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# THESE

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# Abstract

We construct and study the Yang-Mills measure in two dimensions. According to the informal description given by the phisicists, it is a probability measure on the space of connections modulo gauge transformations on a principal bundle with compact structure group. We are interested in the case where the base space of this bundle is a compact orientable surface.

The construction of the measure in a discrete setting, where the base space of the fiber bundle is replaced by a graph traced on a surface, is quite well understood thanks to the work of E. Witten. In contrast, the continuum limit of this construction, which should allow to put a genuine manifold as base space, still remains problematic.

This work presents a complete and unified approach of the discrete theory and of its continuum limit. We give a geometrically consistent definition of the Yang-Mills measure, under the form of a random holonomy along a wide, intrinsic and natural class of loops. This definition allows us to study combinatorial properties of the measure, like its Markovian behaviour under the surgery of surfaces, as well as properties specific to the continuous setting, for example, some of its microscopic properties. In particular, we clarify the links between the Yang-Mills measure and the white noise and show that there is a major difference between the Abelian and semi-simple theories. We prove that it is possible to construct a white noise using the measure as a starting point and vice versa in the Abelian case but we show a result of asymptotic independence in the semi-simple case which suggests that it is impossible to extract a white noise from the measure.

**Keywords :** Gauge theory, Yang-Mills, continuum limit, random holonomy, white noise, zero-one law.

**MSC Classification :** 58D20, 81T13, 81T27, 81Q70, 60F20, 60H40.

# Contents

# Introduction

Cette thèse est consacrée à la construction et à l'étude de la mesure de Yang-Mills en deux dimensions. L'équivalent quadridimensionnel et pseudo-riemannien de cette mesure est utilisée par les physiciens pour rendre compte des interactions fondamentales, dans le cadre des théories de jauge comme l'électrodynamique et la chromodynamique quantiques. Elle apparaît dans des intégrales dites de chemins, qui sont des intégrales dont la description, souvent informelle, ne suffit pas à prouver l'existence. Si l'on s'en tient à cette description, la mesure de Yang-Mills est une mesure de probabilités sur l'espace des connexions modulo transformations de jauge sur un fibré principal dont le groupe de structure est compact. Nous nous intéressons ici au cas où la base de ce fibré est une surface compacte orientable. L'expression informelle de la mesure est alors la suivante:

$$d\mu(\omega) = \frac{1}{Z}\, e^{-\frac{1}{2}S(\omega)}\, D\omega, \tag{1}$$

où $S$ désigne l'action de Yang-Mills, qui est la norme $L^2$ de la courbure. La constante $Z$ est une constante de normalisation et la mesure $D\omega$ une hypothétique mesure invariante par translation sur l'espace des connexions.

La construction de la mesure dans un cadre discret, où la base du fibré est remplacée par un graphe tracé sur une surface, est comprise dans ses grandes lignes depuis les travaux de Witten [Wi]. En revanche, le passage à la limite continue de cette construction, qui doit permettre de considérer une véritable surface comme espace de base, est resté jusqu'à maintenant problématique. Plusieurs travaux menés dans cette direction ont amené des progrès, sans toutefois aboutir à des résultats entièrement satisfaisants.

Ce travail présente une approche complète et unifiée de la théorie discrète et de son passage à la limite continue. Il aboutit à une définition géométriquement cohérente de la mesure de Yang-Mills sous la forme d'une holonomie aléatoire le long d'une grande classe de lacets, intrinsèque et naturelle. Cette définition permet d'étudier aussi bien les propriétés combinatoires de la mesure, comme son comportement markovien lors du découpage et du recollement des surfaces, que ses propriétés plus spécifiques au cadre continu, par exemple des propriétés à l'échelle microscopique. En particulier, on clarifie les liens entre la mesure de Yang-Mills et le bruit blanc et on met en évidence une différence profonde entre les théories à groupe de structure abélien et semi-simple. On montre qu'il est possible de passer de la mesure au bruit blanc et vice versa dans le cas abélien, alors qu'on établit dans le cas semi-simple un résultat d'indépendance asymptotique compte tenu duquel il est certainement impossible d'extraire un bruit blanc de la mesure.

Nous présentons tout d'abord une construction de la mesure, basée sur l'approche combinatoire du problème, initiée par A.A. Migdal en 1975 [Mi] et prolongée par E. Witten en 1991 [Wi]. Nous construisons en premier lieu la théorie discrète, sur un graphe, en complétant la méthode de Witten, puis nous en réalisons le passage à la limite continue, en deux étapes. Nous commençons par prendre la limite projective des mesures associées à une famille de graphes. Nous montrons ensuite que certaines propriétés de régularité permettent, par un procédé d'approximation, d'aboutir à une définition intrinsèque de la mesure de Yang-Mills.

Parallèlement à cette construction générale, nous étudions le cas particulier où le groupe de structure est abélien, en prenant l'exemple du groupe $U(1)$. Ceci nous conduit à proposer une autre construction de la mesure, spécifique au cas abélien, basée sur le caractère gaussien de la théorie en deux dimensions. Cet aspect avait déjà servi de point de départ aux travaux de B. Driver [Dr1, Dr2] et A. Sengupta [Se1, Se2], dans le cas général. Or il nous a semblé que les constructions auxquelles avaient abouti ces travaux n'étaient pas entièrement satisfaisantes et que la raison pouvait en être que l'approche gaussienne était mal adaptée à un groupe de structure non abélien. En étudiant successivement la mesure de Yang-Mills à petite échelle dans les cas abélien et semi-simple, nous mettons en évidence une différence structurelle importante entre ces deux théories, qui explique a posteriori les difficultés auxquelles Driver et Sengupta se sont trouvés confrontés.

