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Étude analytique et numérique de quelques problèmes à frontière libre et modèles de champ de phase, 1995

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A ma femme et mon fils Christine et Edwin ;

à mes parents, Alain et Liliane, et à mon frère Alan ;

à ma presque soeur Sandrine.

Une science uniquement faite en vue des applications est impossible ; les vérités ne sont fécondes que si elles sont enchainées les unes aux autres ; si l'on s'attache seulement à celles dont on attend un résultat immédiat, les anneaux intermédiaires manqueront, et il n'y aura plus de chaîne.

H. Poincaré

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ANALYTICAL AND NUMERICAL STUDY OF SOME FREE BOUNDARY VALUE PROBLEMS AND PHASE FIELD MODELS

Abstract.

In this thesis we study some free boundary problems and phase field models.

The first part of this work is devoted to the study of free boundary problems where the mean curvature explicitely appears in the expression of the problems. We consider in Chapter 1 the stationary flow of a viscous liquid known as Marangoni type flow. The main difficulty here is that the interface between liquid metal and air intersects the boundary of the domain. This leads us to introduce weighted Hölder spaces in order to prove the existence and uniqueness of a smooth solution.

In Chapter 2, we present the numerical study of a one phase Stefan problem with surface tension. The discretization of the heat equation in the liquid part uses a semi-implicit scheme in time and a finite element method in space based on an adaptative mesh algorithm. The computation of the discretized interface uses a front traking method.

The second part of this thesis bears on a study of phase field models from the point of view of dynamical systems. When some small parameters tend to zero, the solution of the Caginalp phase field model converges to the solution of the viscous Cahn-Hilliard equation or to that of the Cahn-Hilliard equation. The purpose here is to obtain related properties for the corresponding maximal attractors. We consider in Chapter 3 the case that the nonlinear function appearing in the equations is of polynomial type and prove that the corresponding maximal attractor is upper-semicontinuous. In Chapter 4 we extend these results to the case of a logarithmic nonlinearity.

Key words : Navier-Stokes equations ; Marangoni effect ; weighted spaces ; existence and uniqueness ; Stefan Problem ; finite element method ; front traking method ; system of second order ; nonlinear parabolic equations ; maximal attractors ; upper-semicontinuity.

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INTRODUCTION

L'objet de cette thèse est l'étude de problèmes à frontière libre et de modèles de transition de phase.

La première partie porte sur des problèmes où la courbure moyenne de la frontière libre apparait explicitement dans l'une des équations qui la décrivent. On étudie tout d'abord un problème à frontière libre stationnaire, connu sous le nom d'écoulement Marangoni. Notre approche s'inspire des travaux de Abergel et Bona [1], Solonnikov [15] dont le cadre plus simple est celui des équations de Navier-Stokes. La difficulté essentielle réside dans la présence d'une paroi rigide qui rencontre l'interface air-métal en des points anguleux. C'est pourquoi nous utilisons des espaces de Hölder pondérés pour démontrer l'existence et l'unicité d'une solution régulière.

On effectue ensuite l'étude numérique d'un problème de Stefan à une phase avec tension superficielle, problème qui apparait en théorie de la corrosion aqueuse. Une méthode d'éléments finis est utilisée pour la discrétisation de l'équation de diffusion dans la phase liquide et une méthode de discrétisation explicite en temps permet de calculer le déplacement du front. La triangulation évolue au cours du temps de telle sorte que l'interface discrète coïncide avec des côtés de triangles, ce qui permet d'obtenir des résultats numériques très précis.

Dans la deuxième partie, on s'intéresse au comportement pour les grands temps des solutions de modèles de champ de phase, c'est-à-dire de systèmes d'équations d'évolution non linéaires couplées pour la température et un paramètre d'ordre. Ces modèles constituent des approximations de problèmes à frontière libre, par exemple du problème de Stefan avec tension superficielle.

La théorie des systèmes dynamiques de dimension infinie développée par R. Temam [17] constitue la base de notre étude. Les modèles considérés étant fortement dissipatifs, ils possèdent un attracteur maximal, c'est-à-dire un ensemble compact, invariant par le flot et qui attire tous les bornés.

Le problème plus spécifique qui nous intéresse ici est d'expliciter la relation entre quelques-uns des modèles les plus standards en transition de phase et en particulier de montrer que l'on peut passer continûment des équations de champ de phase à l'équation de Cahn-Hilliard visqueuse et à l'équation de Cahn-Hilliard. B. Stoth [16] démontre la convergence des solutions du modèle de champ de phase vers celles de l'équation de Cahn-Hilliard pour des temps finis.

En nous inspirant de méthodes dues à Hale et Raugel [8], Debussche [5] et Debussche et Dettori [6], nous démontrons la semi-continuité supérieure de l'attracteur du modèle de champ de phase. Nous considérons tout d'abord le cas où la fonction non linéaire apparaissant dans les équations est de type polynomiale puis le cas où elle est logarithmique.

PREMIERE PARTIE
ETUDE ANALYTIQUE ET NUMERIQUE DE QUELQUES PROBLEMES A FRONTIERE LIBRE

1. Existence d'interfaces stationnaires régulières pour un écoulement de type Marangoni.

On considère l'écoulement stationnaire d'un fluide visqueux contenu dans une cavité et dont la surface libre est soumise à un flux thermique non uniforme. Ce type d'écoulement connu sous le nom d'écoulement Marangoni [4], [13] est caractérisé par l'existence simultanée d'un phénomène de convection libre au sein du fluide et d'une contrainte tangentielle localisée à l'interface. On suppose que l'écoulement satisfait l'approximation de Boussinesq et que l'interface est un graphe.

Le résultat principal de cet article donne l'existence et l'unicité d'interfaces stationnaires régulières dans un voisinage de la solution capillaire, c'est-à-dire la solution obtenue pour un flux nul et un fluide au repos, dans des espaces de Hölder pondérés [10]. Plus précisément, on montre que si les paramètres du problème sont différents d'une suite finie de nombres réels positifs, apparaissant comme une partie des valeurs caractéristiques d'un opérateur de Sturm-Liouville, et si le flux thermique reste suffisamment petit alors le problème possède une solution régulière unique.

Ce résultat est obtenu à l'aide d'un théorème de fonctions implicites. La difficulté essentielle provient de la présence de points anguleux sur la frontière du domaine occupé par le fluide. Les démonstrations s'appuient en particulier sur des résultats obtenus par V.A Solonnikov [14], [15] pour un problème de Stokes avec des points anguleux sur la frontière du domaine, et sur l'étude d'un problème de type Sturm-Liouville dans des espaces de Hölder pondérés. A notre connaissance, le résultat que nous présentons est le premier résultat mathématique ayant trait à l'existence et l'unicité de solutions régulières pour un écoulement de type Marangoni.

2. Etude numérique d'un problème de dissolution-croissance avec tension superficielle.

On considère un problème de Stefan en dimension deux d'espace. Ces équations modélisent l'évolution d'un système physique constitué d'une phase solide formée d'un seul composé en contact avec une phase liquide qui est une solution diluée de ce composé. On décrit et on implémente une méthode numérique simple permettant de suivre l'évolution morphologique de l'interface solide-liquide au cours du temps.

Plus précisément, on considère des problèmes aux limites associés aux équations

$$C_t = D\Delta C \qquad \text{dans la phase liquide ,} \tag{1}$$

$$D\frac{\partial C}{\partial \nu} = \left(\frac{1}{V} - C\right) V_\nu \qquad \text{sur l'interface } \Gamma_t \text{ ,} \tag{2}$$

$$V_\nu = \kappa V\left(C - \alpha \exp(\gamma K)\right) \qquad \text{sur l'interface } \Gamma_t \text{ ,} \tag{3}$$

où $C(x, y, t)$ est la concentration du composant dilué dans le fluide, ν est le vecteur normal à l'interface Γ_t, V_ν est la vitesse normale de l'interface et K sa courbure moyenne. Les paramètres D, V, κ et γ sont respectivement le coefficient de diffusion, le volume molaire du composant solide, une constante cinétique et un coefficient proportionnel à la tension superficielle.

La principale difficulté de ce problème est liée au fait que le domaine spatial varie au cours du temps. L'idée de la méthode est de résoudre successivement à chaque pas de temps l'équation (3) pour déplacer l'interface, puis les équations (1) et (2) pour déterminer la concentration dans le domaine ainsi obtenu.

Deux méthodes différentes ont été adaptées pour le calcul du déplacement de la frontière libre. La première dûe à Ikeda et Kobayashi [9], consiste à déplacer les noeuds de l'interface discrète en leur associant une direction normale approchée et une courbure moyenne approchée. Dans la seconde méthode, dûe à Roosen et Taylor [12], on déplace l'interface discrète en associant à chaque segment une courbure moyenne approchée.

Une discrétisation semi-implicite en temps est alors utilisée pour la résolution de l'équation de diffusion. L'approximation spatiale repose sur une méthode d'éléments finis de degré un dont la triangulation évolue au cours du temps de telle sorte que l'interface coincide avec des côtés de triangles.

Cette étude numérique permet de mieux comprendre les propriétés qualitatives de la solution. Elle montre en particulier que pour des valeurs physiques des paramètres du problème, aucune instabilité morphologique n'apparait. De plus, suivant la condition aux limites imposée sur le bord supérieur du domaine, la solution et la frontière libre convergent soit vers une constante dans le cas de conditions de Neumann, soit vers une onde progressive dans le cas de conditions de Dirichlet quand $t \to +\infty$. Enfin, dans le cas où la tension superficielle est nulle, des points singuliers peuvent apparaitre et se propager au cours du temps.

DEUXIEME PARTIE
SEMI-CONTINUITE SUPERIEURE POUR DES MODELES DE CHAMP DE PHASE

3. Une perturbation singulière des équations de Cahn-Hilliard et Cahn-Hilliard visqueuse.

On considère des problèmes aux limites associés à un modèle de champ de phase constitué d'un système de deux équations paraboliques non linéaires pour un paramètre d'ordre φ et la température u. Ces équations s'écrivent [2]

$$\begin{cases} \delta\varphi_t = \Delta\varphi - g(\varphi) + u & \text{dans } \Omega \times I\!R^+ \ , \quad (4) \\ \varepsilon u_t + \varphi_t = \Delta u & \text{dans } \Omega \times I\!R^+ \ , \quad (5) \end{cases}$$

où Ω est un ouvert borné de $I\!R^n$, $n \leq 3$ de frontière régulière $\partial\Omega$. On suppose que le terme non linéaire $g(\varphi)$ est de la forme

$$g(s) = \sum_{k=1}^{2p-1} a_k s^k \text{ avec } a_{2p-1} > 0 \text{ et } p \geq 2 \ ,$$

et que les fonctions φ et u satisfont des conditions aux limites de Dirichlet homogènes.

Dans le cas physique [3] où $g(s) = s^3 - s$, le problème ainsi obtenu modélise le processus de solidification d'un matériau à l'état liquide ; le paramètre d'ordre φ satisfait $\varphi \simeq 1$ dans la phase liquide, $\varphi \simeq -1$ dans la phase solide et $u = 0$ correspond à la température de fusion du matériau.

Si l'on pose $\varepsilon = 0$, dans le modèle de champ de phase et si l'on substitue l'équation (4) dans l'équation (5) on obtient un problème aux limites associé à l'équation de Cahn-Hilliard visqueuse [11] pour la seule inconnue φ

$$\varphi_t + \Delta\left(\Delta\varphi - g(\varphi) - \delta\varphi_t\right) = 0 \text{ dans } \Omega \times I\!R^+.$$

Quand $\delta = \varepsilon$ et $\varepsilon \downarrow 0$, la solution $(\varphi^\varepsilon, u^\varepsilon)$ du problème de champ de phase converge vers la solution d'un problème aux limites associé à l'équation de Cahn-Hilliard [16], [17]

$$\varphi_t + \Delta\left(\Delta\varphi - g(\varphi)\right) = 0 \text{ dans } \Omega \times I\!R^+ \ .$$

Les problèmes aux limites associés au système de champ de phase, à l'équation de Cahn-Hilliard visqueuse et à l'équation de Cahn-Hilliard possèdent des attracteurs globaux $\mathcal{A}^{\varepsilon\delta}$, $\mathcal{A}^{\delta}$ et $\mathcal{A}$ respectivement.
On démontre la semi-continuité supérieure de l'attracteur $\mathcal{A}^{\varepsilon\delta}$ d'abord en $(\varepsilon, \delta) = (0, \delta)$ pour $\delta > 0$ fixé, puis en $\varepsilon = \delta = 0$.

4. Une perturbation singulière de l'équation de Cahn-Hilliard avec non linéarité logarithmique.

On considère un modèle de champ de phase dont la non linéarité est de type logarithmique. Il s'agit du problème aux limites

$$(Q^\varepsilon)\quad \begin{cases} \varepsilon\varphi_t = \Delta\varphi + \alpha\varphi - g(\varphi) + u & \text{dans } \Omega \times I\!R^+ \ , \quad (6) \\ \varepsilon u_t + \varphi_t = \Delta u & \text{dans } \Omega \times I\!R^+ \ , \quad (7) \\ \varphi = u = 0 & \text{sur } \partial\Omega \times I\!R^+ \\ \varphi(x,0) = \varphi_0 \ , \ u(x,0) = u_0(x) & \text{dans } \Omega \end{cases}$$

où Ω est un ouvert borné de $I\!R^n$, $n \leq 3$ de frontière régulière $\partial\Omega$, $\alpha > 1$ une constante et où la fonction $g(\varphi)$ a pour expression

$$g(s) = \frac{1}{2} ln\left(\frac{1+s}{1-s}\right) \ .$$

On établit dans un premier temps l'existence et l'unicité d'une solution pour le Problème (Q^ε). L'idée essentielle de la démonstration est de considérer une suite de problèmes plus réguliers où la fonction g est remplacée par [6]

$$g_N(s) = \sum_{k=0}^{N} \frac{s^{2k+1}}{2k+1},$$

et de prouver des estimations à priori uniformes en N. On déduit également de ces estimations l'existence d' un attracteur maximal.

Si l'on pose $\varepsilon = 0$, dans le Problème (Q^ε) et si l'on substitue l'équation (6) dans l'èquation (7) on obtient le problème de Dirichlet associé à l'équation de Cahn-Hilliard [6]

$$\varphi_t + \Delta\left(\Delta\varphi + \alpha\varphi - g(\varphi)\right) = 0 \text{ dans } \Omega \times I\!R^+,$$

qui possède un attracteur maximal $\mathcal{A}_l$. On effectue alors dans le Problème (Q^ε) le changement de variable $v = \sqrt{\varepsilon}u$. On déduit des résultats démontrés pour (Q^ε) que ce nouveau problème est bien posé et qu'il admet un attracteur maximal $\mathcal{A}_l^\varepsilon$. On démontre que cet attracteur est semi-continu supérieurement en $\varepsilon = 0$.

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PREMIERE PARTIE

Etude analytique et numérique de quelques problèmes à frontière libre

Existence of smooth, stationary interfaces for Marangoni-type flows

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Abstract. We consider the motion of a fluid, the free surface of which is subject to a non-uniform thermal flux. We use an implicit function theorem in weighted Hölder spaces to prove the existence of a smooth interface for small values of the flux.

Résumé. Nous étudions l'écoulement stationnaire d'un liquide visqueux dont la surface libre est soumise à un flux de chaleur non uniforme. Nous prouvons, par un théorème de fonction implicite, l'existence et l'unicité d'une interface régulière dans des espaces de Hölder avec poids.

AMS codes : 35, 76

Key words : Navier-Stokes equations, Marangoni effect, free surface, weighted spaces, existence and uniqueness

Existence of smooth, stationary interfaces for Marangoni-type flow.

by
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1 Introduction

In this paper, we consider the flow of a viscous liquid, when its free surface is subject to a non uniform thermal flux. This type of flow, known as Marangoni type flow, see e.g [7], [12] and [14], is characterized by both a convection phenomenon within the fluid, and by an additional tangential stress acting at the interface.
Concerning the convection flow, we shall make the classical Boussinesq approximation for variation of the density, so that the really interesting terms in the modeling equations will be provided by the surface law.

Our main result concerns the existence and uniqueness of a stationary interface, in the neighbourghood of the capillary solution, i.e, one that is obtained when the flux is zero and the fluid is at rest. This result is obtained via an adequate implicit function theorem. Contrarily to similar works for the Navier-Stokes systems [2], [1], the main technical difficulties lie in the presence of corners at the boundary of the domain occupied by the fluid, and use is made of the machinery developped in [15], [16]. We also need to study Sturm-Liouville type problems in weighted spaces.

To our knowledge, this is the first mathematical result concerning the Marangoni flow, as far as existence and regularity are concerned. Moreover, we want to mention [4], where a numerical study is performed, and [3] where a bifurcation problem is studied.

The paper is organized as follows : Section 2 contains the main geometrical and functional hypotheses and notations; in Section 3, the physical problem is described, and its mathematical formulation given. Section 4 is devoted to the proof of our main result : after presenting the variations of domain suitable for the problem, we study in Section 4.3 Sturm-Liouville problems in weighted spaces, and finally prove in Section 4.4 the implicit function theorem that we need.

Essentially, we show that, when a certain parameter (the Marangoni number times the Froude number) does not belong to a finite set of real numbers, then, for small enough values of the thermal flux, there exists a unique stationary interface in weighted Hölder spaces.

2 Functional setting

Let Ω be an open bounded set of $I\!R^2$ whose boundary $\partial\Omega$ is a piecewise smooth Jordan curve, and denote by $\mathcal{S}(\Omega)$ the set of points where the tangent vector to $\partial\Omega$ is discontinuous.
We define the following functional spaces :

2.1 The space $\mathcal{C}^{m,\alpha}$ of α-hölderian functions of order m

Let m be a non-negative integer and $\alpha \in (0,1)$ a real number.
$\mathcal{C}^m(\Omega)$ will stand for the space of m-times continuously differentiable functions in Ω.
We then define the space of m-times continuously differentiable functions in Ω whose m-th derivatives satisfy a Hölder condition with exponent α.
We denote this space by $\mathcal{C}^{m,\alpha}(\Omega)$ and endow it with the norm :

$$|u|_{\mathcal{C}^{m,\alpha}(\Omega)} = \sum_{|k|\leq m} \underset{x\in\Omega}{Sup} \left|D^k u(x)\right| + \sum_{|k|=m} \underset{\substack{x,y\in\Omega \\ x\neq y}}{Sup} \frac{\left|D^k u(x) - D^k u(y)\right|}{|x-y|^\alpha} .$$

2.2 The space $\mathcal{C}^l_s$ of weighted hölderian functions of order $[l]$ and exponent $l-[l]$

For $x \in I\!R$, we define $[x]$ as the integer part of x.
For $y \in \Omega$ we set

$$d(y) := \underset{z\in\mathcal{S}(\Omega)}{Inf} \; ||y-z|| ,$$

where $||.||$ stands for the euclidian norm.
Let l, be a positive non-integer number, and s a non-integer real number.
For $s \in (0,l]$, we define the space $\mathcal{C}^l_s(\Omega,\mathcal{S}(\Omega))$ for which the following weighted norm is finite :

$$\begin{aligned} |u|_{\mathcal{C}^l_s(\Omega,\mathcal{S}(\Omega))} &= |u|_{\mathcal{C}^{[s],s-[s]}(\Omega)} + \sum_{s<|k|<l} \underset{x\in\Omega}{Sup} \left|d(x)^{|k|-s} D^k u(x)\right| \\ &+ \sum_{|k|=[l]} \underset{\substack{x,y\in\Omega \\ x\neq y}}{Sup} \left\{ Min(d^{l-s}(x), d^{l-s}(y)) \frac{\left|D^k u(x) - D^k u(y)\right|}{|x-y|^{l-[l]}} \right\} . \end{aligned}$$

and for $s<0$, we equip $\mathcal{C}^l_s(\Omega,\mathcal{S}(\Omega))$ with the norm :

$$
\begin{aligned}
|u|_{C^l_s(\Omega,\mathcal{S}(\Omega))} &= \sum_{0\le|k|<l} \ \underset{x\in\Omega}{Sup} \ \left|d(x)^{|k|-s} D^k u(x)\right| \\
&+ \sum_{|k|=[l]} \ \underset{\substack{x,y\in\Omega \\ x\neq y}}{Sup} \left\{ Min(d^{l-s}(x), d^{l-s}(y)) \frac{\left|D^k u(x) - D^k u(y)\right|}{|x-y|^{l-[l]}} \right\} .
\end{aligned}
$$

In the same way, and for all non-integer s and all positive non-integer l satisfying $s < l$, we define the space $C^l_s(I, \mathcal{S}(I))$ where I is a segment. $\mathcal{S}(I)$ is then composed of the endpoints of I.

2.3 Remarks - Notations

A quite complete study of weigthed hölderian spaces can be found in [13].
We now give some properties of weigthed hölderian spaces which will be useful later.
Let $l \in I\!R^{+*}$ and $s \in I\!R$ be two given non-integer numbers.

i- Let k be a positive integer satisfying $k \le [l]$. We suppose that $s \le l+k$. If $u \in C^{l+k}_s(\Omega, \mathcal{S}(\Omega))$ then $D^k u \in C^l_{s-k}(\Omega, \mathcal{S}(\Omega))$.

ii- We suppose that $s \le l$. If $u \in C^l_s(\Omega, \mathcal{S}(\Omega))$ then if U is a primitive of u, it satisfies $U \in C^{l+1}_{s+1}(\Omega, \mathcal{S}(\Omega))$.

iii- Let be given $(l_1, l_2) \in (I\!R^{+*})^2$ and $(s_1, s_2) \in I\!R^2$ satisfying $s_1 \le l_1$, $s_2 \le l_2$ on one hand, and $l_1 \le l_2$, $s_1 \le s_2$ on the other hand . Then if $u \in C^{l_1}_{s_1}(\Omega, \mathcal{S}(\Omega))$ and $v \in C^{l_2}_{s_2}(\Omega, \mathcal{S}(\Omega))$, we have $(u.v) \in C^{l_1}_{s_1}(\Omega, \mathcal{S}(\Omega))$. Moreover there exists a positive constant C, independant of u and v such that the following estimate holds
$|u.v|_{C^{l_1}_{s_1}(\Omega,\mathcal{S}(\Omega))} \le C |u|_{C^{l_1}_{s_1}(\Omega,\mathcal{S}(\Omega))} |v|_{C^{l_2}_{s_2}(\Omega,\mathcal{S}(\Omega))}$.

Notations
• In the following, when we will consider the space C^l_s , we will always assume that $l > Max(0, s)$.
• $|u|_\infty$ will stand for $\underset{x\in\Omega}{Sup} \ |u(x)|$.

3 Statement of the problem

3.1 The physical problem

We consider a container C, partially filled with a viscous liquid.The boundary of the container ∂C is composed of two half-lines $(x = -1, y > 0)$ and $(x = +1, y > 0)$ which are connected at the points (-1,0) and (+1,0) by a smooth curve ∂C^-.
∂C^- is supposed to lie in the half-plane $(y < 0)$.
The gravity $\vec{g}$, is directed along the vector $-\vec{j}(0, -1)$.

We denote by Ω the part of C containing the fluid. Ω is the open bounded set of $I\!R^2$ defined as :

$\Omega := \{M(x,y) \in C / y < f(x)\}$

and

$\Sigma := \{M(x,y) \in C / -1 < x < +1 \ and \ y = f(x)\}$

is the interface between the fluid and the atmosphere.

Let $\Gamma := \partial\Omega \backslash \Sigma$ and $\mathcal{S}(\Omega) := \overline{\Gamma} \bigcap \overline{\Sigma}$.

Let $I := (-1,+1)$ and $\mathcal{S}(I) := (\{-1\};\{+1\})$.

We suppose that the angles of contact between Γ and Σ are both equal to $\beta \in (0,\pi)$ with the convention that $\beta = 0$ if $\vec{t_\Sigma} = -\vec{j}$ at $x = -1$, where $\vec{t_\Sigma}$ is the unit tangent vector to Σ.

The flow is supposed to be stationary.

The interface is subject to a non uniform thermal flux $\mathcal{I}$. Thus, the temperature gradients induce, on one hand, a superficial stress which generates a Marangoni flow and, on the other hand, a volume force inside of the fluid, which generates a convection flow.

Several authors have considered the physical aspects of this problem, for example, [7], [12], or [14].

3.2 The mathematical formulation

We first make the two following assumptions.

> **A1-** We assume that the Boussinesq approximation holds, namely, on one hand, the external force satisfies $\rho_C(1 - a(T - T_C))\vec{g}$ and on the other hand, the mass density ρ is constant and equal to ρ_C in the volume of the fluid.
> Where $\rho_C = \rho(T_C)$, T_C is the temperature of the boundary of the container (ie. the rigid part of the boundary). T_C is supposed to be a constant, a is a given positive constant.
> **A2-** The surface tension coefficient γ is given as a non-increasing affine function of the temperature.

The boundary conditions are the followings :

> **BC1-** The fluid satisfies a no-slip condition at the boundary of the container.
> **BC2-** The temperature at the boundary of the container is constant and equal to T_C.
> **BC3-** The interface is subject to a non uniform thermal flux $\mathcal{I}$.
> **BC4-** The interface is in thermal and dynamical equilibrium.

We then write the conservation laws of mass, momentum and energy and obtain, in a dimensionless form, the following system of partial differential equations for the unknowns $(\vec{u}, \theta, \sigma, f, C)$:

$$
(S_\varepsilon)\left\{\begin{array}{lll}
\vec{\nabla}.\vec{u} = 0 & in\ \Omega & (1)\\
(\vec{u}.\vec{\nabla})\vec{u} - \vec{Div}\sigma(\vec{u}) - \lambda\theta\vec{j} = 0 & in\ \Omega & (2)\\
\vec{u}.\vec{\nabla}\theta - \frac{1}{Pr.Re}\Delta\theta = 0 & in\ \Omega & (3)\\
\vec{u} = 0\ and\ \theta = 0 & on\ \Gamma & (4)\\
\frac{\partial\theta}{\partial n} = -\varepsilon\mathcal{I}\ and\ \vec{u}.\vec{n} = 0 & on\ \Sigma & (5)\\
\sigma(\vec{u}).\vec{n}.\vec{t} - Ma^*\frac{\partial\theta}{\partial t} = 0 & on\ \Sigma & (6)\\
\sigma(\vec{u}).\vec{n}.\vec{n} + P_a + \frac{1}{Fr}f - Ma^*\alpha(\theta)\left(\frac{f_x}{\sqrt{1+f_x{}^2}}\right)_x - C = 0 & on\ \Sigma & (7)\\
\left(\frac{f_x}{\sqrt{1+f_x{}^2}}\right)(\pm 1) = \pm cos(\beta) & & (8)\\
\int_{-1}^{1}\sigma(\vec{u}).\vec{n}.\vec{n}dx = 0 & & (9)\\
\int_\Omega dxdy = V & & (10)
\end{array}\right.
$$

where :

• $\vec{u}, \theta, \sigma$ are respectively :
the velocity field, the temperature and the stress tensor with
$\sigma(\vec{u}) = -p.Id + \frac{2}{Re}\varepsilon(\vec{u})$
where $\varepsilon(\vec{u}) = \frac{1}{2}(\nabla\vec{u} + \nabla\vec{u}^t)$ and $p + \frac{y}{Fr}$ is the pressure

• C is a constant which has to be determined

• λ is a constant equal to
either 1 if we consider an evaporation problem
or -1 if we consider a condensation problem

• $\mathcal{I} = \mathcal{I}(x)$ is the non-uniform thermal flux

• ε is a small parameter

• P_a is the atmospheric pressure

• $\alpha(\theta) = \theta + \overline{\theta_0}$ is the surface tension, where $\overline{\theta_0}$ is a constant

• Pr is the Prandtl number (cinematic viscosity over diffusion coefficient)

• Re is the Reynolds number (characteristic velocity V^* of the flow times characteristic length L^* over the cinematic viscosity)

• Fr is the Froude number (square of the characteristic velocity over characteristic length times gravity field)

• $Ma^* = \left(\frac{d\gamma}{d\theta}\right)\frac{1}{\rho L^* V^{*2}}$ where $\frac{d\gamma}{d\theta}$ is the derivative of the surface tension with respect to the temperature θ, and ρ is the mass density. Remark that Ma^* is related to the usual Marangoni number Ma, with $Ma^* = \frac{-Ma}{Pr.Re^2}$

• V is the volume occupied by the fluid.

Remark 1 Equations (6) and (7) result from the local decomposition of the vector equation

$$\sigma(\vec{u})\vec{n} + (P_a + \frac{1}{F_T} f)\vec{n} - C\vec{n} + Ma^* \left(\alpha(\theta)H\vec{n} - \nabla\alpha(\theta)\vec{t}\right) = 0 \quad ,$$

which express the dynamical equilibrium of the interface.
$(\sigma(\vec{u}) + P_a + \frac{1}{F_T} f)\vec{n}$ is the superficial density of force describing the action of the fluid on the interface.
$(\alpha(\theta)H\vec{n})$ is, following the classical interpretation of [11], the first variation of the surface energy, and it measures the forces to apply to obtain a deformation at the interface.
Finally, $\left(-\nabla\alpha(\theta)\vec{t}\right)$ is induced by the variation of the surface tension and characterizes the Marangoni type flows.

4 Well-posedness in the neighbourhood of the Capillary solution

4.1 The main result

Our purpose is to prove the following result :

Theorem 4.1 *If Γ belongs to C^{l+2} then :*
a/ There exist a finite sequence of real numbers,
$0 < \lambda_1 < \cdots < \lambda_K < +\infty$
and real numbers $\tilde{V} > 0$, $\varepsilon_0 > 0$ and $s_0 \in (0;1]$ such that if $Ma^* Fr\overline{\theta_0} \neq \lambda_i$ for all $i = 1, \cdots, K$ then

$\forall\, |\varepsilon| < \varepsilon_0$
$\forall s \in (0, s_0)$
$\forall V > \tilde{V}$
$\forall \mathcal{I} \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$
satisfying the condition $\mathcal{I}(M^{(i)}) = 0$
if $\beta = \frac{\pi}{2}$ with $\mathcal{S}(I) = \bigcup_{i=1,2}\{M^i\}$,

there exists a unique solution $(\vec{u}, \theta, \sigma, f, C)$ of (S_ε) satisfying :

$$\begin{array}{l} \vec{u} \in \left(\mathcal{C}^{l+2}_{s}(\Omega, \mathcal{S}(\Omega))\right)^2 \\ \theta \in \mathcal{C}^{l+2}_{s}(\Omega, \mathcal{S}(\Omega)) \\ \sigma \in \left(\mathcal{C}^{l+1}_{s-1}(\Omega, \mathcal{S}(\Omega))\right)^4 \\ f \in \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I)) \\ C \in I\!R. \end{array}$$

b/ Moreover, there exists a real number $\beta_* > \frac{\pi}{2}$ such that if the angle of contact β satisfies $0 < \beta < \beta_*$, then there exists a positive number $s_1 > 1$ such that the conclusion of a/ remains true for every $s \in (1, s_1)$ satisfying $s \leq l + 2$.

The rest of this section is devoted to the proof of Theorem 4.1.

4.2 A particular solution of the problem

Let us suppose $\varepsilon = 0$ and $V = V_0$, V_0 given, respectively in (5) and (10).
In this case, $(\vec{u}, \theta, \sigma) = (0, 0, constant)$ satisfy equations (1) to (6) and thanks to (9), $\sigma = 0$.
Moreover, integrating (7) with respect to x between -1 and +1, we deduce that $C_0 := C = P_a + \frac{V_0 - V_-}{2Fr} - Ma^*\overline{\theta_0}cos(\beta)$ so that (7) and (8) can be written as :

$$\left\{ \begin{array}{l} \frac{1}{Fr}f(x) - Ma^*\overline{\theta_0}\left(\frac{f_x(x)}{\sqrt{1+f_x(x)^2}}\right)_x = \frac{V_0 - V_-}{2Fr} - Ma^*\overline{\theta_0}cos(\beta)\ \forall x \in I \\ \frac{f_x(\pm 1)}{\sqrt{1+f_x(\pm 1)^2}} = \pm cos(\beta) \ , \end{array} \right. \tag{11}$$

where V_- is the volume of the fluid enclosed between the line $(y = 0)$ and the curve ∂C_-.
But $-1 < cos(\beta) < +1$, $\frac{1}{Fr} > 0$ and $Ma^*\overline{\theta_0} > 0$ and therefore thanks to the results of [8], [9] it is well known that there exists a unique solution $\tilde{g} \in \mathcal{C}^\infty(I)$ of (11). This solution is the so-called capillary solution, and determines the interface of the fluid at rest.

Remark 2 If $\tilde{g}$ is the solution of (11) for $V = V_0$, then $\tilde{g} + \frac{V_1}{2}$ is the solution of (11) for $V = V_0 + V_1$. Thus, we will choose V large enough, so that $\tilde{g}(x) \geq \overline{g} > 0$ for all $x \in [-1, +1]$, where $\overline{g}$ is a positive constant.
We will denote by $\tilde{V}$ the smallest volume such that the condition above can be satisfied.

We now state a first proposition :

Proposition 4.2 *Suppose $\beta \in (0, \pi)$ given.*
Let $V = V_0$ where $V_0 \in I\!R^{+*}$ is given and satisfies $V_0 \geq \tilde{V}$.
Then there exists a unique solution $(\vec{u_0}, \theta_0, \sigma_0, \tilde{g}, C_0)$ of problem (S_0) for $(\varepsilon = 0)$.
This solution satisfies :

$$\left\{ \begin{array}{l} (\vec{u_0}, \theta_0, \sigma_0) = (0, 0, 0) \\ C_0 = P_a + \frac{V_0 - V_-}{2Fr} - Ma^*\overline{\theta_0}cos(\beta) \\ \tilde{g} \in \mathcal{C}^\infty(I) \textit{ is the unique solution of (11).} \end{array} \right.$$

Proof of Proposition 4.2
It remains to prove the uniqueness of the solution of problem (S_0).
Multiplying the Equation (3) by θ and integrating over Ω_0, the open bounded set of $I\!R^2$ associated to $\tilde{g}$, we deduce, using (4), that $\theta \equiv 0$ in Ω_0.
Moreover, taking the scalar product of Equation (2) with $\vec{u}$, integrating over Ω_0,and

using Equations (4), (5) and (6), we deduce thanks to Korn's inequality that $\vec{u} \equiv 0$ in Ω_0.
Finally, we deduce from (9) that $\sigma(\vec{u}) \equiv 0$, which completes the proof of Proposition 4.2.

4.3 Preliminary result

We establish in this section a result that will be useful in the sequel.
We denote by $\tilde{g}$ the capillary solution of Equation (11) and we set $x_1 = -1$, $x_2 = +1$.
We introduce the two linear operators $\mathcal{L}$ and $\mathcal{B}$ defined for every function ρ of $C^2(I)$ as

$$\begin{aligned}
\mathcal{L}\rho(x) &:= \frac{\sqrt{1+\tilde{g}_x^2(x)}}{Fr}\rho(x) - Ma^*\overline{\theta_0}H(\rho(x)) \ , \\
\mathcal{B}\rho(x) &:= \left(\rho_x + \frac{\tilde{g}_x\tilde{g}_{xx}}{1+\tilde{g}_x^2}\rho\right)(x) \ ,
\end{aligned}$$

with

$$H(\rho) = \frac{\rho_{xx}}{1+\tilde{g}_x^2} - \frac{\tilde{g}_x\tilde{g}_{xx}}{(1+\tilde{g}_x^2)^2}\rho_x + \left(\frac{\tilde{g}_{xx}^2(1-3\tilde{g}_x^2)}{(1+\tilde{g}_x^2)^3} + \frac{\tilde{g}_x\tilde{g}_{xxx}}{(1+\tilde{g}_x^2)^2}\right)\rho \ .$$

These operators arise naturally in the linearization of Equation (11).
We are now able to deduce the following proposition

Proposition 4.3 *Let $l > 0$ and $s > 0$ be two real numbers satisfying $s \leq l+2$.*
There exists a finite sequence of real numbers
$0 < \lambda_1 < \cdots < \lambda_K < +\infty$
such that if $Ma^*Fr\overline{\theta_0} \neq \lambda_i$ for all $i = 1, \cdots, K$
then, for all $(a_1, a_2) \in I\!R^2$ and all $h \in C_{s-1}^{l+1}(I, \mathcal{S}(I))$, there exists a unique function $\rho \in C_{s+1}^{l+3}(I, \mathcal{S}(I))$ solution of :

$$\begin{cases} \mathcal{L}\rho(x) = h(x) \ \forall x \in I \\ \mathcal{B}\rho(x_i) = a_i \text{ with } i = 1, 2 \end{cases}$$

Proof of Proposition 4.3
Existence of a solution
We first assume that $h \in C^n(I)$ for some non-negative integer n.
Letting

$$\begin{aligned}
R(x) &= \frac{1}{1+\tilde{g}_x^2(x)} \ , \\
r(x) &= Exp\left(\frac{1}{2}\int_{-1}^{x} \frac{R_x}{R}(t)dt\right) \ ,
\end{aligned}$$

$$
\begin{aligned}
p(x) &= -\frac{r(x)}{R^{\frac{3}{2}}(x)}\,, \\
q(x) &= \frac{1}{2} r(x) \frac{R_{xx}}{R}(x)\,, \\
\tilde{h}(x) &= -h(x) \frac{r(x)}{Ma^* \overline{\theta_0} R(x)}\,, \\
\mu &= \frac{1}{Ma^* Fr \overline{\theta_0}}\,,
\end{aligned}
$$

the problem above can be written as a Sturm-Liouville problem :

$$
\begin{cases} (r(x)\rho_x(x))_x + (\mu p(x) - q(x))\,\rho = \tilde{h}(x) \\ \mathcal{B}\rho(x_i) = a_i \text{ (i=1,2)} \ ; \end{cases}
$$

thus, using for example the results of Churchill [5, p.260-264] or Ince [10, chap.IX], we deduce the existence of an unique function $\rho \in \mathcal{C}^{n+2}(I)$, provided μ is not a characteristic number $(\mu_i)_{i \in I\!N}$ of the Sturm-Liouville operator, namely a number μ for which the homogeneous system

$$
\begin{cases} (r(x)\rho_x(x))_x + (\mu p(x) - q(x))\,\rho = 0 \\ \mathcal{B}\rho(x_i) = 0 \text{ (i=1,2)} \ ; \end{cases}
$$

possess a non trivial solution. This yields the condition $Ma^* Fr \overline{\theta_0} \neq \lambda_i$ for all $i \in I\!N$ with $\lambda_i = \frac{1}{\mu_i}$.

But, on one hand $Ma^* \overline{\theta_0}$ and Fr are positive real numbers. Since, on the other hand, no more than a finite number of the λ_i are positive numbers [5, Th. 4 p267], we deduce the existence of the finite sequence mentioned above.

We now come back to the weighted hölderian spaces.

If $s > 1$, which implies $[s] - 1 \geq 0$, we have then $\mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I)) \subset \mathcal{C}^{[s]-1}(I)$, so that the existence of a solution follows.

If $0 < s < 1$, the previous inclusion does not hold; we use the density of $\mathcal{C}^k(I)$ in $\mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$ for all $k \geq [l] + 1$, to prove the existence.

Regularity of the solution

Letting

$$
\rho(x) = w(x) Exp\left(-\frac{1}{4}\int_{-1}^{x} \frac{R_x}{R}(t)dt\right) \ ,
$$

then, w satisfies the following ordinary differential equation

$$
(S_{var}) \quad \begin{cases} w_{xx}(x) + \varphi(x) w(x) = H(x) \ \forall x \in I \\ (w_x + B_i w)(x_i) = A_i \text{ with } i = 1,2 \ , \end{cases}
$$

where, using the previous notations, we have set :

$$
H(x) \ := \ -\frac{h(x)}{Ma^* \overline{\theta_0} R(x)} Exp\left(\frac{1}{4}\int_{-1}^{x} \frac{R_x}{R}(t)dt\right) \ ,
$$

$$
\begin{aligned}
\varphi(x) &:= -\frac{\mu}{R^{\frac{3}{2}}(x)} - \frac{3}{4}\frac{R_{xx}(x)}{R(x)} + \frac{3}{16}\frac{R_x^2(x)}{R^2(x)} , \\
B_i &:= -\frac{3}{4}\frac{R_x}{R}(x_i) \ \text{i=1,2} , \\
A_i &:= a_i Exp\left(\frac{1}{4}\int_{-1}^{x_i}\frac{R_x}{R}(t)dt\right) \ \text{i=1,2} ,
\end{aligned}
$$

We first consider the similar problem with constant coefficient, namely

$$
(S_{cst}) \ \begin{cases} z_{xx}(x) + b.z(x) = H(x) \ \forall x \in I \\ (z_x + B_i z)(x_i) = A_i \text{ with } i = 1,2 . \end{cases}
$$

Let us prove that $z \in \mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))$ for $H \in \mathcal{C}_{s-1}^{l+1}(I, \mathcal{S}(I))$.
For $x \in I$, z can be written as :

$$
z(x) = \gamma z_0(x) + \nu z_1(x) + \frac{1}{2\alpha_1}\int_{-1}^{x}\left(z_0(x-t) - z_1(x-t)\right)H(t)dt
$$

where :
$\alpha_1 = \sqrt{-b}$ if $b \leq 0$, $\alpha_1 = i\sqrt{b}$ else
$z_0(x) = e^{\alpha_1 x}$ and $z_1(x) = e^{-\alpha_1 x}$
γ and ν being the solution of the linear system

$$
\begin{cases} \alpha_1(\gamma e^{-\alpha_1} - \nu e^{\alpha_1}) + B_1(\gamma e^{-\alpha_1} + \nu e^{\alpha_1}) = A_1 \\ \alpha_1(\gamma e^{\alpha_1} - \nu e^{-\alpha_1}) + B_2(\gamma e^{\alpha_1} + \nu e^{-\alpha_1}) = A_2 - \tilde{A}_2 , \end{cases}
$$

with

$$
\tilde{A}_2 = \frac{1}{2}\int_{-1}^{1}\left\{\left(1+\frac{1}{\alpha_1}\right)e^{\alpha_1(1-t)} + \left(1-\frac{1}{\alpha_1}\right)e^{-\alpha_1(1-t)}\right\}H(t)dt .
$$

For notational convenience, we write z in the form

$$
z(x) = \psi(x) + \int_{-1}^{x}\phi(x,t)H(t)dt ,
$$

where

$$
\begin{aligned}
\psi(x) &:= \gamma z_0(x) + \nu z_1(x) , \\
\phi(x,t) &:= \frac{1}{2\alpha_1}\left(z_0(x-t) - z_1(x-t)\right) .
\end{aligned}
$$

Thus, using the fact that

$$
\begin{aligned}
\forall p \in I\!N , \ \frac{\partial^{2p}}{\partial x^{2p}}\phi(x,x) &= 0 , \\
\frac{\partial^{2p+1}}{\partial x^{2p+1}}\phi(x,x) &= \alpha_1^{2p} ,
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
&\forall p \in I\!N \text{ with } 2p+1 \leq [l]+3 , \\
&\frac{\partial^{2p}}{\partial x^{2p}}z(x) = \frac{\partial^{2p}}{\partial x^{2p}}\psi(x) + \sum_{k\in\mathcal{K}_1}\alpha_1^{2(p-1)-k}\frac{\partial^k}{\partial x^k}H(x) + \int_{-1}^{x}\frac{\partial^{2p}}{\partial x^{2p}}\phi(x,t)H(t)dt , \\
&\frac{\partial^{2p+1}}{\partial x^{2p+1}}z(x) = \frac{\partial^{2p+1}}{\partial x^{2p+1}}\psi(x) + \sum_{k\in\mathcal{K}_2}\alpha_1^{2p-1-k}\frac{\partial^k}{\partial x^k}H(x) + \int_{-1}^{x}\frac{\partial^{2p+1}}{\partial x^{2p+1}}\phi(x,t)H(t)dt ,
\end{aligned}
$$

where $\mathcal{K}_1 = \{0, 2, \cdots, 2p-2\}$ and $\mathcal{K}_2 = \{1, 3, \cdots, 2p-1\}$.

