OLIVIER GOUBET

Quelques aspects de l'approximation pour les grands temps des solutions d'équations d'évolution dissipatives, par ondelettes et éléments finis, 1992

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Olivier GOUBET

Sujet :
Quelques aspects de l'approximation pour les grands temps des solutions d'équations d'évolution dissipatives, par ondelettes et éléments finis

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M. R. TEMAM, Président
M. C. JOURON
Mme M. MARION
MM. J.-C. SAUT
H. YSERENTANT
J. BONA, invité

Some questions related to the large time approximation, by wavelets and finite elements, of the solutions of dissipative evolution equations.

Abstract: In this work we consider some methods to approximate the global attractor of the dynamical system defined by the flow of the solutions of a dissipative nonlinear parabolic equation. These methods, that relate to the inertial manifolds theory, are adjusted to wavelets and to new hierarchical finite-elements bases.

In the first part, we construct approximate inertial manifolds (A.I.M.) for a class of evolution equations. A.I.M.'s are smooth finite-dimensional manifolds that contain the attractor into a thin neighborhood. The innovation is that these manifolds are defined as graphs on orthonormal wavelet bases.

In the second part, we first study new hierarchical finite-elements bases. We then use these bases for a nonlinear Galerkin approximation of a reaction-diffusion equation. We prove convergence results for approximate solutions towards the solution of the original problem.

Key words:

Dissipation, nonlinear parabolic equations, approximate inertial manifolds, wavelets, nonlinear Galerkin methods, hierarchical bases, finite-elements.

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RESUME

Dans cette thèse, nous abordons quelques aspects de l'approximation pour les grands temps des solutions d'équations aux dérivées partielles dissipatives. Les méthodes utilisées se situent dans le cadre de la théorie des variétés inertielles, et sont adaptées aux ondelettes et à de nouveaux types d'éléments finis.

En quelques mots, la dissipation de l'E.D.P. se traduit par l'existence d'un attracteur pour le système dynamique associé, i.e. d'un ensemble compact de l'espace ambiant qui attire toutes les orbites quand $t \to +\infty$ (cf [T1]). Pour surmonter les difficultés inhérentes à l'approximation d'un tel ensemble – l'attracteur peut avoir une structure complexe, fractale; la vitesse de convergence des orbites peut être arbitrairement lente – de nouveaux objets mathématiques ont été introduits, les variétés inertielles. Ce sont des variétés régulières de dimension finie, invariantes par le flot des solutions, et qui attirent toutes les orbites avec une vitesse exponentielle (cf [FST]).

En particulier parce que leurs équations plus simples facilitaient l'implémentation d'algorithmes numériques, vinrent ensuite les variétés inertielles approximatives (V.I.A.), qui sont des variétés régulières de dimension finie qui attirent toutes les orbites dans un voisinage mince, en un temps fini et avec une vitesse exponentielle (cf [FMT]). En outre il subsiste des équations pour lesquelles on ne connaît pas de variété inertielle, et pour lesquelles ont été construites des familles de V.I.A. qui approchent l'attracteur avec un ordre de plus en plus élevé (cf [DM], [Fl], [T2]).

A titre d'exemple, l'espace de dimension finie sur lequel on implémente une méthode de Galerkin pour calculer une solution approchée de l'équation considérée, représente la V.I.A. la plus simple, puisque plate, et la plus grossière, puisque fournissant un ordre d'approximation peu élevé. Par ailleurs les V.I.A. d'ordre supérieur sont la base d'algorithmes numériques performants, en vue de l'intégration pour les grands temps des équations d'évolution (cf [DJT], [DJMT], [FJoKSTi], [JRT])

La théorie et la pratique des Méthodes de Galerkin Non-Linéaires est maintenant bien développée dans le cas d'une approximation spectrale des solutions, et s'étend à présent au-delà : cf [MT] pour les éléments finis, [T3] pour les différences finies.

La thèse s'articule dès lors autour de deux axes. La première partie est consacrée à la construction de V.I.A. à l'aide d'ondelettes pour une équation pa-

rabolique dissipative. La seconde partie commence par la description de nouvelles bases d'éléments finis , que l'on utilise ensuite afin d'implémenter un algorithme non linéaire – dont on étudie la convergence – pour une équation de type réaction-diffusion.

1)Construction de variétés inertielles approximatives à l'aide d'ondelettes

Soit une équation qui s'écrit sous forme abstraite

$$\frac{du}{dt} + Au + B(u) = f$$

où l'inconnue $u(t)$ envoie R_+ dans $L^2(T^n)$, T^n étant le tore n-dimensionnel, où A est l'opérateur non borné $(-\Delta)^m$, avec conditions au bord périodiques, et où la nonlinéarité B est choisie bilinéaire avec des hypothèses techniques assurant l'existence d'un attracteur pour le système dynamique associé à l'équation.

Le résultat central établit des estimations pour les grands temps sur la distance, pour différentes topologies, entre toute trajectoire $u(t)$, et l'espace de dimension finie V_j engendré par les 2^{jn} premières ondelettes, rangées dans l'ordre naturel. Dans le langage des variétés inertielles, nous avons établi que, sous réserves de considérer des ondelettes de régularité suffisante, V_j fournit une V.I.A. d'ordre comparable à celle obtenue en utilisant la décomposition spectrale de $u(t)$ suivant les vecteurs propres de A, et ceci pour des espaces de même dimension.

Auparavant nous aurons établi des résultats préliminaires sur les ondelettes, où nous retrouvons à quelles conditions elles fournissent des bases inconditionelles des espaces de Sobolev périodiques (cf [Me]).

Dans le théorème central, nous prouvons en outre des estimations pour les grands temps sur la distance entre $\frac{du}{dt}(t)$ et V_j, ce qui nous permet de dériver des équations de V.I.A. qui fournissent des ordres d'approximation supérieurs à celui de V_j.

2)Méthodes de Galerkin non linéaires à l'aide de bases hiérarchiques presque orthogonales d'éléments finis

Dans cette partie nous envisageons d'étudier un algorithme non linéaire pour approcher les solutions d'une équation de réaction-diffusion de type

$$\frac{\partial u}{\partial t} - \nu\Delta u + R(u) = f,$$

où la non-linéarité R est un polynôme de degré impair dont le terme de plus haut degré est positif.

Dans un premier temps, nous décrivons la construction de nouvelles bases hiérarchiques d'éléments finis associées à la discrétisation de l' opérateur $Id - \Delta$ sur $\Omega = [0,1]^n$ ($n = 1,2$), avec respectivement des conditions aux limites de type Dirichlet, Neumann ou périodiques.

Ensuite on établit des résultats relatifs à l'analyse des fonctions des espaces de Sobolev qui interviennent dans la formulation variationelle du problème initial, en particulier on traite le cas des espaces $L^p(\Omega)$, à l'aide des bases précédemment obtenues.

La dernière partie est consacrée à l'étude de la convergence de la solution du problème semi-discrétisé, la variable t étant gardée continue, vers la solution du problème initial.

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CONSTRUCTION OF APPROXIMATE INERTIAL MANIFOLDS USING WAVELETS

Olivier Goubet

Laboratoire d'Analyse Numérique,

Université de Paris-Sud,

91405 Orsay Cedex, France

Introduction

One of the aims of this article is to make the connection between two recent theories: the theory of inertial manifolds that has emerged from the study of dynamical systems and the theory of orthonormal wavelet bases.

First let us have an overview of the inertial manifolds theory, that relates to the large time study of dissipative evolution equations.

Let us consider a nonlinear P.D.E. that is dissipative; it means that there exists a *global attractor* for the associated dynamical system, i.e. a compact set that is invariant by the flow of the solutions, and that attracts all the orbits when $t \to +\infty$. Nevertheless the convergence of the orbits towards the attractor can be arbitrarily slow, and this one can have a complex structure, and even be a fractal (see [T1]).

Then new Mathematical tools have been introduced ([FST]): The inertial manifolds (I.M.), which are smooth finite-dimensional manifolds, positively invariant by the flow of the solutions, and that attract all the trajectories with exponential speed. From the Physical point of view, I.M.'s modelize the interaction laws between small and large structures of a turbulent flow, and represent its permanent regime; actually on a I.M. small eddies are slaved by large ones, and there is similar results, after a transient time, for a trajectory that is not on the I.M. (see [FST], [FMT]). Nevertheless, until now the existence of I.M. necessitates a very restrictive property, the spectral gap condition (see [T1]).

Hence came the approximate inertial manifolds (A.I.M.). They are smooth finite-dimensional manifolds that attract all the orbits into a thin neighborhood, in a finite time, and with exponential speed. A.I.M.'s are useful when no existence result for I.M. is available, and also since their equations are rather simple, and then make easier the implementation of numerical algorithms (see [FMT], [T5], [DeMa], [Fl])

The theory of A.I.M.'s which first developed in the spectral case has begun to extend beyond. Some nonlinear algorithms have been established for finite elements (see [MT]) and finite differences (see [T3]). The purpose of this paper, following a suggestion of R. Temam, is to construct approximate inertial manifolds using the newly developed concept of orthonormal bases of wavelets.

The improvement featured by wavelets with respect to spectral bases (here the trigonometrical system), is to combine good localization properties in space variable, and good localization properties in frequencies (see [M]). In this paper we are interested in the space periodic case, and we consider the periodic version of wavelet bases of [D], [LM], [L], as described in [M]. However, wavelets are flexible tools that can be adapted to other domains than the n-dimensional torus (see [JM]), and allow us to consider the construction of A.I.M.'s in domains where few information on the spectral bases is available.

Nevertheless, in this paper, for the sake of clarity, we will focus more particularly on the one-dimensional spline wavelet bases of [L], referring the reader to an Appendix for multidimensional results and for other wavelet bases.

The paper is organized as follows. In section 1 we briefly recall some results about spline wavelets in the one-dimensional periodic case. In section 2 we give for the sake of completeness a proof of a result announced in [M] saying that, if spline wavelets are regular enough, then they provide an unconditional basis for periodic Sobolev spaces. In section 3, we describe a class of nonlinear parabolic P.D.E.'s; Then we show how one can use the wavelet expansion of a function to construct several A.I.M.'s for this class of evolution equation. The method follows [T2]; first we define the *induced trajectories*, tools that allow us to estimate the distance, for different topologies, between the orbits and the space spanned by the k first wavelets, ordered in the natural way. These estimates, that hold for large time, are then compared to the ones obtained in the spectral case, for spaces that have the same dimension. The result is that the wavelets provide a flat A.I.M. that provide the same order of approximation than the one obtained using spectral bases. Then we give two examples of nonflat A.I.M.'s that approximate the attractor with higher order than the flat one; once again we match the accuracy obtained in the spectral case. Finally in the appendix we extend the results of section 2 to the constructions of [D] and of [LM] and to the multidimensional case.

Notations: Let $\mathcal{Z}$ (*resp.* $\mathcal{R}$, $\mathcal{C}$) be the set of integers (resp. of real numbers, of complex numbers).

Let $\Pi = \mathcal{R}/\mathcal{Z}$ be the one-dimensional torus. We denote by $C^N(\Pi)$ the space of N times continuously differentiable functions on Π and by $H^s(\Pi)$ the usual periodic Sobolev space.

We denote by $\dot{H}^s(\Pi)$ the space of functions u in $H^s(\Pi)$ such that

$$\int_\Pi u(x)dx = 0.$$

$\dot{H}^s(\Pi)$ is a Hilbert space when endowed with the scalar product

$$(u,v)_s = \sum_{k \in \mathcal{Z}} | k |^{2s} \hat{u}(k)\overline{\hat{v}(k)},$$

where

$$\hat{u}(k) = \int_{\Pi} u(x) e^{-2i\pi kx} dx.$$

Let $| u |_s$ be the corresponding norm,

$$| u |_s = (u,u)_s^{\frac{1}{2}}.$$

When $s = 0$ we write $H^0(\Pi) = L^2(\Pi)$, $| u |_0 = | u |$ and $(u,v)_0 = (u,v)$.

In the following we will denote by C a constant that only depends on the regularity N of the wavelet, and in part 3 on the data of the equation.

1. Spline wavelet bases of $\dot{L}^2(\Pi)$

We consider the finite dimensional space $V_j = \{v \in C^N(\Pi); v$ is a piecewise polynomial function of degree less than or equal to $N+1$, with nodes at $k/2^j; 0 \leq k < 2^j\}$. Then we have the embeddings

$$V_0 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots L^2(\Pi).$$

We define

$$W_j = V_{j+1} \cap (V_j)^{\perp}, \tag{1.1}$$

then we have

$$\dot{L}^2(\Pi) = \bigoplus_{j=0}^{+\infty} W_j, \tag{1.2}$$

the sum being orthogonal.

Let us introduce what are the periodic wavelet bases associated to the W_j's. We first recall the original construction on $\mathcal{R}$; from [L] (see also [B], [M]) we know that, for each integer N, there exists a function ψ_N satisfying

(i)

$$\psi_N \in C^N(R), \tag{1.3}$$

ψ_N being a piecewise polynomial function of degree less than or equal to $N+1$ with nodes at the half integers.

(ii)

$$\exists\ \varepsilon_N > 0/ \text{ for } m \leq N+1$$

$$\mid \frac{\partial^m}{\partial x^m}\psi_N(x) \mid \leq Ce^{-\varepsilon_N |x|}. \tag{1.4}$$

(iii) If $m \leq N+1$

$$\int_{\mathcal{R}} x^m \psi_N(x)dx = 0. \tag{1.5}$$

The wavelets, that are derived from ψ_N by a translation and a dilation as below, satisfy

(iv) The family $\{2^{j/2}\psi_N(2^j x - k)\}_{j,k\in\mathcal{Z}}$ is an orthonormal basis of

$$L^2(\mathcal{R}). \tag{1.6}$$

Remark 1: Formula (1.4) shows the exponential decay of the wavelet $2^{j/2}\psi_N(2^j x - k)$ away from $\frac{k+\frac{1}{2}}{2^j}$. Formulae (1.3) and (1.5) describe the localization in frequencies of the wavelet $2^{j/2}\psi_N(2^j x - k)$ around an anulus of radii $c_1 2^j, c_2 2^j$.

Remark 2: Throughout this paper we shall omit the subscript N on ψ_N.

Following [M] we define the periodic wavelets as

$$\psi_{j,k}(x) = 2^{j/2}\sum_{\ell\in\mathcal{Z}}\psi(2^j x + 2^j\ell - k). \tag{1.7}$$

This periodization transfers to periodic wavelets the localization in frequencies (see Lemma 2 below), and does not deteriorate too much the localization in space variable. Then we have

$$\cdot \text{ The family } \{\psi_{j,k}\}_{1\leq k\leq 2^j} \text{ is an orthonormal basis of } W_j. \tag{1.8}$$

$$\cdot \text{ The family } \{\psi_{j,k}\}_{0\leq j\leq +\infty; 1\leq k\leq 2^j} \text{ is an orthonormal basis of } \dot{L}^2(\Pi). \tag{1.9}$$

2. Preliminary results

2.1 Bernstein inequalities

Proposition 1: *There exists $C > 0$ such that for any v in V_j*

$$| v |_{N+1} \leq C 2^{j(N+1)} | v | . \tag{2.1}$$

Proof: Let b_N be the N^{th} fundamental B-spline, defined from the characteristic function χ of $[-1/2, 1/2]$ by

$$b_N = \chi \underbrace{* \cdots\cdots *}_{N+1 \text{ times}} \chi. \tag{2.2}$$

For v in V_j,

$$v = \sum_{k=1}^{2^j} \alpha_{j,k} 2^{j/2} b_N(2^j x - k). \tag{2.3}$$

It is well known that

$$c_1 (\sum_{k=1}^{2^j} | \alpha_{j,k} |^2) \leq | v |^2 \leq c_2 (\sum_{k=1}^{2^j} | \alpha_{j,k} |^2), \tag{2.4}$$

for some constants c_1, c_2 depending on N.

On the other hand, we have

$$| v |_{N+1}^2 = \sum_{1\leq k,p\leq 2^j} 4^{j(N+1)} \alpha_{j,k} \alpha_{j,p} m_{j,k,p}, \tag{2.5}$$

where

$$m_{j,k,p} = 2^j \int_\Pi \frac{\partial^{N+1}}{\partial x^{N+1}} b_N(2^j x - k) \frac{\partial^{N+1}}{\partial x^{N+1}} b_N(2^j x - p) dx. \tag{2.6}$$

We observe that either $m_{j,k,p} = 0$ if $| k-p | > N+1$ or

$$| m_{j,k,p} | \leq \| \frac{\partial^{N+1}}{\partial x^{N+1}} b_N \|_{L^1(\mathcal{R})} \| \frac{\partial^{N+1}}{\partial x^{N+1}} b_N \|_{L^\infty(\mathcal{R})} . \tag{2.7}$$

It follows

$$| v |_{N+1}^2 \leq C 4^{j(N+1)} \sum_{|k-p| \leq N+1} | \alpha_{j,k} || \alpha_{j,p} | . \tag{2.8}$$

Using

$$\sum_{|k-p| \leq N+1} | \alpha_{j,k} || \alpha_{j,p} | \leq C \sum_{k=1}^{2^j} | \alpha_{j,k} |^2, \tag{2.9}$$

we infer the result from (2.4) and (2.8).

Remark 3: We recall that C is a constant that depends on N, but which is independent of j.

Lemma 1: *Let r, s, t be real numbers, $r \leq s \leq t$. Then for any u in $\dot{H}^t(\Pi)$*

$$| u |_s \leq | u |_r^{\frac{t-s}{t-r}} | u |_t^{\frac{s-r}{t-r}} . \tag{2.10}$$

Proof: This is just a particular case of the interpolation inequality (see [T1] and the references therein).

Corollary 1: *Let s be in $[0, N+1]$; then for any v in V_j*

$$| v |_s \leq C 2^{js} | v | . \tag{2.11}$$

Proof: Thanks to (2.10)

$$| v |_s \leq | v |^{1-(s/(N+1))} | v |_{N+1}^{s/(N+1)} . \tag{2.12}$$

We infer from (2.1) and (2.12)

$$| v |_s \leq C^{s/(N+1)} 2^{js} | v | . \qquad (2.13)$$

Corollary 2: *Let s be in $[-N-1, 0]$; then for any v in V_j*

$$| v | \leq C 2^{-js} | v |_s . \qquad (2.14)$$

Proof: Thanks to (2.10)

$$| v | \leq | v |_s^{(N+1)/(N+1-s)} | v |_{N+1}^{-s/(N+1-s)} . \qquad (2.15)$$

Using (2.1)

$$| v | \leq C^{-s/(N+1-s)} 2^{-js(N+1)/(N+1-s)} | v |^{-s/(N+1-s)} | v |_s^{(N+1)/(N+1-s)} . \qquad (2.16)$$

To conclude we take the $\{\frac{N+1-s}{N+1}\}^{th}$ power of inequality (2.16).