Enfin, nous étudions le comportement de la mesure vis-à-vis de la chirurgie des surfaces. Nous démontrons une propriété de Markov du champ aléatoire qui étend un résultat prouvé dans le cadre discret par C. Becker et A. Sengupta [BS]. Nous étudions ensuite ce que devient la mesure lorsque l'on recolle deux surfaces, ainsi que lorsque l'on coupe ou recolle une anse d'une surface. Cette étude conduit naturellement à celle des fonctions de partition conditionnelles, dont l'importance avait déjà été mise en évidence par Witten [Wi].

**Position du problème.** Le contexte géométrique est le suivant: on se donne une surface $M$, un groupe de Lie $G$ et un fibré principal $P$ sur $M$. La surface $M$ est une variété différentiable réelle de dimension 2, compacte, orientable, avec ou sans bord. Elle est munie d'une mesure $\sigma$ qu'on suppose équivalente à la mesure de Lebesgue dans toute carte, avec une densité strictement positive et lisse. Le groupe $G$ est un groupe de Lie compact connexe, qui sera choisi abélien ou semi-simple dans la plupart des exemples. Le fibré $P$ est un fibré principal sur $M$ de groupe de structure $G$.

Rappelons qu'une connexion sur $P$ est le choix d'une distribution horizontale $G$-invariante dans $P$ et que ce choix peut être représenté par une 1-forme $\omega$ sur $P$ à valeurs dans l'algèbre de Lie $\mathfrak{g}$ de $G$. La courbure de la connexion $\omega$ est la 2-forme $\Omega$ sur $P$ à valeurs dans $\mathfrak{g}$ définie par

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X),\omega(Y)]. \tag{2}$$

Cette 2-forme peut également être vue comme une 2-forme sur $M$ à valeurs dans le fibré $\mathrm{ad}\, P$ associé à $P$ par l'action adjointe de $G$ sur $\mathfrak{g}$. Quitte à choisir une orientation de $M$, on peut alors identifier $\Omega$ avec une section de $\mathrm{ad}\, P$. La donnée d'un produit scalaire sur $\mathfrak{g}$ invariant par adjonction permet de munir le fibré $\mathrm{ad}\, P$ d'une métrique et de définir la norme $\| \Omega \|$ de la courbure, qui ne dépend pas du choix de l'orientation. On définit alors l'action de Yang-Mills sur l'espace $\mathcal{A}$ des connexions sur $P$ par

$$\begin{array}{rcl} S : \mathcal{A} & \longrightarrow & \mathbf{R}_+ \\ \omega & \longmapsto & S(\omega) = \displaystyle\int_M \| \Omega \|^2 \; d\sigma. \end{array}$$

Le problème qui nous occupe maintenant est de donner un sens à l'expression (1). L'obstacle le plus manifeste est qu'il n'existe aucune mesure invariante par translation sur $\mathcal{A}$ qui est un espace affine de dimension infinie. Une autre difficulté provient du fait que l'action de Yang-Mills est invariante par l'action d'un groupe énorme, le groupe de jauge de $P$. Ce groupe, noté $\mathcal{J}$, est le groupe des difféomorphismes de $P$ au-dessus de l'identité de $M$ qui commutent à l'action de $G$. Il agit par pull-back sur $\mathcal{A}$ et préserve $S$, car son action sur la courbure d'une connexion est une conjugaison en chaque point, qui préserve la norme dans le fibré $\operatorname{ad} P$. Du fait de cette invariance, la constante $Z$ doit être proportionnelle au volume de $\mathcal{J}$, donc infinie. On remédie à ce problème en cherchant à construire la mesure non plus sur $\mathcal{A}$ mais sur l'espace quotient $\mathcal{A}/\mathcal{J}$. Ceci implique qu'on ne pourra intégrer contre la mesure de Yang-Mills que des fonctions invariantes par transformations de jauge, en accord avec le principe physique selon lequel toutes les grandeurs mesurables sont invariantes de jauge. En revanche, la structure géométrique de l'espace quotient est beaucoup plus compliquée que celle d'un espace affine, si bien qu'on évite en général de travailler directement dessus, pour lui préférer un espace de fonctions, comme nous verrons un peu plus loin qu'il est possible de le faire. A ce titre, les travaux de D. Fine [Fi1, Fi2] font exception, puisque l'auteur y analyse la structure géométrique du quotient $\mathcal{A}/\mathcal{J}$ pour interpréter plus précisément l'expression (1).

**La mesure de Yang-Mills comme holonomie aléatoire.** La première question qu'on peut se poser est de savoir quelles fonctions on veut être capable d'intégrer contre la mesure de Yang-Mills. La réponse donnée par les physiciens est la suivante: on veut intégrer les fonctions appelées boucles de Wilson.