Thus, using the results of Remark 2.3, we deduce that
$H \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$ implies $z_{xx} \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$.

Then, if $s > 1$, we have $|z|_{\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I))} = |z|_\infty + |z_x|_\infty + |z_{xx}|_{\mathcal{C}^{l+1}_{s-1}(I,\mathcal{S}(I))}$ and $H \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$ implies that $|H|_\infty < +\infty$, so that

$$\begin{aligned} |z(x)| &\leq |\psi(x)| + \left| \int_{-1}^{x} \phi(x,t) H(t) dt \right| \\ &\leq |\psi|_\infty + 2\,|\phi|_\infty\,|H|_\infty \\ &< +\infty\ , \end{aligned}$$

and thus $|z|_\infty < +\infty$. In the same way, we deduce that $|z_x(x)| < +\infty$.
Thus, for $s > 1$, $z \in \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I))$

On the other hand, if $0 < s < 1$, we have $[s] = 0$, and thus

$$|z|_{\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I))} = |z|_\infty + |z_x|_\infty + \underset{\substack{x, y \in I \\ x \neq y}}{Sup} \frac{|z_x(x) - z_x(y)|}{|x-y|^s} + |z_{xx}|_{\mathcal{C}^{l+1}_{s-1}(I,\mathcal{S}(I))}\ .$$

But, for $0 < s < 1$,

$$\int_{-1}^{x} \frac{1}{d^{1-s}(t)} dt < +\infty\ ,$$

and $H \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$ implies $\underset{x \in I}{Sup}\ |d^{1-s}(x) H(x)| < +\infty$,

so that

$$\int_{-1}^{1} |H(t)|\,dt = \int_{-1}^{1} \frac{d^{1-s}(t)}{d^{1-s}(t)} |H(t)|\,dt$$

$$\leq \underset{t \in I}{Sup}\ |d^{1-s}(t) H(t)| \int_{-1}^{1} \frac{1}{d^{1-s}(t)} dt < +\infty\ .$$

Thus, we deduce that

$$\begin{aligned} |z(x)| &\leq |\psi(x)| + \left| \int_{-1}^{x} \phi(x,t) H(t) dt \right| \\ &\leq |\psi|_\infty + |\phi|_\infty \int_{-1}^{1} |H(t)|\,dt \\ &< +\infty\ . \end{aligned}$$

and in the same way, we deduce that $|z_x|_\infty < +\infty$.
Finally, we have

$$\frac{|z_x(x)-z_x(y)|}{|x-y|^s}$$

$$= \frac{\left|\psi_x(x)-\psi_x(y)+\int_{-1}^{x}\frac{\partial}{\partial x}\phi(x,t)H(t)dt-\int_{-1}^{y}\frac{\partial}{\partial x}\phi(y,t)H(t)dt\right|}{|x-y|^s}$$

$$\leq \frac{|\psi_x(x)-\psi_x(y)|}{|x-y|^s}+\frac{\left|\int_{-1}^{x}\left(\frac{\partial}{\partial x}\phi(x,t)-\frac{\partial}{\partial x}\phi(y,t)\right)H(t)dt\right|}{|x-y|^s}$$
$$+\frac{\left|\int_{x}^{y}\frac{\partial}{\partial x}\phi(y,t)H(t)dt\right|}{|x-y|^s}$$

$$\leq 2^{1-s}\left|\psi_{xx}\right|_\infty+\int_{-1}^{x}|H(t)|\,dt \quad \underset{\substack{x,y\in I\\ x\neq y}}{Sup}\quad \underset{t\in I}{Sup}\ \frac{\left|\frac{\partial}{\partial x}\phi(x,t)-\frac{\partial}{\partial x}\phi(y,t)\right|}{|x-y|^s}$$
$$+\left|\frac{\partial}{\partial x}\phi\right|_\infty\frac{1}{|x-y|^s}\left|\int_{x}^{y}H(t)dt\right|$$

$$< +\infty\ .$$

Thus, $H\in\mathcal{C}^{l+1}_{s-1}(I,\mathcal{S}(I))$ implies that z is in $\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I))$, which completes the proof for the regularity of z solution of (S_{cst}).

We now come back to the problem (S_{var}).
Let Υ be the Green's function associated to problem (S_{cst}) and Π, that associated to problem (S_{var}).
We have for all $x\in I$,

$$\begin{aligned} z(x) &= \int_{-1}^{x}\Upsilon(x,t)H(t)dt\ , \\ w(x) &= \int_{-1}^{x}\Pi(x,t)H(t)dt\ . \end{aligned}$$

We then write w in the form

$$w(x)=\int_{-1}^{x}\left(\Pi(x,t)-\Upsilon(x,t)\right)H(t)dt+z(x)\ .$$

Thus, we want to prove that

$$F(x):=\int_{-1}^{x}\left(\Pi(x,t)-\Upsilon(x,t)\right)H(t)dt\in\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I))\ .$$

Using the properties of the Green's functions Υ and Π, see for example [6, p305], and in particular that $(\Upsilon-\Pi)(.,t)$ is $\mathcal{C}^1$ everywhere, and is a $\mathcal{C}^2$ function exept possibly for $x = t$, we deduce, proceeding as in the case of constant coefficient, that $H \in C^{l+1}_{s-1}(I, \mathcal{S}(I))$ implies that $F \in C^{l+3}_{s+1}(I, \mathcal{S}(I))$, and the proof of Proposition 4.3 is completed.

4.4 Proof of Theorem 4.1

To prove the Theorem 4.1, namely the well-posedness of the problem (S_ε) for small enough values of the parameter ε, we use an implicit function theorem.
Thus we consider the problem (S_ε) on a domain Ω_δ obtained as a perturbation of the domain Ω_0 determined by $\tilde{g}$.
We transform the problem (S_ε) on Ω_δ to a problem on the fixed domain Ω_0, and we then apply an implicit function theorem to a mapping defined on a space of suitable deformations.

4.4.1 The space $\mathcal{P}^{l,s}$ of admissible perturbations

Let $\delta \in I\!R^{+*}$.
In the following, Ω_0 will stand for the open bounded set of $I\!R^2$ determined by the capillary solution $\tilde{g}$ of (11).
For $s \in I\!R^{+*}$ and $l \in I\!R^{+*}$ satisfying $s \leq l + 2$ and $(\rho, \eta) \in \left(C^{l+3}_{s+1}(I, \mathcal{S}(I))\right)^2$, we define the transformation

$$M_0 \in \Sigma_0 \longmapsto M_\delta = M_0 + \delta \left(\rho \vec{n}_0 + \eta \vec{t}_0\right) \ ,$$

where $\vec{n}_0$ and $\vec{t}_0$ are respectively the outward unit normal and unit tangent vector to Σ_0 at M_0.
Σ_δ will stand for the set of the points M_δ so obtained.

(ρ, η) cannot, of course, be choosen in an arbitrary way.
Indeed, since the container has vertical rigid walls, $\overrightarrow{M_0 M_\delta}$ and $\vec{j}$ must satisfy a colinearity condition at $x = \pm 1$, namely :

$$\eta = \rho \tilde{g}_x \text{ at } x = \pm 1 \ .$$

Moreover, for simplicity, we choose tangential deformations η such that

$$\eta(x) = \rho(x).\tilde{g}_x(x) \ \forall x \in \overline{I} \ .$$

This choice amounts to considering deformations in the $\vec{j}$ direction only.

This kind of deformations is allowed because of the existence of a representation of Σ_0 in term of a function $\tilde{g}$ of x. For a more general case, namely for a parametric representation of Σ_0, the choice of the deformations is a little bit more complicated but remains possible. Indeed, we would then choose deformations in parametric forms, as graphs over Σ_0.

Then we denote by $\mathcal{P}^{l,s}(I)$, the space of admissible perturbations.

4.4.2 Perturbations of the open bounded set Ω_0

For $\rho \in \mathcal{P}^{l,s}(I)$, we set

$$\Sigma_\delta = \left\{ M_\delta \ / \ M_\delta = M_0 + \delta\rho\left(\vec{n}_0 + \tilde{g}_x \vec{t}_0\right) \ \forall M_0 \in \Sigma_0 \right\}$$

$$= \left\{ \ M_\delta\left(x_\delta, y_\delta\right) \ / \ \left(\begin{array}{l} x_\delta = x_0 \\ y_\delta = \tilde{g}(x_0) + \delta\rho(x_0)\sqrt{1+\tilde{g}_x^2(x_0)} \end{array} \right) \ \forall x_0 \in (-1,1) \ \right\}.$$

We then associate to ρ a global transformation $\chi = \chi(\rho)$ mapping Ω_0 onto Ω_δ and defined as

$$\begin{array}{rccl} \chi = \chi(\rho) : & \Omega_0 & \longrightarrow & \Omega_\delta \\ & M_0 & \longmapsto & \chi(M_0) \ , \end{array}$$

where

$$\begin{aligned} \chi(M_0) = \ & \varphi(x_0, y_0) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ & +(1-\varphi(x_0,y_0)) \begin{pmatrix} x_0 \\ \frac{y_0}{\tilde{g}(x_0)}\left(\tilde{g}(x_0) + \delta\rho(x_0)\sqrt{1+\tilde{g}_x^2(x_0)}\right) \end{pmatrix}, \end{aligned}$$

where φ is a smooth function such that

$$\left\{ \begin{array}{l} \varphi(x,y) = 0 \ \text{ if } \ h_1 < y \leq \tilde{g}(x) \\ 0 \leq \varphi(x,y) \leq 1 \text{ if } \ h_2 \leq y \leq h_1 \\ \varphi(x,y) = 1 \ \text{ if } \ y < h_2 \end{array} \right.$$

where $h_1 > h_2$ are two positive constants.
For δ small enough, we can define the inverse transformation χ^{-1}, which maps Ω_δ onto Ω_0, as

$$\begin{array}{rccl} \chi^{-1} = \chi^{-1}(\rho) : & \Omega_\delta & \longrightarrow & \Omega_0 \\ & M_\delta & \longmapsto & \chi^{-1}(M_0) \ , \end{array}$$

where

$$\begin{aligned} \chi^{-1}(M_\delta) = \ & \varphi(x_\delta, y_\delta) \begin{pmatrix} x_\delta \\ y_\delta \end{pmatrix} \\ & +(1-\varphi(x_\delta,y_\delta)) \begin{pmatrix} x_\delta \\ \dfrac{y_\delta \tilde{g}(x_\delta)}{\tilde{g}(x_\delta) + \delta\rho(x_\delta)\sqrt{1+\tilde{g}_x^2(x_\delta)}} \end{pmatrix}. \end{aligned}$$

Thus for δ small enough, say $\delta < \delta_0$, the transformations χ and χ^{-1} are as smooth as the function ρ.
Moreover, if $\rho \in \mathcal{B}^{l,s}(0, \frac{1}{\delta_0})$, the ball of $\mathcal{P}^{l,s}(I)$ centered at the origin and of radius $\frac{1}{\delta_0}$, we have

$$\begin{array}{l} \chi(\rho) \in \left(\mathcal{C}^{l+3}_{s+1}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2 \\ \chi^{-1}(\rho) \in \left(\mathcal{C}^{l+3}_{s+1}(\Omega_\delta, \mathcal{S}(\Omega_\delta))\right)^2 \end{array}$$

Definition :
Let us suppose that $0 < \delta < \delta_0$.
Ω_δ will be called a *perturbation* of the open bounded set Ω_0, if there exists a function $\rho \in \mathcal{B}^{l,s}(0, \frac{1}{\delta_0})$ such that $\Omega_0 = \chi^{-1}(\Omega_\delta)$.

4.4.3 Transformation of the problem on Ω_δ to a problem on the fixed domain Ω_0

Let Ω_δ be a *perturbation* of Ω_0.
Thus, we want to find $(\vec{u}_\delta, \theta_\delta, \sigma_\delta, f_\delta, C_\delta)$ solution of (S_ε) on Ω_δ.
But, Ω_δ is a *perturbation* of Ω_0, so that this problem is equivalent to finding $(\vec{u}, \theta, \sigma, f, C)$ solution of

$$\left\{\begin{array}{lll}
\check{\nabla}.\vec{u} = 0 & in & \Omega_0 \\
(\vec{u}.\check{\vec{\nabla}})\vec{u} - \overrightarrow{\check{Div}}\sigma(\vec{u}) - \lambda\theta\vec{j} = 0 & in & \Omega_0 \\
\vec{u}.\check{\vec{\nabla}}\theta - \frac{1}{Pr.Re}\check{\Delta}\theta = 0 & in & \Omega_0 \\
 & & \\
\vec{u} = 0 \ and \ \theta = 0 & on & \Gamma_0 \\
\frac{\partial\theta}{\partial\check{\vec{n}}} = -\varepsilon\mathcal{I} \ and \ \vec{u}.\check{\vec{n}} = 0 & on & \Sigma_0 \\
\sigma(\vec{u}).\check{\vec{n}}.\check{\vec{t}} - Ma^*\frac{\partial\theta}{\partial\check{\vec{t}}} = 0 & on & \Sigma_0 \\
\sigma(\vec{u}).\check{\vec{n}}.\check{\vec{n}} + P_a + \frac{1}{Fr}f - Ma^*\alpha(\theta)\left(\frac{f_x}{\sqrt{1+f_x{}^2}}\right)_x - C = 0 & on & \Sigma_0 \\
\frac{f_x(\pm 1)}{\sqrt{1+f_x^2(\pm 1)}} = \pm cos(\beta) & & \\
\int_{-1}^{1} \sigma(\vec{u}).\check{\vec{n}}.\check{\vec{n}}dx = 0 & & \\
\int_{\Omega_0} det(\mathcal{J})dxdy = V & &
\end{array}\right.$$

where
• $\mathcal{J}$ and $\mathcal{J}^{-1}$ respectively stand for the jacobian matrix associated to $\chi(\rho)$ and $\chi^{-1}(\rho)$
• $\check{\vec{\nabla}} = (\mathcal{J}^{-1})^t.\vec{\nabla}$, $\check{\Delta} = \check{\vec{\nabla}}^t.\check{\vec{\nabla}}$, $\overrightarrow{\check{Div}} = \check{\vec{\nabla}}^t$
and where we set :
• $\vec{u} = \vec{u}_\delta o\chi$
• $\theta = \theta_\delta o\chi$
• $\sigma = \sigma_\delta o\chi$
• $f = f_\delta o\chi$
• $\mathcal{I} = \mathcal{I}_\delta o\chi$
• $\check{\vec{n}}(M_0) = \vec{n}(\chi(M_0)) = \vec{n}(M_\delta)$
• $\check{\vec{t}}(M_0) = \vec{t}(\chi(M_0)) = \vec{t}(M_\delta)$.

4.4.4 Using the implicit function Theorem

We define the function $\mathcal{F}$ acting from
$I\!R \times (\tilde{V}, +\infty) \times \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I)) \times I\!R \times \left(\mathcal{C}^{l+2}_{s}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$
into
$\mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I)) \times I\!R \times I\!R \times \left(\mathcal{C}^{l+2}_{s}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2 \times I\!R^2$ as

$$\mathcal{F} : (\varepsilon, V, \rho, C, \vec{w}) \longmapsto \begin{pmatrix} \check{\mathcal{L}}(\varepsilon, V, \rho, \vec{w}) - C \\ \displaystyle\int_{\Sigma_0} \sigma(\vec{u}).\check{\vec{n}}.\check{\vec{n}} dx \\ \displaystyle\int_{\Omega_0} det(\mathcal{J}) dx dy - V \\ \vec{u} - \vec{w} \\ \dfrac{f_x(\pm 1)}{\sqrt{1 + f_x^2(\pm 1)}} \mp cos(\beta) \end{pmatrix},$$

where

$$\check{\mathcal{L}}(\varepsilon, V, \rho, \vec{w}) = \sigma(\vec{u}).\check{\vec{n}}.\check{\vec{n}} + P_a + \frac{1}{Fr} f - Ma^* \alpha(\theta) \left(\frac{f_x}{\sqrt{1 + {f_x}^2}} \right)_x,$$

and $(\vec{u}, \theta, \sigma)$ is the solution of the system

$$\begin{cases} \check{\nabla}.\vec{u} = 0 \ in \ \Omega_0 \\ (\vec{w}.\check{\vec{\nabla}})\vec{u} - \overrightarrow{\check{Div}\sigma(\vec{u})} - \lambda\theta\vec{j} = 0 \ in \ \Omega_0 \\ \vec{w}.\check{\vec{\nabla}}\theta - \dfrac{1}{Pr.Re}\check{\Delta}\theta = 0 \ in \ \Omega_0 \\ \vec{u} = 0 \ and \ \theta = 0 \ on \ \Gamma_0 \\ \dfrac{\partial\theta}{\partial\check{\vec{n}}} = -\varepsilon\mathcal{I} \ and \ \vec{u}.\check{\vec{n}} = 0 \ on \ \Sigma_0 \\ \sigma(\vec{u}).\check{\vec{n}}.\check{\vec{t}} - Ma^* \dfrac{\partial\theta}{\partial\check{\vec{t}}} = 0 \ on \ \Sigma_0 \ . \end{cases}$$

If we set $S_0 := (\varepsilon = 0, V = V_0, \rho = 0, C = C_0, \vec{w} = 0)$, it is clear that $\mathcal{F}(S_0) = 0$. The main result in the course of proving Theorem 4.1, is stated in the following proposition :

Proposition 4.4 *Let us suppose $s > 0$ and $l > 0$ given and satisfying $s \leq l + 2$. There exists a finite sequence of real numbers*
$0 < \lambda_1 < \cdots < \lambda_K < +\infty$
such that if $Ma^* Fr\overline{\theta_0} \neq \lambda_i$ for all $i = 1, \cdots, K$ then

a/ There exists an open neighbourhood W_0 of S_0 in $I\!R \times (\tilde{V}, +\infty) \times \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I)) \times$

$I\!R \times \left(\mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$ *such that* $\mathcal{F}$ *is a* $\mathcal{C}^1$ *function on* W_0.
b/ $D_{(\rho,C,\vec{w})}\mathcal{F}(S_0)$*, the Frechet derivative of* $\mathcal{F}$ *with respect to* $(\rho, C, \vec{w})$ *at* S_0*, is an isomorphism from* $\mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I)) \times I\!R \times \left(\mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$ *onto* $\mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I)) \times I\!R \times I\!R \times \left(\mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))\right)^2 \times I\!R^2$

Theorem 4.1 is a straightforward consequence of Proposition 4.4.
In order to prove Proposition 4.4, we need several auxiliary results.
Therefore, we introduce the following functional spaces

$$
\begin{aligned}
\mathrm{X} &:= \left(\mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))\right)^2 \times \mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0)) \times \left(\mathcal{C}^{l+1}_{s-1}(\Omega_0, \mathcal{S}(\Omega_0))\right)^4 / I\!R \\
\mathrm{Y} &:= \mathcal{C}^{l+1}_{s-1}(\Omega_0, \mathcal{S}(\Omega_0)) \times \left(\mathcal{C}^{l}_{s-2}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2 \times \mathcal{C}^{l}_{s-2}(\Omega_0, \mathcal{S}(\Omega_0)) \\
&\quad \times \left(\mathcal{C}^{l+2}_s(\Gamma_0, \mathcal{S}(\Gamma_0))\right)^2 \times \mathcal{C}^{l+2}_s(\Gamma_0, \mathcal{S}(\Gamma_0)) \times \mathcal{C}^{l+2}_s(\Sigma_0, \mathcal{S}(\Sigma_0)) \\
&\quad \times \mathcal{C}^{l+1}_{s-1}(\Sigma_0, \mathcal{S}(\Sigma_0)) \times \mathcal{C}^{l+1}_{s-1}(\Sigma_0, \mathcal{S}(\Sigma_0)) \ ,
\end{aligned}
$$

and for $S = (\varepsilon, V, \rho, C, \vec{w}) \in I\!R \times (\bar{V}, +\infty) \times \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I)) \times I\!R \times \left(\mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$ we define the linear operator $A = A(S)$ such that

$$
\begin{aligned}
A(S): \ & \mathrm{X} \longrightarrow \mathrm{Y} \\
& (\vec{u}, \theta, \sigma) \longmapsto A(S)(\vec{u}, \theta, \sigma) \ ,
\end{aligned}
$$

where

$$
A(S)(\vec{u}, \theta, \sigma) = \begin{pmatrix}
\check{\nabla}.\vec{u} \\
(\vec{w}.\check{\vec{\nabla}})\vec{u} - \overrightarrow{\check{Div}}\sigma(\vec{u}) - \lambda\theta\vec{j} \\
\vec{w}.\check{\vec{\nabla}}\theta - \dfrac{1}{Pr.Re}\check{\Delta}\theta \\
\vec{u}_{|\Gamma_0} \\
\theta_{|\Gamma_0} \\
\vec{u}.\check{\vec{n}}_{|\Sigma_0} \\
\left(\dfrac{\partial\theta}{\partial\check{n}} + \varepsilon\mathcal{I}\right)_{|\Sigma_0} \\
\left(\sigma(\vec{u}).\check{\vec{n}}.\check{\vec{t}} - Ma^*\dfrac{\partial\theta}{\partial\check{t}}\right)_{|\Sigma_0}
\end{pmatrix} .
$$

Then, we consider the following problem

$$
\left|
\begin{array}{l}
\text{Given } F = (f_i)_{i=1}^8 \in \mathrm{Y} \\
\text{find } (\vec{u}, \theta, \sigma) \in \mathrm{X} \text{ such that} \\
A(S_0)(\vec{u}, \theta, \sigma) = F.
\end{array}
\right.
$$

This problem can be formulated as

Given $F = (f_i)_{i=1}^8 \in \mathrm{Y}$

find$(\vec{u}, \theta, \sigma) \in \mathrm{X}$ such that

$$\left\{\begin{array}{rll} -\dfrac{1}{Pr.Re}\Delta\theta = f_3 & \text{in } \Omega_0 & (12) \\ \theta = f_5 & \text{on } \Gamma_0 & (13) \\ \dfrac{\partial\theta}{\partial n} = f_7 & \text{on } \Sigma_0 & (14) \end{array}\right.$$

and

$$\left\{\begin{array}{rll} \nabla.\vec{u} = f_1 & \text{in } \Omega_0 & (15) \\ -\overrightarrow{Div}\sigma(\vec{u}) = \lambda\theta\vec{j} + f_2 & \text{in } \Omega_0 & (16) \\ \vec{u} = \vec{f_4} & \text{on } \Gamma_0 & (17) \\ \vec{u}.\vec{n} = f_6 & \text{on } \Sigma_0 & (18) \\ \sigma(\vec{u}).\vec{n}.\vec{t} = Ma^*\dfrac{\partial\theta}{\partial t} + f_8 & \text{on } \Sigma_0. & (19) \end{array}\right.$$

Let Ω be an open bounded set of $I\!R^2$. Let Γ be a part of $\partial\Omega$ with endpoints $\mathcal{S}(\Omega) = \bigcup\limits_i \left\{M^{(i)}\right\}$.

Let us suppose that there exist neighbourhoods of $M^{(i)}$ and functions g_i such that, in the local coordinates system $\left(M^{(i)}, \vec{n}(M^{(i)}), \vec{t}(M^{(i)})\right)$, Γ can be represented as $y = g_i(x)$ for $0 \leq x \leq \zeta_i$.
We will say that $\Gamma \in \mathcal{C}^l_s$ for $s \in (0, l)$, if Γ belongs to $\mathcal{C}^{[l], l-[l]}$ and if $g_i \in \mathcal{C}^l_s((0, \zeta_i), \{0\})$.

We have the two following results

Lemma 4.5 *a/ Let $\Gamma_0 \in \mathcal{C}^{l+2}_{a+1}$ and $\Sigma_0 \in \mathcal{C}^{l+3}_{a+1}$ with $a \in (0,1)$.*
There exists a real number $\alpha_0 > 0$, such that if $s \in (0, \alpha_0)$ then, for all $(f_3, f_5, f_7) \in \mathcal{C}^l_{s-2}(\Omega_0, \mathcal{S}(\Omega_0)) \times \mathcal{C}^{l+2}_s(\Gamma_0, \mathcal{S}(\Gamma_0)) \times \mathcal{C}^{l+1}_{s-1}(\Sigma_0, \mathcal{S}(\Sigma_0))$ satisfying the compatibility condition $\dfrac{\partial f_5}{\partial n_\Sigma} = f_7$ at points of $\mathcal{S}(\Omega_0)$,
there exists a unique solution $\theta \in \mathcal{C}^{l+2}_s(\Omega_0, \mathcal{S}(\Omega_0))$ of Equations (12...14).
b/ Moreover, there exists a real number α_1 satisfying
$\alpha_1 > Max(\alpha_0, \frac{3}{2})$ such that, if $\Gamma_0 \in \mathcal{C}^{l+2}_s$ and $\Sigma_0 \in \mathcal{C}^{l+3}_{s+1}$ with $s \in (1, \alpha_1)$ and $s \leq l+2$, then the conclusion of a/ remains true.

Lemma 4.6 *a/ Let $\Gamma_0 \in \mathcal{C}^{l+2}_{a+1}$ and $\Sigma_0 \in \mathcal{C}^{l+3}_{a+1}$ with $a \in (0,1)$.*
There exists a real number $\overline{\alpha_0} > 0$, such that if $s \in (0, \overline{\alpha}_0)$ then :
if $\left(f_1, f_2 - \lambda\theta\vec{j}, \vec{f_4}, f_6, f_8 + Ma^*\frac{\partial}{\partial t}\theta\right) \in Y_1$, there exists a unique solution $(\vec{u}, \sigma) \in$

$\left(\mathcal{C}_s^{l+2}(\Omega_0,\mathcal{S}(\Omega_0))\right)^2 \times \left(\mathcal{C}_{s-1}^{l+1}(\Omega_0,\mathcal{S}(\Omega_0))\right)^4/I\!R$ *of Equations (15...19).*
b/ Moreover, there exists an angle β_* *such that, if* $\beta < \beta_*$ *then there exists a real number* $\overline{\alpha}_1$ *such that, if* $\Gamma_0 \in \mathcal{C}_s^{l+2}$ *and* $\Sigma_0 \in \mathcal{C}_{s+1}^{l+3}$ *with* $s \in (1,\overline{\alpha}_1)$ *and* $s \le l+2$*, then the conclusion of a/ remains true.*

Where Y_1 is the subspace of Y of functions $(g_1, g_2, \vec{g}_4, g_6, g_8) \in \mathcal{C}_{s-1}^{l+1}(\Omega_0,\mathcal{S}(\Omega_0)) \times \left(\mathcal{C}_{s-2}^{l}(\Omega_0,\mathcal{S}(\Omega_0))\right)^2 \times \left(\mathcal{C}_s^{l+2}(\Gamma_0,\mathcal{S}(\Gamma_0))\right)^2 \times \mathcal{C}_s^{l+2}(\Sigma_0,\mathcal{S}(\Sigma_0)) \times \mathcal{C}_{s-1}^{l+1}(\Sigma_0,\mathcal{S}(\Sigma_0))$ satisfying the compatibility conditions :

- $\displaystyle\int_{\Omega_0} g_1(x,y)dxdy = \int_{\Sigma_0} g_6(x)dx + \int_{\Gamma_0} \vec{g}_4.\vec{n}(x(\tau),y(\tau))d\tau$
- $g_6(M_i) = \vec{g}_4.\vec{n}_\Sigma(M_i)$
- If $\beta \neq \frac{\pi}{2}$

$$\begin{aligned} g_1(M_i) = {} & \frac{1}{sin(\beta)}\left[-\vec{n}_\Sigma\frac{\partial}{\partial t_\Gamma}\vec{g}_4 + cos(\beta)\left(\frac{\partial}{\partial t_\Sigma}g_6 - \vec{g}_4.\frac{\partial}{\partial t_\Sigma}\vec{n}_\Sigma\right)\right](M_i) \\ & + \frac{1}{cos(\beta)}\left[\vec{t}_\Sigma\frac{\partial}{\partial t_\Gamma}\vec{g}_4 + sin(\beta)\left(Re.g_8 - \frac{\partial}{\partial t_\Sigma}g_6 + \vec{g}_4.\frac{\partial}{\partial t_\Sigma}\vec{n}_\Sigma\right)\right](M_i) \end{aligned}$$

- If $\beta = \frac{\pi}{2}$

$$\left(-\vec{t}_\Sigma.\frac{\partial}{\partial t_\Gamma}\vec{g}_4\right)(M_i) = \left(Re.g_8 - \frac{\partial}{\partial t_\Sigma}g_6 + \vec{g}_4\frac{\partial}{\partial t_\Sigma}\vec{n}_\Sigma\right)(M_i)$$

where $\mathcal{S}(\Omega) = \displaystyle\bigcup_{i=1}^{2} \{M_i\}$ and where $\vec{t}_\Gamma$ is choosen so that $\vec{t}_\Gamma = -\vec{n}_\Sigma$ for $\beta = \frac{\pi}{2}$.

The proof of Lemma 4.6 is given in [16], and that of Lemma 4.5 follows step by step that of Lemma 4.6.

From these two Lemmas, we deduce the following corollary :

Corollary 4.7 *The assumptions on* Γ_0 *and* Σ_0 *are the same as in Lemmas 4.5 and 4.6.*
a/ There exists s_0 *with* $s_0 \in (0,1)$*, such that, for all* $s \in (0,s_0)$ *and all* $F \in \mathrm{Y}$ *satisfying the compatibility conditions of Lemma 4.5 and 4.6, there exists a unique solution* $(\vec{u},\theta,\sigma) \in \mathrm{X}$ *of Equations (12) to (19).*
b/ Moreover, if $\beta < \beta_*$*, there exists* $s_1 > 1$ *such that for all* $s \in (1,s_1)$ *satisfying* $s \le l+2$ *and all* $F \in \mathrm{Y}$ *satisfying the compatibility conditions of Lemmas 4.5 and 4.6, the conclusion of a/ remains true.*

Therefore, we deduce from the result above that $A(S_0)$ is an isomorphism from X onto the subspace of Y of functions satisfying the compatibility conditions of Lemmas 4.5 and 4.6.

We now give an other useful result.

Lemma 4.8 *Let $\delta_0 > 0$ be as in §4.4.2 .*
For $0 < \delta < \delta_0$,
$S_\delta \longmapsto A(S_\delta)$ is a C^1 function from $I\!R \times (\tilde{V}, +\infty) \times \mathcal{B}^{l,s}(0, \frac{1}{\delta_0}) \times I\!R \times \left(\mathcal{C}_s^{l+2}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$ into $L(\mathcal{X}, \mathcal{Y})$, space of linear continuous operators from $\mathcal{X}$ into $\mathcal{Y}$.

Proof of Lemma 4.8
The operator A is a linear function of $(\varepsilon, V, C, \vec{w})$.
Thus, we have to prove that, for all fixed $(\varepsilon, V, C, \vec{w})$ in $I\!R \times (\tilde{V}, +\infty) \times I\!R \times \left(\mathcal{C}_s^{l+2}(\Omega_0, \mathcal{S}(\Omega_0))\right)^2$, the mapping

$$\tilde{A}(\bullet) = A(\varepsilon, V, \bullet, C, \vec{w}) \; : \; \begin{array}{rcl} \mathcal{B}^{l,s}(0, \frac{1}{\delta_0}) & \longrightarrow & L(\mathcal{X}, \mathcal{Y}) \\ \rho & \longmapsto & \tilde{A}(\rho) \; , \end{array}$$

is $\mathcal{C}^1$.
This mapping can be written as :
$\rho \longmapsto \chi^{-1}(\rho) \longmapsto \tilde{A}(\rho)$
Thus, we first consider the mapping $\rho \longmapsto \chi^{-1}(\rho)$.
Let $(\rho_n)_{n\in I\!N}$ be a sequence of $\mathcal{B}^{l,s}(0, \frac{1}{\delta_0})$ converging in $\mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))$ to ρ belonging to $\mathcal{B}^{l,s}(0, \frac{1}{\delta_0})$.
We have :

$$(\chi^{-1}(\rho_n) - \chi^{-1}(\rho))(x, y) = \begin{pmatrix} 0 \\ \delta\Psi_n(x, y)(\rho - \rho_n)(x, y) \end{pmatrix} ,$$

with $\delta\Psi_n$ bounded in $\mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))$ independently of n.
Thus, there exists $\tilde{C} > 0$ such that

$$|\chi^{-1}(\rho_n) - \chi^{-1}(\rho)|_{\mathcal{C}_{s+1}^{l+3}(\Omega_\delta, \mathcal{S}(\Omega_\delta))} \leq \tilde{C} |\rho_n - \rho|_{\mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))} ,$$

and thus

$$\begin{array}{rcl} \mathcal{B}^{l,s}(0, \frac{1}{\delta_0}) & \longrightarrow & \left(\mathcal{C}_{s+1}^{l+3}(\Omega_\delta, \mathcal{S}(\Omega_\delta))\right)^2 \\ \rho & \longmapsto & \chi^{-1}(\rho) \; , \end{array}$$

is a lipschitz continuous function.
Moreover, and in the same way, it is easily seen that
$\chi^{-1}(\rho_n) - \chi^{-1}(\rho) = L(\rho_n - \rho) + |\rho_n - \rho|_{\mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))} \xi(\rho_n - \rho)$ where $L(.)$ is a linear operator and where $\xi(\rho_n - \rho)$ goes to 0 in $\left(\mathcal{C}_{s+1}^{l+3}(\Omega_\delta, \mathcal{S}(\Omega_\delta))\right)^2$ as ρ tends to ρ_n in $\mathcal{C}_{s+1}^{l+3}(I, \mathcal{S}(I))$. Using for example the results of [2], we infer that

$$\begin{array}{rcl} \mathcal{B}^{l,s}(0, \frac{1}{\delta_0}) & \longrightarrow & \left(\mathcal{C}_{s+1}^{l+3}(\Omega_\delta, \mathcal{S}(\Omega_\delta))\right)^2 \\ \rho & \longmapsto & \chi^{-1}(\rho) \end{array}$$

is $\mathcal{C}^1$.
We now consider the mapping

$$\begin{array}{rcl}\left(\mathcal{C}^{l+3}_{s+1}(\Omega_\delta,\mathcal{S}(\Omega_\delta))\right)^2 & \longrightarrow & \mathrm{L}(\mathrm{X}\ ,\mathrm{Y}\ ) \\ \chi^{-1}(\rho) & \longmapsto & \tilde{A}(\rho)\ .\end{array}$$

$\tilde{A}$ depends on $\chi^{-1}(\rho)$ through its jacobian matrix $\mathcal{J}^{-1}$ in a polynomial way. Then, the fact that the function $\chi^{-1}(\rho) \longmapsto \mathcal{J}^{-1}$ is smooth from $\left(\mathcal{C}^{l+3}_{s+1}(\Omega_\delta,\mathcal{S}(\Omega_\delta))\right)^2$ into $\left(\mathcal{C}^{l+2}_{s-1}(\Omega_\delta,\mathcal{S}(\Omega_\delta))\right)^4$, make the proof of Lemma 4.8 complete.

We can now prove the Proposition 4.4.
The mapping $\mathcal{F}$ can be written

$$S \overset{\mathcal{F}_1}{\longmapsto} (\vec{u}_S,\theta_S,\sigma_S) \overset{\mathcal{F}_2}{\longmapsto} \mathcal{F}(S)(\vec{u}_S,\theta_S,\sigma_S),$$

where $(\vec{u}_S,\theta_S,\sigma_S)$ is solution of $A(S)(\vec{u},\theta,\sigma)=0$.
We set $Q_S := (\vec{u}_S,\theta_S,\sigma_S)$.
Thus, we first want to prove that there exists an open neighbourhood W_0 of S_0 in $I\!R\times(\tilde{V},+\infty)\times\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I))\times I\!R\times\left(\mathcal{C}^{l+2}_{s}(\Omega_\delta,\mathcal{S}(\Omega_\delta))\right)^2$ such that the mapping $\mathcal{F}_1$ is of class $\mathcal{C}^1$ on W_0.
Let $\mathcal{U} := I\!R\times(\tilde{V},+\infty)\times B^{l,s}(0,\frac{1}{\delta_0})\times I\!R\times\left(\mathcal{C}^{l+2}_{s}(\Omega_\delta,\mathcal{S}(\Omega_\delta))\right)^2$
Let Q_{S_0} be the solution of $A(S_0).Q_{S_0}=0$.
Let $\eta\in I\!R^{+*}$.
For $S\in\mathcal{U}$ such that $|S-S_0|_{\mathcal{U}}<\eta$, let Q_S be the solution of $A(S).Q_S=0$.
Then $A(S_0).[Q_{S_0}-Q_S]=(A(S)-A(S_0)).Q_S$.
so that, using Lemma 4.8 :
$A(S_0).[Q_{S_0}-Q_S]$
$= DA(S_0)(S-S_0).[Q_S-Q_{S_0}] + DA(S_0)(S-S_0).Q_{S_0} + |S-S_0|_{\mathcal{U}}\,\xi(S-S_0)$
where $DA(S_0)$ is the derivative of A at point S_0 and $\xi(S-S_0)$ tends to zero in Y when S tends to S_0 in $\mathcal{U}$.
Thus, using that

- $A(S_0)\in\mathrm{L}(\mathrm{X}\ ,\mathrm{Y}\ )$
- $S \longmapsto A(S)$ is a $\mathcal{C}^1$ function
- $(Q_{S_0}-Q_S)$ remains bounded in X ,

we deduce that, when S tends to S_0 in $\mathcal{U}$, then $A(S_0).[Q_{S_0}-Q_S]$ goes to zero in $\mathrm{L}(\mathrm{X}\ ,\mathrm{Y}\ )$.
Thus, using that $A^{-1}(S_0)$ remains bounded in $\mathrm{L}(\mathrm{Y}\ ,\mathrm{X}\ )$ thanks to Corollary 4.7, we deduce that $Q_S-Q_{S_0}=|S-S_0|_{\mathcal{U}}\,\xi(S-S_0)$.
Thus $Q_S = Q_{S_0} - A^{-1}(S_0)(DA(S_0)(S-S_0).Q_{S_0}) + |S-S_0|_{\mathcal{U}}\,\tilde{\xi}(S-S_0)$ where

$\tilde{\xi}(S-S_0)$ tends to zero in X when S tends to S_0 in $\mathcal{U}$, so that $D\mathcal{F}_1(S_0)=0$ and there exists an open neighbourhood W_0 of S_0 in $\mathcal{U}$ in which $S \longmapsto Q_S$ is C^1.

Using the expression of $\mathcal{F}(S).Q_S$, it is now easily seen that the mapping $Q_S \longmapsto \mathcal{F}(S).Q_S$ is C^1.