2.2 Poincaré inequalities

Lemma 2: *Let f be in $L^1(\mathcal{R})$. Let $g(x) = \sum_{\ell \in \mathcal{Z}} f(x+\ell) \in L^1(\Pi)$. Then*

$$\hat{g}(k) = \hat{f}(k). \qquad (2.17)$$

(the left hand side of the equality above represents the k^{th} Fourier coefficient of g, the right hand side denotes the value at the point k of the Fourier transform of f)

Proof: Thanks to Fubini's theorem,

$$\int_{\Pi} g(x) e^{-2i\pi kx} dx = \sum_{\ell \in \mathcal{Z}} \int_{\ell}^{\ell+1} f(x) e^{-2i\pi(k-\ell)x} dx \qquad (2.18)$$

provides the result.

Proposition 2: *There exists $C > 0$ such that for any w in W_j*

$$| w |_{-N-1} \leq C 2^{-j(N+1)} | w | . \tag{2.19}$$

Proof: Let $w \in W_j$; we write

$$w = \sum_{k=1}^{2^j} \alpha_{j,k} \psi_{j,k}. \tag{2.20}$$

We easily infer from (1.7) and (2.17)

$$\hat{\psi}_{j,k}(\ell) = \frac{1}{2^{j/2}} \hat{\psi}(\frac{\ell}{2^j}) exp(-2i\pi \frac{k\ell}{2^j}). \tag{2.21}$$

We set

$$m(\frac{\ell}{2^j}) = 2^{-j/2} \sum_{k=1}^{2^j} \alpha_{j,k} exp(-2i\pi \frac{k\ell}{2^j}). \tag{2.22}$$

This yields

$$\hat{w}(\ell) = m(\frac{\ell}{2^j}) \hat{\psi}(\frac{\ell}{2^j}). \tag{2.23}$$

Then, thanks to Parseval's identity,

$$| w |^2 = \sum_{\ell \in \mathcal{Z}} | \hat{w}(\ell) |^2, \tag{2.24}$$

we obtain

$$| w |^2 = \sum_{\ell \in \mathcal{Z}} | m(\frac{\ell}{2^j}) |^2 | \hat{\psi}(\frac{\ell}{2^j}) |^2 . \tag{2.25}$$

we write

$$| w |^2 = \sum_{k=1-2^{j-1}}^{2^{j-1}} \sum_{\ell \in \mathcal{Z}} | m(\frac{k}{2^j} + \ell) |^2 | \hat{\psi}(\frac{k}{2^j} + \ell) |^2 . \tag{2.26}$$

Observing that m is a one-periodic function, we obtain

$$| w |^2 = \sum_{k=1-2^{j-1}}^{2^{j-1}} | m(\frac{k}{2^j}) |^2 (\sum_{\ell \in \mathcal{Z}} | \hat{\psi}(\frac{k}{2^j} + \ell) |^2). \tag{2.27}$$

Now we need

Lemma 3: *Let f be in $L^2(\mathcal{R}) \cap L^1(\mathcal{R})$ such that $\mid \hat{f}(z) \mid \leq C(1+ \mid z \mid)^{-\alpha}$ with $\alpha > \frac{1}{2}$, and such that the family $\{f(x+\ell)\}_{\ell \in \mathcal{Z}}$ is an orthonormal family in $L^2(\mathcal{R})$. Then, for each z in Π*

$$\sum_{\ell \in \mathcal{Z}} \mid \hat{f}(z+\ell) \mid^2 = 1. \tag{2.28}$$

Proof: We prove that the Fourier coefficients of the one-periodic function

$$1 - \sum_{\ell \in \mathcal{Z}} \mid \hat{f}(z+\ell) \mid^2$$

are equal to zero.

Lemma 2 yields

$$\int_{\Pi} (\sum_{l \in \mathcal{Z}} \mid \hat{f}(z+\ell) \mid^2) e^{-2i\pi kz} dz$$

$$= \int_{\mathcal{R}} \mid \hat{f}(z) \mid^2 e^{-2i\pi kz} dz. \tag{2.29}$$

The result follows, using Plancherel's theorem

$$\int_{\mathcal{R}} \mid \hat{f}(z) \mid^2 e^{-2i\pi kz} dz$$

$$= \int_{\mathcal{R}} f(x) f(x-k) dx. \tag{2.30}$$

Then, we apply Lemma 3 to (2.27) to obtain

$$\mid w \mid^2 = \sum_{k=1-2^{j-1}}^{2^{j-1}} \mid m(k/2^j) \mid^2 . \tag{2.31}$$

On the other hand, by the same computations as above

$$\mid w \mid^2_{-N-1} =$$

$$\frac{1}{4^{j(N+1)}} \sum_{k=1-2^{j-1}}^{2^{j-1}} \mid m(\frac{k}{2^j}) \mid^2 (\sum_{\ell \in \mathcal{Z}} (\ell + \frac{k}{2^j})^{-2N-2} \mid \hat{\psi}(\ell + \frac{k}{2^j}) \mid^2). \tag{2.32}$$

Using (2.31), we observe that it is sufficient to prove that $\exists\, C > 0/$

$$\sum_{\ell \in \mathbb{Z}} \mid (\ell + \frac{k}{2^j}) \mid^{-2N-2} \mid \hat{\psi}(\ell + \frac{k}{2^j}) \mid^2 \leq C < +\infty \tag{2.33}$$

to end the proof of Proposition 2.

First we obtain

$$\sum_{\ell \in \mathbb{Z}^*} (\frac{k}{2^j} + \ell)^{-2N-2} \mid \hat{\psi}(\ell + \frac{k}{2^j}) \mid^2 \leq 4^{N+1}, \tag{2.34}$$

using that for $|k| \leq 2^{j-1}$, we have

$$|\frac{k}{2^j} + \ell|^{-2N-2} \leq 4^{N+1},$$

and thanks to Lemma 3

$$\sum_{\ell \in \mathbb{Z}^*} \mid \hat{\psi}(\ell + \frac{k}{2^j}) \mid^2 \leq 1. \tag{2.35}$$

It remains to majorize

$$(\frac{k}{2^j})^{-2N-2} \mid \hat{\psi}(\frac{k}{2^j}) \mid^2,$$

that is the term corresponding to $\ell = 0$ in (2.33). For that purpose we observe that, thanks to (1.5), $\hat{\psi}(z) = 0(z^{N+1})$ when $z \to 0$. Therefore there exists a constant C such that

$$(\frac{k}{2^j})^{-2N-2} \mid \hat{\psi}(\frac{k}{2^j}) \mid^2 \leq C. \tag{2.36}$$

This fact concludes the proof of Proposition 2.

Then we also have

Corrolary 3: *Let s be in $[-N-1, 0]$; then for any w in W_j*

$$|w|_s \leq C 2^{js} |w|. \tag{2.37}$$

Corrolary 4: *Let s be in $[0, N+1]$; then for any w in W_j*

$$|w| \leq C2^{-js}|w|_s. \tag{2.38}$$

Proof: We prove (2.37) and (2.38) as we established (2.11) and (2.14), using (2.19) instead of (2.1).

2.3 Conclusion

We summarize section 2.1 and section 2.2 by:

Proposition 3: *There exist $C_1, C_2 > 0$ such that, for each s in $[-N-1, N+1]$, for any w_j in W_j, setting*

$$w_j = \sum_{k=1}^{2^j} \gamma_{j,k}\psi_{j,k},$$

we have

$$C_1 \mid w_j \mid_s \leq 2^{js}(\sum_{k=1}^{2^j} \gamma_{j,k}^2)^{1/2} \leq C_2 \mid w_j \mid_s . \tag{2.39}$$

We also have

$$\mid w_j \mid = (\sum_{k=1}^{2^j} \gamma_{j,k}^2)^{1/2}. \tag{2.40}$$

The following theorem includes the main result of this section.

Theorem 1: *Let s be in $(-N-1, N+1)$; let u be in $\dot{H}^s(\Pi)$. We set*

$$\gamma_{j,k} = < u, \psi_{j,k} >_{\dot{H}^s(\Pi), \dot{H}^{-s}(\Pi)},$$

$$w_j = \sum_{k=1}^{2^j} \gamma_{j,k}\psi_{j,k}.$$

Then, $u = \sum_{j=0}^{+\infty} w_j$, where the sum is unconditionally convergent in $\dot{H}^s(\Pi)$.

Actually there exist $C_1(s), C_2(s) > 0$ such that

$$C_1(s) \mid u \mid_s^2 \leq \sum_{j=0}^{+\infty} \mid w_j \mid_s^2 \leq C_2(s) \mid u \mid_s^2 \tag{2.41}$$

Proof: We first prove (2.41) for u regular, such that $u = \sum_{j=0}^{+\infty} w_j$ holds in $\dot{L}^2(\Pi)$ for instance.

Then we conclude by noticing that if $u_n \to u$ in $\dot{H}^s(\Pi)$, then, thanks to Proposition 3, $w_{j,n} \to w_j$ in $\dot{H}^s(\Pi)$. ($w_{j,n}$ defined from u_n in the obvious way)

Now we follow step by step the proof of Theorem 8 chapter 2 of [M]. We write

$$\mid u \mid_s^2 = \sum_{j=0}^{+\infty} \mid w_j \mid_s^2 + 2 \sum_{j<\ell} (w_j, w_\ell)_s. \tag{2.42}$$

Choosing ε such that $\mid s \mid < N - \varepsilon$, we obtain

$$\mid (w_j, w_\ell)_s \mid \leq \mid w_j \mid_{s+\varepsilon} \mid w_\ell \mid_{s-\varepsilon}, \tag{2.43}$$

and thanks to (2.39)

$$\mid (w_j, w_\ell)_s \mid \leq C 2^{-\varepsilon(\ell-j)} \mid w_j \mid_s \mid w_\ell \mid_s . \tag{2.44}$$

Therefore

$$\mid 2 \sum_{j<\ell} \mid (w_j, w_\ell)_s \mid \mid \leq C \sum_{j=0}^{+\infty} (\sum_{\ell \neq j} 2^{-\varepsilon|\ell-j|}) \mid w_j \mid_s^2$$

$$\leq \frac{C}{2^\varepsilon - 1} \sum_{j=0}^{+\infty} \mid w_j \mid_s^2 . \tag{2.45}$$

This yields

$$\mid u \mid_s^2 \leq C(s) \sum_{j=0}^{+\infty} \mid w_j \mid_s^2 . \tag{2.46}$$

On the other hand, setting

$$u_J = \sum_{j=0}^{J} 4^{js} w_j, \tag{2.47}$$

we have

$$(u, u_J) = \sum_{j=0}^{J} 4^{js} \mid w_j \mid^2 . \tag{2.48}$$

Thanks to (2.39)

$$\sum_{j=0}^{J} 4^{js} \mid w_j \mid^2 \geq C \sum_{j=0}^{J} \mid w_j \mid_s^2 . \tag{2.49}$$

Then we infer from (2.48), (2.49)

$$\sum_{j=0}^{J} \mid w_j \mid_s^2 \leq C \mid u \mid_s \mid u_J \mid_{-s} . \tag{2.50}$$

Using the first part of the proof we obtain

$$\mid u_J \mid_{-s}^2 \leq C(s) \sum_{j=0}^{J} \mid 4^{js} w_j \mid_{-s}^2 , \tag{2.51}$$

this yields, using (2.39)

$$\mid u_J \mid_{-s}^2 \leq C(s) \sum_{j=0}^{J} \mid w_j \mid_s^2 . \tag{2.52}$$

We then infer from (2.50) and (2.52)

$$\left(\sum_{j=0}^{J} \mid w_j \mid_s^2\right)^{1/2} \leq C(s) \mid u \mid_s . \tag{2.53}$$

We let $J \to +\infty$ to end the proof.

We summarize Proposition 3 and Theorem 1 by:

Corollary 5: *For each s in $(-N-1, N+1)$, there exist $C_1(s), C_2(s) > 0$ such that, if*

$$u = \sum_{j=0}^{+\infty} \sum_{k=1}^{2^j} \gamma_{j,k} \psi_{j,k} \in \dot{H}^s(\Pi),$$

then

$$C_1(s) \mid u \mid_s^2 \leq \sum_{j,k} 4^{js} \mid \gamma_{j,k} \mid^2 \leq C_2(s) \mid u \mid_s^2 . \tag{2.54}$$

In other words, if $| s | < N + 1$, the family $\{\psi_{j,k}\}_{0 \leq j \leq +\infty;\, 1 \leq k \leq 2^j}$ is an unconditional basis for $\dot{H}^s(\Pi)$. (For equivalent definitions of an unconditional basis in Hilbert spaces see for instance [JM]; see also [M] and the references therein)

3. Approximate Inertial Manifolds

3.1. A class of evolution equations

Let H be the Hilbert space $\dot{L}^2(\Pi)$. The class of evolution equations we shall consider has the form

$$\frac{du}{dt} + Au + B(u,u) = f, \tag{3.1}$$

where $f \in H$, A is the dissipative partial differential operator $(\frac{-\partial^2}{\partial x^2})^m$, acting on H, whose domain is

$$D(A) = \dot{H}^{2m}(\Pi),$$

B is a bilinear operator from $D(A^{1/2}) \times D(A^{1/2})$ into $D(A^{-1/2})$, and the unknown u maps R_+ into H.

We will consider the initial value problem consisting of (3.1) and of initial condition

$$u(0) = u_0 \in H. \tag{3.2}$$

Notations: As usual we write

$$V = D(A^{1/2}) = \dot{H}^m(\Pi), \quad \| \cdot \| = | \cdot |_m$$

and

$$((\cdot,\cdot)) = (\cdot,\cdot)_m.$$

Moreover we assume that the following properties involving B hold: Setting

$$B(u) = B(u,u),$$

$$b(u,v,w) = < B(u,v), w >_{V',V},$$

for

$$u,v,w \in V,$$

we have

$$b(u,v,v) = 0 \tag{3.3}$$

and there exists $C_b > 0$ such that for any u,v,w in V

$$| b(u,v,w) | \leq C_b | u |^{1/2} \| u \|^{1/2} \| v \| | w |^{1/2} \| w \|^{1/2} . \tag{3.4}$$

We also assume that there exists $C_B > 0$ such that for any u,v in $D(A)$

$$| B(u,v) | \leq C_B | u |^{1/2} | Au |^{1/2} \| v \|, \tag{3.5}$$

$$| B(u,v) | \leq C_B | u |^{1/2} \| u \|^{1/2} \| v \|^{1/2} | Av |^{1/2}, \tag{3.6}$$

and that

$$| B(u,v) | \leq C_B \| u \| \| v \| (1 + \mathrm{Log}(\frac{| Au |^2}{\lambda_1 \| u \|^2}))^{1/2}, \tag{3.7}$$

where $\lambda_1 = (2\pi)^{2m}$ is the smallest eigenvalue of A.

Under these assumptions we recall without proofs the following well-known results:

Theorem 2 (well-posed problem)

There exists a unique solution of (3.1) and (3.2) belonging to

$$C(0,+\infty; H) \cap L^2(0,T; D(A^{1/2})),$$

$$\text{for each } T > 0.$$

Moreover if $u_0 \in V$ then u belongs to

$$C(0, +\infty; V) \cap L^2(0, T; D(A)),$$

$$\text{for each } T > 0.$$

Proof: See [T1] and the references therein.

Theorem 3 (Dissipativity. Absorbing sets)

Let us consider initial data u_0 in (3.2) satisfying

$$| u_0 | \leq R_0, \; \| u_0 \| \leq R_1.$$

Then there exists a time t_0 that depends on u_0 through R_0, and on the data $\lambda_1, | f |$ of the equation such that for $t \geq t_0$

$$| u(t) | \leq M_0; \| u(t) \| \leq M_1, \tag{3.8}$$

where M_0, M_1 are independent of u_0, but depend on the other data.

Proof: This is related to the existence of an absorbing set in H and V for the dynamical system (3.1); actually $t_0 = C_1 \log(R_0) + C_2$ is the entrance time in these absorbing sets; see Chapter III Section 2.2 in [T1].

Theorem 4 (Time analycity)

There exists a domain of $\mathcal{C}$ containing

$$\Delta(\| u_0 \|) = \{\zeta \in \mathcal{C}, Re\zeta > 0,$$

$$| Im\zeta | \leq T_0 \text{ if } Re\zeta \geq T_0$$

$$| Im\zeta | \leq Re\zeta \text{ if } Re\zeta \leq T_0\}$$

where u can be extended to an analytic map into $D(A)$; *here* T_0 *depends on* $\lambda_1, | f |, \| u_0 \|$.

Proof: See [FMT], [FT], [P].

Remark 4: We will use Theorem 4 in the following form:

For $t \geq t_0$ (t_0 *as in Theorem 3*), u *can be extended to an analytic map from*

$$\Delta = \{\zeta \in \mathcal{C}, Re\zeta > t_0,$$

$$| Im\zeta | \leq T_0 \;\; \text{if} \;\; Re\zeta \geq t_0 + T_0$$

$$| Im\zeta | \leq Re\zeta - t_0 \;\; \text{if} \;\; Re\zeta \leq t_0 + T_0\}$$

into $D(A)$, *with* T_0 *that depends on* $\lambda_1, | f |$ *but which is independent of* $\| u_0 \|$.

We also recall from the proof of Theorem 4 (see [P] for instance) that for $\zeta \in \Delta$,

$$\| u(\zeta) \| \leq 2(1 + M_1), \tag{3.9}$$

M_1 being as in Theorem 3.

3.2 Induced trajectories lying in the flat manifold

Let V_j (j being fixed in this section) be the flat manifold associated with the (linear) Galerkin approximation of (3.1); (3.2) by periodic piecewise polynomial functions.

Let P_j be the orthogonal projector in H onto V_j; let $Q_j = Id_H - P_j$. We also define $P_{1,j}$ (being provided $m \leq N + 1$ such that $V_j \subset V$ holds) as the orthogonal projector in V onto V_j, and $Q_{1,j} = Id_V - P_{1,j}$.

Remark 5: In the following we shall omit the subscript j on $P_j, O_j, P_{1,j}$ and $O_{1,j}$.

Following the methods developed in [T2] we call induced trajectories (lying in the flat manifold) associated with $u(t)$ solution of (3.1); (3.2) the trajectories $y(t), y_1(t)$ defined by:

$$y(t) = Pu(t)\,;\; y_1(t) = P_1u(t). \tag{3.10}$$

we also set

$$z(t) = Qu(t);\; z_1(t) = Q_1u(t). \tag{3.11}$$

Remark 6: Unlike the spectral case we no longer have $y = y_1$.

We recall from the results of Section 2

Proposition 4: *For $m < N+1$, there exists $c > 0$ such that for any z in QV*

$$| z | \leq \frac{c}{2^{jm}} \| z \| \,. \tag{3.12}$$

Proof: We write

$$z = \sum_{\ell=j}^{+\infty} \sum_{k=1}^{2^\ell} \gamma_{\ell,k}\psi_{\ell,k}. \tag{3.13}$$

Thanks to Corollary 5 there exist constants c_1, c_2 that depend only on m, N such that

$$c_1 \| z \|^2 \leq \sum_{\ell,k} 4^{\ell m} | \gamma_{\ell,k} |^2 \leq c_2 \| z \|^2 \,. \tag{3.14}$$

This yields

$$4^{jm} \sum_{\ell,k} | \gamma_{\ell,k} |^2 \leq c_2 \| z \|^2, \tag{3.15}$$

and the identity

$$| z |^2 = \sum_{\ell,k} | \gamma_{\ell,k} |^2 \tag{3.16}$$

provides the result.