La donnée d'une connexion $\omega$ sur $P$ permet d'y définir un transport parallèle le long de chemins réguliers sur $M$. Etant donné un chemin $c : [0,1] \longrightarrow M$, le transport parallèle, ou holonomie, le long de $c$ est un difféomorphisme $G$-équivariant de la fibre $P_{c(0)}$ dans la fibre $P_{c(1)}$, noté $\operatorname{hol}(\omega, c)$. Si $c$ est un lacet et si on choisit un point $p$ dans la fibre $P_{c(0)}$, ce difféomorphisme peut être représenté par l'élément $g$ de $G$ tel que $\operatorname{hol}(\omega, c)(p) = pg$. Si on choisit un autre point dans $P_{c(0)}$, on trouve un autre élément de $G$ conjugué à $g$. Ainsi, pour toute représentation $\rho$ de $G$ et tout lacet $l$, on peut définir la boucle de Wilson $W_{l,\rho}$ par

$$W_{l,\rho}(\omega) = \operatorname{tr} \rho(\operatorname{hol}(\omega, l)).$$

Les fonctions que l'on veut intégrer sont donc des fonctions centrales de l'holonomie le long des lacets. Nous venons de voir que, lorsqu'on fixe une connexion, l'holonomie le long d'un lacet détermine une classe de conjugaison dans $G$. Il nous faut également prendre en compte l'action du groupe de jauge, qui conjugue par un même élément de $G$ les holonomies de tous les lacets basés en un même point. Notons $LM$ l'ensemble des lacets réguliers sur $M$, $\mathcal{F}(M,G)$ et $\mathcal{F}(LM,G)$ les ensembles des fonctions de $M$ et $LM$ dans $G$. Un élément $j$ du groupe $\mathcal{F}(M,G)$ agit sur un élément $f$ de $\mathcal{F}(LM,G)$ de la façon suivante:

$$j.f(l) = j(l(0))^{-1} f(l) j(l(0)).$$

Il est alors possible de définir une application de $\mathcal{A}/\mathcal{J}$ dans le quotient $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$, qui associe à une connexion la classe de l'holonomie qu'elle définit le long des éléments de $LM$. Un argument de Sengupta [Se1] prouve que cette application est injective. Ceci nous conduit à changer de point de vue: on cherche désormais à construire une mesure sur l'espace $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$, qu'on voit comme un espace de connexions généralisées. En fait, on va construire une mesure sur $\mathcal{F}(LM,G)$ dont on prendra ensuite le quotient par l'action de $\mathcal{F}(M,G)$. Autrement dit, c'est bien une holonomie aléatoire qu'on cherche à construire et non plus une connexion aléatoire. L'avantage est qu'il existe des outils classiques pour construire des mesures de probabilités sur des espaces de fonctions.

A ce stade de l'analyse du problème, il est nécessaire de disposer d'une caractérisation plus précise de la mesure de Yang-Mills. Pour cela, on peut soit examiner l'expression informelle de la mesure de plus près, soit chercher auprès des physiciens d'autres descriptions de la mesure. La seconde approche est celle que ce travail met en œuvre, indépendemment de la nature du groupe de jauge. Elle s'appuie sur la description combinatoire donnée par Migdal et Witten. La première approche en revanche est celle qui met à profit le caractère gaussien de la théorie et il semble qu'elle soit mieux adaptée au cas d'un groupe abélien. C'est l'approche décrite et utilisée par Driver et Sengupta [Dr1, Se1, Se2].

**Interprétation gaussienne : courbure de la connexion aléatoire.** Supposons que $G$ soit abélien, par exemple $G = U(1)$. Dans ce cas, la relation (2) qui définit la courbure d'une connexion devient linéaire en $\omega$. Un changement de variables formel permet de réécrire (1) comme suit:

$$d\mu(\Omega) = \frac{1}{Z'} e^{-\frac{1}{2}\|\Omega\|^2}\, D\Omega. \tag{3}$$

Comme $G$ est abélien, le fibré $\mathrm{ad}\, P$ est trivial, on peut l'identifier à $M \times \mathfrak{g}$. On reconnaît alors dans (3) l'expression d'une mesure gaussienne sur l'espace de Hilbert des fonctions de carré intégrable sur $M$ à valeurs dans $\mathfrak{g}$. Ceci est le cœur de toutes les interprétations de (1) et constitue une sorte de principe heuristique: sous la mesure de Yang-Mills, la courbure aléatoire d'une connexion est distribuée de façon gaussienne, c'est un bruit blanc sur $M$ d'intensité $\sigma$ à valeurs dans l'algèbre de Lie du groupe de structure.

Cet argument dépend bien entendu du fait que le groupe de structure est abélien, puisque la courbure est en général une fonction quadratique de la connexion. Cependant, la théorie en deux dimensions présente la particularité qu'étant donné une connexion, il est toujours possible de se ramener à la situation précédente par un choix de jauge approprié. Développons un peu cet argument.

Si on choisit une trivialisation locale de $P$ sur un ouvert $U \subset M$, c'est-à-dire une section locale $s : U \longrightarrow P$ de $P$, on peut tirer sur $M$ par $s^*$ tous les objets définis sur $P$, en particulier les formes de courbure et de connexion. On note traditionnellement $A = s^*\omega$ et $F = s^*\Omega$. Ces formes sur $M$ vérifient une équation de structure $F = dA + [A, A]$ identique à (2). Soit maintenant $j$ un élément du groupe de jauge $\mathcal{J}$. On peut le faire agir de deux façons dans ce contexte, soit en transformant la section $s$ en $j \circ s$, soit en transformant les formes $\omega$ et $\Omega$ en $j^*\omega$ et $j^*\Omega$. Ces deux actions sont perçues de la même façon sur $M$, puisque $(j \circ s)^* = s^* j^*$. On notera donc par exemple $A^j$ la forme $(j \circ s)^*\omega = s^* j^*\omega$, sans ambiguité.