Taking into account the fact that $(\vec{w_0},\theta_0,\sigma_0)=(0,0,0)$, we can see that the derivative of $\mathcal{F}$ at S_0 evaluated at $(\rho, C, \vec{\overline{w}})$, denoted by $D_{(\rho,C,\vec{\overline{w}})}\mathcal{F}(S_0)(\rho,C,\vec{\overline{w}})$, has the following expression

$$\begin{pmatrix}
\tau.\vec{n_0}.\vec{n_0}+\frac{1}{F_r}\left(\rho\sqrt{1+\tilde{g}_x^2}+D_w f(\vec{\overline{w}})\right)- \\
-Ma^*\left(T\left(\frac{\tilde{g}_x}{\sqrt{1+\tilde{g}_x^2}}\right)_x-\overline{\theta_0}\left(D_\rho H(\tilde{g}).\rho+D_w H(\vec{\overline{w}})\right)\right)-C \\
\int_{\Sigma_0}\tau.\vec{n_0}.\vec{n_0}dx \\
\int_{-1}^{1}\rho\sqrt{1+\tilde{g}_x^2}dx \\
\vec{v}-\vec{\overline{w}} \\
\left(\sqrt{1+\tilde{g}_x^2}\left(\rho_x+\rho\frac{\tilde{g}_x\tilde{g}_{xx}}{1+\tilde{g}_x^2}\right)+\frac{\partial}{\partial x}(D_w f(w))\right)(\pm 1)
\end{pmatrix}$$

where we have set
$\vec{v}=D_\rho\vec{v}(\rho)+D_w\vec{v}(\vec{\overline{w}})$,
$T=D_\rho T(\rho)+D_w T(\vec{\overline{w}})$,
$\tau=D_\rho\tau(\rho)+D_w\tau(\vec{\overline{w}})$,
and where the derivative of the mean curvature H with respect to ρ at $\tilde{g}$, denoted by $D_\rho H(\tilde{g}).\rho$, is given by

$$D_\rho H(\tilde{g}).\rho=-\frac{\rho_{xx}}{1+\tilde{g}_x^2}+\frac{\tilde{g}_x\tilde{g}_{xx}}{(1+\tilde{g}_x^2)^2}\rho_x-\left(\frac{\tilde{g}_{xx}^2(1-3\tilde{g}_x^2)}{(1+\tilde{g}_x^2)^3}+\frac{\tilde{g}_x\tilde{g}_{xxx}}{(1+\tilde{g}_x^2)^2}\right)\rho\ .$$

$(\vec{v},T,\tau)$ are the solution of the linearized problem

$$(Pl_{S_0})\begin{cases}
\vec{\nabla}.\vec{v}=0 \ in\ \Omega_0 \\
-\overrightarrow{Div}(\tau)-\lambda T\vec{j}=0 \ in\ \Omega_0 \\
-\frac{1}{Pr.Re}\Delta T=0 \ in\ \Omega_0 \\
\vec{v}=0 \ and\ T=0 \ on\ \Gamma_0 \\
\vec{v}.\vec{n_0}=0 \ on\ \Sigma_0 \\
\frac{\partial T}{\partial n_0}=0 \ on\ \Sigma_0 \\
\tau.\vec{n_0}.\vec{t_0}-Ma^*\frac{\partial T}{\partial t_0}=0 \ on\ \Sigma_0.
\end{cases}$$

Thus, we are now going to prove that $D_{(\rho,C,\vec{w})}\mathcal{F}(S_0)$ defined above, is an isomorphism from $\mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I)) \times I\!R \times \left(\mathcal{C}^{l+2}_{s}(\Omega_0,\mathcal{S}(\Omega_0))\right)^2$ onto $\mathcal{C}^{l+1}_{s-1}(I,\mathcal{S}(I)) \times I\!R \times I\!R \times \left(\mathcal{C}^{l+2}_{s}(\Omega_0,\mathcal{S}(\Omega_0))\right)^2 \times I\!R^2$.

Therefore, we now show that, under the assumption that , $Ma^*Fr\overline{\theta_0} \neq \lambda_i$ for all $i = 1,\cdots,K$, there exists a unique solution
$(\rho,C,\vec{w}) \in \mathcal{C}^{l+3}_{s+1}(I,\mathcal{S}(I)) \times I\!R \times \mathcal{C}^{l+2}_{s}(\Omega_0,\mathcal{S}(\Omega_0))^2$ satisfying :

$$\begin{cases} L_1(\tau,T) + D_\rho \pounds(\rho) + L_2(\vec{w}) - C = h_1 \ \forall x \in I \\ \displaystyle\int_{\Sigma_0} \tau.\vec{n_0}.\vec{n_0}dx = C_1 \\ \displaystyle\int_{-1}^{1} \rho\sqrt{1+\bar{g}_x^2}dx = C_2 \\ \vec{v} - \vec{w} = \vec{h_2} \end{cases}$$

together with the condition
$\sqrt{1+\bar{g}_x^2}\left(\rho_x + \rho\frac{\bar{g}_x\bar{g}_{xx}}{1+\bar{g}_x^2}\right) + \frac{\partial}{\partial x}(D_w f(w)) = a_{\pm 1} \ at \ x = \pm 1.$

Where $(\vec{v},T,\tau)$ satisfies (Pl_{S_0}), and where we have set
$L_1(\tau,T) = \tau.\vec{n_0}.\vec{n_0} - Ma^*T\left(\frac{\bar{g}_x}{\sqrt{1+\bar{g}_x^2}}\right)_x$,
$D_\rho\pounds(\rho) = \frac{1}{Fr}\rho\sqrt{1+\bar{g}_x^2} + Ma^*\overline{\theta_0}D_\rho H(\rho)$,
$L_2(\vec{w}) = \frac{1}{Fr}D_w f(\vec{w}) + Ma^*\overline{\theta_0}D_w H(\vec{w})$.

First, we remark that for all $(h_1,C_1,C_2,\vec{h_2},a_1,a_{-1}) \in \mathcal{C}^{l+1}_{s-1}(I,\mathcal{S}(I)) \times I\!R \times I\!R \times \left(\mathcal{C}^{l+2}_{s}(\Omega_0,\mathcal{S}(\Omega_0))\right)^2 \times I\!R^2$, there exists a unique solution $(\vec{v},T,\tau)$ satisfying the system (Pl_{S_0}) :
$(\vec{v},T,\tau) = (0,0,constant)$.
Moreover, the condition $\int_{\Sigma_0} \tau.\vec{n_0}.\vec{n_0}dx = C_1$ yields $\tau = \frac{C_1}{Mes(\Sigma_0)}.Id$, where Id states for the identity matrix of $I\!R^4$.
Thus, $\vec{v} = 0$ together with equation $\vec{v} - \vec{w} = \vec{h_2}$, gives $\vec{w} = -\vec{h_2}$, so that in order to prove Proposition 4.4, we only need to study

$$\begin{cases} D_\rho\pounds(\rho) - C = h \ \forall x \in I & (20) \\ \displaystyle\int_{-1}^{1} \rho\sqrt{1+\bar{g}_x^2}dx = C_2 & (21) \\ \left(\rho_x + \rho\frac{\bar{g}_x\bar{g}_{xx}}{1+\bar{g}_x^2}\right)(x_i) = a_i \ for \ i = 1,2 & (22) \\ \qquad\qquad with \ x_1 = -1 & \\ \qquad\qquad\qquad x_2 = +1 \ . & \end{cases}$$

But assuming that ρ is known, then the equation (20) gives
$\rho = (D_\rho\pounds)^{-1}(h+C)$, so that Equation (21) gives C as the implicit solution of $\int_{-1}^{1}(D_\rho\pounds)^{-1}(h-C)(x)\sqrt{1+\bar{g}_x^2(x)}dx = C_2$.

Thus, it only remains to prove, for all function $h \in \mathcal{C}^{l+1}_{s-1}(I, \mathcal{S}(I))$ and all $(a_1, a_2) \in I\!R^2$, the existence of a unique function $\rho \in \mathcal{C}^{l+3}_{s+1}(I, \mathcal{S}(I))$ satisfying

$$\begin{cases} D_\rho \pounds(\rho) = h \ \forall x \in I \\ \left(\rho_x + \rho \dfrac{\tilde{g}_x \tilde{g}_{xx}}{1 + \tilde{g}_x^2}\right)(x_i) = a_i \ \ for \ i = 1, 2 \\ \qquad\qquad\qquad\qquad\qquad with \ x_1 = -1 \\ \qquad\qquad\qquad\qquad\qquad\qquad x_2 = +1 \end{cases}$$

but this is precisely the result given by the Proposition 4.3.
This ends the proof of Proposition 4.4.

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On a dissolution-growth problem with surface tension : a numerical study

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Abstract. We consider a one-phase Stefan problem with surface tension in space dimension two. We show how this problem arises from corrosion phenomena and present a numerical solution, based on a finite element method for the discretization in space and on two alternative methods for tracking the moving free boundary.

Résumé. Nous étudions un problème de Stefan à une phase avec tension superficielle en dimension deux d'espace. Après avoir montré comment ce problème modélise un phénomène de corrosion aqueuse, nous en donnons une résolution numérique basée sur la méthode des éléments finis pour la discrétisation spatiale et sur des techniques fines de déplacement de fronts.

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Key words : Stefan problem with surface tension, free boundary, finite element method, front tracking methods

On a dissolution-growth problem with surface tension : a numerical study

by

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1 Introduction

Many physical processes involve a solid phase in contact with a liquid phase. These phenomena are accompanied by a change of the geometry of the interface between the two phases. We are interested in the evolution in time of these interfaces. This paper is devoted to the study of a dissolution-growth process appearing in corrosion phenomena where typically a metal is in contact with a liquid and where the two phases evolve while exchanging mass. The physical model described in Section 2 leads us to study a one-phase Stefan problem in space dimension two for the concentration $C(x,y,t)$ of the chemical species passed into the liquid phase Ω_t and the interface Γ_t between solid and liquid.

In Section 2 we derive the following basic equations for C and Γ_t :

$$C_t = D\Delta C \qquad \text{in } \Omega_t \;, \tag{1}$$

$$D\frac{\partial C}{\partial \nu} = \left(\frac{1}{V} - C\right) V_\nu \qquad \text{on } \Gamma_t \;, \tag{2}$$

$$V_\nu = \kappa V \left(C - \alpha e^{\gamma K}\right) \qquad \text{on } \Gamma_t \;, \tag{3}$$

where ν is the unit normal vector to the interface Γ_t, V_ν is the normal velocity of Γ_t, K is its mean curvature, D is the diffusion coefficient, V is the molar volume of the solid compound, κ is a kinetic constant, α is the saturation concentration of the solution and γ is proportional to the surface tension of the interface Γ_t. Closely related models have been developed by Conrad & Cournil [6] and Gruy & Cournil [7]. The aim of this paper is to present a numerical solution for corresponding boundary value problems.

In Sections 3 and 4, we describe a numerical algorithm. An essential difficulty is the variation in time of the space domain. The idea is to successively solve at each time step the equation (3) for the interface motion and then the equations (1) and (2) for the concentration. We suppose that $\Gamma^n := \Gamma_{n\Delta t}$, $\Omega^n := \Omega_{n\Delta t}$ and $C^n := C(.,.,n\Delta t)$ are known and we want to compute Γ^{n+1} and C^{n+1}. We proceed in two steps :

(i) we use an explicit scheme for computing Γ^{n+1};

(ii) we discretize the equations (1) and (2) by means of a semi-implicit scheme in order to compute C^{n+1} on Ω^{n+1}.

Section 3 deals with step *(i)*. We adapt two different methods for tracking the interface. The first one, which is due to Ikeda & Kobayashi [8], consists in moving each point of the discrete interface by computing an approximate normal direction and an approximate value of the mean curvature at each vertex of the discrete interface. In the second method, which is due to Roosen [11] and Taylor [17], [18], one displaces the edges of the discrete interface by associating a mean curvature value to those edges.

The fact that we deal with a one-phase problem and that we do not known any phase field approximation makes it necessary for us to use a front tracking method rather than phase field computations as it is done by Caginalp & Socolovsky [5] or a level line method as presented for instance by Osher & Sethian [9] or Sethian [16].

In Section 4 we discretize the equation for the concentration. We use a semi-implicit scheme for the discretization in time and a finite element method with piecewise linear basis functions for the discretization in space. In this paper the triangularization varies in time so that the discrete interface coincides with edges of triangles at each time step, whereas some previous computations were performed with a fixed mesh [13].

We give numerical results in Section 5 and show how they are compatible with the qualitative properties of the solution. In particular in the case that a homogeneous Neumann boundary condition is given on the upper boundary of the space domain, which corresponds to the case of a closed physical system, one numerically verifies that the concentration converges to the saturation concentration α as $t \to +\infty$ and that the integral $\int_{\Omega_t} (1 - C(x,y,t))\, dxdy$, namely the total mass of the solid, is conserved in time. On the other hand in the case that a constant Dirichlet boundary condition is given on the moving upper boundary of the space domain, one observes that the concentration C and the interface Γ_t converge to a travelling wave solution. Finally we remark that in all the cases that we consider the free boundary does not develop dendrites and stabilizes for large time.

Boundary value problems associated to equations (1), (2) and (3) have also been study from an analytical point of view. For the local existence and uniqueness of a solution in the case that the interface is parametrized in the form $y = f(x,t)$, we refer to [14]. For the existence and uniqueness in the neighborhood of a stationary solution we refer to [1], and the local existence and uniqueness of the solution in the case that the interface is a smooth simply connected curve is proven in [2].

2 The physical model

2.1 The basic equations

We consider a system composed of a solid phase of a single compound and an incompressible liquid phase which is a dilute solution of that compound. The time evolution of this system induces mass transfer processes : a homogeneous one which consists in a diffusion process in the fluid and a heterogeneous one, namely a dissolution-growth process, located at the interface between solid and liquid.

Let Ω_t denote the liquid phase and Γ_t the interface between solid and liquid. Let $C(x,y,t)$ represent the concentration of the chemical species passed into solution, depending on the space variables (x,y) and on the time t.

The equations governing the evolution of the concentration and of the interface are deduced from the following physical laws.

i) *Mass transfer*
We suppose that the liquid is at rest and that at every point of the interface the volume decrease of the solid is exactly equal to the volume increase of the liquid, so that the convective velocity can be neglected. Moreover we also disregard all other fluxes (e.g. gravity induced flux, thermal flux, etc...) with respect to the diffusion flux. If we denote by J the diffusion flux, the first Fick's law gives

$$J = -D \text{ grad } C \ , \tag{4}$$

where D is the diffusion coefficient.

Let ω_t be an arbitrary subdomain which can be decomposed into ω_t^l, the liquid part and ω_t^s the solid part, i.e $\omega_t = \omega_t^l \cup \omega_t^s$. By the conservation of mass in ω_t, ω_t contains the same number of particules at each time t :

$$\frac{d}{dt}\left(\int_{\omega_t^l} C dxdy\right) + \frac{d}{dt}\left(\int_{\omega_t^s} \frac{1}{V} dxdy\right) = 0 \ , \tag{5}$$

where V is the molar volume of the solid compound, so that $\frac{1}{V}$ represents the concentration in the solid phase. Let ν^l (respectively ν^s) denote the inward unit normal to $\partial\omega_t^l$ (resp. $\partial\omega_t^s$) and $\mathcal{V}_{\nu^l}$ (resp. $\mathcal{V}_{\nu^s}$) denote the normal velocity of $\partial\omega_t^l$ (resp. $\partial\omega_t^s$). We deduce from (5) that

$$\int_{\omega_t^l} C_t dxdy - \int_{\partial\omega_t^l} C\mathcal{V}_{\nu^l} d\sigma - \int_{\partial\omega_t^s} \frac{1}{V}\mathcal{V}_{\nu^s} d\sigma = 0 \ . \tag{6}$$

Let V_ν denote the normal velocity of the interface and let $\tilde{\Gamma}$ be the part of the interface contained into ω_t, i.e. $\tilde{\Gamma} = \Gamma_t \cap \omega_t$.
Using (6) and the fact that $J.\nu^l = C\,\mathcal{V}_{\nu^l}$, we obtain

$$\int_{\omega_t^l} C_t dxdy + \int_{\tilde{\Gamma}} \left(\frac{1}{V} - C\right) V_\nu d\sigma - \int_{\partial\omega_t^l \setminus \tilde{\Gamma}} J.\nu^l d\sigma = 0 \ . \tag{7}$$

Equation (7) holds for any subdomain ω_t, in particular if we take $\omega_t = \omega_t^l$, that is $\omega_t^s = \emptyset$ or $\tilde{\Gamma} = \emptyset$, equation (7) reduce to :

$$\int_{\omega_t^l} C_t dxdy = \int_{\partial\omega_t^l} J.\nu^l d\sigma \ . \tag{8}$$

Moreover it follows from (4) that

$$\int_{\partial\omega_t^l} J.\nu^l d\sigma = -D\int_{\partial\omega_t^l} \frac{\partial C}{\partial\nu^l} d\sigma = D\int_{\omega_t^l} \Delta C\, dx dy \ , \tag{9}$$

and consequently that

$$\int_{\omega_t^l} C_t dx dy = D\int_{\omega_t^l} \Delta C\, dx dy \ .$$

Hence, we obtain the diffusion equation

$$C_t = D\Delta C \ , \tag{10}$$

in the liquid domain. Then, substituting (10) into (7), we deduce using (9) that

$$\int_{\hat{\Gamma}} \left(\frac{1}{V} - C\right) V_\nu d\sigma + \int_{\hat{\Gamma}} J.\nu d\sigma = 0 \ ,$$

where $\vec{\nu}$ is the unit normal to Γ_t directed towards the fluid. Therefore, we obtain the equation at the interface Γ_t

$$-J.\nu = D\frac{\partial C}{\partial \nu} = \left(\frac{1}{V} - C\right) V_\nu \ . \tag{11}$$

ii) *Dissolution-growth of the solid*

We suppose that the rate of dissolution or growth of the interface, follows the law $V_\nu = h(C,K)$, where h is a kinetic function, depending on the reaction pattern modified by the mean curvature K of the interface.

We consider an interface reaction of first-order and a Gibbs-Thomson law [10] to introduce dependency on the mean curvature. We suppose that the kinetic function is given by

$$h(C,K) = \kappa V\left(C - S_0 e^{\gamma K}\right) \ ,$$

where κ is a kinetic constant, S_0 is the saturation concentration of the solution and γ is proportional to the surface tension of the interface. With the particular choice of the kinetic function h, we obtain (3) which we substitute in (11) to obtain

$$D\frac{\partial C}{\partial \nu} = \kappa V\left(\frac{1}{V} - C\right)\left(C - S_0 e^{\gamma K}\right) . \tag{12}$$

The kinetic law (3) is valid only when the argument of the exponential is not too large which is verified in the experimental context where values of γK do not exceed 2. In practice, it turns out in the numerical experiments that if we choose an initial interface satisfying $\gamma K \simeq 2$, then γK remains of order 2 or less at all later times.

Finally we remark that the choice of the kinetic function h is not unique. Another type of interface reaction could have been used, for example a reaction of second order where the kinetic function h is a quadratic function of the concentration C [4].

2.2 Boundary and initial conditions

We consider two kinds of problems :

i) *The Neumann problem*

The upper boundary Σ of the domain Ω_t is a fixed plane $\{y = M\}$. The concentration C satisfies the Neumann condition $\frac{\partial C}{\partial \nu} = 0$ on Σ.

ii) *The Dirichlet problem*

The liquid domain Ω_t is a diffusion layer. The upper boundary $\Sigma_t = \{y = d(t) + M\}$ of Ω_t moves in such a manner that the area of the liquid domain Ω_t is conserved in time. The concentration C satisfies the Dirichlet condition $C = g$ on Σ_t, where g given.

Moreover we suppose that C and Γ_t are L-periodic in the x-direction. This assumption enables us to transform the problem into an equivalent one where the new domain, which we still denote by Ω_t, is bounded in the x-direction, with x varying in $(0, L)$. Then C and Γ_t satisfy periodicity conditions.

We assume that Γ_t does not have more than one point on $\{x = 0\}$, namely Γ_t can be parametrized by the x-coordinate in a neighborhood of $\{x = 0\}$. Without this assumption, the periodicity condition would have to involve all the points of Γ_t with x-coordinate zero. However the numerical study shows that if the initial interface is parametrized in the form $y = f_0(x)$, then the interface keeps being parametrized by x for all positive times, which justifies this assumption for the dissolution-growth problem.

Then the periodicity conditions are given by :

$$\begin{aligned} &C(0,y,t) = C(L,y,t) \\ &\frac{\partial C}{\partial x}(0,y,t) = \frac{\partial C}{\partial x}(L,y,t) \ , \end{aligned} \tag{13}$$

and if we parametrize the interface Γ_t by its arc length, i.e.

$$\begin{aligned} \Gamma_t : \ [0,l] &\longmapsto I\!R^2 \\ s &\longmapsto (x(s,t), y(s,t)) \ , \end{aligned}$$

then

$$\begin{aligned} &x(l,t) = x(0,t) + L, \quad y(l,t) = y(0,t) \\ &\frac{\partial x}{\partial s}(0,t) = \frac{\partial x}{\partial s}(l,t), \quad \frac{\partial y}{\partial s}(0,t) = \frac{\partial y}{\partial s}(l,t) \ . \end{aligned} \tag{14}$$

Finally, periodic initial conditions are given for C and Γ_t :

$\Gamma_{t=0} = \Gamma_0$ and $C(x,y,0) = C_0(x,y)$, $(x,y) \in \Omega_0$,

where Ω_0 is the initial liquid domain.

2.3 Dimensionless equations

In order to obtain dimensionless equations, we set

$$(\tilde{x},\tilde{y}) := \tfrac{\kappa}{D}(x,y) \;\; ; \;\; \tilde{s} := \tfrac{\kappa}{D}s \;\; ; \;\; \tilde{l} \;:=\; \tfrac{\kappa}{D}l \;\; ;$$
$$\tilde{t} := \tfrac{\kappa^2}{D}t \;\; ; \;\; \tilde{\gamma} := \tfrac{\kappa}{D}\gamma \;\; ; \;\; \alpha := VS_0 \;\; ;$$
$$\tilde{C}(\tilde{x},\tilde{y},\tilde{t}) := VC(x,y,t) \;\; ; \;\; \tilde{\Gamma}_{\tilde{t}}(\tilde{s}) := \tfrac{\kappa}{D}\Gamma_t(s) \; .$$

Some easy computations show that $\tilde{C}$ and $\tilde{\Gamma}_{\tilde{t}}$ satisfy the following rescaled equations, where the tildas have been omitted

$$(P_1) \left\{ \begin{array}{ll} C_t = \Delta C \;\text{ in } Q & (15) \\ \dfrac{\partial C}{\partial \nu} = (1-C)\left(C - \alpha e^{\gamma K}\right) \quad \text{on } \Gamma & (16) \\ \left\{ \begin{array}{l} \dfrac{\partial C}{\partial \nu} = 0 \;\text{ on } \Sigma \;,\; t>0 \\ \text{or} \\ C = g \;\text{ on } \Sigma_t \;,\; t>0 \end{array} \right. & \begin{array}{l} (17) \\ \\ (18) \end{array} \\ C \;\text{ satisfies (13)} & (19) \\ C(x,y,0) = C_0(x,y), \;\; (x,y) \in \Omega_0 & (20) \end{array} \right.$$

$$(P_2) \left\{ \begin{array}{ll} V_\nu = C - \alpha e^{\gamma K} \;\text{ on } \Gamma & (21) \\ \Gamma_t \;\text{ satisfies (14)} & (22) \\ \Gamma_{t=0} = \Gamma_0 \;, & (23) \end{array} \right.$$

where $Q = \{(x,y,t),\ (x,y) \in \Omega_t \;,\; t>0\}$ and $\Gamma = \{(x,y,t),\ (x,y) \in \Gamma_t \;,\; t>0\}$.

2.4 Some bounds on the concentration

With the scaling of Section 2.3, we have that

$$0 \le C_0(x,y) \le 1 \quad \text{for all } (x,y) \in \Omega_0 \;, \tag{24}$$

and

$$0 \le g \le 1 \; . \tag{25}$$

One can formally show, by means of the maximum principle that (24) and (25) imply a similar property for the concentration C, namely that

$$0 \le C \le 1 \quad \text{in } Q \; . \tag{26}$$

From now on we suppose that the conditions (24) and (25) are satisfied so that (26) is satisfied as well.

3 Discretization of the interface equation

This section is devoted to the numerical solution of the equation for the displacement of the interface

$$V_\nu = C - \alpha e^{\gamma K} \qquad \text{on } \Gamma_t \ .$$

We do so by means of an explicit scheme, namely

$$V_\nu = C^n - \alpha e^{\gamma K(n\Delta t)} \qquad \text{on } \Gamma^n,$$

where $\Gamma^n := \Gamma_{n\Delta t}$, $\Omega^n := \Omega_{n\Delta t}$ and $C^n := C(.,.,n\Delta t)$.

Since $V_\nu = \frac{\partial \Gamma_t}{\partial t}(n\Delta t).\vec{\nu}(n\Delta t)$, we compute the interface at time $t^{n+1} = (n+1)\Delta t$ by means of the formula

$$\Gamma^{n+1}.\vec{\nu}(n\Delta t) = \Gamma^n.\vec{\nu}(n\Delta t) + \Delta t V_\nu \ .$$

Hence, the knowledge of Γ^n and the computation of the normal velocity V_ν permit to determine Γ^{n+1}. We now present two methods for moving the interface and computing its curvature.

Let P_i^n be a point of the discretized interface at time $t^n = n\Delta t$ and let $C_i^n = C_h(P_i^n, n\Delta t)$, where C_h is obtained by a discretization in space of the function C (see Section 4 below). Let $\{P_i^n\}_{i=1}^{I+1}$ be $(I+1)$ points of Γ^n saisfying

$$\left.\begin{array}{rcl} x(P_1^n) = 0 & ; & x(P_{I+1}^n) = L; \\ y(P_{I+1}^n) & = & y(P_1^n) \\ \dfrac{y(P_2^n) - y(P_1^n)}{\left|\overrightarrow{P_1^n P_2^n}\right|} & = & \dfrac{y(P_{I+1}^n) - y(P_I^n)}{\left|\overrightarrow{P_I^n P_{I+1}^n}\right|} \end{array}\right\} \qquad (27)$$

The interface Γ^n is then approximated by

$$\Gamma_h^n = \left\{ \bigcup_{i=1}^{I} \left[P_i^n, P_{i+1}^n\right] ; \text{ satisfying } (27) \right\} \ ,$$

where the notation $y(P)$ stands for the y-coordinate of the point P, and with the convention that the liquid part lies on the left side of the interface when one follows Γ^n from $s = 0$ to $s = l$.

Let p be a positive integer and set $\Delta t_1 = \frac{\Delta t}{p}$. We also set $\Gamma_h^{n,0} = \Gamma_h^n$, $C_h^{n,0} = C_h^n$ and for $q = 1, ..., p$ we define

$$\Gamma_h^{n,q} = \left\{ \bigcup_{i=1}^{I} [P_i^{n,q}, P_{i+1}^{n,q}] ; \text{ satisfying the same as } (27) \right\} \ ,$$

and $C_h^{n,q} = C_h\left(P_i^{n,q}, n\Delta t\right)$.

We adapt two methods. In the first one, due to Ikeda and Kobayashi [8], one moves the points $P_i^{n,q}$ so that one has to compute at those points an approximate normal direction and an approximate value of the mean curvature. In the second method, due to Roosen and Taylor [11] (see also [17] [18]), one displaces the edges $[P_i^{n,q}, P_{i+1}^{n,q}]$ of the discrete interface and associates a normal direction and a mean curvature value to those edges.

3.1 Adaptation of the method of Ikeda and Kobayashi.

Motion of the discretized interface

For $q = 0, ..., p-1$, we obtain $\Gamma_h^{n,q+1}$ from $\Gamma_h^{n,q}$ by computing the displacement

$$\overrightarrow{P_i^{n,q} P_i^{n,q+1}} = \Delta t_1 \left[C_i^{n,q} - \alpha e^{\gamma K_{h,i}^{n,q}} \right] \vec{\nu}_{h,i}^{n,q} \tag{28}$$

for $i = 1, \cdots, I+1$, where $K_{h,i}^{n,q}$ is the curvature of the circle circumscribed about the triangle $(P_{i-1}^{n,q}, P_i^{n,q}, P_{i+1}^{n,q})$, and $\vec{\nu}_{h,i}^{n,q}$ is the unit vector at $P_i^{n,q}$ pointing into the liquid phase, and perpendicular to the segment $[P_{i-1}^{n,q}, P_{i+1}^{n,q}]$ as shown in Figure 1 below.

More precisely, $\vec{\nu}_{h,i}$ and $K_{h,i}$ are given by the formulas

$$\vec{\nu}_{h,i} = \frac{J.\overrightarrow{P_{i-1}P_{i+1}}}{\mid \overrightarrow{P_{i-1}P_{i+1}} \mid} , \tag{29}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$K_{h,i} = \frac{2 \det \left(\overrightarrow{P_i P_{i-1}} , \overrightarrow{P_i P_{i+1}} \right)}{\mid \overrightarrow{P_i P_{i-1}} \mid . \mid \overrightarrow{P_i P_{i+1}} \mid . \mid \overrightarrow{P_{i-1} P_{i+1}} \mid} . \tag{30}$$

In view of the orientation that we choose for the interface, we remark that the mean curvature $K_{h,i}$ is positive when the solid part enters the liquid one at point P_i.

Also note that the method is of order 1 for the computation of the normal and of the curvature. (see Appendix A).

Furthermore, we deduce $C_h^{n,q+1}$ from $C_h^{n,q}$ by the formula

$$C_i^{n,q+1} = C_h \left(P_i^{n,q+1}, n\Delta t \right) \simeq C_h \left(P_i^{n,q}, n\Delta t \right) + \overrightarrow{P_i^{n,q} P_i^{n,q+1}} . \overrightarrow{\nabla C_h} \left(P_i^n, n\Delta t \right) ,$$

in which we substitute (28) and in view of the interface condition (16), this yields

$$C_i^{n,q+1} = C_i^{n,q} + \Delta t_1 \left[C_i^{n,q} - \alpha e^{\gamma K_{h,i}^{n,q}} \right]^2 \left(1 - C_i^{n,q} \right) . \tag{31}$$

Hence, $\Gamma_h^{n+1} = \Gamma_h^{n,p}$ follows from solving (28) and (31) for $i = 1, \cdots, I+1$ and $q = 0, \cdots, p-1$.

The time step Δt_1 is chosen in order to avoid numerical instabilities. Moreover we have to control the length of line segments of $\Gamma_h^{n,q}$ in order to prevent some possible self-intersections of the interface. We do so as follows :

Control of the edges
Let $l_{max} > l_{min} > 0$ be two given real numbers.

1. If $\left|\overrightarrow{P_i^{n,q} P_{i+1}^{n,q}}\right| < l_{min}$, we consider the midpoint of $[P_i^{n,q}, P_{i+1}^{n,q}]$ as a new vertex and remove $P_i^{n,q}$ and $P_{i+1}^{n,q}$.

2. If $\left|\overrightarrow{P_i^{n,q} P_{i+1}^{n,q}}\right| > l_{max}$, we introduce a new vertex $P_{i+\frac{1}{2}}^{n,q}$. If $P_{i+\frac{1}{2}}^{n,q}$ would be taken as the midpoint of the edge, it would be a point with zero curvature. Thus, in order to avoid this problem we use an idea due to T.I. Seidman [15] (see Figure 2 below).
We suppose for simplicity that $K_{h,i}^{n,q} \neq 0$ and $K_{i+1}^{n,q} \neq 0$. Let $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ be the circles of radii R_i and R_{i+1} circumscribed respectively about the triangles $(P_{i-1}^{n,q}, P_i^{n,q}, P_{i+1}^{n,q})$ and $(P_i^{n,q}, P_{i+1}^{n,q}, P_{i+2}^{n,q})$. Let $\left(D_{i+\frac{1}{2}}\right)$ be the mediatrice line of $[P_i^{n,q}, P_{i+1}^{n,q}]$. Let $\mathcal{M}_i$ denote $\det\left(\overrightarrow{P_i^{n,q} P_{i-1}^{n,q}}, \overrightarrow{P_i^{n,q} P_{i+1}^{n,q}}\right)$ and $\mathcal{N}(P)$ denote $\det\left(\overrightarrow{PP_i^{n,q}}, \overrightarrow{PP_{i+1}^{n,q}}\right)$.
We then define
$N_i = \left\{\left(D_{i+\frac{1}{2}}\right) \bigcap \mathcal{C}_i\right\}$ with $\mathcal{M}_i \mathcal{N}(N_i) > 0$,
and
$N_{i+1} = \left\{\left(D_{i+\frac{1}{2}}\right) \bigcap \mathcal{C}_{i+1}\right\}$ with $\mathcal{M}_{i+1} \mathcal{N}(N_{i+1}) > 0$.
This condition means that N_i is the intersection point of the mediatrice line $D_{i+\frac{1}{2}}$ with the circle $\mathcal{C}_i$ and that the angles between the vectors $\overrightarrow{P_i^{n,q} P_{i-1}^{n,q}}$ and $\overrightarrow{P_i^{n,q} P_{i+1}^{n,q}}$ and between the vectors $\overrightarrow{N_i P_i^{n,q}}$ and $\overrightarrow{N_i P_{i+1}^{n,q}}$ have the same orientation. Finally, we set the new vertex $P_{i+\frac{1}{2}}$ as the midpoint of $[N_i, N_{i+1}]$.

As it is shown in Appendix B, one can check that the curvature corresponding to point $P_{i+\frac{1}{2}}$ is approximately equal to the average of the curvatures corresponding to the points $P_i^{n,q}$ and $P_{i+1}^{n,q}$.

3.2 Adaptation of the method of Roosen and Taylor.

We propose here an alternative formula for computing the curvature which was proposed by Angenent & Gurtin [3] and Taylor & al. [17], [18] in the case of crystals with polygonal edges.

Motion of the discretized interface
To each edge $[P_i, P_{i+1}]$, we associate $\vec{\nu}_{h,i+\frac{1}{2}}$, the unit normal to the edge pointing into the fluid, and the curvature $K_{h,i+\frac{1}{2}}$.
This is done according to the formulas :

$$\vec{\nu}_{h,i+\frac{1}{2}} = \frac{J.\overrightarrow{P_iP_{i+1}}}{|\overrightarrow{P_iP_{i+1}}|}, \tag{32}$$

and

$$K_{h,i+\frac{1}{2}} = -\frac{(\delta_{i,i-1}\, g_{i,i-1} + \delta_{i,i+1}\, g_{i,i+1})}{|\overrightarrow{P_iP_{i+1}}|}, \tag{33}$$

where

$$g_{i,j} = \frac{1 - \vec{\nu}_{h,i+\frac{1}{2}}.\vec{\nu}_{h,j+\frac{1}{2}}}{\left[1 - \left(\vec{\nu}_{h,i+\frac{1}{2}}.\vec{\nu}_{h,j+\frac{1}{2}}\right)^2\right]^{1/2}},$$

and

$$\begin{aligned} \delta_{i,j} &= +1 \text{ if they are adjacents and the solid enters} \\ &\qquad \text{into the fluid (concave case),} \\ &= -1 \text{ else.} \end{aligned}$$

Note that this method is of order 1 for the computations of the normal and the curvature (see Appendix A).

As mentioned above, the motion by mean curvature of the interface is implemented by moving each of its edges. Thus, w need another description of the discrete interface. We associate to Γ_h^n, the set $\Gamma_{\frac{1}{2}}^n = \left\{\left(P_{i+\frac{1}{2}}^n, l_{i+\frac{1}{2}}^n\right)\ i = 1, \cdots, I\right\}$, where $P_{i+\frac{1}{2}}^n = \frac{1}{2}(P_i^n + P_{i+1}^n)$ and $l_{i+\frac{1}{2}}^n = \left|\overrightarrow{P_i^nP_{i+1}^n}\right|$, and we define the vector $C_{\frac{1}{2}}^n$ of components $(C_{\frac{1}{2}}^n)_i := C_{i+\frac{1}{2}}^n = \frac{1}{2}(C_i^n + C_{i+1}^n)$, $i = 1, ..., I$.
In the same way, we define the set $\Gamma_{\frac{1}{2}}^{n,q}$ and the vector $C_{\frac{1}{2}}^{n,q}$ for $q = 1, \cdots, p$, and we set $\Gamma_{\frac{1}{2}}^{n,0} = \Gamma_{\frac{1}{2}}^n$, $C_{\frac{1}{2}}^{n,0} = C_{\frac{1}{2}}^n$. Moreover, to each edge $\left(P_{i+\frac{1}{2}}^n, l_{i+\frac{1}{2}}^n\right)$ we associate $\vec{\nu}_{h,i+\frac{1}{2}}^{n,q}$ and $K_{h,i+\frac{1}{2}}^{n,q}$ as defined above and we denote by $\left(B_{i+\frac{1}{2}}^{n,q}\right)$ the whole straight line containing this edge.

Then $\Gamma_h^{n+1} = \Gamma_h^{n,p}$ is obtained by solving for $q = 0, \cdots, p-1$,

$$\left\{ \begin{array}{l} \overrightarrow{P_{i+\frac{1}{2}}^{n,q} P_{i+\frac{1}{2}}^{n,q+1}} = \Delta t_1 \left[C_{i+\frac{1}{2}}^{n,q} - \alpha e^{\gamma K_{h,i+\frac{1}{2}}^{n,q}} \right] \vec{\nu}_{i+\frac{1}{2}}^{n,q} \\ \text{for } i = 0, \cdots, I-1 \ , \\ \\ P_i^{n,q+1} = B_{i-\frac{1}{2}}^{n,q+1} \bigcap B_{i+\frac{1}{2}}^{n,q+1} \text{ for } i = 1, \cdots, I \ , \\ \text{and } P_{I+1}^{n,q+1} \text{ is calculated from the periodicity} \\ \text{ condition for the interface} \\ \\ C_i^{n,q+1} \quad = \quad C_i^{n,q} \\ \qquad\qquad + \quad \Delta t_1 \left[C_i^{n,q} - \alpha e^{\gamma K_{h,i+\frac{1}{2}}^{n,q}} \right]^2 (1 - C_i^{n,q}) \\ \text{for } i = 1, \cdots, I+1 \ . \end{array} \right.$$

Control of the edges
As in the case of the previous method we have to control the lengths of the segments. Let $l_{max} > l_{min} > 0$ be two given real numbers.

1. If $\left| \overrightarrow{P_i^{n,q} P_{i+1}^{n,q}} \right| < l_{min}$, we consider the midpoint of the edge as a new vertex and remove $P_i^{n,q}$ and $P_{i+1}^{n,q}$.

2. If $\left| \overrightarrow{P_i^{n,q} P_{i+1}^{n,q}} \right| > l_{max}$, we take the midpoint of the edge as a new vertex.

Moreover, the method requires another kind of test. Indeed, the calculus of $P_i^{n,q}$ as the intersection of the two lines $B_{i-\frac{1}{2}}^{n,q}$ and $B_{i+\frac{1}{2}}^{n,q}$ can generate what Roosen & Taylor [11] call flipped-segments. This corresponds to the case $\vec{\nu}_{h,i+\frac{1}{2}}^{n,q-1} . \vec{\nu}_{h,i+\frac{1}{2}}^{n,q} < 0$. If it happens, the vertex $P_i^{n,q}$ is removed. The Figure 3 below gives an example of such a segment.

4 Discretization of the equations for the concentration

To begin with we give a weak formulation for the diffusion problem (P_1). Since the space domain Ω_t depends on time, we are led to introduce function spaces which depend on time as well. For $t \geq 0$, we set $\mathcal{S}_t = \{v \in H^1(\Omega_t) \ ; \ v|_{x=0} = v|_{x=L}\}$. (For a domain Ω, we denote by $H^1(\Omega)$ the space of square integrable functions with square integrable first derivatives).
We also introduce the spaces $\mathcal{H}_t$ and $\mathcal{V}_t$ defined as follows :
(i) in the case of a Neumann boundary condition,

$$\mathcal{H}_t = \mathcal{V}_t = \mathcal{S}_t,$$

(ii) in the case of a Dirichlet boundary condition,

$$\mathcal{H}_t = \{v \in \mathcal{S}_t;\ v = 0 \text{ on } \Sigma_t\}$$
$$\mathcal{V}_t = \{v \in \mathcal{S}_t;\ v = g \text{ on } \Sigma_t\}\,.$$

Next we define the set
$Q_T := \{(x,y,t),\ 0 < t < T \text{ and } (x,t) \in \Omega_t\}$.

We assume that the interface Γ_t is smooth enough. We multiply equation (15) by $\varphi \in \mathcal{H}_t$ and integrate by parts on Ω_t. We obtain the problem

$$\left\{\begin{array}{l} \text{Find } C \in H^1(Q_T) \text{ with } 0 \leq C \leq 1 \text{ such that} \\ (i)\ C(x,y,0) = C_0(x,y) \qquad\qquad (x,y) \in \Omega_0; \\ (ii)\ C(t) \in \mathcal{V}_t \text{ for a.e. } t \in (0,T); \\ (iii)\ \displaystyle\int_{\Omega_t} C_t\varphi\, dxdy + \int_{\Omega_t} \nabla C \nabla\varphi\, dxdy + \int_{\Gamma_t} C\varphi\, d\sigma - \int_{\Gamma_t} C^2\varphi\, d\sigma \\ \displaystyle +\alpha\int_{\Gamma_t} Ce^{\gamma K}\varphi\, d\sigma = \alpha\int_{\Gamma_t} e^{\gamma K}\varphi\, d\sigma \\ \text{for a.e. } t \in (0,T) \text{ and for all } \varphi \in \mathcal{H}_t\ . \end{array}\right. \qquad (34)$$

4.1 Discretization in time.

Next we show how we discretize in time Problem (34). For all $\varphi \in \mathcal{H}_{(n+1)\Delta t}$ and for all integer $n \in [0, (T-\Delta t)/(\Delta t)]$, we associate to problem (34) the following discretized problem.

$$\left\{\begin{array}{l} \text{Find } C^{n+1} \in \mathcal{V}_{(n+1)\Delta t} \text{ such that} \\ (i)\ \displaystyle\frac{1}{\Delta t}\int_{\Omega^{n+1}} C^{n+1}\varphi\, dxdy + \int_{\Omega^{n+1}} \nabla C^{n+1}\nabla\varphi\, dxdy + \int_{\Gamma^{n+1}} C^{n+1}\left(1 - \tilde{C}^n + \alpha e^{\gamma K^{n+1}}\right)\varphi\, d\sigma \\ \displaystyle = \frac{1}{\Delta t}\int_{\Omega^{n+1}} \tilde{C}^n\varphi\, dxdy + \alpha\int_{\Gamma^{n+1}} e^{\gamma K^{n+1}}\varphi\, d\sigma \\ \text{for all } \varphi \in \mathcal{H}_{(n+1)\Delta t} \text{ and for all integer} \quad n \in [0,(T-\Delta t)/(\Delta t)]\ ; \\ (ii)\ C^0(x,y) = C_0(x,y) \qquad\qquad (x,y) \in \Omega_0\ , \end{array}\right. \qquad (35)$$

where $\tilde{C}^n$ is an extension of C^n to the domain $\overline{\Omega^{n+1}}$. We will explicitely show such an extension after having presented the discretization in space.

