied to other equations.

Finally, in the third part, we study a singular perturbation of the Cahn-Hilliard equation in space dimension one obtained by adding a second order derivative in time whose coefficient ε is small. We prove the existence of attractors for the perturbed equation. Moreover, the Haussdorf semi distance from these attractors to the attractor of the unperturbed equation converges to zero when ε goes to zero.

Key words: Inertial manifold, Sacker's aquation, elliptic regularization, approximate inertial manifold, Cahn-Hilliard equation, reaction diffusion equation, attractor, singular perturbation.

PLAN DE LA THESE :

Introduction.

1ère partie : Variété inertielle et équation de sacker.

2ème partie : Construction de familles de variété inertielle approchée.

3ème partie : Une perturbation singulière de l'équation de Cahn-Hilliard.

INTRODUCTION

Dans cette thèse, nous abordons certains aspects de l'étude du comportement pour les grands temps des solutions des équations d'évolution dissipatives. Ce domaine a connu un essor assez grand depuis quelques temps. Ceci est dû, entre autres, à l'apparition d'ordinateurs de plus en plus puissants capables de travailler sur de tels problèmes.

Une façon d'aborder ces équations est de les considérer comme des systèmes dynamiques de dimension infinie. On peut alors essayer de généraliser les concepts qui existent en dimension finie. C'est ainsi qu'est apparue la notion d'attracteur. Un attracteur associé à un système dynamique est un ensemble invariant qui attire toutes les trajectoires de ce système. Si le semi-groupe associé à l'équation d'évolution possède certaines propriétés de compacité et si on sait qu'il existe un ensemble absorbant borné B alors on peut montrer que l'ensemble oméga limite de B est un attracteur. Cette méthode générale s'applique à de nombreuses équations (voir [1],[2]). L'attracteur paraît donc être l'objet idéal pour décrire le comportement asymptotique des solutions, malheureusement c'est en général un ensemble dont la structure peut être très compliquée (voir l'attracteur étrange de Ruelle et Takens [3] ou le "Smale's horseshoe" [4],[5]), ce qui empêche son étude pratique.

Pour cette raison, C.Foias, G.R.Sell et R.Temam ont récemment introduit la notion de variété inertielle qui généralise celle de variété centrale. Une variété inertielle est une variété régulière de dimension finie positivement invariante qui attire les orbites du système à vitesse exponentielle. Quand elle existe, une variété inertielle permet de restreindre l'étude du système de dimension infinie à sa projection sur cette variété: la forme inertielle. Les techniques classiques de construction de variété centrale (méthode de Lyapunov-Perron, méthode de Hadamard...) ont pu être généralisées pour plusieurs équations.

Toutefois, l'existence de variété inertielle n'a pas été établie pour certaines équations très importantes. En général ces équations possèdent des variétés inertielles approchées. Une variété inertielle approchée est une variété régulière de dimension finie telle que toutes les solutions du système entrent dans un voisinage très mince de celle-ci en un temps fini. Outre le fait d'exister sous des hypothèses très larges, les variétés inertielles approchées présentent l'avange d'être données par des formules explicites.

Cette thèse est composée de trois parties dont nous exposons maintenant le contenu.

1ère partie: Variété inertielle et équation de Sacker.

Dans cette partie, nous proposons une nouvelle méthode de construction de variété inertielle. L'idée est de résoudre une équation aux dérivées partielles hyperbolique dont la solution possède un graphe positivement invariant (l'équation de Sacker).

Plus précisément, supposons que l'on étudie l'équation parabolique suivante

$$du/dt + Au + f(u) = 0, \qquad (1)$$

que l'on suppose bien posée dans un espace de Hilbert H, où A désigne un opérateur du type $(-\Delta)^r$ et f une nonlinéarité. On cherche une fonction $\Phi: H_1 \rightarrow H_2$, où $H = H_1 + H_2$ et $\dim H_1 < \infty$, telle que son graphe soit invariant pour le système dynamique associé à (1). On note P_i le projecteur sur H_i (i=1,2). En projetant (1) sur H_1 et H_2, on voit que si $p(t)+\Phi(p(t))$ est une solution de (1) sur la variété inertielle, on a

$$d\Phi(p(t))/dt + A\Phi(p(t)) + P_2 f(p(t)+\Phi(p(t))) = 0,$$

$$dp/dt + Ap(t) + P_1 f(p(t) + \Phi(p(t))) = 0,$$

et, en éliminant le temps, Φ doit vérifier l'équation introduite par R.J. Sacker [6]

$$-D\Phi(p)(Ap + P_1 f(p+\Phi(p))) + A\Phi(p) + P_2 f(p+\Phi(p)) = 0, \qquad (2)$$

(P_1 et P_2 commutent avec A). On cherche les solutions de (2) dont le support est inclus dans une boule de $P_1 H$.

Pour cela, on résout d'abord une équation approchée obtenue en remplaçant H par un sous espace de dimension finie m et en ajoutant une viscosité artificielle de coefficient ε. Cette équation approchée est une équation elliptique nonlinéaire que l'on résout par une méthode de point fixe dans $W^{1,\infty}$. Les estimations sont obtenues à partir de principes du maximum. Une grosse difficulté vient de l'application de ces principes du maximum aux dérivées de Φ dont on ne connaît pas le comportement à la frontière de la boule. On obtient des estimations suffisantes pour passer à la limite $\varepsilon \rightarrow 0$ et $m \rightarrow \infty$ sous une hypothèse assez forte sur l'opérateur A (hypothèse d'écart spectral). Cette restriction est commune à toutes les méthodes.

2éme partie: Construction de familles de variétés inertielles approchées.

La deuxième partie est consacrée à la construction de familles de variétés inertielles approchées. Ce travail a été fait en collaboration avec M.Marion et fait suite à deux de ses précédents articles [7,8] dans lesquels étaient construites deux variétés inertielles approchées pour l'équation de réaction diffusion et six pour l'équation de Cahn-Hilliard.

On montre pour ces deux équations que la construction peut être généralisée afin d'obtenir une famille infinie de variétés inertielles approchées dont l'ordre d'approximation est croissant. Le principe de la méthode est de décomposer la solution u de l'équation en $u(t) = p(t) + q(t)$ où $p(t)$ représente les grandes structures et $q(t)$ représente les petites (on a encore décomposé $H=H_1+ H_2$ avec $\dim H_1 < \infty$). Pour le cas de la réaction diffusion, on obtient en projetant l'équation

$$dp/dt + Ap = P_1 f(p + q), \tag{3}$$

$$dq/dt + Aq = P_2 f(p + q), \tag{4}$$

où $A= -\Delta$ muni de conditions aux limites, et f est une nonlinéarité à croissance polynomiale. On montre qu'à partir d'un certain temps q, ainsi que toutes ses dérivées, est petit. Ce qui légitime l'approximation de (4) par

$$Aq = P_2 f(p). \tag{5}$$

Si on fixe p, on note $\Phi_1(p)$ la solution de (5). Le graphe de Φ_1 est la premiére variété inertielle approchée. On peut améliorer l'approximation de (4) en introduisant une approximation de dq/dt obtenue en dérivant (4) et en négligeant d^2q/dt^2. Pour obtenir Φ_2 dont le graphe sera la deuxième variété inertielle approchée, on résout successivement

$$p^1 + Ap = P_1 f(p+ \Phi_1(p))$$

$$Aq^1 = P_2 f'(p)(p^1),$$

$$q^1 + Aq = P_2 f(p+ \Phi_1(p)),$$

et on pose

$$\Phi_2(p) = q.$$

On peut répéter ce procédé, la $k^{ième}$ variété inertielle approchée est construite en introduisant une approximation de $d^{k-1}q/dt^{k-1}$ obtenue en négligeant d^kq/dt^k. On montre que les solutions de l'équation de réaction diffusion entrent en un temps fini dans un voisinage de la $k^{ième}$ variété inertielle approchée de taille $K_k\delta^k$, où K_k ne dépend pas de $m= \dim H_1$ et δ est proportionnel à la $(m+1)^{ième}$ valeur propre de A. Pour l'équation de Cahn-Hilliard, on procède de la même manière, mais pour des raisons techniques la construction et les démonstrations sont légérement différentes.

$3^{\text{ième}}$ partie: Une perturbation singulière de l'équation de Cahn-Hilliard.

Dans cette troisième partie, on étudie une perturbation singulière de l'équation de Cahn-Hilliard obtenue en ajoutant une dérivée seconde en temps dont le coefficient ε est petit

$$\varepsilon\, d^2u/dt^2 + \nu\Delta^2 u - \Delta f(u) = 0, \tag{6}$$

où f est un polynôme de degré impair et de coefficient dominant positif. On s'intéresse plus particulièrement à l'existence d'un attracteur pour cette équation et au comportement de celui-ci lorsque ε tend vers zéro.

L'équation de Cahn-Hilliard (ε =0) a déjà été étudiée dans [9], il y est montré qu'il existe un attracteur pour la topologie forte dans $\{u \in L^2 : |m(u)| \leq \alpha\}$. Dans ce travail, on montre en utilisant la méthode classique mentionnée auparavant que (6) possède des attracteurs au sens de la topologie faible dans les espaces $V^s=\{u \in H^s \times H^{s-2} : |m(u)|+|m(du/dt)| \leq \alpha\}$ pour s=1,2 ou 3 (m(u) désigne la moyenne spatiale de u). Pour des raisons techniques, nos résultats ne s'appliquent qu'en dimension d'espace n=1.

On montre que les attracteurs de (6) dans V^3 que l'on note A_ε sont bornés indépendamment de ε dans V^3, ces bornes sont assez longues à obtenir et sont issues d'estimations a priori fines sur les solutions de (6). Ensuite, on définit une injection de l'attracteur de l'équation de Cahn-Hilliard (qui est un sous ensemble de L^2) dans V^2 que l'on note A et on montre que la semi distance de Haussdorf de A à A_ε tend vers zéro lorsque ε tend vers zéro

$$\lim_{\varepsilon \to 0} \; \sup_{(u_\varepsilon, v_\varepsilon) \in A_\varepsilon} \; \inf_{(u,v) \in A} \; \|u-u_\varepsilon\|_{H^2} + \|v-v_\varepsilon\|_{L^2} = 0 .$$

En d'autres termes, l'attracteur de (6) est semi continu supérieurement en ε=0.On emploie le même genre de techniques que dans [10].

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1ère PARTIE:

VARIETE INERTIELLE
ET EQUATION DE SACKER.

INERTIAL MANIFOLDS AND SACKER'S EQUATION

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I. INTRODUCTION

Much progress has been made in the study of dissipative evolution equations because they can be considered as infinite dimensional dynamical systems. For instance, the notion of attractor that existed for finite dimensional systems has been extended and many equations have a global attractor which is compact and connected (see e.g. R. Temam [1] or J.K. Hale [1]).

If $(S(t))_{t\geq 0}$ is the semigroup associated to a dynamical system in a Banach space E, the global attractor is a compact subset of E that attracts all the bounded sets of E :

(1.1) $\qquad \forall\, B \subset E$, B bounded :

$$d(S(t)B, \mathcal{A}) \to 0 \quad \text{when} \quad t \to +\infty,$$

moreover it is invariant :

(1.2) $$\forall\, t \geq 0 \quad , \quad S(t)\mathcal{A} = \mathcal{A} .$$

Such a set is interesting in order to describe the behaviour of the orbits of the system for large time. Indeed, we know that in numerous cases the attractor is finite dimensional and we have very good estimates of its dimension that correspond to the physical situation (see R. Temam [1]).

But attractors are not well adapted to practical purposes (especially numerical utilizations) for two reasons :

- they can attract the orbits very slowly ;

- their geometry can be very complicated (perhaps fractals) and they are not regular object.

For those reasons, the notion of inertial manifold has been introduced by C. Foias, G. Sell and R. Temam [1],[2]). It is defined as follows : let $(S(t))_{t\geq 0}$ be a dynamical system in a Banach space E, a set $\mathcal{M}$ is an inertial manifold for this semigroup if :

(1.3) - $\mathcal{M}$ *is a Lipschitzian manifold,*

- $\mathcal{M}$ *is positively invariant :*

$$\forall\, t \geq 0 \quad , \quad S(t)\,\mathcal{M} \subset \mathcal{M} ,$$

- $\mathcal{M}$ *attracts the orbits of* $(S(t))_{t\geq 0}$ *with an exponential speed,*

- $\mathcal{M}$ *is finite dimensional.*

When an inertial manifold exists, the system is very well approximated by the inertial system that we obtain by restricting $(S(t))_{t\geq 0}$ to $\mathcal{M}$.

We are interested in an equation of the form :

$$(1.4) \qquad \begin{cases} \dfrac{du}{dt} + Au + R(u) = 0 , \\ u(0) = u_0 , \end{cases}$$

in a Hilbert space H ; here A is an unbounded linear operator (for instance $A = (-\Delta)^r$, $r > 0$, with boundary conditions) and R is a locally Lipschitzian non linear function from H to $D(A^{-\gamma})$ $(0 \leq \gamma \leq \frac{1}{2})$.

(In fact, we are not able to prove the existence of an inertial manifold in that case, we will have to modify the function R to avoid problems for large values of the norm of u).

There already exist some methods to construct an inertial manifold. They all look for it as the graph of a Lipschtzian function and the main necessary hypothesis is a spectral gap condition on the operator A ; for instance, we do not know if there exists an inertial manifold for the Navier-Stokes equation (see C. Foias, G. Sell and R. Temam [1],[2] ; C. Foias, B. Nicolaenko, G. Sell and R. Temam [1],[2] ; P. Constantin, C. Foias, B. Nicolaenko and R. Temam [1],[2],[3] ; J. Mallet-Paret and G. Sell [1] ; S.N. Chow and K. Lu [1]). When that spectral gap condition is not satisfied, there often exist approximative inertial manifolds, that object has been introduced recently by C. Foias, O. Manley and R. Temam for the Navier-Stokes equation and has been extended to others (see C. Foias, O. Manley and R. Temam [1],[2] ; Temam [2] ; M. Marion [1],[2]).

In this paper, we are interested in another construction of inertial manifolds based on an equation of R.J. Sacker [1]. In that paper it was observed that if an invariant manifold is the graph of a function Φ, then this function must satisfy an hyperbolic equation. Although the article is devoted to finite dimension, the method applies as well in infinite dimension (with a function Φ taking its values in infinite dimension). Our aim in the present article is to show how one can construct an inertial manifold for a broad class of equations by solving this hyperbolic equation .Classically the hyperbolic equation is resolved by elliptic regularization. We are able to derive inertial manifolds in a constructive manner for the same type of equations as in the references quoted above, with the same type of hypotheses.

In a related work M. Luskin and G. Sell [1],E. Fabes, M. Luskin and G. Sell [1],[2] used this method to construct an inertial manifold, but in their article, the condition on the existence are much stronger than in the others references quoted above as far as the spectral gap condition and as far as the type of equation are concerned.In this article we consider the same class of equations as in the other references and recover the *same* type of gap condition, the method being furthermore construstive.

Let us describe the construction of the hyperbolic equation ; we first modify equation (1.4) by replacing the nonlinear term R by a truncated form R_θ that has a compact support to avoid problem with large values of the norm of u (this will be done precisely in the next section), we search an inertial manifold for the system :

$$\frac{du}{dt} + Au + R_\theta(u) = 0 \,, \tag{1.5}$$

where u lies in a Hilbert space H.

Let P be a finite dimensional orthogonal projector and $Q = Id_H\text{-}P$ (hence we have $H = PH \oplus QH$). We look for a function Φ from PH to QH whose graph is positively invariant for the equation (1.5). We project (1.5) on PH and on QH :

$$\frac{dp}{dt} + Ap + PR_\theta\,(p+q) = 0 \,, \tag{1.6}$$

$$\frac{dq}{dt} + Aq + QR_\theta\,(p+q) = 0 \,, \tag{1.7}$$

where $p(t) = Pu(t)$ and $q(t) = Qu(t)$ (we assumed that P and A commute, that will be satisfied [see section II]).

If u(0) is on the graph of Φ and if its graph is positively invariant :

$$\forall\, t \geq 0 \;:\; q(t) = \Phi(p(t)) \,.$$

Therefore :

$$\frac{d\Phi(p)}{dt} + A\Phi(p) + QR_\theta\,(p+\Phi(p)) = 0 \,. \tag{1.8}$$

Using (1.6), we have :

$$\frac{d\Phi(p)}{dt} = D\Phi(p)\left(\frac{dp}{dt}\right)$$
$$= D\Phi(p)\left(-Ap - PR_\theta\,(p+\Phi(p))\right)$$
$$= -\left(Ap + PR_\theta\,(p+\Phi(p))\right).\nabla\,\Phi(p)\,.$$

We infer from (1.8) that Φ is a solution of :

(1.9) $$-\left(Ap + PR_\theta\,(p+\Phi(p))\right).\nabla\,\Phi + A\Phi + QR_\theta\,(p+\Phi(p)) = 0\ .$$

R_θ has a compact support, let ρ be given such that the ball centered at 0 of radius $\rho/2$ in H contains its support, we will search Φ with a support in the ball centered at 0 of radius ρ in PH (this is natural since we will construct R_θ such that its support contains an absorbing set of the equation), thus we are lead to the Dirichlet boundary condition :

(1.10) $$\Phi = 0 \quad \text{on} \quad \partial B(0,\rho)\,.$$

This construction of a partial differential equation that characterizes invariant surfaces is due to R.J. Sacker [1]. He was interested in a perturbation of a differential equation that possesses an invariant torus, he used that equation to show that the perturbed equation has an invariant surface that is a graph on the torus ; his result is restricted by an asumption that is similar to the spectral gap condition. The method that we will use to solve (1.9) is similar to his but new difficulties appear :

- Φ lies in an infinite dimensional space (R.J. Sacker only considered the finite dimensional case).
- The a priori estimates are based on a maximum principle that is very easy to use when the quantity that we want to estimate has its maximum inside the domain, this is always the case on a torus but not in our case where the domain (which is a ball) has a boundary. This difficulty appears in lemma 4.4 here after.

In the following sections, we will first give the notation and the precise asumptions (section II), then we will give the existence theorem and a few remarks (section III) ; finally, in section IV, we will prove the theorem.

II <u>NOTATION AND HYPOTHESES</u>

Let H be a Hilbert space with the norm $|\,.\,|$ and scalar product $(.,.)$.

A denotes a linear closed unbounded positive self-adjoint operator in H with compact inverse ; under these asumptions, there exists an orthonormal basis of H consisting of eigenvectors of A : $(w_1,\ldots,w_k,\ldots)$, we denote by $(\lambda_1,\ldots,\lambda_k,\ldots)$ the associated eigenvectors :

$$(2.1)\qquad \begin{cases} \forall\, i,j \in \mathbb{N}^* : \\ (w_i,w_j) = \delta_{ij}\ , \\ Aw_i = \lambda_i\, w_i\ , \\ \text{and } \ 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \to +\infty\ . \end{cases}$$

Moreover we can define A^s for every s in $\mathbb{R}$, the domain $D(A^s)$ when endowed with the norm $|A^s.|$ is a Hilbert space.

P denotes the orthogonal projector on the space spanned by $(w_1,\ldots,w_N)$ (N is to be chosen later).

P_j denotes the j^{th} coordinate in PH :

$$(2.2)\qquad \forall\, u \in H\ :\qquad P_j u = (u,w_j)\ .$$

Q (resp. Q_ℓ, $\ell \ge N+1$) denotes the orthogonal projector on the space spanned by $(w_{N+1},\ldots,w_k,\ldots)$ (resp. $(w_{N+1},\ldots,w_\ell)$, $\ell \ge N+1$). $H_\ell = PH \oplus Q_\ell H$ is the space spanned by $(w_1,w_2,\ldots,w_\ell)$, it is endowed with the norm induced by the norm in H.

Let us notice that :

$$(2.3)\qquad \begin{cases} \forall\, s > 0\ ,\ \ \forall\, p \in PH : \lambda_1^s\, |p| \le |A^s p| \le \lambda_N^s\, |p|\ , \\ \forall\, s > 0\ ,\ \ \forall\, q \in QH : \lambda_{N+1}^s\, |q| \le |A^s q|\ . \end{cases}$$

R is a nonlinear operator from H to $D(A^{-\gamma})$ $(0 \le \gamma \le 1/2)$, Lipschitzian on the bounded sets of H and continuously differentiable. As we already said, we will replace R by a truncated form R_θ ; in order to do that, we make some further asumptions on the system (1.4). We assume that for every u_0 in H, (1.4) possesses a unique solution in $\mathcal{C}(\mathbb{R}^+,H) \cap L^2(0,T;D(A^{1/2}))$ for all $T > 0$. Moreover the semigroup $(S(t))_{t\ge 0}$ associated to (S) possesses a bounded absorbing set $\mathcal{B}_0$ in H which is positively invariant (these asumptions are often satisfied, see e.g R. Temam [1] for numerous examples).

Let us choose ρ such that the ball of H centered at 0 of radius $\rho/4$ contains $\mathcal{B}_0$, we choose a C^∞ function θ :

$$(2.4)\qquad \begin{cases} \theta : \mathbb{R}^+ \to [0,1]\ , \\ \theta(s) = 1 \quad \text{for } s \in [0,\rho/3]\ , \\ \theta(s) = 0 \quad \text{for } s \in [\rho/2,+\infty[\ , \\ \sup\limits_{s\ge 0} |\theta'(s)| \le 2\ . \end{cases}$$

We write :

$$R_\theta(u) = \theta(|u|)\, R(u) \quad , \quad \forall\, u \in H\,.$$

R_θ is a C^1 function from H to $D(A^{-\gamma})$ with a compact support ; thus it is bounded together with its differential.

From now on, we will always consider the truncated equation in which R_θ replaces R, thus we will omit the index θ in order to make notation simpler (except in section III.3).
Let us write :

(2.5) $$K_0 = \sup_{u\in H} \left| A^{-\gamma} R(u) \right| ,$$

(2.6) $$K_1 = \sup_{u\in H}\ \sup_{\substack{v\in H\\ |v|=1}} \left| A^{-\gamma} DR(u)(v) \right| .$$

Ω will be the ball in PH centered at 0 and of radius ρ :

$$\Omega = B_{PH}\,(0,\rho)\ .$$

Now we define some functional spaces that we will need :
$H^1_0(\Omega, Q_\ell H)$ is the space of the functions from Ω to $Q_\ell H$ which are L^2 together with their first partial derivatives and which equal zero on $\partial\Omega$, it is a Hilbert space when endowed with the norm :

(2.7) $$|\Phi|_{H^1_0} = \left(\left| A^{1/2}\Phi \right|^2_{L^2} + \left| \nabla\Phi \right|^2_{L^2} \right)^{1/2} ,$$

where

$$\left| A^{1/2}\Phi \right|^2_{L^2} = \int_\Omega \left| A^{1/2}\Phi(p) \right|^2 dp\ .$$

(this norm is equivalent to the usual one : $|\Phi|^2_{L^2} = \int_\Omega |\Phi(p)|^2\, dp$ on $L^2(\Omega, Q_\ell H)$ since $Q_\ell H$ is a finite dimensional space ; of course, this is no longer true on $L^2(\Omega, QH)$)
and

$$\left| \nabla\Phi \right|^2_{L^2} = \sum_{i=1}^{N} \int_\Omega \left| D_i\Phi(p) \right|^2 dp\ .$$

(p is a variable in Ω which is embedded in PH ; we write $p = \sum^{N=1} p_i w_i$, $D_i\Phi(p)$ is the partial derivative of Φ with respect to p_i).

$L^\infty(\Omega,QH)$ (resp. $L^\infty(\Omega,Q_\ell H)$, $L^\infty(\Omega,QD(A^{1/2}))$) is the space of the essentially bounded functions from Ω to QH (resp. $Q_\ell H$, $QD(A^{1/2})$), it is a Banach space when endowed with the norm :

$$|\Phi|_{L^\infty} = \operatorname*{ess\,sup}_{p\in\Omega} |\Phi(p)|$$

$$\Big(\text{resp. } |\Phi|_{L^\infty} = \operatorname*{ess\,sup}_{p\in\Omega} |\Phi(p)|$$
$$|A^{1/2}\Phi|_{L^\infty} = \operatorname*{ess\,sup}_{p\in\Omega} |A^{1/2}\Phi(p)|\Big)$$

$W^{1,\infty}(\Omega,QH)$ (resp. $W^{1,\infty}(\Omega,Q_\ell H)$) is the space of functions in $L^\infty(\Omega,QH)$ (resp. $L^\infty(\Omega,Q_\ell H)$) whose gradients are in $L^\infty(\Omega,QH)^N$ (resp. $L^\infty(\Omega,Q_\ell H)^N$), it is a Banach space when endowed with the norm :

(2.8) $$|\Phi|_{W^{1,\infty}} = |\Phi|_{L^\infty} + |D\Phi|_{L^\infty},$$

where :

$$|D\Phi|_{L^\infty} = \operatorname*{ess\,sup}_{p\in\Omega} \sup_{h\in\mathbb{R}^N} \left(\frac{|h.D\Phi(p)|}{|h|}\right),$$

and for $h = (h_1,\dots,h_N)$ in $\mathbb{R}^N$

$$h.D\Phi(p) = \sum_{i=1}^{N} h_i D_i\Phi(p) \quad , \quad |h| = \left(\sum_{i=1}^{N} h_i^2\right)^{1/2}$$

We will use the subspaces of $W^{1,\infty}(\Omega,Q_\ell H)$ and $W^{1,\infty}(\Omega,QH)$:

$$V_\ell = \{\Phi \in W^{1,\infty}(\Omega,Q_\ell H) : |\Phi|_{L^\infty} \le M_0 \text{ and } |D\Phi|_{L^\infty} \le M_1\},$$
$$V = \{\Phi \in W^{1,\infty}(\Omega,QH) \quad : |\Phi|_{L^\infty} \le M_0 \text{ and } |D\Phi|_{L^\infty} \le M_1\},$$

(M_0 and M_1 will be choosen later).

This section ends with the asumptions on the spectrum of A.
We choose M_0 such that :

(2.9) $$K_0\, \lambda_{N+1}^{\gamma-1} \le M_0 ,$$

(let us recall that $K_0 = \sup_{u\in H} |A^{-\gamma} R(u)|$) and we assume that there exists N such that :

(2.10) $$\lambda_{N+1} - \lambda_N \ge K_1 \left(\lambda_N^{\gamma/2} + \lambda_{N+1}^{\gamma/2}\right)^2 ,$$

(let us recall that $K_1 = \sup_{u\in H} |A^{-\gamma} DR(u)|_{\mathcal{L}(H)}$).

This is a necessary and sufficient condition to the existence of a real M_1 such that :

and $$\begin{cases} M_1 > 0 , \\ -K_1\, \lambda_N^{\gamma} M_1^2 + M_1 \left(\lambda_{N+1} - \lambda_N - K_1(\lambda_N^{\gamma} + \lambda_{N+1}^{\gamma})\right) - K_1\, \lambda_{N+1}^{\gamma} \ge 0 , \end{cases}$$

that implies :

(2.11) $$\lambda_{N+1} - \lambda_N - K_1(1+M_1)\, \lambda_N^{\gamma} > 0 ,$$

(2.12) $$\frac{K_1(1+M_1)\, \lambda_{N+1}^{\gamma}}{\lambda_{N+1} - \lambda_N - K_1(1+M_1)\, \lambda_N^{\gamma}} \le M_1 .$$

We make the further asumption that we can choose M_1 such that :

(2.13) $$\left(\sum_{i=1}^{N} \lambda_i\right) \ge K_1(1+M_1)\left(\sum_{i=1}^{N} \lambda_i^{\gamma}\right) ,$$

(2.14) $$\lambda_{N+1}^{\gamma-1} K_1(1+M_1) < 1 .$$

III <u>MAIN RESULT</u>

We state the main result, this is an existence theorem for the inertial manifold (the proof will be given in section IV). This section ends with some comments on the results.

1) *<u>THEOREM 3.1</u> : Under the asumptions made in section II, there exists a function* Φ *in* $W^{1,\infty}(\Omega,QH) \cap L^{\infty}(\Omega,QD(A^{1/2}))$ *solution of :*

$$(3.1) \qquad \begin{cases} -\big(Ap + PR(p+\Phi(p))\big).\nabla\Phi + A\Phi + QR(p+\Phi(p)) = 0 \quad \text{in } \Omega\,, \\ \Phi = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

Moreover, if we extend Φ *outside* Ω *by setting :* $\Phi|_{PH\setminus\Omega} = 0$, *then its graph is an inertial manifold for the system :*

$$\frac{du}{dt} + Au + R(u) = 0\,.$$

2) (2.10) is a spectral gap condition, this is the most restricting condition among (2.10)-(2.14) ; for instance if $\lambda_N \sim c\,N^{\alpha}$ (this is the case when $A = (-\Delta)^r$ on a n-cube in $\mathbb{R}^n$ with periodic boundary conditions, then $\alpha = \frac{2r}{n}$ (see R. Courant and D. Hilbert [1])), in order to satisfy (2.10) we need $\alpha > \frac{1}{1-\gamma}$ and (2.13), (2.14) are then easily checked. If we consider the Navier-Stockes equation, we have $\gamma = \frac{1}{2}$ and $\alpha = \frac{2}{n}$, thus our result is not sufficient in this case.