Le fait particulier à la dimension deux est que pour toute connexion $\omega$, il existe localement une section $s$ telle que $A = s^*\omega$ vérifie $[A, A] = 0$. Vu depuis $M$ à travers une section locale donnée, ceci se reformule en disant que pour toute connexion $A$, il existe un changement de jauge $j$ tel que $[A^j, A^j] = 0$. En effet, choisissons un ouvert assez petit pour posséder des coordonnées locales $x, y$. Posons $m = (0, 0)$ dans ces coordonnées et choisissons $p$ dans la fibre $P_m$. On définit $s$ le long de l'axe des $y$ comme le relevé horizontal de cet axe partant de $p$. Maintenant, partant d'un point $(0, y_0)$ dans $U$, on définit $s$ le long de la droite $y = y_0$ comme le relevé horizontal de cette droite partant de $s(0, y_0)$. Il en résulte une section lisse et horizontale le long de toutes les droites parallèles à l'axe $(Ox)$. Ainsi, $A = s^*\omega$ s'écrit $A = A_y dy$ et on a $[A, A] = 0$. A travers une telle section $s$, le raisonnement fait au début de ce paragraphe fonctionne, à ceci près qui la section à travers laquelle la courbure est censée être une fonction linéaire de la connexion dépend elle-même de la connexion.

**Suite de l'interprétation gaussienne : de la courbure à l'holonomie.** L'étape suivante consiste à extraire de sa courbure aléatoire l'holonomie aléatoire de la connexion. On s'appuie pour cela sur les liens déterministes entre courbure et holonomie. Supposons encore que le groupe de structure soit $U(1)$ et plaçons-nous non pas sur une surface compacte mais sur $\mathbf{R}^2$. Etant donné une connexion $\omega$ sur le fibré $\mathbf{R}^2 \times U(1)$ et un lacet simple $l$ qui borde un domaine $D$, la formule de Stokes permet d'écrire

$$\mathrm{hol}(\omega, l) = \exp i \oint_l A = \exp i \int_D dA = \exp i \int_D F = \exp i (F, \mathbf{1}_D)_{L^2}.$$

L'avantage de cette formulation est qu'elle s'étend aisément à la situation aléatoire. En effet, choisissons un bruit blanc $W$ sur $\mathbf{R}^2$, c'est-à-dire une isométrie de $L^2(\mathbf{R}^2)$ dans un espace gaussien. On peut alors remplacer formellement $F$ par $W$ dans la dernière expression et définir une holonomie aléatoire $H_l$ le long de $l$ par

$$H_l = \exp iW(\mathbf{1}_D).$$

La construction spécifiquement abélienne que nous présentons au chapitre 3 est une adaptation de ce procédé au cas de surfaces dont la topologie n'est pas toujours triviale et où la notion d'intérieur d'un lacet demande à être précisée.

On peut ici expliquer plus en détail les problèmes rencontrés par Driver et Sengupta. Ils ont cherché à utiliser la méthode que nous venons de décrire lorsque $G$ n'est pas commutatif. Or dans ce cas, l'holonomie ne s'écrit plus $\exp \int_l A$, mais $P \exp \oint_l A$, qui est une notation condensée pour la solution de l'équation différentielle

$$\begin{cases} \dot{h}_t = A(\dot{l}(t))h_t \\ h_0 = 1. \end{cases}$$

Dans ce cadre, la formule de Stokes ne s'applique pas. En quelque sorte, il faudrait choisir l'ordre dans lequel on multiplie les petits éléments de $G$ obtenus par exponentiation de l'intégrale de $F$ sur des petits carrés inclus dans $D$. De fait, la connexion étant fixée, Driver et Sengupta utilisent les coordonnées qui ont permis de construire la section à travers laquelle $[A, A] = 0$ pour choisir l'ordre dans lequel ils intègrent le bruit blanc à l'intérieur de $D$. L'inconvénient est que la classe de lacets le long desquels ils parviennent à définir l'holonomie aléatoire est un peu restreinte et surtout dépend complètement de ce choix de coordonnées.

Notons que L. Gross [Gr] et B. Driver [Dr2] ont introduit un nouvel objet local pour tenter de remplacer le bruit blanc dans ce contexte. Il se peut qu'il y ait là un début de solution à aux problèmes que nous venons d'évoquer.

Bien que ce point ne soit pas abordé dans le travail que nous présentons, il est impossible de finir cette introduction sans mentionner l'importance du passage à la limite semi-classique de la mesure de Yang-Mills. Le phénomène remarquable est que lorsque la surface totale de $M$ tend vers 0, la mesure de Yang-Mills se concentre sur l'espace des connexions plates sur $M$ et tend vers la mesure de volume correspondant à la structure symplectique naturelle sur cet espace. Ce problème est l'objet d'une littérature assez abondante (voir par exemple [Fo, BS, KS, Se3, Liu]), car il est intimement lié à l'étude de la géométrie de certains espaces de modules [AB].

