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Théorèmes limites pour des processus faiblement ou fortement dépendants. Applications statistiques, 1999

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Sujet : **THÉORÈMES LIMITES POUR DES PROCESSUS FAIBLEMENT OU FORTEMENT DÉPENDANTS. APPLICATIONS STATISTIQUES.**

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À mon père, à ma mère...
À mon frère et à mes soeurs...
À mes professeurs...
À tous mes amis...

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Limit Theorems for Short or Long-range Dependent Processes. Statistical Applications.

Abstract. This thesis is devoted to the study of short and long-range dependent processes in a unified approach.

The main subjects of this work are non-linear functional of Gaussian processes. We are interested in functional limit theorems, called also invariance principles, for empirical processes indexed by classes of functions or classes of sets. We have considered the discret time case as well as the continuous time case.

To carry out the proof of these theorems we established new moment inequalities which are of their own interest. Some other applications are given such as a rate of convergence in the strong law of large numbers and a high order asymptotic for the empirical distribution function.

In the last part, we dealt with the density estimation problem under gaussian subordination. In particular, we established the rate of convergence in law of the kernel estimator and we identified the limit law depending on the order of magnitude of the bandwidth and the decay of the covariance function.

Key Words: Short and long-range dependence, Gaussian processes, Empirical processes, Moment inequalities, Functional central limit theorem, Kernel estimators, Local time, Partial sums, Hermite's polynomial, Diagram method.

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Introduction

En statistique, nous sommes souvent face à une suite d'observations et nous souhaitons, à partir de ces observations, extraire des informations sur le système en question. Pour cela, nous avons besoin d'un modèle mathématique pour modéliser le système .

Clairement, le modèle est d'autant plus fiable chaque fois qu'il reflète au mieux les propriétés du système et tient compte de ses spécificités. Ainsi, dans le domaine de l'analyse des séries temporelles : par exemple, modéliser l'évolution d'un système au fil du temps.... ; le modèle d'observations indépendantes semble peu réaliste et peut même fausser les résultats de l'étude. Pour prendre en compte cette structure de dépendance entre le passé et le futur, les probabilistes ont développé des mesures de dépendance. Les premières sont apparues dans les travaux de Rosenblatt. Ainsi, de nouvelles difficultés techniques apparaissent : les théorèmes limites habituelles sous l'hypothèse d'indépendance ne sont plus valides ainsi que les tests et les estimateurs construits à l'aide de ces théorèmes.

Une litterature abondante généralise des résultats existants sous des hypothèses d'indépendance à ce qu'on appelle processus faiblements dépendants : c.à.d. la mesure de dépendance tend vers zéro avec une vitesse suffisante lorsque la distance entre le passé et le futur tend vers l'infini . En d'autres termes, notre système oublie petit à petit, mais relativement vite son passé, une hypothèse qui semble tout à fait naturelle. Il est important de noter que les théorèmes limites ainsi que les outils permettant de les établir ne se distinguent pas dans leurs formes générale de leurs homologues sous des hypothèses d'indépendance. Par suite, l'enjeu était la recherche des hypothèses minimales dites alors optimales qui suffisent pour énoncer ces résultats.

Cependant, des données empiriques ont montré que la covariance des données réelles converge très lentement vers zéro et ne peuvent satisfaire ces hypothèses de dépendance faible. Ces phénomènes qui expriment le faite que notre système a une forte mémoire et oublie difficilement son passé, apparaissent naturellement en hydrologie, économétrie, météorologie et d'autres domaines. Ceux-ci exposent les mathématiciens à des nouvelles difficultés aussi bien sur les techniques de démonstrations que sur les types de résultats eux-mêmes.

Ce phénomène est connu sous le nom de dépendance forte ou encore de longue mémoire. Un intérêt croissant lui a été porté ces dernières années, et c'est dans ce cadre là que s'inscrit notre travail qui se veut être une contribution à l'étude des variables fortement ou faiblement dépendantes dans une approche unifiée.

Évidemment, nous sommes loin de résoudre ce problème dans toute sa généralité. Nous devons donc spécifier notre modèle. Rappelons que dans la littérature existante deux types de modèles sont généralement traités :

- les fonctionnelles des processus gaussiens.
- les processus linéaires.

Nous avons choisi de travailler sur le premier modèle. Plus précisément, nous considérons un processus gaussien que nous supposons stationnaire pour simplifier les calculs et nous proposons d'établir des théorèmes limites pour des fonctionnelles non linéaires du processus.

Pour pouvoir démontrer ces résultats, nous développons des nouvelles inégalités de moments aussi bien en temps continu qu'en temps discret. Notons au passage que nous nous sommes confrontés à une difficulté supplémentaire dans le cas continu due au comportement de la fonction de corrélation au voisinage de zéro.

Résultats

Les deux premiers chapitres sont consacrés au temps continu, les deux suivants au temps discret. Enfin, dans le dernier chapitre, nous étudions une application statistique très courante: l'estimation fonctionnelle. Avant de se lancer dans le détail des résultats obtenus, nous signalons que sous des hypothèses de dépendance forte les résultats diffèrent quantitativement (normalisation, vitesse de convergence...) et qualitativement (nature des lois limites : non gaussiennes en général, ...) du cas de la dépendance faible.

Chapitre 1

Dans le chapitre 1, comme exemple de fonctionnelle non linéaire d'un processus gaussien, nous étudions le temps local d'un processus gaussien stationnaire. En particulier nous donnons un développement du temps local et nous montrons que le reste du développement (partie correspondante à la dépendance faible) converge avec la normalisation habituelle vers un processus gaussien.

Soit $\{X_t, t \in \mathbb{R}\}$ un processus gaussien stationnaire vérifiant

$$\mathbb{E}(X_s X_t) = r(|s-t|), \mathbb{E}(X_s) = 0,\ \mathbb{E}(X_s^2) = 1$$

Supposons que

$$\int_0^t \frac{1}{\sqrt{1-r^2(s)}} ds < \infty, \tag{1}$$

et qu'il existe m tel que

$$\int |r(s)|^m ds < \infty. \tag{2}$$

La condition (1) (voir [9]) assure l'existence du temps local que nous notons $l_t(x)$ et admet le développement suivant dans $L^2(\Omega)$:

$$l_t(x) = p(x) \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} \int_0^t H_k(X_s) ds \tag{3}$$

avec :

$$p(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{et} \quad H_k(x) := (-1)^k \frac{p^{(k)}(x)}{p(x)}$$

où $p(x)$ désigne la densité d'une loi normale centrée réduite et $H_k(x)$ est le $k-$ième polynôme d'Hermite. Écrivons

$$l_t(x) = tp(x) + \sum_{k=1}^{m-1} \frac{H_k(x)p(x)}{k!} \int_0^t H_k(X_s) ds + R_t(x) \tag{4}$$

où

$$R_t(x) := p(x) \sum_{k=m}^{\infty} \frac{H_k(x)}{k!} \int_0^t H_k(X_s) ds. \tag{5}$$

On montre alors

Théorème 1. *Soit* $\{X_t, t \in \mathbb{R}\}$ *un processus gaussien stationnaire. Soit*

$$\Gamma(x,y) = 2\sum_{k=m}^{\infty} \frac{H_k(x)p(x)H_k(y)p(y)}{k!} \int_0^{\infty} r^k(s)ds. \tag{6}$$

(a) *Sous les conditions* (1) (2), l'expression (6) *est bien définie. En plus les lois fini-dimensionnelles du processus*

$$\{\frac{1}{\sqrt{t}}R_t(x), \ x \in \mathbb{R}\}$$

converge vers ceux d'un processus gaussien centré, $(R(x))_{x\in\mathbb{R}}$ *dont la fonction de covariance est donnée par* $(\Gamma(x,y))_{(x,y)\in\mathbb{R}^2}$.

(b) *S'il existe t suffisamment grand tel que*

$$\int_0^t \frac{1}{(1-r^2(s))^p} ds < \infty, \tag{7}$$

pour $p > \frac{1}{2}$, *le processus limite admet une modification en loi ayant des trajectoires continues.*

(c) *Enfin, si la condition* (7) *est satisfaite pour un* $p > 1$, *alors la convergence aura lieu dans* $C(\mathbb{R}, \mathbb{R})$ *muni de la convergence uniforme sur les compacts.*

Chapitre 2

Le temps local qui est par définition la densité de la mesure d'occupation, n'existe pas toujours. Dans le deuxième chapitre nous étudions cette mesure d'occupation.

Soit G une fonction réelle de variable réelle. Posons

$$m = \inf\{k > 0; \quad c_k \neq 0\}, \text{ où } c_k = \mathbb{E}(G(X)H_k(X))$$

m est dit le rang d'Hermite de la fonction G et joue un rôle important dans les lois limites de $\int_0^t G(X_s)ds$. (Voir les travaux de Breuer et Major (1983), Taqqu (1977), Dobrushin et Major (1977)). Étant donnée une fonction $G : \mathbb{R} \to \mathbb{R}$, nous donnons des conditions suffisantes sur $\{X_t : t \in \mathbb{R}\}$ et sur G pour que

$$\left\{\int_0^t G(X_s) - \mathbb{E}[G(X_s)]\, ds\right\}_{t>0}$$

convenablement normalisé converge en distribution. Ces questions ont fait l'objet de plusieurs travaux : Taqqu (1979), Dobrushin et Major (1979), Breuer et Major (1983), Chambers et Slud (1989), Arcones (1994), Csörgő et Mielniczuk (1996).

Nous étudions pour commencer la convergence des processus des sommes partielles. Nous montrons que si la fonction de covariance et les coefficients de la fonction décroissent suffisamment vite vers zéro, alors le processus

$$\left\{\frac{1}{\sqrt{t}}\int_0^{tx}\left(G(X_s)-\mathbb{E}\left[G(X_s)\right]\right)ds : x\in\mathbb{R}\right\}_{t>0},$$

converge dans $\mathcal{C}([0,1];\mathbb{R})$ muni de la convergence uniforme.

Théorème 2. *Soit G une fonction vérifiant $\mathbb{E}\left[G\left(X\right)\right]=0$ et $\mathbb{E}\left[G^2\left(X\right)\right]<\infty$. Soit m son rang d'Hermite*

(i) *Si $\int\left|r^m\left(s\right)\right|ds<\infty$, alors les lois fini-dimensionnelles de $\left(Z_t(\cdot)\right)_{t\geq 0}$ définies par :*

$$Z_t(x)=\frac{1}{\sqrt{t}}\int_0^{tx}G(X_s)ds, \qquad x\in\mathbb{R}$$

tendent vers celles de $\sigma W(\cdot)$ où $W(\cdot)$ est le mouvement brownien standard et

$$\sigma^2=\sum_{k=m}^{\infty}\frac{c_k^2}{k!}\int r^k(s)ds \;<\; \infty. \tag{8}$$

(ii) *Si de plus, l'une des deux conditions suivantes est satisfaite :*

1. Il existe $R>1$ telle que

$$\sum_{k=m}^{\infty}\frac{|c_k|}{\sqrt{k!}}\left(\int|r^k(s)|ds\right)^{1/2}R^k \;<\; \infty, \tag{9}$$

2. Les c_k sont positifs et $\mathbb{E}\left[G^4\left(X\right)\right]<\infty$..

Alors, la convergence aura lieu dans $\mathcal{C}([0,1])$ muni de la convergence uniforme.

Lorsque le processus est fortement dépendant, la question a été résolue par Taqqu (1979), Dobrushin et Major (1979), pour des processus gaussiens à valeurs réelles et généralisée par Arcones 1994 au champs gaussiens à valeurs dans $\mathbb{R}^n$. Dans le cas de la dépendance faible, la question a été étudiée par Chambers et Slud 1989. Ici nous améliorons partiellement leurs résultats. En effet, la condition de Chambers et Slud était

$$\sum_{k=m}^{\infty} \frac{|c_k|}{\sqrt{k!}}(\sqrt{3})^k \;<\; \infty,$$

Le théorème suivant donne des conditions suffisantes pour qu'un théorème limite centrale (abrégé TLC dans la suite) uniforme sur la classe $\mathcal{F} = \{1_{G(\cdot)\leq x} : x \in \mathbb{R}\}$ soit vraie. Dans le cas discret, la convergence uniforme a été étudié par Dehling et Taqqu (1989), dans le cas de la longue mémoire et par Csörgő et Mielniczuk (1996) dans le cas de courte mémoire. Les techniques utilisées ne s'adaptent pas au cas continu. Nous complétons ces résultats existants par le théorème suivant.

Théorème 3. *Soit $(X_s)_{s\geq 0}$ un processus gaussien stationnaire. Soit m le rang d'Hermite de la famille $\mathcal{F} = \{1_{\{G(\cdot)\leq x\}}; x \in \mathbb{R}\}$ et F la fonction de distribution $G(X)$.*

Si $r^m \in \mathbb{L}^1$, alors les lois fini-dimensionnelles de $Z_t(G,.)$ définies par

$$Z_t(G,\cdot) = \frac{1}{\sqrt{t}} \int_0^t \left(1_{\{G(X_0)\leq x\}} - \mathbb{E}\left[1_{\{G(X_0)\leq x\}}\right]\right) ds$$

convergent vers celles d'un processus gaussien centré dont la fonction de covariance est

$$\Gamma(x,y) = 2\int_0^{\infty} \mathrm{Cov}\left(1_{\{G(X_0)\leq x\}}, 1_{\{G(X_s)\leq y\}}\right) ds$$

(i) *Si de plus on suppose que*

$(K1)$ $\int_0^t \frac{1}{\sqrt{1-r^2(s)}} ds < \infty, \forall t > 0$

(K_2) *F est continue et la mesure de Lebesgue de $G^{-1}[0,1]$ est finie.*

Alors, la famille $Z_t(G,\cdot)$ est tendue dans $\mathcal{C}([-\infty,+\infty])$.

(ii) *Si F est continue et si les hypothèses $H1, H2$, et ($H3$ ou $H'3$) ci-dessous sont satisfaites :*

$(H1)$ $X_t = \int a_{t+s} dW_s$

$(H2)$ $v^{2\beta}\int_{|s|>v} a_s^2 ds < \infty, \quad \beta > 1/2.$

$(H3)$ $\sup_k \sup_v k \int_0^v (r_v^*(s))^k ds < \infty.$

$(H'3)$ $\sup_{x,y} v^{\beta}\mathbb{E}\left|1_{\{x<F\circ G(X)\leq y\}} - 1_{\{x<F\circ G(X^v)\leq y\}}\right|^2 < \infty$, *pour un* $\beta > 1/2$.

Alors, la famille $Z_t(G,\cdot)$ *est tendue dans* $\mathcal{C}([-\infty,+\infty])$.

La condition $(K1)$ assure l'existence du temps local. Notons que cette condition est similaire à $(H3)$ puisque toutes les deux sont concernées par le comportement de la fonction de covariance au voisinage de l'origine. De telles considerations n'apparaissent pas dans le cas discret.

Chapitre 3

Dans les deux premiers chapitres, les processus étudiés peuvent être vus comme des cas particuliers de processus empiriques indexés par des classes de fonctions. Dans le troisième chapitre, nous examinons la question de TLC uniforme sur une classe donnée de fonctions. Clairement, nous aurons besoin de contrôler la taille de cette classe. Pour cela nous avons utilisé des conditions sur l'entropie à crochets qui semblent bien adaptés aux variables dépendantes. Nous montrons que les conditions suffisantes assurant un TLC uniforme au sens de Hoffman réalisent un équilibre entre la régularité du processus mesurée avec les structures de dépendance et la taille de la famille en question.

Soit $(X_i)_{i\geq 0}$ une suite stationnaire et $\mathcal{F}$ une classe de fonctions réelles à variable réelle. Soit $l^{\infty}(\mathcal{F})$ l'espace des fonctions bornées définies sur $\mathcal{F}$. Nous définissons l'application

$$\begin{aligned} Z_n \ : \ & \mathcal{F} \longrightarrow \mathbb{R} \\ & f \longmapsto Z_n(f) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(f(X_i) - \mathbb{E}(f(X_i))\right). \end{aligned}$$

Si $\sup_{\mathcal{F}}|f(.) - \mathbb{E}(f(X_0))|$ existe et est fini, alors l'application Z_n est un élément de $l^{\infty}(\mathcal{F})$ et il est donc légitime d'étudier la convergence en loi de Z_n dans $l^{\infty}(\mathcal{F})$ muni de la topologie uniforme. Une famille vérifiant cette convergence est dite *classe de Donsker* (en référence au premier auteur qui a étudié cette question pour la classe des quadrants). D'après les travaux de Pollard, Hoffman, cette convergence est caractérisée par celle des lois marginales plus la tension. Précisément, Z_n converge $l^{\infty}(\mathcal{F})$ si et seulement si :

(i) Pour tout $f_1, ..., f_k$ éléments de $\mathcal{F}$,

$$(Z_n(f_1), ..., Z_n(f_k)) \text{ converge en loi.}$$

(ii) Il existe une pseudo-métrique ρ telle que $(\mathcal{F}, \rho)$ soit totalement bornée et telle que pour tout $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \mathbb{P}\left(\sup_{\rho(f,g)<\delta} |Z_n(f-g)| > \varepsilon \right) = 0. \tag{10}$$

La seconde propriété est connue sous le nom d'équicontinuité stochastique et c'est généralement l'étape difficile dans les preuves des théorèmes fonctionnels. Rappelons qu'en 1987, Ossiander a montré que si les variables sont indépendantes et identiquement distribuées, alors (ii) est satisfaite dès que :

$$\int_0^1 \sqrt{\log N_{[\,]}(\varepsilon, \|\cdot\|_2, \mathcal{F})} d\varepsilon < \infty,$$

où $N_{[\,]}(\varepsilon, \|\cdot\|_2, \mathcal{F})$ est le nombre minimal de $\varepsilon-$crochets suffisant pour recouvrir $\mathcal{F}$. (Voir plus loin pour une définition précise). Ce résultat a été généralisé par Doukhan, Massart et Rio en 1995 aux suites absolument régulières dites encore $\beta-$mélangeantes.

D'autre part, Arcones en 1994 a montré l'équicontinuité stochastique de $\{Z_n(f), f \in \mathcal{F}\}_{n>0}$ lorsque X est gaussien avec une fonction de covariance sommable dès que

$$\int_0^1 N_{[\,]}^{1/2}(x, \|\cdot\|_2, \mathcal{F})\, dx < \infty.$$

Andrews et Pollard 1994 ont montré la tension pour les suites fortement mélangeantes sous les hypothèses

$$\sum_{i>0} i^{p-2} \alpha^{\frac{\gamma}{p+\gamma}}(i) < \infty,$$

$$\sup_{\mathcal{F}} |f| \leq 1 \text{ et } \int_0^1 x^{-\frac{\gamma}{\gamma+2}} N_{[\,]}^{1/p}(x, \|.\|_2, \mathcal{F})\, dx < \infty$$

où $p \geq 2$ et $\gamma > 0$. Leurs preuve était essentiellement fondée sur une inégalité de moment, combinée avec un argument de chainage.

Avant d'énoncer le prochain théorème, nous rappelons la définition de l'entropie à crochets.

Définition 1. *Étant données deux fonctions l et u, le crochet $[l, u]$ est l'ensemble de toutes les fonctions f vérifiant $l \leq f \leq u$. Soit $\|\cdot\|$ une norme définie sur un espace contenant $\mathcal{F}$. Un $\varepsilon-$crochet pour $\|\cdot\|$ est un $[l, u]$ qui vérifie $\|l - u\| < \varepsilon$. L'entropie à crochets $N_{[\,]}(\varepsilon, \|\cdot\|, \mathcal{F})$ est le nombre minimal de $\varepsilon-$crochets permettant de couvrir $\mathcal{F}$.*

Pour $p \geq 2$, nous introduisons deux types d'hypothèses :

$\mathbf{H}(p, X)$: Il existe deux constantes $a(p)$ et $b(p)$ telles que pour tout fonction f mesurable

$$\mathbb{E}\,|Z_n(f)|^p \leq a(p)\,\|f\|_{2,X}^p + b(p)n^{1-p/2}\,\|f\|_\infty^{p-2}\,\|f\|_{2,X}^2 \tag{11}$$

où $\|\cdot\|_{2,X}$ est une norme vérifiant :

* Il existe $C > 0$ telle que $\|\cdot\|_1 \leq C\,\|\cdot\|_{2,X}$.
* $|f| \leq |g| \Rightarrow \|f\|_{2,X} \leq \|g\|_{2,X}$.

$\mathbf{H}(p, \mathcal{F})$: $\mathcal{F}$ est uniformément bornée si $p > 2$ et

$$\int_0^1 N_{[\,]}^{1/p}\left(x, \|\cdot\|_{2,X}, \mathcal{F}\right) dx < \infty. \tag{12}$$

La norme $\|\cdot\|_{2,X}$ est celle qui apparait dans les inegalités de moment du type Rosenthal. L'indice X exprime le fait que cette norme dépend en général du processus. En particulier, pour les variables indépendantes cette norme n'est autre que la norme $\|\cdot\|_2$.

Nous somme maintenant en mesure d'énoncer un théorème qui regroupe trois résultats d'équicontinuité stochastique. Dans le (a), nous traitons les classes de fonctions uniformement bornées. Dans le (b) l'hypothèse selon laquelle $\mathcal{F}$ est uniformément bornée est remplacée par une hypothèse d'intégrabilité de la fonction enveloppe. Le (c) donne un exemple de condition sur le logarithme de l'entropie si les coefficients de mélange décroissent avec une vitesse exponentielle.

Théorème 4. *Soit $(X_i)_{i\geq 0}$ une suite stationnaire de v. a. et $\mathcal{F}$ une classe de fonctions*

(a) *Si les hypothèses H(p, X) et H($p, \mathcal{F}$) sont satisfaites, alors*

$$\forall \varepsilon > 0, \quad \lim_{\delta\to 0}\limsup_{n\to+\infty} \mathbb{P}\left(\sup_{\|f-g\|_{2,X}<\delta} |Z_n(f-g)| > \varepsilon\right) = 0. \tag{13}$$

(b) *Soit $F \geq \sup_{f\in\mathcal{F}}|f|$. Nous supposons que $F \in \mathbb{L}^{r+1}$, pour $r > 1$ et que $\mathcal{F}$ vérifie l'hypothèse H(p, X),*

$$\int_0^1 N_{[\,]}^{\nu/p}\left(x, \|\cdot\|_{2,X}, \mathcal{F}\right) dx < \infty, \tag{14}$$

où $1/\nu = 1 - \frac{1}{r}\left(1 - \frac{2}{p}\right)$. Alors (13) est vérifiée.

(c) *Soit $(X_i)_{i\geq 0}$ une suite stationnaire et $\mathcal{F}$ une famille de fonctions bornées par 1. Si les conditions suivantes*

a) $\alpha(i) \leq c\exp(-\alpha i)$, *où c, α sont des constantes positives et $\alpha(i)$ désigne le coefficient de mélange fort de la suite $(X_i)_{i\geq 0}$.*

b) $\int_0^1 \log^2 N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F})\, d\varepsilon < \infty$.

sont vérifiées, alors $\forall \varepsilon > 0$,

$$\lim_{\delta\to+\infty}\limsup_{n\to+\infty} \mathbb{P}\left(\sup_{\|f-g\|_1\leq\delta} |Z_n(f-g)| > \varepsilon\right) = 0.$$

Ce dernier résultat améliore le résultat de Massart 1987 puisque sa condition sur $\mathcal{F}$ était $\log N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F}) \leq C\left(\frac{1}{\varepsilon}\right)^{\xi}$, avec $\xi < 1/4$. Enfin, nous signalons que Andrews et Pollard ont conjoncturé dans leurs article de 1994 qu'une condition suffisante pourrait être

$$\int_0^1 \varepsilon^{-\frac{\gamma}{\gamma+2}} \log^2 N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F})\, d\varepsilon < \infty,$$

où $\gamma > 0$.

Chapitre 4

Dans le quatrième chapitre, nous étudions les sommes partielles. En particulier, nous établissons des inégalités de moments de type Rosenthal sous des conditions de dépendance très générales. Nous remarquons en particulier que contrairement aux inégalités de moments avec des hypothèses de mélanges, la condition de sommabilité absolue de la série des covariances suffit (en fait une condition plus faible est parfois suffisante) pour avoir des inégalités similaires à celles du cas indépendant, pour les moments d'ordre quelconque.

Soit f une fonction réelle et p un entier pair. Désignons par m le rang d'Hermite de f et par m_p le rang de la famille $\{f, f^2, f^3, ..., f^{p-2}\}$. Définissons aussi

$$\begin{aligned} \|f\|_{2,p} &= \max\left(p^{1/2}\|f\|_2, \sum_{k=m_p}^{\infty} \frac{|d_k(f)|}{\sqrt{k!}}\left(2\sum_{i=1}^{n} |(p-1)\, r(i)|^k\right)^{1/2}\right) \quad (15) \\ r^*(j) &= \sup_{i\geq j} |r(i)|. \\ \alpha(s,i) &= ((s-1)r^*(i))^{1/2} \end{aligned}$$

où $d_k(f) = \mathbb{E}\,|f\,(X)\,H_k(X)|$. On montre

Théorème 5. *Soit f une fonction centrée telle que $\mathbb{E}\,[f^2\,(X)] < \infty$, et p un entier pair*

(i) *Si la fonction f est bornée nous avons :*

$$\mathbb{E}\left|\sum_{i=1}^{n} f(X_i)\right|^p \leq 6^p \left\{\left(\sqrt{n}\,\|f\|_{2,p}\right)^p + p^p n\, \|f\|_{2,p}^2\, \|f\|_\infty^{p-2}\right\} \quad (16)$$

où $\|f\|_{2,p}$ est donnée par (15).

(ii) *Soit m_p est le rang d'Hermite de $\{f, f^2, ..., f^{p-2}\}$. Si r^{m_p} est intégrable alors il existe une constante $K = K(p,r)$ telle que pour $n > 0$,*

$$\mathbb{E}\,|S_n(f)|^p \leq K(p,r)\left(\left(\sqrt{n}\,\|f\|_2\right)^p + n\,\|f\|_p^p\right).$$

(iii) *Si r^m est intégrable où m est le rang d'Hermite de f, alors il existe une constante $K = K(p,r)$ telle que pour tout $n > 0$,*

$$\mathbb{E}\,|S_n(f)|^p \leq K(p,r)\left(\sqrt{n}\,\|f\|_p^p\right)^p.$$

Comme applications de ces inégalités, nous donnons un développement asymptotique de la fonction de répartition empirique dans le cas d'une subordination gaussienne, qui généralise et unifie les résultats de Dehling et Taqqu 1989 (cas de la dépendance forte) et ceux de Csörgő et Mielniczuk (cas de la dépendance faible). Des résultats similaires, mais sans convergence uniforme ont été prouvés récemment par Koul et Surgailis 1997, lorsque le processus est linéaire.

Soit $Y_k(\cdot)$ le processus d'Hermite d'ordre k défini par

$$Y_k(t) = \int_{\mathbb{R}^k} \left[\int_0^t \prod_{i=1}^k (v - u_i)_+^{-\frac{\alpha+1}{2}} dv \right] W(du_1)...W(du_k) \qquad (17)$$

où $v_+ = v \vee 0$ et W est le mouvement Brownien standard. Pour une fonction G mesurable, posons

$$F_n(x) = \frac{1}{n} \sum_{j=1}^n I_{\{G(X_j) \leq x\}}.$$

Soit

$$m(x) := rang(I_{\{G(\cdot) \leq x\}}),\ m := \inf_{x \in \mathbb{R}} m(x) \text{ et } F(x) = \mathbb{P}\left(G(X_j) \leq x\right).$$

Alors

$$F_n(x) - F(x) = \sum_{k=m}^{\infty} \frac{j_k(x)}{k!} \frac{1}{n} \sum_{j=1}^n H_k(X_j)$$

où

$$j_k(x) = \mathbb{E}\left(I_{\{G(X_j) \leq x\}} H_k(X_j)\right). \qquad (18)$$

Pour tout $k^* > m$ on pose

$$R_n(x) := \sum_{k=k^*}^{\infty} \frac{j_k(x)}{k!} \frac{1}{n} \sum_{j=1}^n H_k(Z_j). \qquad (19)$$

A l'aide des inégalités précédentes nous prouvons

Théorème 6. *Soit X_n un processus gaussien stationnaire.*

(1) *Supposons F continue et r^{k^*} intégrable, alors*

$$\sqrt{n} R_n(\cdot) \to R(\cdot) \quad \textit{dans } \mathcal{C}(\mathbb{R}, \mathbb{R})$$

muni de la topologie uniforme, où R est un processus gaussien de fonction de covariance

$$R(x,y) = \sum_{k=k^*}^{\infty} \frac{j_k(x)j_k(y)}{k!} \sum_j r^k(j)$$

(2) *Si de plus $r(n)$ est équivalente à $n^{-\alpha}L(n)$ où $\alpha > 0$ est un réel strictement positif et L est une fonction à variation lente. Alors pour tout k tel que $0 < km < 1$ nous avons*

$$\frac{1}{d_{n,k}} \frac{j_k(x)}{k!} \sum_{j=1}^{n} H_k(Z_j) \Longrightarrow \frac{j_k(x)}{k!} Y_k(1) \qquad \text{dans } \mathcal{D}(\mathbb{R}, \mathbb{R})$$

où $d_{n,k}^2 = \mathbb{E}(\sum_{i=1}^n H_k(X_i))^2$ et $Y_k(\cdot)$ est le processus d'Hermite d'ordre k.

Chapitre 5

Enfin, cette thèse effectuée dans un laboratoire de statistique, ne peut se terminer sans donner des applications statistiques. Aussi, dans le dernier chapitre nous évoquons le problème d'estimation de la densité (classique en statistique) par la méthode du noyau lorsque les observations sont issues d'une fonction de gaussien fortement ou faiblement dépendant. Contrairement à ce que nous attendions, nous montrons qu'il est possible de réduire l'effet de la dépendance, voir même de l'éliminer en jouant sur la taille de la fenêtre d'estimation. Ceci est résumé dans le théorème 7 ci-dessous.

Soit $Y_n = G(X_n)$. Supposons que Y admette une densité marginale f à estimer. Nous proposons d'utiliser l'estimateur à noyau défini par :

$$f_n(x) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{x - G(X_i)}{b_n}\right)$$

où K est un noyau et b_n est la fenêtre d'estimation (c'est à dire une suite de réels positifs tendant vers 0). Motivés par les travaux de Hall et Hart 1990, Csörgő et Mielniczuk 1995, nous étudions les propriétés asymptotiques de l'estimateur à noyau et nous montrons les résultats suivants

Théorème 7. *Soit (X_n) une suite gaussienne stationnaire et $Y_n = G(X_n)$ la suite stationnaire engendrée supposée de densité marginale f continue.*

(i) *Si $r^m \in \mathbb{L}^1$, $\lim b_n = 0$ et $\lim nb_n = +\infty$, alors*

$$\sqrt{nb_n}\left(f_n(x) - \mathbb{E}[f_n(x)]\right) \Longrightarrow \mathcal{N}\left(0, \sigma^2(x)\right). \tag{20}$$

où $\sigma^2(x) = \int K^2(u)du\ f(x)$

(ii) *Si* $r(n) \sim n^{-\alpha}L(n)$, $m\alpha < 1$ *et* $n^{\beta}b_n \to 0$ *pour* $\beta < 1 - m\alpha$ *alors l'assertion* (20) *est encore valide.*

(iii) *Si* $r(n) \sim n^{-\alpha}L(n)$, $m\alpha < 1$ *et* $n^{\beta}b_n \to \infty$ *pour* $\beta < 1 - m\alpha$ *alors,*

$$\frac{n}{d_{n,m}}\mathbb{E}\left(\sup_{x\in\mathbb{R}} |f_n(x) - \mathbb{E}[f_n(x)] - C_{m,n}(x)\, Y_{m,n}|\right) \longrightarrow 0$$

où

$$Y_{m,n} = n^{-1}\sum_{j=1}^{n} H_m(X_j) \text{ et } C_{m,n}(x) = \mathbb{E}\left(\frac{1}{b_n}K\left(\frac{x-G(X)}{b_n}\right)H_m(X)\right),$$

(iv) *Si de plus* $j_m(\cdot)$ *est bornée et uniformément continue,*

$$\frac{n}{d_{n,m}}\left(f_n(x) - \mathbb{E}[f_n(x)]\right) \Longrightarrow \frac{j'_m(\cdot)}{m!}Y_m(1) \quad \text{dans} \quad \mathcal{C}[-\infty, +\infty]. \qquad (21)$$

Pour remplacer $\mathbb{E}[f_n(x)]$ par $f(x)$, il suffit de choisir des noyaux d'ordre supérieur et de supposer des conditions de régularités sur f, afin de contrôler le terme de biais. Cette discussion est bien exposée dans l'article de Csörgő et Mielniczuk 1995, (voir aussi l'article de Bretagnolle et Huber 1979).

Conclusions et Perspectives

À l'issue de cette étude nous pouvons tirer quelques conclusions. Modéliser la dépendance dans les séries temporelles avec les structures de covariances semble une bonne alternative aux coefficients de mélanges, au moins pour les raisons suivantes :

- Elle permet de tenir en compte des processus à longue mémoire, c'est-à-dire non mélangeants.

- La généralisation aux dimensions supérieures pour traiter les champs de vecteurs est naturelle et garde tout son sens.

- Les covariances sont plus simples à évaluer ou à estimer.

Cependant, nous sommes obligés de nous restreindre à des classes de modèles qui, bien qu'assez générales sont loin de tout englober, d'autant plus qu'il est difficile de vérifier ces hypothèses. Des travaux ayant pour perspective d'élargir la classe de modèles existent déjà, (voir les travaux de Giraitis et Surgailis, Avram et Taqqu ... sur les processus linéaires), mais beaucoup reste à faire.

L'objet fondamental de cette thèse était sans doute les processus empiriques ainsi que leurs fonctionnelles. Une généralisation possible consiste à étudier les $U-$statistiques et les $U-$processus. Les travaux de Bretagnolle 1983, Arcones et Giné, peuvent ainsi être revues dans ce cadre de dépendance.

Enfin, d'autre applications statistiques sont envisageables, dans ce cadre de dépendance. Je pense entre autres, au bootstrap, comme procédure de rééchantillonnage très utile, la régression...etc. Ces questions feront l'objet de mes futures recherches.

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Chapitre 1

Central Limit Theorem for the Local Time

ABSTRACT. We prove a central limit theorem for the local time of a real stationary Gaussian process via its expansion in term of Hermite polynomials. The limiting process is Gaussian, and we give conditions ensuring its sample paths continuity. In the later sections we show other new asymptotics for the local time.

Key words: Central limit theorem, Local time, Gaussian process, Hermite polynomials, Edgeworth expansion.