3) Theorem 3.1 gives the existence of an inertial manifold for the truncated system :

$$(3.2) \qquad \frac{du}{dt} + Au + R_\theta(u) = 0\,.$$

Consider an orbit $\{u(t)\}_{t\geq 0}$ of the initial system :

$$(3.3) \qquad \frac{du}{dt} + Au + R(u) = 0\,,$$

as R_θ and R are equal on the absorbing set of (3.3), there exists a time t_0 such that :

$$\forall\, t \geq t_0 \;,\; R(u(t)) = R_\theta(u(t))\,,$$

thus $\{u(t)\}_{t\geq t_0}$ is an orbit of (3.2) and if we denote by $\mathcal{M}$ the inertial manifold of system (3.2) :

either $\quad u(t) \in \mathcal{M}$ for $t \geq t_0$

or $\quad d(u(t),\mathcal{M})$ decays exponentially.

4) (3.1) is an hyperbolic equation in the unknown Φ, it is quite natural to try the method of characteristics to solve it ; we see the term $-(Ap + PR(p+\Phi)).\nabla\Phi$ as a directional derivative. We introduce the curves :

$$\begin{cases} \frac{dp}{dt} = -\,(Ap + PR(p+\Phi))\,, \\ p(0) = p_0\,, \end{cases}$$

on these curves, we have :

$$\frac{d\Phi(p(t))}{dt} + A\Phi(p(t)) + QR(p(t) + \Phi(p(t))) = 0\,,$$

which can be integrated :

$$\Phi(p(t)) = -\int_{t_0}^{t} e^{A(\sigma-t)}QR\big(p(\sigma) + \Phi(p(\sigma))\big)\,d\sigma$$
$$+ e^{A(t_0-t)}\,\Phi(p(t_0))\,.$$

Letting t_0 go to $-\infty$ (assuming Φ is bounded) and taking t equals 0:

$$\Phi(p_0) = -\int_{-\infty}^{0} e^{A\sigma}QR\big(p(\sigma) + \Phi(p(\sigma))\big)\,d\sigma\,.$$

And we can solve this integral equation by a fixed-point method.
This is the principle of the Lyapunov-Perron method developped by C. Foias, G. Sell and R. Temam [1],[2] (see also R. Temam [1] chap. 8 and S.N. Chow and K. Lu [1]).

IV <u>PROOF OF THEOREM 3.1</u>

In order to prove theorem 3.1, we will first solve a Galerkin approximation of equation 3.1 (which is an infinite dimensional hyperbolic system, whose unknowns are the coordinates of Φ in the base $(w_{N+1},\dots,w_k,\dots)$).

Let $\ell \geq N+1$, we search Φ_ℓ from Ω to $Q_\ell H$ solution of:

$$\begin{cases} -\big(Ap + PR(p+\Phi_\ell(p))\big).\nabla\Phi_\ell + A\Phi_\ell + Q_\ell R(p+\Phi_\ell(p)) = 0 \quad \text{in } \Omega\,, \\ \Phi_\ell = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

This is done by regularizing this system with a second order elliptic term that tends to zero. Under the asumptions made in section II (especially (2.10)), we can find a solution of the regularized system through a fixed point method in a subset of $W^{1,\infty}(\Omega,Q_\ell H)$, in the same time we obtain a priori estimates that enable us to pass to the limit to obtain a solution of the Galerkin approximation of equation (3.1), then to make $\ell \to +\infty$ and thus to have a solution of equation (3.1).

<u>*LEMMA 4.1*</u> *: Let* $\psi \in V_\ell$, *then there exists a unique* Φ *in* $H_0^1(\Omega,Q_\ell H)$ *such that :*

$$\text{(4.1)}\qquad \begin{cases} \varepsilon\,\Delta\Phi + (Ap + PR(p+\psi)).\nabla\Phi - A\Phi - Q_\ell R(p+\psi) = 0\,, \\ \Phi = 0 \quad \textit{on} \quad \partial\Omega\,. \end{cases}$$

We write $\Phi = \mathcal{T}_{\varepsilon,\ell}\, \psi.$

(Let us recall that $V_\ell = \left\{\Phi \in W^{1,\infty}(\Omega, Q_\ell H) \,/\, |\Phi|_{L^\infty} \le M_0 \text{ and } |D\Phi|_{L^\infty} \le M_1\right\}$ *).*

Proof

Let $a(\Phi_1,\Phi_2)$ be the bilinear continuous form on $H_0^1(\Omega, Q_\ell H)$ defined by :

$$a(\Phi_1,\Phi_2) = \varepsilon \int_\Omega \sum_{i=1}^{N} (D_i\Phi_1, D_i\Phi_2)\, dp - \int_\Omega \left((Ap + PR(p+\psi)).\nabla\Phi_1, \Phi_2\right) dp + \int_\Omega \left(A^{1/2}\Phi_1, A^{1/2}\Phi_2\right) dp\,,$$

and L the linear continuous form on $H_0^1(\Omega, Q_\ell H)$:

$$L(\Phi_2) = -\int_\Omega \left(A^{-\gamma} Q_\ell R(p+\psi), A^{\gamma}\Phi_2\right) dp\,.$$

We claim that a is coercive :

Let $\Phi \in H_0^1(\Omega, Q_\ell H)$,

$$a(\Phi,\Phi) = \varepsilon \int_\Omega \sum_{i=1}^{N} |D_i\Phi|^2\, dp - \int_\Omega \left((Ap + PR(p+\psi)).\nabla\Phi, \Phi\right) dp + \int_\Omega |A^{1/2}\Phi|^2\, dp\,,$$

We observe that :

$$-\int_\Omega \left((Ap + PR(p+\psi)).\nabla\Phi, \Phi\right) dp = -\frac{1}{2}\int_\Omega (Ap + PR(p+\psi)).\nabla|\Phi|^2\, dp = \frac{1}{2}\int_\Omega \operatorname{div}\left(Ap + PR(p+\psi)\right)|\Phi|^2\, dp\,,$$

and this is positive since :

$$\begin{aligned}\operatorname{div}(Ap + PR(p+\psi)) &= \sum_{i=1}^{N} D_i\big(\lambda_i p_i + P_i R(p+\psi(p))\big)\\ &= \sum_{i=1}^{N} \lambda_i + P_i\, DR(p+\psi)\,(w_i + D_i\psi)\\ &= \sum_{i=1}^{N} \lambda_i + \sum_{i=1}^{N} \lambda_i^{-\gamma} P_i\, DR(p+\psi)\big(\lambda_i^{\gamma}(w_i + D_i\psi)\big)\\ &\geq \sum_{i=1}^{N} \lambda_i - \sum_{i=1}^{N} \left|A^{-\gamma} PDR(p+\psi)\big(\lambda_i^{\gamma}(w_i + D_i\psi)\big)\right|\\ &\geq \sum_{i=1}^{N} \lambda_i - K_1 \sum_{i=1}^{N} \left|\lambda_i^{\gamma}(w_i + D_i\psi)\right|\\ &\geq \sum_{i=1}^{N} \lambda_i - K_1\left(\sum_{i=1}^{N} \lambda_i^{\gamma}\right)(1+M_1) \quad .\end{aligned}$$

(Let us recall that

$$K_1 = \sup_{u\in H} |A^{-\gamma} DR(u)|_{\mathcal{L}(H)}$$

and

$$|D\psi|_{L^\infty} = \operatorname*{ess\,sup}_{p\in\Omega} \sup_{h\in\mathbb{R}^N} \frac{\left|\sum_{i=1}^{N} h_i D_i\psi\right|}{\left(\sum_{i=1}^{N} h_i^2\right)^{1/2}}$$

$$\leq M_1 \quad . \qquad)$$

Thus :

$$a(\Phi,\Phi) \geq \inf(\varepsilon,1)\, |\Phi|^2_{H_0^1} \quad .$$

By the Lax-Milgram theorem, there exists a unique Φ in $H_0^1(\Omega,Q_\ell H)$ such that :

$$a(\Phi,\Phi_1) = L(\Phi_1) \text{ for every } \Phi_1 \text{ in } H_0^1(\Omega,Q_\ell H) \; .$$

Taking Φ_1 C^∞ with a compact support, we have :

$$\varepsilon\,\Delta\Phi + (Ap + PR(p+\psi)).\nabla\Phi - A\Phi - Q_\ell R(p+\psi) = 0 \, ,$$

in the sense of distributions, that implies that Φ is in $H^2(\Omega,Q_\ell H)$ and that (4.1) is satisfied almost everywhere in Ω.

<u>*LEMMA 4.2*</u> *(Continuity of $\mathcal{T}_{\varepsilon,\ell}$) .*

If V_ℓ is endowed with the norm of $L^\infty(\Omega,Q_\ell H)$, then

$$\mathcal{T}_{\varepsilon,\ell} : V_\ell \to H_0^1(\Omega,Q_\ell H)$$

is continuous.

Proof

Let $(\psi_n)_{n\in N}$ be a sequence in V_ℓ such that

$$\psi_n \to \psi \text{ in } L^\infty(\Omega, Q_\ell H).$$

We write $\Phi_n = \mathcal{T}_{\varepsilon,\ell}\, \psi_n$ and $\Phi = \mathcal{T}_{\varepsilon,\ell}\, \psi$ and substract the equations satisfied by Φ_n and Φ :

$$\begin{aligned}
&\varepsilon\, \Delta\big(\Phi_n\text{-}\Phi\big) + (Ap + PR(p+\psi_n)).\nabla\big(\Phi_n\text{-}\Phi\big)\\
&+ (PR(p+\psi_n) - PR(p+\psi)).\nabla\Phi - A\big(\Phi_n\text{-}\Phi\big)\\
&- (Q_\ell\, R(p+\psi_n) - Q_\ell\, R(p+\psi)) = 0\,.
\end{aligned}$$

We take the scalar product with $\Phi_n\text{-}\Phi$ in $Q_\ell H$ and integrate over Ω (by parts in the two first terms, there are no integral over $\partial\Omega$ since $\Phi_n\text{-}\Phi$ is in $H_0^1(\Omega, Q_\ell H)$) :

$$\begin{aligned}
&-\varepsilon\int_\Omega \left|\nabla\big(\Phi_n\text{-}\Phi\big)\right|^2 dp - \frac{1}{2}\int_\Omega \operatorname{div}\big(Ap + PR(p+\psi_n)\big)\left|\Phi_n\text{-}\Phi\right|^2 dp\\
&+ \int_\Omega \Big((PR(p+\psi_n) - PR(p+\psi)).\nabla\Phi, \Phi_n\text{-}\Phi\Big)\, dp\\
&- \int_\Omega \left|A^{1/2}\big(\Phi_n\text{-}\Phi\big)\right|^2 dp\\
&- \int_\Omega \Big(A^{-\gamma}(Q_\ell R(p+\psi_n) - Q_\ell R(p+\psi)), A^{\gamma}\big(\Phi_n\text{-}\Phi\big)\Big)\, dp\\
&= 0\,.
\end{aligned}$$

The second term is negative since :

$$\operatorname{div}\big(Ap + PR(p+\psi_n)\big) \geq 0 \quad \text{(see lemma 1).}$$

Thus :

$$\begin{aligned}
&-\varepsilon\int_\Omega \left|\nabla\big(\Phi_n\text{-}\Phi\big)\right|^2 dp + \int_\Omega \left|A^{1/2}\big(\Phi_n\text{-}\Phi\big)\right|^2 dp\\
&\leq \int_\Omega \Big((PR(p+\psi_n) - PR(p+\psi)).\nabla\Phi, \Phi_n\text{-}\Phi\Big)\, dp\\
&- \int_\Omega \Big(A^{-\gamma}(Q_\ell R(p+\psi_n) - Q_\ell R(p+\psi)), A^{\gamma}\big(\Phi_n\text{-}\Phi\big)\Big)\, dp\\
&\leq \lambda_N^{\gamma} \sup_{p\in\Omega} \left|A^{-\gamma}(PR(p+\psi_n) - PR(p+\psi))\right| \left|\nabla\Phi\right|_{L^2} \left|\Phi_n\text{-}\Phi\right|_{L^2}\\
&+ \lambda_{N+1}^{\gamma-1/2} \sup_{p\in\Omega} \left|A^{-\gamma}(Q_\ell R(p+\psi_n) - Q_\ell R(p+\psi))\right| \big(\text{mes }\Omega\big)^{1/2} \left|A^{1/2}\big(\Phi_n\text{-}\Phi\big)\right|_{L^2}\\
&\leq K_1 \Big(\left|\nabla\Phi\right|_{L^2} + \lambda_{N+1}^{\gamma-1/2}\text{ mes }(\Omega)^{1/2}\Big) \left|\psi_n\text{-}\psi\right|_{L^\infty} \left|A^{1/2}\big(\Phi_n\text{-}\Phi\big)\right|_{L^2}\,.
\end{aligned}$$

Therefore if $|\psi_n - \psi| \to 0$ then

$$\left|A^{1/2}(\Phi_n - \Phi)\right|_{L^2} \to 0$$

and

$$\left|\nabla(\Phi_n - \Phi)\right|_{L^2} \to 0 \ .$$

LEMME 4.3 (L^∞ *estimate)*

Let ψ *in* V_ℓ *and* $\Phi = \mathcal{T}_{\varepsilon,\ell}$ *then* $\Phi \in L^\infty(\Omega, Q_\ell H)$ *and*

(4.3) $$|\Phi|_{L^\infty} \le M_0 \ .$$

Proof

ψ is a Lipschitzian function (it is in $W^{1,\infty}(\Omega, Q_\ell H)$), hence it is in $C^{0,s}(\Omega, Q_\ell H)$ for all $s < 1$ ($C^{0,s}(\Omega, Q_\ell H)$ is the space of Hölderian continuous function of order s).

R is a C^1 function, thus the coefficients of equation (4.1) are in $C^{0,s}(\Omega, Q_\ell H)$, and thanks to the classical results on the regularity of the elliptic equations (see D. Gilbarg and N.S. Trudinger [1]) :

$$\Phi \in C^{2,s}(\Omega, Q_\ell H) \ .$$

We deduce that (4.1) is verified everywhere.

Let us take the scalar product of (4.1) with Φ, and use :

(4.4) $$(\Delta\Phi, \Phi) = \frac{1}{2}\Delta|\Phi|^2 - |\nabla\Phi|^2 \ ,$$

(4.5) $$(\nabla\Phi, \Phi) = \frac{1}{2}\nabla|\Phi|^2 \ .$$

We obtain :

$$\frac{\varepsilon}{2}\Delta|\Phi|^2 - \varepsilon|\nabla\Phi|^2 + \frac{1}{2}(Ap + PR(p+\psi)).\nabla|\Phi|^2$$
$$- \left|A^{1/2}\Phi\right|^2 - \left(A^{-\gamma}Q_\ell R(p+\psi), A^\gamma\Phi\right) = 0 \ .$$

At the point $\bar{p}$ where $|\Phi|^2$ is maximum, we have :

$$\Delta|\Phi(\bar{p})|^2 = 0 \quad \text{and} \quad \nabla|\Phi(\bar{p})|^2 = 0 \ .$$

($\bar{p}$ is in the interior of Ω, since $\Phi = 0$ on $\partial\Omega$).

Hence :

$$\begin{aligned} \left|A^{1/2}\Phi(\bar{p})\right|^2 &\le -\left(A^{-\gamma}Q_\ell R(\bar{p}+\psi(\bar{p})), A^\gamma\Phi(\bar{p})\right) \\ &\le \left|A^{-\gamma}Q_\ell R(\bar{p}+\psi(\bar{p}))\right|\left|A^\gamma\Phi(\bar{p})\right| \\ &\le K_0\,\lambda_{N+1}^{\gamma-1/2}\left|A^{1/2}\Phi(\bar{p})\right| \ , \end{aligned}$$

thanks to (2.5).

We deduce :

$$\begin{aligned}|\Phi|_{L^\infty} = |\Phi(\bar{p})| &\leq \lambda_{N+1}^{-1/2} |A^{1/2}\,\Phi(\bar{p})| \\ &\leq \lambda_{N+1}^{\gamma-1}\,K_0 \ , \\ |\Phi|_{L^\infty} \leq M_0 \ ,\end{aligned}$$

thanks to (2.9).

<u>*LEMMA 4.4*</u> *($W^{1,\infty}$ estimate)*

Let ψ in V_ℓ and $\Phi = \mathcal{T}_{\varepsilon,\ell}\,\psi$ then $\Phi \in W^{1,\infty}(\Omega, Q_\ell H)$ and

$$|D\Phi|_{L^\infty} \leq M_1 \ .$$

Using lemma 4.3, we have :

$$\mathcal{T}_{\varepsilon,\ell}\,V_\ell \subset V_\ell \ .$$

<u>Proof</u>

We want to use the same method as in lemma 4.3 to obtain estimates on the derivatives of Φ, we will differentiate equation (4.1) and use the same type of arguments. But two difficulties appear :

- we do not know wether the derivatives of Φ are sufficiently regular
- the maximum of the derivatives of Φ can occur on the boundary.

The first difficulty is easily overcome by approximating ψ with a sequence of more regular functions.

The second is more difficult to overcome, it is done by looking carefully what happens on the boundary if the maximum occurs there ; the arguments employed are hidden by technical calculous, thus we will first describe the method in a simpler case.

We now begin with the proof.

We first assume that ψ is in $C^1(\Omega, Q_\ell H)$ then we have :

(4.7) $$\Phi \in C^3(\Omega, Q_\ell H) \ .$$

(See D. Gilbarg and N.S. Trudinger [1]).

We differentiate (4.1) with respect to p_i $(1 \leq i \leq N)$:

(4.8) $$\begin{cases} \varepsilon \Delta D_i \Phi + (Ap + PR(p+\psi)).\nabla D_i\Phi + \lambda_i D_i\Phi \\ + \sum_{j=1}^{N} P_j\, DR(p+\psi)\, (w_i + D_i\psi)\, D_j\Phi \\ - A D_i\Phi - Q_\ell\, DR(p+\psi)\, (w_i + D_i\psi) = 0 \ . \end{cases}$$

Thanks to (4.7), (4.8) is satisfied everywhere in Ω.

We choose $h = (h_1,\ldots,h_N)$ in $\mathbb{R}^N$, multiply (4.8) by h_i, make the sum for $i = 1,\ldots,N$ and take the scalar product with $h.D\Phi = \sum_{i=1}^{N} h_i D_i \Phi$, using the analog of (4.4), (4.5) :

$$\left(\Delta\, h.D\Phi, h.D\Phi\right) = \frac{1}{2}\ \Delta\left|h.D\Phi\right|^2 - \left|\nabla h.D\Phi\right|^2 \ ,$$
$$\left(\nabla\, h.D\Phi, h.D\Phi\right) = \frac{1}{2}\ \nabla\left|h.D\Phi\right|^2 \ ,$$

we obtain :

$$\begin{aligned}
&\frac{\varepsilon}{2}\ \Delta\left|h.D\Phi\right|^2 - \varepsilon\left|\nabla h.D\Phi\right|^2 + \frac{1}{2}(Ap + PR(p+\psi)).\nabla\left|h.D\Phi\right|^2 \\
&+\left(\sum_{i=1}^{N} \lambda_i h_i D_i \Phi, h.D\Phi\right) \\
&+\left(\sum_{i,j=1}^{N} h_i P_j DR(p+\psi)\,(w_i+D_i\psi)\, D_j\Phi, h.D\Phi\right) \qquad (4.9)\\
&-\left|A^{1/2}h.D\Phi\right|^2 - \left(\sum_{i=1}^{N} h_i Q_\ell DR(p+\psi)\,(w_i+D_i\psi), h.D\Phi\right) \\
&= 0 \ .
\end{aligned}$$

We will consider two cases :

- <u>1st case</u> : $|h.D\Phi|^2$ attains its maximum at a point $\bar{p}$ in the interior of Ω, then we have :

$$\Delta|h.D\Phi(\bar{p})|^2 \leq 0 \ , \qquad (4.10)$$

$$\nabla|h.D\Phi(\bar{p})|^2 = 0 \ . \qquad (4.11)$$

We estimate the other terms in (4.9) at the point $\bar{p}$:

$$\begin{aligned}
&\left(\sum_{i,j=1}^{N} h_i P_j DR(\bar{p}+\psi)\,(w_i+D_i\psi)\, D_j\Phi, h.D\Phi\right) \qquad (4.12)\\
&= \left(\sum_{j=1}^{N} P_j DR(\bar{p}+\psi)\left(\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right) D_j\Phi, h.D\Phi\right) \\
&\leq \left|\sum_{j=1}^{N} P_j DR(\bar{p}+\psi)\left(\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right) D_j\Phi\right|\, |h.D\Phi| \\
&\leq \left|P\, DR(\bar{p}+\psi)\left(\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right)\right|\, |D\Phi|_{L^\infty}\, |h.D\Phi| \\
&\leq \lambda_N^{\gamma} K_1 \left|\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right|\, |D\Phi|_{L^\infty}\, |h.D\Phi| \\
&\leq \lambda_N^{\gamma} K_1\, |h|\, \left(1+|D\psi|_{L^\infty}\right) |D\Phi|_{L^\infty}\, |h.D\Phi| \\
&\leq \lambda_N^{\gamma} K_1\, |h|\, (1+M_1)\, |D\Phi|_{L^\infty}\, |h.D\Phi| \ .
\end{aligned}$$

$$(4.13)\qquad \left(\sum_{i=1}^{N} \lambda_i\, h_i\, D_i\Phi, h.D\Phi\right) \le \left|\sum_{i=1}^{N} \lambda_i\, h_i\, D_i\Phi\right| |h.D\Phi|$$
$$\le \left(\sum_{i=1}^{N} \lambda_i^2\, h_i^2\right)^{1/2} |D\Phi|_{L^\infty}\, |h.D\Phi|$$
$$\le \lambda_N\, |h|\, |D\Phi|_{L^\infty}\, |h.D\Phi|\ .$$

$$|A^{1/2}h.D\Phi|^2 + \left(\sum_{i=1}^{N} h_i\, Q_\ell\, DR(\overline{p}+\psi)\,(w_i+D_i\psi), h.D\Phi\right)$$
$$\ge |A^{1/2}\, h.D\Phi|^2 - \left|A^{-\gamma}\, Q_\ell\, DR(\overline{p}+\psi)\left(\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right)\right|\, |A^{\gamma}\, h.D\Phi|$$
$$\ge |A^{1/2}\, h.D\Phi|^2 - K_1 \left|\sum_{i=1}^{N} h_i(w_i+D_i\psi)\right| |A^{\gamma}\, h.D\Phi|$$
$$\ge |A^{1/2}\, h.D\Phi|^2 - K_1\, \lambda_{N+1}^{\gamma-1/2}\, |h|\,(1+M_1)\, |A^{1/2}\, h.D\Phi|\ .$$

If $\lambda_{N+1}^{1/2}\,|h.D\Phi| \ge \frac{K_1}{2}\,\lambda_{N+1}^{\gamma-1/2}\,|h|\,(1+M_1)$, as the function $x^2 - K_1\,\lambda_{N+1}^{\gamma-1/2}\,|h|\,(1+M_1)\,x$ is increasing when $x \ge \frac{K_1}{2}\,\lambda_{N+1}^{\gamma-1/2}\,|h|\,(1+M_1)$, we have :

$$(4.14)\qquad |A^{1/2}h.D\Phi|^2 + \left(\sum_{i=1}^{N} h_i\, Q_\ell\, DR(\overline{p}+\psi)\,(w_i+D_i\psi), h.D\Phi\right)$$
$$\ge \lambda_{N+1}\,|h.D\Phi|^2 - K_1\,\lambda_{N+1}^{\gamma}\,|h|\,(1+M_1)\,|h.D\Phi|\ .$$

(4.10) to (4.14) with (4.9) give :

$$\lambda_{N+1}\,|h.D\Phi|^2 \le \lambda_N\,|h|\,|D\Phi|_{L^\infty}\,|h.D\Phi|$$
$$+ \lambda_N^{\gamma}\, K_1\,|h|\,(1+M_1)\,|D\Phi|_{L^\infty}\,|h.D\Phi|$$
$$+ \lambda_{N+1}^{\gamma}\, K_1\,|h|\,(1+M_1)\,|h.D\Phi|\,,$$

$$(4.15)\qquad \lambda_{N+1}\,|h.D\Phi| \le |h|\left(\lambda_N\,|D\Phi|_{L^\infty} + K_1\,\lambda_N^{\gamma}\,(1+M_1)\,|D\Phi|_{L^\infty} + \lambda_{N+1}^{\gamma}\, K_1\,(1+M_1)\right)\ .$$

If $\lambda_{N+1}^{1/2}\,|h.D\Phi| \le \frac{K_1}{2}\,\lambda_{N+1}^{\gamma-1/2}\,|h|\,(1+M_1)$, then $\lambda_{N+1}\,|h.D\Phi| \le \frac{K_1}{2}\,\lambda_{N+1}^{\gamma}\,|h|\,(1+M_1)$ and (4.15) is still satisfied. As $|h.D\Phi|$ attains its maximum at $\overline{p}$, we have :

$$(4.16)\qquad \begin{cases} \forall\, p \in \Omega : \\ \lambda_{N+1}\left|h.D\Phi\right| \le |h|\left(\left(\lambda_N + K_1\lambda_N^{\gamma}(1+M_1)\right)\left|D\Phi\right|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right) . \end{cases}$$

• <u>2nd case</u> : $|h.D\Phi|$ attains its maximum at a point $\bar{p}$ on the boundary.
In order to describe the arguments we will use, we first give a heuristic proof in a simpler case : we replace Ω by the N-cube $[-\rho,\rho]^N$ (the following arguments are not valid since we do not have enough regularity in such a domain).

$\bar{p} = (\bar{p}_1,\bar{p}_2,\ldots,\bar{p}_N)$ is on the boundary if there exists i such that $|\bar{p}_i| = \rho$, we assume, and this is no restriction, that $\bar{p}_1 = \rho$.

Φ is constant on the bounary thus all tangential derivatives are zero at $\bar{p}$:

$$\begin{aligned} &D_i\,\Phi(\bar{p})=0 \quad \text{if } i \ne 1 , \\ &\left|h.D\Phi(\bar{p})\right| = \left|h_1\, D_1\Phi(\bar{p})\right| \\ &\qquad\qquad \le |h|\left|D_1\Phi(\bar{p})\right| . \end{aligned}$$

We claim that $\left|D_1\Phi\right|$ cannot attain its maximum on the boundary, thus we can use the estimate (4.16) of the first case with $h = (1,0,\ldots,0)$:

$$\lambda_{N+1}\left|D_1\,\Phi(\bar{p})\right| \le \left(\left(\lambda_N + K_1\lambda_N^{\gamma}(1+M_1)\right)\left|D\Phi\right|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right),$$

$$\begin{aligned} \lambda_{N+1}\left|h.D\Phi(\bar{p})\right| &\le \lambda_{N+1}|h|\left|\,D_1\Phi\,(\bar{p})\right| \\ &\le |h|\left(\left(\lambda_N + K_1\,\lambda_N^{\gamma}(1+M_1)\right)\left|D\Phi\right|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right) . \end{aligned}$$

And (4.16) is valid. We have to prove the claim ; assume that $\left|\,D_1\Phi\,\right|$ attains its maximum at p_0 on the boundary, p_0 must be on the face $|p_1| = \rho$ (otherwise $\left|D_1\Phi(p_0)\right| = 0$), assume that $p_{0,1} = \rho$, then we have :

$D_i\,\Phi(p_0) = 0$, if $i \neq 1$,
$D_{ii}\,\Phi(p_0) = 0$, if $i \neq 1$,
$R\,(p_0 + \psi(p_0)) = 0$, since $|p_0 + \psi(p_0)| \geq |p_0| \geq \rho$,
$\Phi(p_0) = 0$, since p_0 is on the boundary .

Using equation (4.1) :

(4.17) $$\varepsilon\, D_{1,1}\,\Phi(p_0) + \lambda_1\,\rho\, D_1\,\Phi(p_0) = 0\,,$$

we take the scalar product of (4.17) with $D_1\Phi(p_0)$:

(4.18) $$\frac{\varepsilon}{2}\, D_1|D_1\Phi(p_0)|^2 + \lambda_1\,\rho\,|D_1\Phi(p_0)|^2 = 0\,.$$

But $|D_1\Phi(p_0)|$ is maximum, thus :

$$|D_1\Phi(p_0)| > 0\,,$$
$$D_1|D_1\Phi(p_0)| \geq 0\,.$$

This contradicts (4.18), hence $|D_1\Phi|$ cannot obtain its maximum on the boundary.
The proof on that case is complete but it is not valid since Φ is not regular enough.