## Présentation des résultats

Nous allons maintenant présenter en détail les résultats essentiels de ce travail, profitant du fait que nous ne donnons pas de preuves pour en modifier parfois un peu l'ordre.

**Construction de la mesure sur des graphes.** Notre point de départ est la description combinatoire de la mesure qu'a donnée Witten [Wi] à la suite de Migdal [Mi]. Elle consiste à se ramener à la construction d'une mesure sur un espace de dimension finie, en remplaçant la variété $M$ par un graphe tracé sur $M$. On cherche alors à définir l'holonomie aléatoire le long des chemins qu'on peut parcourir dans ce graphe. On s'intéresse également à des versions conditionnelles de ces mesures, qui serviront à traiter le cas des surfaces à bord dans le cadre continu et qui permettent surtout de définir des objets analytiques très importants, les fonctions de partition conditionnelles. On examine ensuite les liens entre les mesures associées à des graphes différents et on prouve la propriété principale de la théorie discrète qui est l'invariance par subdivision. Cette propriété est un des ingrédients essentiels du passage à la limite continue. Enfin, dans le cadre de la théorie discrète, nous commençons à étudier certaines spécificités du cas où le groupe de structure est abélien.

Nous choisissons une surface compacte $M$, avec ou sans bord, munie d'une mesure de surface $\sigma$. Nous choisissons également un fibré principal $P$ sur $M$ de groupe de structure $G$. Commençons par préciser la classe de chemins avec laquelle nous allons travailler.

**Définition 0.1** *On appelle chemin sur $M$ une classe d'équivalence à reparamétrisation croissante près d'applications $c : [0,1] \longrightarrow M$ qui sont des concaténations de plongements lisses. On note $PM$ l'ensemble des chemins de $M$.*

Pour discrétiser $M$, il nous faut définir une notion de graphe

**Définition 0.2** *On appelle arête sur $M$ un chemin injectif. On appelle graphe un ensemble fini d'arêtes qui ne se rencontrent deux à deux, le cas échéant, qu'en leurs extrémités et tel que les propriétés suivantes soient vérifiées :*
*1. Le support du graphe, c'est-à-dire la réunion des images de ses arêtes est connexe et contient le bord de $M$ si celui-ci n'est pas vide.*
*2. Toutes les faces, c'est-à-dire les composantes connexes du complémentaire du support, sont difféomorphes à des disques.*

La condition 2 implique qu'un graphe rend correctement compte de la topologie de $M$. Plus précisément, l'inclusion du support du graphe dans $M$ induit une surjection des premiers groupes d'homologie entière (cf. 1.2.4).

Soit $\Gamma = \{a_1, \ldots, a_r\}$ un graphe. On note $\Gamma^*$ l'ensemble des chemins qu'on peut parcourir dans $\Gamma$, qui s'identifie avec l'ensemble des mots formés avec les arêtes et leurs inverses. La restriction de $P$ au support de $\Gamma$ est triviale et on l'identifie à $\mathrm{Supp}(\Gamma) \times G$. Une version discrète de l'holonomie consiste à attribuer un élément de $G$ à chaque chemin de $\Gamma^*$. Comme l'holonomie le long du composé de deux chemins est le produit, dans l'ordre inverse, des holonomies le long de chaque chemin, il suffit d'attribuer un élément de $G$ à chaque arête de $\Gamma$. Ainsi, l'espace des connexions discrètes est $G^\Gamma$. A tout chemin $c$ de $PM$ est donc associée une holonomie $h_c : G^\Gamma \longrightarrow G$.

On considère ensuite les transformations de jauge discrètes, celles qui n'agissent que sur les sommets de $\Gamma$. Si $\mathcal{V}(\Gamma)$ désigne l'ensemble des sommets de $\Gamma$, le groupe de jauge discret est $G^{\mathcal{V}(\Gamma)}$ et il agit sur $G^\Gamma$ de la façon suivante: si $g$ est une connexion discrète et $j$ une transformation de jauge,

$$j \cdot g = (j(a_1(1))^{-1} g_1 j(a_1(0)), \ldots, j(a_r(1))^{-1} g_r j(a_r(0))).$$

Nous allons construire la mesure de Yang-Mills discrète sur $G^\Gamma$, invariante par l'action de $G^{\mathcal{V}(\Gamma)}$. Nous partons de la mesure la plus naturelle sur $G^\Gamma$, le produit des mesures de Haar, qui est noté $dg$. La mesure de Yang-Mills va être définie par sa densité par rapport à $dg$. Cette densité est un produit de fonctions centrales de l'holonomie discrète le long de lacets, ce qui en assure l'invariance sous l'action de $G^{\mathcal{V}(\Gamma)}$. Pour toute face $F$ de $\Gamma$, le bord $\partial F$ est défini aux choix près d'une orientation et d'une origine. La fonction $h_{\partial F}$ est donc définie à conjugaison et inversion près. Soit $(p_t)_{t>0}$ la solution fondamentale de l'équation de la chaleur sur $G$ munie de sa métrique biinvariante normalisée afin que son volume total soit égal à 1. La fonction $p_{\sigma(F)}(h_{\partial F})$ est alors bien définie car $p_t$ est précisément invariant par conjugaison et inversion. Nous définissons la densité $D : G^\Gamma \longrightarrow \mathbf{R}_+$ et le nombre $Z$ par :

$$D = \prod_{F \in \mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F}) : G^\Gamma \longrightarrow \mathbf{R}_+^*,$$

$$Z = \int_{G^\Gamma} D \, dg.$$

On définit alors la mesure $P$ sur $(G^\Gamma, \mathrm{Bor}(G^\Gamma))$ par

$$dP = \frac{1}{Z} D \, dg.$$

La loi d'un $n$-uplet de variables $(h_{c_1}, \ldots, h_{c_n})$ sous $P$ est par définition la loi de l'holonomie discrète le long de $c_1, \ldots, c_n$. Cette holonomie satisfait la même propriété de multiplicativité que l'holonomie déterministe.