Introduction

Let $\{X_t, t \in \mathbb{R}\}$ refer to a real valued, Gaussian, stationary process with covariance function $\mathbb{E}(X_s X_t) = r(|s-t|)$, $\mathbb{E}(X_s) = 0$ and $\mathbb{E}(X_s^2) = 1$. We assume that:

$$\int_0^t \frac{1}{\sqrt{1-r^2(s)}} ds < \infty, \tag{1.1}$$

and moreover that there exists some positive integer m with

$$\int |r(s)|^m ds < \infty. \tag{1.2}$$

Condition (1) (see Berman [1], Geman, Horowitz [7] or Doukhan, León [6]) ensures that the local time which we denote by $l_t(x)$ exists and admits the following expansion in $L^2(\Omega)$:

$$l_t(x) = p(x) \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} \int_0^t H_k(X_s) ds \tag{1.3}$$

where for any nonnegative integer k:

$$p(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad H_k(x) \equiv (-1)^k \frac{p^{(k)}(x)}{p(x)}$$

denote the normal density and the k-th order Hermite polynomial. Note that the existence of the local time is related to the condition

$$\frac{1}{\sqrt{1-r^2}} \in L^1_{loc},$$

We will write the expansion (3) as follows:

$$l_t(x) = tp(x) + \sum_{k=1}^{m-1} \frac{H_k(x)p(x)}{k!} \int_0^t H_k(X_s) ds + R_t(x) \tag{1.4}$$

where we set:

$$R_t(x) = p(x) \sum_{k=m}^{\infty} \frac{H_k(x)}{k!} \int_0^t H_k(X_s) ds \tag{1.5}$$

here we prove that under the conditions (1.1) and (1.2) the remainder of the expansion denoted by $R_t(\cdot)$, suitably normalized is asymptotically Gaussian.

The limiting process (in the finite dimensional - fidi - sense) is continuous as soon as,

$$\frac{1}{(1-r^2)^p} \in L^1_{loc} \tag{1.6}$$

for some $p > 1/2$; and the last convergence is functional if the condition (1.6) holds for some $p > 1$.

In (1.4) m is chosen to be the smallest number satisfying (1.2). In the expansion (1.4), note that the k-th term belongs to the k-th order of the chaos relative to the process $(X_t)_{t\in\mathbb{R}}$. Under a long range dependence assumption, this term correctly normalized has a behavior usually non Gaussian (see Taqqu [15], Dobrushin & Major [7] or Doukhan & León [6] for multidimensional versions); if $r(s) \sim |s|^{-\alpha}L(s)$ and $k\alpha < 1$, this normalization is $\sqrt{L^k(t)t^{2-k\alpha}}$. From another hand, in short range dependence the tail of this series (5) has an order $\sqrt{t}$ in probability as it was shown in Doukhan & León [6]. It is called there the Gaussian part of l_t's Edgeworth expansion. For $m = 1$ it may be proved directly that this series is asymptotically Gaussian using Berman technique (see [2]). The aim of this paper is to investigate this Gaussian convergence in the general frame.

The paper is organized as follows: In section 2 we present our main results, while some applications are stated and proved in section 3. For example, we will provide a complete Edgeworth expansion of the local time if $r(s) \sim |s|^{-\alpha}L(s)$. Section 4 is devoted to the proof of the main results.

Main results

In the following we are interested in the asymptotic behavior in law of $R_t(\cdot)$. We shall prove that the limiting process in x, with a convenient normalization is Gaussian. Then sample path properties of the limiting process are investigated. Finally conditions ensuring tightness and functional central limit theorem are discussed.

Before stating ours main results we recall the following definition; we say that $\frac{1}{\sqrt{t}}R_t(\cdot)$ satisfies the functional central limit theorem in $\mathcal{C}_K(\mathbb{R},\mathbb{R})$ if for each compact subset K, $\frac{1}{\sqrt{t}}R_t(x)_{x\in K}$ satisfies the central limit theorem in the space $(\mathcal{C}(K,\mathbb{R}); ||\cdot||_K)$ where $||f||_K = \sup_{x\in K}|f(x)|$.

The expression

$$\Gamma(x,y) = 2\sum_{k=m}^{\infty} \frac{H_k(x)p(x)H_k(y)p(y)}{k!} \int_0^{\infty} r^k(s)ds \tag{1.7}$$

is of a special interest; in the appropriate case, this will be the covariance function of the limiting Gaussian process.

Theorem 1.1. *Assume that assumptions* (1.1) *and* (1.2) *hold then* (1.7) *is well defined. Moreover, the finite dimensional distributions of the random process*

$$\{\frac{1}{\sqrt{t}}R_t(x),\ x \in \mathbb{R}\}$$

converge to those of a Gaussian and centered process $(R(x))_{x\in\mathbb{R}}$ *with the covariance function* $(\Gamma(x,y))_{(x,y)\in\mathbb{R}^2}$.

If there exists some $p > \frac{1}{2}$ *such that*(1.6) *holds then, the limiting process* $R(\cdot)$ *admits a modification in law with almost surely continuous sample paths.*

If in addition, the relation $p > 1$ *holds, then the previous convergence is functional (in the space* $C(\mathbb{R},\mathbb{R})$*).*

In the proof of the theorem above we will first prove the convergence in distribution of the finite dimensional laws of $\{\frac{1}{\sqrt{t}}N_t^k, k \in \mathbb{N}^*\}$ where $\frac{1}{\sqrt{t}}N_t^k = \frac{1}{\sqrt{t}}\int_0^t H_k(X_s)ds$ to those of independent and Gaussian random variables $\{N_k,\ k \in \mathbb{N}^*\}$. Second, we apply the previous result to prove the finite dimensional convergence. The technique of the proof is essentially the method of moments and the diagram formula (cf. Appendix). The idea is motivated by the paper of Breuer and Major, [4]. The assumption (1.6) may be relaxed to

$$\int_0^{t_0} (1 - r^2(s))^{-p} ds < \infty$$

for some t_0 such that $|r(s)| \leq 1/2$ for $s > t_0$.

As examples of behaviors of covariance functions satisfying (1.6) for $p > 1$ we can take $r(s) \sim 1 - c|s|^{\alpha}$ with $\alpha < 1$. The Ornstein Uhlenbeck process (e.g. $r(t) = \exp(-|t|)$) satisfies (1.6) if and only if $p \leq 1$, and thus does not fulfill (1.6) for $p > 1$.

Applications

Other asymptotic for the local time

In this paragraph we deal with the L^2 and almost sure convergence for the local time, we always assume that relation (2) holds. Using the orthogonality of Hermite polynomials we write

$$\mathbb{E}\left[(l_t(x) - tp(x))^2\right] = \sum_{k=1}^{m-1} \frac{H_k^2(x)p^2(x)}{k!} \int_0^t \int_0^t r^k(s,s')dsds' + \mathbb{E}\left[R_t^2(x)\right].$$

Proposition 1.1. *Assume that* (1.1) *and* (1.2) *hold then*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \mathbb{E}(\frac{l_t(x)}{t} - p(x))^2 = 0$$

and

$$\lim_{t\to\infty} \frac{l_t(x)}{t} = p(x) \qquad a.s.$$

Proof. Let $k \in \{1, 2, .., m-1\}$ we have

$$\int_0^t \int_0^t r^k(s,s')dsds' = 2\int_0^t (t-s)r^k(s)ds \le 2t \int_0^t \left|r^k(s)\right| ds$$

hence for some constant $C > 0$

$$\int_0^t \int_0^t r^k(s,s')dsds' \le 2t \left[\int_0^t \left|r^k(s)\right| ds\right]^{k/m} t^{1-k/m} \le Ct^{2-k/m}$$

and

$$\lim_{t\to\infty} \mathbb{E}(\frac{1}{\sqrt{t}} R_t(x))^2 = \sum_{k=m}^{\infty} P_k^2(x) \int_0^\infty r^k(s)ds.$$

where

$$P_k(x) = \frac{H_k(x)p(x)}{\sqrt{k!}} \tag{1.8}$$

then for some C independent of x

$$\mathbb{E}(\frac{1}{\sqrt{t}} R_t(x))^2 \le C,$$

therefore

$$\sup_{x\in\mathbb{R}} \mathbb{E}\left[(\frac{l_t(x)}{t} - p(x))^2\right] \le Cmt^{-1/m} + C/t$$

which tends to zero as t goes to infinity. Now

$$\mathbb{E}\left[\frac{1}{t}\int_0^t H_k(X_s)ds\right]^2 \leq Ct^{-k/m}$$

thus Borel Cantelli lemma yields

$$\lim_{n\to\infty}\frac{1}{n^{2m}}N^k_{n^{2m}} = 0 \text{ and also } \lim_{n\to\infty}\frac{1}{n^{2m}}R_{n^{2m}}(x) = 0 \qquad \text{a.s}$$

Let t_k a sequence tending to infinity and let n_k such that $n_k^{2m} \leq t_k \leq (n_k+1)^{2m}$ using the fact that the local time is a non decreasing function we write

$$\frac{(n_k+1)^{2m}}{{n_k}^{2m}} \times \frac{l_{{n_k}^{2m}}(x)}{{n_k}^{2m}} \leq \frac{l_{t_k}(x)}{t_k} \leq \frac{l_{(n_k+1)^{2m}}(x)}{(n_k+1)^{2m}} \times \frac{{n_k}^{2m}}{(n_k+1)^{2m}}$$

We let k go to infinity in order to conclude that

$$\lim_{n\to\infty}\frac{1}{n^{2m}}l_{n^{2m}}(x) = p(x) \qquad a.s.$$

As a consequence of the previous relation there exists a sequence t_k such that $\lim_{k\to\infty}\frac{l_{t_k}(x)}{t_k} = p(x)$. In fact this holds true for all sequences t_k increasing to infinity and yields the result.

A complete Edgeworth expansion for the local time

In this paragraph we assume that the covariance function is of the form

$$r(s) \sim s^{-\alpha}L(s)$$

for some $\alpha > 0$ and some slowly varying function L. Define

$$k^* = \inf\{k,\ k\alpha \geq 1\}.$$

Set

$$l_t(x) = tp(x) + \sum_{k=1}^{k^*-1}\frac{H_k(x)p(x)}{k!}\int_0^t H_k(X_s)ds + R_t(x) \tag{1.9}$$

with

$$R_t(x) = p(x)\sum_{k=k^*}^{\infty}\frac{H_k(x)}{k!}\int_0^t H_k(X_s)ds. \tag{1.10}$$

Then one obtains the following higher order asymptotic for the local time.

Proposition 1.2. *Assume that the relations* (1.1) *and* (1) *hold, set* $L^*(t) = \int_{-t}^{t} r^{k^*}(s)ds$, *then*

a) *If* $0 < k < k^*$ *:*

$$\frac{1}{d_{k,t}} \int_0^t H_k(X_s)ds \Longrightarrow Y_k(1)$$

where $d_{k,t}^2 = \mathbb{E}(\int_0^t H_k(X_s)ds)^2$ *and* $Y_k(\cdot)$ *denotes the Hermite process of order* k.

b-1) *If* $k^*\alpha > 1$ *:*

$$\frac{1}{\sqrt{t}} R_t(x) \Longrightarrow \mathcal{N}(0, \Gamma(x,x)).$$

b-2) *If* $k^*\alpha = 1$ *then* L^* *is a slowly varying function; if it is not a bounded function then*

$$\frac{1}{\sqrt{tL^*(t)}} R_t(x) \Longrightarrow \mathcal{N}(0, \sigma^{*2}(x)) \text{ where } \sigma^{*2}(x) = \frac{H_{k^*}^2(x)p^2(x)}{k^*!}.$$

b-3) *If* $k^*\alpha = 1$ *and* $L^*(\cdot)$ *is bounded, then*

$$\frac{1}{\sqrt{tL^*(t)}} R_t(x) \Longrightarrow \mathcal{N}(0, \sigma^{*2}(x))$$

where

$$\sigma^{*2}(x) = \frac{H_{k^*}^2(x)p^2(x)}{k^*!} + \frac{2}{L^*} \sum_{k=k^*+1}^{\infty} \frac{H_k^2(x)p^2(x)}{k!} \int_0^{\infty} r^k(s)ds$$

and $L^* = \lim_{t\to\infty} L^*(t)$.

Example. Taking $L(t) = \log^a t$ for $t > 1$ and a real number a we assume that $k^*\alpha = 1$ for some k^* (this k^* is the same as defined above) then

$$L^*(t) = \int_{-t}^{t} r^{k^*}(s)ds \sim \begin{cases} 2\log\log t & \text{if } k^*a = -1 \\ \frac{2}{k^*a+1}\log t^{k^*a+1} & \text{if } k^*a > -1 \\ C & \text{if } k^*a < -1 \end{cases}$$

hence if $k^*a < -1$ we are in the case (b-2), if $k^*a > -1$ we are in the case (b-3), in particular if $k^*a = 0$ then $L^*(t) \sim 2\log t$.

Proof. Point (a) is proved in [15] and point (b-1) is a consequence of Theorem 1. Therefore we have only to prove (b-2) and (b-3). According to [14] $L^*(t)$ is a slowly varying function. Now the same arguments as in Proposition 1 when $r = 1$ apply with small changes yielding

$$\frac{1}{\sqrt{tL^*(t)}} \int_0^t H_{k^*}(X_s)ds \Longrightarrow \mathcal{N}(0, k^*!) \tag{1.11}$$

on the other hand

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{tL^*(t)}\left(p(x)\sum_{k=k^*+1}^{\infty}\frac{H_k(x)}{k!}\int_0^t H_k(X_s)ds\right)^2\right] \longrightarrow 0,$$

and this ends the proof. Now, (b-3) is proved in a similar way.

Proofs of the main results

To prove our theorems we need some results stated as lemmas and propositions; their proofs are postponed to the end of this section.

Proof of Theorem 1.1

Fidi convergence

Proposition 1.3. *Assume that relations* (1.1) *and* (1.2) *hold. Let* $k_1, k_2, ..., k_r$ *be* r *be integers satisfying* $k_i \geq m$ *for* $i = 1, 2, ..., r$ *then the following convergence in distribution holds in* $\mathbb{R}^r$

$$\frac{1}{\sqrt{t}}\,(N_t^{k_1}, ..., N_t^{k_r}) \Longrightarrow (N^{k_1}, ..., N^{k_r})$$

where the N^{k_i} *are independent Gaussian random variables.*

In order to prove Theorem 1, we have to check that $\sigma^2(x) \equiv \Gamma(x, x)$ is finite. For this use that there exists an universal constant C (see [9]) such that for any real x and any integer k

$$\frac{|H_k(x)p(x)|}{\sqrt{k!}} \leq Ck^{-1/4} \tag{1.12}$$

yields

$$\sigma^2(x) = 2\sum_{k=m}^{\infty} \frac{H_k^2(x)p^2(x)}{k!} \int_0^{\infty} r^k(s)ds \le C\sum_{k=m}^{\infty} k^{-1/2} \int_0^{\infty} r^k(s)ds$$

Now remark that the two conditions $\sum_{k=m}^{\infty} k^{-1/2} \int_0^{\infty} r^k(s)ds < \infty$ and (1.1) are equivalent under the assumption (1.2). Indeed

$$\sum_{k=m}^{\infty} k^{-1/2} \int_0^{\infty} r^k(s)ds = \sum_{k=m}^{\infty} k^{-1/2} \int_0^{t} r^k(s)ds + \sum_{k=m}^{\infty} k^{-1/2} \int_t^{\infty} r^k(s)ds$$

Now, choose t such that $|r(s)| \le 1/2$ for $s > t$ use the hypothesis (1.1) and the relation (1.21) below.

Applying proposition 1.3 we conclude that for all $K \ge m$,

$$\frac{1}{\sqrt{t}} R_t^K(x) \equiv p(x) \sum_{k=m}^{K} \frac{H_k(x)}{k!} \int_0^t H_k(X_s)ds \Longrightarrow N_K(x) = \mathcal{N}(0, \sigma_K^2(x))$$

where

$$\sigma_K^2(x) = \sum_{k=m}^{K} \frac{H_k^2(x)p^2(x)}{k!} \int_0^{\infty} r^k(s)ds,$$

and

$$N_K(x) \Longrightarrow \mathcal{N}(0, \sigma^2(x)) \quad \text{since} \quad \sigma_K^2(x) \to \sigma^2(x),$$

on the other hand

$$\mathbb{E}\left[R_t(x) - R_t^K(x)\right]^2 = \sum_{k=K}^{\infty} \frac{H_k^2(x)p^2(x)}{k!} \int_0^t \int_0^t r^k(s,s')dsds'$$

Therefore

$$\limsup_{K\to\infty} \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{\sqrt{t}}(R_t(x) - R_t^K(x))\right]^2 = 0.$$

then from Theorem 4.2, page 25 in [3] we obtain

$$\frac{1}{\sqrt{t}} R_t(x) \Longrightarrow R(x) = \mathcal{N}(0, \sigma^2(x)).$$

Let now, $x_1, x_2, .., x_l$ and $\alpha_1, .., \alpha_l$ real numbers, the expression

$$\frac{1}{\sqrt{t}} \sum_{i=1}^{l} \alpha_i R_t(x_i) = \sum_{k=m}^{\infty} \left(\sum_{i=1}^{l} \alpha_i \frac{H_k(x_i)p(x_i)}{k!} \right) \int_0^t H_k(X_s)ds$$

clearly converges to a normal distribution, and this prove the first point in theorem 1.

Remark If $m = 1$ it is possible to prove this result in a more simple way using an idea of Berman (see [2]). If $r \in \mathbb{L}^1$ then X admits the representation

$$X_t = \int b(t-s)dW_s \qquad (b \in \mathbb{L}^2(\lambda))$$

where W denotes a standard Brownian motion. Following [2] we introduce:

$$X_{t,v} = \int b_v(t-s)dW_s$$

where $b_v(s) = b(s)I_{\{|s| \le v/2\}}$. Now, we define:

$$\begin{aligned} Z_{t,K}(x) &= \frac{1}{\sqrt{t}} \sum_{k=1}^{K} \frac{H_k(x)p(x)}{k!} \int_0^t H_k(X_s)ds \\ Z_{t,K,v}(x) &= \frac{1}{\sqrt{t}} \sum_{k=1}^{K} \frac{H_k(x)p(x)}{k!} \int_0^t H_k(X_{s,v})ds \end{aligned}$$

Using the same notations as in [2] we have:

$$Z_{t,K,v}(x) = \frac{1}{\sqrt{t}} \int_0^t G_K(X_{s,v})ds \Longrightarrow \mathcal{N}(0, \sigma^2_{v,K}(x)) \tag{1.13}$$

since $X_{s,v}$ is a v-dependent process (a process is said to be v-dependent if X_s is independent of $X_{s'}$ whenever $|s - s'| > v$); here

$$\sigma^2_{v,K}(x) = 2 \sum_{k=1}^{K} \frac{H_k^2(x)p^2(x)}{k!} \int_0^\infty r^k_{2,v}(s)ds.$$

Using (lemma 8.4.1, [2]) we write:

$$\lim_{t\to\infty} \sigma^2_{v,K}(x) = \sigma^2_K(x) \equiv 2 \sum_{k=1}^{K} \frac{H_k^2(x)p^2(x)}{k!} \int_0^\infty r^k(s)ds \tag{1.14}$$

Again from (lemma 8.4.1, [2]) we obtain

$$\limsup_{K\to\infty} \limsup_{t\to\infty} \mathbb{E}\left[(Z_{t,K}(x) - Z_{t,K,v}(x))^2\right] = 0 \tag{1.15}$$

From (1.13), (1.14) and (1.15) we conclude that

$$Z_{t,K}(x) = \frac{1}{\sqrt{t}} \int_0^t G_K(X_s)ds \Longrightarrow \mathcal{N}(0, \sigma^2_K(x)) \tag{1.16}$$

On the other hand:

$$\lim_{t\to\infty} \sigma_K^2(x) = \sigma^2(x) \equiv 2\sum_{k=1}^{\infty} \frac{H_k^2(x)p^2(x)}{k!} \int_0^{\infty} r^k(s)ds \tag{1.17}$$

Using again (lemma 8.4.1, [2]) yields

$$\limsup_{K\to\infty} \limsup_{t\to\infty} \mathbb{E}\left[(Z_t(x) - Z_{t,K}(x))^2\right] = 0 \tag{1.18}$$

From (1.16), (1.17) and (1.18) we conclude that:

$$Z_t(x) = \frac{1}{\sqrt{t}}\left(\frac{l_t(x)}{t} - p(x)\right) \Longrightarrow \mathcal{N}(0, \sigma^2(x)) \tag{1.19}$$

and this ends the proof for the one dimensional case, the finite dimensional case may be handled in the same way.

Continuity of the limiting process

We have proved that the family of processes $\left(\frac{1}{\sqrt{t}} R_t(\cdot)\right)_{t>0}$ converges in the finite dimensional sense to a real, zero mean, non stationary Gaussian process $R(\cdot)$ with covariance function given by (1.7). First, remark that (1.6) is equivalent to

$$\sum_{k=m}^{\infty} k^{p-1} \int_0^t r^k(s)ds < \infty. \tag{1.20}$$

indeed

$$(1-x)^{-p} = \sum_{k=0}^{\infty} \frac{(k-1+p)!}{k!} x^k$$

and Stirling formula $\frac{(k-1+p)!}{k!} \sim Ck^{p-1}$ yield with $p = \frac{1}{2} + \delta$

$$\sum_{k=m}^{\infty} k^{-1/2+\delta} \int_0^t r^k(s)ds < \infty \Longleftrightarrow \int_0^t \frac{1}{(1-r^2(u))^{1/2+\delta}} du < \infty \tag{1.21}$$

$$\mathbb{E}[R(x) - R(y)]^{2q} = (q-1)!! \left[\sum_{k=m}^{\infty} Q_k^2(x,y) \int_0^{\infty} r^k(s)ds\right]^q.$$

where $Q_k(x,y) = P_k(x) - P_k(y)$ and $P_k(x)$ is defined by (1.8); but

$$|H_k(x)p(x) - H_k(y)p(y)| = \left|\int_x^y H_{k+1}(z)p(z)\,dz\right|$$

which together with (1.12) entails

$$\begin{aligned} |Q_k(x,y)|^2 &\le Ck^{-1/2}(1 \wedge |x-y|^2(k+1)) \\ &\le Ck^{-1/2}|x-y|^{2\delta}(k+1)^{\delta} \end{aligned} \tag{1.22}$$

hence

$$\mathbb{E}\,[R(x)-R(y)]^{2q} \le C\left[\sum_{k=m}^{\infty} k^{-1/2+\delta}\int_0^{\infty} r^k(s)ds\right]^q |x-y|^{2q\delta}$$

Now, to conclude the proof we consider higher order moments and we use the Kolmogorov-Chentsov criterion.

Remark The law of $R(\cdot)$ is identical to that of the process defined by

$$\sum_{k=m}^{\infty} \frac{H_k(x)p(x)}{\sqrt{k!}}\left(\int_0^{\infty} r^k(s)ds\right)^{1/2} \xi_k \tag{1.23}$$

where $(\xi_k)_k$ is a sequence of i.i.d standard normal variables and the series (1.23) converge a.s and in square mean. This representation offers another possibility of studying the sample path properties of R.

Tightness and functional CLT.

According to Theorem 1.1 it suffices to prove the tightness of the family $\{(\frac{1}{\sqrt{t}}R_t(x))_{x\in K}\ ;\ t>0\}$. But

$$\lim_{t\to\infty}\mathbb{E}\left[\frac{1}{\sqrt{t}}(R(x)-R(y))\right]^2 = \left[\sum_{k=m}^{\infty} Q_k^2(x,y)\int_0^{\infty} r^k(s)ds\right].$$

hence, by (1.12)

$$\begin{aligned} \mathbb{E}\left[\frac{1}{\sqrt{t}}(R(x)-R(y))\right]^2 &\le C\left[\sum_{k=m}^{\infty} k^{-1/2\,+\delta}\int_0^{\infty} r^k(s)ds\right]^p |x-y|^{2\delta} \\ &\le C|x-y|^{2\delta} \end{aligned}$$

where we still set $\delta = p-\frac{1}{2}$. Now, since $2\delta > 1$, using the criterion of tightness in [3]. This concludes the proof of theorem 1.

Proofs

Proof of proposition 1.3

Let $\alpha_1, ..., \alpha_r$ be real numbers, we assume that the k_i's are distinct. We are aimed to prove

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{\sqrt{t}}\sum_{i=1}^{r}\alpha_i N_t^{k_i}\right]^p = \begin{cases} (p-1)!!\,\left(\sum_{i=1}^{r}\alpha_i^2\sigma_{k_i}^2\right)^{p/2} & \text{if } p=2q \text{ is even} \\ 0 & \text{if is odd} \end{cases}$$

Set

$$M(p,t,r) = \mathbb{E}\left[\frac{1}{\sqrt{t}}\sum_{i=1}^{r}\alpha_i N_t^{k_i}\right]^p$$

Then:

$$M(p,t,r) = \qquad M'(p,t,r) \qquad + \qquad M"(p,t,r)$$

where

$$\begin{aligned} M'(p,t,r) &= \frac{1}{t^{p/2}}\sum_{i_1,..,i_p=1}^{r}\alpha_{i_1}..\alpha_{i_p}\sum_{G\in\Gamma'(k_{i_1},...,k_{i_p})} I(G,k,t) \\ M"(p,t,r) &= \frac{1}{t^{p/2}}\sum_{i_1,..,i_p=1}^{r}\alpha_{i_1}...\alpha_{i_p}\sum_{G\in\Gamma"(k_{i_1},...,k_{i_p})} I(G,k,t) \end{aligned}$$

Setting $\alpha_i = \beta_{k_i}$ we write

$$\begin{aligned} M'(p,t,r) &= \frac{1}{t^{p/2}}\sum_{i_1,..,i_p=1}^{r}\alpha_{i_1}..\alpha_{i_p}\sum_{G\in\Gamma'(k_{i_1},...,k_{i_p})} I(G,k,t) \\ &= \frac{1}{t^{p/2}}\sum_{i_1,..,i_p=1}^{r}\sum_{G\in\Gamma'(k_{i_1},...,k_{i_p})}\beta_{k_{i_1}}...\beta_{k_{i_p}} I(G,k,t) \\ &= \sum_{k_{i_1}=1}^{r}\frac{1}{t^{p/2}}\sum_{i_2,..,i_p=1}^{r}\sum_{G\in\Gamma'(k_{i_1},...,k_{i_p})}\beta_{k_{i_1}},..,\beta_{k_{i_p}} I(G,k,t) \end{aligned}$$

The first level with cardinal k_{i_1} is chosen, a second level is needed to define the first component of G, there are $2q-1$ choices, without loss of generality assume that it is the second level, so k_{i_2} must be equal to k_{i_1}. Thus

$$M'(p,t,r) = \sum_{k_{i_1}=1}^{r} \frac{1}{t^{p/2}}(2q-1)\beta_{k_{i_1}}^2 k_{i_1}! \int_0^t \int_0^t r^{k_{i_1}}(s,s')dsds' \times$$
$$\sum_{i_3,..,i_p=1}^{r} \sum_{G\in\Gamma'(k_{i_3},..,k_{i_p})} \beta_{k_{i_3}},..,\beta_{k_{i_p}} I(G,(k_{i_3},...,k_{i_p}),t)$$

repeating this decomposition q times yields

$$M'(p,t,r) = (p-1)!! \sum_{k_{i_1},...,k_{i_q}=1}^{r} \prod_{j=1}^{q} \beta_{k_{i_j}}^2 k_{i_j}! \frac{1}{t} \int_0^t \int_0^t r^{k_{i_j}}(s,s')dsds'$$

letting t goes to infinity we obtain

$$\begin{aligned}
\lim_{t\to\infty} M'(p,t,r) &= (p-1)!! \sum_{i_1,...,i_q=1}^{r} \prod_{j=1}^{q} \alpha_{i_j}^2 k_{i_j}! 2 \int_0^\infty r^{k_{i_j}}(s)ds \\
&= (p-1)!! \sum_{i_1,..,i_q=1}^{r} \alpha_{i_1}^2..\alpha_{i_q}^2 \sigma_{k_{i_1}}^2..\sigma_{k_{i_q}}^2 \\
&= (p-1)!! \left[\sum_{i=1}^{r} \alpha_i^2 \sigma_{k_i}^2\right]^q
\end{aligned}$$

now, we will prove in the following lemma that the contribution of the irregular diagram to the limit is zero. Hence we obtain

$$\lim_{t\to\infty} M"(p,t,r) = 0,$$

and this ends the proof of the proposition.

Lemma 1.1. *Let G be an irregular diagram element of $\Gamma(k_1,k_2,..,k_p)$, then*

$$\lim_{t\to\infty} \frac{1}{t^{p/2}} I(G,(k_1,k_2,..,k_p),t) = 0$$

Proof. The lemma above is a simple adaptation of the Proposition 1 in Major [4].

Appendix

Let $k_1, k_2, ..., k_p$ some integers, V a set of points of cardinality $k_1 + k_2 + ... + k_p$ We call an indirected graph of type $\Gamma(k_1, k_2, ..., k_p)$ an element of $G(V)$ satisfying:

- i) V is a disjointed reunion of p levels with respective cardinalities $k_1, k_2, ..., k_p$

$$V = \cup_{i=1}^{p} L_i, \qquad L_i = \{(i,l),\ l = 1, 2..., k_i\}$$

- ii) Only edges between different levels are allowed

$$w = ((i,l),(i',l')) \Rightarrow i \neq i'$$

- iii) Every point has exactly one edge

$$\forall (i,l) \in V,\ \exists\ (i',l')\ /\ ((i,l);(i',l')) \in G(V)$$

For $w = ((i,l),(i',l'))$ element of $G(V)$ we define $n_1(w) \equiv i$ as the first level of w and $n_2(w) \equiv i'$ as the second one.

Lemma 1.2 (the Diagram Formula). *Let $(X_{s_1}, X_{s_2}, ..., X_{s_p})$ a Gaussian vector centered at expectation and with a covariance matrix given by $(r(s_i, s_j))_{1 \leq i,j \leq p}$*

$$\mathbb{E}\left[\prod_{i=1}^{p} H_k(X_{s_i})\right] = \sum_{G \in \Gamma(k_1,...,k_p)} \prod_{w \in G} r(s_{n_1(w)}, s_{n_2(w)})$$

where $n_1(w)$, $n_2(w)$ are the first and the second level of w

The proof can be found in Giraitis and Surgailis [9] in a more general setting.

We say that a subset of levels $\cup_{i \in I} L_i$ $I \subset \{1, 2..., p\}$ form a *chain* for a diagram G if every vertex in $\cup_{i \in I} L_i$ can be connected by an edge to some other vertex $\cup_{i \in I} L_i$. A graph is said to be *irreducible* if there isn't a chain with less than p levels. As a consequence, each graph can be considered as a reunion of irreducible *subgraphs* called *components* of the graph.

A subgraph is said to be of *degree* k, $k = 2, 3, .., p$ if it forms a chain on k levels; then an irreducible graph is one of degree p. A graph is said to be *regular* if all its components are of degree 2, otherwise he is said to be *irregular*, that is, there exists at least one component of degree greater or equal than 3.

Note that if $p = 2$ the set of irregular diagram is empty, and if p is odd then the set of regular diagrams is empty.

For G element of $\Gamma(k, k, ..., k)$ we set the following definitions: $k_G(i)$ is the number of edges going from level i and $g(i) = \frac{k_G(i)}{k_i}$.

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Chapitre 2

Limit Theorems for Gaussian Processes

ABSTRACT. In this paper we consider two functional limit theorems for non linear functional of stationary Gaussian process satisfying short range dependence conditions: the functional CLT for partial sums processes and the uniform CLT for a special class of functions. To carry out the proofs, we develop Rosenthal type inequalities for functional of Gaussian processes.

Key words: empirical processes; weakly dependent processes; Gaussian processes; functional central limit theorems; Rosenthal-type inequalities.

AMS subject classifications: 60E15; 60F17.

Introduction

Let $\{X_t : t \in \mathbb{R}\}$ be a real valued, Gaussian, stationary process with covariance function $\mathbb{E}(X_0 X_t) = r(|s-t|)$, $\mathbb{E}(X_s) = 0$ and $\mathbb{E}(X_s^2) = 1$. If G is a real function satisfying $\mathbb{E}(G^2(X)) < \infty$ and X is a standard normal variable, then it is well known that G can be expanded as:

$$G(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!} H_k(x), \tag{2.1}$$

where H_k is the kth Hermite polynomial and the series (2.1) is convergent in $\mathbb{L}^2(\Omega)$. Let m be $\inf\{k > 0; \quad c_k \neq 0\}$. m is called the Hermite rank of G and plays a central role in the asymptotic of $\int_0^t G(X_s)ds$ as shown by Breuer and Major (1983), Taqqu (1977), Dobrushin and Major (1977). Given a function $G : \mathbb{R} \to \mathbb{R}$ we will give conditions on the process $\{X_t : t \in \mathbb{R}\}$ and the function G in order that

$$\left\{ \int_0^t G(X_s) - \mathbb{E}\left[G(X_s)\right] ds \right\}_{t>0},$$

suitably normalized, converge in distribution. Such questions have been studied by several authors, both in short and long range dependence, continuous and discrete time processes, for example Taqqu (1979), Dobrushin and Major (1979), Breuer and Major (1983), Chambers and Slud (1989), Arcones (1994), Csörgő and Mielniczuk (1996). Our aim here is to complete some of these results for non linear functional of continuous time, short range dependent, Gaussian processes.

In section 2, we present our main results. Theorem 1 deals with the functional CLT (central limit theorem), for integrated processes. We will see that if the covariance function and the Hermite coefficients of the function G go fast enough to zero, then the process

$$\left\{ \frac{1}{\sqrt{t}} \int_0^{tx} G(X_s) - \mathbb{E}\left[G(X_s)\right] ds : x \in \mathbb{R} \right\}_{t>0},$$

converges in $\mathcal{D}\left[-\infty; +\infty\right]$ endowed with uniform topology on compact sets, to a Brownian motion.

When the process is long range dependent, these questions have been solved by Taqqu (1979), Dobrushin and Major (1979), for real valued Gaussian sequences and generalized by Arcones (1994) to vector valued Gaussian fields. In short range dependence, the same question was also considered by Chambers and Slud (1989) in a more general setting. Here we partially improve some of these results.

In Theorem 2 we will give sufficient conditions for the uniform CLT over classes of functions of the type $\mathcal{F} = \left\{1_{G(\cdot)\leq x} : x \in \mathbb{R}\right\}$ to happen.

Uniform convergence have also been investigated by many authors. Dehling and Taqqu (1989) have proved that the empirical process, suitably normalized, converges under long range dependence conditions, to a degenerate process which is not Gaussian in general. In short range dependence Csörgő and Mielniczuk (1996) round off this question for Gaussian sequences. On the other hand, Arcones (1994) gives sufficient conditions for a family of functions in order to satisfy the uniform CLT. Unfortunately, the two results do not apply in our case.

Finally, we point out that both in Theorems 1 and 2 the limiting process is the same as for mixing or associated processes, (see for example [6] and [13]). Note that here no mixing condition is assumed.