Let us consider the original problem, the calculous are a bit tedious because on the boundary we have to use local coordinates.
Φ is constant on $\partial\Omega$, thus all tangential derivatives are zero on $\partial\Omega$.
We note $\vec{n}(p) = (n_1(p),\ldots,n_N(p))$ the unit outward normal at $p \in \partial\Omega$ and $\frac{\partial\Phi}{\partial r}$ the derivatives in the direction $\vec{n}$, we have :

$$h.D\Phi(p) = \sum_{i=1}^{N} h_i\, D_i\,\Phi(p) = (h.\vec{n}(p))\frac{\partial\Phi}{\partial r}(p)\,, \quad \text{when } p \in \partial\Omega\,.$$

We deduce :

(4.19) $$|h.D\Phi(\overline{p})| \leq |h|\left|\frac{\partial\Phi}{\partial r}(\overline{p})\right|\,.$$

We will estimate $\left|\frac{\partial\Phi}{\partial r}(\overline{p})\right|$. First, we claim that $\left|\frac{\partial\Phi}{\partial r}\right|$ (which is define everywhere in Ω) does not attain its maximum on the boundary. Indeed, if we assume that $\left|\frac{\partial\Phi}{\partial r}\right|$ attains its maximum at p_0 on the boundary. Let $(\tau_1,\ldots,\tau_{n-1})$ be a local coordinate system on $\partial\Omega$ near p_0, then :

$$\Delta\Phi = \sum_{i,j,k} \frac{\partial^2\Phi}{\partial\tau_j\,\partial\tau_k} \frac{\partial\tau_k}{\partial p_i} \frac{\partial\tau_j}{\partial p_i}$$
$$+ \frac{\partial^2\Phi}{\partial r^2} + \frac{N\text{-}1}{r} \frac{\partial\Phi}{\partial r}$$
$$+ 2 \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\,\partial r} \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{r}$$
$$+ \sum_{i,j} \frac{\partial\Phi}{\partial\tau_j} \frac{\partial^2\tau_j}{\partial p_i^{\,2}} \;.$$

Thus, at p_0, we have :

(4.20) $$\Delta\Phi(p_0) = \frac{\partial^2\Phi}{\partial r^2}(p_0) + \frac{N\text{-}1}{\rho} \frac{\partial\Phi}{\partial r}(p_0)$$
$$+ 2 \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\,\partial r}(p_0) \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{\rho} \;.$$

(the others terms are zero since they are tangential derivatives).

Moreover :

(4.21) $$Ap_0.\nabla\Phi(p_0) = \big(Ap_0.\vec{n}(p_0)\big) \frac{\partial\Phi}{\partial r}(p_0) \;,$$

(4.22) $$R(p_0 + \psi(p_0)) = 0 \quad , \quad \big(|p_0 + \psi(p_0)| \geq |p_0| \geq \rho\big) \;,$$

(4.23) $$\Phi(p_0) = 0 \;.$$

Using (4.19) to (4.23) in (4.1) :

(4.24) $$\varepsilon \frac{\partial^2\Phi}{\partial r^2} + 2\varepsilon \sum_{i,j} \frac{\partial^2\Phi}{\partial\tau_j\partial r} \frac{\partial\tau_j}{\partial p_i} \frac{p_i}{\rho}$$
$$+ \left(\varepsilon \frac{N\text{-}1}{\rho} + Ap_0.\vec{n}\right) \frac{\partial\Phi}{\partial r} = 0 \;.$$

We take the scalar product of (4.24) with $\frac{\partial\Phi}{\partial r}(p_0)$:

$$(4.25)\qquad \frac{\varepsilon}{2}\frac{\partial}{\partial r}\left(\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2\right) + \varepsilon \sum_{i,j} \frac{\partial}{\partial \tau_j}\left(\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2\right)\frac{\partial \tau_j}{\partial p_i} \times \frac{p_i}{\rho} + \left(\varepsilon \frac{N-1}{\rho} + Ap_0.\vec{n}\right)\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2 = 0\ .$$

(This is the analog of (4.18)).

We assumed that $\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2$ is maximum :

$$\frac{\partial}{\partial r}\left(\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2\right) \geq 0\ , \quad \frac{\partial}{\partial \tau_j}\left(\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2\right) = 0\ ,$$

$$\left|\frac{\partial \Phi}{\partial r}(p_0)\right|^2 > 0\ ,$$

and this contradicts (4.25). Hence $\left|\frac{\partial \Phi}{\partial r}\right|^2$ cannot have a maximum on the boundary and there exists a point p_1 inside Ω such that $\left|\frac{\partial \Phi}{\partial r}(p_1)\right|^2$ is maximum. We denote by $\vec{m} = (m_1,\ldots,m_N)$ the outward normal at p_1, then :

$$(4.26)\qquad \frac{\partial \Phi}{\partial r}(p_1) = \sum_{i=1}^{N} m_i\, D_i\, \Phi(p_1)\ .$$

Let p be a point on the boundary :

$$\left|\sum_{i=1}^{N} m_i\, D_i\, \Phi(p)\right| = \left|(\vec{m}.\vec{n}(p))\frac{\partial \Phi}{\partial r}(p)\right| \leq \left|\frac{\partial \Phi}{\partial r}(p)\right| < \left|\frac{\partial \Phi}{\partial r}(p_1)\right|\ ,$$

$$(4.27)\qquad \left|\sum_{i=1}^{N} m_i\, D_i\, \Phi(p)\right| < \left|\sum_{i=1}^{N} m_i\, D_i\, \Phi(p_1)\right|\ .$$

We deduce that the maximum of $\left|\sum_{i=1}^{N} m_i D_i \Phi(p)\right|$ occurs inside Ω, we can use the estimate (4.16) with $h = (m_1, \ldots, m_N)$:

(4.28) $$\begin{cases} \forall\, p \in \Omega : \\ \lambda_{N+1} \left|\sum_{i=1}^{N} m_i D_i \Phi(p)\right| \le |m| \left(\left(\lambda_N + K_1(1+M_1)\, \lambda_N^{\gamma}\right) |D\Phi|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right). \end{cases}$$

Using (4.19), (4.26), (4.28) at p_1 and $|m| = 1$, we have for all p in Ω :

$$\begin{aligned} \lambda_{N+1} |h.D\Phi(p)| &\le \lambda_{N+1} |h.D\Phi(\overline{p})| \\ &\le \lambda_{N+1} |h| \left|\frac{\partial \Phi}{\partial r}(\overline{p})\right| \\ &\le \lambda_{N+1} |h| \left|\frac{\partial \Phi}{\partial r}(p_1)\right| \\ &\le \lambda_{N+1} |h| \left|\sum_{i=1}^{N} m_i D_i \Phi(p_1)\right| \\ &\le |h| \left(\left(\lambda_N + K_1(1+M_1)\, \lambda_N^{\gamma}\right) |D\Phi|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right) . \end{aligned}$$

And (4.16) is still valid, it is valid for all h in $\mathbb{R}^N$, we deduce :

$$\begin{aligned} \lambda_{N+1} |D\Phi|_{L^\infty} &= \lambda_{N+1} \sup_{p \in B} \sup_{h \in \mathbb{R}^N} \frac{|h.D\Phi|}{|h|} \\ &\le \left(\left(\lambda_N + K_1(1+M_1)\, \lambda_N^{\gamma}\right) |D\Phi|_{L^\infty} + \lambda_{N+1}^{\gamma} K_1(1+M_1)\right), \end{aligned}$$

thanks to (2.11) :

$$|D\Phi|_{L^\infty} \le \frac{\lambda_{N+1}^{\gamma} K_1(1+M_1)}{\lambda_{N+1} - \lambda_N - K_1(1+M_1)\, \lambda_N^{\gamma}} ,$$

thanks to (2.12) :

$$|D\Phi|_{L^\infty} \le M_1 .$$

Thus the lemma is proved when $\psi \in C^1(\Omega, Q_\ell H)$; if $\psi \in W^{1,\infty}(\Omega, Q_\ell H)$ we choose a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $C^1(\Omega, Q_\ell H) \cap V_\ell$ which converges to ψ, we write $\Phi_n = \mathcal{T}_{\varepsilon,\ell}\, \psi_n$, then :

$$\forall\, n \in \mathbb{N} \ : \ |D\Phi_n|_{L^\infty} \le M_1 .$$

(Φ_n) is bounded in $W^{1,\infty}(\Omega, Q_\ell H)$, it admits a subsequence which converges in $W^{1,\infty}(\Omega, Q_\ell H)$ weak star, thanks to lemma 4.2, the limit must be Φ, we deduce : $\Phi \in W^{1,\infty}(\Omega, Q_\ell H)$,

$$|D\Phi|_{L^\infty} \le \liminf |D\Phi_n|_{L^\infty} \le M_1 .$$

<u>LEMMA 4.5</u> *(fixed point)*

i) V_ℓ *is a Banach space when endowed with the* L^∞ *norm.*

ii) $\mathcal{T}_{\varepsilon,\ell} : V_\ell \to V_\ell$ *is a strict contraction for that norm.*

We deduce that $\mathcal{T}_{\varepsilon,\ell}$ *possesses a fixed point, we denote it by* $\Phi_{\varepsilon,\ell}$, *we have :*

$$(4.29)\qquad \begin{cases} \varepsilon\,\Delta\Phi_{\varepsilon,\ell} + \big(Ap + PR(p+\Phi_{\varepsilon,\ell})\big).\nabla\Phi_{\varepsilon,\ell} - A\Phi_{\varepsilon,\ell} - Q_\ell\, R(p+\Phi_{\varepsilon,\ell}) = 0 \quad \text{in } \Omega, \\ \Phi_{\varepsilon,\ell} = 0 \quad \text{on } \partial\Omega\ . \end{cases}$$

<u>Proof</u>

i) Left to the reader.

ii) Let ψ_i in V_ℓ and $\Phi_i = \mathcal{T}_{\varepsilon,\ell}\,\psi_i$ $(i = 1,2)$.

We know that Φ_1 and Φ_2 are in $C^{2,s}(\Omega, Q_\ell H)$ $(s < 1)$.

We denote by $\Phi = \Phi_1 - \Phi_2$, we have :

$$(4.30)\qquad \begin{aligned} &\varepsilon\Delta\Phi + \big(Ap + PR(p+\psi_1)\big).\nabla\Phi + \big(PR(p+\psi_1) - PR(p+\psi_2)\big).\nabla\Phi_2 \\ &- A\Phi - \big(Q_\ell R(p+\psi_1) - Q_\ell R(p+\psi_2)\big) = 0\ . \end{aligned}$$

We take the scalar product of (4.30) with Φ :

$$\begin{aligned} &\frac{\varepsilon}{2}\,\Delta\left|\Phi\right|^2 - \varepsilon\left|\nabla\Phi\right|^2 + \frac{1}{2}\big(Ap + PR(p+\psi_1)\big).\nabla\left|\Phi\right|^2 \\ &+ \Big(\big(PR(p+\psi_1) - PR(p+\psi_2)\big).\nabla\Phi_2, \Phi\Big) \\ &- \left|A^{1/2}\Phi\right|^2 - \Big(A^{-\gamma}\big(Q_\ell R(p+\psi_1) - Q_\ell R(p+\psi_2)\big), A^{\gamma}\Phi\Big) \\ &= 0\ . \end{aligned}$$

At the point $\bar{p}$ where $|\Phi|^2$ is maximum ($\bar{p}$ is inside Ω since $\Phi = 0$ on $\partial\Omega$) :

$$\begin{aligned} \left|A^{1/2}\Phi(\bar{p})\right|^2 &\le \Big(\big(PR(\bar{p}+\psi_1) - PR(\bar{p}+\psi_2)\big).\nabla\Phi_2\Phi(\bar{p})\Big) \\ &\quad - \Big(A^{-\gamma}\big(Q_\ell R(\bar{p}+\psi_1) - Q_\ell R(\bar{p}+\psi_2)\big), A^{\gamma}\Phi(\bar{p})\Big) \\ &\le \lambda_N^{\gamma}\, K_1 |\psi_1 - \psi_2|_{L^\infty} |D\Phi_2|_{L^\infty} |\Phi(\bar{p})| \end{aligned}$$

$$
\begin{aligned}
& \quad + K_1 |\psi_1 - \psi_2|_{L^\infty} |A^\gamma \Phi(\bar{p})| \\
& \leq \lambda_N^\gamma \lambda_{N+1}^{-1/2} K_1 M_1 |\psi_1 - \psi_2|_{L^\infty} |A^{1/2}\Phi(\bar{p})| \\
& \quad + \lambda_{N+1}^{\gamma-1/2} K_1 |\psi_1 - \psi_2|_{L^\infty} |A^{1/2}\Phi(\bar{p})| \\
& \leq \lambda_{N+1}^{\gamma-1/2} K_1(1+M_1) |\psi_1 - \psi_2|_{L^\infty} |A^{1/2}\Phi(\bar{p})| \\
|\Phi(\bar{p})| & \leq \lambda_{N+1}^{-1/2} |A^{1/2}\Phi(\bar{p})| \\
& \leq \lambda_{N+1}^{\gamma-1} K_1(1+M_1) |\psi_1 - \psi_2|_{L^\infty} , \\
|\Phi_1 - \Phi_2|_{L^\infty} & \leq \lambda_{N+1}^{\gamma-1} K_1(1+M_1) |\psi_1 - \psi_2|_{L^\infty} .
\end{aligned}
$$

Thanks to (2.14), $\mathcal{T}_{\varepsilon,\ell}$ is a strict contraction.

<u>LEMMA 4.6</u> : *(limit $\varepsilon \to 0$ and $\ell \to +\infty$)*

i) The sequence $(\Phi_{\varepsilon,\ell})_{\varepsilon>0}$ possesses a subsequence that converges when $\varepsilon \to 0$ in $W^{1,\infty}(\Omega, Q_\ell H)$ weak star to a function Φ_ℓ that satisfies :

(4.31)
$$
\begin{cases}
-\left(Ap + PR(p+\Phi_\ell)\right).\nabla\Phi_\ell + A\Phi_\ell + Q_\ell R(p+\Phi_\ell) = 0 & \text{in } \Omega , \\
\Phi_\ell = 0 \quad \text{on } \partial\Omega .
\end{cases}
$$

ii) Let $\tilde{\Phi}_\ell$ be the function from Ω to QH defined by :

$$
\begin{aligned}
(\tilde{\Phi}_\ell, w_k) &= (\Phi_\ell, w_k) && \textit{if } k \leq \ell , \\
(\tilde{\Phi}_\ell, w_k) &= 0 && \textit{if } k > \ell ,
\end{aligned}
$$

then the sequence $(\tilde{\Phi}_\ell)_{\ell \geq N+1}$ possesses a subsequence that converges when $\ell \to +\infty$ in $W^{1,\infty}(\Omega, Q H)$ weak star to a function Φ, that satisfies :

(4.32)
$$
\begin{cases}
-\left(Ap + PR(p+\Phi(p))\right).\nabla\Phi + A\Phi + QR(p+\Phi) = 0 & \textit{in } \Omega , \\
\Phi = 0 \quad \textit{on } \partial\Omega .
\end{cases}
$$

Moreover $\Phi \in L^\infty(\Omega, QD(A^{1/2})) \cap W^{1,\infty}(\Omega, QH)$.

<u>Proof</u>

i) For all $\varepsilon > 0$, we have, since $\Phi_{\varepsilon,\ell}$ is in V_ℓ :

$$
|\Phi_{\varepsilon,\ell}|_{W^{1,\infty}} \leq M_0 + M_1 .
$$

Hence there is a subsequence $(\Phi_{\varepsilon'})_{\varepsilon'>0}$ which converges in $W^{1,\infty}(\Omega, Q_\ell H)$ weak star to a function Φ_ℓ, as the imbedding of $W^{1,\infty}(\Omega, Q_\ell H)$ in $L^\infty(\Omega, Q_\ell H)$ is compact, we have :

$$\varepsilon\,\Delta\Phi_{\varepsilon',\ell} \to 0 \quad \text{in} \quad W^{-1,\infty}(\Omega,Q_\ell H)\ ,$$

$$\big(Ap + PR(p+\Phi_{\varepsilon',\ell})\big).\nabla\Phi_{\varepsilon',\ell} \rightharpoonup \big(Ap + PR(p+\Phi_\ell)\big).\nabla\Phi_\ell$$
$$\text{in } L^\infty\big(\Omega,Q_\ell H\big) \text{ weak star}\ ,$$

$$\big(A\Phi_{\varepsilon',\ell} + Q_\ell\, R(p+\Phi_{\varepsilon',\ell})\big) \to A\Phi_\ell + Q_\ell\, R(p+\Phi_\ell)$$
$$\text{in } L^\infty\big(\Omega,Q_\ell H\big) \text{ strong}\ ,$$

Using those limits in (4.29), we obtain (4.31).
Moreover :

$$|\Phi_\ell|_{W^{1,\infty}(\Omega,Q_\ell H)} \le M_0+M_1\ .$$

We take the scalar product of (4.31) with Φ_ℓ :

$$-\Big(\big(Ap + PR(p+\Phi_\ell)\big).\nabla\Phi_\ell,\Phi_\ell\Big) + |A^{1/2}\,\Phi_\ell|^2$$
$$+ \big(A^{-\gamma}\,Q_\ell\, R(p+\Phi_\ell),A^{\gamma}\,\Phi_\ell\big) = 0\ ,$$

$$|A^{1/2}\,\Phi_\ell|^2 \le \Big(\sup_{p\in\Omega}|Ap| + \lambda_N^{\gamma}\,K_0\Big)\,M_1\,\lambda_{N+1}^{-1/2}\,|A^{1/2}\,\Phi_\ell|$$
$$+ \lambda_{N+1}^{\gamma-1/2}\,K_0\,|A^{1/2}\,\Phi_\ell|\ ,$$

$$|A^{1/2}\,\Phi_\ell|_{L^\infty} \le \Big(\sup_{p\in\Omega}|Ap| + \lambda_N^{\gamma}\,K_0\Big)\,M_1\,\lambda_{N+1}^{-1/2} + \lambda_{N+1}^{\gamma-1/2}\,K_0\ .$$

Hence $(\Phi_\ell)_{\ell\ge N+1}$ is bounded in $W^{1,\infty}(\Omega,Q_\ell H) \cap L^\infty(\Omega,Q_\ell\, D(A^{1/2}))$.

ii) We have :

$$|\tilde{\Phi}_\ell|_{W^{1,\infty}} = |\Phi_\ell|_{W^{1,\infty}} \quad \text{and} \quad |A^{1/2}\,\tilde{\Phi}_\ell|_{L^\infty} = |A^{1/2}\,\Phi_\ell|_{L^\infty}\ .$$

The sequence $\big(\tilde{\Phi}_\ell\big)_{\ell\ge N+1}$ is bounded in $W^{1,\infty}(\Omega,QH) \cap L^\infty(\Omega,QD(A^{1/2}))$. Thanks to lemma (4.8) below, there exists a subsequence $\big(\tilde{\Phi}_{\ell_k}\big)$ and a function Φ in $W^{1,\infty}(\Omega,QH) \cap L^\infty(\Omega,QD(A^{1/2}))$ such that :

$$\tilde{\Phi}_{\ell_k} \rightharpoonup \Phi \quad \text{in } W^{1,\infty}(\Omega,QH) \text{ weak star}\ ,$$
$$\tilde{\Phi}_{\ell_k} \to \Phi \quad \text{in } L^\infty(\Omega,QH) \text{ strong}\ .$$

Then, we have :

$$\big(Ap + PR(p+\tilde{\Phi}_{\ell_k})\big).\nabla\tilde{\Phi}_{\ell_k} \rightharpoonup \big(Ap + PR(p+\Phi)\big).\nabla\Phi \quad \text{in } L^\infty(\Omega,QH) \text{ weak star}\ ,$$
$$A\tilde{\Phi}_{\ell_k} \to A\Phi \quad \text{in } L^\infty(\Omega,QD(A^{-1})) \text{ strong}\ ,$$
$$QR(p+\tilde{\Phi}_{\ell_k}) \to QR(p+\Phi) \quad \text{in } L^\infty(\Omega,QD(A^{-\gamma})) \text{ strong}\ .$$

We deduce that Φ satisfies (4.32).

<u>LEMMA 4.7</u> :

If we extend Φ, obtained in lemma 4.6 ii), by zero outside Ω, its graph is an inertial manifold for the system :

$$(4.33)\qquad \begin{cases} \frac{du}{dt} + Au + R(u) = 0 , \\ u(0) = u_0 . \end{cases}$$

<u>Proof</u>

We still denote by Φ the extended function.

Then we have :

$$-\big(Ap + PR(p+\Phi)\big).\nabla\Phi + A\Phi + QR(p+\Phi) = 0 ,$$

almost everywhere in PH.

Let u_0 be in H and $u(t) = S(t)\, u_0$ be the solution in $C(R^+,H) \cap L^2(0,T,D(A^{1/2}))$ for all $T > 0$ of system (4.33) (see the asumptions on (4.33) on section II), we write :

$$p(t) = Pu\,(t) ,$$

$$q(t) = Qu\,(t) ,$$

and $\quad r(t) = q(t) - \Phi(p(t))$.

We have :

$$\frac{dq}{dt} + Aq + QR(p+q) = 0 ,$$

$$\begin{aligned} \frac{d\Phi(p)}{dt} &= \frac{dp}{dt}.\nabla\Phi(p) \\ &= -(Ap + PR(p+q)).\nabla\Phi(p) \\ &= -\big(Ap + PR(p+\Phi)\big).\nabla\Phi(p) \\ &\quad + \big(PR(p+\Phi) - PR(p+q)\big).\nabla\Phi(p) \\ &= \big(PR(p+\Phi) - PR(p+q)\big).\nabla\Phi(p) \\ &\quad - A\Phi(p) - QR(p+\Phi) , \end{aligned}$$

$$(4.34)\qquad \begin{aligned} \frac{dr}{dt} = &-\big(PR(p+\Phi) - PR(p+q)\big).\nabla\Phi(p) \\ &- Ar - \big((QR(p+q) - QR(p+\Phi)\big) . \end{aligned}$$

We take the scalar product of (4.34) with $r(t)$ and use :

$$\left(\frac{dr}{dt}, r\right) = \frac{1}{2}\frac{d}{dt}|r|^2 ,$$

since $r(t) \in L^2(0,T;D(A^{1/2}))$ and $\frac{dr}{dt} \in L^2(0,T;D(A^{-1/2}))$, we obtain :

$$\frac{1}{2}\frac{d|r|^2}{dt}+|A^{1/2}r|^2 = -\Big(\big(PR(p+\Phi)-PR(p+q)\big).\nabla\Phi,r\Big)$$
$$-\Big(A^{-\gamma}\big(QR(p+q)-QR(p+\Phi)\big),A^{\gamma}r\Big)$$
$$\leq |PR(p+\Phi)-PR(p+q)|\,|D\Phi|_{L^\infty}|r|$$
$$+\big|A^{-\gamma}\big(QR(p+q)-QR(p+\Phi)\big)\big|\,|A^{\gamma}r|$$
$$\leq \lambda_N^{\gamma}K_1M_1|r|^2+K_1|r|\,|A^{\gamma}r|$$
$$\leq \lambda_N^{\gamma}K_1M_1\lambda_{N+1}^{-1/2}|r|\,|A^{1/2}r|$$
$$+K_1\lambda_{N+1}^{\gamma-1/2}|r|\,|A^{1/2}r|$$
$$\leq \lambda_{N+1}^{\gamma-1/2}K_1(1+M_1)|r|\,|A^{1/2}r|$$
$$\leq \frac{1}{2}\lambda_{N+1}^{2\gamma-1}K_1^2(1+M_1)^2|r|^2+\frac{1}{2}|A^{1/2}r|^2,$$

$$\frac{d|r|^2}{dt}+\lambda_{N+1}|r|^2 \leq \frac{d|r|^2}{dt}+|A^{1/2}r|^2$$
$$\leq \lambda_{N+1}^{2\gamma-1}K_1^2(1+M_1)^2|r|^2.$$

Thanks to Gronwall's lemma :

(4.35) $$|r(t)|^2 \leq |r(0)|^2 e^{-\lambda_{N+1}\left(1-\lambda_{N+1}^{2\gamma-2}K_1^2(1+M_1)^2\right)t}.$$

We deduce that if u_0 is in the graph of Φ, then $r(0)=0$ and $r(t)=0$ for all $t\geq 0$. The graph is positively invariant.

If u_0 is not the graph of Φ, thanks to (4.35) together with (2.14), $|r(t)|$ (which is larger than $d(u(t),\mathcal{M})$ decays exponentially.)

Theorem 3.1 is proved, except lemma 4.8, used in lemma 4.6.

<u>*LEMMA 4.8*</u> :

Let $(\Phi_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $W^{1,\infty}(\Omega,QH)\cap L^\infty(\Omega,QD(A^{1/2}))$ *then there exists a subsequence* (Φ_{n_k}) *that converges in* $W^{1,\infty}(\Omega,QH)$ *weak star and in* $L^\infty(\Omega,QH)$ *strong.*

<u>Proof</u>

For the convergence in $W^{1,\infty}(\Omega,QH)$ weak star, we use that $L^\infty(\Omega,QH)$ is the dual of $L^1(\Omega,QH)$; for the strong convergence in $L^\infty(\Omega,QH)$, we use Ascoli-Arzela theorem :

- (Φ_n) is bounded in $W^{1,\infty}(\Omega,QH)$, thus it is equicontinuous,
- (Φ_n) is bounded in $L^\infty(\Omega,QD(A^{1/2}))$, thus for all p in Ω, the set $\overline{\{\Phi_n(p)\,/\,n\in\mathbb{N}\}}$ is compact in QH (indeed $A^{-1/2}$ is a compact operator).

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2ème PARTIE:

CONSTRUCTION DE FAMILLES DE VARIETES INERTIELLES APPROCHEES.

On the construction of families of approximate inertial manifolds

Arnaud DEBUSSCHE(1) and Martine MARION(2)

0. Introduction

The aim of this paper is to present a general method for the construction of approximate inertial manifolds (AIM). The concept of AIM has been recently introduced by Foias-Manley-Teman [7] in relationship with several difficulties arising in the theory of inertial manifolds. For a given partial differential equation, an inertial manifold, when it exists, is a smooth finite dimensional invariant manifold which attracts exponentially all the solutions as time goes to infinity. The long-time dynamics can then be described by the solutions of a finite system of ordinary differential equations. Existence results of such manifolds can be found in Foias-Sell-Temam [8], Constantin et al [4], Mallet-Paret and Sell [12], Fabes-Luskin-Sell [6], Debussche [5] (see also the references therein) but there are still many dissipative partial differential equations for which the existence of inertial manifolds is not known ; there are even in some cases non existence results, see Mallet-Paret and Sell [13], Mora and Sola-Morales [18]. Also, although inertial manifolds are much better suited for computations than attractors, their computations remain a very difficult task. These problems have lead to introduce the weaker concept of AIM : AIM are manifolds which attract the orbit in a small (thin) neighborhood exponentially rapidly. It is shown in [7] that the two dimensional Navier-Stokes equations possess AIM while the existence of exact inertial manifold is not known for these equations. Even for equations possessing inertial manifolds, AIM can be useful, especially for practical purposes. Indeed these manifolds are computable and numerical schemes well adapted for long term integration are based on that concept, Marion-Temam [17] ; see Rosier [22], Rosier-Temam [23], for numerical tests of these methods. We also refer to Sell [24] for another definition of AIM.

The existence of AIM has been proved for several partial differential equations (Foias-Manley-Temam [7], Marion [15, 16], Titi [26], Jolly-Kevrekidis-Titi [9]). But in these references the authors restrict themselves to the construction of a finite number of manifolds. Our aim here is to derive a method for constructing infinitely many AIM providing better and better order approximation to the orbits. We will present our techniques in the case of two model problems that one of the authors previously considered in [15, 16] : a reaction-diffusion equation and a fourth order equation borrowed from

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mathematical physics, namely the Cahn-Hilliard equation. But our methods are general and can be adapted to many dissipative partial differential equations. Related results for the Navier-Stokes equations appear in Temam [25].