Etant donné des lacets simples disjoints $L_1, \ldots, L_q$ de $\Gamma^*$, dont l'image est soit intérieure à $M$, soit égale à une composante de son bord, on construit une désintégration de $P$ par rapport à la variable $(h_{L_1}, \ldots, h_{L_q})$ (cf. section 1.5). On la note $(x_1, \ldots, x_q) \mapsto P(x_1, \ldots, x_q)$. Tout comme $P$, elle a une masse naturelle qui n'est pas 1 et qu'on note $Z(x_1, \ldots, x_q)$.

**Définition 0.3** *Etant donné deux graphes* $\Gamma_1$ *et* $\Gamma_2$*, on dit que* $\Gamma_2$ *est plus fin que* $\Gamma_1$ *et on note* $\Gamma_1 < \Gamma_2$ *si toute arête de* $\Gamma_1$ *est un chemin de* $\Gamma_2^*$.

Considérons deux graphes $\Gamma_1$ et $\Gamma_2$. Supposons que $\Gamma_2$ soit plus fin que $\Gamma_1$ et posons $\Gamma_1 = \{a_1 \ldots, a_r\}$. Par définition, chaque arête $a_i$ de $\Gamma_1$ est un chemin de $\Gamma_2^*$ et permet à ce titre de définir une fonction $h_{a_i} : G^{\Gamma_2} \longrightarrow G$. Le $r$-uplet formé par ces fonctions en constitue une nouvelle, $(h_{a_1}, \ldots, h_{a_r}) : G^{\Gamma_2} \longrightarrow G^r = G^{\Gamma_1}$ que nous notons $f_{\Gamma_1\Gamma_2}$. L'invariance par subdivision peut alors être exprimée de la façon suivante:

**Théorème 0.4 (1.6.1)** *1. L'application* $f_{\Gamma_1\Gamma_2} : G^{\Gamma_2} \longrightarrow G^{\Gamma_1}$ *est surjective.*
*2. Elle vérifie de plus :* $(f_{\Gamma_1\Gamma_2})_* P^{\Gamma_2}(x_1, \ldots, x_q) = P^{\Gamma_1}(x_1, \ldots, x_q)$.

Ce résultat essentiel exprime que la loi de l'holonomie discrète le long d'une famille de chemins reste inchangée si on remplace le graphe initial par un graphe plus fin. Il permet également de montrer que les masses naturelles des mesures $P$ et $P(x_1, \ldots, x_q)$, c'est-à-dire les fonctions de partitions, sont invariantes par raffinement du graphe (cf. 1.6.5).

On peut également utiliser l'invariance par subdivision pour établir sans calcul que la loi de l'holonomie discrète le long d'un chemin ouvert est uniforme sur $G$. (cf. paragraphe 1.8.1). D'une façon générale, elle est essentielle dans tous les calculs explicites car elle permet, dans une certaine mesure, de choisir le graphe dans lequel on se place. Ainsi, dans le paragraphe 1.8.2, on estime la loi de l'holonomie le long du bord d'un petit disque. Notons, pour tout élément $x$ de $G$, $\rho(x)$ la distance $d(1, x)$.

**Lemme 0.5 (1.8.3)** $\int_G \rho(g)^4 p_t(g)\, dg = O(t^2)$.

En utilisant cette propriété de régularité du semi-groupe $(p_t)_{t>0}$, on prouve l'estimation fondamentale suivante:

**Proposition 0.6 (1.8.5)** *Soit* $\Gamma$ *un graphe sur* $M$. *Soient* $L_1, \ldots, L_q$ *des lacets simples disjoints de* $\Gamma^*$ *et* $x_1, \ldots, x_q$ *des éléments de* $G$. *Soit* $l$ *le bord d'un disque* $D$ *dont l'adhérence ne contient complètement aucun* $L_i$. *Il existe deux constantes positives* $s$ *et* $C$ *qui dépendent des* $L_i$ *mais pas des* $x_i$ *telles que si* $\sigma(D) \leq s$, *alors*

$$\int_{G^\Gamma} \rho(h_l)\, dP(x_1, \ldots, x_q) \leq C\sqrt{\sigma(D)}.$$

Cette estimation sera une des clés du passage à la limite continue. La mesure vérifie une autre propriété d'invariance, qui est l'invariance par les difféomorphismes qui préservent la surface.