In section 3, we develop a new moment inequality for integrals of functional of Gaussian process. These inequalities which are interesting in themselves, will be repeatedly used in the proofs of the main results. They can be seen as the analogous to those proved in Shao and Yu (1996) [13] for functionals of associated sequences. Our conditions on the function G are weaker than those in[13].

In section 4 we prove our main results. Since we are dealing with functional CLT two steps will be developed: convergence of marginals and tightness. The main tools used in the proofs are the expansion of functions in Hermite polynomials and the diagram formula. In Theorem 2, when proving tightness, we will use an autoregressive representation of the underlying Gaussian process and the moment inequalities developed in section 3.

In the sequel $B, C, D...$ stand for constants with values that may change in each appearance. (X, X') stands for a Gaussian vector, where X and X' denote standard normal variables. G denotes a real function.

Main results

Theorem 2.1. *Let G be a function such that $\mathbb{E}[G(X)] = 0, \mathbb{E}[G^2(X)] < \infty$, with m as Hermite rank. Then:*

(i) *If $\int |r^m(s)|\, ds < \infty$, then the finite dimensional laws of the process $(Z_t(\cdot))_{t\geq 0}$ defined by:*

$$Z_t(x) = \frac{1}{\sqrt{t}} \int_0^{tx} G(X_s)ds, \qquad x \in \mathbb{R}$$

tend to those of $\sigma W(\cdot)$ where $W(\cdot)$ is the standard Brownian motion and

$$\sigma^2 = \sum_{k=m}^{\infty} \frac{c_k^2}{k!} \int r^k(s)ds \ < \ \infty. \tag{2.2}$$

(ii) *Moreover, suppose that one of the following conditions holds:*

1. *there exists $R > 1$ such that*

$$\sum_{k=m}^{\infty} \frac{|c_k|}{\sqrt{k!}} \left(\int |r^k(s)|ds \right)^{1/2} R^k \ < \ \infty, \tag{2.3}$$

2. *the c_k are positive and $\mathbb{E}[G^4(X)] < \infty$.*

Then the convergence stated in (i) holds in $\mathcal{C}(\mathbb{R}, \mathbb{R})$ endowed with the uniform topology on compact sets.

Remarks

1) The condition $\mathbb{E}(G^2(X)) < \infty$ in Theorem 1 can be relaxed to the following one: $\sum_{k=m}^{\infty} (k!)^{-1} c_k \int_0^t H_k(X_s)ds$ is convergent in $\mathbb{L}^2(\Omega)$. Note that this condition is weaker. For example one can take the local time in zero of Gaussian processes. Indeed, under some conditions the local time $l_t(x)$ of a Gaussian process exists and admits the following expansion:

$$l_t(x) = \sum_{k=0}^{\infty} \frac{H_k(x)\, p(x)}{k!} \int_0^t H_k(X_s)ds$$

On the other hand we can show that $(k!)^{-1/2} H_k(0)\, p(0) \sim Ck^{-1/4}$. Note that this condition cannot be removed in the discrete case.

2) As examples of functions satisfying conditions of Theorem 1, one can consider either polynomials or $G(x) = \exp(ax)$ in which case $c_k = \exp(a^2/2)a^k$or $G(x) = \exp\left(1/2ax^2\right)$ for which $c_{2k+1} = 0$ and
$c_{2k} = (1-a)^{-1/2}\left(2^k k!\right)^{-1}(2k)!\left(a/(1-a)\right)^k$.

However the function $G(x) = |x|$ does not fulfill conditions ii) of Theorem 1; indeed, it is easy to check that:

$$|x| = \sqrt{2/\pi} + 2p(0)\sum_{n=1}^{\infty}\frac{1}{2n!}(-1)^{n+1}\frac{(2n-2)!}{2^{n-1}(n-1)!}H_{2n}(x)$$

Thus $(k!)^{1/2}|c_k| \sim Ck^{-5/4}$ and (2.3) does not hold since r is continuous in 0.

3)We recall that Chambers and Slud [5] have proved the functional CLT under the existence of spectral density and condition (2.3) with $R = \sqrt{3}$, without the term involving the correlation function. However, they consider functionals which may depend on infinitely many coordinates of $\{X_t : t \in \mathbb{R}\}$.

4) Clearly, if G is a finite linear combination of functions satisfying 1) or 2) the conclusion remains true.

In the following, we study the uniform CLT for the continuous analogous of the empirical distribution function. For discrete time processes, Csörgő and Mielniczuk [6] show that the functional CLT holds under the continuity of the distribution of $G(X)$ and the condition $\sum_k |r(k)|^m < \infty$ where m is the Hermite rank of the family $\left\{1_{G(.)\leq x}; x \in R\right\}$. However, it seems that their proof cannot be adapted to the continuous case.

Now, in order to state our second result we introduce the following notations and assumptions. Assume that the process admits the following representation

$$X_t = \int a_{t+s}dW_s, \tag{2.4}$$

where W is a standard Brownian motion and $a \in \mathbb{L}^2(\mathbb{R}, \lambda)$. According to Theorem 16.7.2 in [11] this holds if and only if the spectral function of the process is absolutely continuous. In particular, it is the case when the correlation function r belongs to $\mathbb{L}^1$. Now we recall the following definitions from [2].

Definition 2.1. *Let $(X_s)_s$ be a Gaussian process given by* (2.4). *We define X_t^v by:*

$$X_t^v = \int a_{t+s}^v dW_s \qquad a_s^v = \frac{a_s}{\|a\|_v}1_{\{|s|<v/2\}}; \qquad \|a^v\|^2 = \int_{|s|<v/2} a_s^2 ds \tag{2.5}$$

Define $r, r_{1,v}$ and $r_{2,v}$ by:

$$r(s) = \mathbb{E}[X_0 X_s]\,; \qquad r_{1,v}(s) = \mathbb{E}[X_0 X_s^v]\,; \qquad r_{2,v}(s) = \mathbb{E}[X_0^v X_s^v] \tag{2.6}$$

and set: $r_v^*(s) = \sup\left(|r(s)|, |r_{1,v}(s)|, |r_{2,v}(s)|\right)$.

It is easy to check that the process $(X_t; X_t^v)_{t\geq 0}$ is a Gaussian stationary process vector valued with covariance matrix given by:

$$\begin{pmatrix} r(t) & r_{1,v}(t) \\ r_{1,v}(t) & r_{2,v}(t) \end{pmatrix} = \begin{pmatrix} \int a_s a_{t+s} ds & \int a_s^v a_{t+s} ds \\ \int a_s^v a_{t+s} ds & \int a_s^v a_{t+s}^v ds \end{pmatrix}$$

Next we define

Definition 2.2. *Let G be a real function . Let us define the occupation measure $L_t(G,.)$ by*

$$L_t(G,x) = \int_0^t 1_{G(X_x)\leq x} ds$$

and set

$$Z_t(G,.) = \sqrt{t}\left(\frac{L_t(G,.)}{t} - F(.)\right) \tag{2.7}$$

where F is the distribution function of $G(X)$

We will study the convergence in $\mathcal{D}[-\infty,+\infty]$ endowed with the uniform topology of the process defined by $Z_t(G,.)$. Under the terminology of the Definition 2.1, we prove the following

Theorem 2.2. *Let $(X_s)_{s\geq 0}$ be a stationary Gaussian process, let m be the Hermite rank of the family $\mathcal{F} = \left\{1_{\{G(\cdot)\leq x\}}; x \in \mathbb{R}\right\}$ and F the distribution function of $G(X)$.*

We assume that $r^m \in \mathbb{L}^1$, then the finite dimensional laws of $Z_t(G,.)$ defined in (2.7) tend to those of a centered, Gaussian process with covariance function given by:

$$\Gamma(x,y) = 2\int_0^\infty \operatorname{Cov}\left(1_{\{G(X_0)\leq x\}}, 1_{\{G(X_s)\leq y\}}\right) ds$$

(i) *Moreover, if we assume that:*

$(K1)$ $\int_0^t \frac{1}{\sqrt{1-r^2(s)}} ds, \forall t > 0$

(K_2) *F is continuous and the Lebegue measure of $G^{-1}([0,1])$ is finite.*

Then the family $Z_t(G,.)$ is tight in $\mathcal{C}[0,1]$ and thus the convergence takes place in $\mathcal{C}[0,1]$ endowed with the uniform topology.

(ii) *Moreover, if we assume that F is continuous and:*

$(H1)$ $X_t = \int a_{t+s} dW_s$

$(H2)$ $v^{2\beta} \int_{|s|>v} a_s^2 ds < \infty$, *for some* $\beta > 1/2$.

and either

$(H3)$ $\sup_k \sup_v k \int_0^v (r_v^*(s))^k ds < \infty$

or

$(H'3)$ $\sup_{x,y} v^{\beta} \mathbb{E} \left| 1_{\{x<F\circ G(X)\leq y\}} - 1_{\{x<F\circ G(X^v)\leq y\}} \right|^2 < \infty$, *for some* $\beta > 1/2$.

Then the family $Z_t(G,.)$ is tight and thus the convergence takes place in $\mathcal{C}[-\infty,+\infty]$ endowed with the uniform topology.

Examples and comments:

a) The condition $(K1)$ ensures the existence of the local time of the Gaussian process. In fact the proof of tightness in this case relies on the local time. Note that this condition is similar to $(H3)$ in the sense that they are concerned with the behavior of the covariance function near the origin. Such considerations do not appear when dealing with discrete time processes.

b) The hypothesis $(K2)$ will be used in the following form: the function $x \longrightarrow \int 1_{0\leq G(u)\leq x} du$ is defined, continuous on $[0,1]$.

c) Let G be a function such that there exists a finite number of intervals $I_1, I_2..., I_N$ where G is monotone on each one, this is the case when G is continuously differentiable with derivatives vanishing in a finite set of points. In this case

$$\mathbb{E} \left| 1_{\{x<F\circ G(X)\leq y\}} - 1_{\{x<F\circ G(X^v)\leq y\}} \right|^2 = \sum_{k=1}^{\infty} \frac{c_k^2(x,y)}{k!} \left(1 - \|a^v\|^k\right)$$

where

$$c_k(x,y) = \mathbb{E}\left[1_{\{x<F\circ G(X)\leq y\}} H_k(X)\right] = \int 1_{\{x<F\circ G(u)\leq y\}} H_k(u)\, p(u) du.$$

However, $\{u \ / \ x < F \circ G(u) \leq y\}$ is a finite collection of intervals , its number is bounded by N. Now as $\int^x H_k(u)\, p(u)du = -H_{k-1}(x)\, p(x)$, we get

$$|c_k(x,y)| \leq 2N \, \|H_{k-1}(.)\, p(\cdot)\|_\infty$$

Using the bound $\|H_k(.)\, p(\cdot)\|_\infty \leq C\sqrt{k!}k^{-1/4}$ we can write:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{c_k^2(x,y)}{k!}\left(1 - \|a^v\|^k\right) &\leq 2NC\sum_{k=1}^{\infty} k^{-3/2} \inf\left(1, k\left(1 - \|a^v\|\right)\right) \\ &\leq \left(1 - \|a^v\|\right)^\delta 2NC \sum_{k=1}^{\infty} k^{-3/2+\delta}\end{aligned}$$

Hence, provided $\delta < 1/2$, we have

$$\mathbb{E}\left|1_{\{x<F\circ G(X)\leq y\}} - 1_{\{x<F\circ G(X^v)\leq y\}}\right|^2 \leq C\left(\int_{|s|>v/2} a_s^2 ds\right)^\delta .$$

Thus $(H'3)$ holds whenever $(H2)$ holds.

c) For instance the Ornstein-Uhlenbeck process (i.e. $r(s) = \exp - |s|$) fulfill the hypothesis $(H3)$. In this case no condition is assumed on the function G excepting the continuity of the distribution function.

Moment inequalities

In this section we derive a moment inequality for occupation measures when the observations are functional of Gaussian processes. The main idea behind the theorems below is to replace the initial process by a v-dependent one, and then divide the integral into blocks with size q greater than v. So and under the setting of Definition 2.1, we first establish a moment inequality for the process X^v defined by (2.5). This is summarized in the following proposition which is in fact the main technical result of this section.

Proposition 2.1. *Let G be a function such that $\mathbb{E}[G(X)] = 0, \mathbb{E}[G^2(X)] < \infty$, with m as Hermite rank.*

(i) *Let $C_t = 2\int_0^t |r(s)|^m ds$. then*

$$\mathbb{E}\left|\int_0^t G(X_s)\, ds\right|^2 \leq tC_t\mathbb{E}\,|G(X_0)|^2 \tag{2.8}$$

(ii) *Let* $p = 2(1 + s\theta)$; $s > 0, \theta \in]0,1]$, *and assume that* X *admits the following representation:*

$$X_t = \int a_{t+s} dW_s$$

Let $K(p) = 2^{p+1}D(p)$ *where* $D(p)$ *is the Rosenthal's constant in the i.i.d case and*

$$M_2 = \mathbb{E}\left(G^2(X_0)\right), M_{2,s,\theta} = \left(\mathbb{E}\left|G^2(X_0)\right|\right)^{1-\theta}\left(\mathbb{E}\left|G^{2(1+s)}(X_0)\right|\right)^{\theta}.$$

Then, for any $0 < \alpha \leq 1$ *there exists a constant* a *which depends only on* s, θ, C *and* α *such that for* $v(t) = at^{\alpha}$ *we have*

$$\mathbb{E}\left|\int_0^t G(X_s^{v(t)})ds\right|^p \leq K(p)\left[(tCM_2)^{p/2} + t^{1+\alpha\theta(1+2s)}M_{2,s,\theta}\right] \tag{2.9}$$

where $C := 2\sup_{v>0}\int_0^{\infty}|r_{2,v}(s)|^m\, ds$.

Proof. Observe that since X_t^v is v-dependent then $\int_0^{\infty}|r_{2,v}(s)|^m\, ds = \int_0^{v}|r_{2,v}(s)|^m\, ds$, and hence $\sup_{0<v<V}\int_0^{\infty}|r_{2,v}(s)|^m\, ds \leq 2V$. Sufficient conditions ensuring that C is finite are discussed after the proof. Now we turn to the proof of the Proposition. First we note the following

$$\begin{aligned}\mathrm{Cov}\left(G(X), G(X')\right) &= \sum_{k_1,k_2=m}^{\infty}\frac{c_{k_1}c_{k_2}}{k_1!k_2!}\mathrm{Cov}\left(H_k(X), H_k(X')\right) \\ &= \sum_{k=m}^{\infty}\frac{c_k^2}{k!}\mathrm{Cov}^k\left(X, X'\right),\end{aligned}$$

then

$$\left|\mathrm{Cov}\left(G(X), G(X')\right)\right| \leq \sum_{k=m}^{\infty}\frac{c_k^2}{k!}\left|\mathrm{Cov}\left(X, X'\right)\right|^m = \mathbb{E}\left[G^2(X_0)\right]\left|\mathrm{Cov}\left(X, X'\right)\right|^m.$$

Therefore (2.8) follows from

$$\mathbb{E}\left|\int_0^t G(X_s)\, ds\right|^2 \leq 2\int_0^t (t-s)\,|r(s)|^m\, ds\mathbb{E}\left[G^2(X_0)\right]$$

We now prove (2.9) by induction.

- First note that (2.9) is obvious when $0 \leq t \leq 1$, with $K(p) = 1$.

• Assume that there exists a constant $K(p)$ such that the inequality

$$\mathbb{E}\left|\int_0^t G(X_s^v)ds\right|^p \leq K(p)\left[(tCM_2)^{p/2} + t^{1+\alpha\theta(1+2s)}M_{2,s,\theta}\right], \tag{2.10}$$

holds for $t < n$. We shall deduce it for $n \leq t < n+1$. Let $q = at^\alpha$ where $a \leq 1/2$ is some constant to be specified later. Then define

$$U(k) = \int_{2(k-1)q}^{(2k-1)q\wedge t} G(X_s^{v(t)})ds \text{ and } V(k) = \int_{2(k-1)q+q}^{2kq\wedge t} G(X_s^{v(t)})ds$$

for $1 \leq k \leq k_t, k_t = 1 + [t/2q]$ and $q = q(t) = v(t)$. Now, write

$$\mathbb{E}\left|\int_0^t G(X_s^{v(t)})ds\right|^p \leq 2^{p-1}\left(\mathbb{E}\left|\sum_{k=1}^{k_t} U(k)\right|^p + \mathbb{E}\left|\sum_{i=1}^{k_t} V(k)\right|^p\right). \tag{2.11}$$

Since the process (X^v) is v-dependent, the $U(k)$ are independent whenever $q \geq v$. Hence the sequence $M_n = \sum_{k=1}^n U(k)$ is a sum of i.i.d random variables. By Rosenthal's inequality, (see for example [10] or [11]), there exists a constant depending only on p such that:

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^{k_t} U(k)\right|^p &\leq D(p)\left\{\left(\sum_{k=1}^{k_t}\mathbb{E}\left[U^2(k)\right]\right)^{p/2} + \sum_{k=1}^{k_t}\mathbb{E}\left|U(k)\right|^p\right\} \\ &\leq D(p)\left\{\left(k_t\mathbb{E}\left[U^2(1)\right]\right)^{p/2} + k_t\mathbb{E}\left|U(1)\right|^p\right\}. \end{aligned} \tag{2.12}$$

Now we have:

$$\begin{aligned}\mathbb{E}\left[U^2(1)\right] &= \mathbb{E}\left|\int_0^q G(X_s^{v(t)})ds\right|^2 \\ &\leq 2q\int_0^q \left|r_{2,v(t)}(s)\right|^m dsM_2 \\ &\leq CqM_2 \end{aligned} \tag{2.13}$$

Now, in order to apply the induction hypothesis to the second term we must replace $G(X_s^{v(t)})$ by $G(X_s^{v(q)})$. To this end we write:

$$\begin{aligned}\mathbb{E}\left|U(1)\right|^p &\leq 2^{p-1}\left(\mathbb{E}\left|\int_0^q G(X_s^{v(q)})ds\right|^p + \mathbb{E}\left|\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds\right|^p\right) \\ &\leq 2^{p-1}\left(I_{1,p} \quad + \quad I_{2,p}\right). \end{aligned}$$

Applying the induction hypothesis to $I_{1,p}$, we get:

$$I_{1,p} \leq K(p)\left[(qCM_2)^{p/2} + q^{1+\alpha\theta(1+2s)}M_{2,s,\theta}\right]. \tag{2.14}$$

Now to control $I_{2,p}$ we use the following Holder's inequality:

$$\int Z^{2(1+s\theta)}dP \leq \left(\int Z^2 dP\right)^{1-\theta}\left(\int Z^{2(1+s)}dP\right)^{\theta} \tag{2.15}$$

where Z is a random variable defined on a probability space (Ω, P). Applied to $\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds$ the inequality (2.15) writes:

$$\begin{aligned} I_{2,p} &\leq \left(\mathbb{E}\left|\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds\right|^2\right)^{1-\theta} \\ &\qquad \times \left(\mathbb{E}\left|\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds\right|^{2(1+s)}\right)^{\theta} \\ &\leq (4CqM_2)^{1-\theta}\left((2q)^{2(1+s)}\,\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \\ &:= C_2 q^{1+\theta(1+2s)} M_{2,s,\theta} \end{aligned} \tag{2.16}$$

We combine (2.12), (2.13), (2.14) and (2.16) to get

$$\begin{aligned} &\mathbb{E}\left|\sum_{k=1}^{k_t} U(k)\right|^p \\ &\leq D(p)\,(k_t C q M_2)^{p/2} \\ &\quad + D(p)k_t 2^{p-1}\left[K(p)\left((qCM_2)^{p/2} + q^{1+\alpha\theta(1+2s)}M_{2,s,\theta}\right) + C_2 q^{1+\theta(1+2s)}M_{2,s,\theta}\right] \\ &\leq D(p)\,(tCM_2)^{p/2}\left\{1 + 2^{p-1}K(p)a^{p/2-1}t^{(\alpha-1)(p/2-1)}\right\} \\ &\quad + D(p)2^{p-1}M_{2,s,\theta}k_t q\left\{K(p)q^{\alpha\theta(1+2s)} + C_2 q^{\theta(1+2s)}\right\} \\ &\leq D(p)\,(tCM_2)^{p/2}\left\{1 + 2^{p-1}K(p)a^{p/2-1}t^{(\alpha-1)(p/2-1)}\right\} \\ &\quad + D(p)2^{p-1}M_{2,s,\theta}t^{1+\alpha\theta(1+2s)}\left\{K(p)a^{\alpha\theta(1+2s)} + C_2 a^{\theta(1+2s)}\right\} \end{aligned}$$

The same bound applies to $\mathbb{E}\left|\sum_{i=1}^{k_t} V(k)\right|^p$. In conclusion we have:

$$\begin{aligned} \mathbb{E}\left|\int_0^t G(X_s^v)ds\right|^p &\leq 2^{p-1}\left(\mathbb{E}\left|\sum_{k=1}^{k_t} U(k)\right|^p + \mathbb{E}\left|\sum_{i=1}^{k_t} V(k)\right|^p\right) \\ &\leq 2^p D(p)\,(tCM_2)^{p/2}\left\{1 + 2^{p-1}K(p)a^{p/2-1}t^{(\alpha-1)(p/2-1)}\right\} \\ &\quad + 2^p D(p)2^{p-1}M_{2,s,\theta}t^{1+\alpha\theta(1+2s)}\left\{K(p)a^{\alpha\theta(1+2s)} + C_2 a^{\theta(1+2s)}\right\} \end{aligned}$$

Take $K(p) = 2.2^p D(p)$ and choose a such that

$$\begin{aligned} a^{p/2-1}2^{2p-1}D(p) &\leq 1/2 \\ 2^{2p-1}D(p)a^{\alpha\theta(1+2s)} &\leq 1/2 \\ 2^{p-2}C_2 a^{\theta(1+2s)} &\leq 1/2 \end{aligned}$$

allows us to achieve the proof. □

Remark. Clearly, Proposition 2.1 is of interest if we show that $C = \sup_{v>0} \int_0^v |r_{2,v}(s)|^m \, ds < \infty$. In particular,

C is finite if either $r^m \in \mathbb{L}^1$; and $v^{2/m} \int_{|s|>v} a_s^2 ds = O(1)$ or $a \in \mathbb{L}^1$ or $r^m \in \mathbb{L}^1$ or and $a \geq 0$.

In the first case we have

$$\begin{aligned} C^v &= \int_0^v |r_{2,v}(s)|^m \, ds \leq 2^{m-1} \left(\int_0^v |r(s)|^m \, ds + \int_0^v |r_{2,v}(s) - r(s)|^m \, ds \right) \\ &\leq 2^{m-1} \left(\int_0^\infty |r(s)|^m \, ds + v \, \|r_{2,v} - r\|_\infty^m \right) \\ &\leq 2^{m-1} \left(\int_0^\infty |r(s)|^m \, ds + D \left(v^{2/m} \int_{|s|>v} a_s^2 ds \right)^{m/2} \right) \end{aligned} \tag{2.17}$$

In the second case, we write

$$\int_0^v |r_{2,v}(s)|^m \, ds \leq \int_0^v |r_{2,v}(s)| \, ds \leq \|a^v\|^{-2} \left(\int_{|s|<v} |a_s| \, ds \right)^2 .$$

In the third case, we have $|r_{2,v}(s)| \leq \|a^v\|^{-2} \, |r(s)|$.

The next proposition gives a moment inequality for occupations measures under conditions on the coefficients c_k of the function G. Note that here we do not assume a representation (2.4) for $(X_s)_s$.

Proposition 2.2. *Let G be a function such that $\mathbb{E}[G(X)] = 0, \mathbb{E}[G^2(X)] < \infty$, with m as Hermite rank and p be an integer. Then:*

$$\mathbb{E}\left(\int_0^t G(X_s) ds \right)^p \leq (t)^{p/2} \left[\sum_{k=m}^\infty \frac{|c_k|}{\sqrt{k!}} \left(\int_{-t}^t |r^k(s)| ds \right)^{1/2} (p-1)^{k/2} \right]^p$$

Proof. cf. the appendix.

The forthcoming Theorem gives a moment inequality under some conditions on the covariance function, the functional G and the coefficients a_s which define the process $(X_s)_s$.

Theorem 2.3. *Let $p > 2$ and G be a function such that $\mathbb{E}[G(X_0)] = 0$ with m as Hermite rank. Let $(X_s)_s$ a Gaussian process. Assume that*

(i) $X_t = \int a_{t+s} dW_s$.

(ii) $\int |r(s)|^m \, ds < \infty$.

Let us write $p = 2(1+s\theta)$ *with* $s > 0, \theta \in]0,1]$, *and let*

$$\Delta(v) \equiv \mathbb{E}\left[(G(X_0) - G(X_0^v))^2\right] + \mathbb{E}\left[(G(X_0))^2\right]\left(\int_{|s|>v/2} a_s^2 ds\right)^{1/2}.$$

(a) *If*

$$\Delta(v) \leq B_2 v^{-\beta},$$

then, for any $\varepsilon > 0$ *there exists* $K = K(\varepsilon, s, \theta, \beta)$ *such that*

$$\mathbb{E}\left|\int_0^t G(X_s)ds\right|^p \leq K\left(t\mathbb{E}G^2(X_0)\right)^{p/2} + Kt^{1+\max(\theta(1+2s)-\beta(1-\theta);\varepsilon)} B_2^{1-\theta}\mathbb{E}^\theta \left|G^{2(1+s)}(X_0)\right| \quad (2.18)$$

(b) *If moreover we assume that*

$$\mathbb{E}\left|G(X) - G(X^v)\right|^{2(1+s)} \leq B_s \mathbb{E}\left|X - X^v\right|^{2(1+s)},$$

then, for any $\varepsilon > 0$ *there exists* $K = K(\varepsilon, s, \theta, \beta)$ *such that*

$$\mathbb{E}\left|\int_0^t G(X_s)ds\right|^p \leq K\left(t\mathbb{E}G^2(X_0)\right)^{p/2} + Kt^{1+\max(\theta(1+2s)-\beta(1+\theta+2s\theta);\varepsilon)} B(2,s,\theta) \quad (2.19)$$

Where: $B(2,s,\theta) = B_2^{1-\theta} B_s^\theta$

Proof. As in the proof of Proposition 2.1 we will prove the theorem by induction. To this end, the following lemma proved in the Appendix will be useful.

Lemma 2.1. *Let* H *be a function with* K *as Hermite rank. and assume that:*

i) $K \geq 2$

ii) $\max\left(\sup_v \int_0^v |r_{2,v}(s)|^{K-1} ds, \int |r(s)|^{K-1} ds\right) < \infty$.

Then:

$$\mathbb{E}\left[\left(\int_0^t H(X_s) - H(X_s^v)ds\right)^2\right] \leq Ct\mathbb{E}\left[(H(X_0) - H(X_0^v))^2\right] + Ct\mathbb{E}\left[(H(X_0))^2\right]\left(\int_{|s|>v/2} a_s^2 ds\right)^{1/2} \quad (2.20)$$

We only prove (2.18), the proof of (2.19) can be done similarly with small changes.

• Step 1: Assume that $m \geq 2$ and $\sup_{u>0} \int_0^u |r_{2,u}(s)|^{m-1}\, ds < \infty$. Clearly,

$$\mathbb{E}\left|\int_0^t G(X_s)\, ds\right|^p \leq 2^{p-1}\left(E\left|\int_0^t G(X_s^v)\, ds\right|^p + \mathbb{E}\left|\int_0^t G(X_s) - G(X_s^v)\, ds\right|^p\right)$$

To control the first term we proceed as in the proof of Proposition 2.1 with $v = v(t) = at$ until the relation

$$\begin{aligned} I_{2,p} &\leq \left(\mathbb{E}\left|\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds\right|^2\right)^{1-\theta} \\ &\qquad \times \left(\mathbb{E}\left|\int_0^q G(X_s^{v(t)}) - G(X_s^{v(q)})ds\right|^{2(1+s)}\right)^{\theta} \end{aligned}$$

then we use Lemma 2.1 to get:

$$\begin{aligned} I_{2,p} &\leq \left(CqB_2 v(q)^{-\beta}\right)^{1-\theta}\left((2q)^{2(1+s)}\,\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \\ &\leq 2^{2(1+s)\theta}C^{1-\theta}a^{-\beta(1-\theta)}q^{(1-\beta)(1-\theta)+2\theta(1+s)}B_2^{1-\theta}\left(\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \\ &= C_2 a^{-\delta_1}q^{1+\delta_2}B_2^{1-\theta}\left(\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \end{aligned}$$

where $\delta_1 = \beta(1-\theta), \delta_2 = -\beta(1-\theta) + \theta(1+2s)$. Hence:

$$\begin{aligned} k_t I_{2,p} &\leq C_2 a^{-\delta_1} t q^{\delta_2} B_2^{1-\theta}\left(\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \\ &\leq C_2 a^{-\delta_1} t q^{\max(\varepsilon,\delta_2)} B_2^{1-\theta}\left(\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta} \\ &\leq C_2 a^{-\delta_1} t^{1+\max(\varepsilon,\delta_2)} a^{\varepsilon} B_2^{1-\theta}\left(\mathbb{E}\,|G(X_0)|^{2(1+s)}\right)^{\theta}. \end{aligned}$$

Now we continue in the same manner as in the proof of Proposition 2.1 and we choose a and K appropriately to get:

$$\mathbb{E}\left|\int_0^t G(X_s^v)\, ds\right|^p \leq K\left\{\left(t\mathbb{E}\left(G^2(X_0)\right)\right)^{p/2} + t^{1+\max(\varepsilon,\delta_2)}B_2^{1-\theta}\mathbb{E}^{\theta}\left|G^{2(1+s)}(X_0)\right|\right\} \tag{2.21}$$

On the other hand we have:

$$\mathbb{E}\left|\int_0^t G(X_s) - G(X_s^v)\, ds\right|^p \leq C_2 t^{1+\theta(1+2s)-\beta(1-\theta)}B_2^{1-\theta}\mathbb{E}^{\theta}\left|G^{2(1+s)}(X_0)\right|$$

which together with (2.21) proves (2.18).

- Step 2: Now assume only that $\int |r(s)|^m \, ds < \infty$.

Since $\int_{|s|>v/2} a_s^2 < Cv^{-\beta}$, there exists K such that $\sup_{u>0} \int_0^u |r_{2,u}(s)|^K \, ds < \infty$. Now write

$$\begin{aligned} G(X) - \mathbb{E}[G(X)] &= \sum_{k=m}^{K} \frac{c_k}{k!} H_k(X) + \sum_{k=K+1}^{\infty} \frac{c_k}{k!} H_k(X) \\ &= G_1(X) + G_2(X). \end{aligned}$$

Let q an even number greater or equal than p. Proposition 2.2 applied to G_1 gives:

$$\begin{aligned} \mathbb{E}\left|\int_0^t G_1(X_s)ds\right|^p &\leq \left(\mathbb{E}\left|\int_0^t G_1(X_s)ds\right|^q\right)^{p/q} \\ &\leq \left[\sum_{k=m}^{K} \frac{|c_k|}{\sqrt{k!}} \left(t\int_{-t}^{t} |r^k(s)|ds\right)^{1/2} (q-1)^{k/2}\right]^p \\ &\leq \left[K(q-1)^K \mathbb{E}\left(G^2(X_0)\right) t \int |r^m(s)|ds\right]^{p/2}. \qquad (2.22) \end{aligned}$$

According to the first step we can write

$$\begin{aligned} &\mathbb{E}\left|\int_0^t G_2\left(X_s\right) ds\right|^p \\ &\leq K\left\{\left(t\mathbb{E}\left(G_2^2(X_0)\right)\right)^{p/2} + t^{1+\max(\varepsilon,\delta_2)} B_2^{1-\theta} \mathbb{E}^\theta \left|G_2^{2(1+s)}(X_0)\right|\right\} \\ &\leq K\left\{\left(t\mathbb{E}\left(G^2(X_0)\right)\right)^{p/2} + t^{1+\max(\varepsilon,\delta_2)} B_2^{1-\theta} \mathbb{E}^\theta \left|G^{2(1+s)}(X_0)\right|\right\}. \qquad (2.23) \end{aligned}$$

Finally (2.22) and (2.23) yields (2.18).

Proofs

Proof of Theorem 2.1

Proof of Theorem 2.1 (i)

In order to reduce notations we shall restrict our self to the tow dimensional laws. Hence we have to prove that:

$$\begin{aligned} Z_t(x_1, x_2) &\equiv \alpha_1 \frac{1}{\sqrt{t}} \int_0^{tx_1} G(X_s)ds + \alpha_2 \frac{1}{\sqrt{t}} \int_0^{tx_2} G(X_s)ds \\ &\Longrightarrow \alpha_1 \sigma W(x_1) + \alpha_2 \sigma W(x_2) \end{aligned}$$

In Theorem 1 in [4] Breuer & Major proved

Theorem 2.4 (Breuer & Major). *Let G be a function such that $\mathbb{E}[G(X)] = 0, \mathbb{E}[G^2(X)] < \infty$, and let $(X_i)_{i>0}$ a Gaussian sequence. If $\sum |r(k)|^m < \infty$, where m is the Hermite rank of G, then the finite dimensional distributions of the process defined by:*

$$Z_n^k = \frac{1}{\sqrt{n}} \sum_{i=kn}^{(k+1)n} G(X_i)$$

converges to those of $(Z^k)_{k\geq 0}$ where the Z^k are i.i.d with normal distribution.

Now we state the following result which can be seen as the continuous version of the theorem above and can be proved similarly.

Theorem 2.5 (Breuer & Major). *Let G be a function such that $\mathbb{E}[G(X)] = 0, \mathbb{E}[G^2(X)] < \infty$ with m as Hermite rank. Let Z_t^k be defined by:*

$$Z_t^k = \frac{1}{\sqrt{t}} \int_{kt}^{(k+1)t} G(X_s)ds. \tag{2.24}$$

If $\int |r(s)|^m ds < \infty$, then the finite dimensional distributions of $(Z_t^k)_{k\geq 0}$ converge to those of $(Z^k)_{k\geq 0}$ where the Z^k are i.i.d with normal distribution.