We would like to check this kind of inequality for z_1 in Q_1V. For this purpose we need more regularity on the wavelets $\psi_{j,k}$'s :

Proposition 5: *For $2m < N+1$, there exists $c > 0$ such that for any z_1 in Q_1V*

$$| z_1 | \leq \frac{c}{2^{jm}} \| z_1 \| . \tag{3.17}$$

Proof: First we observe that

$$(z_1 \in Q_1V) \Leftrightarrow (Az_1 \in V' \text{ and } < Az_1, \psi_{\ell,k} >_{V',V} = 0 \text{ for } \ell < j)$$

Then if $m < N+1$, thanks to Theorem 1, we write

$$Az_1 = \sum_{\ell=j}^{+\infty} \sum_{k=1}^{2^\ell} \gamma_{\ell,k} \psi_{\ell,k}, \tag{3.18}$$

where

$$c_1(m) | Az_1 |^2_{-m} \leq \sum_{\ell=j}^{+\infty} \sum_{k=1}^{2^\ell} | \gamma_{\ell,k} |^2 \, 4^{-\ell m} \leq c_2(m) | Az_1 |^2_{-m}, \tag{3.19}$$

where $c_1(m), c_2(m)$ as in Corollary 5. We observe that $| Az_1 |^2_{-m} = \| z_1 \|^2$.

On the other hand we would like to estimate $| Az_1 |^2_{-2m} = | z_1 |^2$ with respect to the $\gamma_{\ell,k}$ *'s*. If $2m < N+1$, the sum in (3.18) is unconditionally convergent in $\dot{H}^{-2m}$; then applying Corollary 5 we obtain

$$c_1(2m) | Az_1 |^2_{-2m} \leq \sum_{\ell=j}^{+\infty} \sum_{k=1}^{2^\ell} | \gamma_{\ell,k} |^2 \, 16^{-\ell m}$$

$$\leq c_2(2m) | Az_1 |^2_{-2m}, \tag{3.20}$$

and therefore we infer the result from (3.19) and (3.20).

3.3. Behavior of small eddies

Theorem 5: *For $2m < N+1$, both $z(t), z_1(t)$ satisfy, for t large enough as in Theorem 3, the following inequalities:*

$$| z(t) |, | z_1(t) |, | z'(t) |, | z_1'(t) | \leq c \frac{(jm)^{1/2}}{4^{jm}} \tag{3.21}$$

$$\| z(t) \|, \| z_1(t) \| \leq c\frac{(jm)^{1/2}}{2^{jm}} \tag{3.22}$$

We reinterpret these inequalities saying that the flat manifold V_j associated to splines of order N is an approximate inertial manifold of order $\frac{(jm)^{1/2}}{4^{jm}}$ in H and of order $\frac{(jm)^{1/2}}{2^{jm}}$ in V.

Remark 7: Here we set $z' = \frac{dz}{dt}, z'_1 = \frac{dz_1}{dt}$.

Remark 8: We match here the accuracy established (see [FMT], [T2]) in the spectral case in the following sense: Let $\tilde{P}_j$ be the orthogonal projector onto the space spanned by the first 2^j eigenvectors of $A; \tilde{Q}_j = Id - \tilde{P}_j$; then we have, for t large enough

$$| \tilde{Q}_j u | \leq c\delta L^{1/2},$$

$$\| \tilde{Q}_j u \| \leq c\delta^{1/2} L^{1/2},$$

where $\delta = \frac{c}{4^{jm}}, L$ being a logarithmic correction of δ.

Proof of Theorem 5:

We take the scalar product in H of (3.1) with z_1; using (3.3) and

$$(Au, z_1) = ((u, z_1)) = \| z_1 \|^2 \tag{3.23}$$

we obtain

$$\| z_1 \|^2 = (f, z_1) - (\frac{du}{dt}, z_1) - b(z_1, y_1, z_1) - b(y_1, y_1, z_1). \tag{3.24}$$

Then (3.7) yields

$$| b(y_1, y_1, z_1) | \leq C_B \| y_1 \|^2 (1 + \text{Log}(\frac{| Ay_1 |^2}{\lambda_1 \| y_1 \|^2}))^{1/2} | z_1 | . \tag{3.25}$$

Thanks to (2.1)

$$(1 + \text{Log}(\frac{| Ay_1 |^2}{\lambda_1 \| y_1 \|^2}))^{1/2} \leq c(jm)^{1/2}. \tag{3.26}$$

Hence

$$| b(y_1, y_1, z_1) | \leq C(jm)^{1/2} \| y_1 \|^2 | z_1 | . \tag{3.27}$$

On the other hand we use (3.4) to obtain

$$| b(z_1, y_1, z_1) | \leq C \| y_1 \| \| z_1 \| | z_1 | . \tag{3.28}$$

Then we infer from (3.24), (3.27) and (3.28)

$$\| z_1 \|^2 \leq (| f | + | \frac{du}{dt} | + C(jm)^{1/2} \| y_1 \|^2 + C \| y_1 \| \| z_1 \|) | z_1 | . \tag{3.29}$$

We observe that for t large enough as in Theorem 3 we have both

$$\| y_1 \|, \| z_1 \| \leq \| u \| \leq M_1, \tag{3.30}$$

and that for t large enough as in Theorem 4

$$| \frac{du}{dt} | \leq C, \tag{3.31}$$

that is a consequence of the Cauchy formula applied to u in a ball included in Δ, and of (3.9). We use these facts to obtain

$$\| z_1 \|^2 \leq C(jm)^{1/2} | z_1 |, \tag{3.32}$$

holding for t large enough as above, where C depends on N and on the data $\lambda_1, | f |$ of the equation, but is independent of j. Then Proposition 5 yields

$$\| z_1 \| \leq C \frac{(jm)^{1/2}}{2^{jm}}, \tag{3.33}$$

$$| z_1 | \leq C \frac{(jm)^{1/2}}{4^{jm}}. \tag{3.34}$$

Now we estimate $| z |$ and $\| z \|$; we observe that $z = Qz_1$ and that therefore

$$| z | \leq | z_1 | \leq C \frac{(jm)^{1/2}}{4^{jm}} \tag{3.35}$$

holds; but moreover we have

Proposition 6: *For $m < N+1, Q$, which is an orthogonal projector in H, maps continuously V into itself, and its norm as a linear operator acting in V is bounded independently of j.*

Proof: We write for $v \in V$

$$v = \sum_{\ell=0}^{+\infty} \sum_{k=0}^{2^\ell} \gamma_{\ell,k} \psi_{\ell,k}, \tag{3.36}$$

$$Qv = \sum_{\ell=j}^{+\infty} \sum_{k=0}^{2^\ell} \gamma_{\ell,k} \psi_{\ell,k}. \tag{3.37}$$

Thanks to Theorem 1, for $m < N+1$ we have

$$\begin{aligned} \| Qv \| &\leq \frac{1}{c_1(m)^{1/2}} \sum_{\ell=j}^{+\infty} \sum_{k=0}^{2^\ell} 4^{\ell m} \mid \gamma_{\ell,k} \mid^2 \\ &\leq \frac{1}{c_1(m)^{1/2}} \sum_{\ell=0}^{+\infty} \sum_{k=0}^{2^\ell} 4^{\ell m} \mid \gamma_{\ell,k} \mid^2 \\ &\leq \left(\frac{c_2(m)}{c_1(m)}\right)^{1/2} \| v \|, \end{aligned} \tag{3.38}$$

where $c_1(m), c_2(m)$ are as in Corollary 5.

We apply this result to $z = Qz_1$ to obtain

$$\| z \| \leq C \frac{(jm)^{1/2}}{2^{jm}}. \tag{3.39}$$

To end the proof of Theorem 5, it remains to estimate $\mid z' \mid$ and $\mid z_1' \mid$. For this purpose we observe that z and z_1 are analytic in time in the same domain as u, and then we use Cauchy's formula to get the estimates on z' and z_1' from these on z and z_1. For reader's convenience we give a complete proof below.

First we need to introduce some notations. Let H_C, V_C, V_C^j and $D(A^s)_C$ be the complexifications of H, V, V_j and $D(A^s)$. We recall that if $u_1 + iu_2$ is a typical element of H_C, then we have

$$A(u_1 + iu_2) = Au_1 + iAu_2,$$

$$(u_1 + iu_2, v_1 + iv_2) = (u_1, v_1) + (u_2, v_2)$$

$$+ i[(u_2, v_1) - (u_1, v_2)],$$

and that the multiplication by a complex constant is performed in the natural manner.

We observe that the family $\{\psi_{\ell,k}\}_{0\leq\ell<j;1\leq k\leq 2^j}$ is an orthonormal basis of V_C^j in H_C and that moreover we have

Lemma 4:

$$\textit{The family } \{\psi_{\ell,k}\}_{0\leq\ell\leq+\infty;1\leq k\leq 2^\ell} \textit{ is an orthonormal basis of } H_C, \tag{3.40}$$

and for $m \mid s \mid < N + 1$ *the family*

$$\{\psi_{\ell,k}\}_{0\leq\ell<j;1\leq k\leq 2^\ell} \textit{ is an unconditional basis of } D(A^{s/2})_C \tag{3.41}$$

Proof: The proof is straightforward and is left as an exercise to the reader.

Now we observe that y_1 and z_1 can be extended as time analytic functions in the same domain as u. Let Y_1, Z_1, and U be the extensions of y_1, z_1 and u. Then U satisfies, for $\zeta \in \Delta(\| u_0 \|)$

$$\frac{\partial U}{\partial \zeta} + AU + B(U) = f. \tag{3.42}$$

Taking the scalar product in H_C of (3.42) with Z_1 we obtain

$$\| Z_1 \|^2 \leq \mid \frac{\partial U}{\partial \zeta} \mid\mid Z_1 \mid + \mid b(Y_1, Y_1, Z_1) \mid$$

$$+ \mid b(Y_1, Z_1, Z_1) \mid + \mid b(Z_1, Y_1, Z_1) \mid$$

$$+ \mid b(Z_1, Z_1, Z_1) \mid + \mid f \mid\mid Z_1 \mid . \tag{3.43}$$

Easy computations yield

$$\mid b(Z_1, Z_1, Z_1) \mid \leq \mid Z_1 \mid\mid\mid Z_1 \mid\mid^2, \tag{3.44}$$

$$\mid b(Z_1, Y_1, Z_1) \mid \leq C \mid Z_1 \mid\mid\mid Z_1 \mid\mid\mid\mid Y_1 \mid\mid, \tag{3.45}$$

where C is an absolute constant.

Using (2.1) on $Re(Y_1)$ and $Im(Y_1)$ we obtain

$$\mid b(Y_1, Y_1, Z_1) \mid \leq C \mid\mid Y_1 \mid\mid^2 (jm)^{1/2} \mid Z_1 \mid, \tag{3.46}$$

$$\mid b(Y_1, Z_1, Z_1) \mid \leq C \mid\mid Y_1 \mid\mid\mid\mid Z_1 \mid\mid (jm)^{1/2} \mid Z_1 \mid, \tag{3.47}$$

where C depends on N.

For ζ such that $\mid Im\zeta \mid \leq \frac{T_0}{2}$ and $Re\zeta \geq t_0 + 2T_0$ (with t_0 and T_0 as in Remark 4), we apply Cauchy's formula on a ball B centered at ζ, of radius $\frac{T_0}{2}$, to obtain

$$\mid \frac{\partial U}{\partial \zeta}(\zeta) \mid \leq C \sup_{\eta \epsilon B} \mid U(\eta) \mid, \tag{3.48}$$

and we infer from (3.9)

$$\mid \frac{\partial U}{\partial \zeta} \mid \leq C, \tag{3.49}$$

where C depends on the data of the equation through M_1.

We also use (3.9) to majorize $\mid\mid Y_1 \mid\mid$ and $\mid\mid Z_1 \mid\mid$ by $2(1 + M_1)$, for ζ as above.

All these facts yield

$$\mid\mid Z_1 \mid\mid^2 \leq C(jm)^{1/2} \mid Z_1 \mid . \tag{3.50}$$

On the other hand we observe that Lemma 4 provides analogous forms of Poincaré inequalities (3.12) and (3.17) for Z and Z_1. We also infer from Lemma 4 that the orthogonal projector in H_C onto the orthogonal complement of V_C^j is continuous as a linear operator mapping V_C into itself, the corresponding norm being bounded independently of j.

We apply these remarks to (3.50) to obtain, by the same computations as above,

$$| Z |, | Z_1 | \leq C \frac{(jm)^{1/2}}{4^{jm}}, \tag{3.51}$$

$$\| Z \|, \| Z_1 \| \leq C \frac{(jm)^{1/2}}{2^{jm}}, \tag{3.52}$$

for ζ belonging to a thinner strip than Δ, for example

$$\{Re\zeta \geq t_0 + 2T_0$$

$$| Im\zeta | \leq \frac{T_0}{2}.$$

To end the proof of Theorem 5, we apply Cauchy's formula on a ball B centered at $t \geq t_0 + 3T_0$, of radius $\frac{T_0}{2}$ to obtain

$$| \frac{dz}{dt}(t) | \leq C \sup_{\eta \epsilon B} | Z(\eta) |, \tag{3.53}$$

where C depends on the data of the equation through M_1. We then infer from (3.51) and (3.53)

$$| \frac{dz}{dt} | \leq C \frac{(jm)^{1/2}}{4^{jm}}. \tag{3.54}$$

An analogous result for z_1' concludes the proof.

3.4. Two nonflat approximate inertial manifolds

Following the methods developed in [T2] we provide below two examples of (nonflat) approximate inertial manifolds of higher order than the flat one.

First let us consider the (nonlinear) mapping $\Phi_1 : PV \to QV$ defined as follows: For each y in PV, there exists a unique $\Phi_1(y)$ in QV such that

$$((\Phi_1(y), \tilde{z})) = (f - Ay - B(y), \tilde{z}), \tag{3.55}$$

for any $\tilde{z}$ in QV. Φ_1 is well defined, thanks to a straightforward consequence of the Riesz representation theorem.

Remark 9: Let us notice that the term $(Ay, \tilde{z})$ does not vanish. Actually, unlike the spectral case y and $\tilde{z}$ are orthogonal in H, not in V. This point can also be observed in the nonlinear algorithms described in [MT].

Let $\mathcal{M}_1$ be the graph of Φ_1; then we have

Proposition 7:

$\mathcal{M}_1$ is an approximate inertial manifold for the equation (3.1) of order $\frac{(jm)}{8^{jm}}$ in H and of order $\frac{(jm)}{4^{jm}}$ in V.

Remark 10: We match here the accuracy established in the spectral case (see [FMT], [T2]) in the sense of Remark 8.

Proof: We plan to estimate the gap between the trajectory $u(t)$ and its induced trajectory lying in $\mathcal{M}_1$, $y(t) + \Phi_1(y(t))$, where $y(t) = Pu(t)$ as above. We set

$$\chi_1(t) = \Phi_1(y(t)) - z(t). \tag{3.56}$$

We rewrite (3.55) as

$$QA\Phi_1(y) + QAy + QB(y) = Qf. \tag{3.57}$$

Hence χ_1 satisfies

$$QA\chi_1 + Q(B(y) - B(u)) = \frac{dz}{dt}. \tag{3.58}$$

We take the scalar product in H of (3.58) with χ_1 to obtain

$$\| \chi_1 \|^2 \leq | b(y, z, \chi_1) | + | b(z, y, \chi_1) |$$

$$+ | b(z, z, \chi_1) | + | \frac{dz}{dt} || \chi_1 | . \tag{3.59}$$

On the other hand (3.7) yields

$$| b(y,z,\chi_1) | \leq C_B \| y \| \| z \| (1+\mathrm{Log}(\frac{| Ay |^2}{\lambda_1 \| y \|^2}))^{1/2} | \chi_1 | . \tag{3.60}$$

We infer from (3.8) and (3.38)

$$\| y \| \leq C \| u \| \leq C M_1, \tag{3.61}$$

and from (3.22)

$$\| z \| \leq C \frac{(jm)^{1/2}}{2^{jm}},$$

these estimates holding for t large enough as in Theorem 5.

We apply (3.12) to obtain

$$| \chi_1 | \leq \frac{C}{2^{jm}} \| \chi_1 \|, \tag{3.62}$$

and (2.1) to have

$$(1+\mathrm{Log}(\frac{| Ay |^2}{\lambda_1 \| y \|^2}))^{1/2} \leq C(jm)^{1/2}. \tag{3.63}$$

Then we finally obtain

$$| b(y,z,\chi_1) | \leq C \frac{jm}{4^{jm}} \| \chi_1 \| . \tag{3.64}$$

We also have, using (3.4)

$$| b(z,y,\chi_1) | \leq C_b | z |^{1/2} \| z \|^{1/2} \| y \| | \chi_1 |^{1/2} \| \chi_1 \|^{1/2} . \tag{3.65}$$

We then infer from (3.21), (3.22), (3.61), (3.62) and (3.65)

$$| b(z,y,\chi_1) | \leq C \frac{(jm)^{1/2}}{4^{jm}} \| \chi_1 \|, \tag{3.66}$$

For t large enough as above.

Using (3.3) and (3.4) we obtain as well

$$| b(z,z,\chi_1) | \leq C_b | z | \| z \| \| \chi_1 \|, \tag{3.67}$$

and thanks to (3.21), (3.22)

$$| b(z,z,\chi_1) | \leq C\frac{jm}{8^{jm}} \| \chi_1 \| . \tag{3.68}$$

Using (3.21) to estimate $| \frac{dz}{dt} |$ we finally obtain

$$\| \chi_1 \|^2 \leq [C_1\frac{(jm)}{4^{jm}} + C_2\frac{(jm)^{1/2}}{4^{jm}} \tag{3.69}$$

$$+C_3\frac{(jm)}{8^{jm}} + C_4\frac{(jm)^{1/2}}{8^{jm}}] \| \chi_1 \|$$

This yield

$$\| \chi_1 \| \leq C\frac{(jm)}{4^{jm}}, \tag{3.70}$$

and thanks to (3.62),

$$| \chi_1 | \leq C\frac{(jm)}{8^{jm}}, \tag{3.71}$$

holding for t large enough as above.

Now we define $\mathcal{M}_2$ as the graph of the mapping $\Phi_2 : PV \rightarrow QV$ defined by induction from Φ_1 by:

$$((\Phi_2(y), \tilde{z})) \tag{3.72}$$

$= (QA\Phi_1(y) - QB(y, \Phi_1(y)) - QB(\Phi_1(y), y), \tilde{z})$, for any $\tilde{z}$ in QV.

Existence and uniqueness of $\Phi_2(y)$ are consequences of the Riesz representation theorem. We have

Proposition 8:

$\mathcal{M}_2$ is an approximate inertial manifold for the equation (3.1) of order $\frac{(jm)^{3/2}}{16^{jm}}$ in H and of order $\frac{(jm)^{3/2}}{8^{jm}}$ in V.