4.2 Discretization in space.

We use a finite element method.

Before discretizing in space Problem (35) we introduce some notations. We denote by Ω_h^{n+1} the discrete approximation of the domain $\Omega_{(n+1)\Delta t}$ and by $\mathcal{T}_h^{n+1}$ a

triangularization of Ω_h^{n+1} such that to each point of the boundary $\{x = L\}$ there corresponds one point having the same y-coordinate on the boundary $\{x = 0\}$. Furthermore we denote by N^{n+1} the number of nodes of $\mathcal{T}_h^{n+1}$ of x-coordinate stricty less than L.
Next we introduce some discrete approximations of the function spaces, namely

$$\mathcal{S}_h^{n+1} = \left\{ \begin{array}{l} v_h \in C^0\left(\overline{\Omega_h^{n+1}}\right) \text{ for all } K \in \mathcal{T}_h^{n+1}, \\ v_h \text{ is linear on } K \text{ and } v_h(0,.) = v_h(L,.) \end{array} \right\} .$$

In the case of the Neumann boundary condition, we set

$$\mathcal{H}_h^{n+1} = \mathcal{V}_h^{n+1} = \mathcal{S}_h^{n+1} ,$$

and in the case of the Dirichlet boundary condition we set

$$\mathcal{H}_h^{n+1} = \left\{ v_h \in \mathcal{S}_h^{n+1},\ v_h = 0 \text{ on } \Sigma_h^{n+1} \right\}$$

$$\mathcal{V}_h^{n+1} = \left\{ v_h \in \mathcal{S}_h^{n+1},\ v_h = g \text{ on } \Sigma_h^{n+1} \right\} ,$$

where Σ_h^{n+1} is the upper boundary of Ω_h^{n+1}.

Let $\left\{\varphi_i^{n+1}\right\}_{i=1}^{N^{n+1}}$ be the piecewise linear basis functions of $\mathcal{S}_h^{n+1}$ (they take value one on one node and vanish at all other nodes). We decompose the approximate solution C_h^{n+1} on this basis,

$$C_h^{n+1}(x,y) = \sum_{i=1}^{N^{n+1}} C_i^{n+1}\varphi_i^{n+1}(x,y), \qquad (x,y) \in \overline{\Omega_h^{n+1}},$$

where $C_i^{n+1} = C_h^{n+1}(P_i^{n+1})$. In the same way

$$\begin{aligned} \tilde{C}_h^n(x,y) &= \sum_{i=1}^{N^{n+1}} \tilde{C}_i^n\varphi_i^{n+1}(x,y) \qquad (x,y) \in \overline{\Omega_h^{n+1}}, \\ e^{\gamma K^{n+1}}(x,y) &= \sum_{i=1}^{N^{n+1}} e_i^{n+1}\varphi_i^{n+1}(x,y) , \end{aligned}$$

with

$$e_i^{n+1} = \left\{ \begin{array}{ll} e^{\gamma K_{h,i}^{n+1}} & \text{if the node } P_i^{n+1} \text{ of } \mathcal{T}_h^{n+1} \text{ belongs} \\ & \text{to the moving interface,} \\ 0 & \text{elsewhere.} \end{array} \right.$$

Furthermore we decompose $(\tilde{C}_h^n C_h^{n+1})$ and $(C_h^{n+1}e^{\gamma K^{n+1}})$ according to

$$\tilde{C}_h^n C_h^{n+1}(x,y) = \sum_{i=1}^{N^{n+1}} \tilde{C}_i^n C_i^{n+1}\varphi_i^{n+1}(x,y),$$

$$C_h^{n+1}e^{\gamma K^{n+1}}(x,y) = \sum_{i=1}^{N^{n+1}} C_i^{n+1}e_i^{n+1}\varphi_i^{n+1}(x,y) .$$

In the case that a Neumann boundary condition is prescribed we obtain the linear system :

$$\left\{\begin{array}{l}\sum_{i=1}^{N^{n+1}} C_i^{n+1}\left[\frac{1}{\Delta t}\int_{\Omega_h^{n+1}}\varphi_i^{n+1}\varphi_j^{n+1}\,dxdy+\int_{\Omega_h^{n+1}}\nabla\varphi_i^{n+1}\nabla\varphi_j^{n+1}\,dxdy+\right.\\ \left.+\left(1-\tilde{C}_i^n+\alpha e_i^{n+1}\right)\int_{\Gamma_h^{n+1}}\varphi_i^{n+1}\varphi_j^{n+1}\,d\sigma\right]\\ =\sum_{i=1}^{N^{n+1}}\left[\frac{1}{\Delta t}\tilde{C}_i^n\int_{\Omega_h^{n+1}}\varphi_i^{n+1}\varphi_j^{n+1}\,dxdy+\alpha e_i^{n+1}\int_{\Gamma_h^{n+1}}\varphi_i^{n+1}\varphi_j^{n+1}\,d\sigma\right],\\ \text{for } j=1,\cdots,N^{n+1}\ .\end{array}\right. \qquad (36)$$

One obtains a rather similar system in the case that a Dirichlet boundary condition is prescribed on the upper boundary of the domain which is then moving.

Construction of the extension $\tilde{C}_h^n$ of C_h^n to the domain $\overline{\Omega_h^{n+1}}$.
We now describe the construction of the extension $\tilde{C}_h^n$ of C_h^n to the domain $\overline{\Omega_h^{n+1}}$. Let P_i^{n+1} be a node of Ω_h^{n+1}. Either

(i) $P_i^{n+1}\in\Omega_h^n$ and $C_h(P_i^{n+1})$ is computed by linear interpolation in the triangle of $\mathcal{T}_h^{n+1}$ containing P_i^{n+1},

or

(ii) $P_i^{n+1}\notin\Omega_h^n$ and P_i^{n+1} is located in a neighborhood of the interface Γ_h^n.

First of all we suppose that P_i^n is a node of Γ_h^n and P_i^{n+1} the corresponding node Γ_h^{n+1} obtained by moving the free boundary, i.e. by computing the displacement

$$\overrightarrow{P_i^nP_i^{n+1}}=\Delta t\left[C_i^n-\alpha e^{\gamma K_{h,i}^n}\right]\vec{\nu}_{h,i}^n. \qquad (37)$$

We use the approximation formula

$$C_h^n(P_i^{n+1})\ \simeq\ C_h^n(P_i^n)+\overrightarrow{P_i^nP_i^{n+1}}\,.\,\overrightarrow{\nabla C_h^n}(P_i^n),$$

so that in view of the equation (37) and by means of the interface condition (16), we obtain

$$C_h^n(P_i^{n+1})\simeq C_i^n+\Delta t\left[C_i^n-\alpha e^{\gamma K_{h,i}^n}\right]^2(1-C_i^n).$$

Then we choose

$$\tilde{C}_i^n=C_i^n+\Delta t\left[C_i^n-\alpha\exp\left(\gamma K(P_i^n)\right)\right]^2(1-C_i^n),$$

on the discrete interface Γ_h^{n+1}.
Otherwise, if the node P_i^{n+1} is strictly located between the two interface Γ_h^n and Γ_h^{n+1}, we determine the quadrangle $\left(P_j^n,P_k^n,P_j^{n+1},P_k^{n+1}\right)$, where $P_j^n,P_k^n\in\Gamma_h^n$ and $P_j^{n+1},P_k^{n+1}\in\Gamma_h^{n+1}$, which contains P_i^{n+1} and interpolate in this quadrangle the

value of C_h^n at P_i^{n+1}.

(iii) $P_i^{n+1} \notin \Omega_h^n$ and P_i^{n+1} is located above Σ_h^n.

In the case of a Dirichlet boundary condition on the moving upper boundary Σ_h^{n+1}, when Σ_h^{n+1} is above Σ_h^n, we have to extend C_h^n beyond to Σ_h^n up to Σ_h^{n+1}. In this case we set $\tilde{C}_h^n = g$, in the whole part delimited by Σ_h^{n+1} and Σ_h^n.

Two variants of the conjugate gradient method have been used for solving the linear system (36) : BI-CGSTAB which was introduced by Van Der Vorst [19] and GMRES introduced by Y. Saad & M. Schultz [12]. No significant differences in the results were observed.

At each time step a new triangularization is generated by means of the mesh generator Modulef in such a way that the nodes used to move the interface are degrees of freedom of the problem. An advantage of this method is the possibility to refine the mesh in a neighborhood of the interface without increasing too much the cost of the numerical computations. Furthermore, a part of the triangularization of the domain may be fixed, at least for a number of time steps. The main draw back of this method is that since, at each time step, at least on a part of the domain the triangularization changes, we have to interpolate the value of the concentration there.

5 Numerical results

In this section we present and discuss a number of numerical results; some have been obtained with a homogeneous Neumann boundary condition on the fixed upper boundary of the space domain while others have been obtained with a Dirichlet boundary condition on the moving upper boundary (cf. Section 2.2 *(i)* and *(ii)*). The computations have been performed with the method of Ikeda & Kobayashi for tracking the moving free boundary. In the case of the Neumann boundary condition, we present a comparaison test between the methods of Ikeda & Kobayashi and of Roosen & Taylor.

5.1 Domain of variation of the different parameters.

The main parameters to be chosen are the initial concentration C_0, the initial interface Γ_0, the value of the saturation concentration α and the value of the surface tension σ.

(i) Typically initial concentrations are given by $C_0 = 0$ and $C_0 = 2\alpha$.

(ii) Most of the tests have been performed with taking as initial interface the function

$$y = f_0(x) = b \sin\left(\frac{2\pi x}{L}\right), \qquad x \in [0, L] \tag{38}$$

with $b = 4\ 10^{-6}$ m and $L = 15\ 10^{-6}$ m. More rapidly oscillating or less smooth initial interfaces have also been considered (see test 2 in Section 5.2 and test 4 in Section 5.3).

(iii) α is chosen as a multiple of the value $S_0 = 1.42\ 10^3$ mol/m^3 which corresponds

to the saturation concentration of copper, for example $\alpha = 30 S_0$.
(iv) Two values of the surface tension σ have been chosen, $\sigma = 0$ and $\sigma = 30$. The value of the constant γ is then given by $\gamma = 6\ 10^{-9}\ \sigma$. When there is no surface tension, i.e $\sigma = 0$, singular points may appear on the interface in finite time : this is due to the fact that the interface then satisfies a first order equation.

The other parameters, i.e. the diffusion coefficient D, the molar volume V of the solid and the kinetic constant κ, are fixed and take the values $D = 10^{-9}$ m^2/s, $V = 7.09\ 10^{-6}$ m^3/mol and $\kappa = 7.09\ 10^{-9}$ m/s.

5.2 The case of a homogeneous Neumann boundary condition.

The height M of the fixed upper boundary is given by $M = 6\ 10^{-6}$ m.
Before describing the numerical results, let us make some remarks about the solution.
(i) The concentration satisfies a conservation law, namely the total mass of the solid is preserved. Indeed coming back to the dimensionless equations, integrating (15) by parts over Ω_t and using the boundary conditions (16), (17), (21) we find that

$$\int_{\Omega_t} (1 - C)_t \ dxdy = \int_{\Gamma_t} (1 - C)\, V_\nu \, d\sigma,$$

but since

$$\frac{d}{dt} \int_{\Omega_t} (1 - C) \ dxdy = \int_{\Omega_t} (1 - C)_t \ dxdy - \int_{\Gamma_t} (1 - C)\, V_\nu \, d\sigma,$$

we deduce that for all $t \geq 0$

$$\int_{\Omega_t} (1 - C) \ dxdy = \int_{\Omega_0} (1 - C_0) \ dxdy \ . \tag{39}$$

When performing numerical computations, we systematically compute the quantity $\int_{\Omega_t} (1 - C) \ dxdy$. A numerical observation is that this integral varies slightly in the first steps of the computation and becomes constant afterwards; in order to remedy the variation for small times, we take smaller time steps initially and let them increase with time.

(ii) Suppose that $\gamma > 0$. A numerical observation in the case that Γ_0 is parametrized in the form $y = f_0(x)$ is that the pair $(C,\ \Gamma_t)$ converges to $(\alpha,\ \mathit{constant})$ as $t \to +\infty$. Next we show how one can compute this constant.
Suppose that

$$\lim_{t \to +\infty} C(t) = \alpha.$$

Letting t goes to $+\infty$ in (16), we formally deduce that for $\gamma \neq 0$,

$$\lim_{t \to +\infty} K(t) = 0,$$

which means that the free boundary converges to a plane $\{y = y_\infty\}$ as t tends to $+\infty$.
On the other hand, letting t tend to $+\infty$ in (39) and using that $|\Omega_\infty| = (M - y_\infty)L$ gives

$$y_\infty = M - \frac{1}{(1-\alpha)L}\int_{\Omega_0}(1 - C_0)\,dxdy. \tag{40}$$

This also provides a criterium for checking the validity of the numerical programs.

Next we present the results of some numerical computations.

1. The case that $C_0 = 0$, $\alpha = 30S_0$, $\sigma = 0$.

Applying formal arguments based on the maximum principle one can check that $0 \leq C \leq \alpha$ so that $V_\nu = C - \alpha \leq 0$. Therefore this case only involves the dissolution process.

The curves presented in Figure 4 show the interface at several times starting from time $t = 0$. Clearly the interface decreases in time and converges to some nontrivial stationary state. In fact we remark that every solution (C, f) with $C = \alpha$ and f arbitrary is a stationary solution.

2. The case that $C_0 = 0$, $\alpha = 30S_0$, $\sigma = 30$.

Both dissolution and growth occur here and as it has been discussed above the solution converges to a constant as $t \to +\infty$.

Figure 5 shows the time evolution of the interface, whereas Figure 6 represents the concentration in the all domain at times $t = 10s$, $t = 1700s$ and $t = 4200s$.

The process seems to exhibit two stages : in the first one, only dissolution occurs and the concentration C on the interface stabilizes to the saturation value : Figure 7 shows the time evolution of the mean concentration on the free boundary; in the second stage both dissolution and growth occur and the interface converges to the constant y_∞ as t increases.

We show on Figure 8 the variation in time of the quantity $\int_{\Omega_t}(1 - C)\,dxdy$ for three different sets of time steps : Δt, $\Delta t/2$ and $2\Delta t$. We remark that the variation of $\int_{\Omega_t}(1 - C)\,dxdy$ decreases as the time steps become smaller. The relative error is of order 0.1%.

In Figure 9 we present error estimates, when multiplying and dividing the time steps by 2, namely the quantities

$$\text{(a) } e^f_{2\Delta t} = \frac{\|f_{\Delta t} - f_{2\Delta t}\|_{L^2(0,L)}}{\|f_{\Delta t}\|_{L^2(0,L)}} \quad \text{and} \quad \text{(b) } e^f_{\Delta t/2}$$

where we suppose that the interface Γ_t is given in the form $y = f(x,t)$. Note that $e^f_{2\Delta t}$ and $e^f_{\Delta t/2}$ are respectively of order 1% and 0.5%. Similarly we have computed relative errors $e^C_{\Delta t/2}(t)$ and $e^C_{2\Delta t}(t)$ for the concentration on the interface. The errors are very small since $e^C_{2\Delta t}$ and $e^C_{\Delta t/2}$ respectively have a maximum value of order 0.25% and 0.125%.

Finally we compare the two methods for tracking the front. In Figure 10 we present the relative error of the computation of the interface when using the algorithms based on the methods of Ikeda & Kobayashi and of Roosen & Taylor. Note that the error is at most 0.1%.
We show in Figure 11 the relative error of the computation of the concentration on the interface. This error is at most 1.75%.

Moreover the relative error for the computation of the asymptotic value y_∞ of the interface as t tends to $+\infty$ is equal to 0.8% in the case of Ikeda & Kobayashi, whereas it is equal to 1.6% in the case of Roosen & Taylor.

Two more numerical tests have been performed with different initial data. Figure 12 represents the time evolution of the free boundary in the case that the initial interface is given by

$$y = f_0(x) = b\left[\sin\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{8\pi x}{L}\right)\right], \quad x \in [0, L],$$

with $b = 4\ 10^{-6}$ m, $L = 15\ 10^{-6}$ m and $M = 1\ 10^{-5}$ m.

Figure 13 shows the time evolution when starting from a rather singular initial interface; we note the regularization in time of the free boundary.

5.3 The case of a Dirichlet boundary condition.

This case is characterized by the existence of a planar travelling wave solution where the interface Γ_t is given by $f(x,t) = f_0 + vt$, where $f_0 = constant$ is the initial interface and the concentration is given in the form $C(x,y,t) = U(y - vt - f_0)$. In fact we can compute the expressions of the dimensionless quantities v and U, namely if we set

$$z = y - vt - f_0,$$

then

$$U(z) = 1 + (v + \alpha - 1)e^{-vz}, \tag{41}$$

and v is the unique solution of the algebraic equation

$$(v + \alpha - 1)e^{-vM} = g - 1, \tag{42}$$

where g is the value of the concentration on the upper boundary and M is the height of the initial upper boundary. It turns out that there is growth, i.e. that $v > 0$, when $\alpha < g \leq 1$ whereas there is dissolution, i.e. $v < 0$, in the case that $0 \leq g < \alpha$. If $g = \alpha$, the travelling wave solution reduces to a stationary solution of the form $(C_\infty, \Gamma_\infty) = (\alpha, y = constant)$.

From a chemical point of view, one would expect the travelling wave solution U to be linear in z instead of having the exponential form (41); however for practical purposes it does not matter too much since the profile of U is very close to linear.

The height M of the initial upper boundary is given by $M = 6\ 10^{-6}$ m. The saturation value α is fixed and equal to $30S_0$. On the upper boundary three values of the Dirichlet data g have been chosen, i.e. $g = 0$ which corresponds to a global dissolution process, $g = \alpha$ which in view of (42) implies that $v = 0$ and $g = 2\alpha$

which corresponds to a global growth process. We also choose two values of the surface tension, i.e. $\sigma = 0$ and $\sigma = 30$. We observe that as t increases, the solution converges to the travelling wave solution (U, f, v) computed above.

Next we present the results of our numerical computations.

1. The case that $C_0 = g = 0$, $\alpha = 30S_0$, $\sigma = 30$.

Global dissolution occurs and the solution converges to the travelling wave solution as t increases.

Figure 14 shows the time evolution of the interface and the convergence to a plane.

We show on Figure 15 the time evolution of the difference between the velocity v of the travelling wave and the velocity $v_c(t)$ of the mean plane of the interface. For large enough times, $v_c(t)$ practically coincides with the travelling wave velocity v.

We show on Figure 16 the time evolution of the mean concentration on the interface which asymptotically tends to the dimensionless value $v + \alpha$, namely to the physical value 1.812 mol/m^3.

2. The case that $C_0 = g = \alpha$, $\alpha = 30S_0$, $\sigma = 30$.

Since $g = \alpha$ we deduce from (42) that the asymptotic value of the velocity should be equal to zero.

Figure 17 shows the time evolution of the interface. We observe here that the mean planes of the interfaces do not move, even for small times.

3. The case that $C_0 = g = 2\alpha$, $\alpha = 30S_0$, $\sigma = 30$.

Global growth occurs here and the solution converges to the travelling wave solution as t increases.

Figure 18 shows the time evolution of the interface and the convergence to a plane.

4. The case that $C_0 = g = 2\alpha$, $\alpha = 30S_0$, $\sigma = 0$.

We take as initial interface the function

$$y = f_0(x) = b\left[\sin\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{4\pi x}{L}\right)\right], \quad x \in [0, L].$$

Figure 19 shows the time evolution of the interface. Global growth occurs here and two singular points appear on the interface exactly at the local minima of the initial interface. The two singular points move and converge to a unique singular point.

5.4 Conclusion

For the values of the physical constants considered in this paper and which were chosen in view of the physical context, the dissolution-growth problem (P_1), (P_2) in space dimension two does not exhibit morphological instabilities, i.e. no fingering occurs. In both the cases that a homogeneous Neumann condition and a constant

Dirichlet condition are imposed on the upper boundary of the domain, the numerical interface converges to a constant profile as t increases whenever the surface tension σ is positive. In the case of the Neumann boundary condition the concentration C converges to α and the interface converges to a constant as $t \to +\infty$ whereas in the case of a constant Dirichlet boundary condition the concentration converges to the travelling profile U and the interface converges to a constant which displaces itself at the constant velocity v; if the boundary value is larger than the saturation value α then $v > 0$ and only growth occurs for t large enough while if the boundary value is smaller than the saturation value α then $v < 0$ so that only dissolution occurs for large times. A final remark is that in the case that $\sigma = 0$ singular points may appear on the interface and propagate as t increases.

Appendix A :

Estimation of the numerical error on the computation of the unit normal vector and of the curvature.

We assume for simplicity that the interface is parametrized in the form $y = f(x)$ with f sufficiently smooth.

A1. The case of the method of Ikeda and Kobayashi.

The discrete interface Γ_h is defined by

$$\Gamma_h = \{P_i = (x_i, f_i) \text{ where } f_i = f(x_i),\ i = 1, ..., I\}.$$

Moreover, we suppose that $x_i > x_{i-1}$ and we set $h = \max\limits_{i \in \{2,...,I\}} (x_i - x_{i-1})$. Then the following result holds.

Proposition 1 : *Let $\vec{\nu}_i$ and K_i be respectively the unit normal vector and the curvature of the interface $y = f(x)$ at a point P_i. Let $\vec{\nu}_{h,i}$ be the approximate unit normal vector defined by (29) and $K_{h,i}$ the approximate curvature defined by (30). Then*

$$\|\vec{\nu}_i - \vec{\nu}_{h,i}\| = O(h) \quad \text{and} \quad |K_i - K_{h,i}| = O(h).$$

Proof :

We have

$$\vec{\nu}_{h,i} = \frac{1}{\left[1 + \left(\dfrac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}}\right)^2\right]^{1/2}} \begin{pmatrix} -\dfrac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} \\ 1 \end{pmatrix}.$$

By Taylor expansion up to order 1 for f, we have $f_x = \dfrac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} + \mathrm{O}(h)$, where $f_x \equiv f_x(x_i)$. Hence we have that

$$\vec{\nu}_{h,i} = \frac{1}{[1 + f_x^2]^{1/2}} \begin{pmatrix} -f_x \\ 1 \end{pmatrix} + \overrightarrow{\mathrm{O}}(h).$$

Thus we get

$$\|\vec{\nu}_i - \vec{\nu}_{h,i}\| = \mathrm{O}(h).$$

From (30) we obtain

$$K_{h,i} = \frac{1}{|x_{i-1} - x_i||x_{i+1} - x_i||x_{i+1} - x_{i-1}|} \times$$

$$\frac{2\left[(x_{i-1} - x_i)(f_{i+1} - f_i) - (x_{i+1} - x_i)(f_{i-1} - f_i)\right]}{\left(\left[1 + \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i}\right)^2\right]\left[1 + \left(\frac{f_{i-1} - f_i}{x_{i-1} - x_i}\right)^2\right]\left[1 + \left(\frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}}\right)^2\right]\right)^{1/2}} \tag{43}$$

Using Taylor expansion up to order 2 for f, we get

$$f_{i\pm1} = f_i + (x_{i\pm1} - x_i)f_x + \frac{(x_{i\pm1} - x_i)^2}{2} f_{xx} + \mathrm{O}\left(h^3\right),$$

where the derivatives f_x and f_{xx} are taken at x_i. Thus we have

$$\begin{aligned}&(x_{i-1} - x_i)(f_{i+1} - f_i) - (x_{i+1} - x_i)(f_{i-1} - f_i) \\ &= \frac{1}{2}(x_{i-1} - x_i)(x_{i+1} - x_i)(x_{i+1} - x_{i-1})f_{xx} + \mathrm{O}(h^4).\end{aligned} \tag{44}$$

Remark that $(x_{i-1} - x_i)(x_{i+1} - x_i)(x_{i+1} - x_{i-1}) < 0$, and thus using (44) and expansions for f_x in the expression of $K_{h,i}$, we deduce that

$$K_{h,i} = \frac{-f_{xx} + \mathrm{O}(h)}{\left[1 + (f_x + \mathrm{O}(h))^2\right]^{3/2}} = -\frac{f_{xx}}{(1 + f_x^2)^{3/2}} + \mathrm{O}(h).$$

Thus we have

$$|K_i - K_{h,i}| = \mathrm{O}(h).$$

Remark : For a constant discretisation step, i.e $x_i - x_{i-1} = h$ for all $i = 2, ..., I$, we have the following improved estimation for the normal vector : $\|\vec{\nu}_i - \vec{\nu}_{h,i}\| = \mathrm{O}(h^2)$.

A2. The case of the method of Roosen and Taylor.

The discrete interface Γ_h is defined by

$$\Gamma_h = \left\{ [P_i, P_{i+1}] \;\text{ where } P_i = (x_i, f_i) \text{ with } f_i = f(x_i), \text{ for } i = 1, ..., I \right\}.$$

We also suppose that $x_i > x_{i-1}$ and we set $h = \max_{i \in \{2,...,I\}} (x_i - x_{i-1})$. Then the following result holds

Proposition 2 : *Let $\vec{\nu}_i$ and K_i be respectively the unit normal vector and the curvature of the interface $y = f(x)$ at a point P_i. Let $\vec{\nu}_{h,i+\frac{1}{2}}$ be the approximate unit normal vector associated to the edge $[P_i, P_{i+1}]$, defined by (32) and $K_{h,i+\frac{1}{2}}$ the approximate curvature associated to $[P_i, P_{i+1}]$, defined by (33). Then*

$$\begin{aligned} \|\vec{\nu}_i - \vec{\nu}_{h,i+\frac{1}{2}}\| &= O(h)\,, \\ K_{h,i+\frac{1}{2}} &= \frac{1}{2}(K_i + K_{i+1}) + O(h)\,. \end{aligned}$$

Proof :

We have

$$\begin{aligned} \vec{\nu}_{h,i+\frac{1}{2}} &= \begin{pmatrix} -\dfrac{f_{i+1} - f_i}{[(x_{i+1} - x_i)^2 + (f_{i+1} - f_i)^2]^{1/2}} \\ \dfrac{x_{i+1} - x_i}{[(x_{i+1} - x_i)^2 + (f_{i+1} - f_i)^2]^{1/2}} \end{pmatrix} \\ &= \frac{1}{(1 + f_x^2)^{1/2}} \begin{pmatrix} -f_x \\ 1 \end{pmatrix} + \overrightarrow{\mathrm{O}}\,(h). \end{aligned}$$

Thus we obtain

$$\|\vec{\nu}_i - \vec{\nu}_{h,i+\frac{1}{2}}\| = \mathrm{O}\,(h)\,.$$

Next we compute the approximation order for the computation of the curvature. In order to simplify computations, we assume that $x_i - x_{i-1} = h$ for $i = 2, ..., I$.
We set

$$\begin{aligned} C_{i,i-1} &= \vec{\nu}_{h,i+\frac{1}{2}} . \vec{\nu}_{h,i-\frac{1}{2}} \\ &= \frac{(f_{i+1} - f_i)(f_i - f_{i-1}) + h^2}{[(f_{i+1} - f_i)^2 + h^2]^{1/2}\,[(f_i - f_{i-1})^2 + h^2]^{1/2}} \end{aligned} \tag{45}$$

By Taylor expansion up to order 4 for f, we have

$$f_{i\pm 1} = f_i \pm h f_x + \frac{h^2}{2} f_{xx} \pm \frac{h^3}{6} f_{3x} + \frac{h^4}{24} f_{4x} + \mathrm{O}\,(h^5)$$

where all f derivatives are taken at x_i. Then equation (45) becomes

$$C_{i,i-1} = \frac{\left[1 + f_x^2 + h^2(\frac{1}{3} f_x f_{3x} - \frac{1}{4} f_{xx}^2) + \mathrm{O}\,(h^4)\right]}{d_+\ d_-},$$

where

$$d_\pm = (1 + f_x^2)^{1/2} \left[1 \pm h\, a_1 + h^2\, a_2 \pm h^3\, a_3 + \mathrm{O}\left(h^4\right)\right]^{1/2},$$

with

$$a_1 = \frac{f_x f_{xx}}{1 + f_x^2}, \quad a_2 = \frac{\frac{1}{3} f_x f_{3x} + \frac{1}{4} f_{xx}^2}{1 + f_x^2}, \quad a_3 = \frac{f_{xx} f_{3x} + \frac{1}{2} f_x f_{4x}}{6(1 + f_x^2)}.$$

Using Taylor expansion of $(1+x)^{-1/2}$ up to order three, we get

$$\begin{aligned} d_{\pm}^{-1} = (1+f_x^2)^{-1/2} \Big[& 1 \mp \tfrac{h}{2} a_1 + \tfrac{h^2}{2}\left(\tfrac{3}{4} a_1^2 - a_2\right) \\ & \pm \tfrac{h^3}{2}\left(\tfrac{3}{2} a_1 a_2 - a_3 - \tfrac{5}{8} a_1^3\right) + \mathrm{O}\left(h^4\right) \Big]. \end{aligned}$$

Then we obtain

$$C_{i,i-1} = 1 - \frac{h^2}{2} \frac{f_{xx}^2}{(1+f_x^2)^2} + \mathrm{O}\left(h^4\right).$$

Moreover we get

$$g_{i,i-1} = \left(\frac{1 - C_{i,i-1}}{1 + C_{i,i-1}}\right)^{1/2} = \frac{h |f_{xx}|}{2(1+f_x^2)} + \mathrm{O}\left(h^3\right),$$

and

$$|\overrightarrow{P_i P_{i+1}}| = h(1+f_x^2)^{1/2} + \mathrm{O}\left(h^2\right).$$

Thus $\dfrac{g_{i,i-1}}{|\overrightarrow{P_i P_{i+1}}|} = \dfrac{|f_{xx}|}{2(1+f_x^2)^{3/2}} + \mathrm{O}(h)$ and with the definition of $\delta_{i,j}$ in Section 3.2, we obtain

$$\begin{aligned} \delta_{i,i-1} \frac{g_{i,i-1}}{|\overrightarrow{P_i P_{i+1}}|} &= -\frac{f_{xx}}{2(1+f_x^2)^{3/2}} + \mathrm{O}(h), \\ &= \frac{1}{2} K_i + \mathrm{O}(h). \end{aligned} \qquad (46)$$

In a same way, we have

$$\delta_{i,i+1} \frac{g_{i,i+1}}{|\overrightarrow{P_i P_{i+1}}|} = \frac{1}{2} K_{i+1} + \mathrm{O}(h)\ . \qquad (47)$$

Thus, in view of (46) and (47) we obtain

$$K_{h,i+\frac{1}{2}} = \frac{1}{2}(K_i + K_{i+1}) + \mathrm{O}(h).$$

Appendix B :

Approximate curvature of the new points introduced in the case of the method of Ikeda and Kobayashi.

The aim of this part is to justify the procedure described in section 3.1 which allows to introduce a new vertex $P_{i+\frac{1}{2}}$ with non-zero mean curvature, when the segment $[P_i P_{i+1}]$ is too long. We show that the approximate curvature K at point $P_{i+\frac{1}{2}}$ satisfies $K \simeq \dfrac{K_i + K_{i+1}}{2}$ where K_i and K_{i+1} denote the approximate curvatures respectively at P_i and P_{i+1}. More precisely, if we put $h = \frac{P_i P_{i+1}}{2}$ we have the following result.

Proposition 3 : *Let K_i and K_{i+1} be the approximate curvatures respectively at points P_i and P_{i+1} defined by (30). Let $P_{i+\frac{1}{2}}$ be the point determined as in Section 3.1 and K the corresponding approximate curvature at $P_{i+\frac{1}{2}}$ defined by (30) as well. Then*

$$K = \frac{1}{2}\left(K_i + K_{i+1}\right) + O\left(h^2\right).$$

Proof :

The notations that we use here are those of Figure 2. We assume that $K_i \neq 0$ and $K_{i+1} \neq 0$. Let O_l be the center of $\mathcal{C}_l$ for $l = i, i+1$ and O the center of the circle circumscribed about the triangle $\left(P_i P_{i+\frac{1}{2}} P_{i+1}\right)$. We suppose that the line (O_i, O_{i+1}) is oriented so that

$$\overline{O_l N_l} = \frac{1}{K_l} \text{ with } l = i, i+1 \text{ and } \overline{OP}_{i+\frac{1}{2}} = \frac{1}{K}.$$

By construction of the center O there exists $a > 0$ such that

$$\overline{MP}_{i+\frac{1}{2}} = a\, \overline{OP}_{i+\frac{1}{2}},$$

where M is the midpoint of $[P_i P_{i+1}]$ and since

$$MP_{i+\frac{1}{2}} = OP_{i+\frac{1}{2}} - OM = OP_{i+\frac{1}{2}}\left(1 - \left(1 - h^2K^2\right)^{1/2}\right)$$

we deduce that

$$\begin{aligned} \overline{MP}_{i+\frac{1}{2}} &= \overline{OP}_{i+\frac{1}{2}}\left(1 - \left(1 - h^2K^2\right)^{1/2}\right) \\ &= \frac{1}{K}\left(1 - \left(1 - h^2K^2\right)^{1/2}\right) \\ &= \frac{h^2}{2}K + \mathrm{O}\left(h^4\right) \end{aligned} \qquad (48)$$

In the same way expressing that $MN_l = O_lN_l - O_lM$, we have for $l = i, i+1$

$$\overline{MN}_l = \frac{h^2}{2}K_l + \mathrm{O}\left(h^4\right). \qquad (49)$$

Since $\overline{MP}_{i+\frac{1}{2}} = \frac{1}{2}\left(\overline{MN}_i + \overline{MN}_{i+1}\right)$ we deduce from (48) and (49) that

$$K = \frac{K_i + K_{i+1}}{2} + \mathrm{O}\left(h^2\right)$$

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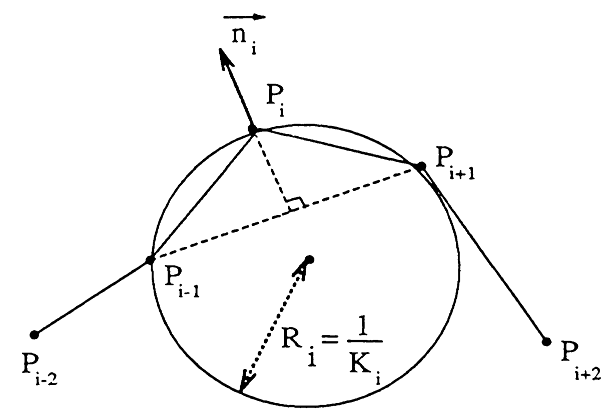


Figure 1. Discrete unit normal and discrete mean curvature at vertex P_i for the method of Ikeda & Kobayashi.

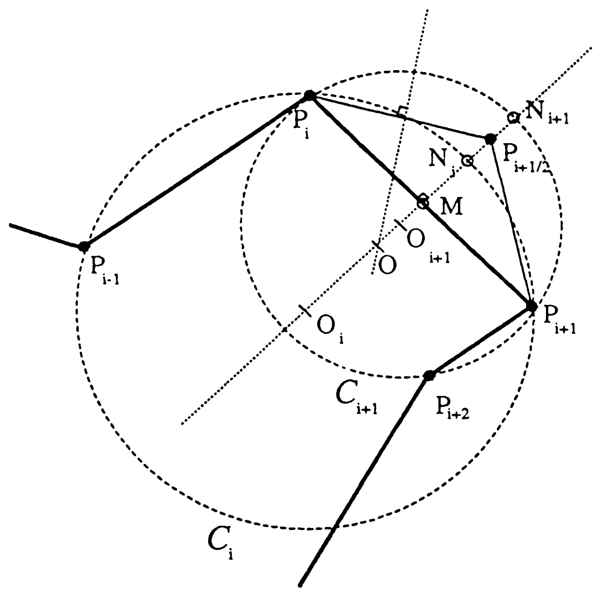


Figure 2. Introduction of a new point as midpoint of $[N_i, N_{i+1}]$, using an idea due to T.I. Seidman.

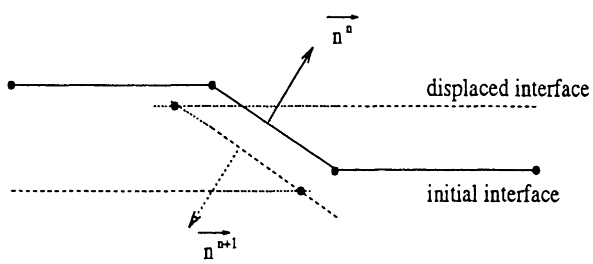


Figure 3. The case of a flipped segment.

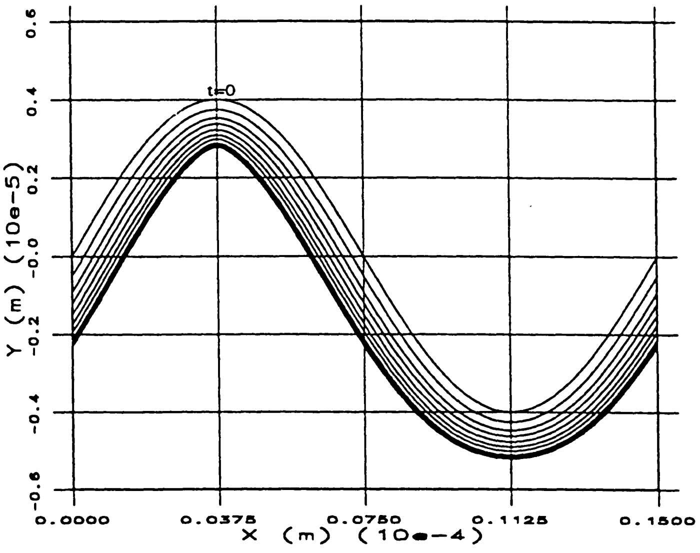


Figure 4. Convergence to a stationary solution in the case of a homogeneous Neumann boundary condition on the upper boundary, with $C_0 = 0$, $\alpha = 30S_0$, $\sigma = 0$.

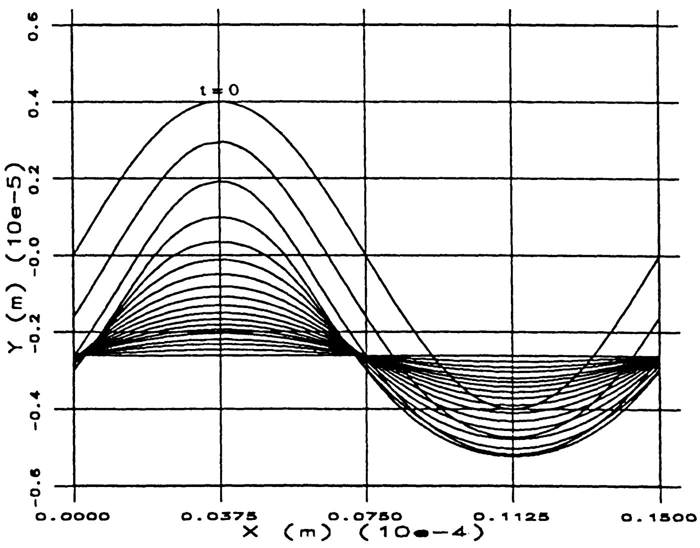
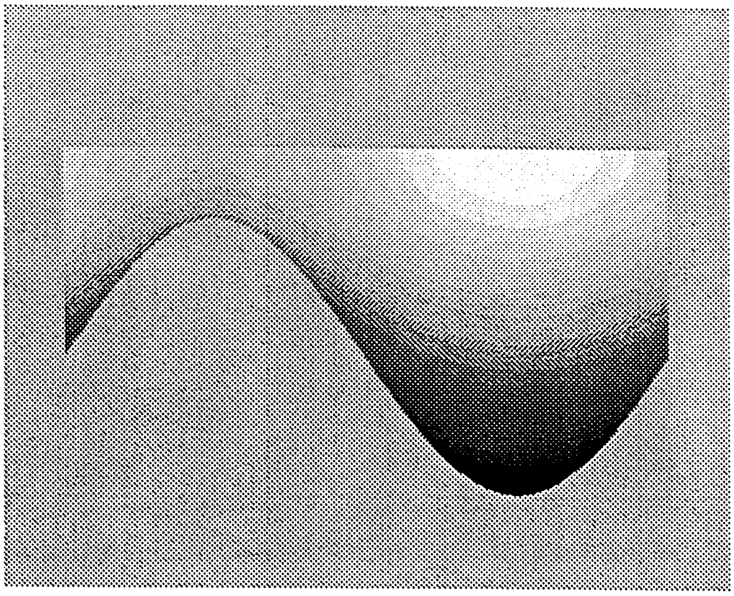
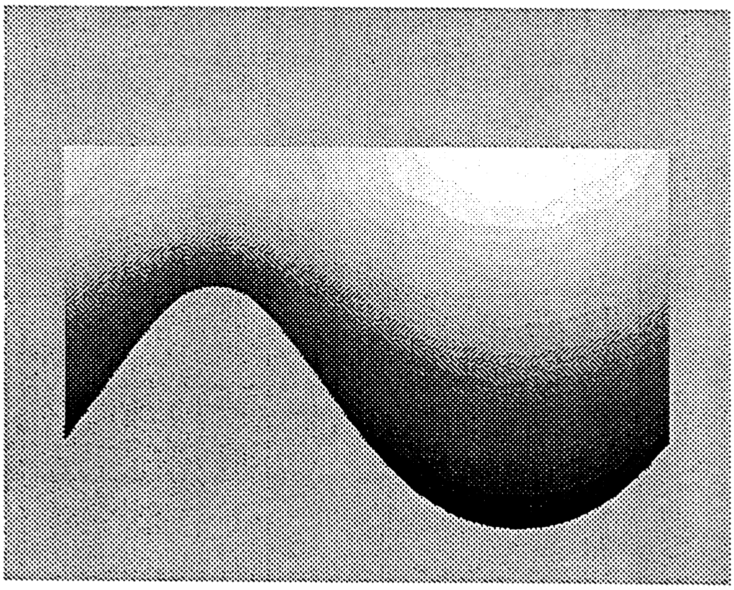


Figure 5. Time evolution of the interface in the case of a homogeneous Neumann boundary condition on the upper boundary, with $C_0 = 0$, $\sigma = 30$, $\alpha = 30S_0$.

t=10s



t=1700s



t=4200s

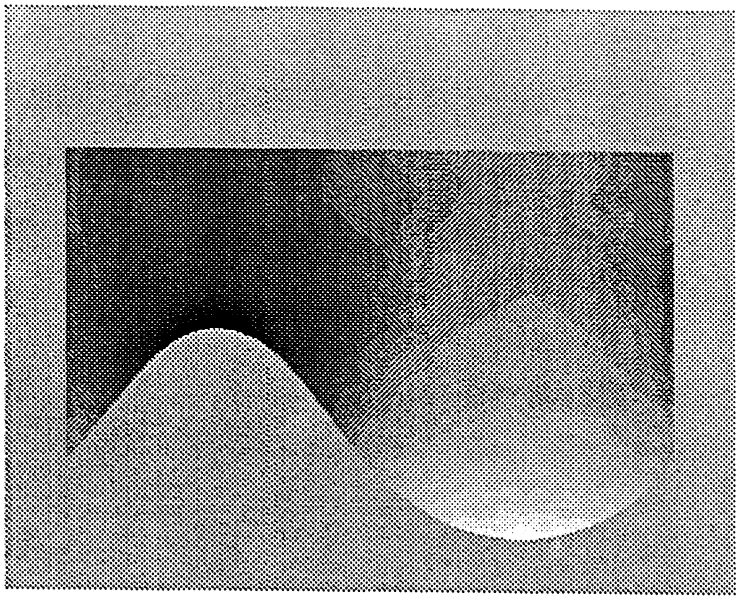


Figure 6. The concentration in the whole domain at different times in the case of the homogeneous Neumann boundary condition, with $C_0 = 0$, $\alpha = 30S_0$, $\sigma = 30$. The dark shaded regions are regions with maximum concentration.