First, note that $Z_t(x_1, x_2)$ can be written as:

$$Z_t(x_1, x_2) = (\alpha_1 + \alpha_2)\frac{1}{\sqrt{t}} \int_0^{tx_1} G(X_s)ds + \alpha_2 \frac{1}{\sqrt{t}} \int_{tx_1}^{tx_2} G(X_s)ds.$$

We will prove the desired convergence in several steps:
-First case: We assume that $x_1 = p$ and $x_2 = q$ where p, q are positive integers. Then:

$$Z_t(p,q) = (\alpha_1 + \alpha_2) \sum_{i=0}^{p-1} \frac{1}{\sqrt{t}} \int_{it}^{t(i+1)} G(X_s)ds + \alpha_2 \frac{1}{\sqrt{t}} \sum_{i=p}^{q-1} \int_{ti}^{t(i+1)} G(X_s)ds.$$

It then follows by the Theorem above that:

$$Z_t(p,q) \longrightarrow \sigma N(0, (\alpha_1 + \alpha_2)^2 x_1 + \alpha_2^2(x_2 - x_1)).$$

-Second case: $x_1 = p_1/q$ and $x_2 = p_2/q$ where p_1, p_2, q are integer.

$$\begin{aligned} Z_t(p_1/q, p_2/q) &= \alpha_1 \frac{1}{\sqrt{t}} \int_0^{tp_1/q} G(X_s)ds + \alpha_2 \frac{1}{\sqrt{t}} \int_0^{tp_2/q} G(X_s)ds \\ &= 1/q \left[\alpha_1 \frac{1}{\sqrt{t}} \int_0^{tp_1} G(X_s^{(q)})ds + \alpha_2 \frac{1}{\sqrt{t}} \int_0^{tp_2} G(X_s^{(q)})ds \right], \end{aligned}$$

where $(X_s^{(q)})$ is a Gaussian stationary process with $r^q(s,t) = r(q^{-1}|s-t|)$ as the correlation function. Then it follows by the case above that:

$$\begin{aligned} Z_t(x_1, x_2) &\longrightarrow \frac{\sigma^{(q)}}{q^2} N(0, (\alpha_1+\alpha_2)^2 p_1 + \alpha_2^2(p_2 - p_1) \\ &= \sigma N(0, (\alpha_1+\alpha_2)^2 x_1 + \alpha_2^2(x_2 - x_1)). \end{aligned}$$

-<u>Third case:</u> x_1, x_2 are real numbers.

Let y_n^1, y_n^2 be two sequences of rational numbers converging respectively to x_1 and x_2. According to the second case we can write:

$$\forall n, \quad Z_t(y_1^n, y_2^n) \longrightarrow \sigma N(0, (\alpha_1+\alpha_2)^2 y_1^n + \alpha_2^2(y_2^n - y_1^n).$$

On the other hand

$$\sigma N(0, (\alpha_1+\alpha_2)^2 y_1^n + \alpha_2^2(y_2^n - y_1^n) \longrightarrow \sigma N(0, (\alpha_1+\alpha_2)^2 x_1 + \alpha_2^2(x_2 - x_1).$$

In addition we have

$$\limsup_{n\to\infty} \limsup_{t\to\infty} \mathbb{E}\,[Z_t(x_1,x_2) \; - \; Z_t(y_1^n, y_2^n)]^2 = 0,$$

Thus the result is proved.□

Proof of Theorem 2.1 (ii)

Let $0 \le \theta \le 1$, we will study $\mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^{2(1+\theta)}$. By Holder's inequality we have

$$\int Z^{2(1+\theta)} dP \le \left(\int Z^2 dP\right)^{1-\theta} \left(\int Z^4 dP\right)^{\theta}. \tag{2.25}$$

Applying (4.18) to $Z = \int_0^t H_k(X_s)ds$ we get:

$$\mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^{2(1+\theta)} \le \left(\mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^2\right)^{(1-\theta)} \left(\mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^4\right)^{\theta}.$$

By Proposition 2.2:

$$\mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^4 \le \left(3^k k! t \int_{-t}^t |r^k(s)|ds\right)^2.$$

It follows that

$$\begin{aligned} \mathbb{E}\left(\int_0^t H_k(X_s)ds\right)^{2(1+\theta)} &\le \left(k!t \int_{-t}^t r^k(s)ds\right)^{(1-\theta)} \left(3^k k!t \int_{-t}^t r^k(s)ds\right)^{2\theta} \\ &\le \left(k!t \int_{-t}^t r^k(s)ds\right)^{(1+\theta)} (3^k)^{2\theta}. \end{aligned}$$

Thus

$$\begin{aligned}\left\|\int_0^t G(X_s)ds\right\|_{2(1+\theta)} &\leq \sum_{m=1}^{\infty}\frac{|c_k|}{k!}\left\|\int_0^t H_k(X_s)ds\right\|_{2(1+\theta)}\\ &\leq \sqrt{t}\sum_{m=1}^{\infty}\frac{|c_k|}{\sqrt{k!}}\left(\int_{-t}^t r^k(s)ds\right)^{1/2}\left(3^{\frac{\theta}{1+\theta}}\right)^k.\end{aligned}$$

Taking θ sufficiently small and tx instead of t we get:

$$\mathbb{E}\left(\frac{1}{\sqrt{t}}\int_0^{tx} G(X_s)ds\right)^{2(1+\theta)} \leq Cx^{(1+\theta)}. \tag{2.26}$$

(C is independent of x), as soon as the hypothesis (2.3) is satisfied. Inequality (2.26) is sufficient to prove tightness; and this proves ii) 1 through the Billingsley criterion of tightness.

We shall prove under the hypothesis of (ii) 2 that $\mathbb{E}\left(t^{-1/2}\int_0^{tx} G(X_s)ds\right)^4 \leq Cx^2$. We write

$$\begin{aligned}\mathbb{E}\left(\int_0^t G(X_s)ds\right)^4 &= \sum_{k_1,..,i_4=m}^{\infty}\prod_{i=1}^{4}\frac{c_{k_i}}{k_i!}\sum_{G\in\Gamma(k_1,\dots,k_4)} I(G,k,t)\\ &\leq \sum_{k_1,..,i_4=m}^{\infty}\prod_{i=1}^{4}\frac{c_{k_i}}{k_i!}|\Gamma(k_1,\dots,k_4)|\left(t^4\prod_{i=1}^{4}\int_{-t}^t |r^{k_i}(s)|ds\right)^{1/2}\\ &\leq \left(\int_{-t}^t |r^m(s)|ds\right)^2 t^2\sum_{k_1,..,i_4=m}^{\infty}\prod_{i=1}^{4}\frac{c_{k_i}}{k_i!}|\Gamma(k_1,\dots,k_4)|\\ &\leq \left(\int_{-t}^t |r^m(s)|ds\right)^2 t^2\mathbb{E}\left[G^4(X_0)\right].\end{aligned}$$

($\Gamma(k_1,\dots,k_4)$, $I(G,k,t)$ are defined in the appendix.)□

Proof of Theorem 2.2

The proof of the finite dimensional convergence can be done in an analogous way to the discrete case, see [6] for example, hence it is omitted. Recalling that

$$Z_t(G,x) = \frac{1}{\sqrt{t}}\int_0^t \left(1_{\{G(X_s)\leq x\}} - F(x)\right)ds,$$

we only have to prove the stochastic equicontinuity of $Z_t(G,.)$ i.e.:

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\mathbb{P}\left(\sup_{|x-y|<\delta}|Z_t(G,x)-Z_t(G,y)|>\varepsilon\right)=0.$$

a) Assume the hypothesis $(K1)$, $(K2)$

Under $(K2)$ the local time of the process $(X_s)_S$ denoted by $l_t(\cdot)$ exists, then

$$\begin{aligned} Z_t(G,y) - Z_t(G,x) &= \frac{1}{\sqrt{t}} \int_0^t \left(1_{\{x \le G(X_s) \le y\}} - (F(y) - F(x))\right) ds \\ &= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \left(1_{\{x \le G(u) \le y\}} \left(l_t(u) - tp(u)\right)\right) du \\ &= \frac{1}{\sqrt{t}} \int_{\mathbb{R}} 1_{\{x \le G(u) \le y\}} \left(\sum_{k=m}^{\infty} \frac{H_k(x)\, p(x)}{k!} \int_0^t H_k(X_s) ds \right) du. \end{aligned}$$

Hence:

$$\begin{aligned} \Delta_t(x,y) &= \left(Z_t(G,y) - Z_t(G,x)\right)^2 \\ &= \frac{1}{t} \left(\int_{\mathbb{R}} 1_{\{x \le G(u) \le y\}} \left(\sum_{k=m}^{\infty} \frac{H_k(u)\, p(u)}{k!} \int_0^t H_k(X_s) ds \right) du \right)^2 \\ &\le \left(\int_{\mathbb{R}} 1_{\{x \le G(u) \le y\}} du \right) \\ &\quad \times \left(\int_{\mathbb{R}} 1_{\{x \le G(u) \le y\}} \frac{1}{t} \left(\sum_{k=m}^{\infty} \frac{H_k(u)\, p(u)}{k!} \int_0^t H_k(X_s) ds \right)^2 du \right) \end{aligned}$$

Since $x, y \in [0,1]$, we get:

$$\begin{aligned} \Delta_t(\delta) &\equiv \sup_{|x-y|<\delta} \left(Z_t(G,y) - Z_t(G,x)\right)^2 \\ &\le \sup_{|x-y|<\delta} \left(\int_{\mathbb{R}} 1_{\{x < G(u) \le y\}} du \right) \\ &\quad \times \int_{\mathbb{R}} 1_{\{0 \le G(u) \le 1\}} \frac{1}{t} \left(\sum_{k=m}^{\infty} \frac{H_k(u)\, p(u)}{k!} \int_0^t H_k(X_s) ds \right)^2 du. \end{aligned}$$

Now under the existence of the local time and if $\int |r^m(s)|\, ds$ is finite one can show that:

$$\sup_{x \in \mathbb{R}} \mathbb{E} \left(\sum_{k=m}^{\infty} \frac{H_k(x)\, p(x)}{k!} \int_0^t H_k(X_s) ds \right)^2 \le Ct.$$

Therefore:

$$\mathbb{E}(\Delta_t(\delta)) \leq \sup_{|x-y|<\delta} \left(\int_{\mathbb{R}} 1_{\{x<G(u)\leq y\}} du \right) \left(\int_{\mathbb{R}} 1_{\{0\leq G(u)\leq 1\}} du \right).$$

Hence:

$$\mathbb{E}(\Delta_t(\delta)) \leq C \sup_{|x-y|<\delta} \left(\int_{\mathbb{R}} 1_{\{x<G(u)\leq y\}} du \right) \left(\int_{\mathbb{R}} 1_{\{0\leq G(u)\leq 1\}} du \right).$$

as the function $x \longmapsto \int_{\mathbb{R}} 1_{\{0\leq G(u)\leq x\}} du$ is continuous by (K_2), we conclude that $\lim_{\delta\to 0} \limsup_{t\to\infty} \mathbb{E}(\Delta_t(\delta)) = 0.$

b) Assume $(H1), (H2)$ and $(H'3)$.

For $f : \mathbb{R} \longrightarrow \mathbb{R}$ we set

$$Z_t(f) = \frac{1}{\sqrt{t}} \int_0^t (f(X_s) - \mathbb{E}[f(X_s)])\, ds,$$

and we define two classes of functions by:

$$\begin{aligned} \mathcal{F} &= \left\{ 1_{\{G(\cdot)\leq x\}} ; x \in \mathbb{R} \right\} \\ \mathcal{F}' &= \left\{ f_x(\cdot) := 1_{\{F\circ G(\cdot)\leq x\}} ; x \in [0,1] \right\}. \end{aligned}$$

Note that since F is continuous, $\{Z_t(f); f \in \mathcal{F}\}$ is tight if it is the case for $\{Z_t(f); f \in \mathcal{F}'\}$. (see [6] for details). Next we recall the following criterion of tightness as it was pointed out in Shao & Yu (1996) [13].

Lemma 2.2. *If there exists reals r, p, q, α such that $r > 0, p > 0, 0 \leq \alpha \leq 1, q, q+\alpha > 1$ and which satisfy:*

$$\mathbb{E}|Z_t(f_x) - Z_t(f_y)|^p \leq K\left(|x-y|^{1+r} + t^{-q/2}|x-y|^{\alpha}\right), \tag{2.27}$$

then the family $\{Z_t(f_x), x \in [0,1]\}$ is tight in $\mathcal{D}[0,1]$.

Now choose θ sufficiently small such that $\beta > \frac{1}{2} + \frac{2\theta}{1-\theta}$, apply (2.18) with $s = 1$ and ε small enough to $1_{\{x\leq F\circ G(\cdot)\leq y\}} - (y-x)$ yields

$$\mathbb{E}|Z_t(f_x) - Z_t(f_y)|^p \leq K\left(|x-y|^{1+\theta} + t^{-\theta+\max(\varepsilon,\theta(1+2)-\beta(1-\theta))}|x-y|^{\theta}\right).$$

then use (2.27) with $q = 2(-2\theta + \beta - \beta\theta), \alpha = r = \theta$ to conclude.

c) Assume $(H1), (H2)$ and $(H3)$.

In this case (2.27) apply again after deriving a moment inequality using Proposition 2.1 and a slight modification of Lemma 2.1. Indeed in this case (2.20) becomes:

$$\mathbb{E}\left[\left(\int_0^t H(X_s) - H(X_s^v)ds\right)^2\right] \le Ct\mathbb{E}\left[(H(X_0))^2\right]\left(\int_{|s|>v/2} a_s^2 ds\right)^{1/2}. \quad (2.28)$$

which combined with Proposition 1 will gives:

$$\mathbb{E}\left|Z_t(f_x) - Z_t(f_y)\right|^{2(1+\theta)} \le K\left(|x-y|^{1+\theta} + t^{-\delta}|x-y|\right).$$

where δ is some positive real.□

Appendix

Let $k_1, k_2, ..., k_p$ denote integers, V a set of points of cardinal $k_1 + k_2 + ... + k_p$. An undirected graph of type $\Gamma(k_1, k_2, ..., k_p)$ is an element of $G(V)$ satisfying:

i) V is the union of disjoint p levels with respective cardinals $k_1, k_2, ..., k_p$

$$V = \cup_{i=1}^p L_i, \qquad L_i = \{(i,l),\ l = 1, 2..., k_i\}$$

ii) Only edges between different levels are allowed

$$w = ((i,l),(i',l')) \Rightarrow i \ne i'$$

iii) Every point has exactly one edge

$$\forall (i,l) \in V,\ \exists!\ (i',l')\ /\ ((i,l);(i',l')) \in G(V)$$

For $w = ((i,l),(i',l'))$ in $G(V)$ we define $n_1(w) \equiv i$ as the first level of w and $n_2(w) \equiv i'$ as the second one.

Lemma 2.3 (DIAGRAM FORMULA). *Let $(X_{s_1}, X_{s_2}, ..., X_{s_p})$ a Gaussian vector centered and with a covariance matrix given by $(r(s_i, s_j))_{1 \le i,j \le p}$. Then:*

$$\mathbb{E}\left[\prod_{i=1}^p H_k(X_{s_i})\right] = \sum_{G \in \Gamma(k_1,...,k_p)} \prod_{w \in G} r(s_{n_1(w)}, s_{n_2(w)})$$

where $n_1(w)$, $n_2(w)$ are the first and the second level of w.

The proof can be found in Giraitis and Surgailis [9] in a more general setting. For G element of $\Gamma(k_1, k_2, ..., k_p)$ we introduce the following notations: $k_G(i)$ is the number of edges going from level i and $g(i) = \frac{k_G(i)}{k_i}$, and $I(G,k,t)$ is defined by:

$$I(G,k,t) = \int_0^t \dots \int_0^t \prod_{i=1}^{p} \prod_{w \in G, n_1(w)=i} r(s_i, i_{n_2(w)}) ds_1 ds_2 \dots ds_p.$$

Proof of Proposition 2.2. We need two inequalities stated in the following lemma. The formula (2.29) is proved by Taqqu in [14]. the second is proved below.

Lemma 2.4. *i) Let X be a standard Gaussian random variable. Then*

$$\mathbb{E}\left(|H_{k_1}(X) \dots H_{k_p}(X)|\right) \leq \prod_{i=1}^{p} (p-1)^{k_i/2} \sqrt{k_i!}. \tag{2.29}$$

ii) If $G \in \Gamma(k_1, ..., k_p)$ then

$$I^2(G,k,t) \leq t^p \prod_{i=1}^{p} \int_{-t}^{t} |r^{k_i}(s)| ds. \tag{2.30}$$

Now let us write

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t G(X_s) ds\right)^p \\
&\leq \sum_{k_1,k_2,\dots,k_p=m}^{\infty} \prod_{i=1}^{p} \cdot \frac{|c_{k_i}|}{k_i!} \int_0^t \dots \int_0^t \mathbb{E}\left(H_{k_1}(X_{s_1}) \dots H_{k_p}(X_{s_p})\right) ds_1 \dots ds_p \\
&\leq \sum_{k_1,k_2,\dots,k_p=m}^{\infty} \prod_{i=1}^{p} \cdot \frac{|c_{k_i}|}{k_i!} \sum_{G \in \Gamma(k_1,k_2,\dots,k_p)} I(G,k,t).
\end{aligned}$$

By (2.29) and (2.30) we conclude that:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t G(X_s) ds\right)^p \\
&\leq \sum_{k_1,k_2,\dots,k_p=m}^{\infty} \prod_{i=1}^{p} \cdot \frac{|c_{k_i}|}{k_i!} |\Gamma(k_1, k_2, \dots, k_p)| \prod_{i=1}^{p} \left(t \int_{-t}^{t} |r^k(s)| ds\right)^{1/2} \\
&\leq (tx)^{p/2} \left[\sum_{k=m}^{\infty} \frac{|c_k|}{k!} (p-1)^{k/2} \sqrt{k!} \left(\int_{-t}^{t} |r^k(s)| ds\right)^{1/2}\right]^p.
\end{aligned}$$

Proof of Lemma 2.4. We assume that $k_1 \leq k_2 \leq .. \leq k_p$. Moreover, without loss of generality assume that edges go from lower levels to higher ones, (by the symmetry of the covariance function). Hence, using Holder's inequality we obtain

$$
\begin{aligned}
I(G,k,t) &= \int_0^t \cdots \int_0^t \prod_{i=1}^{p} \prod_{\{w\in G, n_1(w)=i\}} r(s_i, s_{n_2(w)}) ds_1 ds_2 \dots ds_p \\
&= \int_0^t \cdots \int_0^t \prod_{\{w\in G, n_1(w)=1\}} r(s_1, s_{n_2(w)}) ds_1 \\
&\qquad \times \prod_{i=2}^{p} \prod_{\{w\in G, n_1(w)=i\}} r(s_i, s_{n_2(w)}) ds_2 \dots ds_p \\
&\qquad \times \int_0^t \cdots \int_0^t \prod_{i=2}^{p} \prod_{\{w\in G, n_1(w)=i\}} |r(s_i, s_{n_2(w)})| ds_2 \dots ds_p \\
&\le \int_{-t}^t |r(s_1)|^{k_G(1)} ds_1 \int_0^t \cdots \int_0^t \prod_{i=2}^{p} \prod_{\{w\in G, n_1(w)=i\}} |r(s_i, s_{n_2(w)})| ds_2 \dots ds_p .
\end{aligned}
$$

Repeating the same for $s_2, \dots, s_p$ we get

$$
|I(G,k,t)| \le \prod_{i=1}^{p} \int_{-t}^{t} |r(s)|^{k_G(i)} ds. \tag{2.31}
$$

In order to prove the inequality (2.30), for any graph G we write the two symmetric formulas

$$
\begin{aligned}
|I(G,k,t)| &\le \prod_{i=1}^{p} \int_{-t}^{t} |r(s)|^{k_G(i)} ds, \\
|I(G,k,t)| &\le \prod_{i=1}^{p} \int_{-t}^{t} |r(s)|^{k'_G(i)} ds,
\end{aligned}
$$

where $k_G(i) + k'_G(i) = k_i$. The first relation is (2.31). For the second relation assume that edges go from high levels to lower ones, proceed as in (2.31) considering $n_2(w)$ instead of $n_1(w)$ and begin by integrating out s_p instead of s_1. Hence from Holder's inequality

$$
\begin{aligned}
I^2(G,k,t) &\le \prod_{i=1}^{p} \int_{-t}^{t} \int_{-t}^{t} |r(s)|^{k_G(i)} |r(s')|^{k'_G(i)} ds ds' \\
&\le \prod_{i=1}^{p} \left(\int_{-t}^{t} \int_{-t}^{t} |r(s)|^{k_i} ds ds' \right)^{\frac{k_G(i)}{k_i}} \left(\int_{-t}^{t} \int_{-t}^{t} |r(s')|^{k_i} ds ds' \right)^{\frac{k'_G(i)}{k_i}} \\
&\le \prod_{i=1}^{p} t \int_{-t}^{t} |r^{k_i}(s)| ds.
\end{aligned}
$$

Proof of lemma 2.1.

$$\mathbb{E}\left[\left(\int_0^t H(X_s) - H(X_s^v)ds\right)^2\right] = A(t) + B(t)$$

where:

$$\begin{aligned} A(t) &= \int_{|s-s'|\leq t_0} \mathbb{E}\left[\left(H(X_s) - H(X_s^v)\right)\left(H(X_{s'}) - H(X_{s'}^v)\right)\right] ds ds', \\ B(t) &= \int_{|s-s'|> t_0} \mathbb{E}\left[\left(H(X_s) - H(X_s^v)\right)\left(H(X_{s'}) - H(X_{s'}^v)\right)\right] ds ds'. \end{aligned}$$

On the one hand

$$A(t) \leq t t_0 \mathbb{E}\left[\left(H(X_0) - H(X_0^v)\right)^2\right].$$

On the other hand

$$\begin{aligned} B(t) &= \int_{|s-s'|> t_0} \mathbb{E}\left[\left(H(X_s) - H(X_s^v)\right)\left(H(X_{s'}) - H(X_{s'}^v)\right)\right] ds ds' \\ &\leq 2 \sum_{k\geq K} \frac{c_k}{k!} \int_{t_0}^t (t-s)\left(r^k(s) - 2r_{1,v}^k(s) + r_{2,v}^k(s)\right) ds \\ &= B^1(t,v) \qquad + \qquad B^2(t,v). \end{aligned}$$

where

$$\begin{aligned} B^1(t,v) &= 2 \sum_{k\geq K} \frac{c_k^2}{k!} \int_{t_0}^t (t-s)\left(r^k(s) - r_{1,v}^k(s)\right) ds, \\ B^2(t,v) &= 2 \sum_{k\geq K} \frac{c_k^2}{k!} \int_{t_0}^t (t-s)\left(r_{2,v}^k(s) - r_{1,v}^k(s)\right) ds. \end{aligned}$$

Therefore

$$\begin{aligned} B^1(t,v) &\leq 2t \sum_{k\geq K} \frac{c_k^2}{k!} \int_{t_0}^t \left(|r(s) - r_{1,v}(s)| \sum_{l=1}^{k} \left| r^{k-l}(s) r_{1,v}^{l-1}(s) \right| \right) ds \\ &\leq Ct \left(\|r - r_{1,v}\|_\infty \sum_{k\geq K} \frac{c_k^2}{k!} \sup_k k \int_{t_0}^t r_t^{*k-1}(s) ds \right). \end{aligned}$$

Choose t_0 such that $r_t^*(s) \leq 1/2$ for $s > t_0$, hence

$$B^1(t,v) \leq Ct \, \|r - r_{1,v}\|_\infty \sum_{k\geq K} \frac{c_k^2}{k!}.$$

The same bound holds for $B^2(t,v)$ that is

$$B^2(t,v) \leq Ct \, \|r_{2,v} - r_{1,v}\|_\infty \sum_{k \geq K} \frac{c_k^2}{k!}.$$

Using Lemma 8.4.1 in Berman [2] we can write:

$$\begin{aligned} \|r - r_{1,v}\|_\infty &\leq \frac{1}{\|a\|_v} \int_{|s|>v/2} a_s^2 ds + \left(\int_{|s|>v/2} a_s^2 ds \right)^{1/2} \\ \|r_{2,v} - r_{1,v}\|_\infty &\leq \frac{1}{\|a\|_v} \int_{|s|>v/2} a_s^2 ds + \frac{1}{\|a\|_v} \left(\int_{|s|>v/2} a_s^2 ds \right)^{1/2}, \end{aligned}$$

consequently

$$\|r - r_{1,v}\|_\infty + \|r_{2,v} - r_{1,v}\|_\infty \leq C \left(\int_{|s|>v/2} a_s^2 \right)^{1/2}.$$

Thus (2.20) is proved. □

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Chapitre 3

Uniform CLT for empirical process

ABSTRACT. Empirical processes indexed by classes of functions based on dependent observations are considered. Sufficient conditions in order to satisfy stochastic equicontinuity are given. Conditions are in terms of bracketing numbers with respect to a norm arising from Rosenthal type moment inequality satisfied by the process. Several applications involving mixing sequences, functions of Gaussian sequences are discussed..

Key words: bracketing; chaining; empirical processes; functional central limit theorems; stochastic equicontinuity; weakly dependent processes.

Classification AMS:60F05; 60F17

Introduction

Let $(X_i)_{i\geq 0}$ be a stationary sequence of real random variables defined on a probability space (Ω, A, P), and $\mathcal{F}$ be a class of real valued functions of real variables. Let $l^{\infty}(\mathcal{F})$ denote the space of bounded real functions defined on $\mathcal{F}$. Given a collection $\mathcal{F}$ one can define a map from $\mathcal{F}$ to $\mathbb{R}$ as follows:

$$\begin{aligned} Z_n \ &: \ \mathcal{F} \longrightarrow \mathbb{R} \\ &f \longmapsto Z_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_i)). \end{aligned}$$

If $\sup_{\mathcal{F}} |f(\cdot) - \mathbb{E}(f(X_0))|$ exists and is finite, then the map Z_n is an element of $l^{\infty}(\mathcal{F})$. Consequently, it makes sense to investigate conditions under which the sequences Z_n converge in law in $l^{\infty}(\mathcal{F})$ endowed with the uniform topology. A class $\mathcal{F}$ for which this is true is called a *Donsker class*. Weak convergence in $l^{\infty}(\mathcal{F})$ can be characterized as asymptotic tightness plus convergence of marginals. More precisely, according to Pollard 1990 [11] (see also Van Der Vaart 96 [15]) this convergence is equivalent to the following two conditions.

(i) Convergence of marginals: for all $f_1, ..., f_k$ elements of $\mathcal{F}$,

$$(Z_n(f_1), ..., Z_n(f_k)) \text{ converges in law.}$$

(ii) There exists a pseudo metric ρ such that $(\mathcal{F}, \rho)$ is totally bounded, and for all $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \mathbb{P}\left(\sup_{\rho(f,g)<\delta} |Z_n(f-g)| > \varepsilon \right) = 0. \tag{3.1}$$

The second property is known as the stochastic equicontinuity of the family Z_n. It is useful in proving uniform central limit theorems as well as in other contexts, (see for example Andrews 94 [2].)

Convergence of the finite dimensional distributions is proved for many classes of processes. Roughly speaking, Property (i) is satisfied as soon as the sequence X is sufficiently weakly dependent. Dependence between the past of the process and its future is measured either by mixing coefficients such as α-mixing (strong mixing), ρ, β and ϕ-mixing, or by the decay of covariances for functions of Gaussian, linear processes and associated sequences. Therefore and in order to conclude the uniform CLT, it remains to prove the stochastic equicontinuity. And this will be the main purpose of the present paper.

Several results exists in the literature. We recall some of them in what follows with an emphasis on those which are close to the spirit of this work. In 1987, Ossiander [9] proved that if the variables are i.i.d then (ii) is fulfilled if

$$\int_0^1 \sqrt{\log N_{[\,]}(\varepsilon, \|\cdot\|_2, \mathcal{F})} d\varepsilon < \infty,$$

where $N_{[\,]}(\varepsilon, \|\cdot\|_2, \mathcal{F})$is the minimal number of $\varepsilon-$brackets sufficient to cover $\mathcal{F}$ (exact definition will be given later).

This result has been generalized by Doukhan, Massart and Rio in 1995 [6] to β-mixing sequences under the summability of the sequence of β-mixing and the following condition on the family $\mathcal{F}$:

$$\int_0^1 \sqrt{\log N_{[\,]}\left(\varepsilon, \|.\|_{2,\beta}, \mathcal{F}\right)} d\varepsilon < \infty,$$

where $\|f\|_{2,\beta}^2 = \int_0^1 \beta^{-1}(u) Q_f^2(u) du$. The technique's proof in the two cases was exponential inequalities for independent random variables.

On the other hand, using a moment inequality of order 2, Arcones in 1994 [1] showed the stochastic equicontinuity of $\{Z_n(f), f \in \mathcal{F}\}_{n>0}$ when the process X is Gaussian with summable covariance function and if the family satisfies the condition:

$$\int_0^1 N_{[\,]}^{1/2}(x, \|\cdot\|_2, \mathcal{F})\, dx < \infty.$$

Andrews and Pollard 1994 [3] have concluded the tightness of the empirical process of a strong mixing sequence under the following hypothesis:

$$\sum_{i>0} i^{p-2} \alpha^{\frac{\gamma}{p+\gamma}}(i) < \infty,$$

$$\sup_{\mathcal{F}} |f| \leq 1 \text{ and } \int_0^1 x^{-\frac{\gamma}{\gamma+2}} N_{[\,]}^{1/p}(x, \|\cdot\|_2, \mathcal{F})\, dx < \infty$$

where $p \geq 2$ and $\gamma > 0$. Here also, the main tool was a moment inequality of order p.

In view of these results we can see that the conditions ensuring the tightness of the empirical process is a kind of balance between the regularity of the process on the one hand, expressed here in term of weak dependence, and the size or the complexity of the family $\mathcal{F}$ on the other hand, measured here by the bracketing numbers with respect to a norm induced by the process.

A goal of this work is to give a general approach to this problem which involves some existing results beside other ones. The main result asserts that if the process satisfies a Rosenthal type moment inequality of order p and if $\int_0^1 N_{[\,]}^{1/p}\left(x, \|\cdot\|_{2,X}, \mathcal{F}\right) dx < \infty$ where $\mathcal{F}$ is an uniformly bounded class of functions and $\|\cdot\|_{2,X}$ is an appropriate norm induced by the moment inequality then (ii) is satisfied. The paper is structured as follows, in section 2 we give the main results, several applications are discussed in section 3, section 4 is devoted to the proofs of results.

Main results

Before stating the main result we recall the following definition of bracketing numbers.

Definition 3.1. *Given two functions l and u the bracket $[l, u]$ is the set of all functions f with $l \leq f \leq u$. Given a norm $\|\cdot\|$ on a space containing $\mathcal{F}$, an $\varepsilon-$bracket for $\|.\|$ is a bracket $[l, u]$ with $\|l-u\| < \varepsilon$. The bracketing number $N_{[\,]}(\varepsilon, \|\cdot\|, \mathcal{F})$ is the minimum number of $\varepsilon-$brackets needed to cover $\mathcal{F}$.*

For $p \geq 2$ we define two kind of conditions, the first one on the process, and the second one on the family $\mathcal{F}$.

$\mathbf{H}(p, X)$: There exists $a(p)$ and $b(p)$ constants such that for every measurable f

$$\mathbb{E}\,|Z_n(f)|^p \leq a(p)\,\|f\|_{2,X}^p + b(p) n^{1-p/2}\,\|f\|_\infty^{p-2}\,\|f\|_{2,X}^2 \tag{3.2}$$

where $\|\cdot\|_{2,X}$ is a norm satisfying:

- $\|\cdot\|_1 \leq C\,\|\cdot\|_{2,X}$ for some positive constant C.
- $|f| \leq |g| \Rightarrow \|f\|_{2,X} \leq \|g\|_{2,X}$.

$\mathbf{H}(p, \mathcal{F})$: $\mathcal{F}$ is uniformly bounded if $p > 2$, *and*

$$\int_0^1 N_{[\,]}^{1/p}\left(x, \|\cdot\|_{2,X}, \mathcal{F}\right) dx < \infty. \tag{3.3}$$

We are now able to state our first result.

Theorem 3.1. *Let $(X_i)_{i\geq 0}$ be a strictly stationary sequence of random variables and $\mathcal{F}$ be a class of functions satisfying $H(p,X)$ and $H(p,\mathcal{F})$, then: $\forall\varepsilon > 0$,*

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{\|f-g\|_{2,X}<\delta}|Z_n(f-g)|>\varepsilon\right)=0.$$

The condition under which the family $\mathcal{F}$ is uniformly bounded, may be relaxed by strengthening the condition on the covering numbers and imposing further assumptions on the envelope function of the family. This is what is done in the following theorem.

Theorem 3.2. *Let $(X_i)_{i\geq 0}$ be a strictly stationary sequence of random variables and $\mathcal{F}$ be a class of functions satisfying $H(p,X)$. Let $F \geq \sup_{f\in\mathcal{F}}|f|$ be a measurable function. Assume that $F \in \mathbb{L}^{r+1}$, for some $r > 1$, and*

$$\int_0^1 N_{[\,]}^{\nu/p}\left(x, \|\cdot\|_{2,X}, \mathcal{F}\right) dx < \infty. \tag{3.4}$$

where $1/\nu = 1 - \frac{1}{r}\left(1-\frac{2}{p}\right)$.

Then the conclusion of Theorem 1 holds.

In what follows we are aimed to give sufficient conditions for $\mathcal{F}$ in order to satisfy the stochastic equicontinuity property in the case when the α-mixing coefficient decays exponentially. The result is closely related to the work of Massart 87 [8], however the technique's proof is slightly different. The proof of the next result relies on a Rosenthal type moment inequality, with explicit bounds of the coefficients $a(p)$ and $b(p)$, due to Rio combined, as usual, with a chaining argument.

Theorem 3.3. *Let $(X_i)_{i\geq 0}$ be a stationary sequences and $\mathcal{F}$ be a family of functions bounded by 1. We assume*

(a) $\alpha(i) \leq c\exp(-\alpha i)$, where $c > 0, \alpha > 0$.

(b) $\int_0^1 \log^2 N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F})\, d\varepsilon < \infty$.

Then, $\forall\varepsilon > 0$,

$$\lim_{\delta\to+\infty}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{\|f-g\|_1\leq\delta}|Z_n(f-g)|>\varepsilon\right)=0.$$

The previous theorem improves on Massart's result. Indeed, under the same hypothesis of mixing, the assumption on $\mathcal{F}$ was

$$\log N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F}) \le C\left(\frac{1}{\varepsilon}\right)^{\xi}, \xi < 1/4.$$

We point out however, that Massart shows a rate of convergence for the given weak invariance principle. We note also that Andrews and Pollard [3] conjectured in their paper that the condition implying the stochastic equicontinuity under the same assumption of mixing, may be

$$\int_0^1 \varepsilon^{-\frac{\gamma}{\gamma+2}} \log^2 N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F})\, d\varepsilon < \infty,$$

for some positive constant γ.

Remark. In the independent case, the same method of proof shows that the condition is $\int_0^1 \log^{1/2} N_{[\,]}(\varepsilon, \|\cdot\|_1, \mathcal{F})\, \varepsilon^{-1/2} d\varepsilon < \infty$. This condition is known to be optimal when $\mathcal{F}$ is the class of all subset of $\mathbb{N}$.

Examples of application

In this section we give some examples for which the hypothesis $H(p, X)$ is fulfilled and we compare with some existing results. As for $H(p, \mathcal{F})$, we refer the reader to [8] and [15].