Remark 11: We match here the accuracy established in the spectral case (see [T2]) in the sense of Remark 8.

Proof: Setting

$$\chi_2 = \Phi_2(y) - z, \tag{3.73}$$

and using (3.57) we rewrite (3.72) as

$$QA\Phi_2(y) + QAy + QB(y) \tag{3.74}$$

$$+QB(y, \Phi_1(y)) + QB(\Phi_1(y), y) = Qf.$$

Hence

$$QA\chi_2 + QB(y, \chi_1) + QB(\chi_1, y)$$

$$= \frac{dz}{dt} + QB(z), \tag{3.75}$$

where χ_1 is defined as above. We take the scalar product in H of (3.75) with χ_2 to obtain

$$\| \chi_2 \|^2 \leq | b(y, \chi_1, \chi_2) | + | b(\chi_1, y, \chi_2) |$$

$$+ | b(z, z, \chi_2) | + | \frac{dz}{dt} || \chi_2 | . \tag{3.76}$$

As usual, thanks to (3.7)

$$| b(y, \chi_1, \chi_2) |$$

$$\leq C_B \| y \| \| \chi_1 \| (1 + \text{ Log } (\frac{| Ay |^2}{\lambda_1 \| y \|^2}))^{1/2} | \chi_2 | . \tag{3.77}$$

From previous estimates (3.61), (3.63) and (3.70) we obtain, for t large enough as above

$$| b(y, \chi_1, \chi_2) | \leq C \frac{(jm)^{3/2}}{4^{jm}} | \chi_2 |, \tag{3.78}$$

and thanks to (3.12)

$$| b(y,\chi_1,\chi_2) | \leq C\frac{(jm)^{3/2}}{8^{jm}} \| \chi_2 \| . \tag{3.79}$$

On the other hand, using (3.4)

$$| b(\chi_1,y,\chi_2) | \leq C_b | \chi_1 |^{1/2} \| \chi_1 \|^{1/2} \| y \| | \chi_2 |^{1/2} \| \chi_2 \|^{1/2} . \tag{3.80}$$

We infer from (3.12), (3.61), (3.70) and (3.71)

$$\begin{aligned} | b(\chi_1,y,\chi_2) | &\leq C\frac{(jm)}{(4\sqrt{2})^{jm}} \; \frac{1}{(\sqrt{2})^{jm}} \| \chi_2 \| \\ &\leq C\frac{jm}{8^{jm}} \| \chi_2 \| . \end{aligned} \tag{3.81}$$

By similar computations, using (3.3), (3.4), (3.21) and (3.22)

$$\begin{aligned} | b(z,z,\chi_2) | &\leq C_b | z | \| z \| \| \chi_2 \| \\ &\leq C\frac{jm}{8^{jm}} \| \chi_2 \| \end{aligned} \tag{3.82}$$

We have as well, using (3.12) and (3.21)

$$| \frac{dz}{dt} \| \chi_2 | \leq C\frac{(jm)^{1/2}}{8^{jm}} \| \chi_2 \| . \tag{3.83}$$

We summarize (3.76), (3.79), (3.81), (3.82) and (3.83) by

$$\| \chi_2 \|^2 \leq C\frac{(jm)^{3/2}}{8^{jm}} \| \chi_2 \|, \tag{3.84}$$

holding for t large enough as above. Poincaré inequality (3.12) ends the proof.

APPENDIX

In this appendix we want first to present a proof of Theorem 1 for the multidimensional periodic wavelet bases built from the one-dimensional ones by tensor products. For the sake of simplicity we present this result for the two-dimensional case. The reader could check that it can be extended without difficulties to the d-dimensional case, $d > 2$.

After that we will end the paper explaining in few words how to prove Theorem 1 considering two other important examples of wavelet bases.

In both cases we just have to prove analogous results to Proposition 1 and Proposition 2.

We recall from [L], [M] that, for each $N > 0$, there exists a function φ_N satisfying (i), (ii), such that

$$\int_{\mathcal{R}} \varphi_N(x)dx \neq 0, \qquad (A.1)$$

and that, setting

$$\varphi_{j,k}(x) = 2^{j/2} \sum_{\ell \in \mathcal{Z}} \varphi_N(2^j x + 2^j \ell - k), \qquad (A.2)$$

the family

$$\{\varphi_{j,k}\}_{1 \leq k \leq 2^j} \text{ is an orthonormal basis of } V_j. \qquad (A.3)$$

(We dropped for convenience the subscript N on the $\varphi_{j,k}$'s.)

Let us introduce some notations. $H^s(\Pi^2)$ will be the usual periodic Sobolev space on the two-dimensional torus. $\dot{H}^s(\Pi^2)$ will be the set of functions u in $H^s(\Pi^2)$ such that

$$\iint_{\Pi^2} u(x)dx_1dx_2 = 0.$$

$\dot{H}^s(\Pi^2)$ is a Hilbert space when endowed with the scalar product

$$((u,v))_s = \sum_{k \in \mathcal{Z}^2} \mid k \mid^{2s} \hat{u}(k)\overline{\hat{v}(k)},$$

where

$$| k | = (k.k)^{1/2}, k.\ell = k_1\ell_1 + k_2\ell_2,$$

$$\hat{u}(k) = \iint_{\Pi^2} u(x)e^{-2i\pi k.x}dx.$$

We denote the corresponding norm

$$\| u \|_s = ((u,u))_s^{1/2}.$$

Then let $\mathcal{V}_j$ be the space of functions $v \in L^2(\Pi^2) = H^0(\Pi^2)$ such that both

$$x_1 \longmapsto v(x_1,x_2), x_2 \longmapsto v(x_1,x_2), \text{ belong to } V_j$$

(i.e. $\mathcal{V}_j = V_j \otimes V_j$). For $\alpha = (\alpha_1, \alpha_2)$ in $\mathcal{A}_j = \{1, \dots, 2^j\}^2$, we set

$$\varphi_\alpha(x) = \varphi_{j,\alpha_1}(x_1)\varphi_{j,\alpha_2}(x_2). \quad (A.4)$$

Then we observe that the family $\{\varphi_\alpha\}_{\alpha\epsilon\mathcal{A}_j}$ is an orthonormal basis of $\mathcal{V}_j$.

Now we are ready to claim

Proposition A.1 *There exists $C > 0$ such that for any v in $\mathcal{V}_j$*

$$\| v \|_{N+1} \leq C2^{j(N+1)} \| v \|_0 . \quad (A.5)$$

Remark A.1: As above C is a constant that depends only on N.

Proof: We use the convexity of the function $\lambda \longmapsto \lambda^{N+1}$ to write

$$\begin{aligned} \| u \|_{N+1}^2 &= \sum_{k\epsilon\mathcal{Z}^2} (k_1^2 + k_1^2)^{N+1} | \hat{u}(k) |^2 \\ &\leq 2^N \sum_{k\epsilon\mathcal{Z}^2} (k_1^{2N+2} + k_2^{2N+2}) | \hat{u}(k) |^2 \\ &\leq 2^N [\| (\partial_1)^{N+1}u \|_0^2 + \| (\partial_2)^{N+1}u \|_0^2], \end{aligned} \quad (A.6)$$

setting $(\partial_i)^{N+1}u = \frac{\partial^{N+1}u}{\partial x_i^{N+1}}$; $i = 1,2$.

On the other hand we write for

$$v = \sum_{\alpha \epsilon \mathcal{A}_j} \lambda_\alpha \varphi_\alpha \text{ in } \mathcal{V}_j \tag{A.7}$$

$$\| (\partial_1)^{N+1} v \|_0^2 = \sum_{\alpha,\alpha' \in \mathcal{A}_j} \lambda_\alpha \lambda_{\alpha'} (((\partial_1)^{N+1}\varphi_\alpha, (\partial_1)^{N+1}\varphi_{\alpha'}))_0. \tag{A.8}$$

Thanks to Fubini's theorem

$$(((\partial_1)^{N+1}\varphi_\alpha, (\partial_1)^{N+1}\varphi_{\alpha'}))_0 =$$

$$\left(\int_\Pi \varphi_{\alpha_1}^{(N+1)}(x_1)\varphi_{\alpha_1'}^{(N+1)}(x_1)dx_1\right)\left(\int_\Pi \varphi_{\alpha_2}(x_2)\varphi_{\alpha_2'}(x_2)dx_2\right), \tag{A.9}$$

where we set $\varphi_{\alpha_1}^{(N+1)}(x_1) = \frac{\partial^{N+1}}{\partial x_1^{N+1}}\varphi_{\alpha_1}(x_1)$, and where we dropped the subscript j on the φ_{j,α_i}'s to write φ_{α_i}; $i = 1,2$.

Using $\int_\Pi \varphi_{\alpha_2}(x_2)\varphi_{\alpha_2'}(x_2)dx_2 = 0$ if $\alpha_2 \neq \alpha_2'$ (cf (A.3)), we obtain

$$\| (\partial_1)^{N+1} v \|_0^2 = \sum_{\alpha_2=1}^{2^j} \int_\Pi \left(\sum_{\alpha_1=1}^{2^j} \lambda_\alpha \varphi_{\alpha_1}^{(N+1)}(x_1)\right)^2 dx_1. \tag{A.10}$$

We then apply (2.1) to the function $x_1 \longmapsto \sum_{\alpha_1=1}^{2^j} \lambda_\alpha \varphi_{\alpha_1}^{(N+1)}(x_1)$

$$\int_\Pi \left(\sum_{\alpha_1=1}^{2^j} \lambda_\alpha \varphi_{\alpha_1}^{(N+1)}(x_1)\right)^2 dx_1 \leq C 4^{j(N+1)} \int_\Pi \left(\sum_{\alpha_1=1}^{2^j} \lambda_\alpha \varphi_{\alpha_1}(x_1)\right)^2 dx_1. \tag{A.11}$$

(A.3) yields

$$\int_\Pi \left(\sum_{\alpha_1=1}^{2^j} \lambda_\alpha \varphi_{\alpha_1}(x_1)\right)^2 dx_1 = \sum_{\alpha_1=1}^{2^j} \lambda_\alpha^2 , \tag{A.12}$$

therefore

$$\| (\partial_1)^{N+1} v \|_0^2 \leq C 4^{j(N+1)} \left(\sum_{\alpha \in \mathcal{A}_j} \lambda_\alpha^2\right). \tag{A.13}$$

To conclude we recall that the family $\{\varphi_\alpha\}_{\alpha \in \mathcal{A}_j}$ is an orthonormal basis of $\mathcal{V}_j$, then using (A.6), (A.13), and that (A.13) holds for $\| (\partial_2)^{N+1} v \|_0^2$ as well, we obtain $\| v \|_{N+1}^2 \leq C 4^{j(N+1)} \| v \|_0^2$.

Now we set

$$\mathcal{W}_j = \mathcal{V}_{j+1} \cap (\mathcal{V}_j)^{\perp}. \tag{A.14}$$

$\mathcal{W}_j$ can be viewed as the direct sum of three of its subspaces, namely

$$V_j \otimes W_j; \quad W_j \otimes V_j; \quad W_j \otimes W_j.$$

We observe that the family $\{\varphi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\}_{\alpha\in\mathcal{A}_j}$ is an orthonormal basis of

$$V_j \otimes W_j. \tag{A.15}$$

(we also dropped the subscript j on ψ_{j,α_2} to write ψ_{α_2}).

We claim

Proposition A.2: *There exists $C > 0$ such that for any w in $\mathcal{W}_j$*

$$\| w \|_{-N-1} \leq C 2^{-j(N+1)} \| w \|_0 . \tag{A.16}$$

Proof: For w in $\mathcal{W}_j$ we write $w = w_1 + w_2 + w_3$ where $w_1 \in V_j \otimes W_j; w_2 \in W_j \otimes V_j; w_3 \in W_j \otimes W_j$. We observe that

$$\| w \|_0^2 = \| w_1 \|_0^2 + \| w_2 \|_0^2 + \| w_3 \|_0^2, \tag{A.17}$$

$$\| w \|_{-N-1}^2 \leq 3(\| w_1 \|_{-N-1}^2 + \| w_2 \|_{-N-1}^2 + \| w_3 \|_{-w-1}^2). \tag{A.18}$$

It follows that it is sufficient to check (A.16) on both w_1, w_2, w_3. Because the proofs are similar we present below only the proof for w_1.

Let

$$w_1 = \sum_{\alpha\in\mathcal{A}_j} \lambda_\alpha \varphi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2) \tag{A.19}$$

Using (1.7), (2.17) and (A.2) easy computations yield

$$\hat{w}_1(\ell) = (\frac{1}{2^j} \sum_{\alpha\in\mathcal{A}_j} \lambda_\alpha e^{\frac{-2i\pi\alpha.\ell}{2^j}})\hat{\varphi}(\frac{\ell_1}{2^j})\hat{\psi}(\frac{\ell_2}{2^j}). \tag{A.20}$$

Hence

$$\| w_1 \|_0^2 = \sum_{\ell \in \mathcal{Z}^2} | m(\frac{\ell}{2^j}) |^2 | \hat{\varphi}(\frac{\ell_1}{2^j}) |^2 | \hat{\psi}(\frac{\ell_2}{2^j}) |^2, \tag{A.21}$$

setting

$$m(\frac{\ell}{2^j}) = \frac{1}{2^j} \sum_{\alpha \in \mathcal{A}_j} \lambda_\alpha e^{-2i\pi \frac{\ell.\alpha}{2^j}}. \tag{A.22}$$

m is a $\mathcal{Z}^2$-periodic function. Using this fact we obtain

$$\| w_1 \|_0^2 = \sum_{k \in \Gamma} | m(\frac{k}{2^j}) |^2 \left(\sum_{\ell_1 \in \mathcal{Z}} | \hat{\varphi}(\frac{k_1}{2^j} + \ell_1) |^2\right)\left(\sum_{\ell_2 \in \mathcal{Z}} | \hat{\psi}(\frac{k_2}{2^j} + \ell_2) |^2\right), \tag{A.23}$$

where $\Gamma = \{k \in \mathcal{Z}^2 / 1 - 2^{j-1} \leq k_i \leq 2^{j-1}, i = 1, 2\}$.

Thanks to lemma 3

$$\| w_1 \|_0^2 = \sum_{k \in \Gamma} | m(\frac{k}{2^j}) |^2 . \tag{A.24}$$

On the other hand we write

$$4^{j(N+1)} \| w_1 \|_{-N-1}^2 = \sum_{\ell \in \mathcal{Z}^2_*} | \hat{w}_1(\ell) |^2 | \frac{\ell}{2^j} |^{-2N-2} . \tag{A.25}$$

By the same computations as above

$$4^{j(N+1)} \| w_1 \|_{-N-1}^2$$

$$= \sum_{k \in \Gamma} | m(\frac{k}{2^j}) |^2 \left(\sum_{\ell \in \mathcal{Z}^2} | \frac{k}{2^j} + \ell |^{-2N-2} | \hat{\varphi}(\frac{k_1}{2^j} + \ell_1) |^2 | \hat{\psi}(\frac{k_2}{2^j} + \ell_2) |^2\right). \tag{A.26}$$

we infer from (A.24) and (A.26) that to prove (A.16) for w_1 it is sufficient to majorize the function

$$[-1/2, 1/2]^2 \to \mathcal{R}_+$$

$$z \longmapsto \sum_{\ell \in \mathcal{Z}^2} | z + \ell |^{-2N-2} | \hat{\varphi}(z_1 + \ell_1) |^2 | \hat{\psi}(z_2 + \ell_2) |^2 .$$

observing that $| z + \ell |^{-2N-2} \leq 4^{N+1}$ for z in $[-1/2, 1/2]^2$ and $\ell \neq 0$ we obtain

$$\sum_{\ell \neq 0} | z + \ell |^{-2N-2} | \hat{\varphi}(z_1 + \ell_1) |^2 | \hat{\psi}(z_2 + \ell_2) |^2$$

$$\leq 4^{N+1}(\sum_{\ell_1 \in \mathcal{Z}} \mid \hat{\varphi}(z_1+\ell_1) \mid^2)(\sum_{\ell_2 \in \mathcal{Z}} \mid \hat{\psi}(z_2+\ell_2) \mid^2). \qquad (A.27)$$

We then apply Lemma 3 to obtain

$$\sum_{\ell \neq 0} \mid z+\ell \mid^{-2N-2} \mid \hat{\varphi}(z_1+\ell_1) \mid^2 \mid \hat{\psi}(z_2+\ell_2) \mid^2 \leq 4^{N+1} \qquad (A.28)$$

We have now to majorize

$$\mid z \mid^{-2N-2} \mid \hat{\varphi}(z_1) \mid^2 \mid \hat{\psi}(z_2) \mid^2$$

we infer from (1.5) and (A.1) that

$$\mid \hat{\varphi}(z_1) \mid^2 = 0(1),$$

$$\mid \hat{\psi}(z_2) \mid^2 = 0(\mid z_2 \mid^{2N+2}),$$

when$\mid z \mid \rightarrow 0$. This fact ends the proof.

Let us recall some results about the Daubechies' compactly supported wavelets. (See [D], [M]).

$\forall n \geq 1$, there exists a couple of functions ψ_n, φ_n such that:

$$\cdot \psi_n, \varphi_n \in C^n(\mathcal{R}) \qquad (A.29)$$

$$\cdot \int x^m \psi_n(x)dx = 0 \text{ if } m \leq n \qquad (A.30)$$

$$\cdot \psi_n, \varphi_n \text{ are compactly supported.} \qquad (A.31)$$

(Actually there exist two constant $c_1, c_2 > 0$ such that the width of their support belongs to $[c_1 n, c_2 n]$).

$\cdot$ The family $\{2^{j/2}\psi_n(2^j x - k)\}_{j,k \in \mathcal{Z}}$ is an orthonormal basis of $L^2(\mathcal{R})$. $\qquad (A.32)$

· If we denote by $\tilde{W}_j$ the space spanned by the functions $\psi_n(2^j x - k); k \in \mathcal{Z}$,

then the family $\{2^{j/2}\varphi_n(2^j x - k)\}_{k \in \mathcal{Z}}$ is an orthonormal basis of $\bigoplus_{\ell < j} \tilde{W}_\ell$. $\quad (A.33)$

Now we are able to define the periodic Daubechies wavelet bases and to prove Theorem 1 in this case, being provided n is large enough with respect to s. Actually we just have to replace b_N by φ_n in the proof in Proposition 1 and ψ_N by ψ_n in the proof of Proposition 2. The multidimensional results follow.