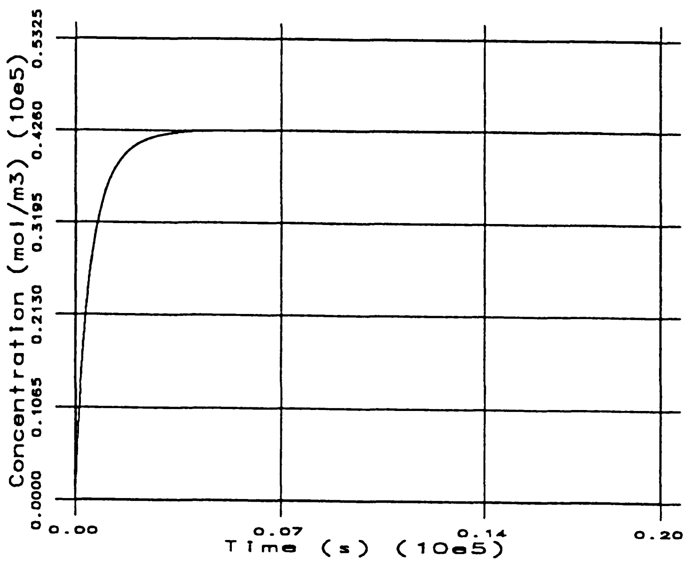


Figure 7. Time evolution of the mean concentration on the interface in the case of a homogeneous Neumann boundary condition on the upper boundary, with $\sigma = 30$ and $\alpha = 30S_0$.

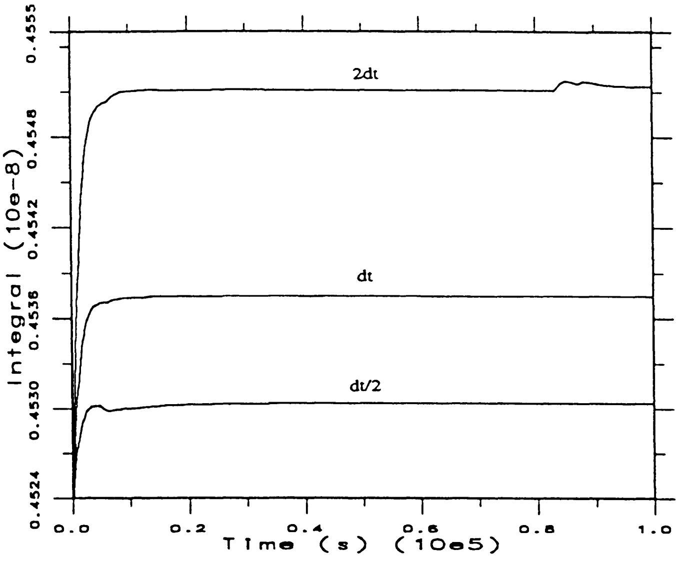


Figure 8. Influence of the time step on the conservation of $\int_{\Omega_t} (1 - C)\, dxdy$ in the case of a homogeneous Neumann boundary condition, with $\alpha = 30S_0$ and $\sigma = 30$.

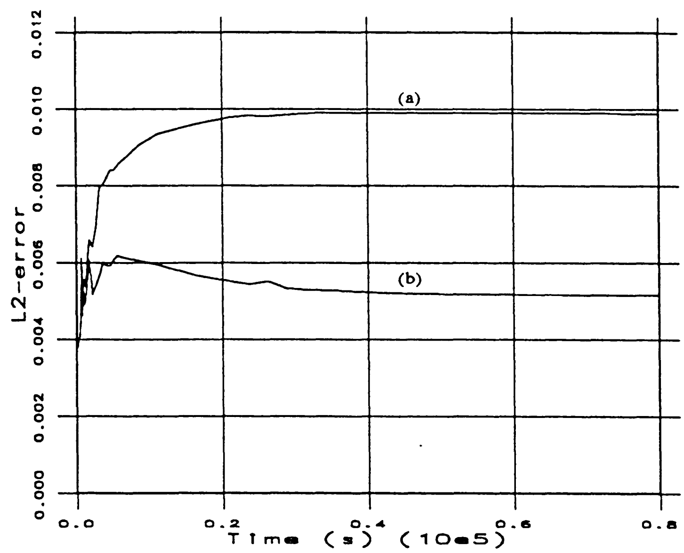


Figure 9. Error in L^2-norm for the computation of the interfaces in the case of a homogeneous Neumann boundary condition with $\alpha = 30S_0$ and $\sigma = 30$, when multiplying and dividing the time steps by 2.

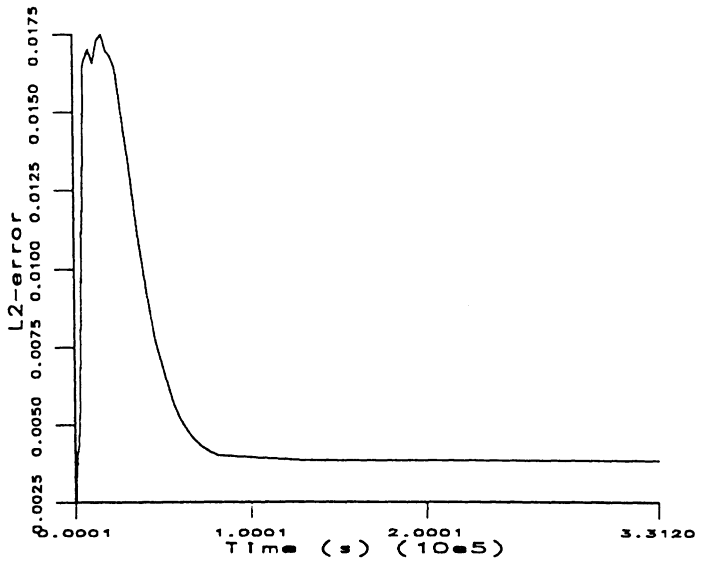


Figure 10. Error in L^2-norm between the algorithms based on the method of Ikeda & Kobayashi and the method of Roosen & Taylor for the computation of the interface.

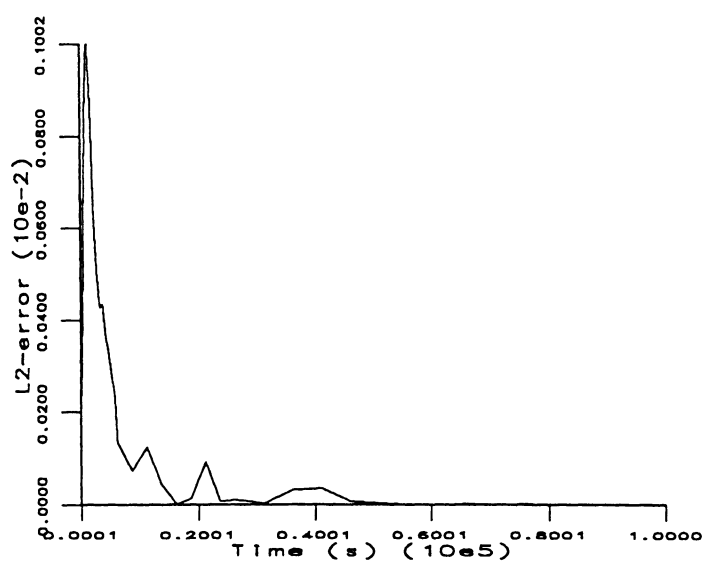


Figure 11. Error in L^2-norm between the algorithms based on the method of Ikeda & Kobayashi and the method of Roosen & Taylor for the computation of the concentration on the interface.

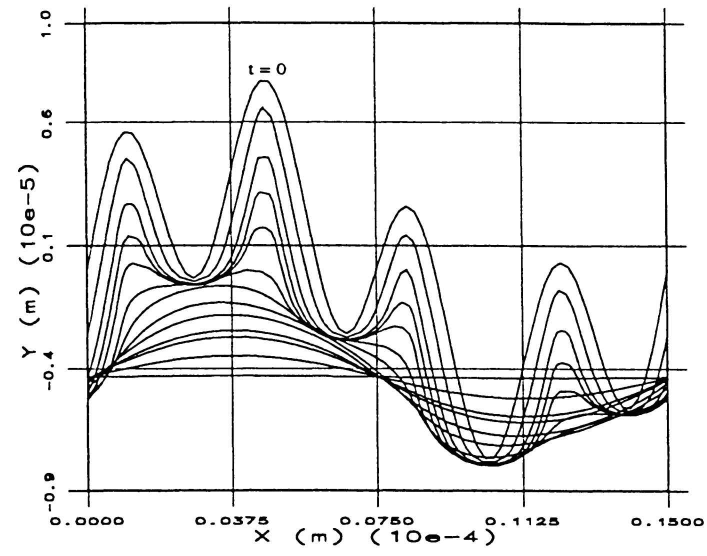


Figure 12. Time evolution of the interface in the case of a homogeneous Neumann boundary condition, with $C_0 = 0$, $\sigma = 30$, $\alpha = 30S_0$.

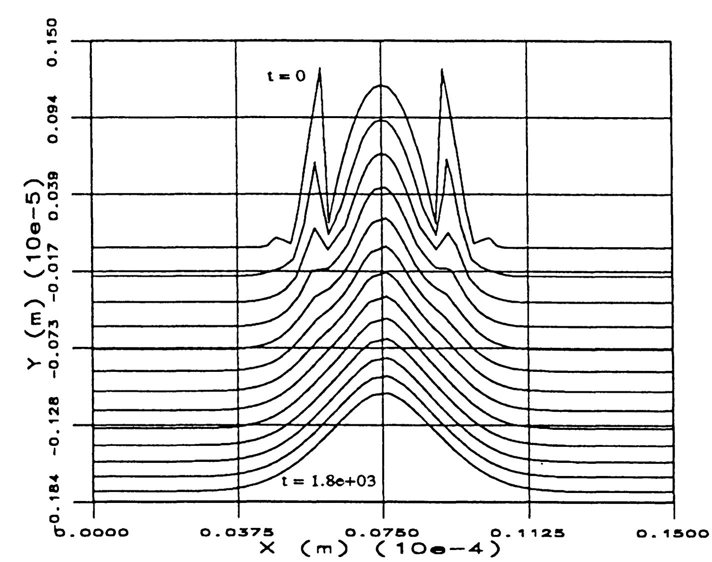


Figure 13. Time evolution of the interface in the case of a homogeneous Neumann boundary condition, with $C_0 = 0$, $\sigma = 30$, $\alpha = 30S_0$.

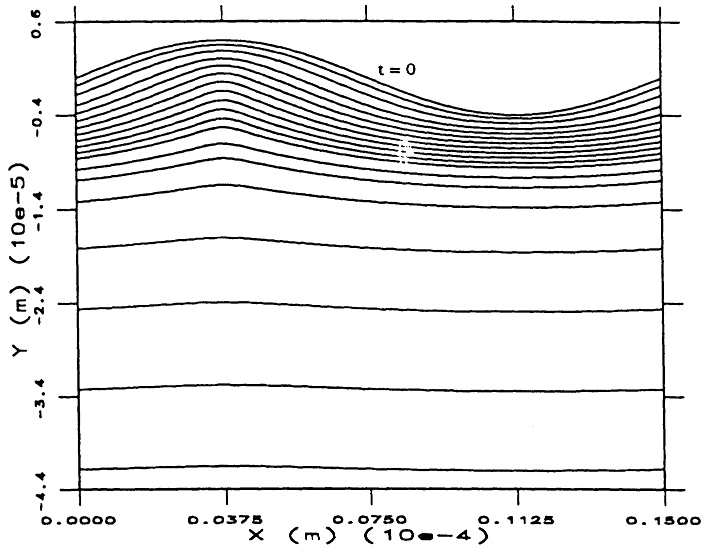


Figure 14. Time evolution of the interface in the case of a Dirichlet boundary condition $g = 0$ on the upper boundary, with $C_0 = g$, $\sigma = 30$, $\alpha = 30S_0$.

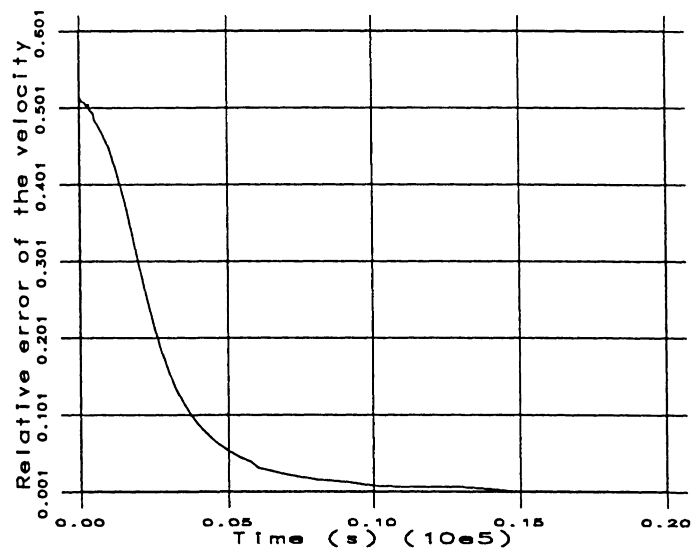


Figure 15. Time evolution of $|v - v_c(t)|$ in the case of a Dirichlet boundary condition $g = 0$ on the upper boundary, with $C_0 = g$, $\sigma = 30$, $\alpha = 30S_0$.

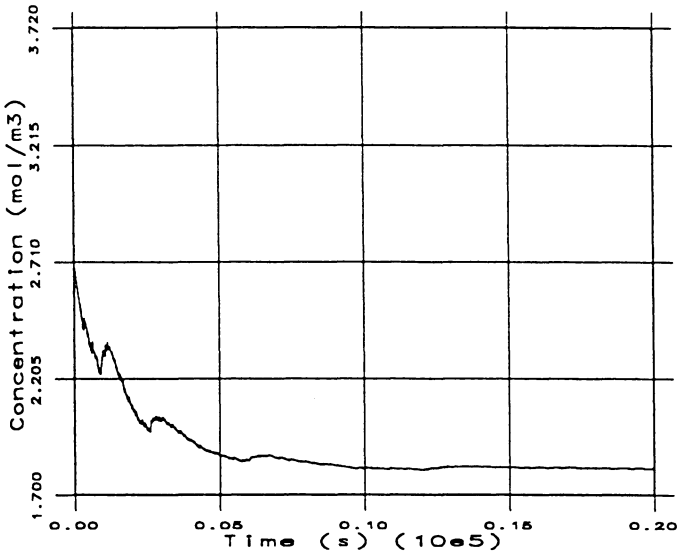


Figure 16. Time evolution of the mean concentration on the interface in the case of a Dirichlet boundary condition $g = 0$ on the upper boundary, with $C_0 = g$, $\sigma = 30$, $\alpha = 30S_0$.

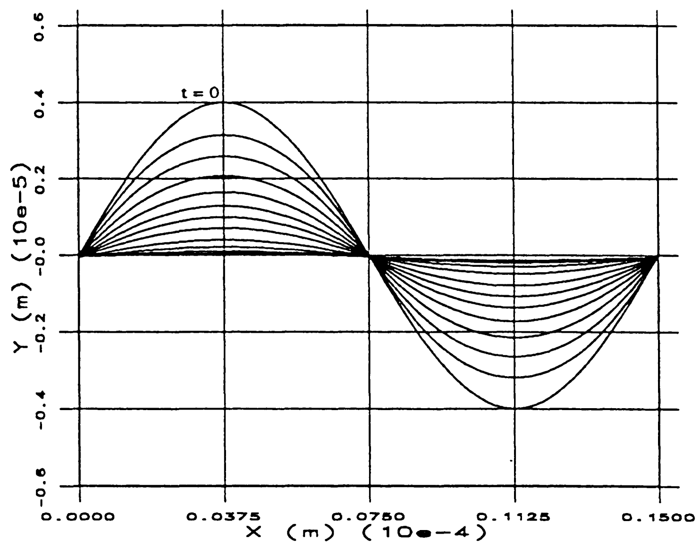


Figure 17. Time evolution of the interface in the case of a Dirichlet boundary condition $g = \alpha$ on the upper boundary, with $C_0 = g$, $\sigma = 30$, $\alpha = 30S_0$.

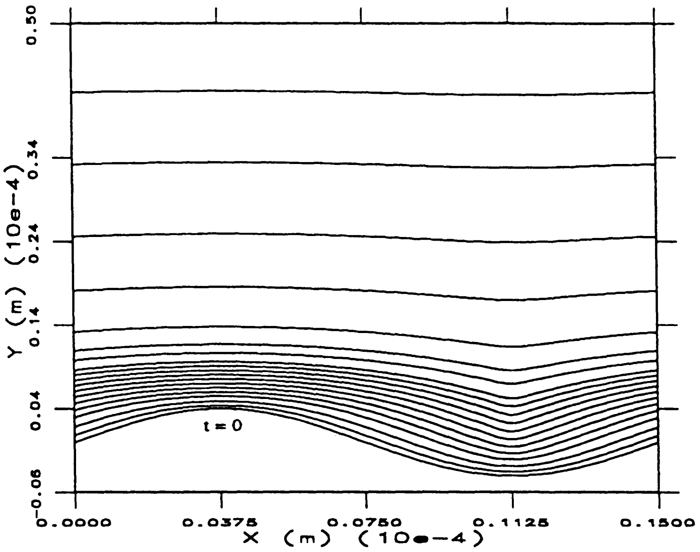


Figure 18. Time evolution of the interface in the case of a Dirichlet boundary condition $g = 2\alpha$ on the upper boundary, with $C_0 = g$, $\sigma = 30$, $\alpha = 30S_0$.

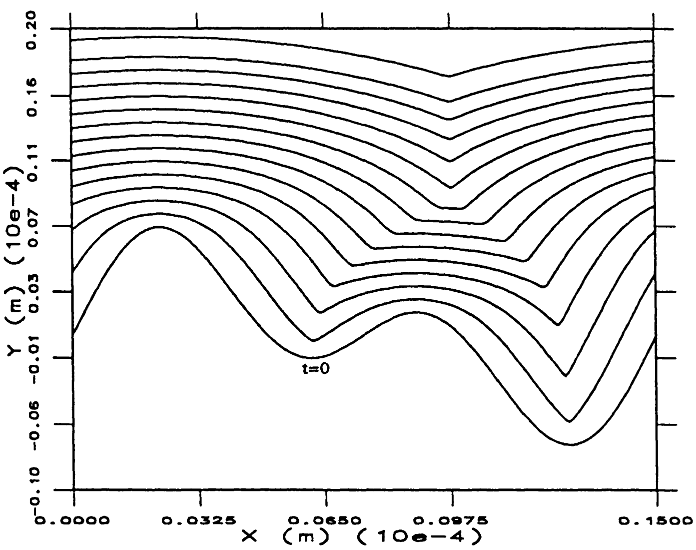


Figure 19. The case of zero surface tension : time evolution of the interface in the case of a Dirichlet boundary condition $g = 0$ on the upper boundary, with $C_0 = g$, $\sigma = 0$, $\alpha = 30S_0$.

DEUXIEME PARTIE

Semi-continuité supérieure pour des modèles de champ de phase

A singularly perturbed phase field model : upper-semicontinuity of the attractor

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Abstract. We consider a singularly perturbed phase field model for the Cahn-Hilliard and the viscous Cahn-Hilliard equations and we prove that the maximal attractor associated to this model is upper-semicontinuous.

Résumé. Nous considérons un modèle de transition de phase qui est une pertubation singulière des équations de Cahn-Hilliard visqueuse et de Cahn-Hilliard et nous montrons que l'attracteur maximal de ce modèle est semi-continu supérieurement.

AMS : 35K50, 35B25

Key words : System of second order, nonlinear parabolic equations, maximal attractor, upper-semicontinuity

A singularly perturbed phase field model : Upper-semicontinuity of the attractor

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1 Introduction

In this paper, we consider a phase field model as well as boundary value problems for two limiting equations, namely the viscous Cahn-Hilliard and the Cahn-Hilliard equations. All three problems possess a global attractor and our concern is the upper-semicontinuity of the global attractor. More precisely, these problems have the following form. The phase field model which we consider [1], [2] is a coupled system for an order parameter φ and the temperature u, namely

$$(PF)\quad \begin{cases} \delta\varphi_t = \Delta\varphi - g(\varphi) + u & \text{in } \Omega \times I\!R^+, \quad (1.1) \\ \varepsilon u_t + \varphi_t = \Delta u & \text{in } \Omega \times I\!R^+, \quad (1.2) \\ \varphi = u = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x)\ ,\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is an open bounded set of $I\!R^n$ $(n \leq 3)$ with smooth boundary $\partial\Omega$, $(\varphi_0, u_0) \in (L^2(\Omega))^2$. We assume that the function g has the form

$$g(s) = \sum_{j=1}^{2p-1} a_j s^j \text{ with } a_{2p-1} > 0\ ,\ p \geq 2\ .$$

In the physical case $g(s) = s^3 - s$ [2].
The problem for the viscous Cahn-Hilliard equation which has been introduced by A. Novick-Cohen [11] is obtained by setting $\varepsilon = 0$ in Problem (PF) and substituting equation (1.1) into equation (1.2) : one then obtains a problem for the single unknown function φ, namely

$$(VCH)\begin{cases} \varphi_t + \Delta\left(\Delta\varphi - g(\varphi) - \delta\varphi_t\right) = 0 & \text{in } \Omega \times I\!R^+, \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x) & x \in \Omega. \end{cases}$$

The problem for the Cahn-Hilliard equation is obtained by setting $\varepsilon = \delta = 0$ in Problem (PF) and substituting equation (1.1) into equation (1.2). One obtains

$$(CH)\begin{cases} \varphi_t + \Delta\left(\Delta\varphi - g(\varphi)\right) = 0 & \text{in } \Omega \times I\!R^+, \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x) & x \in \Omega. \end{cases}$$

In [13], Stoth considers Problem (PF) on a finite time interval in the case that g is a cubic function. In the case that $\varepsilon = \delta$, she proves that as $\varepsilon \downarrow 0$, the solution $(\varphi^\varepsilon, u^\varepsilon)$ of problem (PF) converges to a pair (φ, u) where φ satisfies Problem (CH) and $u = -\Delta\varphi + g(\varphi)$.

Elliott and Kostin [6] and Elliott and Stuart [7] prove continuity properties of the attractor of Problem (VCH).

This paper is organized as follows. In Section 2, we recall existence and uniqueness results for the solutions of the problems (PF), (VCH) and (CH) as well as results concerning the attractors associated to the corresponding semigroups. In Section 3 we derive time independent estimates, uniform with respect to the parameters ε and δ, for the solution of Problem (PF). Finally we prove in Section 4 the upper-semicontinuity of the attractor of Problem (PF) in the space $H^2(\Omega) \times L^2(\Omega)$, first at $(\varepsilon, \delta) = (0, \delta)$ with $\delta > 0$ arbitrary and then with $\varepsilon = \delta$ at $\varepsilon = 0$. Furthermore an immediate consequence of the proofs is the upper-semicontinuity of the attractor of the viscous Cahn-Hilliard equation in the space $H^2(\Omega)$ at $\delta = 0$, which was already proven by Elliott and Stuart [7]. Our methods of proof are partly inspired from methods due to Hale and Raugel [8] and Debussche [3].

The results proven in this paper will be extended to the case of Neumann and periodic boundary conditions by Dupaix, Hilhorst and Laurençot [4] and to the case that g is a logarithmic function by C. Dupaix [5]. The authors also plan to consider the general case where the nonlinear function g involves a maximal monotone operator.

Acknowledgement. The authors wish to thank A. Debussche, I. Kostin and Ph. Laurençot for many inspiring discussions.

2 Preliminaries

In this section we introduce some notations used in this paper and recall existence and uniqueness results as well as results about the existence of a global attractor for the three problems (PF), (VCH) and (CH).

To begin with we recall some properties of the polynomial function g which will be useful in what follows.

(i) There exists a constant C_1 such that

$$g(s)s \geq \frac{3}{4} a_{2p-1} s^{2p} - C_1 \quad \text{for all } s \in I\!R. \tag{2.1}$$

(ii) For every $\eta > 0$, there exists a constant $C_2 = C_2(\eta)$ such that

$$|g(s)| \leq \eta a_{2p-1} s^{2p} + C_2 \quad \text{for all } s \in \mathbb{R}. \tag{2.2}$$

(iii) There exists a positive constant C_3 such that

$$g'(s) \geq -C_3 \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$

In the sequel we will use the scalar product and the norm in $H^{-1}(\Omega) = (H_0^1(\Omega))'$. For $w \in H^{-1}(\Omega)$ we define

$$\psi = Nw$$

as the unique solution in $H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta\psi &= w \text{ in the sense of distributions in } \Omega, \\ \psi &= 0 \text{ on } \partial\Omega. \end{cases}$$

Then if $v,\ w \in H^{-1}(\Omega)$ and if $\psi = Nv,\ \xi = Nw$

$$(v, w)_{H^{-1}(\Omega)} = \int_\Omega \nabla\psi\nabla\xi dx,$$

and

$$||v||^2_{H^{-1}(\Omega)} = \int_\Omega |\nabla\psi|^2 dx.$$

Finally we present known results about the problems (PF), (VCH) and (CH). To begin with, we recall results of Brochet, Chen and Hilhorst [1] about Problem (PF).

Theorem 2.1 *For any $(\varphi_0, u_0) \in (L^2(\Omega))^2$, Problem (PF) has a unique solution $(\varphi^{\epsilon\delta}, u^{\epsilon\delta})$ which satisfies*

$$(\varphi^{\epsilon\delta}, u^{\epsilon\delta}) \in L^\infty\left(0, T; (L^2(\Omega))^2\right) \cap L^2(0, T; (H_0^1(\Omega))^2) \quad , \varphi^{\epsilon\delta} \in L^{2p}(Q_T)$$

for all $T > 0$, where $Q_T := \Omega \times (0, T)$ and

$$(\varphi^{\epsilon\delta}, u^{\epsilon\delta}) \in C\left(\mathbb{R}^+; \left(L^2(\Omega)\right)^2\right).$$

Moreover

$$(\varphi^{\epsilon\delta}, u^{\epsilon\delta}) \in \left(C^\infty(\overline{\Omega} \times (0, +\infty)\right)^2,$$

and the mapping

$$S_{pf}(t) \ : \ (\varphi_0, u_0) \longmapsto (\varphi^{\epsilon\delta}(t), u^{\epsilon\delta}(t))$$

is Lipschitz continuous on $(L^2(\Omega))^2$ for all $t > 0$ and $(S_{pf}(t))_{t\geq 0}$ is a semigroup on $(L^2(\Omega))^2$.

For the maximal attractor of the phase field problem we have that

Theorem 2.2 *The semigroup* $(S_{pf}(t))_{t\geq 0}$ *associated with Problem (PF) possesses in* $\left(L^2(\Omega)\right)^2$ *a maximal attractor* $\mathcal{A}^{\epsilon\delta}$ *that is connected. Moreover* $\mathcal{A}^{\epsilon\delta}$ *is bounded in* $\left(\mathcal{C}^m\left(\overline{\Omega}\right)\right)^2$ *for all* $m \in I\!N$.

The following result [7] holds about the well-posedness of Problem (VCH).

Theorem 2.3 *For any* $\varphi_0 \in L^2(\Omega)$*, Problem (VCH) has a unique solution* φ^δ *satisfying*

$$\varphi^\delta \in L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H_0^1(\Omega)) \bigcap L^{2p}(Q_T)$$

for all $T > 0$*, where* $Q_T := \Omega \times (0,T)$*, and*

$$\varphi^\delta \in \mathcal{C}\left(I\!R^+;L^2(\Omega)\right).$$

Moreover

$$\varphi^\delta \in \mathcal{C}^\infty\left(\overline{\Omega} \times (0,+\infty)\right),$$

and the mapping

$$S_{vch}(t) \;:\; \varphi_0 \longmapsto \varphi^\delta(t)$$

is continuous on $L^2(\Omega)$ *for all* $t > 0$ *and* $(S_{vch}(t))_{t\geq 0}$ *is a semigroup on* $L^2(\Omega)$.

Next we give a result about the existence of a global attractor for the viscous Cahn-Hilliard problem.

Theorem 2.4 *The semigroup* $(S_{vch}(t))_{t\geq 0}$ *associated with Problem (VCH) possesses in* $L^2(\Omega)$ *a maximal attractor* $\mathcal{A}^\delta$ *that is connected. Moreover* $\mathcal{A}^\delta$ *is bounded in* $\mathcal{C}^m(\overline{\Omega})$ *for all* $m \in I\!N$.

Finally, the well-posedness and the existence of a maximal attractor for Problem (CH) follows from Temam [14] and Marion [10].

Theorem 2.5 *For any* $\varphi_0 \in L^2(\Omega)$*, Problem (CH) has a unique solution* φ *satisfying*

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \bigcap L^2(0,T;H^2(\Omega) \bigcap H_0^1(\Omega)) \bigcap L^{2p}(Q_T)$$

for all $T > 0$*, where* $Q_T := \Omega \times (0,T)$*, and*

$$\varphi \in \mathcal{C}\left(I\!R^+;L^2(\Omega)\right).$$

Moreover

$$\varphi \in \mathcal{C}^\infty\left(\overline{\Omega} \times (0,+\infty)\right),$$

and the mapping $S_{ch}(t) \;:\; \varphi_0 \longmapsto \varphi(t)$ *is continuous on* $L^2(\Omega)$ *for all* $t > 0$ *and* $(S_{ch}(t))_{t\geq 0}$ *is a semigroup on* $L^2(\Omega)$.

Theorem 2.6 *The semigroup $(S_{ch}(t))_{t\geq 0}$ associated with Problem (CH) possesses in $L^2(\Omega)$ a maximal attractor $\mathcal{A}$ that is connected. Moreover $\mathcal{A}$ is bounded in $C^m(\overline{\Omega})$ for all $m \in I\!N$.*

3 Time uniform a priori estimates

The main purpose of this section is twofold : show the existence of an absorbing set in $H^3(\Omega) \times H^1(\Omega)$ which does not depend on the small parameters ε and δ; obtain strong enough estimates in order to be able to prove the convergence of orbits.
In what follows we suppose that

$$(i) \quad (\varphi_0, u_0) \in L^2(\Omega) \times L^2(\Omega). \tag{3.1}$$

$$(ii) \quad 0 < \varepsilon \leq 1,\ 0 < \delta \leq 1. \tag{3.2}$$

A natural function to consider [1], [9] is the enthalpy $v^{\varepsilon\delta} = \varepsilon u^{\varepsilon\delta} + \varphi^{\varepsilon\delta}$. We also use the notation $v_0^\varepsilon = \varepsilon u_0 + \varphi_0$.

The first result is the following.

Lemma 3.1 *There exist positive constants a and b such that the function $v^{\varepsilon\delta} = \varepsilon u^{\varepsilon\delta} + \varphi^{\varepsilon\delta}$ satisfies*

$$\delta\|\varphi^{\varepsilon\delta}(t)\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}(t)\|^2_{H^{-1}(\Omega)} \leq \left(\delta\|\varphi_0\|^2_{L^2(\Omega)} + \|w_0^\varepsilon\|^2_{H^{-1}(\Omega)}\right) e^{-at} + b$$

for all $t \geq 0$.

Proof. We rewrite (1.2) as

$$v_t^{\varepsilon\delta} = \Delta u^{\varepsilon\delta},$$

multiply this equation by $N(v^{\varepsilon\delta})$ and integrate on Ω to obtain

$$\int_\Omega v_t^{\varepsilon\delta} N(v^{\varepsilon\delta})dx = \int_\Omega \Delta u^{\varepsilon\delta} N(v^{\varepsilon\delta})dx,$$

that is

$$\int_\Omega -\left(\Delta N(v^{\varepsilon\delta})\right)_t N(v^{\varepsilon\delta})dx = \int_\Omega u^{\varepsilon\delta}\Delta N(v^{\varepsilon\delta})dx,$$

so that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla N(v^{\varepsilon\delta})|^2 dx + \int_\Omega u^{\varepsilon\delta} v^{\varepsilon\delta} dx = 0,$$

which we rewrite as

$$\frac{1}{2}\frac{d}{dt}\|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} + \int_\Omega \left\{\varepsilon(u^{\varepsilon\delta})^2 + \varphi^{\varepsilon\delta}u^{\varepsilon\delta}\right\}dx = 0.$$

By substituting the expression for $u^{\varepsilon\delta}$ in (1.1) in the equality above and using (2.2) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \quad & \left\{\delta\|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)}\right\} \\ & + \int_\Omega \left\{\varepsilon(u^{\varepsilon\delta})^2 + |\nabla\varphi^{\varepsilon\delta}|^2 + \varphi^{\varepsilon\delta}g(\varphi^{\varepsilon\delta})\right\}dx = 0,\end{aligned}$$

which in view of (2.1), gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \quad & \left\{\delta\|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)}\right\} \\ & + \int_\Omega \left\{\varepsilon(u^{\varepsilon\delta})^2 + |\nabla\varphi^{\varepsilon\delta}|^2 + \frac{3}{4}a_{2p-1}(\varphi^{\varepsilon\delta})^{2p}\right\}dx \leq C_1.\end{aligned} \tag{3.3}$$

Using that $\varepsilon \leq 1$ we deduce that

$$\begin{aligned}\|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} \leq \;& C_4\|v^{\varepsilon\delta}\|^2_{L^2(\Omega)} \\ \leq \;& C_5\left(\varepsilon\|u^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)}\right),\end{aligned}$$

and thus since $\delta \leq 1$,

$$\begin{aligned}&\delta\|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} \\ &\leq C_6\left(\varepsilon\|u^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \tfrac{3}{4}a_{2p-1}\int_\Omega \left(\varphi^{\varepsilon\delta}\right)^{2p}dx + 1\right),\end{aligned}$$

which we substitute in (3.3) to deduce that

$$\frac{1}{2}\frac{d}{dt}\left\{\delta\|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)}\right\} + C_7\left(\delta\|\varphi^{\varepsilon\delta}\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)}\right) \leq C_8.$$

The result of Lemma 3.1 then follows from applying Gronwall's Lemma.

Corollary 3.2 *There exist a positive constant D_0 and a time $t_0 = t_0(\|\varphi_0\|_{L^2(\Omega)}, \|u_0\|_{H^{-1}(\Omega)})$ such that*

$$\delta\|\varphi^{\varepsilon\delta}(t)\|^2_{L^2(\Omega)} + \|v^{\varepsilon\delta}(t)\|^2_{H^{-1}(\Omega)} \leq D_0$$

for all $t \geq t_0$.

A key ingredient for the next estimates is the functional

$$V_\varepsilon(\varphi, u) = \int_\Omega \left\{ \frac{1}{2}|\nabla\varphi|^2 + G(\varphi) + \frac{\varepsilon}{2}u^2 \right\} dx,$$

where $G(s) = \int_0^s g(\tau)d\tau$. We show below that it is a Lyapunov functional for Problem (PF).

Lemma 3.3 *For all $t > 0$ and all $r > 0$, the solution $(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})$ of Problem (PF) satisfies*

$$\frac{d}{dt}V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})(t) \leq 0, \tag{3.4}$$

and

$$\begin{aligned} &V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})(t+r) + \delta\int_t^{t+r}\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx ds \\ &+ \int_t^{t+r}\int_\Omega |\nabla u^{\varepsilon\delta}|^2 dx ds = V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})(t). \end{aligned} \tag{3.5}$$

Proof. We have that

$$\frac{d}{dt}V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta}) = \int_\Omega \left\{ \left(-\Delta\varphi^{\varepsilon\delta} + g(\varphi^{\varepsilon\delta})\right)\varphi_t^{\varepsilon\delta} + \varepsilon u^{\varepsilon\delta}u_t^{\varepsilon\delta} \right\} dx,$$

in which we substitute (1.2) to obtain

$$\frac{d}{dt}V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta}) = \int_\Omega \left\{ \left(-\Delta\varphi^{\varepsilon\delta} + g(\varphi^{\varepsilon\delta}) - u^{\varepsilon\delta}\right)\varphi_t^{\varepsilon\delta} - |\nabla u^{\varepsilon\delta}|^2 \right\} dx.$$

In turn substituting equation (1.1) gives

$$\frac{d}{dt}V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta}) = \int_\Omega \left\{ -\delta(\varphi_t^{\varepsilon\delta})^2 - |\nabla u^{\varepsilon\delta}|^2 \right\} dx \leq 0,$$

which gives (3.4). In order to obtain (3.5) one integrates the equality above between t and $t+r$.

The next step is to show that $V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})$ enters an absorbing ball.

Lemma 3.4 *There exist a positive constant D_1 and a time $t_1 = t_1(\|\varphi_0\|_{L^2(\Omega)}, \|u_0\|_{H^{-1}(\Omega)})$ such that*

$$V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})(t) \leq D_1 \text{ for all } t \geq t_1.$$

Consequently $(\varphi^{\varepsilon\delta}, \sqrt{\varepsilon}u^{\varepsilon\delta})$ enters an absobing set of $H^1(\Omega) \times L^2(\Omega)$.

Proof. We deduce from (3.3) and Corollary 3.2 that

$$\int_t^{t+r} V_\varepsilon(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})(s)ds \le C(r, D_0)$$

for all $t \ge t_0$ and $r > 0$. The result of the lemma then follows from (3.4) and the Uniform Gronwall Lemma.

In what follows we show a serie of auxiliary estimates which will allow to prove the existence of an absorbing set in $H^3(\Omega) \times H^1(\Omega)$ for the solution $(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})$ of Problem (PF).

Lemma 3.5 *For all $r > 0$ there exists a constant $C = C(r)$ such that*

$$\int_t^{t+r} \int_\Omega \left(\Delta\varphi^{\varepsilon\delta}\right)^2 dx \le C \text{ for all } t \ge t_1. \tag{3.6}$$

Proof. We multiply equation (1.1) by $\Delta\varphi^{\varepsilon\delta}$ and integrate by parts. This gives

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi^{\varepsilon\delta}|^2 dx + \int_\Omega \left(\Delta\varphi^{\varepsilon\delta}\right)^2 dx = \int_\Omega \left\{g(\varphi^{\varepsilon\delta})\Delta\varphi^{\varepsilon\delta} - u^{\varepsilon\delta}\Delta\varphi^{\varepsilon\delta}\right\} dx,$$

which we rewrite as, using also (2.3),

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi^{\varepsilon\delta}|^2 dx + \frac{1}{2}\int_\Omega \left(\Delta\varphi^{\varepsilon\delta}\right)^2 dx \le C_3 \int_\Omega |\nabla\varphi^{\varepsilon\delta}|^2 dx + \frac{1}{2}\int_\Omega |\nabla u^{\varepsilon\delta}|^2 dx.$$

Next we suppose that $t \ge t_1$ and integrate the inequality above between t and $t + r$. Using also the results of the lemmas 3.3 and 3.4, we deduce the result of Lemma 3.5.

Next we use the lemmas 3.5 to prove the existence of absorbing sets.