The first part of the paper is devoted to the study of a reaction diffusion equation. The precise assumptions are stated in section 1. The problem we investigate can be rewritten as an abstract evolution equation in $H = L^2(\Omega)$:

$$(0.1) \quad \frac{du}{dt} + Au + f(u) = 0 ,$$

where Au is the operator $- d\Delta u + u$, $d > 0$, associated to the appropriate boundary condition (Dirichlet, Neumann, periodic). We consider the orthonormal basis of H consisting of the eigenvectors of A :

$$A w_j = \lambda_j w_j , \; j = 1,2, \ldots$$
$$0 < \lambda_1 < \lambda_2 \ldots \; ; \; \lambda_j \to +\infty \text{ as } j \to +\infty .$$

For a fixed m, let $P = P_m$ denote the projector in H onto the space spanned by $w_1, \ldots, w_m$ and let $Q = Q_m = I - P_m$. We write $p = Pu$, $q = Qu$ so that u is decomposed as the sum

$$u = p + q .$$

We introduce the coupled system of equations for p and q :

$$(0.2) \quad \frac{dp}{dt} + Ap + Pf(p+q) = 0$$

$$(0.3) \quad \frac{dq}{dt} + Aq + Qf(p+q) = 0$$

Then, in section 2, we describe our method for constructing a sequence $\mathcal{M}_i$, i $\in \mathbb{N}$, of AIM. The manifold $\mathcal{M}_i$ is obtained as the graph of a funtion ϕ_i mapping PH into QH. We show that for large time any solution of (0.1) satisfies

$$\text{dist}(u(t), \mathcal{M}_i) \leq c_i (\lambda_1/\lambda_{m+1})^{i+1},$$

where c_i denotes a constant depending on i but independent of m. Therefore the orbits enter a neighborhood of $\mathcal{M}_i$ that can be made arbitrarily small by choosing m large enough. And, for $i_1 < i_2$, if m is large enough, $\mathcal{M}_{i_2}$ provides a better approximation than $\mathcal{M}_{i_1}$. Our method consists in particular in introducing in (0.3) convenient approximation of dq/dt and of p+q. The corresponding technical proofs are given in Section 3. Note that our results hold in space dimension $n \leq 4$, while the existence of exact inertial manifolds for reaction-diffusion equations is generally not known for $n = 3$ and there are non existence results for $n = 4$ [13].

We then address in the second part of the paper similar questions for the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \Delta^2 u + a\,\Delta u - b\,\Delta u^3 = 0 \,,\ a, b > 0 \,.$$

This equation is a model for the spinodal decomposition that occurs when a binary solution is cooled sufficiently [2, 3, 11, 21] . It contains a fourth order dissipative term and a second order anti-dissipative term and can be rewritten in an abstract form :

$$\frac{du}{dt} + A^2 u + Af(u) = 0 \,,\ f(u) = au - b\,u^3 \,,$$

where A denotes the Laplace operator associated to the considered boundary conditions (Neumann or periodic). The equation and its functionnal setting are described in section 4. We recall there results borrowed from Nicolaenko-Scheurer [19], Nicolaenko-Scheurer-Temam [20] , Marion [16]. We then present in section 5 the principle of the construction of AIM. The algebra and the corresponding proofs are different from the ones for (0.1) but the underlying ideas are similar. In particular we again introduce approximations to time derivatives. We obtain a family of manifolds $\mathcal{M}_i$ of dimension m ($m \in \mathbb{N}$ fixed) such that for t sufficiently large,

$$\text{dist}\,(u(t), \mathcal{M}_i) \le c_i\,(\lambda_2/\lambda_{m+1})^{i+2} \,,$$

where c_i is independent of m and $(\lambda_j)\ j \in \mathbb{N}$ denotes the family of eigenvalues of A. Section 6 contains the proofs of technical results concerning time derivatives. Again, our results hold in space dimension $n = 3$, where the existence of inertial manifolds is not known.

CONTENTS

0. Introduction

Part I. Reaction-diffusion equations

1. The equation and some properties of the solution
2. Construction of the family of AIM
3. Proof of the estimates on time derivatives

Part II. The Cahn-Hilliard equation

4. The equation and some properties of the solution
5. Construction of the family of AIM
6. Proof of the estimates on time derivatives

Part I. Reaction-diffusion equations

1. The equation and some properties of the solution

1.1. The equation and the semi-group

We consider the following problem involving a real valued function u(x,t) defined on $\Omega \times \mathbb{R}^+$, where Ω denotes a regular bounded set of $\mathbb{R}^n$ $(n \geq 1)$

(1.1) $$\frac{\partial u}{\partial t} - d\,\Delta u + f(u) = 0\,, \text{ in } \Omega \times \mathbb{R}^+ .$$

The equation is supplemented with the initial condition

(1.2) $$u(x,0) = u_o(x) \quad \text{in } \Omega\,,$$

and one of the three following boundary conditions :

(1.3)
Dirichlet $\quad u = 0 \text{ on } \Gamma = \partial\Omega$,

Neumann $\quad \dfrac{\partial u}{\partial \upsilon} = 0 \text{ on } \Gamma$,

Periodic $\quad \Omega = \prod_{i=1}^{n}]0,L_i[$ and u is Ω periodic.

Here, $d > 0$ is a diffusion coefficient. We assume that the nonlinear term f is infinitely many differentiable from $\mathbb{R}$ into $\mathbb{R}$ and satisfies the growth condition

(1.4) $$c_1 |s|^r - c_3 \leq f(s)s \leq c_2 |s|^r - c_3\,, \forall\, s \in \mathbb{R}\,, \text{ with } r > 2\,,$$

(1.5) $$f'(s) \geq -c_4\,, \forall\, s \in \mathbb{R}\,,$$

where the c_i 's are positive constants.

For the functional setting of the problem, let us introduce the operator $Au = -d\,\Delta u + u$ on $H = L^2(\Omega)$ equipped with its usual scalar product (.,.) and norm $|.|$. Then A is a positive linear unbounded operator on H with domain

$$D(A) = \{ u \in H^2(\Omega)\,,\ (1.3) \text{ holds}\}.$$

As usual, we denote by V the space $D(A^{1/2})$ endowed with the norm $|A^{1/2}.|$. We shall denote by $\|.\|_p$ the norm of $L^p(\Omega)$, $1 \leq p \leq +\infty$ $(\|.\|_2 = |.|)$, while the norm of any other Banach space X is denoted by $\|.\|_X$.

Under assumptions (1.4), (1.5), for u_o given in H , the initial-boundary value problem (1.1)-(1.3) possesses a unique solution u defined for all t > 0 such that

$$u \in C(\mathbb{R}^+; H) \cap L^2(0,T; V) \cap L^r([0,T] \times \Omega), \forall T > 0.$$

Furthermore, if $u_o \in V \cap L^r(\Omega)$, then

$$u \in C(\mathbb{R}^+, V \cap L^r(\Omega)) \cap L^2(0,T; D(A)), \forall T > 0.$$

It is useful here to recall several time uniform estimates satisfied by the solution u of (1.1) - (1.3) borrowed from Marion [14, 15] . Let u_o be given in a ball B(0,R) of H of center 0 and of radius R. Then there exists a time t_o depending only on R (and of course on the data (Ω,d,f)) such that :

(1.6) $|Au(t)| \leq \kappa_o, \forall t \geq t_o,$

(1.7) $\|u(t)\|_\infty \leq \kappa_o, \forall t \geq t_o,$

where κ_o denotes a constant depending only on the data. Alternatively (1.6) (resp. (1.7)) expresses that the ball of center 0 and of radius κ_o is an absorbing set in D(A) (resp. $L^\infty(\Omega)$) for the semi-group associated to (1.1) - (1.3). We recall that the existence of an absorbing set in H means a dissipative property of the problem. Also the existence of an absorbing set in a space compactly imbedded in H yields the existence of the universal attractor associated to (1.1) (1.3) (see [1], [10], [14]).

We will also need in the sequel estimates anolog to (1.6) (1.7) for the time derivatives of u. Hereafter, we write :

$$u^{(j)} = \frac{\partial^j u}{\partial t^j}, \text{ for } j \in \mathbb{N}.$$

Then, according to [15], there exists a time t_j depending on j and R such that

(1.8) $|Au^{(j)}(t)| \leq \kappa_j, \forall t \geq t_j,$

(1.9) $\|u^{(j)}(t)\|_\infty \leq \kappa_j, \forall t \geq t_j,$

where κ_j denotes a constant depending on j and on the data.

It follows from (1.7) that, for $t \geq t_o$, u = u(t) is solution of an evolution equation with a Lipschitz continuous nonlinear term. Indeed let ζ denote a C^∞ truncation function such that

$$\zeta(s) = 1 \text{ if } 0 \le s \le 1 \text{ and } \zeta(s) = 0 \text{ if } s \ge 2.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \zeta\left(\frac{s^2}{\kappa_0^2}\right)(f(s) - s).$$

Then, for $t \ge t_o$, $u(t)$ satisfies

$$\frac{\partial u}{\partial t} - d\,\Delta u + u + f(u) = 0.$$

This equation rewrites as the following abstract differential equation in H

$$(1.10) \qquad \frac{du}{dt} + Au + f(u) = 0.$$

Since we are only interested in long-time behaviours, we will consider from now on (1.10) instead of (1.1).

Remark 1.1 : All the results presented hereafter can be easily extended to more general reaction-diffusion equations. In particular, they apply to the reaction diffusion systems admitting a positively invariant region considered in [14].

1.2. Behaviour of small eddies

Let w_j, $j \in \mathbb{N}^*$, denote the orthonomal basis of H consisting of the eigen vectors of the operator A :

$$Aw_j = \lambda_j w_j , \; j = 1,2,...$$
$$0 < \lambda_1 \le \lambda_2 \le ... ; \; \lambda_j \to +\infty \text{ as } j \to +\infty .$$

We fix an integer $m \in \mathbb{N}$ and denote by $P = P_m$ the projector in H onto the space spanned by $(w_1, ..., w_m)$; we set $Q = Q_m = I - P_m$. In order to simplify the notations, we will write :

$$\lambda = \lambda_m , \; \Lambda = \lambda_{m+1} ,$$

and we introduce the number

$$\delta = \lambda_1/\lambda_{m+1} .$$

Applying P and Q to equation (1.10), we obtain the following coupled system of equations for $p = Pu$ and $q = Qu$:

(1.11) $\frac{dp}{dt} + Ap + Pf(p+q) = 0,$

(1.12) $\frac{dq}{dt} + Aq + Qf(p+q) = 0.$

Here, p represents the "large structures" of size larger lthan $\lambda_m^{-1/2}$ and q represents the "small structures" of size smaller than $\lambda_{m+1}^{-1/2}$. In the following, we will frequently use that for all $\alpha > 0$:

(1.13) $|A^{\alpha+1/2}\, p|^2 \leq \lambda\ |A^{\alpha}p|^2,\ \forall\, p \in PD(A^{\alpha+1/2})$

(1.14) $|A^{\alpha+1/2}\, q|^2 \geq \Lambda\ |A^{\alpha}q|^2,\ \forall\, q \in QD(A^{\alpha+1/2}).$

Hereafter, we denote by κ any constant which depends only on the data Ω , d and f.

We now give an estimate of the size of q and of its derivatives for large time.

Proposition 1.1 : *Assume that (1.4), (1.5) hold. Then, for all j in $\mathbb{N}$ there exists a time t'_j which depends on the data (Ω , d , f) and on R when $|u_o| \leq R$ such that :*

(1.15) $|q^{(j)}(t)| \leq \kappa\,\delta,\ \forall\, t \geq t'_j,$

(1.16) $|A^{1/2}\, q^{(j)}(t)| \leq \kappa\,\delta^{1/2},\ \forall\, t \geq t'_j.$

Proof. Differentiating j times equation (1.12) with respect to t, we obtain:

(1.17) $\frac{d}{dt}\, q^{(j)} + Aq^{(j)} + Q\, f_j\, (u,u^{(1)}, ..., u^{(j)}) = 0,$

where we have set

$$\frac{d^j}{dt^j}(f(u)) = f_j(u,u^{(1)}, ..., u^{(j)}),$$

$$q^{(j)} = \frac{d^j}{dt^j}\, q.$$

By taking the scalar product of (1.17) with $Aq^{(j)}$, we have :

$$\frac{1}{2}\frac{d}{dt}|A^{1/2}q^{(j)}|^2 + |Aq^{(j)}|^2 = -(f_j(u,u^{(1)},\dots,u^{(j)}),\ Aq^{(j)}),$$
$$\le |f_j(u,u^{(1)},\dots,u^{(j)})|\,|Aq^{(j)}|$$
$$\le \frac{1}{2}|f_j(u,u)^{(1)},\dots,u^{(j)})|^2 + \frac{1}{2}|Aq^{(j)}|^2,$$

$$\frac{d}{dt}|A^{1/2}q^{(j)}|^2 + |Aq^{(j)}|^2 \le |f_j(u,u^{(1)},\dots,u^{(j)})|^2.$$

From the construction of f, we see that f together with its derivatives are bounded. Therefore, using (1.8) and (1.9), one easily checks that there exists a constant K such that

$$|f_j(u,u^{(1)},\dots,u^{(j)})|^2 \le \kappa^2,\ \forall t \ge \max_{1\le k\le j} t_k.$$

This gives, thanks to (1.14) :

$$\frac{d}{dt}|A^{1/2}q^{(j)}|^2 + \Lambda\,|A^{1/2}q^{(j)}|^2 \le \kappa^2.$$

Integrating this inequality between $T_j = \max_{1\le k\le j} t_k$ and t , we find

$$|A^{1/2}q^{(j)}(t)|^2 \le \frac{\kappa^2}{\Lambda} + |A^{1/2}q^{(j)}(T_j)|^2 e^{-\Lambda(t-T_j)},\quad \forall t \ge T_j. \tag{1.18}$$

Since

$$|A^{1/2}q^{(j)}(T_j)| \le |A^{1/2}u^{(j)}(T_j)| \le \kappa_j$$

we infer from (1.18) that

$$|A^{1/2}q^{(j)}(t)|^2 \le \frac{2\kappa}{\Lambda},\ \forall t \ge t'_j$$

where $t'_j = \sup_{m\in\mathbb{N}} \max\left(T_j, T_j + \frac{1}{\lambda_{m+1}}\operatorname{Log}\frac{\lambda_{m+1}\kappa_j^2}{\kappa}\right).$

This shows (1.16). Finally, (1.15) follows readily from (1.16) by using (1.14) with $\alpha = 0$.

Proposition 1.1 is proved.

2. Construction of the family of AIM

The aim of this section is to present the algebra of construction of the sequence $\mathcal{M}_i$, $i \in \mathbb{N}$, of AIM. These manifolds are closely related to approximations of the equation for q. These approximations are obtained by using Proposition 1.1, i.e. that q and its time derivatives are small for large time.

The construction of the first two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ follows Marion [15]. The manifold $\mathcal{M}_1$ corresponds to the simplest approximation of (1.12). Thanks to Proposition 1.1, we know that q and $q^{(1)} = dq/dt$ are small for large time. Therefore, we guess that, in (1.12), $q^{(1)}$ is small compared to Aq and q is small compare to p. We are lead to replace (1.12) by the following approximate equation :

(2.1) $Aq + Qf(p) = 0$.

For p given in PH, the resolution of (2.1) is easy and we denote by q_1 its solution:

(2.2) $q_1 = \phi_1(p)$.

The graph of the function $\phi_1 : PH \to QD(A)$ defines a smooth (C^∞) manifold $\mathcal{M}_1$ in H of dimension m. The orbits are attracted by a thin neighborhood of $\mathcal{M}_1$ as shown in the following proposition.

Proposition 2.1. *Assume that (1.4) (1.5) hold. Then, for t sufficiently large, $t \geq t_1^*$, any orbit of (1.1)-(1.3) remains at a distance in H of $\mathcal{M}_1$ bounded by $\kappa_1\delta^2$; κ_1 is an appropriate constant which depends on the data and t_1^* depends on the data and on R when $|u_o| \leq R$.*

Proof. Let u = p+q be an orbit of (1.1)- (1.3). For every t > 0 , we define $q_1(t) = \phi_1(p(t))$. Then, $p(t) + q_1(t)$ lies in $\mathcal{M}_1$ and

$$\text{dist}\,(u(t), \mathcal{M}_1) = \inf_{v \in \mathcal{M}_1} |u(t) - v| \leq |q(t) - q_1(t)|.$$

Therefore, it suffices to evaluate the norm of

$$\chi_1(t) = q_1(t) - q(t)$$

Substracting (1.12) from (2.1) with $q = q_1$ we find

$$A\chi_1 = Q(f(p+q) - f(p)) + q' ,$$

and

$|A\chi_1 \leq |f(p+q) - f(p)| + |q'|$.

Since f' is bounded on $\mathbb{R}$, it follows easily that

$|A\chi_1| \leq \kappa |q| + |q'|$,

which gives thanks to (1.15) for j = 0,1,

(2.3) $|A\chi_1| \leq \kappa\, \delta$, for $t \geq \max(t_o', t_1')$.

Therefore, using (1.14), we obtain :

$|\chi_1| \leq \kappa\, \delta^2$.

This shows Proposition 2.1.

Next, we give as an example the construction of $\mathfrak{M}_2$. Making use of $q_1 = \phi_1(p)$, we now approximate Qf(p+q) by $Qf(p+q_1)$. Also it follows easily from the proof of Proposition 2.1 that we need now to introduce an approximation of $q^{(1)}$. This is obtained by considering the equation for $q^{(1)}$:

(2.4) $q^{(2)} + Aq^{(1)} + Qf'(p+q)(p^{(1)} + q^{(1)}) = 0$.

In (2.4), $p^{(1)}$ given by (1.11) is approximated by

(2.5) $p_1^1 = - Ap - Pf(p)$

Also $q^{(2)}$ is neglected and the nonlinear term $Qf'(p+q)(p^{(1)} + q^{(1)})$ is replaced by $Qf'(p)\, P_1^1$; the approximate value q_1^1 is given by

(2.6) $Aq_1^1 + Qf'(p)\, p_1^1 = 0$.

Hence, (1.12) is now replaced by the equation :

(2.7) $q_1^1 + Aq_2 + Qf(p + q_1) = 0$

The manifold $\mathfrak{M}_2$ is therefore defined as follows. For $p \in PH$, we define q_1 by (2.1). Then we define p_1^1 by (2.5) and the resolution of (2.6) gives q_1^1 . Finally, by solving (2.7), we obtain

$q_2 = \phi_2(p)$

The graph of the function $\phi_2 : PH \to QD(A)$ defines a C^∞ manifold $\mathfrak{M}_2$ of dimension m in H. This manifold provides a better order approximation to the orbits than $\mathfrak{M}_1$ and this is stated in

Theorem 2.2 *: Assume that (1.4) (1.5) hold. Then for t sufficiently large, $t \geq t_2^*$, any orbit of (1.1)-(1.3) remains at a distance in H of $\mathfrak{M}_2$ bounded by $\kappa_2\, \delta^3$; κ_2 is an appropriate constant which depends on the data and t_2^* depends on the data and on R when $|u_o| \leq R$.*

Proof. The proof follows the same steps as that of Theorem 2.3 below and is left to the reader.

In order to construct $\mathfrak{M}_3$, we will need in particular to consider improved approximations of q' and of p+q in (1.12). The approximation of p+q is provided by $p+q_2$. Then, we introduce an approximate value q_1^2 of $q^{(2)}$ by neglecting $q^{(3)}$ in the second derivative of (1.12) and that enables us to improve the approximation of (2.3) and to derive a better value for q'.

More generally, when we construct the manifold $\mathfrak{M}_k$, we construct two families p_{k-i}^i , i = 1, ..., k-1 and q_{k-i}^i , i = 0, ..., k-1 such that p_{k-1}^i provides an approximation of $p^{(i)}$ and q_{k-i}^i of $q^{(i)}$ and that satisfy estimates of the form

$$(2.8) \qquad |p_{k-i}^i(t) - p^{(i)}(t)| \leq \kappa_k\, \delta^{k-i}, \; i = 1, \ldots, \kappa - 1 ,$$

$$(2.9) \qquad |q_{k-i}^i(t) - q^{(i)}(t)| \leq \kappa_k\, \delta^{k-i+1}, \; i = 0, \ldots, \kappa - 1 ,$$

for $t \geq t''_k$, where κ_k depends on the data and k, and t''_k depends on the data, κ and R when $|u_o| \leq R$. In particular, at step κ, we introduce approximations of the time derivatives of p and q up to the order (k-1).

The manifolds $\mathfrak{M}_k$ are defined recursively. For k = 1, $\mathfrak{M}_1$ is given by (2.1). Assume that $\mathfrak{M}_1, \ldots, \mathfrak{M}_{k-1}$ are constructed ($\kappa \geq 2$). We aim now to construct $\mathfrak{M}_k$. We start by computing the sequence p_{k-i}^i , i = 1, ..., k - 1. The sequence is defined recursively for increasing values of i thanks to the following formula :

$$(2.10) \qquad p_{k-i}^i = -A\, p_{k-i+1}^{i-1} - P f_{i-1}(p + q_{k-i-1},\, p_{k-i+1}^1 + q_{k-i}^1, \ldots, p_{k-i+1}^{i-2} + q_{k-i}^{i-2},$$
$$p_{k-i}^{i-1} + q_{k-i-1}^{i-1}).$$

Here, we have set for convenience

$$p_l^o = p, \ \forall\, l \geq 1 ,$$
$$q_o = 0.$$

Note that p_{k-i}^i is defined explicitly and the right hand-side of (2.10) involves either quantities known from the construction of $\mathcal{M}_1, \ldots, \mathcal{M}_{k-1}$, or the term p_{k-i+1}^{i-1}, defined at step (i-1) of the recursive scheme. The formula (2.10) is obtained by considering the equation (1.11) differentiated (i-1) times, that is

$$(2.11) \qquad p^{(i)} + Ap^{(i-1)} + P\, f_{i-1}\,(p+q,\, p^{(1)} + q^{(1)}, \ldots,\ p^{(i-1)} + q^{(i-1)}) = 0 .$$

Then we replace in (2.11) the time derivatives $p^{(l)}$, $q^{(l)}$ by p_α^l , q_β^l for convenient values of α,β.

We then compute the family q_{k-i}^i , i = k-1, ..., 1. The family is again defined recursively, but now for decreasing values of i. At step i, q_{k-i}^i , which is an approximate value of $q^{(i)}$, is given by the resolution of

$$(2.12) \qquad q_{k-i-1}^{i+1} + A\, q_{k-i}^i + Q\, f_i(p + q_{k-i-1} ,\, p_{k-i+1}^1 + q_{k-i}^1 ,\ldots,\ p_{k-i+1}^{i-1} + q_{k-i}^{i-1} ,\, p_{k-i}^i + q_{k-i-1}^i) = 0 ,$$

where we agree that

$$q_o^l = 0 ,\ \forall\, l \geq 1 .$$

Relation (2.12) is obtained by considering equation (1.12) differentiated i times and introducing approximations of the different terms. Note that in (2.12) all terms except q_{k-i}^i are known either from the construction of $\mathcal{M}_1$, ..., $\mathcal{M}_k$ or for p_{k-i}^i from (2.10), or for q_{k-i-1}^{i+1} from the step (i+1) of the recursive scheme. For the first step i =k-1, we have $q_{k-i-1}^{i+1} = q_o^k = 0$; this expresses that the time derivative of order k of q is neglected.

Finally, $q_k = q_k^o$ is defined by the analog of (2.12) for k = 0 that is

$$(2.13) \qquad q_{k-1}^1 + Aq_k + Q\, f(p+q_{k-1}) = 0$$

It is clear that q_k is a function of t through p(t) :

$q_k(t) = \phi_k(p(t))$

The manifold $\mathcal{M}_k$ with equation

$q = \phi_k(p)$, ϕ_k : $PH \rightarrow QD(A)$

is a $\mathcal{C}^\infty$ manifold of dimension m in H. This manifold provides a better order approximation to the orbits than $\mathcal{M}_1$, ... $\mathcal{M}_{k-1}$. This is stated in the following Theorem which is valid in space dimension $n \leq 4$.

Theorem 2.3. *Assume that (1.4) (1.5) hold and $n \leq 4$. Then, for t sufficiently large, $t \geq t_k^*$, any solution of (1.1) -(1.3) remains at a distance in H of $\mathcal{M}_k$ bounded by $\kappa_k \delta^{k+1}$; κ_k is an appropriate constant which depends on the data and k and t_k^* depends on the data, k and R when $|u_o| \leq R$.*

Remark 2.4. As it will appear in the proof of Theorem 2.3, we obtain in fact a stronger result. We show that for any arbit u(t) = p(t) + q(t) of (1.1)-(1.3), the induced orbit on the manifold $\mathcal{M}_k$:

$u_k(t) = p(t) + \phi_k(p(t))$

satisfies for large time

$|u(t) - u_k(t)| \leq \kappa_k \delta^{k+1}$.

Remark 2.5. The constants κ_k and t_k^* are independent of m. Hence, the orbits enter a neighbourhood of $\mathcal{M}_k$ that can be made arbitrarily small by increasing the dimension m of the manifold $\mathcal{M}_k$.

For k and k' given, k < k' , and for m sufficiently large, the orbits are closer from $\mathcal{M}_{k'}$, than from $\mathcal{M}_k$. However since we are not able to compare K_k and ,$\kappa_{k'}$, we are not certain that for k, k', m given, k < k', the orbits are closer to $\mathcal{M}_k$ than to $\mathcal{M}_{k'}$.

Remark 2.6. For a fixed value of k, the schemes (2.11), (2.13) can be improved and simplified by using the explicit forms for the derivatives of the nonlinearity f.

Remark 2.7. The restriction on the space dimension comes from the use of Sobolev imbedding Theorems in the proof of Theorem 2.3. It is easy to see that, for small values of k, the conclusions of Theorem 2.3 hold in space dimension n larger than four. For example, Theorem 2.2 is valid in any space dimension.

Proof of Theorem 2.3. Let u(t) = p(t) + q(t) be a solution of (1.1)-(1.3). We consider the families p^i_{l-i} , q^i_{l-i} , q_l related to the manifolds $\mathfrak{M}_l$, $1 \leq l \leq k$. Our aim is to estimate

$$\chi_k(t) \ = \ q_k(t) - q(t)$$

We admit here that (2.8) (2.9), hold (the proof of these estimates is postponed to section 3). Substracting (1.12) from (2.13), we have :

$$A\,\chi_k = Q\,(f(p+q) - f(p+q_{k-1})) \ + \ q^{(1)} - q^1_{k-1} \ ,$$

$$|A\chi_k| \leq \ |f(p+q) - f(p+q_{k-1})\ | + \ |q^{(1)} - q^1_{k-1}| \ .$$

Therefore, since f is bounded on $\mathbb{R}$,

$$(2.14) \qquad |A\chi_k| \ \leq \ \kappa\,|q - q_{k-1}| \ + \ |q^{(1)} - q^1_{k-1}|\,.$$

Now using (2.10) (recall that $q_{k-1} = q^o_{k-1}$) , we infer from (2.14)

$$|A\chi_k| \ \leq \ \kappa\,\delta^k, \ \ \forall\, t \geq t''_{k-1}\,.$$

Hence,

$$|\chi_k| \ \leq \ \kappa\,\delta^{k+1}\ , \ \ \forall\, t \geq t''_{k-1}.$$

This shows Theorem 2.3.

3. Proof of the estimates on time derivatives

Our aim in this Section is to check the estimates (2.8) and (2.9). We will prove by induction on k that

$$(3.1)_k \quad |p^i_{k-i}(t) - p^{(i)}(t)| \ \leq \ \kappa_k\,\delta^{k-i}\ , \ \ i = 0, ..., k\text{-}1\ ,$$

$$(3.2)_k \quad |A(q^i_{k-i}(t) - q^{(i)}(t))| \ \leq \ \kappa_k\,\delta^{k-i}\ , \ i = 0, ..., k\text{-}1\ ,$$

for $t \geq t''_k$, where κ_k depends on the data , k and R when $|u_o| \ \leq \ R$. We recall that $p^o_l = p$ and $q^o_l = q_l$, $\forall\, l \geq 1$.

i) Initialization of the induction (k = 1). The estimate $(3.2)_o$ restates (2.3) ; while (3.1) is olvious.

ii) The induction argument. We now assume that $(3.1)_l$ and $(3.2)_l$ hold for $0 \leq l \leq k-1$ for $k \geq 2$ given and prove that $(3.1)_k$ and $(3.2)_k$ are satisfied.

Hereafter, we denote by κ any constant which depends on the data and on k and "for t large enough" means for $t \geq T_k$ where T_k depends on the data, k and R when $|u_o| \leq R$.

Following the procedure of definition of p^i_{k-i} and q^i_{k-i} we will first derive $(3.1)_k$ by induction on i with i increasing and then $(3.2)_k$ by induction on i with i decreasing.