We recall now what is the Littlewood-Paley wavelet basis (see [LM], [M]). There exists a couple of functions φ, ψ belonging to the Schwartz class $\mathcal{S}(\mathcal{R})$ satisfying respectively (A.1), (A.30) for each integer m, (A.32) and (A.33). Moreover $\hat{\varphi}$ and $\hat{\psi}$ are compactly supported. Then, with the same notations as above, for w in W_j

$$\hat{w}(\ell) = m(\ell/2^j)\hat{\psi}(\ell/2^j), \tag{A.34}$$

where m is a one-periodic function. This yields, for each s in $\mathcal{R}$

$$\begin{aligned} | w |_s^2 &= \sum_{\ell \in \mathcal{Z}} | \hat{w}(\ell) |^2 | \ell |^{2s} \\ &= \sum_{a2^j \leq \ell \leq b2^j} | \hat{w}(\ell) |^2 | \ell |^{2s}, \end{aligned} \tag{A.35}$$

where a and b are independent of j. This fact yields to Proposition 1 and Proposition 2. We then deduce that the periodic Littlewood-Paley wavelets provide an unconditional basis for all Sobolev spaces $\dot{H}^s(\Pi)$. The multidimensional results follow.

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Nonlinear Galerkin Methods Using Hierarchical Almost-Orthogonal Finite Elements Bases

Olivier GOUBET
Laboratoire d'Analyse Numérique, Université Paris-Sud
Bâtiment 425, 91405 Orsay, France

1. Introduction

New methods have recently emerged in Numerical Analysis in view of a better understanding of nontrivial dynamical situations arising in large time integration of dissipative evolution equations. Actually the set that describes the permanent dynamics of the equation –the attractor– can have a complex structure and even be a fractal (see [T1]). A dynamical approach for the numerical discretization of such problems led to the development of the so-called *Nonlinear Galerkin Methods.*

Their history can be overviewed as follows. A first step was the theory of inertial manifolds (I.M.). The idea was to imbed the attractor into a smooth finite dimensional manifold and to reduce the dynamics to this manifold (see [FST], [T1] and the references therein).

Next came the approximate inertial manifolds (A.I.M.). These manifolds capture all the orbits after a transient time in a small (thin) neighborhood. Hence they represent the large time behavior of the system up to a certain level of accuracy : the order of the A.I.M., that is the width of the neighborhood. Sequences of A.I.M.'s, that approximate the attractor with higher and higher order, have been derived for a broad class of dissipative evolution equations (see [DM], [F], [PT], [T2],...).

As the classical Galerkin method is related to the coarsest of these A.I.M.'s, the flat one, the nonlinear Galerkin methods feature inertial nonlinear algorithms that correspond to A.I.M.'s providing better orders of accuracy. The theory first developed in the spectral case (see [FMT], [MT1]) extends now beyond : for instance see [T3], [CT] for works about finite differences or [G] about wavelets.

In this paper we are interested in finite elements. Hence we go back to the framework of [MT2] : let V_h be a finite elements space corresponding to a triangulation whose mesh size is h. Instead of computing an approximation y_h of a solution u of a dissipative evolution equation as the solution of the approximated problem on V_h, we are looking for a nonlinear approximation $y_h + \phi(y_h)$, where ϕ maps V_h into a suitable supplementary W_h of V_h into $V_{h/2}$.

The algorithms of [MT2] used finite elements spaces that are related to classical finite elements hierarchical bases ; for a description of such bases see [Y]. But for numerical reasons it appears that it is convenient to enforce an orthogonality condition between the spaces V_h and W_h. Here appears the theory of orthonormal wavelet bases (see [M]).

Actually it could be possible to use the wavelets for open subsets of $\mathbb{R}^N$ as described in [JM]. On the other hand an unconvenient feature of these functions is that they are not given in an explicit form, although very interesting asymptotical results are available (see [J1]).

Therefore we are looking for new hierarchical bases whose elements enjoy both properties of finite elements and wavelets :

- Numerical convenience : each function is derived from a single([1]) basic function by dilations and translations (up to a truncation for functions whose support is close to the boundary).
- Localization : each function is compactly supported around a vertex of one grid.
- Orthogonality : each function is orthogonal to all but a finite number (independent of the function) of the others.
- Cancellations : each function oscillates (because the basic function has vanishing moments).

All these properties will allow us to obtain a robust decomposition of functions u as a sum

$$u = y_h + \sum_{j=0}^{+\infty} z_{h_j} \tag{1.1}$$

where y_h is as above and where the incremental variables $z_{h_{j+1}}$ are obtained by successive mesh refinements $h_{j+1} = \frac{h_j}{2}$, with the condition that $z_{h_{j+1}}$ is orthogonal to z_{h_k} for $k < j+1$.

The remainder of the article is organized as follows. In part 2 we describe the construction of the basic function. In part 3 we present the new hierarchical bases and we address some questions related to the robustness of formula (1.1). In part 4 we study a nonlinear Galerkin algorithm that applies to a class of reaction-diffusion equations. This class of equation that does not appear in [MT1], [MT2] necessitates some particular treatment. Other questions related to large time approximation properties of this algorithm will appear in a subsequent paper.

Notations :

- $\Omega : (0,1)^N$, $N = 1,2$; $\partial\Omega = \bar{\Omega}\backslash\Omega$
- $\text{dist}(x,\Omega) = \inf\{|x-\omega|, \omega \in \Omega\}$
- $\delta_{\lambda,\mu}$: Krönecker symbol

([1]) One in space dimension one, $2^N - 1$ in space dimension N.

- $\ell^2(I)$: space of sequences $\{a_\lambda\}$ indexed by I such that

$$\sum_{\lambda \in I} |a_\lambda|^2 < +\infty$$

- $L^{2p}(\Omega)$, $H_0^s(\Omega)$: classical Banach, Sobolev spaces on Ω.
- $|u|_{2p}^{2p} = \int_\Omega |u(x)|^{2p} dx$
- $|u| = |u|_2$, (,) corresponding scalar product.
- $\|u\|^2 = \int_\Omega |\nabla u(x)|^2 dx$; ((,)) corresponding scalar product.
- $\|u\|_* = \sup_{\|\varphi\| = 1} |(u, \varphi)|$
- supp (u) denotes the support of a function u.

2. Construction of the Function θ

For the sake of simplicity we will only describe in the following the construction as it relates to usual Q_1 finite elements in space dimension one or two. The existence of such function for various finite elements is related to the existence of compactly supported functions in multiresolution analysis ; see [L] where this last question is addressed.

2.1. The one-dimensional case

In this paragraph σ is the roof function depicted in Fig. 2.1, up to a multiplication by a constant choosen such that

$$\int_{\mathbb{R}} \sigma(x)^2 dx = 1.$$

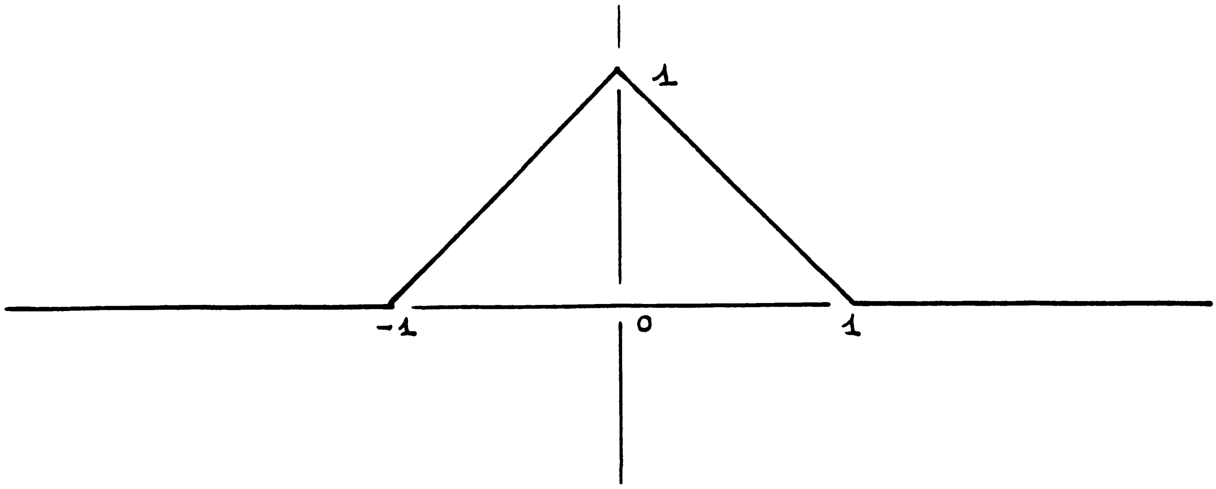

Fig.2.1. The roof function

We are looking for a piecewise affine continuous function θ that is compactly supported, whose nodes are the half-integers, and which satisfies :

$$\int_{\mathbb{R}} \sigma(x)\theta\left(x+\frac{1}{2}-k\right)dx=0, \text{ for any } k \text{ in } \mathbb{Z}. \tag{2.1}$$

A convenient solution is borrowed from [A] ; θ is the function whose graph is

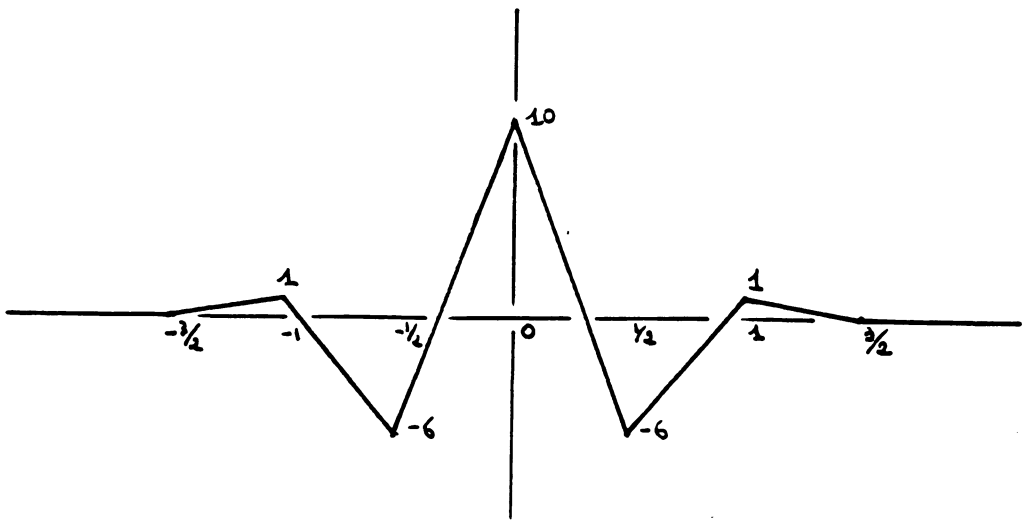

Fig.2.2. θ : *one – dimensional case*

Actually we shall consider the function

$$x \to \frac{1}{|\theta|^{1/2}}\theta(x).$$

Now we prove some preliminary material that we shall use in the next section.

LEMMA 2.1. *Let* $j \geq 0$, $h=\frac{1}{2^j}$. *The (nonvanishing) restrictions to* $(0,1)$ *of the functions* $x \to \theta\left(\frac{x}{h}-\ell-\frac{1}{2}\right)$, $\ell \in \mathbb{Z}$, *are linearly independent.*

Proof.

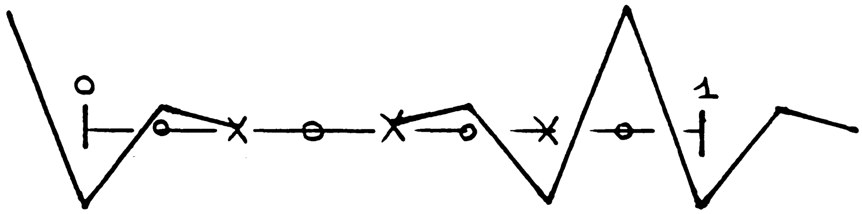

Fig.2.3. : Functions with nonvanishing restrictions. $j = 2$

Remark : In the following we will represent the points that are on the coarse grid by $\times$ symbols, and the uncremental points, that are the points on the fine grid that are not on the coarse one, by $\circ, \triangle, \square, \dots$ symbols.

There are exactly $\frac{1}{h} + 2$ functions $\theta\left(\frac{x}{h} - \ell - \frac{1}{2}\right)$ whose support is not disconnected to (0,1). Let $I = \left\{\lambda = h\left(\ell + \frac{1}{2}\right)\right\}$ be the set of indices of such functions. We observe that all the points of I are in $(0,1)$ except $-\frac{h}{2}$ and $1 + \frac{h}{2}$. We have to prove that, if there exist numbers γ_λ's such that

$$\sum_{\lambda \in I} \gamma_\lambda \theta\left(\frac{x - \lambda}{h}\right) = 0 \tag{2.2}$$

for all x in $(0,1)$, then $\gamma_\lambda = 0$ for all λ in I.

Let μ be such that

$$\gamma_\mu = |\gamma_\mu| = \max_{\lambda \in I} |\gamma_\lambda| .$$

First, if μ is in $(0,1)$, we take $x = \mu$ in (2.2) and we obtain

$$\gamma_\mu \theta(0) \leq \max_{\lambda \in I} |\gamma_\lambda| \left(\sum_{\lambda \neq \mu} \left| \theta\left(\frac{\lambda - \mu}{h}\right) \right| \right)$$

On the other hand, we go back to the definition of θ to observe that for any $\lambda_0 = h\left(\ell_0 + \frac{1}{2}\right)$

$$\theta(0) > \sum_{\lambda \neq \lambda_0} \left| \theta\left(\frac{\lambda - \lambda_0}{h}\right) \right| .$$

These two last inequalities lead to $\gamma_\mu = 0$.

The second case occurs when μ is equal to $-\frac{h}{2}$ (the case $1+\frac{h}{2}$ being similar). We then take $x = 0$ in (2.2) and we obtain

$$\gamma_{-h/2} + \gamma_{h/2} = 0,$$

and we are back to the first case.

2.2. The two-dimensional case : Q_1 elements

For that case we mimic the construction of two-dimensional wavelets by tensorial products (see [M]).

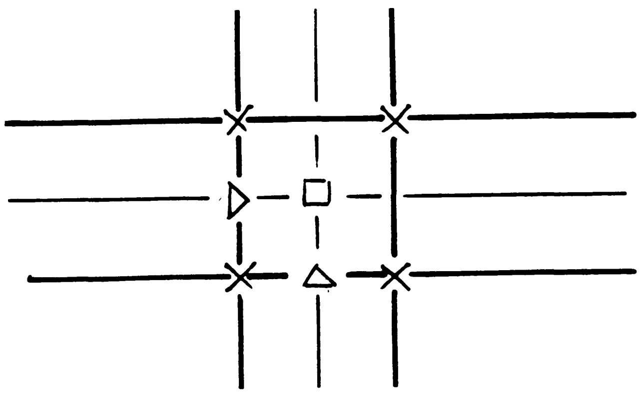

Fig.2.4. Q_1 uniform triangulation

Written with their center at (0,0), the three basic functions are

$$\begin{cases} \theta_\square(x_1,x_2) = \theta(x_1)\theta(x_2), \\ \theta_\triangleright(x_1,x_2) = \sigma(x_1)\theta(x_2), \\ \theta_\triangle(x_1,x_2) = \theta(x_1)\sigma(x_2), \end{cases}$$

where θ, σ are as in the one-dimensional case. For instance the values on the grid of $\theta_\triangle$ are depicted in Fig. 2.5 below ; as usual in the next sections we will chose the L^2

normalization for $\theta_\triangle$.

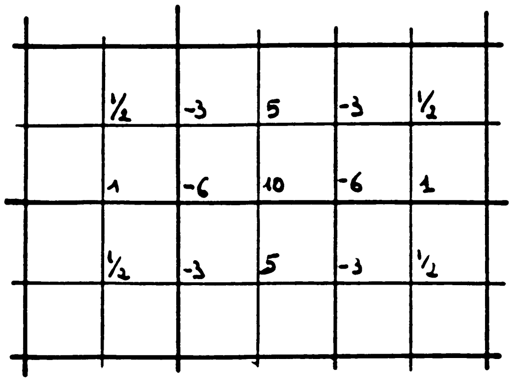

Fig.2.5. $\theta_\triangle$: *two – dimensional* Q_1 *case.*

We also define the two-dimensional σ function as

$$\sigma(x_1, x_2) \;=\; \sigma(x_1)\sigma(x_2),$$

the σ's in the right hand side of the equality above being the one-dimensional one.

Remark. We have to observe that the analog of (2.1) holds thanks to Section 2.1 and to Fubini's Theorem.

3. The New Hierarchical Bases.

This section is organized as follows : first we shall present the construction of several hierarchical bases on Ω =(0,1), in view of the discretization of an elliptic problem with respectively homogeneous Dirichlet, Neumann or periodic boundary conditions. Then we will derive such bases for the unit square $[0,1]^2$ in $\mathbb{R}^2$. We will conclude this section by establishing some properties of these bases.

3.1. The bases ; the one-dimensional case

Let us introduce some notations. We are given a triangulation of $[0,1]$ with intervals of width $h = h_j = \frac{1}{2^j}$, and we denote by N_h the set of the nodes (vertices) of the triangulation, and by V_h the corresponding $\mathcal{P}_1$ finite elements space. For instance for the Dirichlet boundary conditions

$$N_h = \left\{ k.h,\; 1 \leq k \leq \frac{1}{h} - 1,\; k \in \mathbb{N} \right\}.$$

Remark. We have to add to N_h respectively $k = 0$ and $k = \frac{1}{h}$, and $k = 0$ (or $k = \frac{1}{h}$), for respectively the Neumann and periodic problem.

For σ as in Section 2, for κ in N_h, we set

$$\sigma_\kappa(x) = h^{-1/2}\sigma\left(\frac{x-\kappa}{h}\right). \tag{3.1}$$

Remark. Formula (3.1) defines σ_0 as the restriction to (0,1) of $h^{-1/2}\sigma\left(\frac{x}{h}\right)$ in the Neumann case, and as the restriction to (0,1) of $h^{-1/2}\left[\sigma\left(\frac{x}{h}\right)+\sigma\left(\frac{x-1}{h}\right)\right]$ in the periodic one (since 1=0 in that case).

We then recall the following well-known result :

PROPOSITION 3.1. *The family $\{\sigma_\kappa\}_{\kappa\in N_h}$ is a basis for V_h. Moreover there exist two constants $C_1, C_2 > 0$ that are independent of h such that for any*

$$y = y(x) = \sum_{\kappa\in N_h} \alpha_\kappa \sigma_\kappa(x)$$

in V_h, the following inequalities hold :

$$C_1|y|^2 \le \sum_{\kappa\in N_h} |\alpha_\kappa|^2 \le C_2|y|^2. \tag{3.2}$$

Proof. We just compute the Gramm matrix of the σ_κ's and then observe that on each column, the diagonal entry is larger than the sum of the modulus of the other ones.

We then define W_h as the orthogonal complementary (in $L^2(\Omega)$) of V_h in $V_{h/2}$. We are looking for a basis of W_h whose elements are functions that are localized around the λ's belonging to $I_h = N_{h/2}\backslash N_h$. We will derive these functions by truncation, dilation and translation of the function of Section 2. We first observe that, independently of the boundary conditions, card $I_h = \dim W_h = \frac{1}{h}$.