First we recall the following measures of dependence. Suppose $(\Omega, \mathcal{K}, \mathcal{P})$ is a probability space. For any two σ-fields $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{K}$, we define

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup |\mathbb{P}(A \cup B) - \mathbb{P}(A) \cup \mathbb{P}(B)|, \quad A \in \mathcal{A}, \quad B \in \mathcal{B},$$

and

$$\rho(\mathcal{A}, \mathcal{B}) = \sup |\mathrm{Corr}(f, g)|, \quad f \in \mathbb{L}^2(\mathcal{A}), \quad g \in \mathbb{L}^2(\mathcal{B}).$$

The following inequality holds

$$\rho(\mathcal{A}, \mathcal{B}) \le \alpha(\mathcal{A}, \mathcal{B})$$

If (X_k) is a sequence of random variables we define

$$c = \sup_k c(G_1^k, G_{k+n}^{\infty})$$

where $c = \alpha, \rho$ and G_n^m is the σ field generated by $(X_k, n \le k \le m)$.

Case of α-mixing

Let (X_k) a stationary sequence, and for u positive real, set $\alpha(u) = \alpha([u])$, where $[x]$ denote the integer part of x. Denote by Q_f the quantile function of $|f(X_0)|$, which is the inverse of the tail function $t \longrightarrow \mathbb{P}(|f(X_0)| > t)$. The following corollary is an immediate consequence of Theorem 1.

Corollary 3.1. *Let* $\|f\|_{2,X}^2 = \int_0^1 [\alpha^{-1}(u)]^{p-1} Q_f^2(u) du$. *If* $H(p, \mathcal{F})$ *is fulfilled, then* $Z_n(f)$ *converges in* $l^\infty(\mathcal{F})$ *to a Gaussian process indexed by* $\mathcal{F}$ *with* $\|f\|_{2,X}$ *continuous sample paths.*

In particular, this convergence holds whenever the following conditions are satisfied:

$$(H1) : \sum_{i \geq 0} (i+1)^{\frac{p-1}{1-\theta}-1} \alpha(i) < \infty$$

$$(H2) : \int_0^1 N_{[\,]}^{1/p} \left(x, \|.\|_{2/\theta}, \mathcal{F} \right) dx < \infty.$$

To compare with the assumptions of Andrews and Pollard we first note that if $N_r(x) = N_{[\,]}(x, \|\cdot\|_r, \mathcal{F})$, and if $\mathcal{F}$ is bounded above by 1, then $N_r\left(x^{2/r}\right) \leq N_2(x)$. By a change of variable we conclude that $(H2)$ is implied by:

$$(H'2) : \int_0^1 N_{[\,]}^{1/p}(x, \|\cdot\|_2, \mathcal{F})\, x^{-1+\theta} dx < \infty.$$

The assumptions of Andrews and Pollard are:

$$(A1) \quad : \quad \sum_{i \geq 0} (i+1)^{p-2} \alpha(i)^{\frac{2-2\theta}{p\theta+2-2\theta}} < \infty$$

$$(A2) \quad : \quad \int_0^1 N_{[\,]}^{1/p}(x, \|.\|_2, \mathcal{F})\, x^{-1+\theta} dx < \infty.$$

Now $(H1)$ is weaker for $p > 2$, (e.g. for a polynomial rate of mixing, say $\alpha(i) \sim c i^{-a}$, $(H1)$ is satisfied if $a > (p-1) / (1-\theta)$ while $(A1)$ is fulfilled if $a > (p-1)(p\theta + 2 - 2\theta) / (2 - 2\theta)$).

Case of ρ-mixing

The forthcoming corollary considers ρ-mixing sequences. Its proof relies on a moment inequality established by Shao [13] and the CLT for ρ-mixing sequences (see [10] for example).

Corollary 3.2. *Let* (X_k) *be a stationary,* $\rho-$*mixing sequence. Assume that*

$F \in L^{2+\delta}$,

$$\sum_{i=0}^{\infty} \rho\left(2^i\right) < \infty \text{ and } \int_0^1 N_{[\,]}^{\eta}\left(x, \|\cdot\|_2, \mathcal{F}\right) dx < \infty,$$

where η, δ are positive reals.

Then $Z_n(f)$ converges in $l^\infty(\mathcal{F})$ to a Gaussian process indexed by $\mathcal{F}$ with $\|\cdot\|_2$ continuous sample paths.

The corollary applies to the family of quadrants, and generalizes the result of Shao and Yu 96 in the sense that the continuity of the distribution function is not needed here.

Case of Gaussian sequences.

Let $(X_i)_{i\geq 0}$ be a stationary Gaussian sequence satisfying: $\mathbb{E}(X_0) = 0$, $\mathbb{E}(X_0^2) = 1$ and let $r(k) = \mathbb{E}(X_0 X_k)$. To apply Theorem 1, we need a Rosenthal type inequality for partial sums of a function of Gaussian sequences. This is the subject of the following lemma. The lemma handles the particular case when $p = 4$. But we think[1] that similar results can be obtained for even integers, however this will not be considered here. We recall that the *rank* of a real function f is defined by $rankf = \inf\{k > 0 \;\; / \;\; \mathbb{E}(H_k(X)f(X)) \neq 0\}$.

Lemma 3.1. *Let f be a real function and assume that:*

$$\sum_{k\geq 0} |r(k)|^m < \infty,$$

where $m = \inf\left(rankf, rankf^2\right)$. Then there exists a constant $K = K(r(\cdot))$ such that:

$$\mathbb{E}\left|\sum_{i=1}^{n} f\left(X_i\right) - \mathbb{E}f\left(X_i\right)\right|^4 \leq K\left(n^2\left(\mathbb{E}f^2\left(X_i\right)\right)^2 + n\|f\|_\infty^2 \mathbb{E}f^2\left(X_i\right)\right).$$

As a consequence of the previous lemma and Theorem 3.1, we deduce that if r belongs to L^1 and if $\int_0^1 N_{[\,]}^{1/4}(x, \|\cdot\|_2, \mathcal{F})\, dx < \infty$ where $\mathcal{F}$ is a class of family bounded by 1, then (ii) is satisfied. Since the condition r belongs to L^1 is sufficient for convergence of marginals (see for example [4]), we have then proved the following corollary.

Corollary 3.3. *Let (X_i) be a stationary Gaussian sequence such that $\mathbb{E}(X_0) = 0$, $\mathbb{E}(X_0^2) = 1$ and let $r(k) = \mathbb{E}(X_0 X_k)$. Let $\mathcal{F}$ be a family of function bounded*

1. indeed, we later prove it. cf chapter 4

by 1. *If*

$$r \in \mathbb{L}^1 \text{ and } \int_0^1 N_{[\,]}^{1/4}(x, \|\cdot\|_2, \mathcal{F})\, dx < \infty,$$

then $\{Z_n(f), f \in \mathcal{F}\}_{n>0}$ *converge in* $l^\infty(\mathcal{F})$ *to a Gaussian, centered process* G *indexed by* $\mathcal{F}$ *with covariance function given by:*

$$\operatorname{Cov}(G(f), G(g)) = \mathbb{E}(G(f)G(g)) = \sum_{i \in Z} \operatorname{Cov}(f(X_0), g(X_i)).$$

In the particular case when $\mathcal{F} = \left\{1_{G(\cdot)\leq x}; x \in R\right\}$ where G is some measurable function, the condition r belongs to L^1 can be relaxed to the following one:

$$\sum_{k=1}^{n} |r^m(k)| < \infty,$$

where m is the Hermite rank of the family $\tilde{\mathcal{F}} := \mathcal{F} \cup \{1_{.<x}; \mathbb{P}(G(X) = x) \neq 0\}$. Indeed, in this case the moment inequality of order 4 will be applied to $\tilde{\mathcal{F}} - \tilde{\mathcal{F}} := \left\{f - g; (f,g) \in \left(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}\right)\right\}$. And since for $f \in \tilde{\mathcal{F}}$ we have $rank(f^2) \geq rank(f)$ it suffices to have $\sum_{k=1}^{n} |r^m(k)| < \infty$. But $N_{[\,]}(x, \|\cdot\|_2, \mathcal{F}) \leq \frac{C}{x^2}$ for this family. Thus the result applies and this generalizes Theorem 1 in [5] to the case when the distribution function of $G(X)$ is discontinuous.

Proof of Main results

For any expressions A and B let us write $A \preceq B$ if $A \leq KB$ for some absolute constant K. And let $[x]$ stands for the integer part of x.

Proof of Theorem 3.1

By hypothesis (3.3), for all integer k there exists a finite sequence of pair of functions $(f_i^k, \Delta_i^k)_{1\leq i\leq N(k)}$, where $N(k) = N_{[\,]}\left(2^{-k}, \|.\|_{2,X}, \mathcal{F}\right)$ such that:

- $\|\Delta_i^k\|_{2,X} \leq 2^{-k}$
- $\forall f \in \mathcal{F}$ there exists i such that $\left|f - f_i^k\right| \leq \Delta_i^k$.

We set $(\pi_k(f), \Delta_k(f))$ the first pair (f_i^k, Δ_i^k) which satisfies: $|f - f_i^k| \leq \Delta_i^k$. Let q_0, k and q integers verifying $q_0 \leq k \leq q$. Following an idea from Arcones 94 [1] we define a map from $\mathcal{F}$ into a finite subset of $\mathcal{F}$ by:

$$T_k(f) = \pi_k \circ \pi_{k+1} \circ .... \circ \pi_q(f).$$

For $1 \leq i \leq N(q_0)$ let us define:

$$E_i = \{f \in \mathcal{F} \; : \; T_{q_0}(f) = f_i\},$$

then the E_i form a partition of $\mathcal{F}$. For $\delta > 0$ we define:

$$F_{i,j} = \Big\{(f,g) \in \mathcal{F} \times \mathcal{F} \text{ such that } f \in E_i, g \in E_j \text{ and } \|f - g\|_{2,X} \leq \delta\Big\}$$

Let now $\Lambda = \{(i,j) \text{ such that } F_{i,j} \neq \emptyset\}$. For every pair in Λ fix an element of $F_{i,j}$ and denote this pair by $(\Phi_{i,j}, \Psi_{i,j})$.

Let (f,g) a pair satisfying $\|f - g\|_{2,X} \leq \delta$, then necessary $(f,g) \in F_{i,j}$ for some $(i,j) \in \Lambda$. Now we write:

$$f - g = f - T_{q_0}(f) + T_{q_0}(f) - \Phi_{i,j} + \Phi_{i,j} - \Psi_{i,j} + \Psi_{i,j} - T_{q_0}(g) + T_{q_0}(g) - g$$

but $T_{q_0}(f) = T_{q_0}(\Phi_{i,j})$ and $T_{q_0}(g) - T_{q_0}(\Psi_{i,j})$ since $f, \Phi_{i,j}$ are in E_i and $g, \Psi_{i,j}$ are in E_j. Consequently:

$$\sup_{\|f-g\|_{2,X} \leq \delta} |Z_n(f - g)| \leq 4 \sup_{f \in \mathcal{F}} |Z_n(f - T_{q_0}(f))| + \sup_{(i,j) \in \Lambda} |Z_n(\Phi_{i,j} - \Psi_{i,j})|$$

Apply the outer expectation to the previous inequality to get:

$$\begin{aligned} &\mathbb{E}\left[\sup_{\|f-g\|_{2,X} \leq \delta} |Z_n(f - g)|\right] \\ \leq\ & 4\mathbb{E}\left[\sup_{f \in \mathcal{F}} |Z_n(f - T_{q_0}(f))|\right] + \mathbb{E}\left[\sup_{(i,j) \in \Lambda} |Z_n(\Phi_{i,j} - \Psi_{i,j})|\right] \\ :=\ & 4E_1 + E_2. \end{aligned}$$

For the shake of brevity we put $\sup_{f \in \mathcal{F}} |Z_n(f)| = \|Z_n(f)\|_{\mathcal{F}}$. In order to control the two terms of the previous display we shall use the following maximal inequality from Pizier, combined with a chaining argument. For all random variables $Z_1, Z_2, ..., Z_N$

$$\left(\mathbb{E}\left|\max_{1 \leq i \leq N} |Z_i|\right|^p\right)^{1/p} \leq N^{1/p} \max_{1 \leq i \leq N} (\mathbb{E}|Z_i|^p)^{1/p}. \tag{3.5}$$

<u>Control of E_1</u>

For f in $\mathcal{F}$ we write:

$$\begin{aligned} f - T_{q_0}(f) &= f - T_q(f) + \sum_{k=q_0+1}^{q} T_k(f) - T_{k-1}(f) \\ &= f - \pi_q(f) + \sum_{k=q_0+1}^{q} T_k(f) - T_{k-1}(f). \end{aligned}$$

Therefore:

$$\begin{aligned} E_1 &:= \mathbb{E}\|Z_n(f - T_{q_0}(f))\|_{\mathcal{F}} \\ &\leq \mathbb{E}\|Z_n(f - \pi_q(f))\|_{\mathcal{F}} + \sum_{k=q_0+1}^{q} \mathbb{E}\|Z_n(T_k(f) - T_{k-1}(f))\|_{\mathcal{F}} \\ &\leq E_{1,q+1} + 2\sqrt{n}\sup_{f\in\mathcal{F}} \mathbb{E}|\Delta_q(f)| + \sum_{k=q_0+1}^{q} E_{1,k} \qquad (3.6) \end{aligned}$$

where

$$\begin{aligned} E_{1,k} &= \mathbb{E}\|Z_n(T_k(f) - T_{k-1}(f))\|_{\mathcal{F}}, \qquad q_0 + 1 \leq k \leq q \\ E_{1,q+1} &= \mathbb{E}\|Z_n\Delta_q(f)\|_{\mathcal{F}}. \end{aligned}$$

Now observe that $T_k(f) - T_{k-1}(f) = T_k(f) - \pi_{k-1}(T_k(f))$ and $T_k(f)$ range on a finite set with cardinal less or equal than $N(k)$. Using inequality (3.5) we can write:

$$E_{1,k} \leq N(k)^{1/p} \max_{g\in T_k(\mathcal{F})} \|Z_n(g - \pi_{k-1}(g))\|_p \qquad (3.7)$$

Since by hypothesis $\mathcal{F}$ is uniformly bounded when $p > 2$, we may assume that f, $\pi_{k-1}(f)$ and $\Delta_q(f)$ are bounded by 1. Apply hypothesis (3.2) to $h = g - \pi_{k-1}(g)$ to get:

$$\begin{aligned} \|Z_n(h)\|_p &\leq a^{1/p}(p)\|h\|_{2,X} + \left(b(p)n^{1-p/2}\|h\|_\infty^{p-2}\|h\|_{2,X}^2\right)^{1/p} \\ &\leq a^{1/p}(p)2^{-(k-1)} + b^{1/p}(p)n^{1/p-1/2}2^{-\frac{2(k-1)}{p}} \qquad (3.8) \end{aligned}$$

Combining (3.7) and (3.8) yields

$$E_{1,k} \leq 2a^{1/p}(p)N(k)^{1/p}2^{-k} + 2b^{1/p}(p)N(k)^{1/p}2^{-k}\left(n^{-1/2}2^k\right)^{1-2/p}.$$

A similar bound holds for $E_{1,q+1}$. Finally using the fact that $\mathbb{E}|\Delta_q(f)| \leq C\,\|\Delta_q(f)\|_{2,X} \leq C2^{-q}$ we obtain:

$$\begin{aligned}
E_1 &\preceq \sqrt{n}2^{-q} + \sum_{k=q_0+1}^{q+1} E_{1,k} \\
&\preceq \sqrt{n}2^{-q} + a^{1/p}(p) \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k} \\
&\quad + b^{1/p}(p) \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k}\left(n^{-1/2}2^k\right)^{1-2/p} \\
&\preceq \sqrt{n}2^{-q} + a^{1/p}(p) \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k} \\
&\quad + b^{1/p}(p)\left(n^{-1/2}2^q\right)^{1-2/p} \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k}
\end{aligned}$$

Hence:

$$E_1 \preceq \sqrt{n}2^{-q} + c(p) \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k}\left(1 + \left(n^{-1/2}2^q\right)^{1-2/p}\right) \tag{3.9}$$

<u>Control of E$_2$</u>

Noting that $|\Lambda| \leq N^2(q_0)$ and $\|\Phi_{i,j} - \Psi_{i,j}\|_{2,X} \leq \delta$ we get:

$$\begin{aligned}
E_2 &= \mathbb{E}\left[\sup_{(i,j)\in\Lambda} |Z_n(\Phi_{i,j} - \Psi_{i,j})|\right] \\
&\leq N^{2/p}(q_0) \max_{(i,j)\in\Lambda} \|Z_n(\Phi_{i,j} - \Psi_{i,j})\|_p
\end{aligned}$$

Again by $H(p,X)$:

$$\begin{aligned}
E_2 &\preceq N^{2/p}(q_0)\left(a^{1/p}(p)\delta + \left(b(p)n^{1-p/2}\delta^2\right)^{1/p}\right) \\
&\preceq c(p)\left(N(q_0)\,\delta\right)^{2/p}.
\end{aligned} \tag{3.10}$$

Let $W(n,\delta)$ denote $\mathbb{E}\left(\sup_{\|f-g\|_{2,X}<\delta} |Z_n(f-g)|\right)$. From (3.9) and (3.10) it follows that:

$$\begin{aligned}
W(n,\delta) &\leq 4E_1 + E_2 \\
&\preceq \sqrt{n}2^{-q} + c(p) \sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k}\left(1 + \left(n^{-1/2}2^q\right)^{1-2/p}\right) \\
&\quad + c(p)\left(N(q_0)\,\delta\right)^{2/p}
\end{aligned}$$

Let $q_0 = q_0\,(\delta)$ the greatest integer satisfying $N(q_0) \leq \delta^{-1/2}$. Without loss of generality we may assume that $q_0\,(\delta)$ goes to infinity as δ goes to zero. Therfore, if we set $\varepsilon(\delta) = \sum_{k=q_0+1}^{\infty} N(k)^{1/p} 2^{-k}$ we have by $H(p, \mathcal{F})$ that $\varepsilon(\delta) \to 0$ when $\delta \to 0$. Take $q = q(n,\delta) = \left[\frac{1}{2\log 2} \log \frac{n}{\varepsilon(\delta)}\right] + 1$. With this choice $q > q_0$ and $\sqrt{n}2^{-q} < 1$ if $n > n(\delta)$ and for $n > n(\delta)$ we have:

$$W(n,\delta) \preceq \sqrt{\varepsilon(\delta)} + c(p)\sqrt{\varepsilon(\delta)} + c(p)\delta^{1/p}.$$

Consequently:

$$\begin{aligned} \lim_{\delta\to 0} \limsup_{n\to+\infty} W(n,\delta) &\preceq \lim_{\delta\to 0} \sqrt{\varepsilon(\delta)} + c(p)\sqrt{\varepsilon(\delta)} + c(p)\delta^{1/p} \\ &= 0, \end{aligned}$$

and Theorem is 3.1 proved.

Proof of Theorem 3.2

We will follow the same lines of the proof of Theorem 1 with small modifications. Thus notations will be unchanged.

<u>Control of E_1</u>

$$\begin{aligned} E_1 &= \mathbb{E}\,\|\,Z_n\,(f - T_{q_0}(f))\|_{\mathcal{F}} \\ &\leq \mathbb{E}\,\|\,Z_n\,((f - \pi_q(f))\,1_{F\leq M})\|_{\mathcal{F}} + \mathbb{E}\,\|\,Z_n\,(f - \pi_q(f)1_{F>M})\|_{\mathcal{F}} \\ &:= E_{1,M} + E'_{1,M} \end{aligned} \tag{3.11}$$

On the one hand, since $F \in \mathbb{L}^{r+1}$ we can write:

$$\begin{aligned} E'_{1,M} &\leq 2\sqrt{n}\mathbb{E}\,|\,F1_{F>M}| \\ &\leq \frac{\sqrt{n}}{M^r} r(M) \end{aligned}$$

where $r(M)$ goes to zero as M goes to $+\infty$. On the other hand:

$$\begin{aligned} E_{1,M} &\leq \mathbb{E}\,\|\,Z_n\,((f - \pi_q(f))\,1_{F\leq M})\|_{\mathcal{F}} \qquad (3.12) \\ &\quad + \sum_{k=q_0+1}^{q} \mathbb{E}\,\|\,Z_n\,((T_k(f) - T_{k-1}(f))\,1_{F\leq M})\|_{\mathcal{F}} \\ &\leq E_{1,q+1}^M + 2\sqrt{n}\sup_{f\in\mathcal{F}} \mathbb{E}\,|\Delta_q(f)| + \sum_{k=q_0+1}^{q} E_{1,k}^M \qquad (3.13) \end{aligned}$$

where:

$$\begin{aligned} E_{1,k}^M &= \mathbb{E}\| Z_n (T_k(f) - T_{k-1}(f)) 1_{F\leq M}\|_{\mathcal{F}}\,, \qquad q_0+1\leq k\leq q \\ E_{1,q+1}^M &= \mathbb{E}\| Z_n\Delta_q(f)1_{F\leq M}\|_{\mathcal{F}}\,. \end{aligned}$$

Note that when $F \leq M$, we may assume that $T_k(f)$ as well as $\Delta_q(f)$ are bounded above by M. Apply hypothesis $H(p, X)$ to $h := T_k(f) - T_{k-1}(f)$ after applying (3.5) to obtain:

$$\begin{aligned} E_{1,k}^M &\leq N(k)^{1/p}\max_{f\in\mathcal{F}}\|h\|_p \\ &\leq 2a^{1/p}(p)N(k)^{1/p}2^{-k} + 2b^{1/p}(p)N(k)^{1/p}2^{-2k/p}\left(n^{-1/2}M\right)^{1-2/p} \end{aligned} \tag{3.14}$$

A similar bound holds to $E_{1,q+1}^M$ that is:

$$E_{1,q+1}^M \leq 2a^{1/p}(p)N(q)^{1/p}2^{-q} + 2b^{1/p}(p)N(q)^{1/p}2^{-2q/p}\left(n^{-1/2}M\right)^{1-2/p}$$

Therefore:

$$\begin{aligned} E_{1,M} &\preceq \sqrt{n}2^{-q} + \sum_{k=q_0+1}^{q+1} E_{1,k} \\ &\preceq \sqrt{n}2^{-q} + c(p)\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k} \\ &\quad + c(p)\left(n^{-1/2}M\right)^{1-2/p}\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-2k/p}. \end{aligned}$$

Taking $M = n^{1/2r}$, from the estimations of $E_{1,M}$ and $E'_{1,M}$, we deduce that:

$$\begin{aligned} E_1 &\preceq \sqrt{n}2^{-q} + c(p)\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k} \\ &\quad + c(p)\left(n^{-1/2}M\right)^{1-2/p}\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-2k/p} + r'(n). \end{aligned} \tag{3.15}$$

where $r\prime(n) \to 0$. Let R denote the third term in the above display, then:

$$\begin{aligned} R &:= \left(n^{-1/2}M\right)^{1-2/p}\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-2k/p} \\ &\leq \left(n^{-1/2}n^{1/2r}\right)^{1-2/p}\int_{2^{-q}}^{2^{-q_0}} N^{1/p}(x)x^{2/p-1}dx \\ &\leq n^{(-1/2+1/2r)(1-2/p)}\int_{2^{-q}}^{2^{-q_0}} N^{1/p}(x)x^{2/p-1}dx \end{aligned}$$

Apply Hölder's inequality to $f = N^{1/p}, g = x^{2/p-1}$ and with $1/\nu = 1 - 1/r\,(1-2/p)$, to obtain:

$$\begin{aligned}\int_{2^{-q}}^{2^{-q_0}} N^{1/p}(x)x^{2/p-1}dx &\le \left(\int_{2^{-q}}^{2^{-q_0}} N^{\nu/p}(x)dx\right)^{1/\nu}\left(\int_{2^{-q}}^{2^{-q_0}} x^{(2/p-1)\frac{\nu}{\nu-1}}dx\right)^{\frac{\nu-1}{\nu}}\\ &\le \left(\int_{2^{-q}}^{2^{-q_0}} N^{\nu/p}(x)dx\right)^{1/\nu}\left((r-1)^{-1}\left[x^{-r+1}\right]_{2^{-q}}^{2^{-q_0}}\right)^{\frac{\nu-1}{\nu}}\\ &\le \left(\int_{2^{-q}}^{2^{-q_0}} N^{\nu/p}(x)dx\right)^{1/\nu}(r-1)^{-\frac{\nu-1}{\nu}}\left(2^{-q}\right)^{(1-2/p)(1/r-1)}\end{aligned}$$

It follows that:

$$R \le c(p,r)\left(\sqrt{n}2^{-q}\right)^{(1/r-1)(1-2/p)}\left(\int_{2^{-q}}^{2^{-q_0}} N^{\nu/p}(x)dx\right)^{1/\nu}. \tag{3.16}$$

Combine (3.15) and (3.16) to get:

$$\begin{aligned}E_1 &\preceq \sqrt{n}2^{-q} + c(p)\sum_{k=q_0+1}^{q+1} N(k)^{1/p}2^{-k}\\ &\quad + c(p,r)\left(\sqrt{n}2^{-q}\right)^{(1/r-1)(1-2/p)}\left(\int_{2^{-q}}^{2^{-q_0}} N^{\nu/p}(x)dx\right)^{1/\nu} + r'(n)\end{aligned} \tag{3.17}$$

<u>Control of E_2</u>

Similarly we have

$$\begin{aligned}E_2 &= \mathbb{E}\left[\sup_{(i,j)\in\Lambda}\left|Z_n\left(\Phi_{i,j}-\Psi_{i,j}\right)\right|\right]\\ &\le E_{2,M} + E'_{2,M}.\end{aligned}$$

First we write:

$$\begin{aligned}E'_{2,M} &:= \mathbb{E}\left[\sup_{(i,j)\in\Lambda}\left|Z_n\left(\left(\Phi_{i,j}-\Psi_{i,j}\right)1_{F>M}\right)\right|\right]\\ &\le 4\sqrt{n}\mathbb{E}\left|F1_{F>M}\right|\\ &\le r(n),\end{aligned} \tag{3.18}$$

where $r(n)$ goes to zero. Second, apply $H(p,X)$ to $(\Phi_{i,j}-\Psi_{i,j})\,1_{F\le M}$ which satisfies $\|(\Phi_{i,j}-\Psi_{i,j})\,1_{F\le M}\|_{2,X}\le\delta$ to obtain:

$$\begin{aligned} E_{2,M} &:= \mathbb{E}\left[\sup_{(i,j)\in\Lambda}\left|Z_n\left(\Phi_{i,j}-\Psi_{i,j}\right)1_{F\le M}\right|\right] \\ &\le N^{2/p}(q_0)\left(a^{1/p}(p)\delta+\left(b(p)n^{1-p/2}M^{p-2}\delta^2\right)^{1/p}\right) \qquad (3.19)\\ &\le c(p)\left(N(q_0)\,\delta\right)^{2/p}. \qquad (3.20)\end{aligned}$$

From (3.18) and (3.20) we conclude that:

$$E_2\le c(p)\left(N(q_0)\,\delta\right)^{2/p}+r(n). \qquad (3.21)$$

<u>End of the proof</u>

Recalling that $W(n,\delta)=\mathbb{E}\left(\sup_{\|f-g\|_{2,X}<\delta}|Z_n(f-g)|\right)$. Then (3.17) together with (3.21) imply:

$$\begin{aligned} W(n,\delta) &\le 4E_{1+}E_2 \\ &\preceq \sqrt{n}2^{-q}+c(p)\sum_{k=q_0+1}^{q+1}N(k)^{1/p}2^{-k}+c(p)\left(N(q_0)\,\delta\right)^{2/p}+r(n) \\ &\quad +c(p,r)\left(\sqrt{n}2^{-q}\right)^{(1/r-1)(1-2/p)}\left(\int_{2^{-q}}^{2^{-q_0}}N^{\nu/p}(x)dx\right)^{1/\nu}+r'(n)\end{aligned}$$

Put $\beta=-(1/r-1)(1-2/p)$, and let

$$q_0=q_0(\delta)=\max\left\{k,k\in N,/N(k)\le\delta^{-1/2}\right\}.$$

We may and do assume that $q_0(\delta)$ goes to infinity as δ goes to zero. Put

$$\varepsilon(\delta)=\left(\int_0^{2^{-q_0(\delta)}}N^{\nu/p}(x)dx\right)^{1/\nu},$$

we have by (3.4) that $\varepsilon(\delta)\to0$ when $\delta\to0$. Now choose $q=q(n,\delta)$ in such a way that $\sqrt{n}2^{-q}$ and $\left(\sqrt{n}2^{-q}\right)^{(1/r-1)(1-2/p)}\varepsilon(\delta)$ have the same order of magnitude, that is:

$$q=q(n,\delta)=\left[\frac{1}{2\log 2}\log\frac{n}{\varepsilon^{\frac{1}{1+\beta}}(\delta)}\right]+1.$$

With this choice $q > q_0$ if $n > n(\delta)$, and in this case we have:

$$\begin{aligned} W(n,\delta) &\preceq \varepsilon^{\frac{1}{1+\beta}}(\delta)\left(1+c(p,r)\right)+c(p)\sum_{k=q_0+1}^{q+1}N(k)^{1/p}2^{-k} \\ &\quad +c(p)\delta^{1/p}+r(n)+r'(n) \end{aligned}$$

It follows that

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{\|f-g\|_{2,X}<\delta}|Z_n(f-g)|>\varepsilon\right)=0$$

and this conclude the proof of Theorem 2.

Proof of Theorem 3.3

In the sequel all the inequalities are valid up to a multiplicative constant. First we recall the following moment inequality which is a corollary of Theorem 6.3 in Rio 97 [12].

Lemma 3.2. *Let $(\alpha_n)_{n\geq 0}$ be the sequence of strong mixing coefficients of the process $(X_i)_{i\geq 0}$. Let f be a measurable function. Then for all $p \geq 2$.*

$$\mathbb{E}\left|Z_n(f)\right|^p \leq a(p)\left\|f\right\|_{2,\alpha}^p + b(p)n^{1-p/2}\sum_{i=1}^{n}(i+1)^{p-2}\alpha_i\left\|f\right\|_\infty^p \tag{3.22}$$

with:

- *Q_f is the quantile function of $|f(X_0)|$.*
- *$\|f\|_{2,\alpha}^2 = \int_0^1 \alpha^{-1}(u)Q_f^2(u)du$.*
- *$a(p) \leq (Cp)^{p/2}$, $a(p) \leq (Cp)^p$.*

We have assumed that: $\forall f \in \mathcal{F}$, $\|f\|_\infty \leq 1$. Hence without loss of generality, we may assume that $\forall f \in \mathcal{F}, \forall k > 0, \Delta_k(f) \leq 1$. From (3.22) it follows that:

$$\left\|Z_n(f)\right\|_p \leq A(p,f)+B(p,f)$$

with:

$$\begin{aligned} A(p,f) &\preceq \sqrt{p}\,\|f\|_{2,\alpha} \\ B(p,f) &\preceq p^2 n^{-1/2+1/p}\,\|f\|_\infty \end{aligned}$$

Applying Hölder's inequality gives:

$$\begin{aligned}
\|f\|_{2,\alpha}^2 &\le \left(\int_0^1 [\alpha^{-1}(u)]^{\frac{1}{1-\theta}}\, du\right)^{1-\theta} \left(\int_0^1 Q_f^{2/\theta}(u)du\right)^{\theta} \\
&\le \left(\frac{1}{1-\theta}\sum_{i=1}^{n}(i+1)^{\frac{1}{1-\theta}}\alpha(i)\right)^{1-\theta} \|f\|_{2/\theta}^2 \\
&\le \left(\frac{1}{1-\theta}\sum_{i=1}^{n}(i+1)^{\frac{1}{1-\theta}}\alpha(i)\right)^{1-\theta} \|f\|_1^{\theta}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
A(p,f) &\preceq \sqrt{p}\,\|f\|_1^{\theta/2} \\
B(p,f) &\preceq p^2 n^{-1/2+1/p}\,\|f\|_\infty
\end{aligned}$$

We then proceed as in the proof of Theorem 3.1, and thus we keep the same notation.