For i = 0 , $(3.1)_k$ is obvious. Let us assume that the relations $(3.1)_k$ are proved up to the order i - 1 for $i \geq 1$. Substracting (2.11) to (2.10), we have :

$$p^i_{k-i} - p^{(i)} = - A\,(p^{i-1}_{k-i+1} - p^{(i-1)}) - P\,[f_{i-1}\,(p+q_{k-i-1}\,,\ p^1_{k-i+1} + q^1_{k-i}\,, \ldots, p^{i-1}_{k-i} + q^{i-1}_{k-i-1}\,) - f_{i-1}(p+q,\ p^{(1)} + q^{(1)}, \ldots, p^{(i-1)} + q^{(i-1)})]\,.$$

Thanks to the relation $(3.1)_k$ at the order (i - 1) and (1.13), we obtain :

$$|A(p^{i-1}_{k-i+1} - p^{(i-1)})| \leq \lambda\, |p^{i-1}_{k-i+1} - p^{(i-1)}| \leq \kappa\,\lambda\,\delta^{k-i+1} \leq \kappa\,\delta^{k-i}\,.$$

We infer from (1.8) (1.9) that

$$p+q,\ p^{(1)} + q^{(1)}\,, \ldots, p^{(i+1)} + q^{(i-1)}$$

are bounded in D(A) and $L^\infty(\Omega)$. Moreover, we easily deduce from (3.1), (3.2) at the preceeding steps that $p+q_{k-i-1}$ is bounded in $L^2(\Omega)$, $p^1_{k-i+1} + q^1_{k-i}\,, \ldots, p^{i-2}_{k-i+1} + q^{i-2}_{k-i}$ are bounded in D(A) and $p^{i-1}_{k-i} + q^{i-1}_{k-i-1}$ in $L^2(\Omega)$ for t large enough. Therefore, Lemma 3.1 below applies and the second term of (3.5) is majorized by

$$\kappa(\,|q_{k-i-1} - q| + |A\,(p^1_{k-i+1} - p^{(1)}| + |A(q^1_{k-i} - q^{(1)})| + \ldots |A(p^{i-2}_{k-i+1} - p^{(i-2)})| + |A(q^{i-2}_{k-i} - q^{(i-2)})| + |p^{i-1}_{k-i} - p^{(i-1)}| + |q^{i-1}_{k-i-1} - q^{(i-1)}|\,),$$

for t large enough. Now, we use (1.13), (1.14) together with the induction hypotheses to conclude that this is bounded by $\kappa\,\delta^{k-i}$ for t large enough. Therefore $(3.1)_k$ is proved at the order i, and this concludes the induction on i . Thus $(3.1)_k$ is shown.

Let us now prove that $(3.2)_k$ is true by induction on i with i decreasing. For $i = k-1$, q_1^{k-1} is defined by

$$Aq_1^{k-1} + Qf_{k-1}(p, p_2^1 + q_1^1, ..., p_2^{k-2} + q_1^{k-2}, p_1^{k-1}) = 0$$

And, $q^{(k-1)}$ satisfies

$$q^{(k)} + Aq^{(k-1)} + Qf_{k-1}(p+q, p^{(1)} + q^{(1)}, ..., p^{(k-2)} + q^{(k-2)}, p^{(k-1)} + q^{(k-1)}) = 0 .$$

Therefore

$$|A(q_1^{k-1} - q^{(k-1)})| \leq |q^{(k)}| + |f_{k-1}(p, p_2^1 + q_1^1, ..., p_2^{k-2} + q_1^{k-2}, p_1^{k-1})$$

$$- f_{k-1}(p+q, p^{(1)} + q^{(1)}, ..., p^{(k-2)} + q^{(k-2)}, p^{(k-1)} + q^{(k-1)})| .$$

Thanks to proposition 1.1, we have

$$|q^{(k)}| \leq \kappa\delta,$$

for t large enough. Next we use Lemma 3.1 below to estimate the second term

$$|f_{k-1}(p, p_2^1 + q_1^1, ..., p_2^{k-2} + q_1^{k-2}, p_1^{k-1}) - f_{k-1}(p + q, p^{(1)} + q^{(1)}, ..., p^{(k-2)} +$$

$$q^{(k-2)}, p^{(k-1)} + q^{(k-1)})| \leq \kappa(|q| + |A(p_2^1 - p^{(1)})| + |A(q_1^1 - q^{(1)})| + ... +$$

$$|A(p_2^{k-2} - p^{(k-2)})| + |A(q_1^{k-2} - q^{(k-2)})| + |p_1^{k-1} - p^{(k-1)}| + |q^{(k-1)}|) .$$

and we infer from Proposition 1.1, (1.13), $(3.1)_k$ and the induction hypotheses that this is majorized by $\kappa\delta$ for t large enough. Therefore

$$|A(q_1^{k-1} - q^{(k-1)})| \leq \kappa\,\delta ,$$

for t large enough. And $(3.2)_k$ is proved for $i = k-1$. Let us assume that $(3.2)_k$ is proved at the orders $k-1, k-2, ..., i+1$ for $i \geq 0$. By substracting (2.12) to the i^{th} derivative of (1.12), we obtain

$$|A(q_{k-i}^i - q^{(i)})| \leq |q_{k-i-1}^{i+1} - q^{(i+1)}| + |f_i(p+q_{k-i-1}, p_{k-i+1}^1 + q_{k-i}^1 ..., p_{k-i+1}^{i-1} +$$

$$q_{k-i}^{i-1}, p_{k-i}^i + q_{k-i-1}^i) - f_i(p+q, p^{(1)} + q^{(1)}, ..., p^{(i-1)} + q^{(i-1)}, p^{(i)} + q^{(i)})| .$$

The induction hypothesis together with (1.14) gives

$$|q^{i+1}_{k-i-1} - q^{(i+1)}| \le \kappa\, \delta^{k-i},$$

for t large enough. Similar arguments as above yield also that the second term is bounded by K δ^{k-i} for t large enough. Therefore

$$|A(q^{i}_{k-i} - q^{(i)})| \le \kappa\, \delta^{k-i},$$

and $(3.2)_k$ is proved for i and, thanks to the induction argument on i, $(3.2)_k$ is true for all i . In the same time, the induction on k is finished and $(3.1)_k$, $(3.2)_k$ hold for all integer k. It remains to state and prove Lemma 3.1.

Lemma 3.1. *Assume that $n \le 4$. Then , for $(u, u_1, \dots, u_{l-1}, u_l)$ in $L^2(\Omega) \times (D(A) \cap L^\infty\Omega))^{l-1} \times (L^2(\Omega) \cap L^\infty(\Omega))$ and $(v, v_1, \dots, v_{l-1}, v_l)$ in $L^2(\Omega) \times D(A)^{l-1} \times L^2(\Omega)$, we have :*

$$|f_l(u, u_1, \dots, u_{l-1}, u_l) - f_l(v, v_1, \dots, v_{l-1}, v_l)| \le$$
$$C(R)\,(|u-v| + |A(u_1 - v_1)| + \dots + |A(u_{l-1} - v_{l-1})| + |u_l - v_l|),$$

when $|u| \le R$, $|v| \le R$, $|u_i|_\infty \le R$, $|Au_i| \le R$, $|Av_i| \le R$, for $i = 1, \dots, l-1$, and $|u_l|_\infty \le R$, $|u_l| \le R$, $|v_l| \le R$ where $C(R)$ depends on the data, l and R.

Proof. The proof is lenghty and we omit the details.

We first decompose f_l in two terms :

$$f_l(u, u_1, \dots, u_{l-1}, u_l) = g_l(u, u_1, \dots, u_{l-1}) + f'(u)\, u_l,$$

where g_l is a polynomial in $u_1, \dots, u_{l-1}$ whose coefficient depends on $f^{(k)}(u)$ for $k \le l$. Then, we write :

$$f_l(u, u_1, \dots, u_{l-1}, u_l) - f_l(v, v_1, \dots, v_{l-1}, v_l) = g_l(u, u_1, \dots, u_{l-1}) -$$
$$g_l(v, u_1, \dots, u_{l-1}) + g_l(v, u_1, \dots, u_{l-1}) - g_l(v, v_1, u_2, \dots, u_{l-1}) + \dots$$
$$g_l(v, v_1, \dots, v_{l-2}, u_{l-1}) - g_l(v, v_1, \dots, v_{l-1}) + f'(u)\, u_l - f'(v)\, v_l.$$

The first term is a sum of monomials in $u_1, \dots, u_{l-1}$ whose coefficients are a constant times $f^{(k)}(u) - f^{(k)}(v)$. Therefore since all the derivatives of f are Lipschitz and $u_1, \dots, u_{l-1}$ are bounded in $L^\infty(\Omega)$, we have :

$$|g_l(u, u_1, \dots, u_{l-1}) - g_l(v, u_1, \dots, u_{l-1})| \le \kappa(R)\, |u-v|.$$

The second term is of the type :

$$\sum_k a_k g^{(k)}(v)\, u_2^{\alpha_{2;k}} \dots u_{l-1}^{\alpha_{l-1;k}} (u_1^{\alpha_{1;k}} - v_1^{\alpha_{1;k}}).$$

Using hölder inequality and the boundedness of the derivatives of g , for $\beta_{i,k} = \alpha_{i,k}\, p_i$ and $\frac{1}{p_1} + \dots + \frac{1}{p_{l-1}} = 1$, we obtain

$$|g_l(v,u_1, \dots, u_{l-1}) - g_l(v, v_1, u_2, \dots, u_{l-1})|$$

$$\leq \sum_k |a_k| (\sup |g^{(k)}|)\, |u_1 - v_1|^{\alpha_{1;k}}_{L^{\beta_{1;k}}} |u_2|^{\alpha_{2;k}}_{L^{\beta_{2;k}}} \dots |u_{l-1}|^{\alpha_{l-1;k}}_{L^{\beta_{l-1;k}}},$$

$$\leq \sum_k |a_k| (\sup |g^{(k)}|)(|u_1|_{L^{\beta_{1;k}}} + |v_1|_{L^{\beta_{1;k}}})^{\alpha_{1;k}-1} |u_2|^{\alpha_{2;k}}_{L^{\beta_{2;k}}} \dots$$

$$|u_{l-1}|^{\alpha_{l-1;k}}_{L^{\beta_{l-1;k}}} |u_1 - v_1|_{L^{\beta_{1,k}}}),$$

$$\leq \kappa(R)\ |A(u_1 - v_1)|,$$

since, for all p, $L^p(\Omega) \subset H^2(\Omega)$ and the H^2 norm is equivalent to the norm $|A|$. We treat the other terms in a similar way except for the last one.

$$\begin{aligned} |f'(u)\, u_l - f'(v)\, v_l| &\leq |(f'(u) - f'(v))\, u_l| + |f'(v)\,(u_l - v_l)|, \\ &\leq \kappa\ |u-v|\ |u_l|_\infty + \kappa |u_l - v_l|, \\ &\leq \kappa(R)\,(|u - v| + |u_l - v_l|). \end{aligned}$$

All these estimates prove Lemma 3.1.

Remark 3.1. We used $n \leq 4$ only in Lemma 3.1. If we look carefully at the proof we can see that Lemma 3.1 is true for every n if $l = 0$ or 1, for $n \leq 8$ if $l = 2$, $n \leq 6$ if $l = 3$ and $n \leq 5$ for $l = 4$. Thus theorem 2.3 is true for larger values of n when k = 1,2,3,4,5.

Part II : The Cahn-Hilliard equation

4. The equation and some properties of the solution

4.1. The equation and the semi-group

We now consider the Cahn Hilliard equation whose unknown is a real valued function u(x,t) defined on $\Omega \times \mathbb{R}^+$ where Ω denotes a regular bounded set of $\mathbb{R}^n (n \leq 3)$:

$$(4.1) \quad \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0 \,, \text{ in } \Omega \times \mathbb{R}^+.$$

Here, f is a polynomial of odd degree with positive leading coefficient

$$f(u) = \sum_{j=1}^{2r-1} a_j u^j \,, \quad a_{2r-1} > 0 \,,$$

and we assume that

$$r \geq 2 \text{ if } n = 1,2 \text{ and } r = 2 \text{ if } n = 3.$$

Strictly speaking, the Cahn-Hilliard equation corresponds to $f(u) = au^3 - bu$, a, $b > 0$, see [23] .

This equation is supplemented with the initial condition :

$$(4.2) \quad u(x,0) = u_0(x) \,, \text{ in } \Omega \,,$$

and with one of the two following boundary conditions :

$$(4.3) \quad \text{Neumann : } \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on } \Gamma = \partial\Omega \,,$$

$$\text{Periodic : } \quad \Omega = \prod_{i=1}^{n} \,]0;L_i[\text{ and } u \text{ is } \Omega \text{ periodic}$$

For the mathematical setting of the equation, it is convenient to introduce the operator $A = -\Delta$ on $L^2(\Omega)$ equipped with its usual scalar product (.,.) and norm $|\,.\,|$. Then A is a positive linear unbounded operator on H with domain

$$D(A) = \{u \in H^2(\Omega), \, (4.3) \text{ holds}\}.$$

Using this abstract setting, (4.1) (4.2) (4.3) rewrites :

(4.4) $\frac{du}{dt} + A^2u + Af(u) = 0,$
$u(0) = u_o$.

As shown in [19], for u_o given in $L^2(\Omega)$, the initial boundary problem (4.1) (4.3) possesses a unique solution u defined for all $t > 0$ such that

$$u \in C(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; D(A)), \ \forall T > 0$$

Furthemore, if $u_o \in D(A) \cap L^{2r}(\Omega)$, then

$$u \in C(\mathbb{R}^+; D(A) \cap L^{2r}(\Omega)) \cap L^2(0,T; D(A^2)), \ \forall T > 0.$$

A particular feature of equation (1.1) is that the average of the solution is conserved, for all $t > 0$:

$$m(u(t)) = \frac{1}{|\Omega|} \int_\Omega u(x,t)\,dx = \frac{1}{|\Omega|} \int_\Omega u_o(x)dx = m(u_o).$$

Thus, there can not exist bounded absorbing sets in the whole space $L^2(\Omega)$. Therefore, following [19,20], we let the semigroup operate on the subset of $L^2(\Omega)$

$$H_\alpha = \{u \in L^2(\Omega) / |m(u)| \leq \alpha\},$$

for some fixed $\alpha > 0$.

We now recall some time uniform estimates borrowed from [19,20] . Let u_o be given in a ball $B(0,R)$ in H_α . Then there exists a time t_o depending on the data, on α and on R such that.

(4.5) $|Au(t)| \leq \kappa_o, \ \forall t \geq t_o$

where κ_o is a constant depending on the data and α. This estimate provides the existence of a bounded absorbing set in $D(A) \cap H_\alpha$. Since $D(A) \cap H_\alpha$ is compactly imbedded in H_α , this yields also the existence of a universal attractor in H_α (see [19,20]).

The estimate (4.5) is extended in [16] to the time derivatives of u :

$$u^{(j)} = \frac{\partial^j u}{\partial t^j}, \ j \in \mathbb{N}.$$

For every $j \geq 1$, there exists a time t_j depending on the data, α, j and R (when $|u_o| \leq R$) such that :

(4.6) $|Au^{(j)}(t)| \leq \kappa_j \ , \ \forall t \geq t_j$,

where κ_j is a constant depending on the data and α.

4.2. Behaviour of small eddies

Let $(w_j)_{j\in \mathbb{N}}$ denote the orthonormal basis of H consisting of the eigenvectors of A and $(\lambda_j)_{j\in \mathbb{N}}$ the associated eigenvalues

$$A w_j = \lambda_j w_j \ , \ j = 1, 2, \ldots$$
$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \ ; \lambda_j \to +\infty \ , \ \text{as } j \to \infty \ .$$

As in Section 1.2, we fix an integer $m \in \mathbb{N}$ and denote by $P = P_m$ the projector in H onto the space spanned by $w_1, \ldots, w_m$; we set $Q = Q_m = I - P_m$ and for the sake of simplicity

$$\lambda = \lambda_m \ , \Lambda = \lambda_{m+1} \ .$$

We also introduce :

$$\delta = \lambda_2 / \lambda_{m+1}$$

Note that :

(4.7) $|A^{\beta+1/2} p|^2 \leq \lambda |A^{\beta} p|^2 \ , \ \forall p \in PD(A^{\beta+1/2})$,

(4.8) $|A^{\beta+1/2} q|^2 \geq \Lambda |A^{\beta} q|^2 \ , \ \forall q \in QD(A^{\beta+1/2})$.

By projection of (4.1) on PH and QH, it comes that $p = Pu$, $q = Qu$ satisfy the following coupled system

(4.9) $\dfrac{dp}{dt} + A^2 p + PA\, f(p+q) = 0$,

(4.10) $\dfrac{dq}{dt} + A^2 q + QA\, f(p+q) = 0$.

The next proposition states that q and all its time derivatives remain small for large time.

Proposition 4.1. *For all j in $\mathbb{N}$, there exists a time t'_j which depends on the data, α , j and R when $|u_0| \leq R$ and $u_0 \in H_\alpha$ such that , for $t \geq t'_j$*

(4.11) $|q^{(j)}(t)| \leq \kappa_j \delta^2$,

(4.12) $|Aq^{(j)}(t)| \leq \kappa_j \delta$, *where κ_j is a constant depending on the data, α and j.*

We omit to give the proof of that proposition that can be found in [16] . It is very similar to the proof of Proposition 1.1.

5. Construction of the family of AIM

We present in this Section the method of construction of a sequence $\mathcal{M}_i$, $i \in N$, of AIM. As in Part I, the manifolds correspond to approximations of equation (4.10) for q. Also, these approximations are improved by considering approximations of an increasing number of time derivatives of q. However since the nonlinear term in (4.1) contains derivatives, the algebra of construction differs from the one in Part I.

We start with the construction of $\mathcal{M}_1$ and $\mathcal{M}_2$ which is borrowed from [16]. Proposition 4.1 indicates that q and $\frac{dq}{dt}$ are small for large time. Therefore we are lead to replace in (4.10) p+q by p while $\frac{dq}{dt}$ is neglected. This gives the following approximate equation :

$$(5.1) \quad A^2 q + Q A f(p) = 0 .$$

For p given in PH since A is injective on QH , (5.1) possesses a unique solution $q_1 \in Q\, D(A^2)$. The manifold $\mathcal{M}_1$ with equation $q_1 = \Phi_1(p)$ is an analytic manifold in H of dimension m. The orbits are attracted by a thin neighborhood of $\mathcal{M}_1$ as shown in the following Theorem.

Theorem 5.1. *There exists a time t_1^* depending on the data, α and R when $|u_0| \leq R$ and $u_0 \in H_\alpha$ such that the orbits of (4.1)-(4.3) remain when $t \geq t_1^*$ at a distance in H of $\mathcal{M}_1$* bounded by $\kappa_1\, \delta^3$; *κ_1 is an appropriate constant which depends on the data and α .*

Proof. Let u = p+q be an orbit of (4.1)-(4.3). For t > 0 , we define

$$q_1(t) = \Phi_1(p(t)).$$

Then, $p(t) + q_1(t)$ lies in $\mathcal{M}_1$ and

$$\text{dist}\,(u(t), \mathcal{M}_1) \leq |q(t) - q_1(t)| .$$

Substracting (4.10) from (5.1), we obtain

$$(5.2) \quad A^2(q_1 - q) = QA(f(p+q) - f(p)) + q^{(1)} .$$

Since, for $n \leq 3$, $H^2(\Omega)$ is an algebra, it is easy to see that the mapping $u \rightarrow f(u)$ from $H^2(\Omega)$ into $H^2(\Omega)$ is lipschitzian on the bounded sets of $H^2(\Omega)$. Therefore, by (4.5), we have :

$$|A\, f(p+q) - A\, f(p)| \leq \kappa \|q\|_{H^2(\Omega)}.$$

Then on the subspace

$$\{u \in H^2(\Omega),\ \int_\Omega u dx = 0\},$$

the norm of $H^2(\Omega)$ is equivalent to the norm $|Au|$. Thus, using also (4.12), we obtain

$$|A\, f(p+q) - A\, f(p)| \leq \kappa\delta ,$$

for t large enough. The second term in the right hand side of (5.2) is bounded by using (4.11)

$$|q^{(1)}| \leq \kappa\, \delta^2, \text{ for large t.}$$

Finally, we have obtaind that

$$|A^2(q_1 - q)| \leq \kappa\, \delta + \kappa\, \delta^2, \tag{5.3}$$

and, thanks to (4.8) , this gives

$$|q_1 - q| \leq \kappa\, \delta^3 .$$

Theorem 5.1 is proved .

It comes from the proof of Theorem 5.1 (especially (5.3)) that the approximation of the time derivative by 0 is of better order than the one of the nonlinearity. Therefore, we can improve the approximation of (4.10) by replacing (5.1) by

$$A^2 q + QA\, f(p+\Phi_1(p)) = 0 . \tag{5.4}$$

For $p \in P H$, we define

$\Phi_2(p) = q_2$, the solution of (5.4).

Let $\mathcal{M}_2$ be the analytic manifold of dimension m in H with equation $q = \Phi_2(p)$. Then

Theorem 5.2 . *There exists a time t_2^* depending on the data, α and R when $|u_o| \le R$ and $u_o \in H_\alpha$ such that the orbits of (4.1)-(4.3) remain when $t \ge t_2^*$ at a distance in H of $\mathcal{M}_2$, bounded by $\kappa_2\delta^4$; κ_2 is an appropriate constant depends on the data and α .*

The proof of this Theorem can be found in [16].

Next, in order to construct $\mathcal{M}_3$, we introduce an approximation of $q^{(1)}$. This is done by differentiating (4.10) with respect to t and neglecting $q^{(2)}$. The method is analog to the one in Section 2, but again the approximations of the different terms in (4.10) will be too crude and we will be able to improve them in a fourth step with $q^{(2)}$ still neglected (the details of the computations for the first six manifolds can be found in [16]). Thus, we see the main difference between the two schemes. For the Cahn-Hilliard, when we neglect $q^{(k)}$ we consider two successive approximations of (4.10) (and therefore two manifolds).

Now, we describe the algebra. The manifolds are defined recursively. Assume that we have obtained $\mathcal{M}_1$, $\mathcal{M}_2$, ..., $\mathcal{M}_{2k-3}$, $\mathcal{M}_{2k-2}$ for some $k \ge 2$ (this has been done for k=2). We aim to construct $\mathcal{M}_{2k-1}$ and $\mathcal{M}_{2k}$. We start by $\mathcal{M}_{2k-1}$; $\mathcal{M}_{2k-1}$ is defined thanks to two sequences $p^i_{2k-2i-1}$, i = 1 , ..., k-1 and $q^i_{2k-2i-1}$, i = 0, ..., k-1, where $p^i_{2k-2i-1}$ provides an approximation of $p^{(i)}$ and q^i_{2k-2i} of $q^{(i)}$. These families satisfy estimates of the form

$$|p^i_{2k-2i-1} - p^{(i)}| \le \kappa_k \delta^{2k-2i}, \tag{5.5}$$

$$|q^i_{2k-2i-1} - q^{(i)}| \le \kappa_k \delta^{2k-2i+1}, \tag{5.6}$$

for $t \ge t''_k$ where κ_k depends on the data, α and k ; t''_k depends on the data, α, R and k when $|u_o| \le R$ and $u_o \in H_\alpha$ (this will be shown in Section 6 below).

We start by defining $p_{2k-2i-1}$ by induction on i, i increasing. We denote :

$$F_l(u,u^{(1)}, ..., u^{(l)}) = \frac{d^l}{dt^l} F(u).$$

Then, $p^i_{2k-2i-1}$, is given by

$$p^i_{2k-2i-1} = - A^2 p^{i-1}_{2k-2i+1} - PA\, f_{i-1}\,(p+q_{2k-2i-1}, p^1_{2k-2i} + q^1_{2k-2i-1}, ..., p^{i-1}_{2k-2i} + q^{i-1}_{2k-2i-1}), \tag{5.7}$$

where we have set

$$p_l^o = p \, , \ \forall l \, .$$

Note that (5.7) is an explicit formula and all the terms in the right-hand side of (5.7) are known either from the construction of the previous manifolds or from step (i-1) of the recursive scheme. Also (5.7) is obtained as an approximation of equation (4.9) for p differentiated (i-1) times.

Then, the sequence $q^i_{2k-2i-1}$, i = k-1, ..., 1, is defined recursively on i for i decreasing thanks to the formula

$$(5.8) \quad q^{i+1}_{2k-2i-3} + A^2 q^i_{2k-2i-1} + QA\, f_i\,(p+q_{2k-2i-2}\, , p^1_{2k-2i-1} + q^1_{2k-2i-2}\, , \ldots, \\ p^i_{2k-2i-1} + q^i_{2k-2i-2}) = 0 \, .$$

We agreed that

$$q^l_o = q^l_{-1} = 0 \, , \ \forall \, l \geq 1 \, .$$

It is easy to check that all necessary terms are known in (5.8). This formula is obtained by considering equation (4.19) for q differentiated i times. Note that for i = k-1, we have $q^{i+1}_{2k-2i-3} = 0$; this means that the derivative $q^{(k)}$ is neglected in the construction of $\mathcal{M}_{2k-1}$.

Finally, $q_{2k-1} = q^o_{2k-1} = \Phi_{2k-1}(p)$ is given by the analog of (5.8) for i=0 :

$$q^1_{2k-3} + A^2 q_{2k-1} + Q\,A\, f(p+q_{2k-2}) = 0 \, .$$

We now proceed to the construction of $\mathcal{M}_{2k}$ .For that purpose, we introduce two families p^i_{2k-2i} i=1, ..., k-1 and q^i_{2k-2i} , i=0, ..., k-1 which satisfy the estimates

$$(5.9) \quad | p^i_{2k-2i} - p^{(i)} | \leq \kappa_k \, \delta^{2k-2i+1} \, ,$$

$$(5.10) \quad | q^i_{2k-2i} - q^{(i)} | \leq \kappa_k \, \delta^{2k-2i+2} \, ,$$

$\forall \, t \geq t''_k$ where κ_k and t''_k are as in (5.5) (5.6).

The construction is analogous to the one of $\mathcal{M}_{2k-1}$, so that we only give the formulas. We first define p^i_{2k-2i} by

$$p^{i}_{2k-2i} = - A^2 p^{i-1}_{2k-2i+2} - PA\, f_{i-1}(p+q_{2k-2i},\ p^{1}_{2k-2i+1} + q^{1}_{2k-2i}, \dots, p^{i-1}_{2k-2i+1} + q^{i-1}_{2k-2i}),$$

for i = 1, ..., k-1. Then q^{i}_{2k-i}, i = k-1, ..., 1 is given by

$$(5.11)\quad q^{i+1}_{2k-2i-2} + A^2 q^{i}_{2k-2i} + QA\, f_i(p+q_{2k-2i-1},\ p^{1}_{2k-2i} + q^{1}_{2k-2i-1}, \dots, p^{i}_{2k-2i} + q^{i}_{2k-2i-1}) = 0$$

Again it follows from (5.11) for i = k-1 that $q^{(k)}$ is neglected in the construction of $\mathcal{M}_{2k}$.

Lastly for p given in PH, $\Phi_{2k}(p) = q_{2k}$ is the solution of

$$q^{1}_{2k-2} + A^2 q_{2k} + QA\, f(p + q_{2k-1}) = 0 .$$

To conclude, we have defined the mapping $\Phi_k : PH \to QD(A^2)$ for all $k \in \mathbb{N}^*$. The graph of this mapping is an analytic manifold $\mathcal{M}_k$ of dimension m in H. The $\mathcal{M}_k$'s provide a better and better order approximation to the orbits as k increases.

Theorem 4.3. *There exists a time t_k^* depending on the data, k, α and R when $|u_0| \le R$ and $u_0 \in H_\alpha$ such the orbits of (4.1)-(4.3) remain when $t \ge t_k^*$ at a distance in H of $\mathcal{M}_k$ bounded by $\kappa_k\, \delta^{k+2}$; κ_k is an appropriate constant depending on the data, k and α.*

Remark 4.4. Remarks similar to Remark 2.4 and 2.5 can be made here. The details are left to the reader.

Remark 5.4. On the contrary of the case of reaction-diffusion equations, the schemes that we have just defined can not be improved.

Proof of Theorem 4.3. We postpone to Section 6 the proof of estimates (5.5) (5.6) (5.9) (5.10). Let us show here that these estimates yield Theorem 4.3.

Let u(t) = p(t) + q(t) be a solution of (4.1)-(4.3) and let $p(t) + q_k(t)$ the induced trajectory on $\mathcal{M}_k$. We aim to estimate

$$\chi_k(t) = q_k(t) - q(t).$$

It follows from (5.9) and (5.13) that q_k satisfies

$$q^{1}_{k-2} + A^2 q_k + Q\, A\, f(p + q_{k-1}) = 0$$

By substracting (4.10) to this equation, we obtain

(5.14) $$A^2\chi_k = QA\,(f(p+q) - f(p+q_{k-1})) + q^{(1)} - q^1_{k-2}\,.$$

It is clear from (4.6) and (5.6) (5.10) that p, q and q_{k-1} are bounded for large t by constants depending only on the data, k and α. Therefore, since f(u) is lipschitzian on the bounded sets of $H^2(\Omega)$, (5.14) yields

(5.15) $$|A^2\chi_k| \leq \kappa\, |A(q - q_{k-1})| + |q^{(1)} - q^1{}_{k-2}|,$$

and using (5.6) (5.10),

$$|A^2\chi_k| \leq \kappa\,\delta^k\,,$$

$$|\chi_k| \leq \kappa\,\delta^{k+2}\,.$$

Theorem 4.3 is proved.