3.1.1. Dirichlet boundary conditions

For λ in I_h such that supp $\theta\left(\frac{x-\lambda}{h}\right) \subset \bar{\Omega}$ we set

$$\theta_\lambda(x) = h^{-1/2}\theta\left(\frac{x-\lambda}{h}\right). \tag{3.3}$$

Hence we already have $\frac{1}{h} - 2$ functions ; it remains to define $\theta_{h/2}$ (and $\theta_{1-(h/2)}$). For that purpose we shall modify $\theta\left(\frac{x}{h} - \frac{1}{2}\right)$ in order to enforce the condition $\theta_{h/2}(0) = 0$. We then set

$$\theta_{h/2}(x) = h^{-1/2}\left(\theta\left(\frac{x}{h} - \frac{1}{2}\right) - \theta\left(\frac{x}{h} + \frac{1}{2}\right)\right) \tag{3.4}$$

$$\theta_{1-(h/2)}(x) = h^{-1/2}\left(\theta\left(\frac{x}{h} - \frac{1}{h} + \frac{1}{2}\right) - \theta\left(\frac{x}{h} - \frac{1}{h} - \frac{1}{2}\right)\right), \tag{3.5}$$

being understood that we consider the restrictions to (0,1) of the functions involved in (3.4),(3.5).

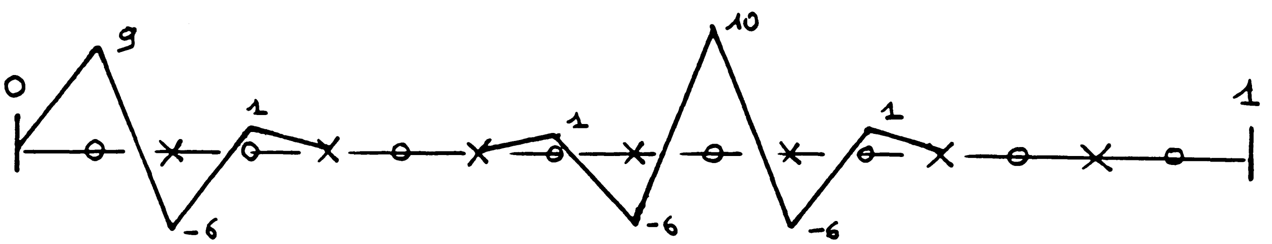

*Fig.*3.1. *Dirichlet boundary conditions*;
$j = 4$; $\theta_{1/16}$, $\theta_{5/16}$

We will prove that this family provides a basis (for W_h) in Proposition 3.2 below.

3.1.2. Other boundary conditions

For λ in $I_h \setminus \left\{\frac{h}{2}, 1 - \frac{h}{2}\right\}$, we define θ_λ as in (3.3). Let us first consider the Neumann case. For a reason that will appear in the proof of Proposition 3.2 below, we set

$$\theta_{h/2}(x) = h^{-1/2}\left(\theta\left(\frac{x}{h} - \frac{1}{2}\right) + \theta\left(\frac{x}{h} + \frac{1}{2}\right)\right), \tag{3.6}$$

$$\theta_{1-(h/2)} = h^{-1/2}\left(\theta\left(\frac{x}{h} - \frac{1}{h} + \frac{1}{2}\right) + \theta\left(\frac{x}{h} - \frac{1}{h} - \frac{1}{2}\right)\right), \tag{3.7}$$

being understood that we consider the restrictions to $[0,1]$ of the functions involved in

(3.6), (3.7).

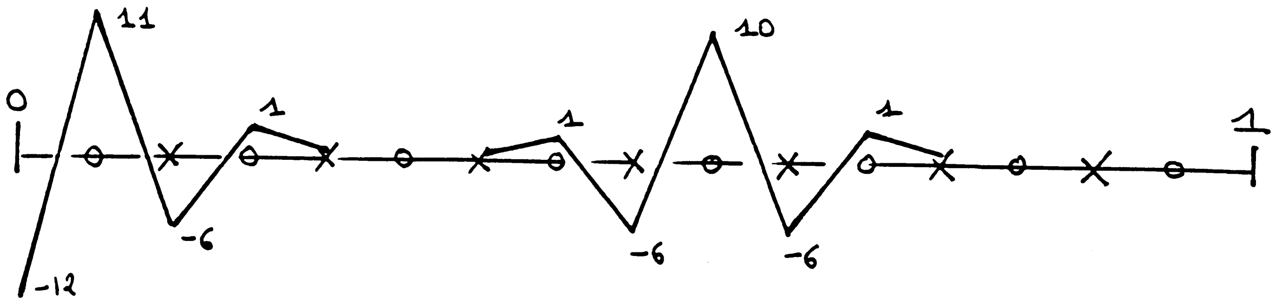

Fig.3.2. Neumann boundary conditions;
$j = 4,\ \theta_{1/16},\ \theta_{5/16}$.

For the periodic case, we set

$$\theta_{h/2}(x) = h^{-1/2}\left(\theta\left(\frac{x}{h}-\frac{1}{2}\right)+\theta\left(\frac{x}{h}+\frac{1}{h}-\frac{1}{2}\right)\right) \tag{3.8}$$

$$\theta_{1-(h/2)}(x) = h^{-1/2}\left(\theta\left(\frac{x}{h}-\frac{1}{h}+\frac{1}{2}\right)+\theta\left(\frac{x}{h}+\frac{1}{2}\right)\right), \tag{3.9}$$

being understood that we consider the restrictions to [0, 1] of the functions involved in (3.8), (3.9)

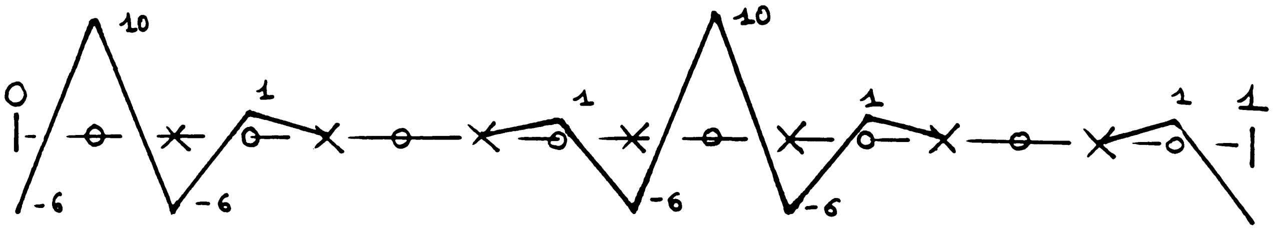

Fig.3.3. Periodic boundary conditions;
$j = 4,\ \theta_{1/16},\ \theta_{5/16}$.

3.1.3. Basic properties

PROPOSITION 3.2. *The family* $\{\theta_\lambda\}_{\lambda\in I_h}$ *defined as above provides a basis for* W_h. *Moreover there exist two constants* $C_3, C_4 > 0$ *that are independent of* h, *such that for any*

$$z = z(x) = \sum_{\lambda\in I_h} \gamma_\lambda \theta_\lambda(x)$$

in W_h, *the following inequalities hold*

$$C_3|z|^2 \leq \sum_{\lambda\in I_h} |\gamma_\lambda|^2 \leq C_4|z|^2. \tag{3.10}$$

Proof. The first step of the proof is to check that each θ_λ belongs to W_h, i.e. that θ_λ is orthogonal to σ_κ for all κ in N_h.

Independently of the boundary conditions, this is a straightforward consequence of (2.1) for any λ in I_h such that supp $\theta_\lambda \subset [0,1]$.

Actually it remains to prove this result for $\theta_{h/2}$ (the case of $\theta_{1-(h/2)}$ being similar) defined by (3.4) or (3.6). Let us first consider the Dirichlet case. For any κ in N_h, we have

$$\begin{aligned}\int_\Omega \theta_{h/2}(x)\sigma_\kappa(x)dx = \frac{1}{h}\Bigg[\int_{\mathbb{R}} \theta\left(\frac{x}{h}-\frac{1}{2}\right)\sigma\left(\frac{x-\kappa}{h}\right) \\ - \int_{\mathbb{R}} \theta\left(\frac{x}{h}+\frac{1}{2}\right)\sigma\left(\frac{x-\kappa}{h}\right)dx\Bigg].\end{aligned} \tag{3.11}$$

Thanks to (2.1), the two integrals involved in the right hand side of (3.11) vanish, and the result follows.

For the Neumann case, an analogous proof gives the result for all κ in N_h except $\kappa = 0$. On the other hand we have

$$\begin{aligned}\int_\Omega \theta_{h/2}(x)\sigma_0(x)\,dx = \frac{1}{h}\Bigg[\int_{x\geq 0} \theta\left(\frac{x}{h}-\frac{1}{2}\right)\sigma\left(\frac{x}{h}\right)dx \\ + \int_{x\geq 0} \theta\left(\frac{x}{h}+\frac{1}{2}\right)\sigma\left(\frac{x}{h}\right)dx\Bigg].\end{aligned} \tag{3.12}$$

Hence, since θ and σ are even functions, we have

$$\int_{x\geq 0} \theta\left(\frac{x}{h}+\frac{1}{2}\right)\sigma\left(\frac{x}{h}\right)dx = \int_{x\leq 0} \theta\left(\frac{x}{h}-\frac{1}{2}\right)\sigma\left(\frac{x}{h}\right)dx$$

and then, thanks to (2.1), the right hand side of (3.12) vanishes.

Let us now prove the first assertion of Proposition 3.2, i.e. that the θ_λ's provide a basis for W_h. We just have to establish that these functions are linearly independent. This point is a straightforward consequence of the definition of the θ_λ's and of Lemma 2.1.

Let us prove now the second part of Proposition 3.2. Inequality

$$|z|^2 \leq C \sum_{\lambda \in I_h} |\gamma_\lambda|^2 \tag{3.13}$$

is straightforward to establish by expanding

$$|z|^2 \leq \sum_{\lambda,\mu \in I_h} |\gamma_\lambda| \, |\gamma_\mu| \left| \int_\Omega \theta_\lambda(x)\theta_\mu(x)dx \right|,$$

and observing that the integrals involved in this sum, that are anyway bounded by 1, vanish if $|\lambda - \mu| \geq Ch$, for some constant C.

For the reverse inequality, we rather prove directly the following stronger result :

THEOREM 3.1. *For any* $h_j = \frac{h_0}{2^j}$, $j \geq 0$, *let us define the family* $\{\theta_\lambda\}_{\lambda \in I_{h_j}}$ *as above. Then the family*

$$\{\sigma_\kappa\}_{\kappa \in N_{h_0}} \cup \left(\bigcup_{j \geq 0} \{\theta_\lambda\}_{\lambda \in I_{h_j}} \right)$$

is a Riesz basis for $L^2(\Omega)$.

<u>Remark</u>. The meaning of the theorem is that any function u in $L^2(\Omega)$ can be written in a unique way

$$u(x) = \sum_{\kappa \in N_{h_0}} \alpha_\kappa \sigma_\kappa(x) + \sum_{j \geq 0} \sum_{\lambda \in I_{h_j}} \gamma_\lambda \theta_\lambda(x),$$

such that the quantity

$$\left(\sum_\kappa |\alpha_\kappa|^2 + \sum_{j,\lambda} |\gamma_\lambda|^2 \right)^{1/2}$$

defines a norm that is equivalent to the L^2 one ; in other words a Riesz basis is a basis that is isomorphic to an orthonormal one.

<u>Proof of Theorem 3.1</u>. Thanks to the density of $\bigcup_{j \geq 0} V_{h_j}$ in $L^2(\Omega)$ we have

$$L^2(\Omega) = V_{h_0} \oplus \left(\bigoplus_{j=0}^{+\infty} W_{h_j} \right), \tag{3.14}$$

the sum being orthogonal.

Let denote by I the set of indices $N_{h_0} \cup \left(\bigcup_{j \geq 0} I_{h_j} \right)$. We then define e_κ as the sequence $(\delta_{\mu,\kappa})_{\mu \in I}$ of $\ell^2(I)$. Then we define a linear operator L from $\ell^2(I)$ into $L^2(\Omega)$ by setting

$$L(e_\kappa) = \sigma_\kappa \text{ if } \kappa \in N_{h_0},$$
$$L(e_\lambda) = \theta_\lambda \text{ if } \lambda \in I \backslash N_{h_0}.$$

Thanks to (3.14), the family $\{\sigma_\kappa\}_{\kappa \in N_{h_0}} \cup \{\theta_\lambda\}_{\lambda \in I \setminus N_{h_0}}$ is complete in $L^2(\Omega)$, and therefore L is onto. On the other hand, thanks to (3.2), (3.13) and (3.14), L is bounded.

Moreover L is one-to-one : if ω in $\ell^2(I)$ is such that $T(\omega) = 0$, then we consider the L^2 projection of $T(\omega)$ on either V_{h_0} or W_{h_j}, and we use Proposition 3.1 and the first part of Proposition 3.2 to obtain $\omega = 0$.

Therefore the Banach open mapping theorem applies and Theorem 3.1 follows.

3.1.4. The two-dimensional case

The construction of bases for the unit square follows the method of tensorial products (see Section 2.2). As an example, we give below some functions related to a problem with mixed boundary conditions.

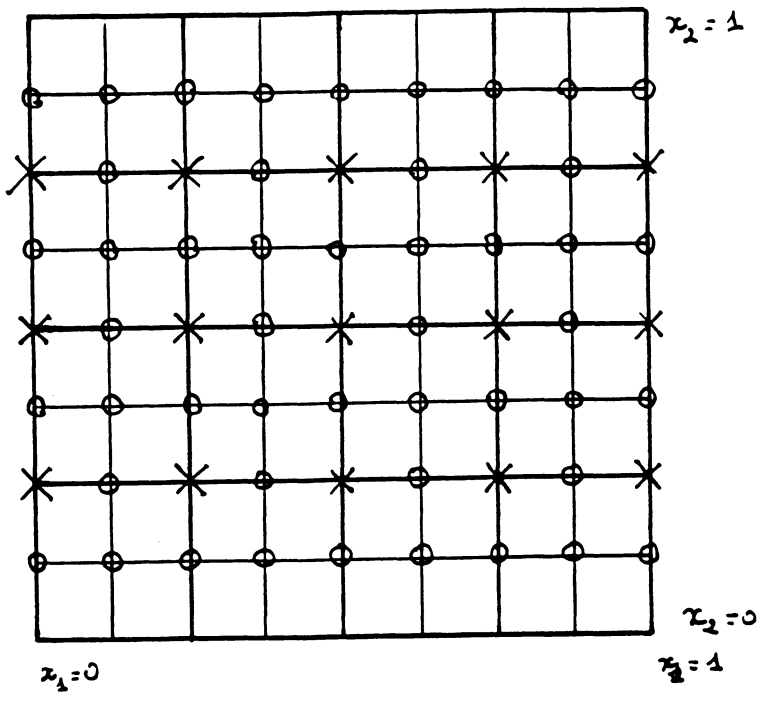

Fig.3.4. Mixed boundary conditions

Let us consider a problem with Neumann boundary conditions on $x_1 = 0$ and $x_1 = 1$, and Dirichlet boundary conditions on $x_2 = 0$ and $x_2 = 1$ (see Fig. 3.4. above).

Remark. As usual, we consider the restrictions to $[0,1]^2$ of the functions presented below.

· For $\lambda = (\lambda_1, \lambda_2)$ as in Fig. 3.4, we have

$$\theta_\lambda(x_1, x_2) = h^{-1}\theta\left(\frac{x_1 - \lambda_1}{h}\right)\theta\left(\frac{x_2 - \lambda_2}{h}\right).$$

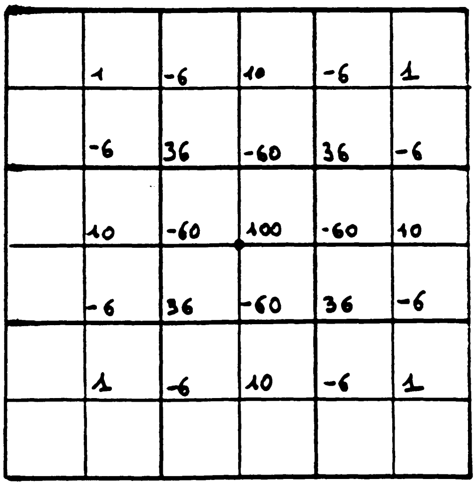

Fig.3.5. θ_λ for λ far away from $\partial\Omega$

· For $\mu = \left(\lambda_1, 1 - \frac{h}{2}\right)$ as in Fig. 3.4, we have

$$\theta_\mu(x_1, x_2) = h^{-1}\theta\left(\frac{x_1 - \lambda_1}{h}\right) \cdot \left[\theta\left(\frac{x_2 - 1}{h} + \frac{1}{2}\right) - \theta\left(\frac{x_2 - 1}{h} - \frac{1}{2}\right)\right]$$

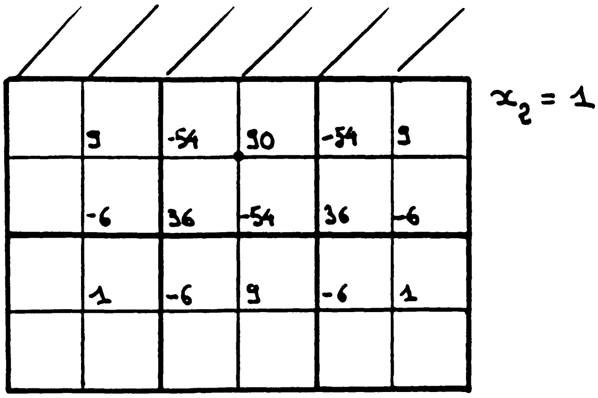

Fig.3.6. θ_μ for μ close to $x_2 = 1$

· For $\omega = \left(\frac{h}{2}, \lambda_2\right)$ as in Fig. 3.4, we have

$$\theta_\omega(x_1, x_2) = h^{-1}\left[\theta\left(\frac{x_1}{h} - \frac{1}{2}\right) + \theta\left(\frac{x_1}{h} + \frac{1}{2}\right)\right]\theta\left(\frac{x_2 - \lambda_2}{h}\right).$$

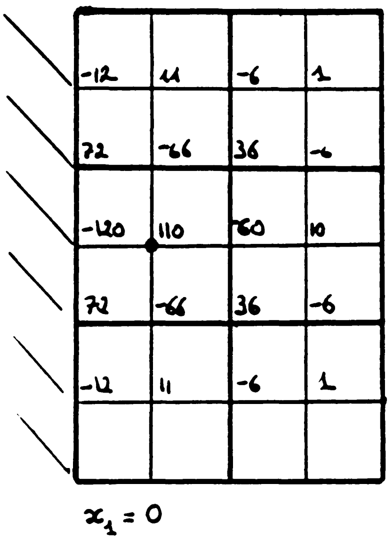

Fig.3.7. θ_ω for ω close to $x_1 = 0$

Remark. We have presented the functions in Fig. 3.5, Fig. 3.6 and Fig. 3.7 without L^2 normalization.

3.2. Sobolev space and the new hierarchical bases

For the sake of clarity, we will establish the results above for the bases related to a Dirichlet problem. Then we will briefly indicate the modifications to do in order to obtain similar results about the Neumann case (Results about the periodic case can be found in [M]).

The previous theorem shows that our bases are convenient to analyze the behavior of functions in $L^2(\Omega)$. In this section we will study some properties of the bases in $H_0^s(\Omega)$ spaces up to the natural edge that is $s = 1$.