Lemma 3.6 *There exist a positive constant D_2, a time $t_2 > t_1$ and a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\text{(i)} \quad \int_\Omega \left\{\delta(\varphi_t^{\varepsilon\delta}(t))^2 + |\nabla u^{\varepsilon\delta}(t)|^2\right\} dx \le D_2 \quad \text{for all } t \ge t_2; \tag{3.7}$$

$$\text{(ii)} \quad \int_{t_2}^{+\infty} \int_\Omega \left\{|\nabla\varphi_t^{\varepsilon\delta}|^2 + \varepsilon(u_t^{\varepsilon\delta})^2\right\} dx ds \le D_2. \tag{3.8}$$

Proof. We differentiate equation (1.1) with respect to t, multiply the result by $\varphi_t^{\varepsilon\delta}$ and integrate by parts to obtain

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx + \int_\Omega |\nabla\varphi_t^{\varepsilon\delta}|^2 dx = \int_\Omega \left\{-g'(\varphi^{\varepsilon\delta})(\varphi_t^{\varepsilon\delta})^2 + u_t^{\varepsilon\delta}\varphi_t^{\varepsilon\delta}\right\}dx. \tag{3.9}$$

Multiplying equation (1.2) by $u_t^{\varepsilon\delta}$ and integrating by parts gives

$$\int_\Omega \left\{\varepsilon(u_t^{\varepsilon\delta})^2 + \varphi_t^{\varepsilon\delta}u_t^{\varepsilon\delta}\right\}dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^{\varepsilon\delta}|^2 dx = 0. \tag{3.10}$$

Hence adding up (3.9) and (3.10) gives, in view of (2.3),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{\delta(\varphi_t^{\varepsilon\delta})^2 + |\nabla u^{\varepsilon\delta}|^2\right\}dx + \int_\Omega \left\{|\nabla\varphi_t^{\varepsilon\delta}|^2 + \varepsilon(u_t^{\varepsilon\delta})^2\right\}dx \\
&\le C_3\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx \\
&\le C_3\left(\frac{1}{2}\int_\Omega |\nabla\varphi_t^{\varepsilon\delta}|^2 dx + \tilde{C}\|\varphi_t^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)}\right).
\end{aligned} \tag{3.11}$$

Furthermore, we deduce from (1.2) that

$$\begin{aligned}
\|\varphi_t^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} &= \|\Delta u^{\varepsilon\delta} - \varepsilon u_t^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} \\
&\le 2\|\Delta u^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} + 2\|\varepsilon u_t^{\varepsilon\delta}\|^2_{H^{-1}(\Omega)} \\
&\le 2\|\nabla u^{\varepsilon\delta}\|^2_{L^2(\Omega)} + 2C\varepsilon^2\|u_t^{\varepsilon\delta}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.12}$$

Using (3.12) we deduce from (3.11) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{\delta(\varphi_t^{\varepsilon\delta})^2 + |\nabla u^{\varepsilon\delta}|^2\right\}dx + \frac{1}{2}\int_\Omega \left\{|\nabla\varphi_t^{\varepsilon\delta}|^2 + \varepsilon(u_t^{\varepsilon\delta})^2\right\}dx \\
&\le C\int_\Omega |\nabla u^{\varepsilon\delta}|^2 dx,
\end{aligned} \tag{3.13}$$

provided that ε is small enough say, $\varepsilon < \varepsilon_0$. The result of the lemma then follows from (3.5), Lemma 3.4 and the Uniform Gronwall Lemma.

From now on, we suppose that $\varepsilon \in (0, \varepsilon_0)$.

Lemma 3.7 *There exists a positive constant D_3 such that*

$$\|\varphi^{\varepsilon\delta}(t)\|_{H^2(\Omega)} \le D_3 \tag{3.14}$$

for all $t \ge t_2$.

Proof. We multiply (1.1) by $\Delta\varphi^{\varepsilon\delta}$ and integrate on Ω. This gives

$$\int_\Omega (\Delta\varphi^{\varepsilon\delta})^2 dx = \int_\Omega \left\{\delta\varphi_t^{\varepsilon\delta}\Delta\varphi^{\varepsilon\delta} + g(\varphi^{\varepsilon\delta})\Delta\varphi^{\varepsilon\delta} - u^{\varepsilon\delta}\Delta\varphi^{\varepsilon\delta}\right\}dx,$$

which we integrate by parts to deduce that, using also (2.3),

$$\frac{1}{2}\int_\Omega (\Delta\varphi^{\varepsilon\delta})^2 dx \leq C\int_\Omega \left\{\delta(\varphi_t^{\varepsilon\delta})^2 + |\nabla\varphi^{\varepsilon\delta}|^2 + |\nabla u^{\varepsilon\delta}|^2\right\}dx.$$

The result then follows from the lemmas 3.4 and 3.6.

In order to be able to show in Section 4 that the trajectories on the attractor are compact in $C([-T,T]; H^2(\Omega))$ for all $T > 0$, we need the following result which is a direct consequence of (3.8).

Corollary 3.8 *There exist a time t_2 and a positive constant C such that*

$$\|\varphi_t^{\varepsilon\delta}\|_{L^2(t_2,+\infty;H^1(\Omega))} + \sqrt{\varepsilon}\|u_t^{\varepsilon\delta}\|_{L^2(t_2,+\infty;L^2(\Omega))} \leq C. \tag{3.15}$$

Lemma 3.9 *There exist a time $t_3 > t_2$ and a positive constant $\tilde{D}$ such that*

(i) $$\int_\Omega \left\{\delta|\nabla\varphi_t^{\varepsilon\delta}(t)|^2 + \delta\varepsilon(u_t^{\varepsilon\delta}(t))^2\right\}dx \leq \tilde{D} \quad \textit{for all } t \geq t_3; \tag{3.16}$$

(ii) $$\delta^2\int_{t_3}^{+\infty}\int_\Omega \left(\varphi_{tt}^{\varepsilon\delta}\right)^2 dx \leq \tilde{D}.$$

Proof. We differentiate (1.1) with respect to t, multiply the resulting equation by $\varphi_{tt}^{\varepsilon\delta}$ and integrate by parts to obtain

$$\begin{aligned}
\delta\int_\Omega (\varphi_{tt}^{\varepsilon\delta})^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi_t^{\varepsilon\delta}|^2 dx &= -\int_\Omega g'(\varphi^{\varepsilon\delta})\varphi_t^{\varepsilon\delta}\varphi_{tt}^{\varepsilon\delta}dx + \int_\Omega u_t^{\varepsilon\delta}\varphi_{tt}^{\varepsilon\delta}dx \\
&\leq \|g'(\varphi^{\varepsilon\delta})\|_{L^\infty(\Omega)}\left(\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega (\varphi_{tt}^{\varepsilon\delta})^2 dx\right)^{\frac{1}{2}} \\
&\quad + \int_\Omega u_t^{\varepsilon\delta}\varphi_{tt}^{\varepsilon\delta}dx.
\end{aligned}$$

Using (3.14), we deduce that for all $t \geq t_2$, $g'(\varphi^{\varepsilon\delta})$ is bounded in $L^\infty(\Omega)$ so that

$$\begin{aligned}
\delta\int_\Omega (\varphi_{tt}^{\varepsilon\delta})^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi_t^{\varepsilon\delta}|^2 dx &\leq \frac{\delta}{2}\int_\Omega (\varphi_{tt}^{\varepsilon\delta})^2 dx + \frac{C}{2\delta}\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx \\
&\quad + \int_\Omega u_t^{\varepsilon\delta}\varphi_{tt}^{\varepsilon\delta}dx.
\end{aligned} \tag{3.17}$$

Next we differentiate (1.2) with respect to t, multiply the result by $u_t^{\varepsilon\delta}$ and integrate by parts to deduce that

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega (u_t^{\varepsilon\delta})^2 dx + \int_\Omega \varphi_{tt}^{\varepsilon\delta} u_t^{\varepsilon\delta} dx + \int_\Omega |\nabla u_t^{\varepsilon\delta}|^2 dx = 0. \tag{3.18}$$

Adding up (3.17) and (3.18) and multiplying then by δ gives

$$\delta\frac{d}{dt}\int_\Omega \left\{|\nabla\varphi_t^{\varepsilon\delta}|^2 + \varepsilon(u_t^{\varepsilon\delta})^2\right\}dx + \delta^2\int_\Omega (\varphi_{tt}^{\varepsilon\delta})^2 dx \le C\int_\Omega (\varphi_t^{\varepsilon\delta})^2 dx. \tag{3.19}$$

Using also (3.15), we deduce Lemma 3.9 (i) by means of the Uniform Gronwall Lemma. Part (ii) of Lemma 3.9 then follows from integrating (3.19) in time between t_3 and t and letting t tends to $+\infty$.

Lemma 3.10 *There exists a positive constant $\hat{D}$ such that*

$$\|\varphi^{\varepsilon\delta}(t)\|_{H^3(\Omega)} \le \hat{D} \text{ for all } t \ge t_3.$$

Proof. The result of the Lemma follows from (1.1) together with (3.8), (3.6) and (3.9).

One can summarize the previous results as follows

Theorem 3.11 *There exist a time t_3 and a positive constant D such that*

$$\sqrt{\delta}\|\varphi_t^{\varepsilon\delta}(t)\|_{H^1(\Omega)} + \|\varphi^{\varepsilon\delta}(t)\|_{H^3(\Omega)} + \sqrt{\delta\varepsilon}\|u_t^{\varepsilon\delta}(t)\|_{L^2(\Omega)} + \|u^{\varepsilon\delta}(t)\|_{H^1(\Omega)} \le D$$

for all $t \ge t_3$.

4 Upper-semicontinuity of the attractor

As it has been recalled in Section 2, the semigroups corresponding to the problems (PF), (VCH) and (CH) possess global attractors that we denote by $\mathcal{A}^{\varepsilon\delta}$, $\mathcal{A}^\varepsilon$ and $\mathcal{A}$ respectively. In this section we prove the upper-semicontinuity of the attractor $\mathcal{A}^{\varepsilon\delta}$ first at $\varepsilon = 0$ for $\delta > 0$ fixed and then at $\varepsilon = \delta = 0$. The upper-semicontinuity of the attractor $\mathcal{A}^\delta$ of Problem (VCH) at $\delta = 0$ then follows.

4.1 Upper-semicontinuity of the attractor for the perturbed viscous Cahn-Hilliard equation

To begin with we define a convenient embedding $\mathcal{A}^{0\delta}$ of $\mathcal{A}^{\delta}$ into a product space. It follows from setting $\varepsilon = 0$ in equations (1.1)-(1.2) that the viscous Cahn-Hilliard equation can be written as the system

$$\begin{cases} \delta\varphi_t^\delta = \Delta\varphi^\delta - g(\varphi^\delta) + u^\delta \\ \varphi_t^\delta = \Delta u^\delta, \end{cases}$$

which gives

$$-\delta\Delta u^\delta + u^\delta = -\Delta\varphi^\delta + g(\varphi^\delta).$$

This leads us to set

$$\mathcal{A}^{0\delta} \;=\; \left\{(\varphi, \mathcal{L}_\delta(-\Delta\varphi + g(\varphi))\;,\; \varphi \in \mathcal{A}^\delta\;\right\},$$

where the operator $f \longmapsto v = \mathcal{L}_\delta(f)$ is defined as the unique solution of the Dirichlet problem

$$\begin{cases} -\delta\Delta v + v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Next we prove the following result.

Theorem 4.1 *Let δ be a fixed constant. The attractor $\mathcal{A}^{\varepsilon\delta}$ is upper-semicontinuous at $\varepsilon = 0$, i.e. the Hausdorff semidistance $d\left(\mathcal{A}^{\varepsilon\delta}, \mathcal{A}^{0\delta}\right)$ converges to zero as $\varepsilon \downarrow 0$:*

$$\lim_{\varepsilon\downarrow 0}\; \underset{(\varphi^{\varepsilon\delta},u^{\varepsilon\delta})\in\mathcal{A}^{\varepsilon\delta}}{Sup}\; \underset{(\varphi^{\delta},u^{\delta})\in\mathcal{A}^{0\delta}}{Inf} \left(\|\varphi^{\varepsilon\delta} - \varphi^\delta\|^2_{H^2(\Omega)} + \|u^{\varepsilon\delta} - u^\delta\|^2_{L^2(\Omega)}\;\right)^{\frac{1}{2}} = 0.$$

Proof. Let $\eta > 0$ be arbitrary and let $\left(\psi^{\varepsilon\delta}, v^{\varepsilon\delta}\right) \in \mathcal{A}^{\varepsilon\delta}$ be such that

$$\underset{(\psi^{\delta},v^{\delta})\in\mathcal{A}^{0\delta}}{Inf} \left(\|\psi^{\varepsilon\delta} - \psi^\delta\|^2_{H^2(\Omega)} + \|v^{\varepsilon\delta} - v^\delta\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \geq d\left(\mathcal{A}^{\varepsilon\delta}, \mathcal{A}^{0\delta}\right) - \eta, \qquad (4.1)$$

and let $(\varphi^{\varepsilon\delta}(t), u^{\varepsilon\delta}(t))_{t\in\mathbb{R}}$ be the complete orbit in $\mathcal{A}^{\varepsilon\delta}$ such that

$$\left(\varphi^{\varepsilon\delta}(0), u^{\varepsilon\delta}(0)\right) = \left(\psi^{\varepsilon\delta}, v^{\varepsilon\delta}\right). \qquad (4.2)$$

We deduce from the invariance of $\mathcal{A}^{\varepsilon\delta}$ and from Theorem 3.11 that there exists a positive constant $C = C(\delta)$ such that

$$\|\varphi_t^{\varepsilon\delta}(t)\|_{H^1(\Omega)} + \|\varphi^{\varepsilon\delta}(t)\|_{H^3(\Omega)} + \sqrt{\varepsilon}\|u_t^{\varepsilon\delta}(t)\|_{L^2(\Omega)} + \|u^{\varepsilon\delta}(t)\|_{H^1(\Omega)} \leq C \tag{4.3}$$

for all $t \in I\!R$. Since by Lemma 3.9 we also have that

$$\|\varphi_{tt}^{\varepsilon\delta}\|_{L^2(\Omega\times(-T,T))} \leq C \text{ for all } T > 0,$$

we deduce from Simon [12] that the set $\left\{\varphi^{\varepsilon\delta}\right\}_{\varepsilon\in(0,\varepsilon_0)}$ is precompact in $\mathcal{C}\left([-T,T]; H^2(\Omega)\right)$ for all $T > 0$ and that the set $\left\{\varphi_t^{\varepsilon\delta}\right\}_{\varepsilon\in(0,\varepsilon_0)}$ is precompact in $\mathcal{C}\left([-T,T]; L^2(\Omega)\right)$ for all $T > 0$. Therefore there exists a subsequence of $\left\{\varphi^{\varepsilon\delta}\right\}_{\varepsilon\in(0,\varepsilon_0)}$ which we denote again by $\left\{\varphi^{\varepsilon\delta}\right\}$ and a function φ^δ belonging to $\{\varphi \in \mathcal{C}_{loc}\left(I\!R; H^2(\Omega)\right),\ \varphi_t \in \mathcal{C}_{loc}\left(I\!R; L^2(\Omega)\right)\}$ such that

$$\begin{aligned} \varphi^{\varepsilon\delta} &\longrightarrow \varphi^\delta \text{ in } \mathcal{C}([-T,T]; H^2 \textstyle\bigcap H_0^1(\Omega)) \\ \varphi_t^{\varepsilon\delta} &\longrightarrow \varphi_t^\delta \text{ in } \mathcal{C}([-T,T]; L^2(\Omega)), \end{aligned} \tag{4.4}$$

as $\varepsilon \downarrow 0$. For $n \leq 3$ one has the embedding $H^2(\Omega) \subset L^\infty(\Omega)$ [14, p. 44-47] so that as $\varepsilon \downarrow 0$

$$g(\varphi^{\varepsilon\delta}) \longrightarrow g(\varphi^\delta) \text{ in } \mathcal{C}([-T,T]; L^2(\Omega)). \tag{4.5}$$

Therefore, using the parabolic equation

$$u^{\varepsilon\delta} = \delta\varphi_t^{\varepsilon\delta} - \Delta\varphi^{\varepsilon\delta} + g(\varphi^{\varepsilon\delta}), \tag{4.6}$$

and (4.4) we deduce that as $\varepsilon \downarrow 0$

$$u^{\varepsilon\delta} \longrightarrow u^\delta := \delta\varphi_t^\delta - \Delta\varphi^\delta + g(\varphi^\delta), \tag{4.7}$$

in $\mathcal{C}\left([-T,T]; L^2(\Omega)\right)$. Moreover we deduce from (4.3) that $g(\varphi^{\varepsilon\delta})$ is bounded in $L^\infty(-T,T; H_0^1(\Omega))$ so that by (4.6) and (4.3), as $\varepsilon \downarrow 0$

$$\Delta\varphi^{\varepsilon\delta} \rightharpoonup \Delta\varphi^\delta \text{ weakly in } L^2(-T,T; H_0^1(\Omega)). \tag{4.8}$$

Note that also by (4.3) , as $\varepsilon \downarrow 0$

$$u^{\varepsilon\delta} \rightharpoonup u^\delta \text{ weakly in } L^2(-T,T; H_0^1(\Omega)). \tag{4.9}$$

Next we show that the function φ^δ satisfies Problem (VCH). It follows from (1.1)-(1.2) that the pair $(\varphi^{\varepsilon\delta}, u^{\varepsilon\delta})$ satisfies the integral equation

$$\int_\Omega \left\{\varepsilon u_t^{\varepsilon\delta} + \varphi_t^{\varepsilon\delta}\right\} \mathcal{X} dx = \int_\Omega \left\{\delta\varphi_t^{\varepsilon\delta} - \Delta\varphi^{\varepsilon\delta} + g(\varphi^{\varepsilon\delta})\right\} \Delta\mathcal{X} dx, \tag{4.10}$$

for all $t \in (-T,T)$ and all $\mathcal{X} \in \mathcal{D}(\Omega)$. Letting $\varepsilon \downarrow 0$ in (4.10) we deduce in view of (4.3), (4.4) and (4.5) that the function φ^δ satisfies the viscous Cahn-Hilliard equation

$$\varphi_t^\delta = \Delta\left(\delta\varphi_t^\delta - \Delta\varphi^\delta + g(\varphi^\delta)\right) \quad \text{in } \mathcal{C}\left([-T,T]; L^2(\Omega)\right). \tag{4.11}$$

Also by (4.4) and (4.11), $\varphi^\delta = \Delta\varphi^\delta = 0$ on $\partial\Omega \times (-T,T)$.
Substituting (4.7) into (4.11) and then (4.11) into (4.7) and also using (4.9) we deduce that u^δ satisfies

$$\left\{\begin{array}{l} -\delta\Delta u^\delta + u^\delta = -\Delta\varphi^\delta + g(\varphi^\delta) \text{ in } \mathcal{C}([-T,T]; L^2(\Omega)) \\ u^\delta = 0 \text{ on } \partial\Omega. \end{array}\right. \tag{4.12}$$

Next note that by (4.3), $\varphi^\delta \in \mathcal{BC}(I\!R; H^2(\Omega))$ where $\mathcal{BC}$ stands for bounded continuous. Therefore, the complete trajectory $(\varphi^\delta(t))_{t\in I\!R}$ belongs to $\mathcal{A}^\delta$. Since also u^δ satisfies (4.12), it follows from the definition of $\mathcal{A}^{0\delta}$ that the pair $(\varphi^\delta(t), u^\delta(t))$ belongs to $\mathcal{A}^{0\delta}$ for each $t \in I\!R$. Moreover, it follows from (4.4) and (4.7) that $(\psi^{\varepsilon\delta}\ v^{\varepsilon\delta}) = (\varphi^{\varepsilon\delta}(0), u^{\varepsilon\delta}(0))$ converges to $(\varphi^\delta(0), u^\delta(0)) \in \mathcal{A}^{0\delta}$ in $H^2(\Omega) \times L^2(\Omega)$ as $\varepsilon \downarrow 0$. This implies that

$$\lim_{\varepsilon\downarrow 0} \quad \underset{(\psi^\delta, v^\delta)\in\mathcal{A}^{0\delta}}{Inf} \left(\|\psi^{\varepsilon\delta} - \psi^\delta\|^2_{H^2(\Omega)} + \|v^{\varepsilon\delta} - v^\delta\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = 0,$$

which, in view of (4.1), implies that for each $\eta > 0$

$$0 \le \lim_{\varepsilon\downarrow 0} \ \sup\ d\left(\mathcal{A}^{\varepsilon\delta}, \mathcal{A}^\delta\right) \le \eta\ .$$

Therefore

$$\lim_{\varepsilon\downarrow 0} d\left(\mathcal{A}^{\varepsilon\delta}, \mathcal{A}^\delta\right) = 0.$$

4.2 Upper-semicontinuity of the attractor for the perturbed Cahn-Hilliard equation

As in the case of the viscous Cahn-Hilliard equation we define a convenient embedding of $\mathcal{A}$ into a product space, namely

$$\mathcal{A}^0 \ = \ \{(\varphi, -\Delta\varphi + g(\varphi)),\ \varphi \in \mathcal{A}\}\,.$$

In what follows we prove the following result.

Theorem 4.2 *The attractor* $\mathcal{A}^\varepsilon := \mathcal{A}^{\varepsilon\varepsilon}$ *is upper-semicontinuous at* $\varepsilon = 0$, *i.e. the Hausdorff semidistance* $d(\mathcal{A}^\varepsilon, \mathcal{A}^0)$ *converges to zero as* $\varepsilon \downarrow 0$ *:*

$$\lim_{\varepsilon\downarrow 0} \; \underset{(\varphi^\varepsilon,u^\varepsilon)\in\mathcal{A}^\varepsilon}{Sup} \; \underset{(\varphi,u)\in\mathcal{A}^0}{Inf} \left(\|\varphi^\varepsilon - \varphi\|^2_{H^2(\Omega)} + \|u^\varepsilon - u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = 0.$$

Proof. Let $\eta > 0$ be arbitrary and let $(\psi^\varepsilon, v^\varepsilon) \in \mathcal{A}^\varepsilon$ be such that

$$\underset{(\psi,v)\in\mathcal{A}^0}{Inf} \left(\|\psi^\varepsilon - \psi\|^2_{H^2(\Omega)} + \|v^\varepsilon - v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \geq d\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) - \eta, \tag{4.13}$$

and let $(\varphi^\varepsilon(t), u^\varepsilon(t))_{t\in I\!R}$ be the complete orbit in $\mathcal{A}^\varepsilon$ such that

$$(\varphi^\varepsilon(0), u^\varepsilon(0)) = (\psi^\varepsilon, v^\varepsilon). \tag{4.14}$$

The invariance of $\mathcal{A}^\varepsilon$ together with Theorem 3.11 imply that there exists a positive constant C such that

$$\sqrt{\varepsilon}\|\varphi^\varepsilon_t(t)\|_{H^1(\Omega)} + \|\varphi^\varepsilon(t)\|_{H^3(\Omega)} + \|u^\varepsilon(t)\|_{H^1(\Omega)} \leq C \tag{4.15}$$

for all $t \in I\!R$. Since by the Corollary 3.8 we have that for all $T > 0$

$$\|\varphi^\varepsilon_t\|_{L^2(-T,T;H^1(\Omega))} + \sqrt{\varepsilon}\|u^{\varepsilon\delta}_t\|_{L^2(\Omega\times(-T,T))} \leq C, \tag{4.16}$$

it follows from Simon [12] that there exists a subsequence of $\{\varphi^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ that we denote again by $\{\varphi^\varepsilon\}$ and a function $\varphi \in L^\infty(-T,T;H^2(\Omega)\bigcap H^1_0(\Omega))$ such that

$$\varphi^\varepsilon \longrightarrow \varphi \text{ in } \mathcal{C}([-T,T];H^2(\Omega)), \tag{4.17}$$

as $\varepsilon \downarrow 0$. In turn, (4.17) and (4.15) imply that as $\varepsilon \downarrow 0$

$$g(\varphi^{\varepsilon\delta}) \longrightarrow g(\varphi) \text{ in } \mathcal{C}([-T,T];L^2(\Omega)) \text{ and weakly in } L^2(-T,T;H^1_0(\Omega)). \tag{4.18}$$

Next we use that

$$u^\varepsilon - \varepsilon\varphi^\varepsilon_t = -\Delta\varphi^\varepsilon + g(\varphi^\varepsilon), \tag{4.19}$$

to deduce from (4.15) and (4.17)-(4.18) that there exists $u \in \mathcal{C}([-T,T];L^2(\Omega))$ such that as $\varepsilon \downarrow 0$

$$u^\varepsilon \longrightarrow u \text{ in } \mathcal{C}([-T,T];L^2(\Omega)) \text{ and weakly in } L^2(-T,T;H^1_0(\Omega)), \tag{4.20}$$

and

$$u = -\Delta\varphi + g(\varphi) \ \text{ in } \mathcal{C}\left([-T,T]; L^2(\Omega)\right) \text{ and in } L^2(-T,T; H_0^1(\Omega)). \qquad (4.21)$$

Also by (4.21), (4.18) and (4.20) one has the boundary condition $\Delta\varphi \in L^2(-T,T;H_0^1(\Omega))$. Since the pair $(\varphi^{\epsilon\delta}, u^{\epsilon\delta})$ satisfies the equation

$$\varepsilon u_t^\epsilon + \varphi_t^\epsilon = \Delta\left(\varepsilon\varphi_t^\epsilon - \Delta\varphi^\epsilon + g(\varphi^\epsilon)\right),$$

it is clear that it satisfies as well the integral equation

$$\int_{-T}^{T}\int_\Omega (\varepsilon u_t^\epsilon + \varphi_t^\epsilon)\mathcal{X}\,dxds = \int_{-T}^{T}\int_\Omega \{\varepsilon\varphi_t^\epsilon - \Delta\varphi^\epsilon + g(\varphi^\epsilon)\}\,\Delta\mathcal{X}\,dxds, \qquad (4.22)$$

for all $\mathcal{X} \in \mathcal{D}(\Omega\times(-T,T))$. Next we let $\varepsilon \downarrow 0$ in (4.22) and use (4.15), (4.17), (4.18) and (4.16) to deduce that the function φ satisfies the Cahn-Hilliard equation

$$\varphi_t + \Delta\left(\Delta\varphi - g(\varphi)\right) = 0,$$

in $L^2(-T,T;H^1(\Omega))$, together with the boundary conditions $\varphi = \Delta\varphi = 0$.

Moreover it follows from (4.15) that $\varphi \in \mathcal{B}C\,(I\!R; H^2(\Omega))$. Therefore, the complete trajectory $(\varphi(t))_{t\in I\!R}$ belongs to $\mathcal{A}$. Since also $u \in \mathcal{B}C\,(I\!R; L^2(\Omega))$ and satisfies (4.21), it follows that the pair $(\varphi(t), u(t)) \in \mathcal{A}^0$ for each $t \in I\!R$. Finally it follows from (4.17) and (4.20) that $(\psi^\epsilon, v^\epsilon) = (\varphi^\epsilon(0), u^\epsilon(0))$ converges to $(\varphi(0), u(0))$ in $H^2(\Omega)\times L^2(\Omega)$ as $\varepsilon \downarrow 0$.

This implies that

$$\lim_{\epsilon\downarrow 0} d\left(\mathcal{A}^\epsilon, \mathcal{A}^0\right) = 0.$$

4.3 Upper-semicontinuity of the attractor for viscous Cahn-Hilliard equation in the limit of the Cahn-Hilliard equation

Up to now we have been interested in the upper-semicontinuity of the attractor of the phase field model. However, the upper-semicontinuity of the attractor of the viscous Cahn-Hilliard equation appears to be an immediate consequence of our estimates, as we will see below. Note that this result has already been shown by Elliott and Stuart [7].

Theorem 4.3 *The attractor $\mathcal{A}^\delta$ is upper-semicontinuous at $\delta = 0$, i.e. the Hausdorff semidistance $d\left(\mathcal{A}^\delta, \mathcal{A}\right)$ converges to zero as $\delta \downarrow 0$:*

$$\lim_{\delta\downarrow 0}\ \underset{\varphi^\delta\in\mathcal{A}^\delta}{Sup}\ \underset{\varphi\in\mathcal{A}}{Inf}\, \|\varphi^\delta - \varphi\|_{H^2(\Omega)} = 0.$$

Proof. Let $\eta > 0$ be arbitrary and let $\psi^\delta \in \mathcal{A}^\delta$ be such that

$$\underset{\varphi\in\mathcal{A}}{Inf}\, \|\psi^\delta - \varphi\|_{H^2(\Omega)} \geq d\left(\mathcal{A}^\delta, \mathcal{A}\right) - \eta, \tag{4.23}$$

and let $(\varphi^\delta(t))_{t\in I\!R}$ be the complete orbit in $\mathcal{A}^\delta$ such that $\varphi^\delta(0) = \psi^\delta$. The invariance of $\mathcal{A}^\delta$ together with Theorem 3.11 imply that there exists a positive constant C such that

$$\sqrt{\delta}\|\varphi_t^\delta(t)\|_{H^1(\Omega)} + \|\varphi^\delta(t)\|_{H^3(\Omega)} + \|u^\delta(t)\|_{H^1(\Omega)} \leq C \tag{4.24}$$

for all $t \in I\!R$.
Since by Corollary 3.9 we have that for all $T > 0$

$$\|\varphi_t^\delta\|_{L^2(-T,T;H^1(\Omega))} \leq C, \tag{4.25}$$

it follows as in section 4.2 that there exists a subsequence of $\left\{\varphi^\delta\right\}_{\delta\in(0,1)}$ that we denote again by $\left\{\varphi^\delta\right\}$ and a function $\varphi \in L^\infty(-T,T;H^2(\Omega)\bigcap H_0^1(\Omega))$ such that as $\delta \downarrow 0$

$$\varphi^\delta \longrightarrow \varphi \text{ in } \mathcal{C}([-T,T];H^2(\Omega)), \tag{4.26}$$

and that

$$g(\varphi^\delta) \longrightarrow g(\varphi) \text{ in } \mathcal{C}([-T,T];L^2(\Omega)) \text{ and weakly in } L^2(-T,T;H_0^1(\Omega)). \tag{4.27}$$

Next we use that

$$u^\delta - \delta\varphi_t^\delta = -\Delta\varphi^\delta + g(\varphi^\delta), \tag{4.28}$$

to deduce from (4.24), (4.26) and (4.27) that there exists $u \in \mathcal{C}([-T,T];L^2(\Omega))$ such that as $\delta \downarrow 0$

$$u^\delta \longrightarrow u \text{ in } \mathcal{C}([-T,T];L^2(\Omega)) \text{ and weakly in } L^2(-T,T;H_0^1(\Omega)), \tag{4.29}$$

and

$$u = -\Delta\varphi + g(\varphi) \;\text{ in } \mathcal{C}\left([-T,T];L^2(\Omega)\right) \text{ and in } L^2(-T,T;H_0^1(\Omega)). \tag{4.30}$$

Furthermore, it follows from (4.30), (4.27) and (4.29) together with equality (4.28) that $\Delta\varphi \in L^2(-T,T;H_0^1(\Omega))$.
Next we note that φ^δ satisfies the integral equation

$$\int_{-T}^{T}\int_\Omega \left\{\varphi^\delta \mathcal{X}_t + (\delta\varphi_t^\delta - \Delta\varphi^\delta + g(\varphi^\delta))\Delta\mathcal{X}\right\} dxds = 0, \tag{4.31}$$

for all $\mathcal{X} \in \mathcal{D}(\Omega \times (-T,T))$, we let $\delta \downarrow 0$ in (4.31) and use (4.24), (4.26), (4.27) and (4.25) to deduce that the function φ satisfies the Cahn-Hilliard equation in $L^2(-T,T;H^1(\Omega))$, together with the boundary conditions of Problem (CH).

Moreover it follows from (4.24) that $\varphi \in \mathcal{BC}(I\!R;H^2(\Omega))$ so that the complete trajectory $(\varphi(t))_{t\in I\!R}$ belongs to $\mathcal{A}$. Finally (4.26) implies that $\psi^\delta = \varphi^\delta(0)$ converges to $\varphi(0)$ in $H^2(\Omega)$ as $\delta \downarrow 0$. This implies the result of Theorem 4.3.

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A singularly perturbed phase field model with a logarithmic nonlinearity

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Abstract. We consider a phase field model with a logarithmic nonlinearity. We prove that this model is well-posed and that it possesses a maximal attractor that is upper-semicontinuous.

Résumé. Nous considérons un modèle de champ de phase avec terme non linéaire de type logarithmique. Nous montrons que ce modèle possède une solution unique et qu'il admet un attracteur maximal. Nous prouvons que cet attracteur est semi-continu supérieurement.

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Key words : System of second order, nonlinear parabolic equations, maximal attractor, upper-semicontinuity

A singularly perturbed phase field model with a logarithmic nonlinearity

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1 Introduction

In this paper we consider a phase field model and a Cahn-Hilliard equation with a logarithmic nonlinearity. More precisely we consider the Dirichlet problem

$$(P^\varepsilon)\quad \begin{cases} \varepsilon\varphi_t = \Delta\varphi + \alpha\varphi - g(\varphi) + u & \text{in } \Omega \times I\!R^+, \quad (1.1) \\ \varepsilon u_t + \varphi_t = \Delta u & \text{in } \Omega \times I\!R^+, \quad (1.2) \\ \varphi = u = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x)\ ,\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where Ω is an open bounded set of $I\!R^n$ with smooth boundary $\partial\Omega$, $\varepsilon \in (0, M)$ for some given constant M, $\alpha > 1$ is a given constant and the function g has the form

$$g(s) = \frac{1}{2} ln\left(\frac{1+s}{1-s}\right).$$

The function φ can be interpreted as an order parameter whereas the function u stands for the temperature. We suppose furthermore that $(\varphi_0, u_0) \in \mathcal{K} \times L^2(\Omega)$ where

$$\mathcal{K} = \left\{\psi \in L^2(\Omega),\ -1 \le \psi \le 1 \text{ a.e. in } \Omega\right\}.$$

Setting $\varepsilon = 0$ and substituting equation (1.1) into equation (1.2), one obtains the Cahn-Hilliard equation for the single unknown function φ. This leads us to consider as well the Dirichlet problem

$$(P)\begin{cases} \varphi_t + \Delta\left(\Delta\varphi + \alpha\varphi - g(\varphi)\right) = 0 & \text{in } \Omega \times I\!R^+, \quad (1.3) \\ \varphi = \Delta\varphi = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x) & x \in \Omega. \end{cases}$$

Elliott and Luckhaus [4] prove the existence and uniqueness of the solution of an extension of Problem (P) with a vector unknown. Debussche and Dettori [2] give an alternative proof for the existence and uniqueness of the solution of Problem (P) and show the existence of a maximal attractor of finite Hausdorff dimension.

The purpose of this paper is twofold : (i) show that Problem (P^ε) is well-posed and possesses a maximal attractor $\mathcal{A}^\varepsilon$ in arbitrary space dimension; (ii) prove an upper-semicontinuity property for this attractor at $\varepsilon = 0$ in the case that $n \leq 3$.

Following an idea of Debussche and Dettori [2] we consider the approximation of the function g

$$g_N(s) = \sum_{k=0}^{N} \frac{s^{2k+1}}{2k+1}, \tag{1.4}$$

and the boundary value problem (P_N^ε) that one obtains by replacing g by g_N in Problem (P^ε).

In Section 2, we recall the precise results of [2] about Problem (P) and the results of Brochet, Chen and Hilhorst [1] about Problem (P_N^ε).

In Section 3, we give a priori estimates which do not depend on N for the solution pair $(\varphi_N^\varepsilon, u_N^\varepsilon)$ of Problem (P_N^ε). Many of the estimates are also independent of ε.

These estimates permit to establish in Section 4 the existence and uniqueness of the solution $(\varphi^\varepsilon, u^\varepsilon) = S^\varepsilon(t)(\varphi_0, u_0)$ of Problem (P^ε) such that $S^\varepsilon(t)$ satisfies the usual semigroup properties and such that for each $t > 0$, $S^\varepsilon(t)$ is a continuous mapping from $\mathcal{K} \times L^2(\Omega)$ into itself.

We prove the existence of a maximal attractor $\mathcal{A}^\varepsilon$ in Section 5.

The function pair $(\varphi, v) = (\varphi, \sqrt{\varepsilon} u)$ satisfies the rescaled boundary value problem

$$(P_r^\varepsilon) \quad \begin{cases} \varepsilon\varphi_t = \Delta\varphi + \alpha\varphi - g(\varphi) + \dfrac{1}{\sqrt{\varepsilon}} v & \text{in } \Omega \times I\!R^+, \quad (1.5) \\ \sqrt{\varepsilon} v_t + \varphi_t = \dfrac{1}{\sqrt{\varepsilon}} \Delta v & \text{in } \Omega \times I\!R^+, \quad (1.6) \\ \varphi = v = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x)\ ,\ v(x,0) = v_0(x) := \sqrt{\varepsilon} u_0(x) & x \in \Omega. \end{cases}$$

In Section 6, we reformulate the results of the sections 3, 4 and 5 in terms of the function pair (φ, v), namely we state an existence and uniqueness result for the solution of Problem (P_r^ε) and a result about the existence of a maximal attractor $\mathcal{A}_r^\varepsilon$. We then show the upper-semicontinuity of $\mathcal{A}_r^\varepsilon$ at $\varepsilon = 0$.

This paper extends similar results obtained by Dupaix, Hilhorst and Laurençot [3] and Brochet, Chen and Hilhorst [1] in the case that the nonlinear function g is a polynomial function.

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2 Preliminaries

In this section we introduce some notations used in this paper and recall results of Debussche and Dettori [2] about the existence and uniqueness of the solution of Problem (P) and about the existence of a maximal attractor of Problem (P). We then give a theorem due to Brochet, Chen and Hilhorst [1] about the existence and uniqueness of a smooth solution of Problem (P_N^ε).

In the sequel we will use the scalar product and the norm in $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$. For $w \in H^{-1}(\Omega)$ we define

$$\psi = \mathcal{N}w$$

as the unique solution in $H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta\psi &= w \text{ in the sense of distributions in } \Omega, \\ \psi &= 0 \text{ on } \partial\Omega. \end{cases}$$

Then if $v,\ w \in H^{-1}(\Omega)$ and if $\psi = \mathcal{N}v$, $\xi = \mathcal{N}w$

$$(v, w)_{H^{-1}(\Omega)} = \int_\Omega \nabla\psi\nabla\xi dx,$$

and

$$\|v\|^2_{H^{-1}(\Omega)} = \int_\Omega |\nabla\psi|^2 dx.$$

Finally we recall results about the Cahn-Hilliard Problem (P) and the phase field model (P_N^ε).

Theorem 2.1 *[2] For any $\varphi_0 \in \mathcal{K}$, Problem (P) has a unique solution φ satisfying*

$$\varphi \in L^\infty\left(0,T; L^2(\Omega)\right) \bigcap L^2(0,T; H^2(\Omega))$$

for all $T > 0$, and

$$\varphi \in C\left(I\!R^+; L^2(\Omega)\right).$$

Moreover for all $t > 0$,

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1$$

and the set $\{x \in \Omega,\ |\varphi(x,t)| = 1\}$ has measure zero.
The mapping $S(t)\ :\ \varphi_0 \longmapsto \varphi(t)$ is continuous on $\mathcal{K}$ endowed with the topology of $L^2(\Omega)$ for all $t > 0$ and $(S(t))_{t\geq 0}$ is a semigroup on $\mathcal{K}$.

Theorem 2.2 *[2] The semigroup $(S(t))_{t\geq 0}$ associated with Problem (P) possesses in $\mathcal{K}$ a maximal attractor $\mathcal{A}$ that is connected.*

Theorem 2.3 *[1] For any $(\varphi_0, u_0) \in (L^2(\Omega))^2$, Problem (P_N^ε) has a unique solution $(\varphi_N^\varepsilon, u_N^\varepsilon)$ which satisfies*

$$(\varphi_N^\varepsilon, u_N^\varepsilon) \in L^\infty\left(0,T;(L^2(\Omega))^2\right) \cap L^2(0,T;(H_0^1(\Omega))^2), \quad \varphi_N^\varepsilon \in L^{2p}(Q_T)$$

for all $T > 0$, where $Q_T := \Omega \times (0,T)$ and

$$(\varphi_N^\varepsilon, u_N^\varepsilon) \in \mathcal{C}\left(I\!R^+;\left(L^2(\Omega)\right)^2\right).$$

Moreover

$$(\varphi_N^\varepsilon, u_N^\varepsilon) \in \left(\mathcal{C}^\infty(\overline{\Omega} \times (0,+\infty))\right)^2,$$

and the mapping

$$S_N^\varepsilon(t) \ : \ (\varphi_0, u_0) \longmapsto (\varphi_N^\varepsilon(t), u_N^\varepsilon(t))$$

is Lipschitz continuous on $(L^2(\Omega))^2$ for all $t > 0$ and $(S_N^\varepsilon(t))_{t\geq 0}$ is a semigroup on $(L^2(\Omega))^2$.

3 Uniform a priori estimates for the solutions of Problem (P_N^ε)

The main purpose of this section is twofold : obtain uniform a priori estimates with respect to N for the solutions of (P_N^ε); obtain uniform a priori estimates with respect to N and ε for these solutions. The first kind of estimates permits to prove results concerning the well-posedness of Problem (P^ε) and the existence of a maximal attractor whereas the second kind is used to deduce the upper-semicontinuity of the attractor of Problem (P_τ^ε).
Problem (P_N^ε) can be written as

$$(P_N^\varepsilon) \quad \begin{cases} \varepsilon\varphi_t = \Delta\varphi + \alpha\varphi - g_N(\varphi) + u & \text{in } \Omega \times I\!R^+, \quad (3.1) \\ \varepsilon u_t + \varphi_t = \Delta u & \text{in } \Omega \times I\!R^+, \quad (3.2) \\ \varphi = u = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \varphi(x,0) = \varphi_0(x)\ ,\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where g_N is given by (1.4).