<u>Control of E_1</u>

We recall that if $N(k) = N_{[\,]}\left(2^{-k}, \|\cdot\|_1, \mathcal{F}\right)$ then:

$$E_1 \le \mathbb{E}\,\|Z_n\Delta_q(f)\|_{\mathcal{F}} + 2\sqrt{n}\sup_{f\in\mathcal{F}}\mathbb{E}\,|\Delta_q(f)| + \sum_{k=q_0+1}^{q}\mathbb{E}\,\|Z_nT_k(f) - T_{k-1}(f)\|_{\mathcal{F}} \tag{3.23}$$

we recall also that :

$$\begin{aligned}
E_{1,k} &= \mathbb{E}\,\|Z_nT_k(f) - T_{k-1}(f)\|_{\mathcal{F}}\,, \qquad q_0+1 \le k \le q \\
E_{1,q+1} &= \mathbb{E}\,\|Z_n\Delta_q(f)\|_{\mathcal{F}}
\end{aligned}$$

From the hypothesis and inequality (3.5) we have:

$$E_{1,k} \le N(k)^{1/p}\max_{g\in T_k(\mathcal{F})}\|Z_n\left(g - \pi_{k-1}(g)\right)\|_p \tag{3.24}$$

We now apply the moment inequality to $g - \pi_{k-1}(g)$ which is bounded by $\Delta_{k-1}(g)$ to obtain:

$$\begin{aligned}
E_{1,k} &\le N(k)^{1/p}\left[\max_{g\in T_k(\mathcal{F})} A\left(p, g-\pi_{k-1}(g)\right) + \max_{g\in T_k(\mathcal{F})} B\left(p, g-\pi_{k-1}(g)\right)\right] \\
&\preceq N(k)^{1/p}\left(\sqrt{p}2^{-k\theta/2} + p^2 n^{-1/2+1/p}\right) \\
&\preceq N(k)^{1/p}\left(\sqrt{p}2^{k(1-\theta/2)}2^{-k} + p^2\left(n^{-1/2}2^k\right)^{1-2/p}2^{2k/p}2^{-k}\right) \qquad (3.25)
\end{aligned}$$

Therefore if $p > 2, n^{-1/2}2^q \geq 1$, we get:

$$E_{1,k} \preceq N(k)^{1/p} \left(\sqrt{p}2^{k(1-\theta/2)}2^{-k} + \left(n^{-1/2}2^q\right) p^2 2^{2k/p}2^{-k}\right)$$

Let $p = k + \log N(k)$, then

$$E_{1,k} \preceq \left(\sqrt{k} + \sqrt{\log N(k)}\right) 2^{k(1-\theta/2)}2^{-k} + \left(n^{-1/2}2^q\right)\left(k^2 + \log^2 N(k)\right) 2^{-k}$$

A similar bound holds for $E_{1,q+1}$. Hence if we assume that :

$$\int_0^1 \log^2 N_{[\,]}\left(\varepsilon, \|\cdot\|_2, \mathcal{F}\right) d\varepsilon < \infty,$$

and

$$\int_0^1 \log^{1/2} N_{[\,]}\left(\varepsilon, \|\cdot\|_2, \mathcal{F}\right) x^{\theta/2-1} d\varepsilon < \infty$$

for some $0 < \theta < 1$. Then there exists a positive sequence $l(k)$ satisfying $\sum l(k) < \infty$, such that for all $k, q_0 \leq k \leq q+1$, if $n^{-1/2}2^q \geq 1$, we have:

$$E_{1,k} \preceq \left(n^{-1/2}2^q + 1\right) l(k).$$

Since $\int_0^1 \log^2 N_{[\,]}\left(\varepsilon, \|\cdot\|_2, \mathcal{F}\right) d\varepsilon < \infty$ implies $\int_0^1 \log^{1/2} N_{[\,]}\left(\varepsilon, \|\cdot\|_2, \mathcal{F}\right) \varepsilon^{\theta/2-1} d\varepsilon < \infty$, for some convenient θ, we conclude that under the hypothesis of the theorem we have: $\forall q \geq q_0$ such that $n^{-1/2}2^q \geq 1$,

$$E_1 \preceq \sqrt{n}2^{-q} + 2n^{-1/2}2^q \sum_{k=q_0+1}^{q+1} l(k) \tag{3.26}$$

<u>Control of E$_2$</u>

Recall that $|\Lambda| \leq N^2(q_0)$ hence:

$$\begin{aligned} E_2 &= \mathbb{E}\left[\sup_{(i,j)\in\Lambda} \left|Z_n\left(\Phi_{i,j} - \Psi_{i,j}\right)\right|\right] \\ &\leq N(q_0) \max_{(i,j)\in\Lambda} \left\|Z_n\left(\Phi_{i,j} - \Psi_{i,j}\right)\right\|_2 \end{aligned}$$

Using a moment inequality of order 2 :

$$\mathbb{E}\left|Z_n(f)\right|^2 \leq C(\theta', \alpha) \|f\|_1^{\theta'}$$

where $0 < \theta' < 1/2$. Applying this to $\Phi_{i,j} - \Psi_{i,j}$ which satisfies $\left\|\Phi_{i,j} - \Psi_{i,j}\right\|_1 \leq \delta$ we get:

$$E_2 \leq C(\theta', \alpha) N(q_0)\, \delta^{\theta'/2} \tag{3.27}$$

End of the proof

Let $W(n,\delta)$ denote $\mathbb{E}\left(\sup_{\|f-g\|_1<\delta}|Z_n(f-g)|\right)$. Combining (3.26) and (3.27) gives:

$$W(n,\delta) \preceq \sqrt{n}2^{-q} + 2n^{-1/2}2^q \sum_{k=q_0+1}^{q+1} l(k) + C(\theta',\alpha)N(q_0)\,\delta^{\theta'/2}.$$

Take $\theta' = 1/3$ for example and let $q_0 = q_0(\delta)$ the greatest integer satisfying $N(q_0) \le \delta^{-1/12}$. Without loss of generality we may assume that $q_0(\delta)$ tends to infinity as δ tends to zero. Therfore, if we set $\varepsilon(\delta) = \sum_{k=q_0+1}^{\infty} l(k)$ we have that $\varepsilon(\delta) \to 0$ when $\delta \to 0$. Take $q = q(n,\delta) = \left[\frac{1}{2\log 2}\log\frac{n}{\varepsilon(\delta)}\right] + 1$. Note that $q > q_0$ and $\sqrt{n}2^{-q} < 1$ for n sufficiently large, say $n > n(\delta)$ and hence for $n > n(\delta)$

$$W(n,\delta) \preceq \varepsilon^{1/2}(\delta) + C(\theta',\alpha)\delta^{1/12}.$$

Consequently:

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{n\to+\infty} W(n,\delta) &\preceq \lim_{\delta\to 0}\varepsilon^{1/2}(\delta) + C(\theta',\alpha)\delta^{1/12}\\ &= 0,\end{aligned}$$

and this conclude the proof of Theorem 3.3.

Other proofs

Proof of corollary 1

From Rio 97 [12] Theorem 6.3 we infer that:

$$\begin{aligned}\mathbb{E}|Z_n(f)|^p &\le a(p)\left(\int_0^1 \alpha^{-1}(u)Q_f^2(u)du\right)^{p/2}\\ &\quad + b(p)n^{1-p/2}\int_0^1 \left[\alpha^{-1}(u)\right]^{p-1} Q_f^p(u)du \qquad (3.28)\end{aligned}$$

where Q_f is the quantile function of $|f(X_0)|$. Assume moreover that $\|f\|_\infty \le M$, then (3.28) can be written:

$$\begin{aligned}\mathbb{E}|Z_n(f)|^p &\le a(p)\left(\int_0^1 \alpha^{-1}(u)Q_f^2(u)du\right)^{p/2}\\ &\quad + b(p)n^{1-p/2}M^{p-2}\int_0^1 \left[\alpha^{-1}(u)\right]^{p-1} Q_f^2(u)du.\end{aligned}$$

Therefore, we can apply Theorem 1 with $\|f\|_{2,X}^2 = \int_0^1 [\alpha^{-1}(u)]^{p-1} Q_f^2(u)du$. Now $H(p,\mathcal{F})$ implies that for $f \in \mathcal{F}$, $\int_0^1 \alpha^{-1}(u)Q_f^2(u)du < \infty$, and this implies (i) according to Doukhan and al. (see [7]).

- Using Hölder's inequality, we get, for any θ in $(0,1)$:

$$\int_0^1 [\alpha^{-1}(u)]^{p-1} Q_f^2(u)du \le \left(\int_0^1 [\alpha^{-1}(u)]^{\frac{p-1}{1-\theta}} du\right)^{1-\theta} \left(\int_0^1 Q_f^{2/\theta}(u)du\right)^{\theta}$$

Since $\int_0^1 [\alpha^{-1}(u)]^q du \le q\sum_{i\ge 0}(i+1)^{q-1}\alpha(i)$ and $Q_f(U) \stackrel{law}{=} |f(X)|$ if U is uniformly distributed on $[0,1]$, we deduce that:

$$\int_0^1 [\alpha^{-1}(u)]^{p-1} Q_f^2(u)du \le \left(\frac{p-1}{1-\theta}\sum_{i\ge 0}(i+1)^{\frac{p-1}{1-\theta}-1}\alpha(i)\right)^{1-\theta} \left(\|f\|_{2/\theta}\right)^2.$$

Hence the following hypotheses are sufficient to imply (ii):

$$(H1) \quad : \quad \sum_{i\ge 0}(i+1)^{\frac{p-1}{1-\theta}-1}\alpha(i) < \infty$$

$$(H2) \quad : \quad \int_0^1 N_{[\,]}^{1/p}\left(x, \|.\|_{2/\theta}, \mathcal{F}\right)dx < \infty.$$

and this proves the second part of the corollary.

Proof of corollary 2

First we recall the following result from Shao 95 [13]. $\forall p \ge 2, \quad \exists \quad K = K(\rho(\cdot), p)$ such that for every measurable f:

$$\begin{aligned}\mathbb{E}\,|Z_n(f)|^p \quad &\le \quad K\exp\left(\sum_{i=0}^{[\log n]}\rho\left(2^i\right)\right)\|f(X)\|_2^p \\ &\qquad + Kn^{1-p/2}\exp\left(K\sum_{i=0}^{[\log n]}\rho^{2/p}\left(2^i\right)\right)\|f(X)\|_p^p.\end{aligned}$$

In particular, if $\sum_{i=0}^{[\log n]}\rho\left(2^i\right) < \infty$, then $\exp\left(K\sum_{i=0}^{[\log n]}\rho^{2/p}\left(2^i\right)\right)$ is a slowly varying function for every p. Hence, $\forall p \ge 2, \forall \varepsilon > 0 \quad \exists \quad K = K(\rho(\cdot), p, \varepsilon)$ such that for every measurable f:

$$\mathbb{E}\,|Z_n(f)|^p \le K\,\|f(X)\|_2^p + Kn^{1+\varepsilon-p/2}\,\|f(X)\|_p^p. \tag{3.29}$$

Arguing as in the proof of Theorem 2, it is easy to see that under (3.29) (ii) is satisfied as soon as F, the envelop function belongs to L^{r+1}, for some $r > 1$ and

$$\int_0^1 N_{[\,]}^{1/p(1-1/r(1-2/p)-2\varepsilon/p)}\left(x, \|\cdot\|_2, \mathcal{F}\right) dx < \infty.$$

Since p can be chosen arbitrary large and ε arbitrary small, (ii) follows under our hypothesis. The proof of (i) follows from Theorem 1 in [10] for example.

Proof of Lemma 3.1

We will take back the proof of a similar result given in Csörgő and Mielniczuk 96 [5] (inequality 3.2) with small changes . In particular, we recall that for $k = 1, 2, 3, 4$:

$$\mathbb{E}\left(S_{kn}\right) = \sum_{1 \le i_1 \ne i_2 \dots \ne i_k \le n} \mathbb{E}\left[\left(f\left(X_{i_1}\right) - \mathbb{E}f\left(X_{i_1}\right)\right) \dots \left(f\left(X_{i_k}\right) - \mathbb{E}f\left(X_{i_k}\right)\right)\right]$$

We first assume that $R := \sup_{k \ge 1} |r(k)| < 1/3$, then, we proceed as in [5] to handle the general case.

- $\mathbb{E}\left(S_{1n}\right) = n\mathbb{E}\left(f\left(X_0\right) - \mathbb{E}f\left(X_0\right)\right)^4 \preceq n \|f\|_\infty^2 \mathbb{E}f^2\left(X_i\right).$
-

$$\mathbb{E}\left(S_{2n}\right) = 3 \sum_{1 \le i_1 \ne i_2 \le n} \mathbb{E}\left[\left(\overline{f}\left(X_{i_1}\right)\right)^2 \left(\overline{f}\left(X_{i_2}\right)\right)^2\right] + 4 \sum_{1 \le i_1 \ne i_2 \le n} \mathbb{E}\left[\left(\overline{f}\left(X_{i_1}\right)\right)^3 \overline{f}\left(X_{i_2}\right)\right]$$

where $\overline{f} = f - \mathbb{E}f\left(X_0\right)$. The first term is bounded by:

$$n^2 \left(\mathbb{E}f^2\left(X_i\right)\right)^2 + n \sum_{i=1}^{n} \left|r^{m(f^2)}(i)\right| \mathbb{E}\left|f\left(X_i\right)\right|^4,$$

and the second one is bounded by:

$$n \sum_{i=1}^{n} \left|r^{m(f)}(i)\right| \mathbb{E}^{1/2}\left|f\left(X_i\right)\right|^2 \mathbb{E}^{1/2}\left|f\left(X_i\right)\right|^6.$$

Hence:

$$\mathbb{E}\left(S_{2n}\right) \preceq n^2 \left(\mathbb{E}f^2\left(X_i\right)\right)^2 + n \|f\|_\infty^2 \mathbb{E}f^2\left(X_i\right).$$

- Using a lemma of Taqqu stated as Lemma 3 in [5], we have

$$\begin{aligned} \mathbb{E}\left(S_{3n}\right) &\preceq n^{3/2}\mathbb{E}^{1/2}\left|f\left(X_i\right)\right|^2 \mathbb{E}^{1/2}\left|f\left(X_i\right)\right|^4 \\ &\preceq n^2 \left(\mathbb{E}f^2\left(X_i\right)\right)^2 + n\mathbb{E}f^4\left(X_i\right) \\ &\preceq n^2 \left(\mathbb{E}f^2\left(X_i\right)\right)^2 + n \|f\|_\infty^2 \mathbb{E}f^2\left(X_i\right). \end{aligned}$$

- Again by Lemma 3 we have:

$$\mathbb{E}(S_{4n}) \preceq n^2\mathbb{E}^2 |f(X_i)|^2 .$$

This comletes the proof.

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Chapitre 4

Moment inequalities for partial sums

ABSTRACT. Rosenthal type inequalities for partial sums of functions of gaussian sequences are obtained. Several applications involving almost sure convergence, functional central limit theorems are then discussed.

AMS subject classifications: 60E15; 60F17.

Key words: Partial sums; Gaussian subordination; Gaussian processes; Rosenthal-type inequalities; Maximal inequalities; functional central limit theorems; Almost sure convergence.

Introduction

Moment inequalities play an important role in the proofs of limit theorems in probability. When the sequence is iid several sharp estimates are available in the literature, see for example Petrov 1992. For optimal moment inequalities in the independent framework we refer the reader to the paper of Penilis 1994. Talagrand 1994 also provides optimal constant for Rosenthal type moment inequalities when we do not separate the term of the variance and the terms which involve the moment of order p. However, in statistics the hypothesis of independence is in many cases very strong and does not fit the models especially when we are dealing with times series in which case the dependence between the past of the process and its future is not weak enough to assume that they are independent. To take this structure of dependence into account, several measures of dependence have been developed. Basically there are two kinds of measure: those which involve the sigma algebra of the past and the future and those constructed with a covariance structure. The first class includes mixing coefficients, and in the second we can find associated sequences, functions of gaussians and linear sequences. Under mixing assumptions, that is a suitable decay of the mixing coefficient, many works have established moment inequalities for partials sums, both Rosenthal's type and M-Z inequalities. These inequalities are more or less similar to the iid case. For strong mixing we cite the work of Doukhan et al. 1983, Rio 1994 and Shao 1996. For $\rho-$mixing sequences, nice inequalities are available in the work of Shao 95. Concerning $\phi-$mixing similar results are proved by Utev 1991.

It is worth noting that except for the case of $\phi-$mixing which is a rather strong condition, (Gaussian sequences which are $\phi-$ mixing are shown to be m independent) all the moment inequalities, up to our knowledge, require a decay of the mixing coefficients which increases with the order of moments. Moreover, many sequences fail to be mixing (even strong mixing) and it is in general difficult to compute the mixing coefficients. For example when $r(j) \sim j^{-a}$, where $a < 1$, then we can show (see Withers 1981), that the sequence is not mixing.

Processes with a correlation converging slowly to 0 naturally appear in meteorology, economics and others fields (we refer the reader to the Beran 1992 for more about this). In other words, such sequences can not be handled in the frame of weak dependence assumptions. Such processes are said to be strong dependent or long range dependent. In the last twenty years much more attention has been devoted to the study of such phenomena. We cite for example, the works of Taqqu, Dubroshin and Major, Csörgő and Mielnuzuck, Giraitis and Surgailis and many others authors. We also point out that the results obtained in this framework are generally, qualitatively different from the weakly dependent case.

In this work, we aim to give estimates of the moments of partials sums when the observations are a function of Gaussian sequences of a possibly long range dependents sequence. To be more precise let $\{X_n : n \in \mathbb{N}\}$ be a real valued, Gaussian, stationary sequence with covariance function

$$\mathbb{E}(X_m X_n) = r(|n-m|),\ \mathbb{E}(X_n) = 0 \ \text{ and } \ \mathbb{E}(X_n^2) = 1. \tag{4.1}$$

Let f be a real function satisfying $\mathbb{E}(f^2(X)) < \infty$ where X is a standard normal variable. A quantity of primary interest in this work is the sum defined by

$$S_n = S_n(f) = \sum_{i=1}^{n} f(X_i) \tag{4.2}$$

We first recall the following expansion of f in terms of Hermite's polynomials which is convergent in the $\mathbb{L}^2(\Omega)$ sense

$$f(x) = \sum_{k=0}^{\infty} \frac{c_k(f)}{k!} H_k(x), \tag{4.3}$$

where H_k is the kth Hermite polynomial and $c_k(f) = \mathbb{E}(f(X) H_k(X))$. We often write c_k for $c_k(f)$ when no confusion can be made. Let

$$m = m(f) = \inf\{k > 0,\ /\ c_k \neq 0\}, \tag{4.4}$$

m is called the Hermite rank of f. We also define the Hermite rank of the family as the smallest of the ranks of its elements.

The paper is structured as follows, in section 2 we give Rosenthal and M-Z inequalities for partial sums. This is in fact the main part of this work. Section 3 is devoted to deriving maximal inequalities from the moments inequalities proved in the previous section. Clearly, these inequalities are useful in the proofs of almost sure properties. Several applications of our results are then presented in section 4. In particular, we gives rates of convergence in the SLLN, we prove the Donsker property for the partial sum process and finally we gives a high order expansion of the empirical distribution function. Since the main technical tool in our proofs is the diagram formula and related techniques, an appendix is devoted to these notions.

Mor-ent inequalities

In this section we derive a moment inequality for partial sums when the observations are functional of Gaussian processes. A key ingredient in the proof of the forthcoming result is presented in Lemma 1. Since this lemma deals with sums on different indices, we will write $(S_n(f))^p$ as sum of terms as in the lemma, then we conclude with repeated applications of Lemma 1 and Hölder's inequalities.

Notations. Let f denotes a real function and p an even integer. Let m denotes the Hermite rank of f and m_p stands for the hermite rank of $\{f, f^2, f^3, ..., f^{p-2}\}$. We then define the following quantities.

$$\|f\|_{2,p} = \max\left(p^{1/2}\|f\|_2, \sum_{k=m_p}^{\infty} \frac{|d_k(f)|}{\sqrt{k!}}(p-1)^{k/2}\left(2\sum_{i=1}^{n}|r^k(i)|\right)^{1/2}\right) \quad (4.5)$$

where $d_k(f) = \mathbb{E}\,|f(X)\,H_k(X)|$, and let

$$r^*(j) = \sup_{i\geq j}|r(i)|. \quad (4.6)$$

$$\alpha(s,i) = ((s-1)r^*(i))^{1/2} \quad (4.7)$$

We first prove the following

Proposition 4.1. *Let f be a function such that $\mathbb{E}[f(X)] = 0, \mathbb{E}[f^2(X)] < \infty$.*

(i) *Let $C_n = 2\sum_{i=0}^{n}|r(i)|^m$ then*

$$\mathbb{E}\left|\sum_{i=1}^{n} f(X_i)\right|^2 \leq nC_n\mathbb{E}\,|f(X_0)|^2 \quad (4.8)$$

(ii) *Let $p \geq 2$, for every centered, bounded f we have:*

$$\mathbb{E}\left|\sum_{i=1}^{n} f(X_i)\right|^p \leq 6^p\left\{\left(\sqrt{n}\,\|f\|_{2,p}\right)^p + p^p n\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right\} \quad (4.9)$$

where $\|f\|_{2,p}$ is given by (4.5).

(iii) *Let p an even integer, then*

$$\mathbb{E}\left|\sum_{i=1}^{n} f(X_i)\right|^p \leq n^{p/2}\mathbb{E}\left|f^p(X)\right| \left(1 + \sum_{s>p/2} (s!)^{-1} s^p 2^s \left(C(m,s,r,n)\right)^s\right)$$

where

$$C(m,s,r,n) = 1 + \sum_{k=m}^{\infty}(\alpha(s,1))^k \left(\max\left(2n^{-1}\sum_{i=1}^{n}|r'(i)|; 2\sum_{i=1}^{n}|r'(i)|^m\right)\right)^{1/2}$$

where $r'(i) = r(i)/r^*(1)$ and m is the Hermite rank of f.

We note that if the (X_i) are l-independent then the constants are – up to a multiplicatif constant – optimal. In particular we can derive exponential inequalities for the tail function.

Proof. The proof of (i) is straightforward. We now prove (4.9). To this end we introduce the following notation. Let

$$\begin{aligned} Q(p) &= \{t = (t_1, t_2, ..., t_s),\ t_1 + t_2 + ... + t_s = p;\ t_i > 0\} \\ N(s) &= \{i = (i_1, i_2, ..., i_s),\ 1 \leq i_k \leq n;\ i_k \neq i_l \ if\ k \neq l\}. \end{aligned} \tag{4.10}$$

Hence

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{n} f(X_i)\right)^p &= \sum_{i_1,i_2,...,i_p=1}^{n} \mathbb{E}\left(\prod_{l=1}^{p} f(X_{i_l})\right) \\ &= \sum_{t\in Q(p)} C(t) \sum_{(i_1,i_2,...,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right) \end{aligned}$$

where $C(t) = (s!t_1!t_2!....t_s!)^{-1}\, p!$. Let us set

$$I(t,n) = \sum_{(i_1,i_2,...,i_s)\in N(s)} \left|\mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right)\right|$$

To control the above we need the following lemma, proved in the appendix.

Lemma 4.1. *Let $f_1, f_2, ...f_p$, be real, centered functions, and set $m_i = m(f_i)$. Let $\|f\|_{r,p}$ defined by*

$$\|f\|_{r,p} = \sum_{k=m(f)}^{\infty} \frac{|c_k(f)|}{\sqrt{k!}}(p-1)^{k/2}\left(2\sum_{i=1}^{n}|r^k(i)|\right)^{1/2} \tag{4.11}$$

Then

$$\sum_{i\in N(p)}\left|\mathbb{E}\left(\prod_{l=1}^{p} f_l\left(X_{i_l}\right)\right)\right| \le \prod_{l=1}^{p}\|f_l\|_{r,p}. \tag{4.12}$$

To apply the previous lemma we have to center the functions, (note that they are already centered when $t_i = 1$). We also set $\overset{\circ}{h} = h - \mathbb{E}(h(X))$ for any measurable function h. With this notation we have

$$I(t,n) \le \sum_{(i_1,i_2,\dots,i_s)\in N(s)}\left|\mathbb{E}\left(\prod_{l=1}^{s}\left(\overset{\circ}{f^{t_l}}\left(X_{i_l}\right)+\mathbb{E}f^{t_l}\left(X\right)\right)\right)\right|.$$

In the following we will bound the terms appearing in the last sum. Note that there is at most 2^s terms. They are all of the form

$$I(J,t,n) = \sum_{(i_1,i_2,\dots,i_s)\in N(s)}\left|\left(\prod_{l\in J}\mathbb{E}f^{t_l}\left(X\right)\right)\left(\prod_{l\in J^C}\left(\overset{\circ}{f^{t_l}}\left(X_{i_l}\right)\right)\right)\right|,$$

where $\{J, J^C\}$ is a partition of $\{1,2,\dots,s\}$. Assume for simplicity that $J = \{1,2,\dots,k\}$. In this case $I(J,t,n) = I_k(t,n)$ where

$$I_k(t,n) = \sum_{(i_1,i_2,\dots,i_s)\in N(s)}\left|\left(\prod_{l=1}^{k}\mathbb{E}f^{t_l}\left(X\right)\right)\mathbb{E}\left(\prod_{l=k+1}^{s}\left(\overset{\circ}{f^{t_l}}\left(X_{i_l}\right)\right)\right)\right|. \tag{4.13}$$

Before proceeding further, observe that $I_k(t,n) = 0$, whenever $t_i = 1$ for some $i \le k$. Therefore we may assume that $t_i > 1$ for $i \le k$. Since $\|f\|_{r,s} \le (s/p)^{1/2}\|f\|_{r,p}$, and $\left\|f^k\right\|_{r,s} \le \|f\|_{2,s}\|f\|_{\infty}^{k-1}$, we get by (4.12) of the lemma

$$\begin{aligned}
I_k(t,n) &\le \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}\left(X\right)\right|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\left\|f^{t_l}\right\|_{r,s-k}\right)\right) &(4.14)\\
&\le \left(\sqrt{n}\|f\|_2\right)^{2k}\|f\|_{\infty}^{t_1+\dots t_k-2k}\left(\sqrt{n}\|f\|_{2,s-k}\right)^{s-k}\|f\|_{\infty}^{t_{k+1}+\dots+t_s-s+k}\\
&\le p^{-p+s}\left(\sqrt{n}\|f\|_{2,p}\right)^{s}\left(\max\left(\sqrt{n}\|f\|_{2,p}, p\|f\|_{\infty}\right)\right)^{p-s}. &(4.15)
\end{aligned}$$

Therefore

$$\begin{aligned}
M_p &:= \mathbb{E}\left(\sum_{i=1}^{n} f\left(X_i\right)\right)^p\\
&\le \sum_{t\in Q(p)} C(t)I(t,n)\\
&\le \sum_{t\in Q(p);|t|=1} C(t)I(t,n) + \sum_{s=2}^{p}\sum_{t\in Q(p);|t|=s} C(t)I(t,n)\\
&:= M_{1,p} + M_{2,p}.
\end{aligned}$$

Clearly

$$\begin{aligned} M_{1,p} &\leq n\mathbb{E}\,|f^p(X)| \\ &\leq \left(\sqrt{n}\,\|f\|_{2,p}\right)^2 (p\,\|f\|_\infty)^{p-2} \end{aligned} \tag{4.16}$$

From (4.15) we deduce that

$$\begin{aligned} M_{2,p} &\leq \sum_{s=2}^{p} \sum_{t\in Q(p);|t|=s} C(t)2^s \max_k I_k(t,n) \\ &\leq \sum_{s=2}^{p} \sum_{t\in Q(p);|t|=s} C(t)p^{-p+s} \left(2\sqrt{n}\,\|f\|_{2,p}\right)^s \left(\max\left(\sqrt{n}\,\|f\|_{2,p}, p\,\|f\|_\infty\right)\right)^{p-s}. \end{aligned}$$

But $\sum_{t\in Q(p);|t|=s} C(t) \leq (s!)^{-1}\, s^p$, It follows that

$$\begin{aligned} M_{2,p} &\leq \sum_{s=1}^{p} (s!)^{-1}\, s^p p^{-p+s} \left(2\sqrt{n}\,\|f\|_{2,p}\right)^s \left(\max\left(\sqrt{n}\,\|f\|_{2,p}, p\,\|f\|_\infty\right)\right)^{p-s} \\ &\leq \sum_{s=1}^{p} 2^p e^p \left(\sqrt{n}\,\|f\|_{2,p}\right)^s \left(\max\left(\sqrt{n}\,\|f\|_{2,p}, p\,\|f\|_\infty\right)\right)^{p-s} \\ &\leq (p-2)\,2^p e^p \left\{\left(\sqrt{n}\,\|f\|_{2,p}\right)^p + \left(\sqrt{n}\,\|f\|_{2,p}\right)^2 (p\,\|f\|_\infty)^{p-2}\right\} \end{aligned} \tag{4.17}$$

From (4.16) and (4.17), we obtain

$$M_p \leq p2^p e^p \left\{\left(\sqrt{n}\,\|f\|_{2,p}\right)^p + \left(\sqrt{n}\,\|f\|_{2,p}\right)^2 (p\,\|f\|_\infty)^{p-2}\right\}$$

And this concludes the proof of (ii).

Now we turn to the proof (iii).

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{n} f(X_i)\right)^p &= \sum_{t\in Q(p)} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right) \\ &= \sum_{t\in Q(p),|t|\leq p/2} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right) \\ &\quad + \sum_{t\in Q(p),|t|>p/2} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right) \\ &= \sum + \sum{}' . \end{aligned}$$

On the one hand, by Hölder's inequality

$$\begin{aligned} \sum &:= \sum_{t\in Q(p),|t|\leq p/2} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right) \\ &\leq n^{p/2}\mathbb{E}|f^p(X)| \end{aligned}$$

On the other hand the terms of the second sum are bounded by some $I_k(t,n)$. Firstly, note that $I_k(t,n) = 0$ as soon as $t_i = 1$ for some $i \le k$, since f is centered by hypothesis. Now we will argue on the number of i such that $t_i = 1$. Since $t_1 + \ldots + t_p = 1$ and $t_i > 0$, we infer that there is at least $(2s-p)$ t_i which take the value 1. According to what precede, we may assume that $i > k$, when $t_i = 1$. With this in mind and assuming for simplicity of notation that $t_{k+1} = \ldots. = t_{k+2s-p} = 1$, we apply Lemma 1 to get

$$\begin{aligned} I_k(t,n) &\le \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\left\|f^{t_l}\right\|_{r,s-k}\right)\right). \\ &\le \prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right| \quad \left(\sqrt{n}\left\|f\right\|_{r,s-k}\right)^{2p-s}\left(\prod_{l=k+2s-p+1}^{s}\left(\sqrt{n}\left\|f^{t_l}\right\|_{r,s-k}\right)\right). \end{aligned}$$

Recalling that

$$\begin{aligned} \|f\|_{r,p} &= \sum_{k=m}^{\infty}\frac{|c_k(f)|}{\sqrt{k!}}(p-1)^{k/2}\left(2\sum_{i=1}^{n}\left|r^k(i)\right|\right)^{1/2} \\ &\le \|f\|_2\sum_{k=m}^{\infty}(p-1)^{k/2}\left(2\sum_{i=1}^{n}\left|r^k(i)\right|\right)^{1/2} \\ &\le \|f\|_2\sum_{k=m}^{\infty}(p-1)^{k/2}r^*(1)^{k/2}\left(2\sum_{i=1}^{n}\left|r(i)/r^*(1)\right|^m\right)^{1/2}. \end{aligned}$$

Here m is the Hermite rank of f. Set $R_n^m = 2\sum_{i=1}^{n}|r(i)/r^*(1)|^m$. Hence

$$\begin{aligned} I_k(t,n) \le{}& \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right|\right)\left(\sqrt{n}\,\|f\|_2\sum_{k=m}^{\infty}(s-1)^{k/2}r^*(1)^{(k-m)/2}\left(R_n^m\right)^{1/2}\right)^{2s-p} \\ &\times\prod_{l=k+2s-p+1}^{s}\sqrt{n}\left\|f^{t_l}\right\|_2\sum_{k=m_l}^{\infty}(p-1)^{k/2}r^*(1)^{(k-m_l)/2}\left(2\sum_{i=1}^{n}\left|r^{m_l}(i)\right|\right)^{1/2} \end{aligned}$$

where m_l is the Hermite rank of f^{t_l}. Therefore, we get

$$\begin{aligned} I_k(t,n) &\le n^{k+s-p/2+s-k-2s+p}\mathbb{E}\left|f^p(X)\right|\left(\sum_{k=m}^{\infty}(\alpha(s,1))^{k/2}\left(R_n^m\right)^{1/2}\right)^{2s-p} \\ &\quad\times\prod_{l=k+2s-p+1}^{s}\left(\sum_{k=m_l}^{\infty}(\alpha(s,1))^{k/2}\left(n^{-1}R_n^1\right)^{1/2}\right) \\ &\le n^{p/2}\mathbb{E}\left|f^p(X)\right|\left(\sum_{k=m}^{\infty}(s-1)^{k/2}r^*(1)^{(k-m)/2}\left(\max\left(n^{-1}R_n^1;R_n^m\right)\right)^{1/2}\right)^{s-k} \end{aligned}$$

From the last inequality we infer that

$$\sum{}' = \sum_{t\in Q(p),|t|>p/2} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^{s} f^{t_l}(X_{i_l})\right)$$

$$\leq n^{p/2}\mathbb{E}\,|f^p(X)| \sum_{t\in Q(p),|t|>p/2} C(t)2^{|t|} \max_{0\leq k\leq s}\left(\sum_{k=m}^{\infty}(\alpha(s,1))^{k/2}\left(n^{-1}R_n^1 \vee R_n^m\right)^{1/2}\right)^{s-k}$$

$$\leq n^{p/2}\mathbb{E}\,|f^p(X)| \times \sum_{s>p/2}(s!)^{-1} s^p 2^s \left(1+\sum_{k=m}^{\infty}(\alpha(s,1))^{k/2}\left(n^{-1}R_n^1 \vee R_n^m\right)^{1/2}\right)^{s}$$

And this ends the proof of (iii).

Remark. In order to extend the previous results to real p. If p is not an even integer, we write $p = 2(1+s\theta)$ and we use the following inequality

$$\mathbb{E}Z^{2(1+s\theta)} \leq \left(\mathbb{E}Z^2\right)^{1-\theta}\left(\mathbb{E}Z^{2(s+1)}\right)^{\theta}, \tag{4.18}$$

which is true for any random variable Z and every θ in $[0,1]$.

The next theorem is an application of the proposition. The basis of the proof is to divide the sample of size n in to blocks in such a way that within the same block different indices are at least T-distant, where T is an integer to be chosen. Then Proposition 4.1 is applied to each block.

Theorem 4.1. *Let $p>2$, an even integer, and f be a real function. Assume that $\mathbb{E}f(X)=0$.*

(a) *For any $n\geq 1$, for every T satisfying $r^*(T)<(p-1)^{-1}$,*

$$\mathbb{E}\,|S_n(f)|^p \leq (Tp)^{p/2}\left(\sqrt{n}\,\|f\|_2\,(1+K(p,r,T))\right)^p + (Tp)^p(1+K(p,r,T))^2\, n\,\|f\|_2^2\,\|f\|_\infty^{p-2}$$

where

$$K(r,p,T) = \frac{(p-1)^{(m_p-1)/2}}{1-\sqrt{r^*(T)(p-1)}}\left(\sum_{i=1}^{[n/T]+1}|r(iT)|^{m_p}\right)^{1/2}.$$

m_p is the Hermite rank of $\{f, f^2, \ldots, f^{p-2}\}$.

(b) *If p is even and $r^{m_p}\in\mathbb{L}^1$, where m_p is the Hermite rank of $\{f, f^2, \ldots, f^{p-2}\}$, then there exists a constant $K=K(p,r)$ such that for all $n>0$,*

$$\mathbb{E}\,|S_n(f)|^p \leq K(p,r)\left(\left(\sqrt{n}\,\|f\|_2\right)^p + n\,\|f\|_p^p\right).$$

(c) *If p is even and $r^m \in \mathbb{L}^1$, where m is the Hermite rank of f, then there exists a constant $K = K(p,r)$ such that for all $n > 0$,*

$$\mathbb{E}\,|S_n(f)|^p \le K(p,r)\left(\sqrt{n}\,\|f\|_p^p\right)^p.$$

Proof. Let $T > 0$. Assume without loss of generality that $n = KT$.

$$S_N = S_n(f) = \sum_{l=1}^{T}\sum_{i=0}^{K-1} f\left(X_{iT+l}\right),$$

• Proof of (a): Applying Proposition 4.1 and the stationarity assumption, we obtain

$$\begin{aligned}\mathbb{E}\,|S_n(f)|^p &\le T^p\mathbb{E}\left|\sum_{i=0}^{K-1} f\left(X_{iT+1}\right)\right|^p \\ &\le T^p\left[\left(\sqrt{K}\,\|f\|_{2,p}\right)^p + p^pK\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right]\end{aligned}$$

where $\|f\|_{2,p}$ is given by

$$\|f\|_{2,p} = \max\left(p^{1/2}\,\|f\|_2\,,\ \sum_{k=m_p}^{\infty}\frac{|d_k(f)|}{\sqrt{k!}}(p-1)^{k/2}\left(2\sum_{i=1}^{[n/T]+1}|r^k(Ti)|\right)^{1/2}\right)$$

Therefore

$$\mathbb{E}\,|S_n(f)|^p \le T^{p/2}\left(\sqrt{TK}\,\|f\|_{2,p}\right)^p + (Tp)^p\,K\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2},$$

but

$$\begin{aligned}&\sum_{k=m}^{\infty}(p-1)^{k/2}\left(\sum_{i=1}^{K}|r^k(iT)|\right)^{1/2} \\ \le\ & (r^*(T))^{-m_p/2}\sum_{k=m}^{\infty}((p-1)r^*(T))^{k/2}\left(2\sum_{i=1}^{K}|r^{m_p}(iT)|\right)^{1/2} \\ \le\ & \frac{(p-1)^{m_p/2}}{1-((p-1)r^*(T))^{1/2}}\left(2\sum_{i=1}^{K}|r^{m_p}(iT)|\right)^{1/2}. \\ =\ & p^{1/2}K(p,r,T)\end{aligned}$$

hence

$$\begin{aligned}\mathbb{E}\,|S_n(f)|^p \le\ & (Tp)^{p/2}\left(\sqrt{n}\,\|f\|_2\,(1+K(p,r,T))\right)^p \\ & + (Tp)^p\,(1+K(p,r,T))^2\,n\,\|f\|_2^2\,\|f\|_\infty^{p-2}\end{aligned}$$

and this conclude the proof of the first relation.