First let us have a look at the properties of the θ_λ's for the H_0^1 norm. The following result describes the cancellations of these functions.

PROPOSITION 3.3. *Let $h = h_j = \frac{h_0}{2^j}$ be as above. There exist two constants $C_5, C_6 > 0$, that are independent of h, such that for any*

$$z = z(x) = \sum_{\lambda \in I_h} \gamma_\lambda \theta_\lambda(x)$$

in W_h the following inequalities hold :

$$C_5 \|z\|^2 \le \frac{1}{h^2} \sum_{\lambda \in I_h} |\gamma_\lambda|^2 \le C_6 \|z\|^2. \tag{3.15}$$

Proof. We derive

$$\|z\|^2 \le \frac{C}{h^2} \sum_{\lambda \in I_h} |\gamma_\lambda|^2 \tag{3.16}$$

as we obtained (3.13), observing that $\|\theta_\lambda\| \leq \frac{C}{h}$.

Remark. We have to observe that this so-called Bernstein inequality holds also for the σ_κ's. Namely there exists a constant $C > 0$ independent of h such that for any y in V_h the following inequality holds :

$$\|y\| \leq \frac{C}{h}|y|. \tag{3.17}$$

Let us now prove the reverse inequality in (3.16). Thanks to (3.10) and to the interpolation inequality

$$|z|^2 \leq \|z\|\|z\|_*,$$

we just have to prove

LEMMA 3.1. *There exists a constant $C > 0$, that is independent of h, such that for any z in W_h the following inequality holds :*

$$\|z\|_* \leq Ch|z|. \tag{3.18}$$

Proof of the Lemma : Let u in $H_0^1(\Omega)$ be such that $\|u\| = 1$. We have to estimate

$$|(z,u)|^2 \leq \left(\sum_{\lambda \in I_h} |\gamma_\lambda|^2\right)\left(\sum_{\lambda\in I_h} |(\theta_\lambda, u)|^2\right),$$

setting as usual

$$z = z(x) = \sum_{\lambda \in I_h} \gamma_\lambda \theta_\lambda(x).$$

Hence, thanks to (3.10), inequality

$$\sum_{\lambda \in I_h} \left|\int_\Omega \theta_\lambda(x)u(x)dx\right|^2 \leq Ch^2\|u\|^2 \tag{3.19}$$

yields (3.18).

Let us now prove (3.19). The first observation is that because $\int \theta(x)dx$ vanishes there exists W compactly supported such that $W(x) - W(x+1/2) = \theta(x)$. This point comes from the fact that we can apply the Paley-Wiener Theorem to $\frac{\hat{\theta}(\xi)}{1-e^{i\frac{\xi}{2}}}$ and that therefore

$$W(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{\hat{\theta}(\xi)}{1-e^{i\frac{\xi}{2}}} e^{ix\xi} d\xi$$

is compactly supported.

On the other hand, we observe that any θ_λ writes as $\theta\left(\frac{x-\lambda}{h}\right)$, or as a sum of two or four functions $\theta\left(\frac{x-\lambda'}{h}\right)$ (restricted to Ω). Hence setting

$$\tilde{u} = \tilde{u}(x) = \begin{cases} u(x) & \text{if } x \text{ is in } \Omega \\ 0 & \text{elsewhere} \end{cases}$$

we have

$$\begin{aligned} \int_\Omega \theta_\lambda(x) u(x) dx &= h^{-N/2} \sum_{\lambda'} \int_{\mathbb{R}^N} W\left(\frac{x-\lambda'}{h}\right)\left(\tilde{u}(x) - \tilde{u}\left(x - \frac{h}{2}\right)\right) dx \\ &\leq C \sum_{\lambda'} \left(\int_{\omega_{\lambda'}} \left|\tilde{u}(x) - \tilde{u}\left(x - \frac{h}{2}\right)\right|^2 dx\right)^{1/2}, \end{aligned} \tag{3.20}$$

where $\omega_{\lambda'} = \left\{x \in \mathbb{R}^N / \frac{x-\lambda'}{h} \in \operatorname{supp} W\right\}$. Then

$$\begin{aligned} \sum_{\lambda \in I_h} |(\theta_\lambda, u)|^2 &\leq C \int_{\mathbb{R}^N} \left|\tilde{u}(x) - \tilde{u}\left(x - \frac{h}{2}\right)\right|^2 dx \\ &\leq Ch^2 \int_{\mathbb{R}^N} |\nabla \tilde{u}(x)|^2 dx, \end{aligned} \tag{3.21}$$

that concludes the proof of Lemma 3.1.

Remark. For the Neumann case, we have to replace $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ by

$$\dot{H}^1(\Omega) = \left\{u \in H^1(\Omega);\ \int_\Omega u = 0\right\}$$

and its dual space.

If we still denote by $\|z\|_*$ the quantity $\sup\limits_{u \in \dot{H}^1(\Omega)} \frac{|(u,z)|}{\|u\|}$, then (3.18), and therefore Proposition 3.3, hold.

The idea of the proof is to set $\tilde{u} = \tilde{u}(x) = Pu(x)$, where P is the prolongation operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^N)$ defined by 2^N successive reflections (and a truncation) (see [B] Remarque 10, p.160).

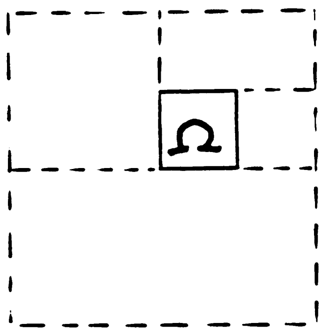

Fig.3.8. The prolongation operator

Going back to the definition of the θ_λ's, we then observe that

$$P^*\left(\theta\left(\frac{x-\lambda}{h}\right)\right) = h^{N/2}\theta_\lambda(x),$$

and that therefore

$$\int_\Omega u(x)\theta_\lambda(x)dx = h^{-N/2}\int_{\mathbb{R}^N}(Pu)(x)\theta\left(\frac{x-\lambda}{h}\right)dx.$$

The same proof as the proof of Lemma 3.1 above provides the estimate

$$\sum_{\lambda\in I_h}|(\theta_\lambda,u)|^2 \leq Ch^2(|u|^2+\|u\|^2),$$

and the conclusion follows the remark that since $\int_\Omega u = 0$ we can apply Poincaré inequality to obtain $|u| \leq C\|u\|$.

Hence let us give two interesting features of our bases that are consequence of Proposition 3.3. The first one is related to the discretization of the operator $-\Delta + Id$ on V_h using a fully hierarchical basis : we already know a basis for V_h that is

$$\{\sigma_\kappa\}_{\kappa\in N_{h_0}} \cup \left(\bigcup_{h_j<h}\{\theta_\lambda\}_{\lambda\in I_{h_j}}\right);$$

We look upon the coarse grid Ω_{h_0} as a superposition of coarser grids $\Omega_{2^j h_0}$, $j \geq 1$, and we define the corresponding spaces $V_{2^j h_0}$. We observe that $V_{2^j h_0} = \{0\}$ if j is large enough (because Ω is bounded). We then define by induction $W_{h_{-j}}$ as the orthogonal of $V_{2^{j+1} h_0}$ in $V_{2^j h_0}$ and we consider the corresponding bases $\{\theta_\lambda\}_{\lambda \in I_{h_{-j}}}$. The basis we shall consider for V_h is

$$\bigcup_{j \geq 1} \{\theta_\lambda\}_{\lambda \in I_{h_{-j}}} .$$

PROPOSITION 3.4. *Let M^h be the matrix whose entries are the $((\theta_\lambda, \theta_\mu)) + (\theta_\lambda, \theta_\mu)$, for λ, μ in $\bigcup_{j \geq 1} I_{h_{-j}}$. Then there exists an explicit diagonal preconditioning L^h such that the condition number K_h of the matrix $(L^h)^{-1} M^h (L^h)^{-1}$ grows (with respect to h) like $|\mathrm{Log}\ h|^2$.*

Remarks

. For analogous studies for wavelets and for the classical hierarchical bases see respectively [J2] and [Y].

. The reason why we do not obtain $0(1)$ instead of $0(|\mathrm{Log}\ h|^2)$ is because the function θ lacks a little smoothness.

Proof : First we observe that $I_{h_{-j}}$ is empty for $j \geq C(\Omega)|\mathrm{Log}\ h|$, where $C(\Omega)$ is a constant that depends on the diameter of Ω. Therefore if we define a function y of V_h as

$$y = \sum_{j \geq 1} z_{-j} \tag{3.22}$$

where

$$z_{-j} = z_{-j}(x) = \sum_{\lambda \in I_{h_{-j}}} \gamma_\lambda \theta_\lambda(x),$$

we obtain both

$$\|y\|^2 \leq C|\mathrm{Log}\ h| \sum_{j \geq 1} \|z_{-j}\|^2 , \tag{3.23}$$

$$\|y\|_*^2 \leq C|\mathrm{Log}\ h| \sum_{j \geq 1} \|z_{-j}\|_*^2 . \tag{3.24}$$

Using (3.15) and (3.23) we obtain

$$\|y\|^2 \leq C|\mathrm{Log}\ h| \sum_{j \geq 1} \left(\left(\frac{1}{2^j h} \right)^2 \left(\sum_{\lambda \in I_{h_{-j}}} |\gamma\lambda|^2 \right) \right) . \tag{3.25}$$

Let us prove now a kind of reverse inequality. We set

$$\tilde{y}(x) = \sum_{j \geq 1} \left(\frac{1}{2^j h} \right)^2 z_{-j}(x), \tag{3.26}$$

where z_{-j} is as above. Hence we have

$$\sum_{j\geq 1}\left(\frac{1}{2^j h}\right)^2 |z_{-j}|^2 = (y,\tilde{y})$$
$$\leq \|y\|\,\|\tilde{y}\|_*\,,$$

and then using (3.10), (3.18), (3.24) we obtain

$$\sum_{j\geq 1}\left(\frac{1}{2^j h}\right)^2 |z_{-j}|^2 \leq C|\mathrm{Log}\ h|^{1/2}\left(\sum_{j\geq 1}\left(\frac{1}{2^j h}\right)^2 |z_{-j}|^2\right)^{1/2}\|y\|, \tag{3.27}$$

and therefore

$$|\mathrm{Log}\ h|^{-1}\sum_{j\geq 1}\left(\left(\frac{1}{2^j h}\right)^2 \sum_{\lambda\in I_{h_{-j}}}|\gamma\lambda|^2\right) \leq C\|y\|^2 \tag{3.28}$$

follows.

We reinterpret (3.25), (3.28) saying that for

$$L^h = \begin{pmatrix} \frac{1}{2h} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{2h} & & & \\ & & & \frac{1}{4h} & & \\ & & & & \ddots & \\ & & & & & \frac{1}{4h} \\ & & & & & & \ddots \\ & & & & & & & \frac{1}{2^j h} \end{pmatrix}$$

then $K_h = O(|\mathrm{Log}\ h|^2)$.

The second application of Proposition 3.3 that we give below extends the robustness of Formula (1.1) beyond the L^2 case. We shall prove that this formula holds for any function or distribution in $H_0^s(\Omega)$ or in $H^{-s}(\Omega)$, for any s in (0,1) except $s = 1/2$. (In that case we cannot define $H_0^{1/2}(\Omega)$ by interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$; see [LM]).

PROPOSITION 3.5. *The family* $\bigcup_{j\in\mathbb{Z}}\{\theta_\lambda\}_{\lambda\in I_{h_j}}$ *is an unconditional basis for the spaces* $H_0^s(\Omega)$ *and* $H^{-s}(\Omega)$, $|s|<1$, $|s|\neq 1/2$. *Namely any* u *in respectively* $H_0^s(\Omega)$ *or* $H^{-s}(\Omega)$ *can be written in a unique way as*

$$u(x)=\sum_{j\in\mathbb{Z}}\sum_{\lambda\in I_{h_j}}\gamma_\lambda\theta_\lambda(x)$$

where the sum is convergent in respectively $H_0^s(\Omega)$ *and* $H^{-s}(\Omega)$. *Moreover*

$$\sum_{j\in\mathbb{Z}}\left((h_j)^{2s}\sum_{\lambda\in I_{h_j}}|\gamma_\lambda|^2\right)$$

defines a norm on respectively $H_0^s(\Omega)$ *and* $H^{-s}(\Omega)$ *that is equivalent to the usual one.*

Remark. We recall that in Prop.3.5. actually

$$u(x)=\sum_{j>j_*}\sum_{\lambda\in I_{h_j}}\gamma_\lambda\theta_\lambda(x),$$

where j_* is a negative number that depends on the diameter of Ω.
Remark. Analogous results hold for the Neumann case, if we replace H_0^s (and H^{-s}) by the spaces defined by interpolation from $\dot{H}^1$ and its dual.

Proof. The proof is classical for orthogonal splittings of L^2 in a sum of subspaces that enjoy both (3.10), (3.15), (3.18). See [M, Th. 8, Ch. 2]. See also [G].

3.3. $L^p(\Omega)$ spaces and the new hierarchical bases

The aim of this section is to establish some properties we will use in the next section. At the same time we will extend the meaning of formula (1.1) beyond the Hilbertian L^2 case.

In this paragraph we will use the localization properties of the θ_λ's and the σ_κ's around respectively λ and κ ; we will also use the fact that the function of the dual bases enjoy similar properties. Namely for any λ or κ in respectively I_h and N_h, if we define by θ_λ^* and σ_κ^* the unique functions in respectively W_h and V_h such that

$$(\theta_{\lambda'},\theta_\lambda^*)=\delta_{\lambda,\lambda'},\ \text{for any } \lambda \text{ in } I_h, \tag{3.29}$$

$$(\sigma_{\kappa'},\sigma_\kappa^*)=\delta_{\kappa,\kappa'},\ \text{for any } \kappa \text{ in } N_h, \tag{3.30}$$

then there exist two absolute constants $\alpha_*, C_*>0$ such that

$$|\theta_\lambda^*(x)|\le C_* h^{-N/2}\exp\left(-\alpha_*\frac{|x-\lambda|}{h}\right), \tag{3.31}$$

$$|\sigma_\kappa^*(x)|\le C_* h^{-N/2}\exp\left(-\alpha_*\frac{|x-\kappa|}{h}\right), \tag{3.32}$$

hold for any h, x, λ, κ.

For a proof of (3.31), (3.32) we refer the reader to [JM]. The idea is that, for instance for (3.31), because the θ_λ's satisfy (3.10), the inverse G^{-1} of their Gramm matrix enjoys good decay properties for the entries away from the main diagonal. Hence by $\theta_\lambda^* = G^{-1}\theta_\lambda$, the localization properties of the θ_λ's transfer to the θ_λ^*'s.

LEMMA 3.2. *The orthogonal projectors $L^2(\Omega) \to V_h$ are uniformly bounded (independently of h) as operators acting on $L^p(\Omega)$, $1 \le p \le +\infty$.*

Proof : Thanks to (3.32), the estimate

$$\begin{aligned} &\sup_{x\in\Omega}\left|\sum_{\kappa\in N_h}(u,\sigma_\kappa^*)\sigma_\kappa(x)\right| \\ &\le |u|_\infty.\left(\sup_{\kappa\in N_h}\int_\Omega |\sigma_\kappa^*(x)|\,dx\right).\left(\sup_{x\in\Omega}\left(\sum_{\kappa\in N_h}\sigma_\kappa(x)\right)\right) \end{aligned} \tag{3.33}$$

provides the L^∞ bound. Since the projector is self-adjoint we then derive the L^1 bound, and therefore the L^p ones by classical interpolation results.

We then order the θ_λ's in the natural way (the functions of W_{h_j} before those of $W_{h_{j+1}}$). We have

COROLLARY 3.1. *The family $\bigcup_{j\in\mathbb{Z}}\{\theta_\lambda\}_{\lambda\in I_{h_j}}$ ordered as above provides a Schauder basis for $L^p(\Omega)$, $1 \le p < +\infty$ and for $C_0(\Omega)$.*

Remarks.

. $C_0(\Omega)$ denotes the subspace of $L^\infty(\Omega)$ whose functions are continuous on Ω and vanish on its boundary.

. Corollary 3.1 means that any function u in $L^p(\Omega)$, $1 \le p < +\infty$, or $C_0(\Omega)$ can be written uniquely as

$$u(x) = \sum_{j=-\infty}^{+\infty}\sum_{\lambda\in I_{h_j}}\gamma_\lambda\theta_\lambda(x),$$

the partial sum being convergent for the corresponding norm.

· As usual we have to replace $C_0(\Omega)$ by $C(\Omega)$ for the Neumann problem.

Proof of Corollary 3.1 : We observe that thanks to Lemma 3.2 and to the fact that the θ_λ's present the same features that the σ_κ's, rewriting the partial sums as

$$\sum_{\kappa\in N_h}\alpha_\kappa\sigma_\kappa(x) + \sum_{\lambda\in I_h}\gamma_\lambda\theta_\lambda(x),$$

we bound them (uniformly with respect to h) in the L^p's. Hence the conclusion results from the density of $\bigcup_{j\ge 0} V_{h_j}$ respectively in $L^p(\Omega)$, $p < +\infty$, and in $C_0(\Omega)$.

Remark : For $1 < p < +\infty$ we have a stronger result for the Dirichlet case. Actually the θ_λ's provide an unconditional basis for $L^p(\Omega)$. We do not want to develop this point here. Nevertheless the idea is to check that the operator from $D(\mathbb{R}^N)$ into $D'(\mathbb{R}^N)$ whose kernel is

$$\sum_{j\in\mathbb{Z}} \sum_{\lambda\in I_{h_j}} \theta_\lambda(x)\theta^*_\lambda(y)$$

belongs to the Calderon-Zygmund class (CZO) and therefore is bounded on $L^p(\mathbb{R}^N)$, $1 < p < +\infty$. (See [M, Chap. 6 and Chap. 7]). The conclusion follows the observation that both the restriction operator

$$u \in L^p(\mathbb{R}^N) \to u/\Omega \in L^p(\Omega)$$

and the prolongation operator

$$u \in L^p(\Omega) \to \begin{cases} \tilde{u} = u & \text{in } \Omega \\ \tilde{u} = 0 & \text{elsewhere} \end{cases} \in L^p(\mathbb{R}^N)$$

are bounded.

4. A Nonlinear Galerkin Algorithm for a Reaction-Diffusion Equation Using the New Hierarchical Bases

In this section, we will only consider a Dirichlet boundary conditions problem.

The results that follow can be also proved for Neumann or periodic problems, with obvious modifications in the statements of the theorems ($H^1(\Omega)$ or $H^1_{\text{per}}(\Omega)$ instead of $H^1_0(\Omega)$), and minor changes in their proofs (for instance we have to work with $\nu A + Id$ instead of νA for some a priori estimates).