A key ingredient for the forthcoming estimates is the functional

$$V_N^\varepsilon(\varphi, u) = \int_\Omega \left\{\frac{1}{2}|\nabla\varphi|^2 - \frac{\alpha}{2}\varphi^2 + G_N(\varphi) + \frac{\varepsilon}{2}u^2\right\}dx,$$

where $G_N(s) = \int_0^s g_N(\tau)d\tau = \sum_{k=0}^{N} \frac{s^{2k+2}}{(2k+2)(2k+1)}$.

Note that if $\varphi \in \mathcal{K}$ then $G_N(\varphi) \leq G(\varphi)$ where G is defined by

$$G(s) = \int_0^s g(\tau)d\tau = \sum_{k=0}^{+\infty} \frac{s^{2k+2}}{(2k+2)(2k+1)} = \frac{1}{2}\left((1+s)ln(1+s) + (1-s)ln(1-s)\right)$$

We remark that $0 \leq G(s) \leq ln(2)$ and that $g'(s) \geq 2$ for all $s \in (-1,1)$. Therefore

$$\begin{aligned} V_N^\varepsilon(\varphi,u) \leq V^\varepsilon(\varphi,u) \; &:= \; \int_\Omega \left\{\frac{1}{2}|\nabla\varphi|^2 - \frac{\alpha}{2}\varphi^2 + G(\varphi) + \frac{\varepsilon}{2}u^2\right\}dx \\ &\leq \; \tfrac{1}{2}\|\varphi\|^2_{H_0^1(\Omega)} + \tfrac{\varepsilon}{2}\|u\|^2_{L^2(\Omega)} + ln(2)|\Omega|. \end{aligned}$$

In particular this implies that the estimates for which the right-hand-side only depends on $V_N^\varepsilon(\varphi_0, u_0)$ are uniform with respect to N.

We now prove a result that will be useful to deduce estimates for $(\varphi_N^\varepsilon, \sqrt{\varepsilon}u_N^\varepsilon)$ in $H_0^1(\Omega) \times L^2(\Omega)$.

Lemma 3.1 *Let $(\varphi,u) \in \left(H_0^1(\Omega) \bigcap L^{2N+2}(\Omega)\right) \times L^2(\Omega)$. Then*

$$\frac{1}{2}\|\varphi\|^2_{H_0^1(\Omega)} \; + \; \frac{\varepsilon}{2}\|u\|^2_{L^2(\Omega)} \leq \frac{3}{2}\alpha^2|\Omega| + V_N^\varepsilon(\varphi,u). \tag{3.3}$$

Proof. We have that

$$\begin{aligned} &\int_\Omega \left\{-\frac{\alpha}{2}\varphi^2 + G_N(\varphi)\right\}dx \\ &= \int_\Omega \left\{-\frac{\alpha}{2}\varphi^2 + \frac{1}{24}\varphi^4 + \frac{1}{2}\varphi^2 + \frac{1}{24}\varphi^4 + \sum_{k=2}^{N} \frac{\varphi^{2k+2}}{(2k+2)(2k+1)}\right\}dx. \end{aligned}$$

Thus using the inequality

$$-\frac{\alpha}{2}s^2 + \frac{1}{24}s^4 \geq -\frac{3}{2}\alpha^2,$$

it follows that

$$\int_\Omega \left\{-\frac{\alpha}{2}\varphi^2 + G_N(\varphi)\right\}dx \geq -\frac{3}{2}\alpha^2|\Omega| + \frac{1}{2}\int_\Omega G_N(\varphi)dx, \tag{3.4}$$

so that

$$\begin{aligned} V_N^\varepsilon(\varphi,u) &\geq -\tfrac{3}{2}\alpha^2|\Omega| + \tfrac{1}{2}\int_\Omega \left(|\nabla\varphi|^2 + G_N(\varphi) + \varepsilon u^2\right)dx \\ &\geq -\tfrac{3}{2}\alpha^2|\Omega| + \tfrac{1}{2}\|\varphi\|^2_{H_0^1(\Omega)} + \tfrac{\varepsilon}{2}\|u\|^2_{L^2(\Omega)}. \end{aligned}$$

3.1 Uniform estimates with respect to N

In this section we prove estimates for the solution of Problem (P_N^ε). They will permit to prove the existence of solutions for Problem (P^ε) in Section 4.

Throughout this section, we assume that the initial functions (φ_0, u_0) satisfy

$$(\varphi_0, u_0) \in \left(\mathcal{K} \bigcap H_0^1(\Omega)\right) \times L^2(\Omega).$$

Next we prove the following result.

Lemma 3.2 *There exists a positive constant C such that for all $T > 0$, that*

$$\|\varphi_N^\varepsilon(t)\|^2_{L^2(\Omega)} + \|u_N^\varepsilon(t)\|^2_{L^2(\Omega)} \leq \frac{C}{\varepsilon^2}\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right)\exp\left(\frac{C}{\varepsilon^2}T\right) \tag{3.5}$$

for all $t \in (0,T)$;

$$\begin{aligned}\|\varphi_N^\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} &+ \varepsilon\|u_N^\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &+ \|G_N(\varphi_N^\varepsilon)\|_{L^1(0,T;L^1(\Omega))} \leq C\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right)\exp\left(\frac{C}{\varepsilon^2}T\right).\end{aligned} \tag{3.6}$$

Proof. We multiply the equation (3.1) by $\frac{1}{\varepsilon}\varphi_N^\varepsilon$, use the inequality

$$\varphi_N^\varepsilon g_N(\varphi_N^\varepsilon) \geq G_N(\varphi_N^\varepsilon)$$

multiply the equation (3.2) by $\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon$, and add the two resulting inequalities to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{(\varphi_N^\varepsilon)^2 + (\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon)^2\right\}dx + \frac{1}{2}\int_\Omega\left\{\frac{1}{\varepsilon}|\nabla\varphi_N^\varepsilon|^2 + \varepsilon|\nabla u_N^\varepsilon|^2 + \frac{1}{\varepsilon}G_N(\varphi_N^\varepsilon)\right\}dx \\
&\leq \frac{1}{\varepsilon}\int_\Omega\left\{\alpha(\varphi_N^\varepsilon)^2 + \varphi_N^\varepsilon u_N^\varepsilon\right\}dx \\
&\leq \frac{\alpha}{\varepsilon}\int_\Omega(\varphi_N^\varepsilon)^2dx + \frac{1}{\varepsilon^2}\int_\Omega(\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon - \varphi_N^\varepsilon)\varphi_N^\varepsilon dx, \qquad (3.7)\\
&\leq \frac{\alpha}{\varepsilon}\int_\Omega(\varphi_N^\varepsilon)^2dx + \frac{1}{2\varepsilon^2}\int_\Omega(\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon)^2dx, \\
&\leq \frac{C_1}{\varepsilon^2}\int_\Omega\left\{(\varphi_N^\varepsilon)^2 + (\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon)^2\right\}dx,
\end{aligned}$$

so that using Gronwall's Lemma we deduce that

$$\begin{aligned}\int_\Omega\left\{(\varphi_N^\varepsilon(t))^2 + (\varepsilon u_N^\varepsilon(t) + \varphi_N^\varepsilon(t))^2\right\}dx &\leq \int_\Omega\left(\varphi_0^2 + (\varepsilon u_0 + \varphi_0)^2\right)dx\exp\left(\frac{2C_1}{\varepsilon^2}t\right) \\ &\leq \int_\Omega\left(3\varphi_0^2 + 2\varepsilon^2u_0^2\right)dx\exp\left(\frac{2C_1}{\varepsilon^2}t\right).\end{aligned}$$

Moreover we have that

$$\begin{aligned}\int_\Omega \left\{(\varphi_N^\varepsilon)^2 + (u_N^\varepsilon)^2\right\}dx &= \int_\Omega \left\{(\varphi_N^\varepsilon)^2 + \frac{1}{\varepsilon^2}(\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon - \varphi_N^\varepsilon)^2\right\}dx \\ &\leq \int_\Omega \left\{(1+\frac{2}{\varepsilon^2})(\varphi_N^\varepsilon)^2 + \frac{2}{\varepsilon^2}(\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon)^2\right\}dx \\ &\leq \frac{C_2}{\varepsilon^2}\int_\Omega \left\{(\varphi_N^\varepsilon)^2 + (\varepsilon u_N^\varepsilon + \varphi_N^\varepsilon)^2\right\}dx,\end{aligned}$$

and thus

$$\begin{aligned}\int_\Omega \left\{(\varphi_N^\varepsilon)^2 + (u_N^\varepsilon)^2\right\}dx &\leq \frac{C_2}{\varepsilon^2}\int_\Omega \left(3\varphi_0^2 + 2\varepsilon^2 u_0^2\right)dx \exp\left(\frac{2C_1}{\varepsilon^2}t\right) \\ &\leq \frac{C_3}{\varepsilon^2}\int_\Omega \left(\varphi_0^2 + u_0^2\right)dx \exp\left(\frac{2C_1}{\varepsilon^2}T\right)\end{aligned}$$

which implies (3.5). (3.6) then follows from integrating (3.7) in time.

Lemma 3.3 *There exists a positive constant C such that For all $T > 0$*

$$\int_0^T V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)ds \leq C\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right)\exp\left(\frac{C}{\varepsilon^2}T\right). \tag{3.8}$$

Proof. We have that

$$\int_0^T V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)ds \leq \int_0^T \int_\Omega \left\{|\nabla\varphi_N^\varepsilon|^2 + G_N(\varphi_N^\varepsilon) + \varepsilon(u_N^\varepsilon)^2\right\}dxds,$$

which in view of (3.6) implies (3.8).

The result below expresses the fact that V_N^ε is a Lyapunov functional for Problem (P_N^ε).

Lemma 3.4 *For all $t \geq 0$ and all $r > 0$, the solution $(\varphi_N^\varepsilon, u_N^\varepsilon)$ of Problem (P_N^ε) satisfies*

(i) $$\frac{d}{dt}V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(t) \leq 0; \tag{3.9}$$

(ii) $$\begin{aligned} V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(t+r) + \int_t^{t+r}\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dxds \\ = V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(t).\end{aligned} \tag{3.10}$$

Proof. We multiply the equation (3.1) by $(\varphi_N^\varepsilon)_t$, the equation (3.2) by u_N^ε, integrate by parts and add the resulting equations to obtain

$$\frac{d}{dt}V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon) = \int_\Omega \left\{-\varepsilon(\varphi_N^\varepsilon)_t^2 - |\nabla u_N^\varepsilon|^2\right\}dx, \tag{3.11}$$

which implies (3.9) and (3.10).

Lemma 3.5 *There exists a positive constant C such that for all $T > 0$*

$$\begin{aligned}\int_0^T \int_\Omega t\left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dxds \\ \leq C\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)} + T\right)\exp\left(\frac{C}{\varepsilon^2}T\right).\end{aligned} \tag{3.12}$$

Proof. We multiply the equation (3.11) by t to obtain

$$\frac{d}{dt}(tV_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)) + \int_\Omega t\left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dxdt \leq V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon).$$

Next we integrate this inequality in time between r and t using also (3.8) and (3.10) to deduce that

$$\begin{aligned}tV_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(t) + & \int_r^t \int_\Omega s\left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dxds \\ \leq & \int_0^T V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(s)ds + rV_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(r) \\ \leq & \ C\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\right)\exp\left(\frac{C}{\varepsilon^2}T\right) + rV_N^\varepsilon(\varphi_0, u_0),\end{aligned}$$

and let then $r \downarrow 0$ to deduce (3.12) in view of (3.3).

Lemma 3.6 *There exists a positive constant C such that for all $r > 0$*

$$\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t) + |\nabla u_N^\varepsilon(t)|^2\right\}dx \leq C\left(\frac{1}{r} + \frac{1}{\varepsilon}\right)\ (V_N^\varepsilon(\varphi_0, u_0) + 1) \tag{3.13}$$

for all $t \geq r > 0$.

Proof. We differentiate equation (3.1) with respect to t, multiply the result by $(\varphi_N^\varepsilon)_t$, integrate by parts and use the monotonicity of g_N to obtain

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega (\varphi_N^\varepsilon)_t^2 dx + \int_\Omega |\nabla(\varphi_N^\varepsilon)_t|^2 dx \leq \int_\Omega \left\{\alpha(\varphi_N^\varepsilon)_t^2 + (u_N^\varepsilon)_t(\varphi_N^\varepsilon)_t\right\}dx. \tag{3.14}$$

Multiplying equation (3.2) by $(u_N^\varepsilon)_t$ and integrating by parts gives

$$\int_\Omega \left\{\varepsilon(u_N^\varepsilon)_t^2 + (\varphi_N^\varepsilon)_t(u_N^\varepsilon)_t\right\}dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_N^\varepsilon|^2 dx = 0. \tag{3.15}$$

Hence adding up (3.14) and (3.15) we obtain

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dx + \int_\Omega \left\{|\nabla(\varphi_N^\varepsilon)_t|^2 + \varepsilon(u_N^\varepsilon)_t^2\right\}dx\\ &\leq \alpha\int_\Omega (\varphi_N^\varepsilon)_t^2 dx.\end{aligned} \tag{3.16}$$

Using (3.10) and (3.3) we can apply the Uniform Gronwall's Lemma to (3.16) and thus we deduce that for all $s > 0$

$$\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t+s) + |\nabla u_N^\varepsilon(t+s)|^2\right\}dx \leq C\left(\frac{1}{s}+\frac{\alpha}{\varepsilon}\right)\,(V_N^\varepsilon(\varphi_0,u_0)+1) \tag{3.17}$$

for all $t \geq 0$, which completes the proof of (3.13).

Lemma 3.7 *There exists a positive constant C such that for all $T > 0$ and all $t \in (0,T)$*

$$\begin{aligned}&t^2\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t) + |\nabla u_N^\varepsilon(t)|^2\right\}dx\\ &\qquad\leq \tfrac{C}{\varepsilon}(T+1)\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)} + T\right)\exp\left(\frac{C}{\varepsilon^2}T\right).\end{aligned} \tag{3.18}$$

Proof. Multiplying inequality (3.16) by t^2 we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega t^2\left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dx \leq t^2\alpha\int_\Omega (\varphi_N^\varepsilon)_t^2 dx + t\int_\Omega\left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dx.$$

Next we integrate between r and t, using also (3.12) and (3.13)

$$\begin{aligned}&\tfrac{t^2}{2}\int_\Omega\left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t) + |\nabla u_N^\varepsilon|^2(t)\right\}dx\\ &\leq \int_r^t\int_\Omega\left\{t^2\alpha(\varphi_N^\varepsilon)_t^2 + t\left(\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right)\right\}dx + \frac{r^2}{2}\int_\Omega\left\{\varepsilon(\varphi_N^\varepsilon)_t^2(r) + |\nabla u_N^\varepsilon|^2(r)\right\}dx\\ &\leq C_1(\tfrac{T}{\varepsilon}+1)\left(\|\varphi_0\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)} + T\right)\exp\left(\frac{C_1}{\varepsilon^2}T\right) + C_2\, r^2(\frac{1}{\varepsilon}+\frac{1}{r})\left(1 + V_N^\varepsilon(\varphi_0,u_0)\right).\end{aligned}$$

The result of Lemma 3.7 then follows from letting $r \downarrow 0$.

Lemma 3.8 *There exists a positive constant C such that for all $r > 0$*

$$\int_\Omega (\Delta\varphi_N^\varepsilon(t))^2 dx \leq C(\frac{1}{r}+\frac{1}{\varepsilon})(V_N^\varepsilon(\varphi_0,u_0)+1) \textit{ for all } t \geq r > 0. \tag{3.19}$$

Proof. We multiply equation (3.1) by $\Delta\varphi_N^\varepsilon$ and integrate by parts. This gives using the monotonicity of g_N

$$\frac{1}{2}\int_\Omega (\Delta\varphi_N^\varepsilon)^2 dx \le \frac{\varepsilon^2}{2}\int_\Omega (\varphi_N^\varepsilon)_t^2 dx + C\int_\Omega \left\{|\nabla\varphi_N^\varepsilon|^2 + |\nabla u_N^\varepsilon|^2\right\}dx, \qquad (3.20)$$

and (3.19) follows from (3.13), (3.3) and (3.10).

Lemma 3.9 *For all $r > 0$ there exists a positive constant $C = C(r)$ such that*

$$\begin{aligned}\int_t^{t+r}\int_\Omega (\Delta\varphi_N^\varepsilon)^2 dx &\le C(V_N^\varepsilon(\varphi_0, u_0) + 1) \qquad (3.21)\\ &\le C(V^\varepsilon(\varphi_0, u_0) + 1)\end{aligned}$$

for all $t \ge 0$.

Proof. We integrate (3.20) between t and $t + r$ and use (3.3) and (3.10) to deduce (3.21).

Lemma 3.10 *There exists a positive constant C such that for all $T > 0$ and all $t \in (0, T)$*

$$t^2\int_\Omega g_N^2(\varphi_N^\varepsilon(t))dx \le \frac{C}{\varepsilon^2}(T+1)^2\left(\|\varphi_0\|_{L^2(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 + T\right)\exp\left(\frac{C}{\varepsilon^2}T\right). \qquad (3.22)$$

Proof. We multiply the equation (3.1) by $g_N(\varphi_N^\varepsilon)$ to obtain

$$\begin{aligned}&\int_\Omega g_N^2(\varphi_N^\varepsilon)dx\\ &\le \int_\Omega\left\{-g_N'(\varphi_N^\varepsilon)|\nabla\varphi_N^\varepsilon|^2 + \alpha\varphi_N^\varepsilon g_N(\varphi_N^\varepsilon) + u_N^\varepsilon g_N(\varphi_N^\varepsilon) - \varepsilon(\varphi_N^\varepsilon)_t g_N(\varphi_N^\varepsilon)\right\}dx \qquad (3.23)\\ &\le C\int_\Omega\left\{(\varphi_N^\varepsilon)^2 + (u_N^\varepsilon)^2 + \varepsilon^2(\varphi_N^\varepsilon)_t^2\right\}dx + \frac{1}{2}\int_\Omega g_N^2(\varphi_N^\varepsilon)dx,\end{aligned}$$

Multiplying (3.23) by t^2, we deduce (3.22) from (3.5) and (3.18).

Lemma 3.11 *For all $r > 0$ there exists a positive constant $C = C(r)$ such that*

$$\begin{aligned}\int_t^{t+r}\int_\Omega g_N^2(\varphi_N^\varepsilon)dxds &\le C\left(V_N^\varepsilon(\varphi_0, u_0) + 1\right) \qquad (3.24)\\ &\le C\left(V^\varepsilon(\varphi_0, u_0) + 1\right)\end{aligned}$$

for all $t \ge 0$.

Proof. The result follows from integrating (3.23) in view of (3.3) and (3.10).

Lemma 3.12 *There exists a constant C such that for all $r > 0$*

$$\int_\Omega g_N^2(\varphi_N^\varepsilon(t))dx \leq C(\frac{1}{r} + \frac{1}{\varepsilon})(V_N^\varepsilon(\varphi_0, u_0) + 1) \text{ for all } t \geq r > 0. \tag{3.25}$$

Proof. (3.25) is a direct consequence of inequality (3.23) together with (3.13), (3.10) and (3.3).

3.2 Time uniform estimates

In this section we prove time uniform estimates, independent on N, for the solution of Problem (P_N^ε). The bounded absorbing sets depending on ε will permit to prove the existence of a maximal attractor of Problem (P^ε) whereas the bounded absorbing sets that do not depend on ε will be used to prove the upper-semicontinuity of the attractor.

Throughout this section we suppose that $(\varphi_0, u_0) \in \mathcal{K} \times L^2(\Omega)$.

A natural function to consider [1] is the enthalpy $z_N^\varepsilon = \varepsilon u_N^\varepsilon + \varphi_N^\varepsilon$. We also use the notation $z_0^\varepsilon = \varepsilon u_0 + \varphi_0$. We first prove the following result.

Lemma 3.13 *There exist positive constants a and b that do not depend on N and ε such that the function $z_N^\varepsilon = \varepsilon u_N^\varepsilon + \varphi_N^\varepsilon$ satisfies*

$$\varepsilon\|\varphi_N^\varepsilon(t)\|^2_{L^2(\Omega)} + \|z_N^\varepsilon(t)\|^2_{H^{-1}(\Omega)} \leq \left(\varepsilon\|\varphi_0\|^2_{L^2(\Omega)} + \|z_0^\varepsilon\|^2_{H^{-1}(\Omega)}\right) e^{-at} + b$$

for all $t \geq 0$.

Proof. We rewrite (3.2) as

$$(z_N^\varepsilon)_t = \Delta u_N^\varepsilon,$$

multiply this equation by $\mathcal{N}(z_N^\varepsilon)$ and integrate on Ω to obtain

$$\int_\Omega (z_N^\varepsilon)_t \mathcal{N}(z_N^\varepsilon)dx = \int_\Omega \Delta u_N^\varepsilon \mathcal{N}(z_N^\varepsilon)dx,$$

that is

$$\int_\Omega -(\Delta\mathcal{N}(z_N^\varepsilon))_t \mathcal{N}(z_N^\varepsilon)dx = \int_\Omega u_N^\varepsilon \Delta\mathcal{N}(z_N^\varepsilon)dx,$$

so that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mathcal{N}(z_N^\varepsilon)|^2 dx + \int_\Omega u_N^\varepsilon z_N^\varepsilon dx = 0,$$

which we rewrite as

$$\frac{1}{2}\frac{d}{dt}\|z_N^\varepsilon\|^2_{H^{-1}(\Omega)} + \int_\Omega \left\{\varepsilon(u_N^\varepsilon)^2 + \varphi_N^\varepsilon u_N^\varepsilon\right\}dx = 0.$$

By substituting the expression for u_N^ε in (3.1) in the equality above we obtain, using also (1.4),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} &\left\{\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|^2_{H^{-1}(\Omega)}\right\} \\ &+ \int_\Omega \left\{\varepsilon(u_N^\varepsilon)^2 + |\nabla\varphi_N^\varepsilon|^2 - \alpha(\varphi_N^\varepsilon)^2 + \sum_{k=0}^{N}\frac{(\varphi_N^\varepsilon)^{2k+2}}{2k+1}\right\}dx = 0.\end{aligned} \tag{3.26}$$

Then using the inequality $-ax^2 \geq -\frac{1}{6}x^4 - \frac{3}{2}a^2$, we deduce that

$$\begin{aligned}&\int_\Omega \left\{-\frac{\alpha}{2}(\varphi_N^\varepsilon)^2 + \sum_{k=0}^{N}\frac{(\varphi_N^\varepsilon)^{2k+2}}{2k+1}\right\}dx \\ &\geq -\frac{1}{6}\int_\Omega(\varphi_N^\varepsilon)^4dx - \frac{3}{8}\alpha^2|\Omega| + \int_\Omega\left((\varphi_N^\varepsilon)^2 + \frac{1}{3}(\varphi_N^\varepsilon)^4 + \sum_{k=2}^{N}\frac{(\varphi_N^\varepsilon)^{2k+2}}{2k+1}\right)dx \\ &\geq -\frac{3}{8}\alpha^2|\Omega| + \int_\Omega\left(\frac{1}{2}(\varphi_N^\varepsilon)^2 + \frac{1}{12}(\varphi_N^\varepsilon)^4 + \sum_{k=2}^{N}\frac{(\varphi_N^\varepsilon)^{2k+2}}{(2k+2)(2k+1)}\right)dx \\ &\geq -\frac{3}{8}\alpha^2|\Omega| + \int_\Omega G_N(\varphi_N^\varepsilon)dx.\end{aligned}$$

Thus we deduce from (3.26) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} &\left\{\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|^2_{H^{-1}(\Omega)}\right\} \\ &+ \int_\Omega \left\{\varepsilon(u_N^\varepsilon)^2 + |\nabla\varphi_N^\varepsilon|^2 - \frac{\alpha}{2}(\varphi_N^\varepsilon)^2 + G_N(\varphi_N^\varepsilon)\right\}dx \leq \frac{3}{8}\alpha^2|\Omega|.\end{aligned} \tag{3.27}$$

Since

$$\begin{aligned}\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|^2_{H^{-1}(\Omega)} &\leq \varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + C_1\|z_N^\varepsilon\|^2_{L^2(\Omega)} \\ &\leq 2\varepsilon^2C_1\|u_N^\varepsilon\|^2_{L^2(\Omega)} + 4(\varepsilon + 2C_1)\frac{\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)}}{4},\end{aligned}$$

using that $\varepsilon \leq M$ we deduce that

$$\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|^2_{H^{-1}(\Omega)} \leq C_2\left(\varepsilon\|u_N^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega G_N(\varphi_N^\varepsilon)dx\right),$$

which we substitute in (3.27) using also (3.4) to deduce that

$$\frac{1}{2}\frac{d}{dt}\left\{\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|^2_{H^{-1}(\Omega)}\right\} + C_3\left(\varepsilon\|\varphi_N^\varepsilon\|^2_{L^2(\Omega)} + \|z_N^\varepsilon\|_{H^{-1}(\Omega)}\right)^2 \leq C_4.$$

The result of Lemma 3.13 then follows from applying Gronwall's Lemma.

A direct consequence of Lemma 3.13 is the following result.

Corollary 3.14 *There exist a positive constant D_0 which does not depend on N and ε and a time $t_0 = t_0(\|\varphi_0\|_{L^2(\Omega)}, \|\varepsilon u_0\|_{H^{-1}(\Omega)})$ such that*

$$\varepsilon\|\varphi_N^\varepsilon(t)\|^2_{L^2(\Omega)} + \|z_N^\varepsilon(t)\|^2_{H^{-1}(\Omega)} \leq D_0$$

for all $t \geq t_0$.

The next step is to show that $V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)$ enters an absorbing ball which neither depends on N nor on ε.

Lemma 3.15 *There exist a time $t_1 = t_1(\|\varphi_0\|_{L^2(\Omega)}, \|\varepsilon u_0\|_{H^{-1}(\Omega)})$ and a positive constant D_1 which neither depend on N nor on ε such that*

$$\text{(i)} \qquad V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(t) \leq D_1 \qquad \text{for all } t \geq t_1; \tag{3.28}$$

$$\text{(ii)} \qquad \|\varphi_N^\varepsilon(t)\|^2_{H_0^1(\Omega)} + \varepsilon\|u_N^\varepsilon(t)\|^2_{L^2(\Omega)} \leq D_1 \qquad \text{for all } t \geq t_1; \tag{3.29}$$

$$\text{(iii)} \quad \int_{t_1}^{+\infty}\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dxds \leq D_1. \tag{3.30}$$

Proof. We deduce from (3.27) and Corollary 3.14 that

$$\int_t^{t+r} V_N^\varepsilon(\varphi_N^\varepsilon, u_N^\varepsilon)(s)ds \leq C(r, D_0)$$

for all $t \geq t_0$ and $r > 0$. (3.28) then follows from (3.9) and the Uniform Gronwall Lemma. (3.29) then follows from Lemma 3.1, whereas (3.30) is a direct consequence of (3.10).

The next result gives the existence of a bounded absorbing set for u_N^ε in $H_0^1(\Omega)$.

Lemma 3.16 *There exist a positive constant D_2, a time $t_2 > t_1$ that do not depend on N and ε and there exists a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\text{(i)} \quad \int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t) + |\nabla u_N^\varepsilon(t)|^2\right\}dx \leq D_2 \text{ for all } t \geq t_2; \tag{3.31}$$

$$\text{(ii)} \qquad \int_{t_2}^{+\infty}\int_\Omega \left\{|\nabla(\varphi_N^\varepsilon)_t|^2 + \varepsilon(u_N^\varepsilon)_t^2\right\}dxds \leq D_2. \tag{3.32}$$

Proof. Our starting point is the inequality (3.16). Its right hand side can be estimated as follows

$$\alpha \int_\Omega (\varphi_N^\varepsilon)_t^2 dx \le \frac{\alpha^2}{2} \|(\varphi_N^\varepsilon)_t\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\int_\Omega |\nabla(\varphi_N^\varepsilon)_t|^2 dx. \tag{3.33}$$

Using equation (3.2) we deduce that

$$\begin{aligned}
\|(\varphi_N^\varepsilon)_t\|^2_{H^{-1}(\Omega)} &= \|\Delta u_N^\varepsilon - \varepsilon(u_N^\varepsilon)_t\|^2_{H^{-1}(\Omega)} \\
&\le 2\|\Delta u_N^\varepsilon\|^2_{H^{-1}(\Omega)} + 2\varepsilon^2\|(u_N^\varepsilon)_t\|^2_{H^{-1}(\Omega)} \\
&\le 2\|\nabla u_N^\varepsilon\|^2_{L^2(\Omega)} + 2C\varepsilon^2\|(u_N^\varepsilon)_t\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus in view of (3.33), we deduce from (3.16) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2 + |\nabla u_N^\varepsilon|^2\right\}dx + \frac{1}{2}\int_\Omega \left\{|\nabla(\varphi_N^\varepsilon)_t|^2 + \varepsilon(u_N^\varepsilon)_t^2\right\}dx \\
&\le \alpha^2 \int_\Omega |\nabla u_N^\varepsilon|^2 dx,
\end{aligned} \tag{3.34}$$

provided ε is small enough. (3.31) then follows from (3.30), (3.34) and the Uniform Gronwall's Lemma. In order to show (3.32) we integrate (3.34) between t_2 and t and let then $t \uparrow +\infty$.

We prove in the next Lemma a similar result as in Lemma 3.16 but for arbitrarily large values of ε.

Lemma 3.17 *There exist a positive constant $\widetilde{D_2}$ that do not depend on N and ε such that*

$$\int_\Omega \left\{\varepsilon(\varphi_N^\varepsilon)_t^2(t) + |\nabla u_N^\varepsilon(t)|^2\right\}dx \le \frac{\widetilde{D_2}}{\varepsilon} \textit{ for all } t \ge t_2. \tag{3.35}$$

Proof. Using (3.30), we can apply the Uniform Gronwall's Lemma to inequality (3.16) which implies (3.35).

The next lemma implies the existence of an absorbing set for φ_N^ε in $H^2(\Omega)$.

Lemma 3.18 *There exists a positive constant D_3 which does not depend on N and ε such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\int_\Omega (\Delta\varphi_N^\varepsilon(t))^2 dx \le D_3 \textit{ for all } t \ge t_2. \tag{3.36}$$

Proof. This is a direct consequence of inequality (3.20) in view of (3.31) and Lemma 3.15.

Lemma 3.19 *There exists a positive constant $\widetilde{D_3}$ which does not depend on N and ε such that*

$$\int_\Omega (\Delta\varphi_N^\varepsilon)^2 dx \leq \frac{\widetilde{D_3}}{\varepsilon} \text{ for all } t \geq t_2. \tag{3.37}$$

Proof. Here again we use inequality (3.20) and deduce (3.37) from (3.35) and Lemma 3.15.

The next result gives the existence of an absorbing set for the nonlinear term $g_N(\varphi_N^\varepsilon)$ in Problem (P_N^ε).

Lemma 3.20 *There exists a positive constant D_4 which does not depend on N and ε such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\int_\Omega g_N^2(\varphi_N^\varepsilon(t))dx \leq D_4 \text{ for all } t \geq t_2. \tag{3.38}$$

Proof. (3.38) follows from (3.23), (3.31) and Lemma 3.15.

One can summarize the previous results concerning the existence of bounded absorbing sets for the solutions of Problem (P_N^ε) as follows.

Theorem 3.21 *There exist a positive constant C, and a time t_2 which do not depend on N and ε and a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$*

(i) $\varepsilon\|(\varphi_N^\varepsilon)_t(t)\|^2_{L^2(\Omega)} + \|\varphi_N^\varepsilon(t)\|^2_{H^2(\Omega)} + \|g_N(\varphi_N^\varepsilon(t))\|^2_{L^2(\Omega)} + \|u_N^\varepsilon(t)\|^2_{H_0^1(\Omega)} \leq C$ *for all* $t \geq t_2$;

(ii) $\displaystyle\int_{t_2}^{+\infty} \left\{\|(\varphi_N^\varepsilon)_t\|^2_{H_0^1(\Omega)} + \varepsilon\|(u_N^\varepsilon)_t\|^2_{L^2(\Omega)}\right\} \leq C.$

Finally for $\varepsilon > 0$ arbitray the following result holds.

Theorem 3.22 *There exists a positive constant $\widetilde{C}$, times t_1 and $t_2 > t_1$, which do not depend on N such that for all $\varepsilon > 0$*

(i) $\|\varphi_N^\varepsilon(t)\|^2_{H_0^1(\Omega)} + \|u_N^\varepsilon(t)\|^2_{L^2(\Omega)} \leq \frac{\widetilde{C}}{\varepsilon}$ *for all* $t \geq t_1$;

(ii) $\|\varphi_N^\varepsilon(t)\|^2_{H^2(\Omega)} + \|u_N^\varepsilon(t)\|^2_{H_0^1(\Omega)} \leq \frac{\widetilde{C}}{\varepsilon}$ *for all* $t \geq t_2$.

4 Well-posedness of Problem (P^ε)

In this section, we prove the existence of a unique solution of Problem (P^ϵ) that we construct as the limit as $N \longrightarrow +\infty$ of the solution $(\varphi_N^\epsilon, u_N^\epsilon)$ of Problem (P_N^ϵ).

More precisely we prove the following result

Theorem 4.1
(i) For every $(\varphi_0, u_0) \in \mathcal{K} \times L^2(\Omega)$, *Problem* (P^ϵ) *has a unique solution* $(\varphi^\epsilon, u^\epsilon)$ *which satisfies*

$$(\varphi^\epsilon, u^\epsilon) \in L^\infty\left(0, T; (L^2(\Omega))^2\right) \bigcap L^2(0, T; (H_0^1(\Omega))^2)$$

for all $T > 0$, *and*

$$(\varphi^\epsilon, u^\epsilon) \in \mathcal{C}\left(I\!R^+; \left(L^2(\Omega)\right)^2\right).$$

(ii) If $(\varphi_0, u_0) \in (\mathcal{K} \bigcap H_0^1(\Omega)) \times L^2(\Omega)$ *then*

$$(\varphi^\epsilon, u^\epsilon) \in L^\infty\left(0, T; H_0^1(\Omega) \times L^2(\Omega)\right) \bigcap L^2(0, T; H^2(\Omega) \times H_0^1(\Omega))$$

for all $T > 0$, *and*

$$(\varphi^\epsilon, u^\epsilon) \in \mathcal{C}\left(I\!R^+; H_0^1(\Omega) \times L^2(\Omega)\right).$$

Moreover for all $t > 0$, $\|\varphi^\epsilon(t)\|_{L^\infty(\Omega)} \leq 1$, *the set* $\{x \in \Omega,\ |\varphi^\epsilon(x,t)| = 1\}$ *has measure zero and the mapping* $S^\epsilon(t)\ :\ (\varphi_0, u_0) \longmapsto (\varphi^\epsilon(t), u^\epsilon(t))$ *is Lipschitz continuous on* $\mathcal{K} \times L^2(\Omega)$ *endowed with the topology of* $(L^2(\Omega))^2$. *Furthermore* $(S^\epsilon(t))_{t \geq 0}$ *is a semigroup on* $\mathcal{K} \times L^2(\Omega)$.

4.1 Continuity of the semigroup - Uniqueness of the solution

Let (φ_0, u_0) and $(\psi_0, \tilde{u}_0)$ be pairs of initial functions in $\mathcal{K} \times L^2(\Omega)$ and let $(\varphi^\epsilon, u^\epsilon)$ and $(\psi^\epsilon, \tilde{u}^\epsilon)$ denote the corresponding solutions of Problem (P^ϵ). We set $\phi^\epsilon = \varphi^\epsilon - \psi^\epsilon$ and $w^\epsilon = u^\epsilon - \tilde{u}^\epsilon$, then

$$\begin{cases} \varepsilon\phi_t^\epsilon = \Delta\phi^\epsilon + \alpha\phi^\epsilon - (g(\varphi^\epsilon) - g(\psi^\epsilon)) + w^\epsilon & \text{in } \Omega \times I\!R^+, \quad (4.1) \\ \varepsilon w_t^\epsilon + \phi_t^\epsilon = \Delta w^\epsilon & \text{in } \Omega \times I\!R^+, \quad (4.2) \\ \phi^\epsilon = w^\epsilon = 0 & \text{on } \partial\Omega \times I\!R^+, \\ \phi^\epsilon(x,0) = \phi_0(x)\ ,\ w^\epsilon(x,0) = w_0(x) & x \in \Omega, \end{cases}$$

where we have introduced the notations $\phi_0 := \varphi_0 - \psi_0$ and $w_0 := u_0 - \tilde{u}_0$.
We multiply the equation (4.1) by $\frac{1}{\varepsilon}\phi^\varepsilon$, the equation (4.2) by $\varepsilon w^\varepsilon + \phi^\varepsilon$, integrate by parts and add the resulting equations using the monotonicity of the function g to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big\{(\phi^\varepsilon)^2 + (\varepsilon w^\varepsilon + \phi^\varepsilon)^2\Big\}dx \\ &\leq -\int_\Omega |\frac{1}{\sqrt{2\varepsilon}}\nabla\phi^\varepsilon + \sqrt{\frac{\varepsilon}{2}}\nabla w^\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_\Omega \Big\{\alpha(\phi^\varepsilon)^2 + w^\varepsilon\phi^\varepsilon\Big\}dx,\end{aligned}$$

and thus proceeding as in the proof of Lemma 3.2 it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big\{(\phi^\varepsilon)^2 + (\varepsilon w^\varepsilon + \phi^\varepsilon)^2\Big\}dx \leq \frac{C}{\varepsilon^2}\int_\Omega \Big\{(\phi^\varepsilon)^2 + (\varepsilon w^\varepsilon + \phi^\varepsilon)^2\Big\}dx.$$

Using the Gronwall's Lemma we then deduce that

$$\int_\Omega \Big\{(\phi^\varepsilon(t))^2 + (\varepsilon w^\varepsilon(t) + \phi^\varepsilon(t))^2\Big\}dx \leq \exp\left(\frac{Ct}{\varepsilon^2}\right)\int_\Omega \Big\{\phi_0^2 + (\varepsilon w_0 + \phi_0)^2\Big\}dx$$

for all $t > 0$. This in particular implies that

$$\int_\Omega \Big\{(\phi^\varepsilon(t))^2 + (w^\varepsilon(t))^2\Big\}dx \leq \frac{C_1}{\varepsilon^2}\exp\left(\frac{Ct}{\varepsilon^2}\right)\int_\Omega \Big\{\phi_0^2 + w_0^2\Big\}dx \tag{4.3}$$

for all $t > 0$. Thus we deduce on one hand the uniqueness for Problem (P^ε) and on the other hand the Lipschitz continuity of the mapping $S^\varepsilon(t)$ from $\mathcal{K} \times L^2(\Omega)$ into itself.

4.2 Existence of solutions

Throughout this section, we suppose that ε is a fixed positive constant.