• Proof of (b): After the proof of (i) of Proposition 4.1 at the relation (4.14), then we write

$$I_k(t,n) \le \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\left\|f^{t_l}\right\|_{r,s-k}\right)\right)$$

But

$$\begin{aligned} \left|c_k(f^t)\right| &= \left|\mathbb{E}f^t(X)H_k(X)\right| \\ &\le \|f\|_2\,\|f\|_p^{t-1}\,\|H_k\|_{2p/p-t-1} \\ &\le \|f\|_2\,\|f\|_p^{t-1}\,\|H_k\|_{2p} \\ &\le \|f\|_2\,\|f\|_p^{t-1}\,\sqrt{k!}\,(2p)^{k/2} \end{aligned} \tag{4.19}$$

where in the last inequality we have used relation (2.29) in the appendix. Applying (4.18) with $\theta = t - 2/(p-2)$, $s = p/2 - 1$ we get

$$n\left|\mathbb{E}f^t(X)\right| \le \left(\sqrt{n}\,\|f\|_2\right)^{2(1-\theta)}\left(n^{1/p}\,\|f\|_p\right)^{p\theta}. \tag{4.20}$$

From (4.19) and (4.20) we infer

$$\begin{aligned} I_k(t,n) &\le \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\left\|f^{t_l}\right\|_{r,s-k}\right)\right) \\ &\le \left(\prod_{l=1}^{k}\left(\max\left(\sqrt{n}\,\|f\|_2\,;n^{1/p}\,\|f\|_p\right)\right)^{t_l}\right) \\ &\quad\times \prod_{l=k+1}^{s}\sqrt{n}\,\|f\|_2\,\|f\|_p^{t_l-1}\sum_{k=m_p}^{\infty}\left(2p^2r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|r(i)/r^*(1)\right|^m\right)^{1/2} \\ &\le \left(\max\left(\sqrt{n}\,\|f\|_2\,;n^{1/p}\,\|f\|_p\right)\right)^{t_1+\dots+t_s} \\ &\quad\times\left(\sum_{k=m_p}^{\infty}\left(2p^2r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|r(i)/r^*(1)\right|^m\right)^{1/2}\right)^{s-k}. \end{aligned}$$

Hence (b) is proved if $2p^2r^*(1) < 1$. In the general case we proceed as in (a).

• Proof of (c): This follows from (iii) of Proposition 4.1 in a routine fashion.

Maxir-al inequalities

From the results of the previous section we can now prove maximal inequalities for partial sums. These inequalities are useful for proving almost sure properties. The idea goes back to Serfling 1970, and maximal inequalities can be deduced by combining the results of section 2 and the Serfling maximal identity. However, some time sharper results are needed. This is the subject of the following theorem. Before stating the result we introduce the following notation

$$S_n = \sum_{i=1}^{n} f(X_i), \; M(m,N) = \max_{m<k\leq N} \left| \sum_{i=m+1}^{k} f(X_i) \right| \text{ and } M_N = \max_{k\leq N} |S_k| .$$

We will prove

Theorem 4.2. *Let $p > 2$ even, $\varepsilon > 0$ and f be a real centered function.*

(i) *There exists a constant $K = K(p,r,\varepsilon)$ such that for all measurable functions and every $N \geq 1$ we have:*

$$\mathbb{E}|M_N|^p \leq K(p,r,\varepsilon)\left[\left(\sqrt{N}\,\|f\|_{2,p}\right)^p + N^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right]. \quad (4.21)$$

(ii) *If r^{m_p} is summable where m_p is the Hermite rank of $\{f, f^2, ..., f^{p-2}\}$. Then there exists $K = K(p,r,\varepsilon)$ such that*

$$\mathbb{E}|M_N|^p \leq K(p,r,\varepsilon)\left[\left(\sqrt{N}\,\|f\|_2\right)^p + N^{1+\varepsilon}\,\|f\|_p^p\right] \quad (4.22)$$

(iii) *If r^m is summable where m is the Hermite rank of f. Then there exists $K = K(p,r)$ such that*

$$\mathbb{E}|M_N|^p \leq K(p,r)\left(\sqrt{N}\,\|f\|_p\right)^p \quad (4.23)$$

Proof. For $0 < m < N$ and $m < n \leq N$, we write:

$$\begin{aligned} |S_n|^p &\leq |S_m + M(m,N)|^p \\ &\leq |S_m|^p + |M(m,N)|^p + \sum_{k+l=p,k,l\neq 0} C_{k,l}\,|S_m|^k\,|M(m,N)|^l . \end{aligned}$$

Since for $n \leq m$ we have $|S_m|^p \leq |M_m|^p$, we conclude that

$$|M_N|^p \leq |M_m|^p + |M(m,N)|^p + \sum_{k+l=p,k,l\neq 0} C_{k,l}\,|S_m|^k\,|M(m,N)|^l .$$

By the stationarity of the underlying sequence we obtain by the mean of Hölder's inequality

$$\mathbb{E}|M_N|^p \leq \mathbb{E}|M_m|^p + \mathbb{E}|M_{N-m}|^p + \sum_{k+l=p,k,l\neq 0} C_{k,l}\left(\mathbb{E}|S_m|^p\right)^{k/p}\left(\mathbb{E}|M_{N-m}|^p\right)^{l/p} .$$

Assuming that for every $\varepsilon > 0$, there exists a constant K such that for $n < N$

$$\mathbb{E}\,|M_n|^p \leq K^p \left[\left(\sqrt{n}\,\|f\|_{2,p}\right)^p + n^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right]$$

we shall prove that the relation remains true for N. Applying the induction hypothesis to $\mathbb{E}\,|M_m|^p\,,\mathbb{E}\,|M_{N-m}|^p$ and using Theorem 1 we get

$$\begin{aligned}\mathbb{E}\,|M_N|^p \;\leq\; & K^p \left[\left(\sqrt{m}\,\|f\|_{2,p}\right)^p + m^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right] \\ & +K^p \left[\left(\sqrt{N-m}\,\|f\|_{2,p}\right)^p + (N-m)^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right] \\ & +K^{p-1}\sum_{k+l=p} C_{k,l}\left[\left(\sqrt{m}\,\|f\|_{2,p}\right)^p + m\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right]^{k/p} \\ & \left[\left(\sqrt{N-m}\,\|f\|_{2,p}\right)^p + (N-m)^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right]^{l/p}.\end{aligned}$$

therefore

$$\begin{aligned}\mathbb{E}\,|M_N|^p \;\leq\; & K^p \left[\left(\sqrt{m}\,\|f\|_{2,p}\right)^p + m^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right] \\ & +K^p \left[\left(\sqrt{N-m}\,\|f\|_{2,p}\right)^p + (N-m)^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\right] \\ & +2^p K^{p-1}\left(\sqrt{m}\,\|f\|_{2,p}\right)^p + m\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2} \\ & +2^p K^{p-1}\left(\sqrt{N-m}\,\|f\|_{2,p}\right)^p + (N-m)^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\end{aligned}$$

Let $m = N/2$

$$\begin{aligned}\mathbb{E}\,|M_N|^p \;\leq\; & \left(\sqrt{N}\,\|f\|_{2,p}\right)^p\left[K^p\left(2^{1-p}+2K^{-1}\right)\right] \\ & +N^{1+\varepsilon}\,\|f\|_{2,p}^2\,\|f\|_\infty^{p-2}\left[K^p\left(2^{-\varepsilon}+2^{p-\varepsilon}K^{-1}\right)\right]\end{aligned}$$

Choosing K such that $2^{1-p}+2K^{-1} \leq 1, 2^{-\varepsilon}+2^{p-\varepsilon}K^{-1} \leq 1$, ends the proof of (a).

Comment. Comparing the inequalities stated in section 2 to those stated in section 3, we note that there is a loss of n^ε. We think that this is due to the method of proof.

Applications

Several applications can be found in the literature see for example Serfling 1970, Utev 1991, Shao 1995, and the references therein. Here, we will focus on some applications.

Almost sure convergence

By the mean of Theorems 1, 2 one can obtain the following results which are concerned about rate of convergence in the SLLN, when the observations are drawn from an instantaneous filter of some Gaussian sequences. It is worth noticing that the result may hold even when the sequence is not mixing.

Corollary 4.1. *Let X_n be a Gaussian sequence of a real random variable and f be a real centered function. Assume that $f \in \mathbb{L}^{p+\eta}$ for $1 \leq p < 2, \eta > 0$ and $r \in \mathbb{L}^1$. Then*

$$\forall \varepsilon > 0, \quad \sum_{i=1}^{\infty} n^{-1}\mathbb{P}\left(M_n > \varepsilon n^{1/p}\right) < \infty.$$

In particular, we have

$$\lim_{n \to \infty} n^{-1/p} S_n(f) = 0.$$

Proof. Let $S_n^*(f) = \max_{k \leq n} \left| S_k(\mathring{f}) \right|$ .Then

$$\begin{aligned} \mathbb{P}\left(M_n > \varepsilon n^{1/p}\right) &\leq \mathbb{P}\left(S_n^*\left(f 1_{|f| \leq M}\right) > \varepsilon/2n^{1/p}\right) \\ &\quad + \mathbb{P}\left(S_n^*\left(f 1_{|f| > M}\right) > \varepsilon/2n^{1/p}\right) \end{aligned}$$

Let $M = n^{1/p}$,

$$\mathbb{P}\left(M_n\left(f 1_{|f|>M}\right) > \varepsilon/2n^{1/p}\right) \leq 4\varepsilon^{-1} n^{-1/p} n \mathbb{E}\left|f 1_{|f|>M}\right|$$

Therefore, $\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(M_n\left(f 1_{|f|>M}\right) > \varepsilon/2n^{1/p}\right) < \infty$. as soon as $f \in \mathbb{L}^p$. Applying Theorem 1 to $f 1_{|f| \leq M}$ yields

$$\begin{aligned} \mathbb{E}\left(S_n^*\left(f 1_{|f| \leq M}\right)\right)^4 &\leq K\left(\sqrt{n}\left\|f 1_{|f| \leq M}\right\|_2\right)^4 + \left(\sqrt{n}\left\|f 1_{|f| \leq M}\right\|_2\right)^2 M^2 n^{\delta} \\ &\leq K\left(n\mathbb{E} f^2 1_{|f| \leq M}\right)^2 + n^{1+\delta}\mathbb{E}\left(f^2 1_{|f| \leq M} M^2\right) \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(M_n\left(f1_{|f|\leq M}\right) > \varepsilon/2n^{1/p}\right)$$

$$\leq \quad (2/\varepsilon)^4 K \sum_{n=1}^{\infty} n^{-1-4/p}\left[\left(n^{1+\delta}\mathbb{E}f^2 1_{|f|\leq M}\right)^2 + n\mathbb{E}f^2 1_{|f|\leq M}M^2\right]$$

$$\leq \quad (2/\varepsilon)^4 K \sum_{k=1}^{\infty}\left(\mathbb{E}f^2 1_{k\leq|f|^p\leq k+1}\right)^2 \sum_{n\geq k} n^{-1-4/p+2+\delta}$$

$$+ (2/\varepsilon)^4 K \sum_{k=1}^{\infty} \mathbb{E}f^2 1_{k\leq|f|^p\leq k+1} \sum_{n\geq k} n^{-2/p}$$

$$\leq \quad (2/\varepsilon)^4 K \sum_{k=1}^{\infty}\left(k\mathbb{E}1_{k\leq|f|^p\leq k+1}\right)^2 + + (2/\varepsilon)^4 K \sum_{k=1}^{\infty} \mathbb{E}1_{k\leq|f|^p\leq k+1}k^{1+\delta}$$

where K is constant independent of n but depends on f , δ and r. Since δ can be choosen arbitrary small, the two series are convergent under the hypothesis $f \in \mathbb{L}^{p+\eta}$.

The functional CLT

For $t \in [0,1]$, let $S_n(f,t) = \sum_{i=1}^{[nt]} f(X_i)$. The following corollary is an easy consequence of the theorems stated above. We need only to prove tightness, which follows by the maximal inequalities of section 3 via Theorem 12.8 in Billingsley 1968.

Corollary 4.2. *Let X_n be a stationary Gaussian sequence of a real random variable, and f be a real function. Assume that $f \in \mathbb{L}^p$ for some $p > 2$, and $r \in \mathbb{L}^1$. Then*

$$\{S_n(f,t), t \in [0,1]\}$$

converge in distribution to the process

$$\{\sigma(f)W_t, t \in [0,1]\}$$

where W is a standard Brownian motion and

$$\sigma^2(f) = \sum_{i\in\mathbb{Z}} \mathrm{Cov}(f(X_0); f(X_{|i|}))$$

Remark. We think that it is possible to reduce the moment condition to $p = 2$, but in this case the proof would be too lengthy. We also recall that Csörgő and Mielnicuck proved the tightness under the moment condition $f \in \mathbb{L}^4$.

High order asymptotic for the empirical distribution function.

In this paragraph we will give a higher order asymptotic for the empirical distribution function under Gaussian subordination. Similar results for linear processes can be found in Koul and Surgailis 1997 . Precisely, let $Y_k(\cdot)$ denotes the Hermite process of order k which is defined by

$$Y_k(t) = \int_{R^k} \left[\int_0^t \prod_{i=1}^{k} (v - u_i)_+^{-\frac{\alpha+1}{2}} dv \right] W(du_1)...W(du_k) \tag{4.24}$$

where $v_+ = v \vee 0$ and W is a standard Brownian motion. For G measurable and $x \in \mathbb{R}$ let

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} I_{\{G(X_j) \leq x\}},$$

and define $m(x) = rank(I_{\{G(\cdot)\leq x\}})$, $m = \inf_{x\in\mathbb{R}} m(x)$ and $F(x) = \mathbb{P}\left(G(X_j) \leq x\right)$, then

$$\begin{aligned} F_n(x) - F(x) &= \frac{1}{n}\sum_{j=1}^{n}(I_{\{G(X_j)\leq x\}} - F(x)) \\ &= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=m}^{\infty}\frac{j_k(x)}{k!}H_k(X_j)) \\ &= \sum_{k=m}^{\infty}\frac{j_k(x)}{k!}\frac{1}{n}\sum_{j=1}^{n}H_k(X_j) \end{aligned}$$

where $j_k(x)) = E\left(I_{\{G(X_j)\leq x\}}H_k(X_j)\right)$. Hence, for any $k^* > m$

$$F_n(x) = F(x) + \sum_{k=m}^{k^*-1}\frac{j_k(x)}{k!}n^{-1}\sum_{j=1}^{n}H_k(Z_j) + R_n(x)$$

with

$$R_n(x) := \sum_{k=k^*}^{\infty}\frac{j_k(x)}{k!}\frac{1}{n}\sum_{j=1}^{n}H_k(Z_j). \tag{4.25}$$

Applying the previous results, one obtains the following high order asymptotic for the empirical distribution function.

Proposition 4.2. *Let X_n be a stationary Gaussian sequence and G some Borel function.*

(1) *Assume that F is continuous and r^{k^*} is integrable. Then*

$$\sqrt{n}R_n(\cdot) \to R(\cdot) \quad \text{in } C(\mathbb{R},\mathbb{R})$$

endowed with the uniform topology. R is a continuous Gaussian process with covariance function given by:

$$R(x,y) = \sum_{k=k^*}^{\infty} \frac{j_k(x) j_k(y)}{k!} \sum_j r^k(j)$$

(2) *Assume that $r(n) \sim n^{-\alpha} L(n)$ for some $\alpha > 0$ and some slowly varying function L. Then for every k such that $0 < km < 1$ we have*

$$\frac{1}{d_{n,k}} \frac{j_k(x)}{k!} \sum_{j=1}^{n} H_k(X_j) \Longrightarrow \frac{j_k(x)}{k!} Y_k(1) \qquad in\ C(\mathbb{R}, \mathbb{R})$$

where $d^2_{n,k} = \mathbb{E}\left(\sum_{j=1}^n H_k(X_j)\right)^2$ and $Y_k(\cdot)$ denotes the Hermite process of order k.

Proof. We have only to prove (1), since (2) is proved in Taqqu 1977 or Dubroshin and Major 1977. The finite dimensional convergence follows from [4]. To prove tightness we apply Lemma 1 to obtain

$$\mathbb{E}(\sqrt{n}(R_n(x) - R_n(y)))^4 \leq K(F(x) - F(y))^2 + n^{-1}|F(x) - F(y)|.$$

(See Lemma 5.3 below). The last relation is sufficient to prove tightness with usual chaining arguments.

Appendix

In this part we prove Lemma 1. Before doing this, we recall the diagram formula and some related notions in the following.

The diagram technique

Let $k_1, k_2, ..., k_p$ denote integers, V a set of points of cardinal $k_1 + k_2 + ... + k_p$. An undirected graph of type $\Gamma(k_1, k_2, ..., k_p)$ is an element of $G(V)$ satisfying:

i) V is the union of disjoint p levels with respective cardinals $k_1, k_2, ..., k_p$

$$V = \cup_{i=1}^p L_i, \qquad L_i = \{(i,l);\ l = 1, 2..., k_i\}$$

ii) Only edges between different levels are allowed

$$w = ((i,l),(i',l')) \Rightarrow i \neq i'$$

iii) Every point has exactly one edge

$$\forall (i,l) \in V,\ \exists!\ (i',l')\ /\ ((i,l);(i',l')) \in G(V)$$

For $w = ((i,l),(i',l'))$ in $G(V)$ we define $n_1(w) \equiv i \vee i'$ as the first level of w and $n_2(w) \equiv i \wedge i'$ as the second one.

Lemma 4.2 (DIAGRAM FORMULA). *Let $(X_{s_1}, X_{s_2}, ..., X_{s_p})$ be a Gaussian vector centered and with a covariance matrix given by $(r(s_i,s_j))_{1\leq i,j\leq p}$. Then*

$$\mathbb{E}\left[\prod_{i=1}^{p} H_k(X_{s_i})\right] = \sum_{G\in\Gamma(k_1,...,k_p)} \prod_{w\in G} r(s_{n_1(w)}, s_{n_2(w)})$$

where $n_1(w)$, $n_2(w)$ are the first and the second level of w.

The proof can be found in Giraitis and Surgailis [9] in a more general setting.

Proof of Lemma 4.1

To prove the lemma, we need two inequalities stated in what follows. The formula (2.29) is proved by Taqqu in [14]. The second is proved below. We write $\mathbf{k}$ for $(k_1, k_2, ..., k_p)$. For G element of $\Gamma(k_1,k_2,...,k_p)$ we introduce the following notations: $k_G(i)$ is the number of edges going from level i and $g(i) = \frac{k_G(i)}{k_i}$ and $I(G,\mathbf{k},n)$ is defined by:

$$I(G,\mathbf{k},n) = \sum_{i\in N(p)} \prod_{w\in G} \left| r(i_{n_1(w)}, i_{n_2(w)}) \right|, \tag{4.26}$$

where $N(p) = \{i = (i_1, i_2, ..., i_p),\ 1 \leq i_k \leq n;\ i_k \neq i_l\ if\ k \neq l\}$.

Lemma 4.3. *i) Let X be a standard Gaussian random variable. Then*

$$\mathbb{E}\left(|H_{k_1}(X)...H_{k_p}(X)|\right) \leq \prod_{i=1}^{p}(p-1)^{k_i/2}\sqrt{k_i!}. \tag{4.27}$$

ii) If $G \in \Gamma(k_1,...,k_p)$ then

$$I^2(G,\mathbf{k},n) \leq n^p \prod_{l=1}^{p} 2\sum_{i=1}^{n} |r^{k_l}(i)|. \tag{4.28}$$

Now let us write

$$\sum_{i\in N(p)}\left|\mathbb{E}\left(\prod_{l=1}^{p} f_l\left(X_{i_l}\right)\right)\right|$$
$$\leq \sum_{k_1,k_2,\ldots,k_p=1,k_l\geq m_l}^{\infty}\prod_{l=1}^{p}\cdot\frac{\left|c_{k_l(f_l)}\right|}{k_l!}\sum_{i\in N(p)}\left|\mathbb{E}\left(H_{k_1}(X_{i_1})\ldots H_{k_p}(X_{i_p})\right)\right|$$
$$\leq \sum_{k_1,k_2,\ldots,k_p=1,k_l\geq m_l}^{\infty}\prod_{l=1}^{p}\cdot\frac{\left|c_{k_l(f_l)}\right|}{k_l!}\sum_{G\in\Gamma(k_1,k_2,\ldots,k_p)} I(G,\mathbf{k},n).$$

By (4.27) and (4.28) we conclude that:

$$\sum_{i\in N(p)}\left|\mathbb{E}\left(\prod_{l=1}^{p} f\left(X_{i_l}\right)\right)\right|$$
$$\leq \sum_{k_1,k_2,\ldots,k_p=1,k_l\geq m_l}^{\infty}\prod_{l=1}^{p}\cdot\frac{\left|c_{k_l(f_l)}\right|}{k_l!}\left|\Gamma(k_1,k_2,\ldots,k_p)\right|\sup_{G\in\Gamma(k_1,k_2,\ldots,k_p)} I(G,\mathbf{k},n).$$
$$\leq \sum_{k_1,k_2,\ldots,k_p=1 k_l\geq m_l}^{\infty}\prod_{l=1}^{p}\frac{\left|c_{k_l(f_l)}\right|}{\sqrt{k_l!}}(p-1)^{k_l/2}\left(2n\sum_{i=1}^{n}\left|r^{k_l}(i)\right|\right)^{1/2}.$$

Proof of Lemma 4.3. We assume that $k_1 \leq k_2 \leq .. \leq k_p$. Moreover, without loss of generality assume that edges go from lower levels to higher ones, (by the symmetry of the covariance function). Hence

$$\begin{aligned}
I(G,\mathbf{k},n) &= \sum_{i_1=1}^{n}\cdots\sum_{i_p=1}^{n}\prod_{w\in G}\left|r(i_{n_1(w)},i_{n_2(w)})\right| \\
&= \sum_{i_1=1}^{n}\cdots\sum_{i_p=1}^{n}\prod_{l=1}^{p}\prod_{\{w\in G,n_1(w)=l\}}\left|r(i_l,i_{n_2(w)})\right| \\
&= \sum_{i_1=1}^{n}\cdots\sum_{i_p=1}^{n}\prod_{\{w\in G,n_1(w)=1\}}\left|r(i_1,i_{n_2(w)})\right| \\
&\qquad\times\prod_{l=2}^{p}\prod_{\{w\in G,n_1(w)=l\}}\left|r(i_l,i_{n_2(w)})\right|.
\end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned} I(G,\mathbf{k},n) &\le \sum_{i_1=1}^{n} \prod_{\{w\in G, n_1(w)=1\}} |r(i_1, i_{n_2(w)})| \\ &\qquad \times \sum_{i_2=1}^{n} .. \sum_{i_p=1}^{n} \prod_{l=2}^{p} \prod_{\{w\in G, n_1(w)=l\}} |r(i_l, i_{n_2(w)})| \\ &\le 2\sum_{i_1=1}^{n} |r(i_1)|^{k_G(1)} \sum_{i_2=1}^{n} .. \sum_{i_p=1}^{n} \prod_{l=2}^{p} \prod_{\{w\in G, n_1(w)=l\}} |r(i_l, i_{n_2(w)})|. \end{aligned}$$

Repeating the same for $i_2, ..., i_p$ we get

$$|I(G,\mathbf{k},n)| \le \prod_{l=1}^{p} 2 \sum_{i=1}^{n} |r(i)|^{k_G(l)}. \tag{4.29}$$

In order to prove the inequality (4.28), for any graph G we write the two symmetric formulas

$$\begin{aligned} |I(G,\mathbf{k},n)| &\le \prod_{l=1}^{p} 2 \sum_{i=1}^{n} |r(i)|^{k_G(l)} \\ |I(G,\mathbf{k},n)| &\le \prod_{l=1}^{p} 2 \sum_{i=1}^{n} |r(i)|^{k'_G(l)}, \end{aligned}$$

where $k_G(i) + k'_G(i) = k_i$. The first relation is (4.29). For the second relation assume that edges go from high levels to lower ones, proceed as in (4.29) considering $n_2(w)$ instead of $n_1(w)$ and begin by integrating out i_p instead of i_1. Hence by Hölder's inequality

$$\begin{aligned} I^2(G,\mathbf{k},n) &\le \prod_{l=1}^{p} \sum_{i=1}^{n} \sum_{i'=1}^{n} |r(i)|^{k_G(l)} |r(i')|^{k'_G(l)} \\ &\le \prod_{i=1}^{p} \left(n \sum_{i=1}^{n} |r(i)|^{k_l} \right)^{\frac{k_G(l)}{k_l}} \left(n \sum_{i=1}^{n} |r(i)|^{k_l} \right)^{\frac{k'_G(l)}{k_l}} \\ &\le \prod_{l=1}^{p} 2n \sum_{i=1}^{n} |r^{k_l}(i)|. \end{aligned}$$

and this concludes the proof of the second display of the lemma.

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Chapitre 5

Density estimation for short and long-range dependent sequences

ABSTRACT. Let $Y_n = G(X_n)$ where X is a stationary gaussian process and G is some borel function. Assume that Y has a marginal density f which is to be estimated. We investigate the asymptotic properties of the density process based on a kernel estimator of the marginal density f. In particular, We study its variance and we establish limit theorems for the density process.

The work is motivated by previous results of Hall and Hart 1990, Csörgő and Mielniczuk 1995. We shall see that the asymptotic behavior depends on the decay of the correlation function and the bandwidth. Roughly speaking, if they decrease sufficiently fast to zero our estimator behaves as in the i.i.d case. Otherwise, the limit will be similar to the long memory case.

Key words: Density estimation, Gaussian subordination , asymptotic normality. Short and long memory processes.

Introduction

Nonparametric density estimation was thoroughly studied during the last years. (see for example Silverman 1986, Bosq 1996, Roussas 1990 and the reference therein.) Much more attention was devoted to the kernel density estimator. This will be the main subject of this work. Precisely, let $Y_1, Y_2, \ldots . Y_n$, be observations from a stationary process. Assume Y has a marginal density f which is to be estimated. In 1956, Rosenblatt introduced the kernel estimator defined by:

$$f_n(x) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{x - Y_i}{b_n}\right) \tag{5.1}$$

K is a kernel, satisfying some conditions that will be specified later; b_n is the bandwidth. It is quite reasonable to look for Criteria for good estimator such as its variance, MSE, MISE, Strong consistency, rate of convergence... .

When the data is i.i.d, the problem is well studied, however the independence hypothesis is in many cases not satisfied by the observations. A great interest last years was given to dependent observations. As a consequence, many results exist under weak dependence conditions. All these results have more or less the following form: looking for minimal hypothesis on mixing conditions ensuring similar behavior to the i.i.d case. However, experience has indicated that correlation of many empirical series does not converge slowly enough to zero in order to satisfy short mixing conditions. For instance, if X is gaussian with $r(n) = \mathbb{E}(X_0 X_n) \sim n^{-\alpha}$, $\alpha < 1$, then is not mixing. Conditions on linear process to be mixing or long range dependent are given in Deniau, Oppenheim and Viano 95. Such sequences naturally appear in econometric, hydrology, Traffic communication ... . For more about the practical aspect of long range dependence, we refer to Beran 1992. Hence, a natural question arises: *what happens if the sequence is not weakly dependent enough?*

Unfortunately, the question can not be solved in its great generality, and some restrictions have to be made. We first specify our model.

The model

From now on, assume that the observations are drawn from a stationary process:

$$Y_n = G(X_n) \quad , n = 0, 1, 2..., \tag{5.2}$$

where X is a stationary gaussian process and G is some measurable function. Note that the model is quite general, in particular any distribution function can be written as a function of a normal random variable. Suppose F, the distribution function of $G(X)$, has a Lebesgue density f which is to be estimated from the observations $G(X_1), ..., G(X_n)$ using the kernel estimator defined by

$$f_n(x) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{x - G(X_i)}{b_n}\right)$$

where K is real function and b_n is the bandwidth. Let D_n stands for the density process defined by

$$D_n(x) = f_n(x) - \mathbb{E}(f_n(x)). \tag{5.3}$$

Noting that $D_n(x)$ can be written as a functional of the empirical distribution function

$$D_n(x) = \frac{1}{b_n} \int K(\frac{x-y}{b_n}) d\{F_n(y) - F(y)\}$$

where $F_n(y) = n^{-1} \sum_{j=1}^{n} I_{\{G(X_j) \leq y\}}$ and $F(y)$ is the distribution function of $G(X)$. Then the behavior of the density process is heavily related to that of the empirical process. First, we recall the following result proved by Dehling and Taqqu. Assume that $r(n) = n^{-\alpha} L(n), \alpha > 0$; and L is a slowly varying function. If $\alpha m < 1$, where $m = \inf m(x)$ and $m(x)$ is the Hermite rank of $1_{G(\cdot)<x}$, then

$$\frac{n}{d_{n,m}} \left(F_n(\cdot) - F(\cdot)\right) \Longrightarrow \frac{j_m(\cdot)}{m!} Y_m(1) \qquad in \quad \mathcal{D}[-\infty, +\infty]$$

where $d_{n,m}^2 = var(\sum_{i=1}^{n} H_m(X_i)) \sim (2m!)^{-1} (1 - m\alpha)(2 - m\alpha) n^{2-m\alpha} L^m(n)$.

Csörgő and Mielniczuk 1995 proved a kind of derivative result for the density process. Namely, if $\alpha m < 1$, and the bandwidth is sufficiently large. Then

$$\frac{n}{d_{n,m}} D_n(x) \Longrightarrow \frac{j'_m(\cdot)}{m!} Y_m(1) \qquad in \quad \mathcal{C}[-\infty, +\infty]$$

On the other hand, Ho 1995 considered the question of rate of convergence for different decays of the correlation function and different bandwidth. However, his hypothesis on the functional G and the underlying process are rather restrictive, (he assumed that the joint density of any finite vector exists and is uniformly bounded).

We aim at complementing these results with an emphasis on the influence of the dependence on the asymptotic behavior of the density process.

Hypothesis

We first set some conditions which are assumed through the paper.

(A) $b_n \to 0$ and $nb_n \to +\infty$

(B) $K(s) = 0$ for $s \notin (-A, A)$, K is derivable.

(C) The density f exists and is continuous.

(D) $m = \inf m(x)$ where $m(x)$ is the Hermite rank of $1_{G(\cdot)<x}$.

(E) If r^m is not summable, then we assume that

$$r(n) = n^{-\alpha} L(s)$$

where $\alpha > 0$, and L is a slowly varying function.

Related results

Firstly note that

$$D_n(x) = \frac{1}{b_n} \int K(\frac{x-y}{b_n}) d\{F_n(y) - F(y)\}$$

where $F_n(y) = \frac{1}{n}\sum_{j=1}^{n} I_{\{Y_j \le y\}}$ and $F(y)$ is the distribution function of Y: We recall some results about the empirical process

$$E_n(x) = F_n(x) - F(x).$$

The question was solved by Dehling and Taqqu when the process is a long memory one and by Csörgő and Mielniczuk in the short memory case. More generally we have proved the following high order asymptotic for the empirical distribution function. Let $Y_k(\cdot)$ denotes the Hermite process of order k which is defined by

$$Y_k(t) = \int_{\mathbb{R}^k} \left[\int_0^t \prod_{i=1}^{k} (v - u_i)_+^{-\frac{\alpha+1}{2}} dv \right] W(du_1)...W(du_k) \qquad (5.4)$$

where $v_+ = v \vee 0$ and W is a standard Brownian motion. For G measurable and $x \in \mathbb{R}$ let $m(x) = rank(I_{\{G(\cdot)\leq x\}})$, $m = \inf_{x\in\mathbb{R}} m(x)$ and $F(x) = \mathbb{P}\left(G(X_j) \leq x\right)$, then

$$\begin{aligned} F_n(x) - F(x) &= \frac{1}{n}\sum_{j=1}^{n}(I_{\{G(X_j)\leq x\}} - F(x)) \\ &= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=m}^{\infty}\frac{j_k(x)}{k!}H_k(X_j) \\ &= \sum_{k=m}^{\infty}\frac{j_k(x)}{k!}\frac{1}{n}\sum_{j=1}^{n}H_k(X_j) \end{aligned}$$

where $j_k(x) = E\left(I_{\{G(X_j)\leq x\}}H_k(X_j)\right)$. Hence, for any $k^* > m$

$$F_n(x) = F(x) + \sum_{k=m}^{k^*-1}\frac{j_k(x)}{k!}n^{-1}\sum_{j=1}^{n}H_k(Z_j) + R_n(x)$$

with

$$R_n(x) := \sum_{k=k^*}^{\infty}\frac{j_k(x)}{k!}\frac{1}{n}\sum_{j=1}^{n}H_k(Z_j). \qquad (5.5)$$

In the previous chapter, we proved

Proposition 5.1. *Let X_n be a stationary Gaussian sequence and G some Borel function.*

(1) *Assume that F is continuous and r^{k^*} is integrable. Then*

$$\sqrt{n}R_n(\cdot) \Longrightarrow R(\cdot) \qquad in\ \mathcal{D}(\mathbb{R},\mathbb{R})$$

endowed with the uniform topology. R is a continuous Gaussian process with covariance function given by:

$$R(x,y) = \sum_{k=k^*}^{\infty}\frac{j_k(x)j_k(y)}{k!}\sum_{j\in\mathbb{Z}}r^k(|j|)$$

(2) *Assume that $r(n) \sim n^{-\alpha}L(n)$ for some $\alpha > 0$ and some slowly varying function L . Then for every k such that $0 < km < 1$ we have*

$$\frac{j_k(x)}{d_{n,k}k!}\sum_{j=1}^{n} H_k(Z_j) \Longrightarrow \frac{j_k(x)}{k!}Y_k(1) \quad \text{in } \mathcal{D}(\mathbb{R},\mathbb{R})$$

where $Y_k(\cdot)$ denotes the Hermite process of order k.

The Density process

In what follows we will study the asymptotic properties of the density process $D_n(x)$. Let $f_n^0(x)$ denotes the estimator based on independent observations with the same law as $G(X_0)$.

The variance

In order to understand the asymptotic order of the density process, we begin by an evaluating of its variance. This is formulated in the following lemma.