4.1. An equation with a polynomial nonlinearity

The evolution equation we shall consider can be written in the abstract form

$$\frac{du}{dt} + \nu Au + R(u) = f \tag{4.1}$$

where the unknown function u maps $[0,+\infty)$ into $L^2(\Omega)$; A denotes the unbounded operator on $L^2(\Omega)$ that is the Laplacian with homogeneous Dirichlet boundary condition; ν is a positive parameter ; f is in $L^2(\Omega)$; and the nonlinearity R is a polynomial

$$R(\xi) = \sum_{k=0}^{2p-1} b_k \xi^{2k-1}, \quad \text{with } b_{2p-1} > 0.$$

For existence and uniqueness results of solutions u of (4.1) and of initial value condition

$$u(0) = u_0 \text{ in } L^2(\Omega), \tag{4.2}$$

we refer the reader to [T1]; see also [Ma].

From the references above, we also recall the existence of two nonnegative constants $t_0 = t_0(\Omega, f)$, M_0, that are independent of u_0, such that for $t \geq t_0$

$$\nu\|u(t)\|^2 + |u(t)|^2 + |u(t)|_{2p}^{2p} \leq M_0 < +\infty \tag{4.3}$$

holds. Here M_0 is a constant that behaves with respect to the data of the equation like

$$M_0 = C(p)\left[|\Omega| + |f|^2\right].$$

The estimate (4.3) is related to the existence of absorbing sets and of an attractor for the dynamical system given by (4.1).

4.2. The nonlinear algorithm

The method developed here follows the framework of the nonlinear Galerkin finite-elements methods (see [MT2]). Nevertheless two slight differences appear :

- First, here the new hierarchical bases replace the classical ones.
- Then, the equation that links y_h and z_h is given in an implicit form (see (4.5) below) ; actually this is because we want the semidiscrete problem to have the same monoticity properties as the original one, which is a gradient-like system.

Let us now present the algorithm. Let V_h, W_h be as in Section 3. Let $u_h = y_h + z_h$ be the (nonlinear) Galerkin approximation of u defined as follows : (y_h, z_h) is the solution in $V_h \times W_h$ of

$$\left(\frac{dy_h}{dt}, \tilde{y}_h\right) + \nu((y_h + z_h, \tilde{y}_h)) + (R(y_h + z_h), \tilde{y}_h) = (f, \tilde{y}_h), \tag{4.4}$$

$$\nu((y_h + z_h, \tilde{z}_h)) + (R(y_h + z_h), \tilde{z}_h) = (f, \tilde{z}_h), \tag{4.5}$$

these relations holding for any $(\tilde{y}_h, \tilde{z}_h)$ in $V_h \times W_h$, and of initial condition

$$y_h(0) = u_{h,0},$$

where $u_{h,0}$ is for instance the L^2 projection of u_0 on V_h.

Remark : Because (4.5) is given in an implicit form, then some preliminary work will be needed to ensure that the application $y_h \to z_h$ is well defined and smooth ; this point is the aim of Section 4.3 below.

Before we go further on the properties of this algorithm, we indicate some hypotheses we need on the spaces V_h and W_h :

- $(H1)$ A density result :

$$\bigcup_{j\geq 0} V_{h_j} \text{ is dense in } H_0^1(\Omega).$$

- $(H2)$ The so-called enhanced Cauchy-Schwarz inequality : There exists $\delta > 0$, that is independent of h, such that for any y_h, z_h in $V_h \times W_h$ the following inequality holds

$$|((y_h, z_h))| \leq (1-\delta) \|y_h\| \|z_h\|. \tag{4.6}$$

- $(H3)$ The L^{2p} version of the enhanced Cauchy-Schwarz inequality : There exists $\delta_p > 0$, that is independent of h, such that for any y_h, z_h in $V_h \times W_h$ the following inequality holds

$$\delta_p \left(|y_h|_{2p}^{2p} + |z_h|_{2p}^{2p} \right) \leq |y_h + z_h|_{2p}^{2p}. \tag{4.7}$$

- $(H4)$ The improved Poincaré inequality : There exists C that is independent of h such that for any z_h in W_h we have

$$|z_h| \leq Ch \|z_h\|. \tag{4.8}$$

Let us explain why these assertions hold. $(H1)$ is well known. $(H3)$ follows from the fact that the L^2 projector onto V_h is uniformly bounded with respect to h as an operator acting on $L^{2p}(\Omega)$ (see Lemma 3.2). $(H4)$ follows from Propositions 3.2 and 3.3. $(H2)$ can be seen as a consequence of Lemma 3.1 and of (3.17) that actually show that the L^2 projector onto W_h is uniformly bounded as an operator acting on $H_0^1(\Omega)$. (For another proof, see Lemma 4.2 below).

4.3. The fixed point problem

Here we could apply a Banach fixed point theorem. But it is more complicated than the method we present below and it leads to a worse stability condition on h with respect to the parameter ν.

4.3.1. *A uniqueness result*

Let us denote by P_h (respectively Q_h) the projector in $L^2(\Omega)$ onto V_h (respectively W_h). Let us denote by P_h^* (respectively Q_h^*) the mapping from $H^{-1}(\Omega)$ into $L^2(\Omega)$ defined by

$$(Q_h^* u, v) = \langle u, Q_h v \rangle_{H^{-1}, H_0^1}$$

(respectively

$$(P_h^* u, v) = \langle u, P_h v \rangle_{H^{-1}, H_0^1})$$

for any u, v in $H^{-1}(\Omega) \times L^2(\Omega)$. Hence we can rewrite (4.5) as

$$\nu Q_h^* A(y_h + z_h) + Q_h^*(R(y_h + z_h) - f) = 0. \tag{4.9}$$

Now let z_h^1, z_h^2 be two elements of W_h that satisfy (4.9). We set

$$z_h = z_h^2 - z_h^1,$$

and we easily derive from (4.9) that z_h must satisfy

$$\nu \|z_h\|^2 + (R(y_h + z_h^2) - R(y_h + z_h^1), z_h) = 0. \tag{4.10}$$

On the other hand, we go back to the definition of R to prove without difficulty that there exists a constant C such that for any u, v in $L^{2p}(\Omega)$

$$(R(u) - R(v), u - v) + C|u - v|^2 \geq 0 \tag{4.11}$$

holds.

Hence we infer from (4.8), (4.10) and (4.11) that, if h satisfies

$$h < h_0(\nu) = C\nu^{1/2}, \tag{4.12}$$

where C is an absolute constant, then $z_h = 0$, that is the uniqueness result.

4.3.2. *An existence result*

We define T_h as the nonlinear mapping acting on the finite dimensional space W_h by

$$T_h(z_h) = \nu Q_h^* A(y_h + z_h) + Q_h^*(R(y_h + z_h) - f). \tag{4.13}$$

We write

$$\begin{aligned}(T_h(z_h), z_h) = \nu \|z_h\|^2 &+ (R(y_h + z_h) - R(y_h), z_h) \\ &+ \nu((y_h, z_h)) + R(y_h) - f, z_h).\end{aligned} \tag{4.14}$$

Hence we use (4.8), (4.11) to obtain from (4.14) :

$$(T_h(z_h), z_h) \geq \nu \|z_h\| \left[\|z_h\| \left(1 - \left(\frac{h}{h_0}\right)^2\right) - \left(\|y_h\| - \frac{\|f\|_* + \|R(y_h)\|_*}{\nu}\right)\right], \tag{4.15}$$

h_0 being as in (4.12). We then apply

LEMMA 4.1. *Let* $[\ ,\ ]$ *be a scalar product on* $\mathbb{R}^N$, *and* T *a continuous mapping on* $\mathbb{R}^N$ *such that*

$$[T(\xi), \xi] > 0 \text{ for } [\xi] = R;$$

then there exists ξ_0, $[\xi_0] \leq R$, *such that*

$$T(\xi_0) = 0.$$

Proof : See Chapter 12, Lemma 3 in [T4].

Therefore, if we assume that (4.12) holds, (4.8), (4.15) and Lemma 4.1 provide the existence result.

4.3.3. *A regularity result*

Let F_h be the nonlinear mapping from $V_h \times W_h$ into W_h defined by

$$F_h(y_h, z_h) = \nu Q_h^* A(y_h + z_h) + Q_h^*(R(y_h + z_h) - f). \tag{4.16}$$

F_h is obviously a smooth mapping and if we compute its partial differential with respect to z_h we obtain

$$D_{z_h} F_h(y_h, z_h) w_h = \nu Q_h^* A w_h + Q_h^* R'(y_h + z_h) w_h, \tag{4.17}$$

holding for any w_h in W_h.

On the other hand we observe (see (4.11)) that there exists a constant C such that, for any u in $L^{2p}(\Omega)$

$$R'(u) + C \geq 0 \tag{4.18}$$

holds for x almost everywhere in Ω. Therefore (4.17) and (4.18) lead to

$$(D_{z_h} F_h(y_h, z_h) w_h, w_h) + C\, |w_h|^2 \geq \nu \, \|w_h\|^2 . \tag{4.19}$$

Then, using (4.8) and assuming that assertion (4.12) holds, we apply the implicit function theorem to F_h and we deduce that the mapping $y_h \to z_h$ defined according to Sections 4.3.1 and 4.3.2 is of class C^∞.

4.3.4. *Conclusion*

PROPOSITION 4.1. *If* (4.12) *holds, for any* y_h *in* V_h *there exists a unique* $z_h = \phi_h(y_h)$ *in* W_h *satisfying* (4.9), *and moreover the mapping* ϕ_h *is of class* C^∞.

4.4. A priori estimates

Using the previous section, we rewrite (4.4), (4.5) as the O.D.E. in V_h

$$\frac{dy_h}{dt} + \nu P_h^* A(y_h + \phi_h(y_h)) + P_h^*(R(y_h + \phi_h(y_h)) - f) = 0, \tag{4.20}$$

implemented with the initial condition

$$y_h(0) = P_h u_0.$$

Owing to the smoothness of ϕ_h, the solution $y_h(t)$ of (4.20) exists, by classical results, on a maximal interval of time $[0, t_h)$. As usual the a priori estimates below show that actually $t_h = +\infty$.

4.4.1. *A priori estimates for finite time intervals*

We take $\tilde{y}_h = y_h$, $\tilde{z}_h = z_h$ in (4.4), (4.5) and we add these equations to obtain, setting $u_h = y_h + z_h$,

$$\frac{1}{2}\frac{d}{dt}\left|y_h\right|^2 + \nu\left\|u_h\right\|^2 + (R(u_h), u_h) \leq |(f, u_h)| \tag{4.21}$$

On the other hand, we go back to the definition of R and we observe that there exists a constant C such that, for any u in $L^{2p}(\Omega)$, the following inequality holds :

$$(R(u), u) + C \geq \frac{b_{2p-1}}{2}|u|_{2p}^{2p}. \tag{4.22}$$

We then use (4.6), (4.7), (4.21), (4.22) to obtain, by straightforward computations

$$\frac{d}{dt}\left|y_h\right|^2 + \nu\delta(\left\|y_h\right\|^2 + \left\|z_h\right\|^2) + b_{2p-1}\delta_p(|y_h|_{2p}^{2p} + |z_h|_{2p}^{2p}) \leq C, \tag{4.23}$$

where C is a constant that depends on $|\Omega|$ and $|f|$.

Then using classical methods (see [T1]) we deduce that :

- The sequence y_h remains in a bounded set of

$$L^\infty(0, +\infty; L^2(\Omega)). \tag{4.24}$$

- Both sequences y_h, z_h remain in a bounded set of

$$L^2(0, T; H_0^1(\Omega)) \cap L^{2p}([0, T] \times \Omega), \quad \forall T > 0. \tag{4.25}$$

4.4.2. *A priori estimates for large time intervals*

We consider the following discrete Lyapunov functional :

$$\Lambda_h(t) = \frac{\nu}{2}\left\|u_h(t)\right\|^2 + \int_\Omega G(u_h(t, x))dx - (f, u_h(t)) + K, \tag{4.26}$$

where G satsifies $G' = R$ and where the constant K is choosen such that

$$\Lambda_h(t) \geq \frac{\nu}{4}\left\|u_h(t)\right\|^2 + \frac{b_{2p-1}}{4}\left|u_h(t)\right|_{2p}^{2p}. \tag{4.27}$$

Then we use (4.4), (4.5) and the smoothness of ϕ_h to obtain that

$$\frac{d}{dt}\Lambda_h(t) + \left|\frac{dy_h}{dt}(t)\right|^2 = 0 \tag{4.28}$$

holds.

On the other hand, from (4.21), (4.22), we obtain that, for $r > 0$ and $t \geq 0$, we have

$$\int_t^{t+r} \Lambda_h(s)ds \leq Cr + C' \left|y_h(t)\right|^2. \tag{4.29}$$

We go back to (4.23) and we apply Lemma 5.1 of [T1] to see that we also have

$$|y_h(t)| \leq M_1$$

for t larger than a t_0^* which is independent of u_0, and for M_1 that is like M_0 in (4.3). We then apply this inequality to (4.29), and using the uniform Gronwall Lemma (see Lemma 1.1 in [T1]), we infer from (4.28), (4.29) that for $t \geq t_1$, $t_1 = t_0 + 1$ for instance,

$$\Lambda_h(t) \leq C < +\infty \tag{4.30}$$

holds for some absolute constant C.

We reinterpret (4.6), (4.7), (4.27), (4.30) by

PROPOSITION 4.2. *There exists $t_1 > 0$ such that for any u_0 in $L^2(\Omega)$, both sequences y_h, z_h remain in a bounded set of*

$$L^\infty(t_1, +\infty; H_0^1(\Omega) \cap L^{2p}(\Omega)). \tag{4.31}$$

4.5. Convergence results

In this section we shall prove that, when the mesh size h tends to 0 –being provided $h \leq h_0$ as above– we have

THEOREM 4.1. *Let u be the solution of problem* (4.1), (4.2). *Then for each $T > 0$*

$$u_h \to u \ \text{ in } L^2(0,T; H_0^1(\Omega)) \text{ strongly,} \qquad (4.33)$$

$$u_h \to u \ \text{ in } L^{2p}([0,T] \times \Omega) \text{ strongly,} \qquad (4.33)$$

and fro $T > t_1$, t_1 being as in Proposition 4.2,

$$u_h \to u \ \text{ in } L^\infty(t_1, +\infty; H_0^1(\Omega)) \text{ weakly star,} \tag{4.34}$$

$$u_h \to u \ \text{ in } L^q([t_1, T] \times \Omega) \text{ strongly, for } q < +\infty. \tag{4.35}$$

Proof. First we prove that

$$\frac{dy_h}{dt} = -P_h^*(Au_h + R(u_h) - f)$$

remains in a bounded set of

$$L^{2p/2p-1}(0,T;H^{-1}(\Omega)). \tag{4.36}$$

In order to establish (4.36) we use

LEMMA 4.2. *The operator P_h is uniformly bounded with respect to h as an operator acting on $H_0^1(\Omega) \cap L^{2p}(\Omega)$.*

Proof. It remains to prove the H_0^1 bound. For that purpose we introduce the elliptic projector P_h^1 that is the orthogonal projector in $H_0^1(\Omega)$ onto V_h. Then we assume that the following well-known error estimate holds :

$$\left|u - P_h^1 u\right| \leq Ch\|u\|. \tag{4.37}$$

Hence using (3.17) we have

$$\|P_h u\| \leq \left\|P_h^1 u\right\| + \frac{C}{h}\left|P_h^1 u - P_h u\right|. \tag{4.38}$$

We then observe that

$$\left|P_h^1 u - P_h u\right| \leq \left|P_h^1 u - u\right|$$

and therefore (4.37), (4.38) conclude the proof of Lemma 4.2.

Let us go back to the proof of (4.36). First, using (4.25) and Lemma 4.2 we observe that the sequence $P_h^* A u_h$ remains in a bounded set of $L^2(0,T;H^{-1}(\Omega))$. On the other hand, using (4.25), Lemma 4.2 and the fact that

$$R \;:\; L^{2p}(\Omega) \rightarrow L^{2p/2p-1}(\Omega)$$

is bounded, we obtain that the sequence $P_h^* R(u_h)$ remains in a bounded set of $L^{\frac{2p}{2p-1}}([0,T] \times \Omega)$. Therefore

$$\frac{dy_h}{dt} = -\nu P_h^* A u_h - P_h^* R(u_h) + P_h f$$

remains in a bounded set of $L^{2p/2p-1}(0,T;H^{-1}(\Omega))$.

From (4.25) and (4.36), applying a classical compactness argument (see [LM]) we observe that the sequence y_n remains in a compact set of $L^2(0,T;L^2(\Omega)) = L^2([0,T] \times \Omega)$. On the other hand, we infer from (4.8), (4.25) that z_h strongly tends to 0 in $L^2([0,T] \times \Omega)$.

Hence we have a subsequence of u_h, still denoted by u_h, such that :

- u_h weakly converges to a function u^* in $L^2(0,T;H_0^1(\Omega)) \cap L^{2p}([0,T] \times \Omega)$, and strongly in $L^2([0,T] \times \Omega)$.

- y_h converges to u^* in $L^{+\infty}(0,+\infty;L^2(\Omega))$ weak star.
- $R(u_h)$ weakly converges to a function φ in $L^{2p/2p-1}([0,T]\times\Omega)$.

To prove that $u=u^*$, and that therefore the whole sequence u_h converges to $u^*=u$ as written above, it is sufficient to check that actually $\varphi=R(u^*)$ holds. It can be done by classical compactness-monotonicity arguments : Let v be a test function. We set

$$0\leq X_h=\int_0^T(R(u_h)-R(v),u_h-v)dt+C\int_0^T|u_h-v|^2\,dt, \tag{4.39}$$

with C as in (4.11).

Using (4.4), (4.5) and the convergence results above we obtain

$$\limsup_{h\to 0} X_h\leq\int_0^T(\varphi-R(v),u^*-v)dt+C\int_0^T|u^*-v|^2\,dt. \tag{4.40}$$

We take $v=u^*-\lambda w,\ \lambda>0$, in (4.40) and we let $\lambda\to 0$ to obtain $\varphi=R(u^*)$.

The strong convergence result (4.32) is obtained as in [MT1] proving that

$$\begin{aligned}Y_h=\nu\int_0^T\|u-u_h\|^2\,dt+\int_0^T(R(u)-R(u_h),u-u_h)dt\\+C\int_0^T|u_h-u|^2\,dt+\frac{1}{2}|y_h(T)-u(T)|^2\end{aligned}$$

tends to zero when $h\to 0$. From $Y_h\to 0$ when $h\to 0$, we also derive that

$$\int_0^T(R(u_h),u_h)dt\to\int_0^T(R(u),u)dt$$

and that moreover

$$\int_0^T|u_h|_{2p}^{2p}\,dt\to\int_0^T|u|_{2p}^{2p}dt, \tag{4.41}$$

thanks to the weak convergence results in $L^{2p}([0,T]\times\Omega)$. We then apply Prop. III.30 in [B] to obtain (4.33).

On the other hand, (4.34) results from (4.31), and (4.35) is obtained as follows : first, (4.32) provides that u_h converges to u in $L^2([t_1,T]\times\Omega)$; moreover, thanks to (4.31), u_h remains in a bounded set of $L^q([t_1,T]\times\Omega)$ for any $q<+\infty$; therefore (4.35) results from classical interpolation results between the L^q's.

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