**Proposition 0.7 (1.7.1)** *Soient* $(M_1, \sigma_1)$ *et* $(M_2, \sigma_2)$ *deux surfaces et* $\phi : M_1 \longrightarrow M_2$ *un difféomorphisme tel que* $\phi_*\sigma_1 = \sigma_2$. *Soit* $\Gamma_1$ *un graphe sur* $M_1$ *et* $\Gamma_2 = \phi(\Gamma_1)$ *le graphe correpondant sur* $M_2$. *En notant encore* $\phi : G^{\Gamma_1} \longrightarrow G^{\Gamma_2}$ *la bijection induite, on a:*

$$\phi_* P^{\Gamma_1}(x_1, \ldots, x_q) = P^{\Gamma_2}(x_1, \ldots, x_q).$$

L'importance de cette propriété vient du fait que le groupe des difféomorphismes qui préservent la surface est très gros: la théorie ne dépend de $M$ que par ses invariants sous l'action de ce groupe, qui sont son genre, le nombre de composantes de son bord et sa surface totale. Ceci jouera un rôle notable dans l'étude des fonctions de partition conditionnelles.

**Etude de l'holonomie discrète lorsque $G = U(1)$.** Dans ce cas, la fonction $h_c$ associée à un chemin $c \in \Gamma^*$ ne dépend que du nombre de fois où $c$ parcourt chaque arête de $\Gamma$. Plus formellement, elle ne dépend que de l'image de $c$ par le morphisme de monoïdes $\Gamma^* \longrightarrow \mathbf{Z}^\Gamma$ qui envoie l'arête $a_i$ sur $(0, \ldots, 1, \ldots, 0)$, avec un 1 en $i$-ième place. Réciproquement, tout élément de $\mathbf{Z}^\Gamma$ permet de définir sans ambiguité une fonction de $U(1)^\Gamma$ dans $U(1)$. C'est pourquoi nous nous intéressons à l'holonomie aléatoire le long d'éléments de $\mathbf{Z}^\Gamma$, plus précisément le long des cycles, qui sont les combinaisons linéaires de lacets. On note $C\Gamma$ l'ensemble de ces cycles et $C_0\Gamma$ l'ensemble des cycles homologues à zéro.

Notons $g$ le genre de $M$. Si le bord de $M$ s'écrit $\partial M = N_1 \cup \ldots \cup N_p$, il existe $2g$ lacets $\ell_1, \ldots, \ell_{2g}$ sur $M$ tels que $\ell_1, \ldots, \ell_{2g}, N_1, \ldots, N_{p-1}$ forment un système de représentants d'une base de $H_1(M; \mathbf{Z})$ (cf. 1.9.1). Un cycle $c \in C\Gamma$ s'écrit alors de façon unique

$$c = \lambda_1\ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1} + c^\perp, \tag{4}$$

avec $c^\perp \in C_0\Gamma$.

**Proposition 0.8 (1.9.2)** *Si $M$ n'a pas de bord (resp. a un bord), les bords de toutes les faces de $\Gamma$ sauf une quelconque (resp. de toutes les faces de $\Gamma$) forment une base du sous-module $C_0\Gamma$ de $C\Gamma$.*

Notons $F_1, \ldots, F_n$ les faces de $\Gamma$. Nous pouvons, d'après ce résultat, encore décomposer $c$:

$$c = \lambda_1\ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1} + \mu_1\partial F_1 + \ldots + \mu_n\partial F_n,$$

d'une façon qui n'est pas unique si $M$ est fermée. Quoi qu'il en soit, la multiplicativité de l'holonomie montre que la loi de la famille $(h_c)_{c\in C\Gamma}$ est déterminée par celle de la famille finie suivante, qu'on appellera système fondamental:

$$(h_{\ell_1}, \ldots, h_{\ell_{2g}}, h_{N_1}, \ldots, h_{N_{p-1}}, h_{\partial F_1}, \ldots, h_{\partial F_n}).$$

**Proposition 0.9 (1.9.4)** *Posons $x = x_1 \ldots x_p$ si $M$ a un bord et $x = 1$ sinon. Pour toute fonction $f$ continue sur $G^{2g+n+p}$, on a*

$$\int_{G^\Gamma} f(h_{\ell_1}, \ldots, h_{\ell_{2g}}, h_{N_1}, \ldots, h_{N_{p-1}}, h_{\partial F_1}, \ldots, h_{\partial F_n})\, dP(x_1, \ldots, x_p) =$$

$$\int_{G^{2g+n}} f(u_1, \ldots, u_{2g}, x_1, \ldots, x_{p-1}, v_1, \ldots, v_n) p_{\sigma(F_1)}(v_1) \ldots p_{\sigma(F_n)}(v_n)\, du_1 \ldots du_{2g}\, d\nu_x^n(v_1, \ldots, v_n),$$

*où $d\nu_x^n$ est le produit des mesures de Haar sur $U(1)^n$ conditionné à ce que le produit des facteurs vaille $x$.*

Ce résultat montre en particulier que l'holonomie le long de cycles homologiquement non nuls est indépendante de celle le long de cycles homologiquement nuls. Il montre également que la partie la plus intéressante de la loi est celle qui concerne l'holonomie le long du bord des faces de $\Gamma$. Une étude plus détaillée de cette holonomie va mettre en évidence le caractère gaussien de la mesure de Yang-Mills dans ce cadre commutatif.