Lemma 5.1. **(a)** *Assume that $r^{m^*} \in \mathbb{L}^1$ for some $m^* > 1$, then:*

$$var\left[f_n(x))\right] = var\left[f_n^0(x))\right] + \sum_{k=m}^{m*-1} \frac{C_{k,n}^2(x)}{k!}\frac{2}{n^2}\sum_{i=1}^{n}(n-i)\, r^k(i) + o\left(\frac{1}{nb_n}\right)$$

(b) *In particular under (E)*

$$var\left[f_n(x)\right] = var\left[f_n^0(x))\right] + \frac{C_{m,n}^2(x)}{m!}\frac{2}{n^2}\sum_{i=1}^{n}(n-i)\, r^m(i) + o\left(\frac{1}{nb_n} + \frac{1}{n}\sum_{i=1}^{n}|r^m(i)|\right)$$

(c) *If moreover, we assume that $j_m'(x)$ exists and is continuous, then $C_{m,n}(x) \to j_m'(x)$ and*

$$var\left[f_n(x)\right] = var\left[f_n^0(x))\right] + \left(j_m'(x)\right)^2 \frac{d_{n,m}^2}{(m!n)^2} + o\left(\frac{1}{nb_n} + \frac{1}{n}\sum_{i=1}^{n}|r^m(i)|\right)$$

As a consequence

- If $r^m \in \mathbb{L}^1$, then $var\,[f_n(x))] \sim var\,[f_n^0(x))]$.
- If $m\alpha < 1$ and $n^\beta b_n \to 0$ for $\beta > 1 - m\alpha$, then $var\,[f_n(x)] \sim var\,[f_n^0(x))]$
- If $m\alpha < 1$ and $n^\beta b_n \to \infty$ for $\beta < 1 - m\alpha$, then

$$var\,[f_n(x))] \sim var\left[\frac{1}{n}\sum_{i=1}^{n} H_m(X_i)\right][j'_m(x)]^2..$$

Comment: It can be seen that the asymptotic variance is a sum of two terms:

- the first is the same as for i.i.d observations
- the second is due to dependence.

We conclude that when the covariance or the bandwidth converges fast enough to zero, the density process behaves in the variance sense as in the i.i.d case. Otherwise the estimator behaves like the empirical process.

Proof. Expanding the function $b_n^{-1}K(\frac{x-G(X)}{b_n})$ in term of Hermite's polynomials $D_n(x)$ gives

$$D_n(x) = \sum_{k=m}^{\infty} \frac{C_{k,n}(x)}{k!}\frac{1}{n}\sum_{j=1}^{n} H_k(Z_j),$$

where,

$$C_{k,n}(x) = \mathbb{E}\left(H_k(X)\frac{1}{b_n}K(\frac{x-G(X)}{b_n})\right) = \frac{1}{b_n}\int j_k(x-b_nu)dK(u)$$

Hence, $C_{k,n}(x) = 0$ for $0 < k < m$. Therefore if r^{m^*} is summable, then

$$\begin{aligned}
var\,[f_n(x))] &= \sum_{k=m}^{\infty} \frac{C_{k,n}^2(x)}{k!}\frac{1}{n^2}\sum_{ji,=1}^{n} r^k(i-j) \\
&= \sum_{k=m}^{\infty} \frac{C_{k,n}^2(x)}{k!}\frac{1}{n^2}\left(n + 2\sum_{i=1}^{n}(n-i)\,r^k(i)\right) \\
&= \frac{1}{n}\sum_{k=m}^{\infty}\frac{C_{k,n}^2(x)}{k!} + \sum_{k=m+1}^{m*-1}\frac{C_{k,n}^2(x)}{k!}\frac{2}{n^2}\sum_{i=1}^{n}(n-i)\,r^k(i) \\
&\quad + \sum_{k=m*+1}^{\infty}\frac{C_{k,n}^2(x)}{k!}\frac{2}{n^2}\sum_{i=1}^{n}(n-i)\,r^k(i)
\end{aligned}$$

Let ε_n denote the last term in the previous display, then

$$\begin{aligned}\varepsilon_n &\leq \sum_{k=m*+1}^{\infty} \frac{C_{k,n}^2(x)}{k!} \left(\max_{i>0}|r(i)|\right)^{k-m^*} \frac{2}{n}\sum_{i=1}^{n}|r^{m^*}(i)| \\ &\leq \frac{2}{n}\sum_{k=m*+1}^{\infty} \frac{C_{k,n}^2(x)}{k!} \left(\max_{i>0}|r(i)|\right)^{k-m^*} \sum_{i=1}^{\infty}|r^{m^*}(i)|\end{aligned}$$

but if $p^{-1}+q^{-1}=1$, then

$$\begin{aligned}|C_{k,n}(x)| &= \left|\mathbb{E}\left(H_k(X)\frac{1}{b_n}K(\frac{x-G(X)}{b_n})\right)\right| \\ &\leq \frac{1}{b_n}\|H_k\|_p \left\|K(\frac{x-G(\cdot)}{b_n})\right\|_q \\ &\leq b_n^{-1+1/q}\sqrt{k!}\,(p-1)^{k/2}\end{aligned}$$

where in the last inequality we used (2.29). Therefore, choosing $p>2$ such that $(p-1)\max_{i>0}|r(i)|<1$ allows us to write

$$\varepsilon_n \leq \frac{Cb_n^{-1+2/q}}{nb_n} = o(\frac{1}{nb_n}).$$

The proof of (b) and (c) can be done analougsly.

Weak limit theorems

The idea behind the theorems below is to extend the result on variance to the rate of convergence in law. The first theorem deals with either short memory process or long memory and small bandwidth processes. As suggested by the variance study, in both cases we prove a clt in the finite dimensional sense with the usual normalization.

Theorem 5.1. *Let $Y_n = G(X_n)$ be a stationary process with a marginal density f.*

(i) *Assume that $r^m \in \mathbb{L}^1$, then*

$$\sqrt{nb_n}\,(f_n(x)-\mathbb{E}[f_n(x)]) \Longrightarrow \mathcal{N}\left(0,\sigma^2(x)\right). \qquad (5.6)$$

where , $\sigma^2(x) = \int K^2(u)du\ f(x)$

(ii) *Assume that $m\alpha<1$ and $n^\beta b_n \to 0$ for $\beta < 1-m\alpha$ then the convergence* (5.6) *holds.*

Note that even when the process is long range dependent, we can still obtain convergence similar to the i.i.d case by choosing a small bandwidth.

The last result deals with large bandwidth and long memory processes. In this case the density process satisfies a sort of derivative result of the empirical process behavior. Setting

$$Y_{k,n} = n^{-1}\sum_{j=1}^{n} H_k(X_j) \text{ and } C_{m,n}(x) = \mathbb{E}\left(\frac{1}{b_n}K(\frac{x-G(X)}{b_n})H_m(X)\right),$$

we prove the following

Theorem 5.2. *Let $Y_n = G(X_n)$ be a stationary process with a marginal density f. Let $D_n(x)$ be defind by* (5.3).

(i) *Assume that $r(n) \sim n^{-\alpha}L(n)$ and $m\alpha < 1$. If $n^{\beta}b_n \to \infty$ for $\beta < 1 - m\alpha$ then*

$$\frac{n}{d_{n,m}}\mathbb{E}\left|\sup_{x\in\mathbb{R}}\left(D_n(x) - C_{m,n}(x)\,Y_{m,n}\right)\right| \Longrightarrow 0$$

(ii) *Assume moreover that the derivative of $j_m(\cdot)$ is bounded and uniformly continuous. Then*

$$\frac{n}{d_{n,m}}D_n(\cdot) \Longrightarrow \frac{j'_m(\cdot)}{m!}Y_m(1) \quad in \quad \mathcal{D}[-\infty,+\infty]. \tag{5.7}$$

Remark As was mentioned in the introduction, Csörgő et al. 1995 proved, by refining the results of Dehling and Taqqu 1989 that if $n^{\beta}b_n \to \infty$ for $\beta < 1/2\inf(D, 1-mD)$, then (5.7) holds. They also gives a law of the iterated logarithm. These results can be rewritten with our new hypothesis on the bandwidth. We also recall that we can replace $\mathbb{E}(f_n(x))$ by $f(x)$ if we assume further regularity conditions on f and the kernel. The question is well discussed in Csörgő and Mielniczuk, (see also the paper by Hall et al. 1994 for relevant considerations on the bandwidth problem). No further development in this direction will be done in the present paper since dependence does not affect the biais term. We finally point out that a similar result to Theorem 5.2 is proved by Ho and Hsing 1996 when Y is linear.

Proofs

This section is devoted to the proofs of the main results stated in the previous sections. The technique proof is moment's method when establishing a central limit result, while the technique is a weak reduction principle and a bound of the oscillation behavior of continuity modulus of a remainder term in the development of the empirical distribution function.

Proof of Theorem 5.1

The proof is based upon the method of moment. We shall conclude if we show

$$\lim_{n\to\infty} \mathbb{E}\left[\sqrt{nb_n}D_n(x)\right]^p = \begin{cases} (p-1)!!\sigma^p(x) & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

The main tool to achieve this will be the moment inequalities established in chapter 4. To this end we first recall some notations. So, let

$$\begin{aligned} Q(p) &= \{\mathbf{t} = (t_1, t_2, ..., t_s),\ t_1 + t_2 + ... + t_s = p;\ t_i > 0\} \\ N(s) &= \{\mathbf{i} = (i_1, i_2, ..., i_s),\ 1 \le i_k \le n;\ i_k \ne i_l \text{ if } k \ne l\}. \\ \mathcal{P}(s) &= \{A : A \text{ a subset of } \{1, 2, ..., s\}\}. \end{aligned}$$

For A a subset of $\{1, 2, ..., s\}$ and $T > 0$, integer, we define

$$\begin{aligned} Q_{A,T}(p) &= \{\mathbf{t} = (t_1, t_2, ..., t_s),\ \mathbf{t}/A = t_A\ : t_1 + t_2 + ... + t_s = p;\ t_i > 0\} \\ N_{A,T}(s) &= \{\mathbf{i} = (i_1, i_2, ..., i_s) \in N(s),\ \forall i \in A, \exists j \in A \text{ such that } |i-j| \le T\} \\ t_A &= (t_i)_{i\in A} \end{aligned}$$

Hence if $K_n(.) := (nb_n)^{-1/2}\left(K\left(b_n^{-1}(x - G(\cdot))\right) - \mathbb{E}K\left(b_n^{-1}(x-G(\cdot))\right)\right)$, then

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^n K_n(X_i)\right)^p &= \sum_{i_1,i_2,...,i_p=1}^n \mathbb{E}\left(\prod_{l=1}^p K_n(X_{i_l})\right) \\ &= \sum_{\mathbf{t}\in Q(p)} C(\mathbf{t}) \sum_{(i_1,i_2,...,i_s)\in N(s)} \mathbb{E}\left(\prod_{l=1}^s K_n^{t_l}(X_{i_l})\right) \end{aligned}$$

where $C(\mathbf{t}) = (s!t_1!t_2!....t_s!)^{-1}\,p!$. But

$$N(s) = \cup_{A\in\mathcal{P}(\{1,2,...,s\})} N_{T,A}(s)$$

hence $N(s) = N_{T,\emptyset}(s) + \cup_{A\neq\emptyset} N_{T,A}(s)$. We write $N_T(s)$ for $N_{T,\emptyset}(s)$ in the sequel. The proof will be divided in three steps.

- **1st step:**

$$\lim_{n\to\infty} \sum_{(i_1,i_2,\ldots,i_s)\in N_{T,\emptyset}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right) = (p-1)!!\sigma^p(x)1_{p=2q}$$

- **2nd step:** For all $A \neq \emptyset$, for all t_A

$$\lim_{n\to\infty} \sum_{t_{A^c}\in Q_A(p), t=(t_A,t_A)\in Q(p)} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N_{T,A}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right) = 0$$

- **3rd step:** From the second step we infer that

$$\lim_{n\to\infty} \sum_{t\in Q(p)} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in\cup_{A\neq\emptyset} N_{T,A}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right) = 0 \tag{5.8}$$

Indeed, $N(s) = \cup_{A\in\mathcal{P}(\{1,2,\ldots,s\})} N_{T,A}(s)$, one **i** may belongs to diffirent $N_{T,A}(s)$, however the multiplicity of the overcount is bounded and independent of n. Using the second step, we conclude that (5.8) holds true.

a) Proof of the first step: Let us set

$$I(t,n) = \sum_{(i_1,i_2,\ldots,i_s)\in N_T(s)} \left|\mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right)\right|$$

To control the last display we need the following lemma proved in the previous chapter when $T = 1$. The proof for general T can be done analogously, hence we omit it.

Lemma 5.2. *Let $f_1, f_2, \ldots f_p$, be real, centered functions which may depend on n , and set $m_i = m(f_i)$. Let $\|f\|_{r,p}$ defined by*

$$\|f\|_{r,p} = \sum_{k=m_i}^{\infty} \frac{|c_k(f)|}{\sqrt{k!}}(p-1)^{k/2}\left(2\sum_{i=T}^{n}|r^k(i)|\right)^{1/2} \tag{5.9}$$

Then

$$\sum_{i\in N_T(p)} \left|\mathbb{E}\left(\prod_{l=1}^{p} f_l(X_{i_l})\right)\right| \le \prod_{l=1}^{p} \sqrt{n}\,\|f_l\|_{r,p}. \tag{5.10}$$

Now we apply the previous lemma to $K_n^{t_l}$. To this end, we have to center the functions. (note that they are already centered when $t_i = 1$). So, we set $\overset{\circ}{h} = h - \mathbb{E}(h(X))$ for any measurable function h. With these notations we have

$$I(t,n) \leq \sum_{(i_1,i_2,\ldots,i_s)\in N_T(s)} \left| \mathbb{E}\left(\prod_{l=1}^{s} \left(\overset{\circ}{K_n^{t_l}}(X_{i_l}) + \mathbb{E}K_n^{t_l}(X) \right) \right) \right| .$$

In the following we will treat the terms appearing in the last sum. Note that there is at most 2^s terms. They are all of the form

$$I(J,t,n) = \sum_{(i_1,i_2,\ldots,i_s)\in N_T(s)} \left| \left(\prod_{l\in J} \mathbb{E}K_n^{t_l}(X) \right) \mathbb{E}\left(\prod_{l\in J^C} \left(\overset{\circ}{K_n^{t_l}}(X_{i_l}) \right) \right) \right| ,$$

where $\{J, J^C\}$ is a partition of $\{1,2,\ldots,s\}$. Assume for simplicity that $J = \{1,2,\ldots,k\}$. In this case $I(J,t,n) = I_k(t,n)$ where

$$I_k(t,n) = \sum_{(i_1,i_2,\ldots,i_s)\in N_T(s)} \left| \left(\prod_{l=1}^{k} \mathbb{E}K_n^{t_l}(X) \right) \mathbb{E}\left(\prod_{l=k+1}^{s} \left(\overset{\circ}{K_n^{t_l}}(X_{i_l}) \right) \right) \right| . \qquad (5.11)$$

Before proceeding further, observe that $I_k(t,n) = 0$, whenever $t_i = 1$ for some $i \leq k$. Therefore we may assume that $t_i > 1$ for $i \leq k$. Since $\|K_n\|_{r,s} \leq (nb_n)^{-t_l/2} C(p,T) b_n^{1-\delta}$, for T sufficiently large, we get by (5.10) of the lemma

$$\begin{aligned} I_k(t,n) &\leq \left(\prod_{l=1}^{k} n \left| \mathbb{E}K_n^{t_l}(X) \right| \right) \left(\prod_{l=k+1}^{s} \left(\sqrt{n} \left\| K_n^{t_l} \right\|_{r,s-k} \right) \right) \\ &\leq \left(\prod_{l=1}^{k} b_n^{t_l-2} \right) (\theta(n))^{s-k} \end{aligned}$$

where $\theta(n) \to 0$. Consequently, the contributing terms to the limit must satisfy: $t_l - 2 = 0, k = s$, but since $t_1 + t_2 + \ldots + t_s = p$, then $s = p/2$. In other words

$$\lim_{n\to+\infty} I_k(t,n) = \begin{cases} (\sigma(x))^p \text{ if } t = (2,2,\ldots,2), \text{ and } k = p/2 \\ 0 \text{ otherwise} \end{cases}$$

and this end the proof of the first step.

b) <u>Proof of the 2^{st} step:</u> For all $A \neq \emptyset$

$$\lim_{n\to\infty} \sum_{t\in Q_A(p)} C(t) \sum_{(i_1,i_2,\ldots,i_s)\in N_{T,A}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l}) \right) = 0$$

First of all note that $|A| \geq 2$. Then, for simplicity but without loss of generality assume that $A = \{i_1, i_2, \dots, i_{|A|}\}$and write $\mathbf{t} = (t_A, t_{A^c})$.

$$\begin{aligned}
R(t_A, n) &= \left| \sum_{\mathbf{t}\in Q_A(p)} C(t) \sum_{(i_1,i_2,\dots,i_s)\in N_{T,A}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right)\right| \\
&= \left| \sum_{(i_1,\dots,i_{|A|});\mathbf{i}\in N_{T,A}(s)} \sum_{\mathbf{t}\in Q_A(p)} C(t) \sum_{(i_{|A|+1},\dots,i_s);\mathbf{i}\in N_{T,A}(s)} \mathbb{E}\left(\prod_{l=1}^{s} K_n^{t_l}(X_{i_l})\right)\right| \\
&\leq D(p, t_A) \sum_{(i_1,\dots,i_{|A|});\mathbf{i}\in N_{T,A}(s)} \mathbb{E}\prod_{l=1}^{|A|} \left|K_n^{t_l}(X_{i_l})\right| \times \\
&\qquad \left| \sum_{t\in Q_A(p)} C(t_{A^c}) \sum_{(i_{|A|+1},\dots,i_s);\mathbf{i}\in N(s-|A|)} \left(\prod_{l=|A|+1}^{s} K_n^{t_l}(X_{i_l})\right)\right|
\end{aligned}$$

therefore

$$R(t_A, n) \leq D(p, t_A) E \sum_{(i_1,\dots,i_{|A|});\mathbf{i}\in N_{T,A}(|A|)} \prod_{l=1}^{|A|} \left|K_n^{t_l}(X_{i_l})\right| \times \left| \left(\sum_{i=1}^{n} K_n(X_i)\right)^{p-|A|}\right|$$

Now define

$$\begin{aligned}
A_1 &= \{j \in A :, |i_1 - i_j| \leq T\}, \\
A_2 &= \{j \in A \backslash A_1 :, |i_2 - i_j| \leq T\}, \\
A_3 &= \{j \in A \backslash (A_1 \cup A_2) :, |i_3 - i_j| \leq T\} \dots
\end{aligned}$$

Apply Hölder's inequality to get:

$$R(t_A, n) \leq \left(\mathbb{E}\left| \sum_{\mathbf{i}\in N_{T,A}(|A|)} \prod_{l=1}^{|A|} \left|K_n^{t_l}(X_{i_l})\right|\right|^q\right)^{1/q} \times \mathbb{E}\left|\left(\sum_{i=1}^{n} K_n(X_i)\right)^{(p-|A|)r}\right|^{1/r}.$$

On the one hand the second term is bounded in view of the moment inequality. Indeed, since the Hermite rank of $K_n^t(X_i) \geq m$, for any $t \geq 1$, integer we deduce under the hypothesis of (i) or (ii) that $\|K_n^{t_l}\|_{2,2q} = O(1)$. Hence,

$$\mathbb{E}\left|\left(\sum_{i=1}^{n} K_n(X_i)\right)^{2q}\right| = O(1)$$

for any integer q. On the other hand

$$\mathbb{E}\left|\sum_{\mathbf{i}\in N_{T,A}(|A|)}\prod_{l=1}^{|A|}\left|K_n^{t_l}(X_{i_l})\right|\right|^q \le |N_{T,A}(|A|)|^q \max_{\mathbf{i}\in N_{T,A}(|A|)} \mathbb{E}\prod_{l=1}^{|A|}\left|K_n^{t_l q}(X_{i_l})\right|$$

Assume that $A = A_1 + A_2 + ..A_l$. Assume also that $A_1 = A_2 = .. = A_l = 2$. If l, k belong to different A_i's, then $|i_k - i_l| \ge T$. By choosing T large enough, arguing as in lemma 3.1 of Taqqu 1979 the joint density of $(X_{i_l})_{l\in A}$ exists and we have

$$\mathbb{E}\prod_{l=1}^{|A|}\left|K_n^{t_l}(X_{i_l})\right| \le \frac{Cb_n^{(1/2+\delta)|A|}}{(nb_n)^{|A|/2}}$$

for δ sufficiently small. Hence

$$\begin{aligned}
\mathbb{E}\left|\sum_{\mathbf{i}\in N_{T,A}(|A|)}\prod_{l=1}^{|A|}\left|K_n^{t_l}(X_{i_l})\right|\right|^q &\le |nT|^{|A|q/2} \max_{\mathbf{i}\in N_{T,A}(|A|)} \mathbb{E}\prod_{l=1}^{|A|}\left|K_n^{t_l q}(X_{i_l})\right| \\
&\le C\,|nT|^{|A|q/2}\,\frac{b_n^{(1/2+\delta)|A|}}{(nb_n)^{|A|/2}}
\end{aligned}$$

The proof of the second step is achieved if $A_1 = A_2 = .. = A_l = 2$ by choosing q sufficiently close to 1. If one of the $|A_i| > 2$, say for example $|A_1| = 3$ and $A_1 = \{i_1, i_2, i_3\}$, then we write

$$\left|K_n^{t_1 q}(X_{i_1})\,K_n^{t_2 q}(X_{i_2})\right| \le \left|K_n^{2t_1 q}(X_{i_1})\right| + \left|K_n^{2t_2 q}(X_{i_2})\right|$$

and we continue the proof in the same manner.

Proof of Theorem 5.2

Recall that

$$D_n(x) = \frac{1}{b_n}\int K\left(\frac{x-y}{b_n}\right)d\left\{F_n(y) - F(y)\right\}$$

but if K is derivable then

$$\begin{aligned}
D_n(\cdot) &= \frac{1}{b_n^2}\int K'\left(\frac{x-y}{b_n}\right)(F_n(y) - F(y))\,dy \\
&= \frac{1}{b_n}\int K'(u)\,(F_n(x - b_n u) - F(x - b_n u))\,du
\end{aligned}$$

But

$$
\begin{aligned}
F_n(y) - F(y) &= \frac{1}{n}\sum_{j=1}^{n} I_{\{G(X_j)\le y\}} - F(y) \\
&= \sum_{k=m}^{k^*-1} \frac{j_k(x)}{k!} n^{-1} \sum_{j=1}^{n} H_k(X_j) + R_n(x)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D_n(\cdot) &= \frac{1}{b_n}\int K'(u)\left(\sum_{k=m}^{k^*-1} \frac{j_k(x-b_n u)}{k!} n^{-1}\sum_{j=1}^{n} H_k(Z_j) + R_n(x-b_n u)\right) du \\
&= \frac{1}{b_n}\sum_{k=m}^{k^*-1}\int K'(u)\frac{j_k(x-b_n u)}{k!} n^{-1}\sum_{j=1}^{n} H_k(Z_j)du + \theta(n,x)
\end{aligned}
$$

Hence if $\int K'(u)du = 0$ we obtain

$$
\theta(n,x) = \frac{1}{b_n}\int K'(u)R_n(x-b_n u)du \tag{5.12}
$$

$$
\theta(n,x) = \frac{1}{b_n}\int K'(u)\left(R_n(x-b_n u) - R_n(x)\right) du \tag{5.13}
$$

$$
\le \frac{1}{b_n}\int |K'(u)| \sup_{x\in R} |R_n(x-b_n u) - R_n(x)|\, du \tag{5.14}
$$

Hence, we have to study the oscillations behavior of the empirical process. This is what is done in the forthcoming proposition

Theorem 5.3. *Assume that r^{k^*} is integrable and let*

$$
w(n,u) = \sup_{x\in\mathbb{R}} n^{1/2}|R_n(x+u) - R_n(x)|
$$

Then for every $\delta > 0$, the following inequality holds

$$
\mathbb{E}\,|w(n,u)| \le C(r,\delta)u^{1/2-\delta} + n^{-1/2+\delta} \tag{5.15}
$$

Proof. First, observe that without the maximum in x, we can easily have

$$
\begin{aligned}
\mathbb{E}\left(n^{1/2}|R_n(x+u) - R_n(x)|\right) &\le |F(x+u) - F(x)|^{1/2} \\
&\le \sup_{y\in\mathbb{R}} |f(y)|u^{1/2}
\end{aligned}
$$

Now, to control the maximum we use a chaining argument combined with a refined moment inequality stated in the following lemma.

Lemma 5.3. *Let f be a real, centered function with m as a Hermite rank. Assume that $r^m \in \mathbb{L}^1$, then for any p even integer, for any a large enough, there exists a constant $K = K(r,p,a)$ such that*

$$\mathbb{E}(S_n(f))^p \le K(r,p,a)\left\{\left(\sqrt{n}\,\|f\|_2\right)^p + n\,\|f\|_p^p + \left(\sqrt{n}\,\|f\|_2\right)^2\|f\|_{ap}^{p-2}\right\}. \quad (5.16)$$

In particular, we can take a such that:

$$\frac{pm-2(m-1)}{apm} = \frac{1}{2m}.$$

The proof of the lemma will be postponed to the end of this section. Now, we turn to the proof of the proposition. Let $(x_i^k)_{i\le N(k)}$ be a finite subdivision of $\mathbb{R}$ such that $\left|F(x_{i+1}^k) - F(x_i^k)\right| \le 2^{-2k}$. We can assume that these subdivisions are made from successive refinement. $N(k)$ is in fact the bracketing number of the family $\mathcal{F} = \{f_x = I_{G(\cdot)\le x};\, x\in\mathbb{R}\}$ with respect to the L^2 norm, hence $N(k)\le 2^{2k}$. Setting $\Delta_n(x,u) = R_n(x+u) - R_n(x)$ we write for any q_0 and q satisfying $q_0 < q$

$$\Delta_n(x,u) = \Delta_n(x,u) - \Delta_n(x_q,u) + \sum_{k=q_0+1}^{q}\Delta_n(x_k,u) - \Delta_n(x_{k-1},u) + \Delta_n(x_{q_0},u).$$

where $x_q = \pi_q(x)$ and $x_k = T_k(x)$, (see chapter 3 for notation). Therefore,

$$\begin{aligned}\mathbb{E}\,|w(n,u)| &= \mathbb{E}\,\|\Delta_n(x,u) - \Delta_n(x_q,u)\|_{\mathcal{F}} + \mathbb{E}\,\|\Delta_n(x_{q_0},u).\|_{\mathcal{F}} \\ &\quad + \sum_{k=q_0+1}^{q}\mathbb{E}\,\|\Delta_n(x_k,u) - \Delta_n(x_{k-1},u)\|_{\mathcal{F}} \\ &\le E_{1,q+1} + \mathbb{E}\,\|\Delta_n(x_{q_0},u).\|_{\mathcal{F}} + \sum_{k=q_0+1}^{q}E_{1,k}\end{aligned}$$

where

$$\begin{aligned}E_{1,k} &= \mathbb{E}\,\|\Delta_n(x_k,u) - \Delta_n(x_{k-1},u)\|_{\mathcal{F}}\,, \qquad q_0+1\le k\le q \\ E_{1,q+1} &= \mathbb{E}\,\|\Delta_n(x,u) - \Delta_n(x_q,u)\|_{\mathcal{F}}\,.\end{aligned}$$

Using inequality (3.5) we can write:

$$E_{1,k} \le N(k)^{1/p}\max_x\|\Delta_n(x_k,u) - \Delta_n(x_{k-1},u)\|_p \quad (5.17)$$

From (5.16) we infer that

$$\begin{aligned}E_{1,k} &\le K(r,p,a)N(k)^{1/p}\max_x\left(\sqrt{n}\,\|\Delta_n(x_k,u) - \Delta_n(x_{k-1},u)\|_2\right) \\ &\quad + K(r,p,a)N(k)^{1/p}\max_x\left(n\,\|\Delta_n(x_k,u) - \Delta_n(x_{k-1},u)\|_2^2\right)^{1/p} \\ &\le K(r,p,a)\left[2^{2k/p}\inf(\sqrt{u},2^{-k}) + n^{-1/2+1/p}\right]\end{aligned}$$

Similarly, we obtain the following for $E_{1,q+1}$

$$E_{1,q+1} \preceq \sqrt{n}2^{-q} + \leq K(r,p,a)\left[2^{2q/p}\inf(\sqrt{u},2^{-q}) + n^{-1/2+1/p}\right] \qquad (5.18)$$

Finally, choosing q such that $q \sim (\ln n)^2$ yields

$$\mathbb{E}\,|w(n,u)| \leq K(r,p,a)\left[2^{2q_0/p}\inf(\sqrt{u},2^{-q_0}) + qn^{-1/2+1/p}\right]$$

Since p can be arbitrary large the result follows.

Now, combining (5.14), (5.15) we conclude that

$$\sup_{x\in\mathbb{R}} |\theta(n,x)| = o(1).$$

Hence the first part of Theorem 5.2 follows. For the second part we argue as in Csörgő et al. 1995.

Proof of Lemma 5.3

We will argue as in the previous chapter, so we pick up the proof of Proposition 4.1 at the relation at the relation (4.14), then we write

$$I_k(t,n) \leq \left(\prod_{l=1}^{k} n\,|\mathbb{E}f^{t_l}(X)|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\,\|f^{t_l}\|_{r,s-k}\right)\right)$$

Let $r_1, r_2, r_3 > 0 : r_1^{-1} + r_2^{-1} + r_3^{-1} = 1$. Applying Hölder's inequality to f_1, f_2, f_3 with

$$\begin{aligned} r_1 &= m/m-1; f_1 = f^{\theta t} : \theta t r_1 = 2 \\ r_2 &= \frac{tm-2m+2}{apm}; f_2 = f^{(1-\theta)t} \\ r_3 &= \left(1 - r_1^{-1} + r_2^{-1}\right)^{-1}; f_3 = H_k \end{aligned}$$

we obtain for $t \geq 2$,

$$\begin{aligned} \sqrt{n}\,|c_k(f^t)| &= |\mathbb{E}f^t(X)H_k(X)| \\ &\leq \left(\sqrt{n}\,\|f\|_2\right)^{2-2/m}\|f\|_{ap}^{t-2+2/m}\|H_k\|_{r_3} \\ &\leq \left(\sqrt{n}\,\|f\|_2\right)^{2-2/m}\|f\|_{ap}^{t-2+2/m}\sqrt{k!}\,(r_3)^{k/2} \qquad (5.19) \end{aligned}$$

where in the last inequality, we have used relation (4.27) in the appendix. On the other hand,

$$\sum_{i=1}^{n}|r(i)/r^*(1)| \leq n^{1-1/m}\left(\sum_{i=1}^{n}|r(i)/r^*(1)|^m\right)^{1/m}.$$

Applying (4.18) with $\theta = t - 2/(p-2)$, $s = p/2 - 1$ we get

$$n\left|\mathbb{E}f^t(X)\right| \le \left(\sqrt{n}\,\|f\|_2\right)^{2(1-\theta)}\left(n^{1/p}\,\|f\|_p\right)^{p\theta}. \tag{5.20}$$

(5.19) together with (5.20) yield for $s \ge 2$,

$$\begin{aligned}
I_k(t,n) &\le \left(\prod_{l=1}^{k} n\left|\mathbb{E}f^{t_l}(X)\right|\right)\left(\prod_{l=k+1}^{s}\left(\sqrt{n}\,\|f^{t_l}\|_{r,s-k}\right)\right)\\
&\le \left(\prod_{l=1}^{k}\left(\sqrt{n}\,\|f\|_2 \max\left(\sqrt{n}\,\|f\|_2\,;n^{1/p}\,\|f\|_p\right)\right)^{t_l-1}\right)\\
&\quad\times \prod_{l=k+1}^{s}\left(\sqrt{n}\,\|f\|_2\right)^{2-2/m}\|f\|_{ap}^{t_l-2+2/m}\sum_{k=m_p}^{\infty}\left(r_3r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|\frac{r(i)}{r^*(1)}\right|^m\right)^{1/2}\\
&\le \left(\sqrt{n}\,\|f\|_2\right)^2\left(\max\left(\sqrt{n}\,\|f\|_2\,;n^{1/p}\,\|f\|_p\,,\|f\|_{ap}\right)\right)^{t_1+\dots+t_s-2}\\
&\quad\times\left(\sum_{k=m_p}^{\infty}\left(r_3r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|\frac{r(i)}{r^*(1)}\right|^m\right)^{1/2}\right)^{s-k}.
\end{aligned}$$

We may assume that $s \ge 2$, indeed we handle easily the case $s = 1$ separately. Hence

$$\begin{aligned}
I_k(t,n) &\le \prod_{l=k+1}^{s}\left(\sqrt{n}\,\|f\|_2\right)^{2-2/m}\|f\|_{ap}^{t_l-2+2/m}\sum_{k=m_p}^{\infty}\left(r_3r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|\frac{r(i)}{r^*(1)}\right|^m\right)^{1/2}\\
&\le \left(\sqrt{n}\,\|f\|_2\right)^2\left(\max\left(\sqrt{n}\,\|f\|_2\,;n^{1/p}\,\|f\|_p\,,\|f\|_{ap}\right)\right)^{t_1+\dots+t_s-2}\\
&\quad\times\left(\sum_{k=m_p}^{\infty}\left(r_3r^*(1)\right)^{k/2}\left(2\sum_{i=1}^{n}\left|r(i)/r^*(1)\right|^m\right)^{1/2}\right)^{s-k}.
\end{aligned}$$

(5.16) is proved if $r_3r^*(1) < 1$. In the general case we proceed as in (a) of Theorem 4.1 in Chapter 4.

Conclusion

Several conclusions can be derived from the previous study.

1. The asymptotic behavior of a kernel density estimator with dependent data depends both on the dependence structure and the bandwidth:

If the process is sufficiently weakly dependent or the bandwidth is small, then the density process behaves like the i.i.d observations. Otherwise, the density process behaves like the empirical process.

2. When measuring dependence with mixing coefficients, we can only obtain sufficient conditions for behavior similar to the i.i.d case. Indeed, one can construct two function G_1 and G_2 with respective Hermite rank 1 and 2, generating the same σ algebra and hence the same mixing coefficients. Therefore we may obtain different rates according the decay of the correlation function. Consequently, the rate of convergence will depend on G_i if the underlying process is strong dependent enough.

3. When dealing with covariance we can have a complete answer. However, we have restrictions on the model. One possible way to generalize these results to other classes is to consider linear models and their functionals. Several significant works have already been done in this area, especially for empirical process and some statistical applications. We cite for example the works of Giraitis and Surgailis, Avram and Taqqu , Robinson and probably many other authors. Finally we point out that similar results under more restrictive conditions, are proved in [4] for the density process of a kernel estimator based on linear process.

These results will probably allow us to estimate the rate of dependence, and to test heavy dependence against short dependence. These questions will be investigated in a subsequent paper.

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