We first consider initial functions (φ_0, u_0) that belong to $(\mathcal{K} \bigcap H_0^1(\Omega)) \times L^2(\Omega)$. It follows from the lemmas 3.1, 3.4, 3.9 and 3.11 that for all $T > 0$ there exists a positive constant $C = C(T, \|\varphi_0\|_{H_0^1(\Omega)}, \|u_0\|_{L^2(\Omega)})$ which does not depend on N such that

$$\|\varphi_N^\varepsilon(t)\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C \tag{4.4}$$

$$\|\Delta\varphi_N^\varepsilon\|_{L^2(Q_T)} \leq C \tag{4.5}$$

$$\|g_N(\varphi_N^\varepsilon)\|_{L^2(Q_T)} \leq C \tag{4.6}$$

$$\|u_N^\varepsilon(t)\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{C}{\sqrt{\varepsilon}} \tag{4.7}$$

$$\|u_N^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{4.8}$$

$$\|(\varphi_N^\varepsilon)_t\|_{L^2(Q_T)} \leq \frac{C}{\sqrt{\varepsilon}} \tag{4.9}$$

where $Q_T = \Omega\times(0,T)$. Thus we deduce that there exist a subsequence of $\{(\varphi_N^\varepsilon, u_N^\varepsilon)\}_{N\geq 0}$ that we denote again by $\{(\varphi_N^\varepsilon, u_N^\varepsilon)\}$ and functions $(\varphi^\varepsilon, u^\varepsilon)$ which satisfy

$$(\varphi^\varepsilon, u^\varepsilon) \in L^2\left(0,T;(H^2(\Omega)\bigcap H_0^1(\Omega))\times H_0^1(\Omega)\right)$$

such that as $N \longrightarrow +\infty$

$$\varphi_N^\varepsilon \longrightarrow \varphi^\varepsilon \quad \text{weakly in } L^2(Q_T), \tag{4.10}$$

$$\Delta\varphi_N^\varepsilon \longrightarrow \Delta\varphi^\varepsilon \quad \text{weakly in } L^2(Q_T), \tag{4.11}$$

$$g_N(\varphi_N^\varepsilon) \longrightarrow g^* \quad \text{weakly in } L^2(Q_T), \tag{4.12}$$

$$u_N^\varepsilon \longrightarrow u^\varepsilon \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{4.13}$$

$$(\varphi_N^\varepsilon)_t \longrightarrow \varphi_t^\varepsilon \quad \text{weakly in } L^2(Q_T). \tag{4.14}$$

Letting $N \longrightarrow +\infty$ in the equation (3.1) we deduce from (4.14), (4.11), (4.10), (4.12) and (4.13) that the pair $(\varphi^\varepsilon, u^\varepsilon)$ satisfies

$$\varepsilon\varphi_t^\varepsilon = \Delta\varphi^\varepsilon + \alpha\varphi^\varepsilon - g^* + u^\varepsilon \text{ in } L^2(Q_T). \tag{4.15}$$

Next we let $N \longrightarrow +\infty$ in the equation (3.2). Therefore we remark that using (4.8) and (4.9) together with equation (3.2) it follows that

$$||(u_N^\varepsilon)_t||_{L^2(0,T;H^{-1}(\Omega))} \leq C, \tag{4.16}$$

and thus

$$(u_N^\varepsilon)_t \longrightarrow u_t^\varepsilon \text{ weakly in } L^2(0,T;H^{-1}(\Omega)) \tag{4.17}$$

as $N \longrightarrow +\infty$. From equation (3.2) we deduce that

$$\langle \varepsilon(u_N^\varepsilon)_t + (\varphi_N^\varepsilon)_t \ , \ \mathcal{X}\rangle = -\langle \nabla u_N^\varepsilon \ , \ \nabla\mathcal{X}\rangle \, ,$$

for all $\mathcal{X} \in \mathcal{D}((0,T)\times\Omega)$ where $\langle \ , \rangle$ denote the duality product between $\mathcal{D}'((0,T)\times\Omega)$ and $\mathcal{D}((0,T)\times\Omega)$, and let then $N \longrightarrow +\infty$ to deduce from (4.17), (4.14) and (4.13) that

$$\varepsilon u_t + \varphi_t = \Delta u \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \tag{4.18}$$

Moreover using Simon [6], (4.4), (4.9) on one hand and (4.7), (4.16) on the other hand imply that as $N \longrightarrow +\infty$

$$\varphi_N^\varepsilon \longrightarrow \varphi^\varepsilon \quad \text{in } \mathcal{C}([0,T];L^2(\Omega)), \tag{4.19}$$

$$u_N^\varepsilon \longrightarrow u^\varepsilon \quad \text{in } \mathcal{C}([0,T];H^{-1}(\Omega)),$$

so that in particular $\varphi^\varepsilon(x,0)=\varphi_0$ and $u^\varepsilon(x,0)=u_0$ in Ω.
Note also that using for example Temam [7, Lemma 3.2, p.69], it follows from (4.8) and (4.16) that

$$u^\varepsilon \in \mathcal{C}([0,T];L^2(\Omega)). \tag{4.20}$$

Hence the functions φ^ε and u^ε satisfy

$$\begin{cases} \varepsilon\varphi^\varepsilon_t = \Delta\varphi^\varepsilon + \alpha\varphi^\varepsilon - g^* + u^\varepsilon & \text{in } L^2(Q_T) \\ \varepsilon u^\varepsilon_t + \varphi^\varepsilon_t = \Delta u^\varepsilon & \text{in } L^2(0,T;H^{-1}(\Omega)) \\ \varphi^\varepsilon = u^\varepsilon = 0 & \text{on } \partial\Omega\times(0,T) \\ \varphi^\varepsilon(x,0)=\varphi_0,\ u^\varepsilon(x,0)=u_0 & \text{in } \Omega. \end{cases} \tag{4.21}$$

We now prove that $g^* = g(\varphi^\varepsilon)$ and the set $\{x\in\Omega,\ |\varphi^\varepsilon(x,t)|=1\}$ has measure zero and that $(\varphi^\varepsilon,u^\varepsilon)\in\mathcal{C}(I\!R^+;H^1_0(\Omega)\times L^2(\Omega))$.

The usual monotonicity argument does allow here to prove that $g^* = g(\varphi^\varepsilon)$. Therefore we adapt a method introduced by Debussche and Dettori [2]. For an arbitrary small $\eta\in(0,1)$ and for all $t\in(0,T)$, we set

$$E^N_\eta(t) \ = \ \{x\in\Omega,\ |\varphi^\varepsilon_N(x,t)|>1-\eta\}\,,$$

and we denote by $|E^N_\eta(t)|$ its measure namely $|E^N_\eta(t)| = Meas(E^N_\eta(t)) = \int_{E^N_\eta(t)} dx$ and by $\mathcal{X}^N_\eta(t)$ its characteristic function :

$$\mathcal{X}^N_\eta(x,t) = \begin{cases} 1 \text{ if } x\in E^N_\eta(t) \\ 0 \text{ elsewhere .} \end{cases}$$

Using Lemma 3.12 we deduce that

$$t\|g_N(\varphi^\varepsilon_N(t))\|^2_{L^2(\Omega)} \le C\ \frac{T+1}{\varepsilon} \text{ for all } t\in(0,T), \tag{4.22}$$

and thus

$$\begin{aligned} \left\{\int_{E^N_\eta(t)} g^2_N(\varphi^\varepsilon_N)dx\right\}^{1/2} &\ge \left|E^N_\eta(t)\right|^{1/2}\left\{\begin{array}{c} Inf \\ x\in E^N_\eta(t)\end{array} \left(\sum_{k=0}^N \frac{(\varphi^\varepsilon_N(x))^{2k+1}}{2k+1}\right)^2\right\}^{1/2} \\ &\ge \left|E^N_\eta(t)\right|^{1/2} \begin{array}{c} Inf \\ x\in E^N_\eta(t)\end{array} \sum_{k=0}^N \frac{|\varphi^\varepsilon_N(x)|^{2k+1}}{2k+1} \\ &\ge \left|E^N_\eta(t)\right|^{1/2} \sum_{k=0}^N \frac{(1-\eta)^{2k+1}}{2k+1}, \end{aligned}$$

which implies that

$$\left|E_\eta^N(t)\right|^{1/2} \leq \frac{C(T,\varepsilon)}{\sqrt{t}\sum_{k=0}^{N}\frac{(1-\eta)^{2k+1}}{2k+1}}. \tag{4.23}$$

Thus letting $N \longrightarrow +\infty$ we deduce from (4.19), (4.23) and Fatou's Lemma that

$$\begin{aligned}|E_\eta(t)| = \int_\Omega \mathcal{X}_\eta(t)dx \leq \int_\Omega \lim_{N\longrightarrow+\infty} inf \mathcal{X}_\eta^N(t)dx &\leq \lim_{N\longrightarrow+\infty} inf \int_\Omega \mathcal{X}_\eta^N(t)dx \\ &\leq \lim_{N\longrightarrow+\infty} inf \left|E_\eta^N(t)\right| \\ &\leq \frac{4C}{t\ ln^2\left(\frac{2-\eta}{\eta}\right)},\end{aligned}$$

where $|E_\eta(t)|$ and $\mathcal{X}_\eta(t)$ respectively stand for the measure of the set $\{x \in \Omega,\ |\varphi^\epsilon(x,t)| > 1-\eta\}$ and for its characteristic function. Letting then $\eta \downarrow 0$, it follows that for all $t \in (0,T)$

$$Meas\{x \in \Omega,\ |\varphi^\epsilon(x,t)| \geq 1\} = 0. \tag{4.24}$$

Next we show that (4.19) and (4.24) implies that as $N \longrightarrow +\infty$, for all $t \in (0,T)$

$$g_N(\varphi_N^\epsilon(t)) \longrightarrow g(\varphi^\epsilon(t)) \text{ a.e. in } \Omega. \tag{4.25}$$

It follows respectively from (4.19) and (4.24) that for all $t \in (0,T)$ and almost every $x \in \Omega$

$$\varphi_N^\epsilon(x,t) \longrightarrow \varphi^\epsilon(x,t), \tag{4.26}$$

$$|\varphi^\epsilon(x,t)| < 1. \tag{4.27}$$

Fix $(x,t) \in \Omega \times (0,T)$ such that $\varphi_N^\epsilon(x,t)$ converges to $\varphi^\epsilon(x,t)$ as $N \longrightarrow +\infty$. It follows from (4.27) that there exists a positive constant N_0 such that for all $N > N_0$,

$$|\varphi_N^\epsilon(x,t)| < 1. \tag{4.28}$$

Then we have that

$$\begin{aligned}|g_N(\varphi_N^\epsilon(x,t)) - g(\varphi^\epsilon(x,t))| \leq\ & |g_N(\varphi_N^\epsilon(x,t)) - g(\varphi_N^\epsilon(x,t))| \\ &+|g(\varphi_N^\epsilon(x,t)) - g(\varphi^\epsilon(x,t))|.\end{aligned} \tag{4.29}$$

It follows from (4.26), (4.27) and (4.28) that the right term of the right-hand-side of (4.29) converges to zero as $N \longrightarrow +\infty$. Concerning the remaining term we have that

$$g_N(\varphi_N^\epsilon(x,t)) - g(\varphi_N^\epsilon(x,t)) = \sum_{k=N+1}^{+\infty} \frac{(\varphi_N^\epsilon(x,t))^{2k+1}}{2k+1}. \tag{4.30}$$

In view of (4.28), there exists $a \in (0,1)$ such that $|\varphi_N^\epsilon(x,t)| \le 1-a$. Hence for $M \ge N+1$

$$
\begin{aligned}
|\sum_{k=N+1}^{M} \frac{(\varphi_N^\epsilon)^{2k+1}}{2k+1}| &\le \frac{1}{2N+3}\sum_{k=N+1}^{M} |\varphi_N^\epsilon|^{2k+1} \\
&\le \frac{1}{2N+3}\sum_{k=N+1}^{M} (1-a)^{2k+1} \\
&\le \frac{(1-a)^{2N+3}}{2N+3} \quad \frac{1-(1-a)^{2(M-N)}}{1-(1-a)^2}.
\end{aligned}
$$

Thus letting $M \longrightarrow +\infty$ we deduce that

$$
|\sum_{k=N+1}^{+\infty} \frac{(\varphi_N^\epsilon)^{2k+1}}{2k+1}| \le \frac{(1-a)^{2N+3}}{2N+3} \quad \frac{1}{1-(1-a)^2},
$$

which implies in view of (4.30) that $g_N(\varphi_N^\epsilon(x,t))$ converges to $g(\varphi_N^\epsilon(x,t))$ as $N \longrightarrow +\infty$. This completes the proof of (4.25).

Then using Lions [5, Lemma 1.3, p.12] it follows from (4.6) and (4.25) that

$$
g_N(\varphi_N^\epsilon) \longrightarrow g(\varphi^\epsilon) \text{ weakly in } L^2(\Omega \times (0,T)). \tag{4.31}
$$

so that $g^* = g(\varphi^\epsilon)$.

Next we prove that $(\varphi^\epsilon, u^\epsilon) \in \mathcal{C}([0,T]; H_0^1(\Omega) \times L^2(\Omega))$. Since we already know that $(\varphi^\epsilon, u^\epsilon) \in \mathcal{C}([0,T]; (L^2(\Omega))^2)$, it only remains to prove that $\varphi^\epsilon \in \mathcal{C}([0,T]; H_0^1(\Omega))$. We deduce from Lemma 3.8 that for all $\delta > 0$ there exists a positive constant $C = C(\delta)$ which does not depend on N and such that

$$
\|\varphi_N^\epsilon(t)\|_{H^2(\Omega)} \le C \text{ for all } t \ge \delta > 0.
$$

Thus using (4.9), this in particular implies that the limiting function φ^ϵ is in $\mathcal{C}([\delta,T]; H_0^1(\Omega))$ for all $\delta > 0$ so that in turn $\varphi^\epsilon \in \mathcal{C}((0,T]; H_0^1(\Omega))$.

In what follows we first prove that $G(\varphi^\epsilon) \in \mathcal{C}([0,T]; L^1(\Omega))$, then that the mapping $t \longmapsto V(\varphi^\epsilon, u^\epsilon)(t)$ is continuous on $[0,T]$. Since the functions φ^ϵ and u^ϵ are continuous from $[0,T]$ into $L^2(\Omega)$, this implies that $\varphi^\epsilon \in \mathcal{C}([0,T]; H_0^1(\Omega))$.

Let $t_0 \in [0,T]$ and let t_k be a sequence in $[0,T]$ converging to t_0 as $k \longrightarrow +\infty$. We deduce using (4.19) and (4.24) that $G(\varphi^\epsilon(t_k))$ converges to $G(\varphi^\epsilon(t_0))$ as $k \longrightarrow +\infty$ almost everywhere in Ω, since also $|G(\varphi^\epsilon(t_k))| \le ln(2)$ it follows that $G(\varphi^\epsilon(t_k))$ converges to $G(\varphi^\epsilon(t_0))$ in $L^1(\Omega)$ as $k \longrightarrow +\infty$ by the dominate convergence theorem.

Next we prove that $t \longmapsto V(\varphi^\epsilon, u^\epsilon)(t)$ is continuous at $t = 0$. We deduce from the Lemma 3.4 that

$$
V^\epsilon(\varphi^\epsilon, u^\epsilon)(t) \le V^\epsilon(\varphi_0, u_0),
$$

and thus

$$\limsup_{t\downarrow 0} V^\varepsilon(\varphi^\varepsilon, u^\varepsilon)(t) \le V^\varepsilon(\varphi_0, u_0). \tag{4.32}$$

But since $\varphi^\varepsilon \in L^\infty(0,T;H_0^1(\Omega))\bigcap \mathcal{C}([0,T];L^2(\Omega))$, we deduce using Temam [8, Lemma 1.4, p.263] that the function φ^ε is weakly continuous on $[0,T]$ with values in $H_0^1(\Omega)$. Hence it follows that

$$V^\varepsilon(\varphi_0, u_0) \le \liminf_{t\downarrow 0} V^\varepsilon(\varphi^\varepsilon, u^\varepsilon)(t),$$

which in view of (4.32) implies that

$$\lim_{t\downarrow 0} V^\varepsilon(\varphi^\varepsilon, u^\varepsilon)(t) = V^\varepsilon(\varphi_0, u_0),$$

and thus

$$\varphi^\varepsilon \in \mathcal{C}([0,T];H_0^1(\Omega)). \tag{4.33}$$

This conclude the existence part of Theorem 4.1 (ii).

In order to prove part (i) of Theorem 4.1, we consider initial functions $(\varphi_0, u_0) \in \mathcal{K} \times L^2(\Omega)$ and we define a sequence $\{(\varphi_{0k}, u_{0k})\}_{k\ge 1}$ such that $(\varphi_{0k}, u_{0k}) \in (\mathcal{K}\bigcap H_0^1(\Omega)) \times L^2(\Omega)$ for all $k \ge 1$ and $(\varphi_{0k}, u_{0k}) \longrightarrow (\varphi_0, u_0)$ in $(L^2(\Omega))^2$ as

$k \longrightarrow +\infty$. Let $(\varphi_p^\varepsilon, u_p^\varepsilon)$ and $(\varphi_q^\varepsilon, u_q^\varepsilon)$ be two solutions of Problem (P^ε) with initial functions (φ_{0p}, u_{0p}) and (φ_{0q}, u_{0q}) respectively. We deduce from (4.3) that

$$\int_\Omega \Big\{(\varphi_p^\varepsilon - \varphi_q^\varepsilon)^2(t) + (u_p^\varepsilon - u_q^\varepsilon)^2(t)\Big\}dx \le \frac{C_1}{\varepsilon^2}\exp\left(\frac{Ct}{\varepsilon^2}\right)\int_\Omega \Big\{(\varphi_{0p}^\varepsilon - \varphi_{0q}^\varepsilon)^2 + (u_{0p}^\varepsilon - u_{0q}^\varepsilon)^2\Big\}dx.$$

Thus $\{(\varphi_k^\varepsilon, u_k^\varepsilon)\}_{k\ge 1}$ is a Cauchy sequence in $\mathcal{C}([0,T];(L^2(\Omega))^2)$ and there exists $(\varphi^\varepsilon, u^\varepsilon) \in \mathcal{C}([0,T];(L^2(\Omega))^2)$ such that as $k \longrightarrow +\infty$

$$(\varphi_k^\varepsilon, u_k^\varepsilon) \longrightarrow (\varphi^\varepsilon, u^\varepsilon) \text{ in } \mathcal{C}([0,T];(L^2(\Omega))^2). \tag{4.34}$$

It remains to show that $(\varphi^\varepsilon, u^\varepsilon)$ satisfies Problem (P^ε). It follows from the uniform estimates with respect to N of Section 3.1 and in particular the lemmas 3.2 and 3.10 that

$$\|(\varphi_k^\varepsilon, u_k^\varepsilon)\|^2_{L^2(0,T;(H_0^1(\Omega))^2)} \le \frac{C}{\varepsilon^2}\|(\varphi_{0k}, u_{0k})\|^2_{(L^2(\Omega))^2}\exp\left(\frac{C}{\varepsilon^2}T\right) \tag{4.35}$$

$$\|tg(\varphi_k^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} \le \frac{C}{\varepsilon}(\|(\varphi_{0k}, u_{0k})\|^2_{(L^2(\Omega))^2} + \sqrt{T})(T+1)\exp\left(\frac{C}{\varepsilon^2}T\right). \tag{4.36}$$

Thus we deduce from (4.35) that as $k \longrightarrow +\infty$

$$(\varphi_k^\varepsilon, u_k^\varepsilon) \longrightarrow (\varphi^\varepsilon, u^\varepsilon) \text{ weakly in } L^2(0,T;(H_0^1(\Omega))^2). \tag{4.37}$$

Moreover it follows from the existence proof above that for all $t \in (0,T)$ the set $\{x \in \Omega,\ |\varphi_k^\epsilon(x,t)| \geq 1\}$ has measure zero. By a similar proof as above, one can show as well that for all $t \in (0,T)$ the set $\{x \in \Omega,\ |\varphi^\epsilon(x,t)| \geq 1\}$ has measure zero. Therefore using again (4.34)

$$g(\varphi_k^\epsilon) \longrightarrow g(\varphi^\epsilon) \text{ a.e in } \Omega \text{ and for all } t \in (0,T) \text{ as } k \longrightarrow +\infty. \tag{4.38}$$

Next we prove that as $k \longrightarrow +\infty$

$$g(\varphi_k^\epsilon) \longrightarrow g(\varphi^\epsilon) \text{ in } L^\infty(0,T;L^1(\Omega)).$$

To that purpose for arbitrarily small $\eta \in (0,1)$ and all $t \in (0,T)$, we introduce the sets

$$\begin{aligned} E_\eta^k(t) &= \{x \in \Omega,\ |\varphi_k^\epsilon(x,t)| > 1-\eta\},\ F_\eta^k(t) &= \overline{\Omega} \setminus E_\eta^k(t), \\ E_\eta(t) &= \{x \in \Omega,\ |\varphi^\epsilon(x,t)| > 1-\eta\},\ F_\eta(t) &= \overline{\Omega} \setminus E_\eta(t), \end{aligned}$$

and the associated functions $\mathcal{Z}_\eta^k$ and $\mathcal{Z}_\eta$ defined by

$$\mathcal{Z}_\eta^k(x,t) = \begin{cases} 1 \text{ if } x \in F_\eta^k(t) \\ 0 \text{ if } x \in E_\eta^k(t), \end{cases}$$

$$\mathcal{Z}_\eta(x,t) = \begin{cases} 1 \text{ if } x \in F_\eta(t) \\ 0 \text{ if } x \in E_\eta(t). \end{cases}$$

Note that using similar technics as before one can deduce from (4.36) that

$$|E_\eta^k(t)|^{1/2} \leq \frac{C}{t\ g(1-\eta)} = \frac{2C}{t\ ln\left(\frac{2-\eta}{\eta}\right)}. \tag{4.39}$$

Next we have that for all $t \in (0,T)$

$$\begin{aligned} &\int_\Omega |g(\varphi_k^\epsilon(t)) - g(\varphi^\epsilon(t))|dx \leq \int_\Omega \left|g(\varphi_k^\epsilon(t))(1-\mathcal{Z}_\eta^k(x,t))\right|dx \\ &+ \int_\Omega \left|g(\varphi_k^\epsilon(t))\mathcal{Z}_\eta^k(x,t) - g(\varphi^\epsilon(t))\mathcal{Z}_\eta(x,t)\right|dx + \int_\Omega |g(\varphi^\epsilon(t))(1-\mathcal{Z}_\eta(x,t))|dx \\ &:= (E)\ +\ (F)\ +\ (G), \end{aligned} \tag{4.40}$$

and we estimate the three terms of the right-hand-side of (4.40) : (E), (F) and (G) as follows.

We deduce from (4.36) and (4.39) that

$$\begin{aligned}
(E) &\leq \left\{\int_\Omega g^2(\varphi_k^\varepsilon)dx\right\}^{1/2}\left\{\int_\Omega (1-\mathcal{Z}_\eta^k(x,t))dx\right\}^{1/2} \\
&\leq \frac{C(T)}{t}\left|E_\eta^k(t)\right|^{1/2} \\
&\leq \frac{2C(T)}{t^2\, ln\left(\frac{2-\eta}{\eta}\right)}
\end{aligned} \tag{4.41}$$

Next we show that (F) tends to zero as $k \longrightarrow +\infty$. Using (4.34) and (4.38), we deduce that for all $t \in (0,T)$ as $k \longrightarrow +\infty$

$$g(\varphi_k^\varepsilon(t))\mathcal{Z}_\eta^k(x,t) \longrightarrow g(\varphi^\varepsilon(t))\mathcal{Z}_\eta(x,t) \text{ a.e. in } \Omega.$$

Moreover g is a non-decreasing function and

$$|g(\varphi_k^\varepsilon(t))\mathcal{Z}_\eta^k(x,t)| \leq g(1-\eta) = \frac{1}{2}ln\left(\frac{2-\eta}{\eta}\right),$$

and thus for all $t \in (0,T)$

$$g(\varphi_k^\varepsilon(t))\mathcal{Z}_\eta^k(x,t) \longrightarrow g(\varphi^\varepsilon(t))\mathcal{Z}_\eta(x,t) \text{ in } L^1(\Omega), \tag{4.42}$$

as $k \longrightarrow +\infty$ by the dominate convergence theorem.

Finally we consider the term $(G) = \int_\Omega |g(\varphi^\varepsilon(t))(1-\mathcal{Z}_\eta(x,t))|dx$. We deduce from (4.36), (4.38) and Fatou's Lemma that

$$t\|g(\varphi^\varepsilon(t))\|_{L^2(\Omega)} \leq C(T) \text{ for all } t \in (0,T),$$

and proceeding as for the term (E), we deduce that

$$(G) \leq \frac{2C(T)}{t^2\, ln\left(\frac{2-\eta}{\eta}\right)}. \tag{4.43}$$

Thus we deduce from (4.40), (4.41), (4.42) and (4.43) that for all $t \in (0,T)$

$$g(\varphi_k^\varepsilon(t)) \longrightarrow g(\varphi^\varepsilon(t)) \text{ in } L^1(\Omega) \text{ as } k \longrightarrow +\infty. \tag{4.44}$$

Note that the pair $(\varphi_k^\varepsilon, u_k^\varepsilon)$ satisfies the integral equations

$$-\varepsilon\int_0^T\int_\Omega \varphi_k^\varepsilon \mathcal{X}_t dxdt = -\int_0^T\int_\Omega \nabla\varphi_k^\varepsilon\nabla\mathcal{X}dxdt + \int_0^T\int_\Omega \{\alpha\varphi_k^\varepsilon - g(\varphi_k^\varepsilon) + u_k^\varepsilon\}\mathcal{X}dxdt$$
$$\int_0^T\int_\Omega \{\varepsilon u_k^\varepsilon + \varphi_k^\varepsilon\}\mathcal{X}_t dxdt = \int_0^T\int_\Omega \nabla u_k^\varepsilon\nabla\mathcal{X}dxdt$$

for all $\mathcal{X} \in \mathcal{D}((0,T) \times \Omega)$. Letting $k \longrightarrow +\infty$ in this equations we deduce from (4.34), (4.37) and (4.44) that the pair $(\varphi^\epsilon, u^\epsilon)$ satisfies Problem (P^ϵ) in $\mathcal{D}'((0,T)\times\Omega)$ together with the initial and boundary conditions of Problem (P^ϵ).

5 Existence of a maximal attractor of Problem (P^ε)

The aim of this section is twofold : prove that the nonliear term $g(\varphi^\epsilon)$ enters an absorbing set of $L^2(\Omega)$ and show the existence of a maximal attractor for Problem (P^ϵ).

5.1 Existence of a bounded absorbing set for the nonlinear term of Problem (P^ε)

We show in this section that the nonlinear term $g(\varphi^\epsilon)$ enters an absorbing set of $L^2(\Omega)$. More precisely we prove the following result.

Lemma 5.1 *There exist a positive constant D and a time t_2 which do not depend on ε and a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\|g(\varphi^\epsilon(t))\|^2_{L^2(\Omega)} \leq D \text{ for all } t \geq t_2. \tag{5.1}$$

Proof. We deduce from Theorem 3.21 that there exist a positive constant C and a time t_2 which do not depend on N and ε and a positive constant ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the solution $(\varphi^\epsilon_N, u^\epsilon_N)$ of Problem (P^ϵ_N) satisfies

$$\|\varphi^\epsilon_N(t)\|^2_{H^2(\Omega)} + \|g_N(\varphi^\epsilon_N(t))\|^2_{L^2(\Omega)} \leq C \text{ for all } t \geq t_2; \tag{5.2}$$

$$\int_{t_2}^{+\infty} \|(\varphi^\epsilon_N)_t\|^2_{H^1_0(\Omega)} \leq C. \tag{5.3}$$

Thus we deduce that there exists a subsequence of $\{\varphi^\epsilon_N\}_{N\geq 0}$ which we denote again by $\{\varphi^\epsilon_N\}$ and a function φ^ϵ such that for all $T > t_2$,

$$\varphi^\epsilon_N \longrightarrow \varphi^\epsilon \text{ in } C([t_2, T]; H^1_0(\Omega)),$$

as $N \longrightarrow +\infty$. In particular

$$\varphi^\epsilon_N \longrightarrow \varphi^\epsilon \text{ a.e. in } \Omega \text{ and for all } t \in (t_2, T) \tag{5.4}$$

as $N \longrightarrow +\infty$. Note that by Section 4.2 the function φ^ϵ satisfies Problem (P^ϵ).

Moreover it follows from (5.2) that on one hand there exists a function g^* such that

$$g_N(\varphi_N^\epsilon) \longrightarrow g^* \text{ in } L^\infty(t_2, +\infty; L^2(\Omega)) \text{ weak } * \tag{5.5}$$

as $N \longrightarrow +\infty$, and

$$\|g^*(t)\|^2_{L^2(\Omega)} \leq C \text{ for all } t \geq t_2, \tag{5.6}$$

and on the other hand that for all $\eta \in (0,1)$ the measure of the set $E_N^{\eta,\epsilon}(t) := \{x \in \Omega, \ |\varphi_N^\epsilon(x,t)| > 1 - \eta\}$ satisfies

$$|E_N^{\eta,\epsilon}(t)|^{1/2} \leq \frac{C}{\displaystyle\sum_{k=0}^{N} \frac{(1-\eta)^{2k+1}}{2k+1}} \text{ for all } t \geq t_2. \tag{5.7}$$

We denote by $\mathcal{X}_N^{\eta,\epsilon}(t)$ the characteristic function of this set. Using (5.4), (5.7) and Fatou's Lemma it follows that for all $t \in (t_2, T)$ and all $\eta \in (0,1)$

$$\begin{aligned}|E^{\eta,\epsilon}(t)| = \int_\Omega \mathcal{X}^{\eta,\epsilon}(x,t)dx \leq \lim_{N\longrightarrow+\infty} \inf \int_\Omega \mathcal{X}_N^{\eta,\epsilon}(x,t)dx &= \lim_{N\longrightarrow+\infty} \inf |E_N^{\eta,\epsilon}(t)| \\ &\leq \frac{4C}{ln^2\left(\frac{2-\eta}{\eta}\right)},\end{aligned}$$

where $|E^{\eta,\epsilon}(t)|$ denote the measure of the set $\{x \in \Omega, \ |\varphi^\epsilon(x,t)| > 1 - \eta\}$ and $\mathcal{X}^{\eta,\epsilon}(x,t)$ its characteristic function. Letting then $\eta \downarrow 0$ it follows that for all $t \in (t_2, T)$

$$Meas \{x \in \Omega, \ |\varphi^\epsilon(x,t)| \geq 1\} = 0. \tag{5.8}$$

Using (5.4) and (5.8) one can show in a similar way as it has been done in Section 4.2 that as $N \longrightarrow +\infty$

$$g_N(\varphi_N^\epsilon) \longrightarrow g(\varphi^\epsilon) \text{ a.e. in } \Omega \text{ and for all } t \in (t_2, T). \tag{5.9}$$

Using Lions [5, Lemma 1.3, p.12] it follows from (5.2) and (5.9) that for all $T > t_2$ as $N \longrightarrow +\infty$

$$g_N(\varphi_N^\epsilon) \longrightarrow g(\varphi^\epsilon) \text{ weakly in } L^2(t_2, T; L^2(\Omega)),$$

which in view of (5.5) implies that $g^* = g(\varphi^\epsilon)$. Therefore (5.1) then follows from (5.6).

5.2 Existence of a maximal attractor

Next we prove the following result about the existence of a maximal attractor for Problem (P^ϵ).

Theorem 5.2 *The semigroup $(S^\epsilon(t))_{t\geq 0}$ associated with Problem (P^ϵ) maps $\mathcal{K}\times L^2(\Omega)$ into itself. It possesses in $\mathcal{K}\times L^2(\Omega)$ a maximal attractor $\mathcal{A}^\epsilon$ that is connected. Moreover $\mathcal{A}^\epsilon$ is bounded in $(\mathcal{K}\bigcap H^2(\Omega))\times H^1(\Omega)$.*

Proof. It follows from the existence theorem that the semigroup $\{S^\epsilon(t)\}_{t\geq 0}$ is continuous on $\mathcal{K}\times L^2(\Omega)$. From Theorem 3.22 (i) we deduce that the solution $(\varphi^\epsilon(t), u^\epsilon(t))$ of Problem (P^ϵ) enters an absorbing ball B of $H_0^1(\Omega)\times L^2(\Omega)$ for $t\geq t_1$. Moreover Theorem 3.22 (ii) implies that there exists a time $t_2 > t_1$ such that $S^\epsilon(t)B$ is bounded in $H^2(\Omega)\times H^1(\Omega)$ and thus relatively compact in $H_0^1(\Omega)\times L^2(\Omega)$ for all $t\geq t_2$. The existence of the maximal attractor then follows from Temam [7, Theorem 1.1, p.23].

6 Upper-semicontinuity of the attractor

From Theorem 4.1 we deduce that Problem (P_r^ε) is well-posed and that the mapping $S_r^\varepsilon(t)\,:\,(\varphi_0, v_0)\longmapsto(\varphi^\epsilon(t), v^\epsilon(t))$ where $(\varphi^\epsilon, v^\epsilon)$ is the solution of Problem (P_r^ε) with initial functions (φ_0, v_0), defines a continuous semigroup on $\mathcal{K}\times L^2(\Omega)$. Moreover it follows from Theorem 5.2 that the semigroup $\{S_r^\varepsilon(t)\}_{t\geq 0}$ possesses in $\mathcal{K}\times L^2(\Omega)$ a maximal attractor $\mathcal{A}_r^\varepsilon$.
Note that there exist simple one to one relations between the two semigroups $S^\epsilon(t)$ and $S_r^\varepsilon(t)$ and the two corresponding attractors, namely

$$S_r^\varepsilon(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} S^\epsilon(t) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\varepsilon}} \end{pmatrix};$$

and

$$\mathcal{A}_r^\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} \mathcal{A}^\epsilon.$$

We also introduce the set $\mathcal{A}^0 = \mathcal{A}\times\{0\}$ and prove the following result.

Theorem 6.1 *For $\Omega\subset I\!R^n$ with $n\leq 3$, the attractor $\mathcal{A}_r^\varepsilon$ is upper-semicontinuous at $\varepsilon = 0$, i.e. the Hausdorff semidistance $d(\mathcal{A}_r^\varepsilon, \mathcal{A}^0)$ converges to zero as $\varepsilon\downarrow 0$:*

$$\lim_{\epsilon\downarrow 0}\ \mathop{Sup}_{(\varphi^\epsilon, v^\epsilon)\in\mathcal{A}_r^\varepsilon}\ \mathop{Inf}_{(\varphi,0)\in\mathcal{A}^0}\left(\|\varphi^\epsilon-\varphi\|^2_{H_0^1(\Omega)} + \|v^\epsilon\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = 0.$$

Proof. Let $\eta > 0$ be arbitrary and let $(\psi^\varepsilon, w^\varepsilon) \in \mathcal{A}_\tau^\varepsilon$ be such that

$$\underset{(\psi,0)\in\mathcal{A}^0}{Inf} \left(\|\psi^\varepsilon - \psi\|^2_{H_0^1(\Omega)} + \|w^\varepsilon\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \geq d\left(\mathcal{A}_\tau^\varepsilon, \mathcal{A}^0\right) - \eta. \tag{6.1}$$

Let $(\varphi^\varepsilon(t), v^\varepsilon(t))_{t\in \mathbb{R}}$ be the complete orbit in $\mathcal{A}_\tau^\varepsilon$ such that

$$(\varphi^\varepsilon(0), v^\varepsilon(0)) = (\psi^\varepsilon, w^\varepsilon) \tag{6.2}$$

We deduce from the invariance of $\mathcal{A}_\tau^\varepsilon$, from Theorem 3.21 and Lemma 5.1 that there exists a positive constant C which does not depend on ε such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\sqrt{\varepsilon}\|\varphi_t^\varepsilon(t)\|^2_{L^2(\Omega)} + \|\varphi^\varepsilon(t)\|^2_{H^2(\Omega)} + \|g(\varphi^\varepsilon(t))\|^2_{L^2(\Omega)} \leq C, \tag{6.3}$$

and

$$\|v^\varepsilon(t)\|^2_{H_0^1(\Omega)} \leq C\varepsilon \tag{6.4}$$

for all $t \in \mathbb{R}$. Moreover

$$\|\varphi_t^\varepsilon\|^2_{L^2(-T,T;H_0^1(\Omega))} + \|v_t^\varepsilon\|^2_{L^2(-T,T;L^2(\Omega))} \leq C \text{ for all } T > 0. \tag{6.5}$$

Thus we deduce from Simon [6] that the set $\{\varphi^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is precompact in $\mathcal{C}([-T,T]; H^{2-\gamma}(\Omega))$ for all $\gamma \in (0,1]$ and all $T > 0$ and that the set $\{v^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is precompact in $\mathcal{C}([-T,T]; L^2(\Omega))$ for all $T > 0$. Next we also use that if $\gamma \in (0, 1/2)$ the embedding $H^{2-\gamma}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ holds [7, p. 44-47] to deduce that there exists a subsequence of $\{\varphi^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ which we denote again by $\{\varphi^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ and a function $\varphi \in L^\infty(-T,T; (H^2(\Omega) \cap H_0^1(\Omega)))$ such that as $\varepsilon \downarrow 0$

$$\varphi^\varepsilon \longrightarrow \varphi \text{ in } \mathcal{C}([-T,T]; H_0^1(\Omega)) \bigcap \mathcal{C}(\overline{\Omega} \times [-T,T]). \tag{6.6}$$

Moreover it follows from (6.4) that

$$v^\varepsilon \longrightarrow 0 \text{ in } \mathcal{C}([-T,T]; L^2(\Omega)),$$

as $\varepsilon \downarrow 0$. Also note that if $n \leq 3$ and if $\gamma \in (0, 1/2)$ then the embedding $H^{2-\gamma}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ holds [7, p. 44-47]. Therefore we deduce that as $\varepsilon \downarrow 0$

$$\varphi^\varepsilon \longrightarrow \varphi \text{ in } \mathcal{C}([-T,T] \times \overline{\Omega}). \tag{6.7}$$

Next we show that as $\varepsilon \downarrow 0$

$$g(\varphi^\varepsilon) \longrightarrow g(\varphi) \text{ weakly in } L^2(-T,T; L^2(\Omega)).$$

The method of proof is similar to that of Section 4. Let $\xi \in (0,1)$ be arbitrary small. For all $t \in I\!R$, we introduce the sets

$$E_\xi^\varepsilon(t) = \{x \in \Omega,\ |\varphi^\varepsilon(x,t)| > 1-\xi\}, \qquad E_\xi(t) = \{x \in \Omega,\ |\varphi(x,t)| > 1-\xi\},$$

and we denote by $|E_\xi^\varepsilon(t)|$ and $|E_\xi(t)|$ their measures and by $\mathcal{X}_\xi^\varepsilon(t)$ and $\mathcal{X}_\xi(t)$ the associated characteristic functions. It follows from (6.3) that for all $t \in I\!R$

$$|E_\xi^\varepsilon(t)| \leq \frac{4C}{ln^2\left(\dfrac{2-\xi}{\xi}\right)}. \tag{6.8}$$

Then letting $\xi \downarrow 0$ we deduce that for all $t \in I\!R$

$$Meas\,\{x \in \Omega,\ |\varphi^\varepsilon(x,t)| \geq 1\} = 0. \tag{6.9}$$

Moreover we deduce from (6.7), (6.8) together with Fatou's Lemma that for all $t \in (-T,T)$

$$|E_\xi(t)| = \int_\Omega \mathcal{X}_\xi(x,t)dx \leq \lim_{\varepsilon\downarrow 0}\ \inf \int_\Omega \mathcal{X}_\xi^\varepsilon(x,t)dx = \lim_{\varepsilon\downarrow 0}\ \inf |E_\xi^\varepsilon(t)| \leq \frac{4C}{ln^2\left(\dfrac{2-\xi}{\xi}\right)},$$

so that letting $\xi \downarrow 0$ it follows that for all $t \in (-T,T)$

$$Meas\,\{x \in \Omega,\ |\varphi(x,t)| \geq 1\} = 0. \tag{6.10}$$

Thus using (6.6), (6.9) and (6.10) we deduce that for all $t \in (-T,T)$

$$g(\varphi^\varepsilon) \longrightarrow g(\varphi) \text{ a.e. in } \Omega. \tag{6.11}$$

It then follows from (6.3), (6.11) and Lions [5, Lemma 1.3, p.12] that

$$g(\varphi^\varepsilon) \rightharpoonup g(\varphi) \text{ weakly in } L^2(-T,T;L^2(\Omega)). \tag{6.12}$$

Next we rewrite the parabolic equation (1.5) in the form

$$\Delta\varphi^\varepsilon - g(\varphi^\varepsilon) = \varepsilon\varphi_t^\varepsilon - \alpha\varphi^\varepsilon - \frac{1}{\sqrt{\varepsilon}}v^\varepsilon,$$

and use (6.5), (6.3) and (6.4) to deduce that

$$\|\Delta\varphi^\varepsilon - g(\varphi^\varepsilon)\|_{L^2(-T,T;H_0^1(\Omega))} \leq C$$

so that using also (6.3) and (6.12) it follows that as $\varepsilon \downarrow 0$

$$\Delta\varphi^\varepsilon - g(\varphi^\varepsilon) \rightharpoonup \Delta\varphi - g(\varphi) \text{ weakly in } L^2(-T,T;H_0^1(\Omega)) \tag{6.13}$$

and thus since $g(0) = 0$, we deduce from (6.7) that φ satisfies the boundary condition $\Delta\varphi = 0$.

Next we prove that the limiting function φ satisfies Problem (P). To that purpose we rewrite the equations (1.5) and (1.6) in the form

$$\sqrt{\varepsilon} v_t^\varepsilon + \varphi_t^\varepsilon = \Delta\left(-\Delta\varphi^\varepsilon + g(\varphi^\varepsilon) - \alpha\varphi^\varepsilon + \varepsilon\varphi_t^\varepsilon\right),$$

and remark that the pair $(\varphi^\varepsilon, v^\varepsilon)$ satisfies the integral equation

$$\int_{-T}^{T}\int_\Omega \left\{\sqrt{\varepsilon} v_t^\varepsilon + \varphi_t^\varepsilon\right\} \mathcal{X} dx = \int_{-T}^{T}\int_\Omega \left\{-\Delta\varphi^\varepsilon + g(\varphi^\varepsilon) - \alpha\varphi^\varepsilon + \varepsilon\varphi_t^\varepsilon\right\} \Delta\mathcal{X} dx \quad (6.14)$$

for all $\mathcal{X} \in \mathcal{D}((0,T)\times\Omega)$. Letting $\varepsilon \downarrow 0$ in (6.14), we deduce in view of (6.5), (6.13) and (6.6) that the function φ satisfies the Cahn-Hilliard equation

$$\varphi_t + \Delta\left(\Delta\varphi + \alpha\varphi - g(\varphi)\right) = 0$$

in $L^2(-T,T;H_0^1(\Omega))$, together with the boundary conditions $\varphi = \Delta\varphi = 0$.

Next note that by (6.3) and (6.10) $\varphi \in BC\left(I\!R; H_0^1(\Omega)\bigcap\mathcal{K}\right)$ where BC stands for bounded continuous. Therefore, the complete trajectory $(\varphi(t))_{t\in I\!R}$ belongs to $\mathcal{A}$. Moreover, it follows from (6.6) that the pair $(\psi^\varepsilon, w^\varepsilon) = (\varphi^\varepsilon(0), v^\varepsilon(0))$ converges to $(\varphi(0), 0) \in \mathcal{A}^0$ in $H_0^1(\Omega)\times L^2(\Omega)$ as $\varepsilon \downarrow 0$. This implies that

$$\lim_{\varepsilon\downarrow 0} \ \underset{(\psi,0)\in\mathcal{A}^0}{Inf} \left(\|\psi^\varepsilon - \psi\|_{H_0^1(\Omega)}^2 + \|w^\varepsilon\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} = 0,$$

which, in view of (6.1), implies that for each $\eta > 0$

$$0 \le \lim_{\varepsilon\downarrow 0} \ \sup\ d\left(\mathcal{A}_\tau^\varepsilon, \mathcal{A}^0\right) \le \eta\ .$$

Therefore

$$\lim_{\varepsilon\downarrow 0} d\left(\mathcal{A}_\tau^\varepsilon, \mathcal{A}^0\right) = 0.$$

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