6. Proof of the estimates on time derivatives

The aim of this Section is to derive estimates (5.5) (5.6) (5.9) (5.10). We will prove by induction on k that

$(6.1)_k$ $$|p^i_{2k-2i-1} - p^{(i)}| \leq \kappa_k\,\delta^{2k-2i},\ i = 0, \ldots, k-1,$$

$(6.2)_k$ $$|A(q^i_{2k-2i-1} - q^{(i)})| \leq \kappa_k\,\delta^{2k-2i},\ i = 0, \ldots, k-1,$$

$(6.3)_k$ $$|p^i_{2k-2i} - p^{(i)}| \leq \kappa_k\,\delta^{2k-2i+1},\ i = 0, \ldots, k-1,$$

$(6.4)_k$ $$|A(q^i_{2k-2i} - q^{(i)})| \leq \kappa_k\,\delta^{2k-2i+1},\ i = 0, \ldots, k-1,$$

for $t \geq t_k$ where κ_k depends on the data, k, α and t_k depends on the data, k, α and R when $|u_o| \leq R$ and $u_o \in H_\alpha$. We recall that $p^o{}_l = p$, $\forall\, l$.

For k = 1, $(6.1)_1$ and $(6.3)_1$ are obvious. Also (5.3) along with (4.8) yield $(6.2)_1$. Finally, $(6.4)_1$ states the result of approximation for the manifold $\mathcal{M}_2$ that can be found in [16] and is derived with techniques similar to the ones used here.

Next assume that $(6.1)_k$-$(6.4)_k$ are true up to order k-1 for k>1. We aim to prove them at order k. Following the schemes of construction of the manifolds, the relations $(6.1)_k$ $(6.2)_k$ $(6.3)_k$ $(6.4)_k$ will be proved successively.

We start by $(6.1)_k$ and argue by induction on i, for i increasing. For i = 0, relation $(6.1)_k$ is obvious.

Assume it is true for 1, 2, ..., i-1. By substracting (5.7) from the (i-1)th derivative of (4.9), we obtain

$$(6.5)\quad |p^{i}_{2k-2i-1} - p^{(i)}| \le |A^2(p^{i-1}_{2k-2i+1} - p^{(i-1)})| + |A(f_{i-1}(p+q_{2k-2i-1}, p^{1}_{2k-2i} + q^{1}_{2k-2i-1}, \dots, p^{i-1}_{2k-2i} + q^{i-1}_{2k-2i-1}) - f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}))|.$$

Thanks to the induction assumption for i-1 and (4.7), we have for large t

$$(6.6)\quad |A^2(p^{i-1}_{2k-2i+1} - p^{(i-1)})| \le \lambda^2 |p^{i-1}_{2k-2i+1} - p^{(i-1)}| \le \kappa\lambda^2 \delta^{2k-2i+2} \le \kappa\,\delta^{2k-2i}.$$

Next, for majorizing the second term in the right handside of (6.5), we use Lemma 6.1 below. Indeed, thanks to (4.6) and the induction assumption, it is easy to check that

$$p+q_{2k-2i-1},\ p^{\alpha}_{2k-2i} + q^{\alpha}_{2k-2i-1},\ p+q,\ p^{(\alpha)} + q^{(\alpha)},\ 1 \le \alpha \le i-1$$

are bounded in $H^2(\Omega)$ by constants depending only on the data, k and α for large time. Therefore Lemma 6.1 applies and yields

$$(6.7)\quad |A(f_{i-1}(p+q_{2k-2i-1}, p^{1}_{2k-2i} + q^{1}_{2k-2i-1}, \dots, p^{i-1}_{2k-2i} + q^{i-1}_{2k-2i-1}) - f_{i-1}(p+q, p^{(1)} + q^{(1)}, \dots, p^{(i-1)} + q^{(i-1)}))| \le \kappa\{\|p^{1}_{2k-2i} - p^{(1)}\|_{H^2(\Omega)} + \dots + \|p^{i-1}_{2k-2i} - p^{(i-1)}\|_{H^2(\Omega)} + \|q_{2k-2i-1} - q\|_{H^2(\Omega)} + \|q^{1}_{2k-2i} - q^{(1)}\|_{H^2(\Omega)} + \dots + \|q^{i-1}_{2k-2i-1} - q^{(i-1)}\|_{H^2(\Omega)}\}.$$

For $1 \le \alpha \le i-1$, we have :

$$\|p^{\alpha}_{2k-2i} - p^{(\alpha)}\|_{H^2(\Omega)} \le c_1\{|p^{\alpha}_{2k-2i} - p^{(\alpha)}| + |A(p^{\alpha}_{2k-2i} - p^{(\alpha)})|\},$$

with c_1 a universal constant,

$\le$ (thanks to (4.7)),

$$\le c_1(1+\lambda)\,|p^{\alpha}_{2k-2i} - p^{(\alpha)}|.$$

Therefore using $(6.3)_{k-i+\alpha}$

$$(6.8)\quad \|p^{\alpha}_{2k-2i} - p^{(\alpha)}\|_{H^2(\Omega)} \le c_1(1+\lambda)\,\kappa\,\delta^{2k-2i+1}, \quad \le \kappa\,\delta^{2k-2i}.$$

Also, for $0 \leq \alpha \leq i-1$, we see that

$$\|q^{\alpha}_{2k-2i-1} - q^{(\alpha)}\|_{H^2(\Omega)} \leq c_2 \,|\, A(q^{\alpha}_{2k-2i-1} - q^{(\alpha)}) \,|\,,$$

where c_2 is a universal constant and thanks to $(6.2)_{k-i+\alpha}$

(6.9) $\|q^{\alpha}_{2k-2i-1} - q^{(\alpha)}\|_{H^2(\Omega)} \leq c_2\, \kappa\, \delta^{2k-2i}$

To conclude, we infer from (6.5)-(6.9) that

$$|\, p^{i}_{2k-2i-1} - p^{(i)} \,| \leq \kappa\, \delta^{2k-2i}, \text{ for large } t,$$

which is $(6.1)_k$.

The estimates $(6.2)_k$, $(6.3)_k$, $(6.4)_k$ are proved thanks to similar arguments, except that the induction is for i decreasing for $(6.2)_k$ and $(6.4)_k$. The details of these very technical estimates are left to the reader.

To conclude, there remains to check the following Lemma.

Lemma 6.1. *The mapping*

$F_k : H^2(\Omega)^{k+1} \to H^2(\Omega)$,

$(u_0, \ldots, u_k) \to F_k(u_0, u_1, \ldots, u_k)$,

is lipschitzian on the bounded sets of $H^2(\Omega)^{k+1}$.

The proof is easy since f_k is a polynomial and $H^2(\Omega)$ is an algebra (recall that $\Omega \subset \mathbb{R}^n$, $n \leq 3$). The details are left to the reader.

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3ème PARTIE:

UNE PERTURBATION SINGULIERE DE L'EQUATION DE CAHN-HILLIARD.

A SINGULAR PERTURBATION OF THE CAHN-HILLIARD EQUATION

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INTRODUCTION.

Let us consider the Cahn-Hilliard equation :

$$\frac{\partial u}{\partial t} + \nu\Delta^2 u - \Delta f(u) = 0, \text{ in } \Omega \times \mathbb{R}^+, \tag{0.1}$$

where $u = u(x,t)$ is a real-valued function, Ω is an open set in $\mathbb{R}^n$ and the nonlinear function f is a polynomial of even order whose leading coefficient is positive.

We know that (0.1) (together with Neumann or periodic boundary conditions) possesses solutions defined for all time and those solutions converge to a global attractor $\mathcal{A}$ (see R. Temam [1] and the references quoted there).

We are interested in a singular perturbation of that equation that we obtain by adding a second order time derivative with a small coefficient :

$$\epsilon\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial u_\epsilon}{\partial t} + \nu\Delta^2 u_\epsilon - \Delta f(u_\epsilon) = 0, \tag{0.2}$$

we want to show the same type of results for (0.2) as for (0.1) : existence of solutions and of a global attractor $\mathcal{A}_\epsilon$. Moreover, we want to study the behaviour of $\mathcal{A}_\epsilon$ when ϵ goes to zero and to know whether $\mathcal{A}_\epsilon$ "converges" to $\mathcal{A}$. More precisely, since $\mathcal{A}$ lies in $L^2(\Omega)$ and $\mathcal{A}_\epsilon$ in $H^2(\Omega) \times L^2(\Omega)$ (indeed, (0.2) is a second order equation thus the solution is $(u, \frac{\partial u}{\partial t})$ and lies in a product space), we will first define a convenient embedding $\mathcal{A}^*$ of $\mathcal{A}$ in $H^2(\Omega) \times L^2(\Omega)$ (an element u in $\mathcal{A}$ will be identified to the pair $\{u, v\}$ where v is the derivative at time 0 of a solution of (0.1) passing through u at time 0, then we will show that the Hausdorff semidistance $\delta(\mathcal{A}_\epsilon, \mathcal{A}^*)$ for the topology of $H^2(\Omega) \times L^2(\Omega)$ converges to zero ; in other words the attractor $\mathcal{A}_\epsilon$ is upper semicontinuous at $\epsilon = 0$. For that purpose, we will need a priori estimates independent of ϵ for the solution of (0.2). Such a priori estimates are not easy to get and that will be the object of Section II. Unfortunately, for technical reasons which will be mentionned below, our results are only valid in space dimension one. In space dimension two or three, we do not know whether there exists an attractor $\mathcal{A}_\epsilon$ for (0.2) while in the unperturbed case the existence of the attractor is known in space dimension one, two or three.

In a related situation, J.K. Hale and G. Raugel [1] proved the upper semicontinuity of the attractor for a singular perturbation of a reaction diffusion equation.

The lower semicontinuity of $\mathcal{A}_\epsilon$ is a problem of a totally different nature (see J.K. Hale and G. Raugel [2]) ; it needs very precise information on the attractor and we intend to adress it eventually.

This work is organized as follows : in Section I, we set the notation and state the theorems of existence of solutions for (0.1) and (0.2) ; thus we can define the corresponding semigroups. Concerning (0.1), we do not give the proof that the reader can find in R. Temam [1] ; concerning (0.2), the proof are classical and are sketched in appendix A. We end Section I with a few remarks on Lyapunov functions for (0.1) and (0.2). In Section II, we derive a priori estimates on the solutions of (0.2). We will prove the existence of bounded absorbing sets in different spaces. Moreover, these absorbing sets are independent of ϵ. In Section III, we use the results of Section II to prove the existence of attractors for (0.2), these are attractors in the weak topology (whereas the attractor of (0.1) is in the strong topology). In Section IV, we prove the upper semicontinuity of the attractor at $\epsilon = 0$. We use the same type of methods as J.K. Hale and G. Raugel [1].

I. Preliminaries

a) Notations.

Ω is the interval $[0, L]$ in $\mathbb{R}$. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and scalar product in $L^2(\Omega)$. For each u in $L^2(\Omega)$, we denote by $m(u)$ its average :

$$m(u) = \frac{1}{L}\int_\Omega u(x)dx. \tag{1.1}$$

$\dot{L}^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions whose average is zero :

$$\dot{L}^2(\Omega) = \{u \in L^2(\Omega)/m(u) = 0\}.$$

For each u in $L^2(\Omega)$, $\tilde{u}$ is its projection on $\dot{L}^2(\Omega)$:

$$\tilde{u} = u - m(u). \tag{1.2}$$

We introduce the unbounded linear operator given by

$$A = \Delta^2,$$

$$D(A) = \{u \in H^4(\Omega) / \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \tag{1.3$_N$}$$

if we consider the Neumann boundary conditions for (0.1), (0.2), or

$$D(A) = H^4_{per}(\Omega), \tag{1.3$_P$}$$

if we consider periodic boundary conditions ($H^4_{per}(\Omega)$ is the space of functions of $H^4(\Omega)$ whose derivatives of order less than three are equal at 0 and L). A is a positive self-adjoint operator, it possesses an orthonormal basis of eigenvectors $(w_j)_{j\in\mathbb{N}}$. We denote by $(\lambda_j)_{j\in\mathbb{N}}$ the associated eigenvalues, we then have :

$$\begin{cases} \forall i,j \in \mathbb{N} \ : \ (w_i, w_j) = \delta_{i,j} \\ \qquad A w_i = \lambda_i w_i \\ 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \to +\infty. \end{cases}$$

Moreover, w_0 is the constant function equal to $\frac{1}{L}$ and for all u in $L^2(\Omega)$:

$$m(u) = (u, w_0). \tag{1.4}$$

For all s in $\mathbb{R}$, we recall that A^s is defined by

$$A^s u = \sum_{j=1}^{+\infty} \lambda_j^s u_j w_j, \quad \forall u = \sum_{j=0}^{+\infty} u_j w_j.$$

We denote its domain by

$$V_s = D(A^{s/4}). \tag{1.5}$$

V_s(1) is endowed with the seminorm and semiscalar product :

$$| u |_s = | A^{s/4} u |, \tag{1.6}$$

$$(u, v)_s = (A^{s/4} u, A^{s/4} v), \tag{1.7}$$

and with the norm :

$$\| u \|_s = (| u |_s^2 + m(u)^2)^{1/2}. \tag{1.8}$$

(1) Usually, V_s denotes the domain of $A^{s/2}$. Here, we prefer to use the domain of $A^{s/4}$, so that V_s is embedded in the Sobolev space $H^s(\Omega)$ (when s is positive).

Moreover, for $u = \sum_{j=0}^{+\infty} u_j w_j$, we have

$$u \in V_s \text{ iff } \sum_{j=1}^{+\infty} \lambda_j^{s/2} u_j^2 < +\infty,$$

and then :

$$| u |_s = \left(\sum_{j=1}^{+\infty} \lambda_j^{s/2} u_j^2 \right)^{1/2} , \quad m(u) = u_0. \tag{1.9}$$

When s is positive, V_s is a subspace of $H^s(0, L)$ and $\| \ \|_s$ is a norm equivalent to the usual norm of $H^s(0, L)$.

We easily deduce from (1.9) the interpolation inequality :

$$| u |_{\lambda s_1 + (1-\lambda) s_2} \leq | u |_{s_1}^{\lambda} | u |_{s_2}^{1-\lambda}, \text{ for all } u \text{ in } V_{s_2},$$
$$s_1 < s_2 \text{ and } \lambda \in [0, 1].$$

Let f denote a polynomial of even order whose leading coefficient is positive and vanishing at 0 :

$$f(x) = \sum_{i=1}^{2p-1} a_i x^i, \ a_{2p-1} > 0. \tag{1.10}$$

We denote by g the primitive of f vanishing at 0 :

$$g(x) = \sum_{i=2}^{2p} \frac{a_{i-1}}{i} x^i.$$

b) The Cahn Hilliard equation.

The unknown function is a scalar $u(x, t)$, $x \in [0, L]$, $t \in \mathbb{R}^+$, and the equation reads :

$$\frac{\partial u}{\partial t} + \nu \Delta^2 u - \Delta f(u) = 0 \text{ in } [0, L] \times \mathbb{R}^+, \tag{1.11}$$

where ν is a strictly positive real parameter. We supplement (1.11) with an initial condition:

$$u(x, 0) = u_0(x) \text{ for } x \text{ in } [0, L], \tag{1.12}$$

and with boundary conditions that can be one of two types. Either the Neumann boundary conditions :

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ at } 0 \text{ and } L, \tag*{$(1.13)_N$}$$

or the periodic boundary conditions :

$$\frac{\partial^\alpha u}{\partial x^\alpha}(0,\cdot) = \frac{\partial^\alpha u}{\partial x^\alpha}(L,\cdot) \text{ for } \alpha \leq 3. \tag{1.13$_P$}$$

(We will write (1.13) for $(1.13)_N$ or $(1.13)_p$; there will be no confusion since the two cases are treated in the same way. The domain of A will be $(1.3)_N$ if we consider $(1.13)_N$ and $(1.3)_P$ if we consider $(1.13)_P$).

We then have an existence and uniqueness theorem for solutions whose proof can be found in R. Temam [1] :

Theorem 1.1. *For all u_0 in V_0, the equation* (1.11)(1.12)(1.13) *has a unique solution u which belongs to*

$$C([0,T];V_0) \cap L^2(0,T;V_2) \cap L^{2p}(0,T;L^{2p}(0,L))$$

for all $T > 0$.

The mapping $S(t) : u_0 \to u(t)$ is continuous in V_0 and the familly $(S(t))_{t\geq 0}$ is a semigroup.

The function $J(u) = \frac{\nu}{2} \mid u \mid_1^2 + \int_\Omega g(u)dx$ decays along the orbits.

Moreover, if u_0 is in V_2, then :

$$u \in C([0,T];V_2) \cap L^2(0,T;V_4)$$

for all $T > 0$.

Remark. Theorem 1.1 is still true in space dimension $n = 2$ and, if we assume $p = 2$, in space dimension $n = 3$.

c) The perturbed Cahn-Hilliard equation.

The unknown function is a scalar $u_\epsilon(x,t)$, $x \in [0,L]$, $t \in \mathbb{R}^+$, and the equation reads:

$$\epsilon\frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial u_\epsilon}{\partial t} + \nu\Delta^2 u_\epsilon - \Delta f(u_\epsilon) = 0 \text{ in } [0,L] \times \mathbb{R}^+, \tag{1.14}$$

where ϵ is a strictly positive real parameter which is supposed to be small. f and ν are as in section b). We supplement (1.14) with the initial conditions

$$\begin{cases} u_\epsilon(x,0) = u_0(x), \\ \dfrac{\partial u_\epsilon(x,0)}{\partial t} = u_1(x), \text{ for all } x \in \Omega, \end{cases} \tag{1.15}$$

and with the same boundary conditions $(1.13)_N$ or $(1.13)_p$.

For the existence and uniqueness of solutions, we have the following theorem whose proof can be found in Appendix A.

Theorem 1.2. *For $s = 1$, 2 or 3 and for all (u_0, u_1) in $V_s \times V_{s-2}$, equation* (1.14)(1.15)(1.13) *possesses a unique solution u_ϵ and*

$$\left(u_\epsilon, \frac{\partial u_\epsilon}{\partial t}\right) \in C([0,T], V_s \times V_{s-2}),$$

for all $T > 0$. The mapping :

$$S_\epsilon^s(t) \; : \; (u_0, u_1) \to \left(u_\epsilon(t), \frac{\partial u_\epsilon}{\partial t}(t)\right)$$

is continuous from $V_s \times V_{s-2}$ into itself and the family $(S_\epsilon^s(t)_{t\geq 0}$ forms a semigroup.

Remark. We can easily prove that $(S_\epsilon^s(t)$ is in fact an homeomorphism from $V_s \times V_{s-2}$ into itself and is defined for all t in $\mathbb{R}$. Therefore $(S_\epsilon^s(t)_{t\geq 0}$ is in fact a group. Thus there is a difference with the semigroup introduced in the preceding section for the Cahn-Hilliard equation that has a regularizing effect and thus cannot be invertible.

d) Evolution of the average.

In order to study the long time behaviour of the solutions of equation (1.14), it is useful to rewrite the equation as a system for its space average $m(u_\epsilon)$ and $\tilde{u}_\epsilon = u_\epsilon - m(u_\epsilon)$. We take the scalar product of (1.19) with $w_0 = \frac{1}{L}$ (this amounts to taking the average of (1.14)); we obtain

$$\epsilon \frac{d^2}{dt^2} m(u_\epsilon) + \frac{d}{dt} m(u_\epsilon) = 0. \tag{1.17}$$

We substract (1.17) from (1.14), and obtain

$$\epsilon \frac{d^2}{dt^2} \tilde{u}_\epsilon + \frac{d}{dt} \tilde{u}_\epsilon + \nu \Delta^2 \tilde{u}_\epsilon - \Delta f(\tilde{u}_\epsilon + m(u_\epsilon)) = 0. \tag{1.18}$$

We supplement (1.17)(1.18) with the obvious initial conditions :

$$m(u_\epsilon)(0) = m(u_0), \; m\left(\frac{du_\epsilon}{dt}\right)(0) = m(u_1), \tag{1.19}$$

$$\tilde{u}_\epsilon(0) = \tilde{u}_0, \; \frac{\partial \tilde{u}_\epsilon}{\partial t}(0) = \tilde{u}_1. \tag{1.20}$$

Then (1.14)(1.15)(1.13) is equivalent to the system (1.17)(1.20) (supplemented by boundary conditions for $\tilde{u}_\epsilon$).

Moreover, (1.17)(1.18) can be solved explicitely for $m(u_\epsilon)$, and we obtain

$$m(u_\epsilon) = m(u_0) + \epsilon m(u_1)(1 - e^{-t/\epsilon}).$$

We easily deduce :

Proposition 1.1. Let u be the solution of (1.14)(1.15)(1.13) with $(u_0, u_1) \in V_s \times V_{s-2}$ ($s = 1, 2$ or 3), then $m(u_\epsilon) + \epsilon m\left(\frac{\partial u_\epsilon}{\partial t}\right)$ is independent of t and

$$m(u_\epsilon) \to m(u_0) + \epsilon m(u_1) \text{ when } t \to +\infty,$$
$$m\left(\frac{\partial u_\epsilon}{\partial t}\right) \to 0 \text{ when } t \to +\infty.$$

For the Cahn Hilliard equation, the average of the solution is constant and the equivalent system formally reduces to the preceding case with $\epsilon = 0$.

e) Lyapunov functions.

Thanks to Theorem 1.1, the function $J(u) = \frac{\nu}{2} \mid u \mid_1^2 + \int_\Omega g(u)dx$ is a Lyapunov function on V_0 for the Cahn-Hilliard equation. For the perturbed equation (1.14), we do not have a Lyapunov function on the whole space $V_s \times V_{s-2}$. We take the scalar product of (1.14) with $A^{-1/2}\frac{du_\epsilon}{dt}$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon\left|\frac{du'_\epsilon}{dt}\right|_{-1}^2 + \nu \mid u_\epsilon \mid_1^2\right) + \left|\frac{du_\epsilon}{dt}\right|_{-1}^2 + \left(f(u_\epsilon), \frac{du_\epsilon}{dt}\right)_0 = 0.$$

But,

$$\left(f(u_\epsilon), \frac{du_\epsilon}{dt}\right)_0 = \left(f(u_\epsilon), \frac{du_\epsilon}{dt}\right) - \left(f(u_\epsilon), m\left(\frac{du_\epsilon}{dt}\right)\right)$$
$$= \frac{d}{dt}\int_\Omega g(u_\epsilon)dx - \left(f(u_\epsilon), m\left(\frac{du_\epsilon}{dt}\right)\right).$$

Thus :

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon\left|\frac{du'_\epsilon}{dt}\right|_{-1}^2 + \nu \mid u_\epsilon \mid_1^2 + 2\int_\Omega g(u_\epsilon)dx\right) + \left|\frac{du_\epsilon}{dt}\right|_{-1}^2 - \left(f(u_\epsilon), m\left(\frac{du_\epsilon}{dt}\right)\right) = 0.$$

We cannot conclude in general that the function $J_\epsilon(u, v) = \epsilon \mid v \mid_{-1}^2 + \nu \mid u \mid_1^2 + 2\int_\Omega g(u)dx$ decays along the orbits of (1.14)(1.15)(1.13) (we do not know the sign of $\left(f(u_\epsilon), m(\frac{du_\epsilon}{dt})\right)$).

If we restrict the semigroup $(S_\epsilon^s(t))_{t\geq 0}$ to the subspace :

$$\mathcal{X}_0 = \{(u,v) \in V_s \times V_{s-2} / m(v) = 0\}$$

(this space is stable for the semigroup thanks to Proposition 1.1), then $J_\epsilon(u,v)$ decays along the orbits in χ_0. Thus, we have a Lyapunov function on the subspace χ_0.

II. Time uniform estimate for the perturbed Cahn-Hilliard equation.

a) In this section, we prove the existence of bounded absorbing sets for the equation (1.14)(1.15)(1.13) in the spaces X_s^α that we define hereafter. As we will study in Section IV the behaviour when ϵ converges to zero, we will always make precise the dependence of our estimate on ϵ, and the constants that appear are independent of ϵ. ϵ is supposed to be in a fixed interval $]0,\epsilon_0[$.

Due to Proposition 1.1 (in particular to the fact that $m(u_\epsilon)+\epsilon m\left(\frac{du_\epsilon}{dt}\right)$ is independent of t), it is impossible to find bounded absorbing sets in the whole space $V_s \times V_{s-2}$, thus we define for $s = 1,2$ or 3 and for all α in $\mathbb{R}^+$:

$$X_s^\alpha = \{(u,v) \in V_s \times V_{s-2} : \mid m(u) \mid +\epsilon_0 \mid m(v) \mid \leq \alpha\}.$$

Thanks to Proposition 1.1, X_s^α is stable for the semigroup $(S_\epsilon^s(t))_{t\geq 0}$, we will prove the existence of bounded absorbing sets for the restriction of $S_\epsilon^s(t)$ to X_s^α. We first give a few lemmas that we will need.

Lemma 2.1. *For $s = 1,2$ or 3, let $u \in L^2(0,T;V_s)$ satisfy :*

$$\frac{du}{dt} \in L^2(0,T;V_{s-2})$$

and

$$\epsilon\frac{d^2u}{dt^2} + \nu Au \in L^2(0,T;V_{s-2}),$$

then

$$\left(\epsilon\frac{\partial^2 u}{\partial t^2} + \nu\Delta u, A^{s/2-1}\frac{\partial u}{\partial t}\right) = \frac{1}{2}\frac{d}{dt}\left(\epsilon\left|\frac{\partial u}{\partial t}\right|_{s-2}^2 + \mid u \mid_s^2\right),$$

in the sense of distribution on $]0,T[$.

This lemma is classical, the reader can find the proof in J.M. Ghidaglia and R. Temam [1].

Lemma 2.2. Let $0 \leq \lambda \leq min\left(\frac{\nu}{2}\lambda_1, \frac{1}{2\epsilon_0}\right)$. Then for all (u,v) in $V_s \times V_{s-2}$:

$$(1-\lambda\epsilon) \mid v \mid_{s-2}^2 + \lambda(\lambda\epsilon - 1)(u,v)_{s-2} + \lambda\nu \mid u \mid_s^2 \geq \frac{1}{4} \mid v \mid_{s-2}^2 + \frac{\lambda\nu}{2} \mid u \mid_s^2 .$$

The proof of this lemma is easy, we use the fact that for all u in V_{s-2} :

$$\mid u \mid_{s-2} \leq \frac{1}{\sqrt{\lambda_1}} \mid u \mid_s .$$

We now define the three following functionnals :

$$\begin{cases} \text{For all } \varphi, \psi \text{ in } V_1 \times V_{-1} \ : \\ E_1(\varphi,\psi) = \epsilon \mid \psi + \lambda\varphi \mid_1^2 + \nu \mid \varphi \mid_{-1}^2 + 2\int_\Omega g(\varphi)dx \end{cases}$$

$$\begin{cases} \text{For all } \varphi, \psi \text{ in } V_2 \times V_0 \ : \\ E_2(\varphi,\psi) = \epsilon \mid \psi + \lambda\varphi \mid_0^2 + \nu \mid \varphi \mid_1^2 + 2\int_\Omega f'(\varphi) \mid \nabla\varphi \mid^2 dx \end{cases}$$

$$\begin{cases} \text{For all } \varphi, \psi \text{ in } V_1 \times V_{-1}, u \text{ in } V_2 \ : \\ E_3(\varphi,\psi) = \epsilon \mid \psi + \lambda\varphi \mid_{-1}^2 + \nu \mid \varphi \mid_1^2 + 2\int_\Omega f'(u)\tilde{\varphi}^2 dx. \end{cases}$$

(λ is the constant appearing in Lemma 2.2).

Then we have :

Lemma 2.3. *There exist some constants $c_1, ..., c_6$, $c_1', ..., c_6'$ independent of ϵ such that :*

$$C_1(\epsilon \mid \psi \mid_{-1}^2 + \mid \varphi \mid_1^2) - C_2 \leq E_1(\varphi,\psi) \leq C_1'(\epsilon \mid \psi \mid_{-1}^2 + \mid \varphi \mid_1^2) + C_2' \parallel \varphi \parallel_1^{2p},$$
$$C_3(\epsilon \mid \psi \mid_0^2 + \mid \varphi \mid_2^2) - C_3 \leq E_2(\varphi,\psi) \leq C_3'(\epsilon \mid \psi \mid_0^2 + \mid \varphi \mid_2^2) + C_4' \parallel \varphi \parallel_1^{2p},$$
$$C_5(\epsilon \mid \psi \mid_{-1}^2 + \mid \varphi \mid_1^2) - C_6 \leq E_3(\varphi,\psi) \leq C_5'(\epsilon \mid \psi \mid_{-1}^2 + \mid \varphi \mid_1^2) + C_6' \parallel u \parallel_1^{2p-2} \mid \varphi \mid_0^2 .$$

Proof of Lemma 2.3 : Let φ, ψ in $V_1 \times V_{-1}$, then

$$\begin{aligned} \epsilon \mid \psi + \lambda\varphi \mid_{-1}^2 &\leq 2\left(\epsilon \mid \psi \mid_{-1}^2 + \epsilon\lambda^2 \mid \varphi \mid_{-1}^2\right) \\ &\leq 2\left(\epsilon \mid \psi \mid_{-1}^2 + \frac{\epsilon}{4\epsilon_0^2} \mid \varphi \mid_{-1}^2\right) \\ &\leq 2\left((\epsilon \mid \psi \mid_{-1}^2 + \frac{1}{2\epsilon_0\lambda_1} \mid \varphi \mid_1^2\right) \end{aligned}$$

g is a polynomial of even order with a positive leading coefficient, therefore we can find two positive constants C_2 and k such that

$$k \mid x \mid^{2p} \geq g(x) \geq -\frac{C_2}{2 \mid \Omega \mid}, \text{ for all real } x.$$

We deduce :

$$2\int_\Omega g(\varphi)dx \geq -C_2,$$

and the left hand side of the first inequalities follows.

Thanks to Sovolev inequalities, we know that $L^\infty(\Omega)$ is embedded in $H^1(\Omega)$ and we can find a positive constant C such that :

$$\mid \varphi \mid_{L^\infty} \leq C \parallel \varphi \parallel_1 .$$

We deduce :

$$\begin{aligned} 2\int_\Omega g(\varphi)dx &\leq 2k \mid \varphi \mid_{L^\infty}^{2p} \mid \Omega \mid \\ &\leq 2kC \mid \Omega \mid \parallel \varphi \parallel_1^{2p}, \end{aligned}$$

and the right hand side of the first inequality follows.

For the two other inequalities, we use the same type of arguments.

b) Absorbing set in X_1^α :

Proposition 2.1. *There exist two constants R_1, R_2 independent of ϵ such that for all u_ϵ solution of* (1.14)(1.15)(1.13) *with* (u_0, u_1) *in* X_1^α, *we have*

$$\epsilon \left| \frac{du_\epsilon}{dt}(t) \right|_{-1}^2 + \mid u_\epsilon(t) \mid_1^2 \leq R_1 + R_2(\epsilon \mid u_1 \mid_{-1}^2 + \mid u_0 \mid_1^2 + \parallel u_0 \parallel_1^{2p})e^{-\frac{\lambda}{2}t}.$$

We easily deduce, thanks to Proposition 1.1 :

Corollary 2.1. *The semigroup $(S_\epsilon^1(t))_{t\geq 0}$ possesses a bounded absorbing set in X_1^α for all α.*

Proof of Proposition 2.1 : Let λ be a positive real number smaller than $inf\left(\frac{\nu}{2}\lambda_1, \frac{1}{2\epsilon_0}\right)$ (see Lemma 2.2) and u_ϵ be the solution of (1.14)(1.15)(1.13) where (u_0, u_1) is in X_1^α. We set

$$v = \frac{du_\epsilon}{dt} + \lambda u_\epsilon,$$

and rewrite (1.14) as follows :

$$\epsilon\frac{dv}{dt}+(1-\lambda\epsilon)v+\lambda(\lambda\epsilon-1)u_\epsilon+\nu\Delta^2u_\epsilon-\Delta f(u_\epsilon)=0. \tag{2.1}$$

We take the scalar product of (2.1) with $A^{-1/2}v$ in L^2 :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\epsilon\mid v\mid_{-1}^2+\nu\mid u_\epsilon\mid_1^2+2\int_\Omega g(u_\epsilon)dx)+(1-\lambda\epsilon)\mid v\mid_{-1}^2\\&+\lambda(\lambda\epsilon-1)(u_\epsilon,v)_{-1}+\lambda\nu\mid u_\epsilon\mid_1^2+\lambda(f(u_\epsilon),u_\epsilon)+(f(u_\epsilon),m(v))=0.\end{aligned} \tag{2.2}$$

The leading term of $g(u)$ is $\frac{a_{2p-1}}{2p}u^{2p}$, the one of $f(u)u$ is $a_{2p-1}u^{2p}$, therefore :

$$\exists C_1>0\ :\ \forall u\in\mathbb{R},\quad f(u)u\geq g(u)-\frac{C_1}{\lambda\mid\Omega\mid}$$

(recall that a_{2p-1} is positive). We deduce

$$\lambda(f(u_\epsilon),u_\epsilon)\geq\lambda\int_\Omega g(u_\epsilon)dx-C_1. \tag{2.3}$$

Moreover, since $(u_\epsilon,\frac{du_\epsilon}{dt})$ belongs to X^1_α :

$$\begin{aligned}m(v)&=m\left(\frac{du_\epsilon}{dt}\right)+\lambda m(u_\epsilon)\\&\leq\alpha\left(\lambda+\frac{1}{\epsilon_0}\right).\end{aligned}$$

There exists a constant C_2 such that :

$$\forall u\in\mathbb{R}\ :\mid f(u)\mid\leq\frac{1}{\alpha\left(\lambda+\frac{1}{\epsilon_0}\right)}\left(\frac{\lambda}{2}g(u)+\frac{C_2}{\mid\Omega\mid}\right),$$

which gives :

$$\mid(f(u_\epsilon),m(v))\mid\leq\frac{\lambda}{2}\int_\Omega g(u_\epsilon)dx+C_2. \tag{2.4}$$

We use (2.3), (2.4) in (2.2), we denote $C_3=2(C_1+C_2)$ and we use Lemma 2.2, we find that for all time t :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\epsilon\mid v\mid_{-1}^2+\nu\mid u_\epsilon\mid_1^2+2\int_\Omega g(u_\epsilon)dx\right)\\&+\frac{1}{4}\mid v\mid_{-1}^2+\frac{\lambda\nu}{2}\mid u_\epsilon\mid_1^2+\frac{\lambda}{2}\int_\Omega g(u_\epsilon)dx\leq\frac{C_3}{2}.\end{aligned}$$

Thus :

$$\frac{d}{dt}E_1\left(u_\epsilon, \frac{du_\epsilon}{dt}\right) + \frac{\lambda}{2}E_1\left(u_\epsilon, \frac{du_\epsilon}{dt}\right) \leq C_3.$$

Thanks to Gronwall's Lemma :

$$E_1\left(u_\epsilon, \frac{du_\epsilon}{dt}\right) \leq \frac{2C_3}{\lambda}(1 - e^{-\frac{\lambda}{2}t}) + E_1(u_0, u_1)e^{-\frac{\lambda}{2}t}.$$

We deduce Proposition 2.1, thanks to Lemma 2.3.

c) Absorbing set in X_2^α :

Proposition 2.2 : *There exist two constants R_3, R_4 independent of ϵ such that for all bounded set B in X_2^α, there exists a time $t_0(B)$ (independent of ϵ) such that for all solutions of* (1.14), (1.15), (1.13) *with* (u_0, u_1) *in* B *and for all time* $t \geq t_0(B)$:

$$\epsilon\left|\frac{du_\epsilon}{dt}(t)\right|_0^2 + \mid u_\epsilon(t) \mid_2^2 \leq R_3 + R_4\left(\epsilon\left|\frac{du_\epsilon}{dt}(t_0)\right|_0^2 + \mid u_\epsilon(t_0) \mid_2^2\right)e^{-\frac{\lambda}{2}(t-t_0)}.$$

We easily deduce, thanks to Proposition 1.1 :

Corollary 2.2 : *The semigroup $(S_\epsilon^2(t))_{t\geq 0}$ possesses a bounded absorbing set in X_2^α for all α.*

Proof of Proposition 2.2.

We take the scalar product in $L^2(\Omega)$ of (2.2) with $\tilde{v}$, we get :

$$\frac{1}{2}\frac{d}{dt}(\epsilon \mid v \mid_0^2 + \nu \mid u_\epsilon \mid_2^2) + (1 - \lambda\epsilon) \mid v \mid_0^2 + \lambda(\lambda\epsilon - 1)(u_\epsilon, v)_0 + \lambda\nu \mid u_\epsilon \mid_2^2 - (\Delta f(u_\epsilon), v) = 0. \quad (2.5)$$

We rewrite the last term as follows :

$$-(\Delta f(u_\epsilon), v) = (f'(u_\epsilon)\nabla u_\epsilon, \nabla\frac{du_\epsilon}{dt}) + \lambda(f'(u_\epsilon)\nabla u_\epsilon, \nabla u_\epsilon),$$

$$-(\Delta f(u_\epsilon), v) = \frac{1}{2}\int_\Omega f'(u_\epsilon)\frac{d}{dt} \mid \nabla u_\epsilon \mid^2 dx + \lambda\int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx,$$

$$
\begin{aligned}
-(\Delta f(u_\epsilon), v) = &\frac{1}{2}\frac{d}{dt}\left(\int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx\right) \\
&+ \lambda \int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx \\
&- \frac{1}{2}\int_\Omega f''(u_\epsilon)\frac{du_\epsilon}{dt} \mid \nabla u_\epsilon \mid^2 dx.
\end{aligned} \tag{2.6}
$$

We estimate the last term of (2.6) :

$$
\begin{aligned}
&\left|\frac{1}{2}\int_\Omega f''(u_\epsilon)\frac{du_\epsilon}{dt} \mid \nabla u_\epsilon \mid^2 dx\right| \\
&\leq \frac{1}{2} \mid f''(u_\epsilon) \mid_{L^\infty} \left|\frac{du_\epsilon}{dt}\right|_{L^2} \mid \nabla u_\epsilon \mid^2_{L^4};
\end{aligned}
$$

we use Sobolev inequalities and interpolation inequalities :

$$
\begin{aligned}
&\exists C_1 > 0 \ : \mid f''(u_\epsilon) \mid_{L^\infty} \leq C_1(\| u_\epsilon \|_1^{2p-3} + 1), \\
&\exists C_2 > 0 \ : \mid \nabla u_\epsilon \mid^2_{L^4} \quad \leq C_2 \mid u_\epsilon \mid_1^{3/2} \mid u_\epsilon \mid_2^{1/2} .
\end{aligned}
$$

(Recall that the space dimension is 1 ; therefore $H^{1/4}(\Omega)$ and a fortiori $H^{1/2}(\Omega)$ is embedded in $L^4(\Omega)$). We deduce :

$$
\begin{aligned}
&\left|\frac{1}{2}\int_\Omega f''(u_\epsilon)\frac{du_\epsilon}{dt} \mid \nabla u_\epsilon \mid^2 dx\right| \\
&\leq C_1C_2(\| u_\epsilon \|_1^{2p-3} + 1) \mid u_\epsilon \mid_1^{3/2} \mid u_\epsilon \mid_1^{1/2} \left|\frac{du_\epsilon}{dt}\right|_{L^2} \\
&\leq C_5((\| u_\epsilon \|_1^{2p-3} + 1) \mid u_\epsilon \mid_1^{3/2})^4 + \frac{1}{8}\left|\frac{du_\epsilon}{dt}\right|^2_{L^2} + \frac{\lambda\nu}{4} \mid u_\epsilon \mid_2^2 \\
&\leq C_5((\| u_\epsilon \|_1^{2p-3} + 1) \mid u_\epsilon \mid_1^{3/2})^4 + \frac{\lambda}{8} \mid u_\epsilon \mid_0^2 + \frac{\alpha^2}{8\epsilon_0} \\
&\quad + \frac{1}{8} \mid v \mid_0^2 + \frac{\lambda\nu}{4} \mid u_\epsilon \mid_2^2 .
\end{aligned} \tag{2.7}
$$

We use (2.6) and (2.7) in (2.5), we obtain thanks to Lemma 2.2:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\epsilon \mid v \mid_0^2 + \nu \mid u_\epsilon \mid_2^2 + \int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx \\
&+ \frac{1}{8} \mid v \mid_0^2 + \frac{\lambda\nu}{4} \mid u_\epsilon \mid_2^2 + \lambda \int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx \\
&\leq C_5((\| u_\epsilon \|_1^{2p-3} + 1) \mid u_\epsilon \mid_1^{3/2})^4 + \frac{\lambda}{8} \mid u_\epsilon \mid_0^2 + \frac{\alpha^2}{8\epsilon_0^2}
\end{aligned} \tag{2.8}
$$

There exists a constant C_6 such that :

$$
\forall u \in \mathbb{R} \quad \lambda f'(u) \geq \frac{\lambda}{4} f'(u) - C_6 \ ;
$$

therefore :

$$\lambda \int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx \geq \frac{\lambda}{4} \int_\Omega f'(u_\epsilon) \mid \nabla u_\epsilon \mid^2 dx$$
$$- C_6 \mid u_\epsilon \mid_1^2 . \tag{2.9}$$

Let B be a bounded set in X_2^α, then B is bounded in X_1^α. Thanks to Corollary 2.1, there exists a constant C_7 (independent of B) and a time $t_0(B)$ such that for all time $t \geq t_0(B)$:

$$2C_5((\| u_\epsilon \|_1^{2p-3} + 1) \mid u_\epsilon \mid_1^{3/2})^4 + \frac{\lambda}{4} |u_\epsilon|_0^2 + \frac{\alpha^2}{4\epsilon_0^2} + 2C_6 \mid u_\epsilon \mid_1^2 \leq C_7. \tag{2.10}$$

(2.9), (2.10) in (2.8) give :

$$\frac{d}{dt} E_2 \left(u_\epsilon, \frac{du_\epsilon}{dt} \right) + \frac{\lambda}{2} E_2 \left(u_\epsilon, \frac{du_\epsilon}{dt} \right) \leq C_7,$$

for all time $t \geq t_0(B)$. Thanks to Gronnwall's lemma :

$$E_2 \left(u_\epsilon(t), \frac{du_\epsilon}{dt}(t) \right) \leq \frac{2C_7}{\lambda} (1 - e^{-\frac{\lambda}{2}(t-t_0)}) + E_2 \left(u_\epsilon(t_0), \frac{du_\epsilon}{dt}(t_0) \right) e^{-\frac{\lambda}{2}(t-t_0)}.$$

We deduce the Proposition 2.2, thanks to Lemma 2.3.

d) Absorbing set in X_3^α :

Proposition 2.3 : *There exists two constants* R_5, R_6 *independent of* ϵ *such that for all bounded set* B *in* X_3^α, *there exists a time* $t_1(B)$ *(independent of* ϵ*) such that for all solutions of* (1.14),(1.15), (1.13) *with* (u_0, u_1) *in* B *and for all time* $t \geq t_1(B)$:

$$\epsilon \left| \frac{d^2 u_\epsilon}{dt^2}(t) \right|_{-1}^2 + \left| \frac{du_\epsilon}{dt}(t) \right|_1^2 + \mid u_\epsilon(t) \mid_3^2$$
$$\leq R_5 + \frac{R_6}{\epsilon} \left(\left| \frac{du_\epsilon}{dt}(t_1) \right|_1^2 + \mid u_\epsilon(t_1) \mid_3^2 + 1 \right) e^{-\frac{\lambda}{2}(t-t_1)}.$$

We easily deduce, thanks to Proposition 1.1 :

Corollary 2.3 : *The semigroup* $(S_\epsilon^3(t))_{t\geq 0}$ *possesses a bounded absorbing set in* X_3^α *for all* x.

In order to prove the Proposition 2.3, we need two lemmas :

Lemma 2.4. *Let u_ϵ be the solution of (1.14)(1.15)(1.13) with (u_0, u_1) in X_1^α, then there exist a constant K_1 (independent of (u_0, u_1) and ϵ) and a time $t_0(| u_1 |_{-1}, | u_0 |_1, m(u_1))$ such that for all time $t \geq t_0$:*

$$\int_{t_0}^{t} \left| \frac{du_\epsilon}{dt} \right|_{-1}^{2} ds \leq K_1.$$

Lemma 2.5. *Let f, g be two functions defined on $\mathbb{R}^+$ such that :*

(i) *g is positive,*

(ii) *there exists k_1, k_2, k_3, k_4 in $\mathbb{R}^+$ and a time t_0 such that for all time $t \geq t_0$:*

$$\frac{df}{dt} + k_1 f \leq k_2 + k_3 g f + k_4 g,$$

(iii) *there exists k_5 in $\mathbb{R}^+$ such that for all time $t \geq t_0$: $\int_{t_0}^{t} g(s)ds \leq k_5$, then, for all time $t \geq t_0$,*

$$f(t) \leq \left(\frac{k_2}{k_1} + k_4 k_5\right) e^{k_3 k_5} + \left(f(t_0) - \frac{k_2}{k_1}\right) e^{k_3 k_5 - k_1(t - t_0)}.$$

Proof of Lemma 2.4. We take the scalar product of (1.14) with $A^{-1/2}\frac{du_\epsilon}{dt}$:

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{du_\epsilon}{dt}\right|_{-1}^{2} + \nu \mid u_\epsilon \mid_1^2 + \int_\Omega g(u_\epsilon)dx\right) + \left|\frac{du_\epsilon}{dt}\right|_{-1}^{2} - \int_\Omega f(u_\epsilon) m\left(\frac{du_\epsilon}{dt}\right) dx = 0. \tag{2.11}$$

Thanks to Proposition 2.1, we can find a constant C_1' and a time $t_0(| u_0 |_{-1}, | u_1 |_1)$ such that, for all $t \geq t_0$:

$$\epsilon \left|\frac{du_\epsilon}{dt}(t)\right|_{-1}^{2} + \nu \mid u_\epsilon(t) \mid_1^2 + \int_\Omega g(u_\epsilon(t))dx \leq C_1'. \tag{2.12}$$

Thanks to Sobolev inequalities (the space dimension is one) :

$$\begin{aligned} \left|\int_\Omega f(u_\epsilon)dx\right| &\leq C_2'(\| u_\epsilon \|_1^{2p-1} + 1) \\ &\leq C_3', \end{aligned} \tag{2.13}$$

for all $t \geq t_0$. We use (2.13) in (2.11), we obtain since $m\left(\frac{du_\epsilon}{dt}\right) = m(u_1)e^{-t/\epsilon}$:

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{du_\epsilon}{dt}\right|_{-1}^{2} + \nu \mid u_\epsilon \mid_1^2 + \int_\Omega g(u_\epsilon)dx\right) + \left|\frac{du_\epsilon}{dt}\right|_{-1}^{2} \leq C_3' \frac{\alpha}{\epsilon_0} e^{-t/\epsilon}. \tag{2.14}$$

We integrate (2.14) from t_0 to t and use (2.12) :

$$\int_{t_0}^{t} \left| \frac{du_\epsilon}{dt} \right|_{-1}^{2} ds \le \frac{1}{2}C_1' + C_3'\alpha(e^{-t_0/\epsilon} - e^{-t/\epsilon})$$
$$\le \frac{1}{2}C_1' + C_3'\alpha.$$

<u>Proof of Lemma 2.5</u> :

We have for all $t \ge t_0$:

$$\frac{df}{dt} + k_1 f \le k_2 + k_3 g f + k_4 g, \tag{2.15}$$

We multiply (2.15) by $e^{k_1 t - k_3 \int_{t_0}^{t} g(s)ds}$:

$$\begin{aligned} &\frac{d}{dt}\left(e^{k_1 t - k_3 \int_{t_0}^{t} g(s)ds} f(t)\right) \\ &\le (k_2 + k_4 g) e^{k_1 t - \int_{t_0}^{t} g(s)ds} \\ &\le (k_2 + k_4 g + k_1 k_4 \int_{t_0}^{t} g(s)ds) e^{k_1 t} \\ &\le \frac{d}{dt}\left(\frac{k_2}{k_1} + k_4 \int_{t_0}^{t} g(s)ds\right) e^{k_1 t}. \end{aligned} \tag{2.16}$$

We integrate (2.16) from t_0 to t :

$$e^{k_1 t - k_3 \int_{t_0}^{t} g(s)ds} f(t) - e^{k_1 t_0} f(t_0) \le$$
$$\left(\frac{k_2}{k_1} + k_4 \int_{t_0}^{t} g(s)ds\right) e^{k_1 t} - \frac{k_2}{k_1} e^{k_1 t_0},$$

$$\begin{aligned} f(t) &\le \left(\frac{k_2}{k_1} + k_4 \int_{t_0}^{t} g(s)ds\right) e^{k_3 \int_{t_0}^{t} g(s)ds} \\ &\quad + \left(f(t_0) - \frac{k_2}{k_1}\right) e^{-k_1(t-t_0) - k_3 \int_{t_0}^{t} g(s)ds} \\ &\le \left(\frac{k_2}{k_1} + k_4 k_5\right) e^{k_3 k_5} + \left(f(t_0) - \frac{k_2}{k_1}\right) e^{k_3 k_5} e^{-k_1(t-t_0)}. \end{aligned}$$

<u>Proof of Proposition 2.3</u> :

We differentiate (2.2) with respect to time and we denote by

$$W = \frac{dv}{dt} = \frac{d^2 u_\epsilon}{dt^2} + \lambda \frac{du_\epsilon}{dt},$$
$$w = \frac{du_\epsilon}{dt},$$

we obtain :

$$\epsilon\frac{dW}{dt}+(1-\lambda\epsilon)W+\lambda(\lambda\epsilon-1)w+\nu\Delta^2 w-\Delta f'(u_\epsilon)w=0. \tag{2.17}$$

We take the scalar product of (2.17) with $A^{-1/2}W$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\epsilon\mid W\mid_{-1}^2+\nu\mid w\mid_1^2)+(1-\lambda\epsilon)\mid W\mid_{-1}^2+\lambda(\lambda\epsilon-1)(w,W)_{-1}\\ +\lambda\nu\mid w\mid_1^2+(f'(u_\epsilon)w,W)_0=0\end{aligned} \tag{2.18}$$

We rewrite the last term as follows :

$$\begin{aligned}(f'(u_\epsilon)w,W)_0&=(f'(u_\epsilon)\tilde{w},\tilde{w}')\\ &+\lambda(f'(u_\epsilon)\tilde{w},\tilde{w})\\ &+(f'(u_\epsilon)m(w),\tilde{W}),\end{aligned}$$

$$\begin{aligned}(f'(u_\epsilon)w,W)_0&=\frac{1}{2}\frac{d}{dt}\left(\int_\Omega f'(u_\epsilon)\tilde{w}^2dx\right)\\ &+\lambda\int_\Omega f'(u_\epsilon)\tilde{w}^2dx\\ &+m(w)(f'(u_\epsilon),\tilde{W})\\ &-\frac{1}{2}\int_\Omega f''(u_\epsilon)\tilde{w}^3dx\\ &-\frac{1}{2}m(w)\int_\Omega f''(u_\epsilon)\tilde{w}^2dx.\end{aligned} \tag{2.19}$$

We now estimate the last three terms of (2.19). Since $H^1(\Omega)$ is an algebra ($\Omega\subset\mathbb{R}$) and f'' is a polynomial of order $2p-3$ there exists a constant c_1 such that :

$$\mid f''(u_\epsilon)\mid_1\leq C_1(\| u_\epsilon\|_1^{2p-3}+1).$$

Therefore :

$$\begin{aligned}\mid m(w)(f''(u_\epsilon),\tilde{W})\mid&\leq\mid m(w)\mid\mid f''(u_\epsilon)\mid_1\mid W\mid_{-1}\\ &\leq C_1\mid m(w)\mid(\| u_\epsilon\|_1^{2p-3}+1)\mid W\mid_{-1}\\ &\leq C_2 m(w)^2(\| u_\epsilon\|_1^{2p-3}+1)^2+\frac{1}{8}\mid W\mid_{-1}^2 .\end{aligned} \tag{2.20}$$

Thanks to the embedding of $H^1(\Omega)$ into $L^\infty(\Omega)$ there exist two constants C_3, C_4 such that:

$$\begin{aligned}\mid f''(u_\epsilon)\mid_{L^\infty}&\leq C_3(\mid u_\epsilon\mid_{L^\infty}^{2p-3}+1)\\ &\leq C_4(\| u_3\|_1^{2p-3}+1).\end{aligned}$$

Now we use the embedding $L^3(\Omega) \subset L^6(\Omega) \subset V_{1/3}$ and an interpolation inequality :

$$\begin{aligned} |\, \tilde{w} \,|_{L^3}^3 &\leq C_5 \,|\, w \,|_{H^{1/3}}^3 \\ &\leq C_5 \,|\, w \,|_{-1} |\, w \,|_1^2 . \end{aligned}$$

Thus :

$$\begin{aligned} \left| \int_\Omega f''(u_\epsilon)\tilde{w}^3 dx \right| &\leq C_4 C_5 (\| \, u_\epsilon \, \|_1^{2p-3} + 1) \,|\, w \,|_{-1} |\, w \,|_1^2 \\ &\leq C_6 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 |\, w \,|_1^2 + \frac{\lambda\nu}{8} \,|\, w \,|_1^2 . \end{aligned} \tag{2.21}$$

Similarly, there exists a constant C_7 such that :

$$\left| m(w) \int_\Omega f''(u_\epsilon)\tilde{w}^2 dx \right| \leq C_7 m(w)^2 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 + \frac{\lambda\nu}{8} \,|\, w \,|_1^2 . \tag{2.22}$$

We use (2.20)-(2.22) in (2.19) :

$$\begin{aligned} (f'(u_\epsilon), W)_0 \geq & \frac{1}{2}\frac{d}{dt} \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx \\ & + \lambda \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx \\ & - C_2 m(w)^2 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 - \frac{1}{8} \,|\, W \,|_{-1}^2 \\ & - C_6 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 |\, w \,|_1^2 - \frac{\lambda\nu}{8} \,|\, w \,|_1^2 \\ & - C_7 m(w)^2 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 - \frac{\lambda\nu}{8} \,|\, w \,|_1^2 . \end{aligned} \tag{2.23}$$

Now, we use (2.23) and the Lemma 2.2 to obtain from (2.18) :

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\epsilon \,|\, W \,|_{-1}^2 + \nu \,|\, w \,|_1^2 + \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx) \\ & + \frac{1}{2} \,|\, W \,|_{-1}^2 + \frac{\lambda\nu}{2} \,|\, w \,|_1^2 + \lambda \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx \\ & \quad \leq C_2 m(w)^2 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \\ & + C_6 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 |\, w \,|_1^2 \\ & + C_7 m(w)^2 (\| \, u_\epsilon \, \|_1^{2p-3} + 1)^2 \,|\, w \,|_{-1}^2 \\ & + \frac{1}{8} \,|\, W \,|_{-1}^2 + \frac{\lambda\nu}{4} \,|\, w \,|_1^2 . \end{aligned}$$

Finally, since there exists a constant C_8 such that :

$$\lambda \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx = \frac{\lambda}{4} \int_\Omega f'(u_\epsilon)\tilde{w}^2 dx - C_8,$$

we have:

$$\frac{d}{dt}\left(E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)\right)+\frac{\lambda}{2}E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)$$
$$\leq 2C_2m(w)^2(\| u_\epsilon \|_1^{2p-3}+1)^2+2C_8$$
$$+\frac{2C_6}{\nu}(\| u_\epsilon \|_1^{2p-3}+1)^2 \mid w \mid_{-1}^2 E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)$$
$$+2C_7m(w)^2(\| u_\epsilon \|_1^{2p-3}+1)^2 \mid w \mid_{-1}^2 .$$

Let B be a bounded set in X_3^α, then B is bounded in X_1^α and thanks to Corollary 2.1, there exist constants independent of B and of ϵ, k_2, k_3, k_4, and a time $t_1(B)$ such that for all (u_0, u_1) in B and all time $t \geq t_1(B)$, we have :

$$\frac{d}{dt}E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)+\frac{\lambda}{2}E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)$$
$$\leq k_2+k_3 \mid w \mid_{-1}^2 E_3\left(\frac{du_\epsilon}{dt},\frac{d^2u_\epsilon}{dt^2}\right)+k_4 \mid w \mid_{-1}^2 .$$

Thanks to Lemma 2.4, there exists a time $t_2(B)$ such that for all $t \geq t_2(B)$:

$$\int_{t_1}^{t} \mid w \mid_{-1}^2 ds \leq K_1.$$

Thanks to Lemma 2.5, for all $t \geq t_0(B) = \sup(t_1(B), t_2(B))$:

$$E_3\left(\frac{du_\epsilon}{dt}(t),\frac{d^2u_\epsilon}{dt^2}(t)\right) \leq \left(2\frac{k_2}{\lambda}+k_4K_1\right)e^{k_3K_1}$$
$$+\left(E_3\left(\frac{du_\epsilon}{dt}(t_0),\frac{d^2u_\epsilon}{dt^2}(t_0)\right)-2\frac{k_2}{\lambda}\right)e^{k_3K_1-\frac{\lambda}{2}(t-t_0)}.$$

Thanks to Lemma 2.3, there exists k_1', k_2' constants such that for all $t \geq t_0$:

$$\epsilon\left|\frac{d^2u_\epsilon}{dt^2}(t)\right|_{-1}^2+\left|\frac{du_\epsilon}{dt}(t)\right|_1^2 \leq k_1'+k_2'\left(\epsilon\left|\frac{d^2u_\epsilon}{dt^2}(t_0)\right|_{-1}^2+\left|\frac{du_\epsilon}{dt}(t_0)\right|_1^2\right.$$
$$\left.+\| u_\epsilon(t_0) \|_1^{2p-2}\left|\frac{du_\epsilon}{dt}(t_0)\right|_0^2\right)e^{-\frac{\lambda}{2}(t-t_0)}. \tag{2.24}$$

We assume that t_0 is sufficiently large so that :

$$\| u_\epsilon(t_0) \|_1^{2p-2}\left|\frac{du_\epsilon}{dt}(t_0)\right|_0^2 \leq k_3', \tag{2.25}$$

where k_3' is a constant independent of B (we use Corollary 2.2). Moreover, thanks to (1.14):

$$\begin{aligned} \epsilon\left|\frac{d^2u_\epsilon}{dt^2}(t_0)\right|_{-1} &\leq \left|\frac{du_\epsilon}{dt}(t_0)\right|_{-1} + |u_\epsilon(t_0)|_3 + |f(u_\epsilon(t_0)|_1 , \\ \epsilon\left|\frac{d^2u_\epsilon}{dt^2}(t_0)\right|_{-1}^2 &\leq \frac{k_4'}{\epsilon}\left(\left|\frac{du_\epsilon}{dt}(t_0)\right|_{-1}^2 + |u_\epsilon(t_0)|_3^2 + 1\right), \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} |u_\epsilon(t)|_3 &\leq \epsilon\left|\frac{d^2u_\epsilon}{dt^2}(t)\right|_{-1} + \left|\frac{du_\epsilon}{dt}(t)\right|_{-1} + |f(u_\epsilon(t))|_1 , \\ |u_\epsilon(t)|_3^2 &\leq k_5'\left(\epsilon^2\left|\frac{d^2u_\epsilon}{dt^2}(t)\right|_{-1}^2 + \left|\frac{du_\epsilon}{dt}(t)\right|_1^2 + 1\right). \end{aligned} \tag{2.27}$$

From (2.24)-(2.27), we deduce the Proposition 2.3.

e) Propositions 1.1 and 2.1 are still true if we consider (1.14) on a bounded set in $\mathbb{R}^2$ or $\mathbb{R}^3$, but we do not know whether Propositions (2.2) and (2.3) are still true. If we try to obtain estimates in $V_2 \times V_0$ by the same techniques as in Proposition 2.2, we have to estimate the last term of (2.6) :

$$\frac{1}{2}\int_\Omega f''(u_\epsilon)\frac{du_\epsilon}{dt} \mid \nabla u_\epsilon \mid^2 dx.$$

Even if $n = 2$, this term is stronger than $E_2\left(u_\epsilon, \frac{du_\epsilon}{dt}\right)$. For instance, we can majorize it by:

$$|f''(u_\epsilon)|_{L^\infty}\left|\frac{du_\epsilon}{dt}\right|_{L^2} \mid \nabla u_\epsilon \mid_{L^4}^2 . \tag{2.28}$$

If $n = 2$, $L^4(\Omega)$ is embedded in $H^{1/2}(\Omega)$ and $L^\infty(\Omega)$ in $H^{1+\nu}(\Omega)$ for all $\eta > 0$; therefore using interpolation inequalities, (2.28) is majorized by

$$C(\| u_\epsilon \|_1^{1-\eta} \| u_\epsilon \|_2^\eta + 1) \| u_\epsilon \|_0 | u_\epsilon |_1 | u_\epsilon |_2$$

and this is too strong (we would need $\eta \leq 0$ and this is not possible). If $n = 3$, (2.28) is stronger. For the Proposition 2.3, we have the same type of problems.

III. EXISTENCE OF ATTRACTORS.

a) In this section, we first give a result of B. Nicolaenko, B. Scheurer and R. Temam [1] concerning the existence of a maximal attractor for the Cahn-Hilliard equation (1.11)(1.12)-(1.13). Then using a general theorem, we prove the existence of attractors for the perturbed Cahn-Hilliard equation (1.14), (1.15), (1.13) in the spaces X_s^α endowed with their weak topology. The results are valid for the Cahn-Hilliard equation in space dimension $n = 1, 2$ or 3, but for the perturbed equation, since we do not have any absorbing set in space dimension higher than two, the results are restricted to the space dimension $n = 1$.

It is known that the attractor for the Cahn-Hilliard equation has finite Hausdorf and fractal dimension. We think that the method developed in J.M. Ghidaglia [1] (which is a slight modification of the usual method presented in R. Temam [1] for instance) can be adapted successfully to the perturbed equation in order to show that the attractors constructed in secton b) hereafter have finite fractal and Hausdorf dimension.

b) The Cahn-Hilliard equation.

As we already mentionned in section Id), the average of a solution of (1.11) is constant. Therefore, there does not exist any absorbing set in the whole space V_0 and we have to introduce the subsets of V_0 :

$$H_\alpha = \{u \in V_0 : \mid m(u) \mid \leq \alpha\}.$$

One can show (see B. Nicolaenko, B. Scheurer and R. Temam [1] or R. Temam [1]) the following result :

Theorem 3.1. *The restriction to H_α of the semigroup $(S(t))_{t\geq 0}$ defined in Theorem 1.1 possesses a maximal attractor $\mathcal{A}_\alpha$ in H_α endowed with the strong topology of V_0. Furthermore, A_α is compact and connected (for the strong topology of V_0).*

Thanks to Theorem 1.1, we know that if u_0 is in V_0, then for $t > 0$,u(t) is in V_2. As the attractor $\mathcal{A}_2$ is in invariant set, we deduce :

$$\mathcal{A}_2 \subset V_2.$$

Using the second assertion of Theorem 1.1 and the same argument as above :

$$A_2 \subset V_4.$$

c) The perturbed Cahn-Hilliard equation.

To show the existence of an attractor for the equation (1.14), (1.15), (1.13), we use a general existence theorem whose proof can be found in R. Temam [1] or Hale [1] :

Theorem 3.2. *Let (E,d) be a metric space and $(S(t))_{t\geq 0}$ a semigroup on E such that for each t, $S(t)$ is continuous.*

Suppose that there exists a bounded absorbing set B_a such that $\bigcup_{t\geq t_0} S(t)B_a$ is relatively compact in E for some $t_0>0$. Then

$$\mathcal{A} = \omega(B_a) = \bigcap_{s\geq 0} \mathrm{cl}\left(\bigcup_{t\geq s} S(t)B_a\right)$$

is a maximal attractor for the semigroup $(S(t))_{t\geq 0}$:

- $\forall\, t \in \mathbb{R}\ :\ S(t)\mathcal{A} = \mathcal{A}$
- *for all bounded set in E :*

$$d(S(t)B,\mathcal{A}) \xrightarrow[t\to+\infty]{} 0.$$

Moreover $\mathcal{A}$ is compact and connected.

Our aim is to apply Theorem 3.2 to the semigroups $(S^s_\epsilon(t))_{t\geq 0}$ on X^s_α endowed with the weak topology. In section II, we obtained bounded absorbing sets in X^s_α endowed with the strong topology, we denote them by $B^s_{\alpha,\epsilon}$. Let t_0 be such that

$$S^s_\epsilon(t)B^s_{\alpha,\epsilon} \subset B^s_{\alpha,\epsilon} \quad \text{for} \quad t \geq t_0;$$

then $\bigcup_{t\geq t_0} S^s_\epsilon(t)B^s_{\alpha,\epsilon} \subset B^s_{\alpha,\epsilon}$ and since B^s_α is compact in X^s_α weak, $\bigcup_{t\geq t_0} S^s_\epsilon(t)B^s_{\alpha,\epsilon}$ is relatively compact in X^s_α weak.

In order to apply Theorem 3.2, it remains to check the weak continuity for the semigroups $(S^s_\epsilon(t))_{t\geq 0}$. We have

Proposition 3.1 : *For α in $\mathbb{R}^+$, $s=1,2$ or 3 and $t\geq 0$, the mapping $S^s_\epsilon(t)$ is weakly continuous on X^s_α.*

Proof : We first consider the case $s=2$ or 3. Let (u_n,v_n) be a sequence in X^α_s weakly convergent to (u,v) in X^α_s. Then (u_n,v_n) converges strongly in X^α_{s-1} and since $S^{s-1}_\epsilon(t)$ is strongly continuous $S^s_\epsilon(t)(u_n,v_n)$ converges to $S^s_\epsilon(t)(u,v)$ in X^α_{s-1} strong. On the other hand, (u_n,v_n) is bounded in X^α_s and consequently $S^s_\epsilon(t)(u_n,v_n)$ too, therefore there exists a subsequence $S^s_\epsilon(t)(u_{n_k},v_{n_k})$ that converges weakly in X^α_s to a limit (U,V). This subsequence is strongly convergent in X^α_{s-1}, thus the limit (U,V) must be equal to $S^s_\epsilon(t)(u,v)$. This shows that the whole sequence weakly converges to $S^s_\epsilon(t)(u,v)$.

We now consider the case $s = 1$. Let (u_n, v_n) be a sequence in X_s^α weakly convergent to (u,v) in X_1^α. We denote by u_ϵ^n (resp. u_ϵ) the solution of (1.19), (1.15), (1.13) with $u_0 = u_n$, $u_1 = v_n$ (resp. $u_0 = u$, $u_1 = v$). We substract the equations satisfied by u_ϵ^n and u_ϵ, we obtain :

$$\epsilon \frac{d^2 w_\epsilon^n}{dt^2} + \frac{dw_\epsilon^n}{dt} + \nu \Delta^2 w_\epsilon^n - \Delta f(u_\epsilon^n) + \Delta f(u_\epsilon) = 0, \tag{3.1}$$

where $w_\epsilon^n = u_\epsilon^n - u_\epsilon$.

We take the scalar product in L^2 of (3.1) with $A^{-1}\frac{dw_\epsilon^n}{dt}$:

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2 + \nu \left|w_\epsilon^n\right|_0^2\right) + \left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2$$
$$+ \left((f(u_\epsilon^n) - f(u_\epsilon), A^{-1/2}\frac{dw_\epsilon^n}{dt}\right) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2 + \nu \left|w_\epsilon^n\right|_0^2\right) + \left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2$$
$$\leq \frac{1}{2}\left|f(u_\epsilon^n) - f(u_\epsilon)\right|_{L^2}^2 + \frac{1}{2}\left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2,$$

$$\frac{d}{dt}\left(\epsilon \left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2 + \nu \left|w_\epsilon^n\right|_0^2\right) \leq \left|f(u_\epsilon^n) - f(u_\epsilon)\right|_{L^2}^2. \tag{3.2}$$

Since (u_ϵ^n) is bounded in $H^1(\Omega)$ and consequently in $L^\infty(\Omega)$ and since f is Lipschitzian on the bounded sets of $\mathbb{R}$, we have

$$\begin{aligned} &\exists\, C(u_\epsilon^n, u_\epsilon) \text{ such that} \\ \left|f(u_\epsilon^n) - f(u)\right|_{L^2}^2 &\leq C\left|u_\epsilon^n - u\right|_{L^2}^2 \\ &\leq C\left|w_\epsilon^n\right|_{L^2}^2 \\ &\leq C(\left|w_\epsilon^n\right|_0^2 + m(w_\epsilon^n)^2). \end{aligned} \tag{3.3}$$

We take the average of (3.1) :

$$\epsilon \frac{d^2}{dt^2} m(w_\epsilon^n) + \frac{d}{dt} m(w_\epsilon^n) = 0,$$

and that implies :

$$m(w_\epsilon^n) = m(u_n - u) + \epsilon m(v_n - v)(1 - e^{-t/\epsilon}), \tag{3.4}$$

$$m\left(\frac{dw_\epsilon^n}{dt}\right) = m(v_n - v)e^{-t/\epsilon}. \tag{3.5}$$

We deduce from (3.2)-(3.5) :

$$\frac{d}{dt}\left(\epsilon\left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2+\nu\,|w_\epsilon^n|_0^2\right)\le |w_\epsilon^n|_0^2$$
$$+C(m(u_n-u)+\epsilon m(v_n-v)(1-e^{-t/\epsilon})).$$

Thanks to Gronwall's Lemma, we have :

$$\epsilon\left|\frac{dw_\epsilon^n}{dt}\right|_{-2}^2+\nu\,|w_\epsilon^n|_0^2\le e^{Ct}\left(\epsilon\,|v_n-v|_{-2}^2+\nu\,|u_n-u|_0^2\right)$$
$$+C\int_0^t e^{C(t-s)}(m(u_n-u)+\epsilon m(v_n-v)(1-e^{-t/\epsilon}))dt. \tag{3.6}$$

As $\{v_n-v,u_n-u\}$ converges weakly to zero in X_1^α and the injection of X_1^α into X_0^α is compact, (3.6) shows that $\left\{\frac{dw_\epsilon^n}{dt}(t),w_\epsilon^n(t)\right\}$ strongly converges to zero in X_0^α. In the same manner as in the case $s=2$ or 3, this implies that $S_\epsilon^1(t)\{u_n,v_n\}$ weakly converges to $S_\epsilon^1(t)\{u,v\}$ in X_1^α.

Now, Propositions 2.1, 2.2, 2.3 and 3.1 allow us to apply Theorem 3.2 in the three cases $s=1$, 2 and 3. Thus, we have

<u>Theorem 3.3</u> : *For $s=1,2$ or 3 and α in $\mathbb{R}^+$, the restriction of $S_\epsilon^s(t)$ to X_α^s possesses a maximal attractor $\mathcal{A}_{\epsilon,\alpha}^s$ compact and connected in the space X_α^s endowed with the weak topology.*

<u>Remark</u>. 1) $\mathcal{A}_{\epsilon,\alpha}^3$ is a bounded set in X_α^3 and consequently in X_α^2. As it is an invariant set, we have :

$$\mathcal{A}_{\epsilon,\alpha}^3\subset\mathcal{A}_{\epsilon,\alpha}^2.$$

In the same way :

$$\mathcal{A}_{\epsilon,\alpha}^2\subset\mathcal{A}_{\epsilon,\alpha}^1.$$

Unfortunately, we do not knwo whether $\mathcal{A}_{\epsilon,\alpha}^2$ and $\mathcal{A}_{\epsilon,\alpha}^1$ are bounded in X_α^3 and X_α^2 and we do not know whether the other inclusions hold.
2) Using the compactness of the injection of X_α^s into $X_\alpha^{s'}$ for all $s'<s$, one can see that the bounded sets of X_α^s are attracted by $\mathcal{A}_{\epsilon,\alpha}^s$ in the strong topology of $X_\alpha^{s'}$.

3) Let (u,v) be in $\mathcal{A}_{\epsilon,\alpha}^s$, then there is a complete orbits $\left(u_\epsilon(t),\frac{du_\epsilon(t)}{dt}\right)_{t\in\mathbb{R}}$ in $\mathcal{A}_{\epsilon,\alpha}$ such that

$$u_\epsilon(0)=u,\quad v_\epsilon(0)=v.$$

For all time t, we have :

$$m\left(\frac{du_\epsilon}{dt}(t)\right) = m(v)e^{-t/\epsilon};$$

$$m(v) = m\left(\frac{du_\epsilon}{dt}(t)\right)e^{t/\epsilon}.$$

Since $m\left(\frac{du_\epsilon}{dt}(t)\right)$ is bounded, by letting $t \to -\infty$ we have

$$m(v) = 0.$$

Using section Ia), for the perturbed Cahn-Hilliard equation there exists a Lyapunov function defined on a set that contains the attractors. We deduce that the attractors consist of the unstable set of the set of the fixed points.

4) Using the technics developped in J.M. Ghidaglia and R. Temam [2], one can generalize Theorem 3.3 to the case where the right hand side of (1.14) contains a time periodic forcing term.

IV. Convergence of the attractor in $\mathbf{X}_3^\alpha$ when $\epsilon \to 0$.

a) In this section, we prove that when ϵ is small, $\mathcal{A}^3_{\epsilon,\alpha}$ is close to $\mathcal{A}_\alpha$. More precisely, we first define a convenient embedding of $\mathcal{A}_\alpha$ (which is included in V_4) into X_2^α :

$$\mathcal{A}^*_\alpha = \{(u, -\nu\Delta^2 u + \Delta f(u))/u \in \mathcal{A}_\alpha\}.$$

If $(u,v) \in \mathcal{A}^*_\alpha$, and if $(u(t))_{t\in\mathbb{R}}$ is a complete orbit in $\mathcal{A}_\alpha$ such that $u(0) = u$, then v is the time derivative of $u(t)$ at the time 0 : $v = \frac{du}{dt}(0)$. We will show that the Hausdorf semidistance $\delta(\mathcal{A}^3_{\epsilon,\alpha}, \mathcal{A}^*_\alpha)$ converges to zero.

b) <u>Theorem 4.1</u>. *The Hausdorf semidistance $\delta(\mathcal{A}^3_{\epsilon,\alpha}, \mathcal{A}^*_\alpha)$ converges to zero when $\epsilon \to 0$:*

$$\lim_{\epsilon\to 0}\ \sup_{(u_\epsilon,v_\epsilon)\in\mathcal{A}_{\epsilon,\alpha}}\ \inf_{(u,v)\in\mathcal{A}^*_\alpha} \left(\| u_\epsilon - u \|_2^2 + \| v_\epsilon - v \|_0^2\right)^{1/2} = 0.$$

<u>Remark</u>. As we already mentioned in the introduction, Theorem 4.1 gives the uppersemicontinuity of the attractor at zero. The lower continuity is much more difficult and we need to have sharp informations on the structure of the attractors (consisting of the stationary solutions and their unstable manifolds).

Proof of Theorem 4.1.

We know that $\mathcal{A}^3_{\epsilon,\alpha}$ is an invariant set. Let $(u_\epsilon(t))_{t\in\mathbb{R}}$ be an orbit in $\mathcal{A}^3_{\epsilon,\alpha}$, then using Proposition 2.4 we have :

$$\epsilon\left|\frac{d^2u_\epsilon(t)}{dt^2}\right|^2_{-1}+\left|\frac{du_\epsilon(t)}{dt}\right|^2_1+|u_\epsilon(t)|^2_3\le R_5, \tag{4.1}$$

for all time t, with R_5 a constant independent of ϵ.

Moreover we saw in the third remark of the preceding section that:

$$m\left(\frac{du_\epsilon}{dt}\right)=0 \tag{4.2}$$

and

$$m(u_\epsilon)\text{ is independent of } t. \tag{4.3}$$

Let η be a real positive number. For each ϵ positive, we choose (u_ϵ, v_ϵ) in X^α_3 such that :

$$\inf_{u,v\in\mathcal{A}^*_\alpha}\left(\| u_\epsilon-u\|^2_2+\| v_\epsilon-v\|^2_0\right)\ge\delta(\mathcal{A}^3_{\epsilon,\alpha},\mathcal{A}^*_\alpha)-\eta.$$

We denote by $\left(u_\epsilon(t),\frac{du_\epsilon}{dt}(t)\right)_{t\in\mathbb{R}}$ a complete orbit in $\mathcal{A}^3_{\epsilon,\alpha}$ such that

$$u_\epsilon(0)=u_\epsilon,\quad \frac{du_\epsilon}{dt}(0)=v_\epsilon.$$

Then thanks to (4.1), (4.2), (4.3) we have

$$(u_\epsilon)_{\epsilon>0}\text{ is bounded in } L^\infty(\mathbb{R},V_3), \tag{4.4}$$

$$\left(\frac{du_\epsilon}{dt}\right)_{\epsilon>0}\text{ is bounded in } L^\infty(\mathbb{R},V_1), \tag{4.5}$$

$$\left(\sqrt{\epsilon}\frac{d^2u_\epsilon}{dt^2}\right)_{\epsilon>0}\text{ is bounded in } L^\infty(\mathbb{R},V_{-1}), \tag{4.6}$$

Thus, using classical compactness theorem, (4.4) and (4.5) imply that there exists a subsequence $(u_{\epsilon'})_{\epsilon'>0}$ and a function u in $C^0(\mathbb{R},V_2)$ such that for all $T>0$, $u_{\epsilon'}$ converges to u in $C^0([-T,T],V_2)$. Moreover (4.4) implies that u belongs to $C^0_b(\mathbb{R},V_2)$.

We know that $\frac{du_\epsilon}{dt}$ is in $C^0(\mathbb{R},V_0)$. And (4.5) implies that there is a subsequence (that we still denote by $(u_{\epsilon'})_{\epsilon'>0}$) such that

$$\frac{du_{\epsilon'}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^\infty(\mathbb{R}, V_1) \text{ weak star.} \tag{4.7}$$

Thanks to (4.6), we have :

$$\epsilon \frac{d^2 u_\epsilon}{dt^2} \to 0 \text{ in } L^\infty(\mathbb{R}, V_{-1}) \text{ strong} \tag{4.8}$$

We rewrite (1.14) as follows :

$$\frac{du_{\epsilon'}}{dt} = -\epsilon \frac{d^2 u_{\epsilon'}}{dt^2} - \nu \Delta^2 u_{\epsilon'} + \Delta f(u_{\epsilon'}). \tag{4.9}$$

Since the mapping $u \to f(u)$ is continuous from V_2 to V_2 we deduce from (4.8) and the convergence of $u_{\epsilon'}$ to u in $C^0([-T,T], V_2)$ for all $T > 0$ that $\frac{du_{\epsilon'}}{dt}$ converges to $-\nu\Delta^2 u + \Delta f(u)$ in $C^0([-T,T], V_{-2})$ for all $T > 0$.

(4.7) implies that :

$$\frac{du}{dt} = -\nu \Delta^2 u + \Delta f(u), \tag{4.10}$$

and (4.5) implies that $\frac{du_{\epsilon'}}{dt}$ converges to $\frac{du}{dt}$ in $C^0([-T,T], V_0)$ for all $T > 0$ and that $\frac{du}{dt}$ belongs to $C_b^0(\mathbb{R}, V_0)$.

Thus, u is solution of the equation (1.11) that belongs to $C_b^0(\mathbb{R}, V_2)$. From the definition of $\mathcal{A}_\alpha$, for all t $u(t)$ belongs to $\mathcal{A}_\alpha$. And from (4.10) :

$$\left(u(t), \frac{du}{dt}(t)\right) \in \mathcal{A}_\alpha^*.$$

Since $\left(u_{\epsilon'}, \frac{du_{\epsilon'}}{dt}\right)$ converges to $(u, \frac{du}{dt})$ in $C^0([-T,T], V_2 \times V_0)$ for all $T > 0$, we have :

$$(u_{\epsilon'}, v_{\epsilon'}) = \left(u_{\epsilon'}(0), \frac{du_{\epsilon'}}{dt}(0)\right) \text{ converges to } \left(u(0), \frac{du}{dt}(0)\right) \text{ in } (V_2 \times V_0).$$

And that implies :

$$\lim_{\epsilon \to 0} \inf_{(u,v) \in \mathcal{A}_\alpha^*} (\| u_\epsilon - u \|_2^2 + \| v_\epsilon - v \|_0^2) = 0,$$

$$0 \leq \limsup_{\epsilon \to 0} \delta(\mathcal{A}_{\epsilon,\alpha}^3, \mathcal{A}_\alpha^*) \leq \eta.$$

η is arbitrary small, therefore :

$$\lim_{\epsilon \to 0} \delta(\mathcal{A}_{\epsilon,\alpha}^3, \mathcal{A}_\alpha^*) = 0.$$

APPENDIX
Proof of Theorem (1.2)

The proof of this theorem is very standard, we use a Faedo-Galerkin method. Let $(w_j)_{j\in\mathbb{N}}$ be the orthonormal basis in $L^2(0,L)$ of eigenvectors of A. We first look for u_m in $Sp(w_1,...,w_m)$ such that

$$\epsilon\frac{d^2u_m}{dt^2}+\frac{du_m}{dt}+\nu\Delta^2u_m-\Delta P_mf(u_m)=0, \qquad (A.1)$$

$$u_m(0)=P_mu_0, \qquad (A.2)$$

$$\frac{du_m}{dt}(0)=P_mu_1. \qquad (A.3)$$

(P_m is the orthogonal projector on $Sp(w_1,...,w_m)$).

The existence and uniqueness for u_m on an interval $[0,T_m[$ follow from classical theorem on ordinary differential equations. Using the same arguments as in the proof of proposition 1.1, 2.1, 2.2, 2.3 one easily prove that, for $s=1,2$ or 3, if (u_0,u_1) belongs to $V_s\times V_{s-2}$ then, for all $T\geq 0$:

$$(u_m)_{m\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;V_s), \qquad (A.4)$$

$$\left(\frac{du_m}{dt}\right)_{m\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T;V_{s-2}). \qquad (A.5)$$

(in fact, we can prove (A.4), (A.5) with simpler calculous since we are not interested in the dependence on ϵ and we do not need time uniform estimates). ((A.4) and (A.5) implies that $T_m=+\infty$).

Now, using classical compactness theorems, we can take the limit in (A.1), (A.2), (A.3). There exists a function u in $L^\infty(0,T;V_s)$ with $\frac{du}{dt}$ in $L^\infty(0,T;V_{s-2})$ for all $T>0$ such that:

$$\epsilon\frac{d^2u}{dt^2}+\frac{du}{dt}+\nu\Delta^2u-\Delta f(u)=0, \qquad (A.6)$$

$$u(0)=u_0, \qquad (A.7)$$

$$\frac{du}{dt}(0)=u_1. \qquad (A.8)$$

Now, we prove the uniqueness in $V_1\times V_{-1}$ together with the continuity of $S_\epsilon^1(t)$.

Let (u_0, u_1), (v_0, v_1) in $V_1 \times V_{-1}$ and u, v the corresponding solution of (A.6), (A.7), (A.8). We denote by

$$w = u - v.$$

We have

$$\epsilon \frac{d^2 w}{dt^2} + \frac{dw}{dt} + \nu \Delta^2 w - \Delta(f(u) - f(v)) = 0. \tag{A.9}$$

We take the scalar product of (A.9) with $A^{-1/2}\frac{dw}{dt}$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{dw}{dt}\right|_{-1}^2 + \nu |w|_1^2\right) + \left|\frac{dw}{dt}\right|_{-1}^2 + \left(f(u) - f(v), \frac{dw}{dt}\right)_0 = 0,$$

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \left|\frac{dw}{dt}\right|_{-1}^2 + \nu |w|_1^2\right) + \left|\frac{dw}{dt}\right|_{-1}^2 \leq |f(u) - f(v)|_1 \left|\frac{dw}{dt}\right|_{-1}.$$

We use a Young inequality :

$$\frac{d}{dt}\left(\epsilon \left|\frac{dw}{dt}\right|_{-1}^2 + \nu |w|_1^2\right) \leq |f(u) - f(v)|_1^2.$$

Since f is a polynomial, V_1 is an algebra and u, v are in $L^\infty(0, T; V_1)$, we know that there exists a constant C_1 (depending on $\sup_{[0,T]} |u|_1$ and $\sup_{[0,T]} |v|_1$) such that

$$|f(u) - f(v)|_1 \leq C_1 |u - v|_1.$$

Therefore :

$$\frac{d}{dt}\left(\epsilon \left|\frac{dw}{dt}\right|_{-1}^2 + \nu |w|_1^2\right) \leq C_1^2 |w|_1^2.$$

An application of Gronwall's lemma leads to :

$$\epsilon \left|\frac{dw}{dt}(t)\right|_{-1}^2 + \nu |w(t)|_1^2 \leq e^{C_1 t}\left(\epsilon |u_1|_{-1}^2 + \nu |u_0|_1^2\right). \tag{A.10}$$

We now take the average of (A.9) :

$$\epsilon \frac{d^2}{dt^2} m(w) + \frac{d}{dt} m(w) = 0.$$

Therefore :

$$m(w) = (m(u_0) - m(v_0)) + \epsilon(m(u_1) - m(v_1))(1 - e^{-t/\epsilon}), \tag{A.11}$$

$$m\left(\frac{dw}{dt}\right) = (m(u_1) - m(v_1))e^{-t/\epsilon}. \tag{A.12}$$

(A.10), (A.11) and (A.12) imply the unicity of the solution in $V_1 \times V_{-1}$ and the continuity of $S_\epsilon^1(t)$.

The continuity of $S_\epsilon^2(t)$ and $S_\epsilon^1(t)$ are left to the reader. We use the same type of arguments as above. It remains to prove the regularity of the solution, that is :

$$\left(u, \frac{du}{dt}\right) \in C\left([0,T], V_s \times V_{s-2}\right).$$

We notice that u is the solution of the nonhomogeneous linear problem :

$$\epsilon\frac{d^2v}{dt^2} + \frac{dv}{dt} + \nu\Delta^2 v = g,$$

supplemented with the same initial and boundary conditions as (A.6) and with :

$$g = \Delta f(u) \in L^\infty(0,T; V_{s-2}).$$

On can find the proof of the regularity for that type of linear problem in Lions-Magenes [1] for instance.

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