**Proposition 0.10 (1.9.5)** *Soient $Y_1, \ldots, Y_n$ des variables gaussiennes réelles indépendantes telles que $Y_i \sim \mathcal{N}(0, \sigma(F_i))$. Soit $S = Y_1 + \ldots + Y_n$ leur somme. Pour tout $i = 1, \ldots, n$, posons*

$$X_i = Y_i - \frac{\sigma(F_i)}{\sigma(M)} S.$$

*Soit $T$ une v.a. réelle indépendante des $Y_i$, dont la loi, discrète, est la suivante:*

$$P(T = t) = \begin{cases} \left( \sum_{s, e^{is} = x} e^{-\frac{s^2}{2\sigma(M)}} \right)^{-1} e^{-\frac{t^2}{2\sigma(M)}} & \text{si } e^{it} = x \\ 0 \text{ sinon}, \end{cases}$$

*où, comme précédemment,* $x = x_1 \ldots x_p$ *si* $M$ *a un bord et* $x = 1$ *sinon. Alors, pour toute fonction* $f$ *continue sur* $G^n$ *on a,*

$$\int_{G^\Gamma} f(h_{\partial F_1}, \ldots, h_{\partial F_n})\, dP(x_1, \ldots, x_p) = E\, f\left(e^{i\left(X_1 + \frac{\sigma(F_1)}{\sigma(M)}T\right)}, \ldots, e^{i\left(X_n + \frac{\sigma(F_n)}{\sigma(M)}T\right)}\right).$$

Notons que la loi de $T$ est celle d'une variable $\mathcal{N}(0, \sigma(M))$ conditionnée à prendre ses valeurs dans $\exp^{-1}(x)$. Ce résultat montre une réalisation possible de la loi de l'holonomie le long des bords des faces de $\Gamma$ à partir de variables gaussiennes réelles. Il est cependant possible d'aller plus loin en montrant comment ces variables gaussiennes sont naturellement associées aux faces de $\Gamma$. Pour cela, nous remarquons que les fonctions

$$u_i = \mathbf{1}_{F_i} - \frac{\sigma(F_i)}{\sigma(M)}$$

définies pour $i = 1, \ldots, n$ forment un $n$-uplet $(u_1, \ldots, u_n)$ isométrique à $(X_1, \ldots, X_n)$. Soit alors $W$ un bruit blanc sur $M$ in dépendant de $T$, c'est-à-dire une isométrie

$$\begin{aligned} W : L^2(M, \sigma) &\longrightarrow \mathcal{G} \\ u &\longmapsto W(u) \end{aligned}$$

à valeurs dans un espace vectoriel $\mathcal{G}$ de variables aléatoires gaussiennes indépendantes de $T$. Nous pouvons traduire la proposition 0.10 comme suit:

**Proposition 0.11 (1.9.9)** *On a l'égalité en loi suivante:*

$$(h_{\partial F_1}, \ldots, h_{\partial F_n}) \stackrel{\mathcal{L}}{=} \left(e^{i\left(W(u_{\partial F_1}) + \frac{\sigma(F_1)}{\sigma(M)}T\right)}, \ldots, e^{i\left(W(u_{\partial F_n}) + \frac{\sigma(F_n)}{\sigma(M)}T\right)}\right).$$

Nous allons généraliser ce résultat à tous les cycles de $C_0\Gamma$. Pour cela, il faut étendre la définition de la fonction $u_i$ à tous les cycles. Ceci nécessite le choix sur $M$ d'une métrique riemannienne dont le volume coïncide avec la mesure $\sigma$. Il en existe, car $\sigma$ est équivalente à la mesure de Lebesgue avec une densité lisse dans chaque carte. Une telle métrique permet de définir sur $M$ un opérateur de Hodge $*$ sur les 1-formes différentielles et un laplacien $\Delta$. Ce laplacien possède une fonction de Green, c'est-à-dire qu'il existe une fonction lisse $G : M \times M \longrightarrow \mathbf{R}_+$ définie et lisse hors de la diagonale, symétrique, qui vérifie les propriétés suivantes:

$$\begin{cases} \Delta G(x, \cdot) = \delta_x - \frac{1}{\sigma(M)} \ \ \forall x \in M \\ \int_M G(x,y)\, d\sigma(y) = 0 \ \ \forall x \in M \ \text{ lorsque } \partial M = \emptyset \\ *dG_x = 0 \ \text{ sur } \partial M \ \ \forall x \in M \ \text{ lorsque } \partial M \neq \emptyset, \end{cases}$$

où $G_x$ désigne la fonction $y \mapsto G(x, y)$.

**Définition 0.12 (1.9.6)** *Soit* $c$ *un chemin sur* $M$*. On appelle potentiel de double couche de* $c$ *la fonction* $u_c$ *définie sur* $M$ *hors de l'image de* $c$ *par :*

$$u_c(x) = \int_c *dG_x.$$

**Lemme 0.13 (1.9.8)** *Le vecteur* $(u_1, \ldots, u_n)$ *est égal à* $(u_{\partial F_1}, \ldots, u_{\partial F_n})$.

Le potentiel de double couche est la généralisation la plus naturelle au cas d'une surface de la notion classique d'indice d'un lacet autour d'un point dans le plan. Pour généraliser la proposition 0.11 à tous les cycles de $C_0\Gamma$, il faut encore généraliser le terme $\sigma(F_i)/\sigma(M)$. On définit ainsi $\sigma_{\text{int}}(c)$ pour tout $c \in C_0\Gamma$ par $\sigma_{\text{int}}(c) = \sigma(\alpha)/\sigma(M)$, où $\alpha$ est une 2-chaîne bordée par $c$. On a alors le résultat suivant:

**Proposition 0.14 (1.9.10)** *Soient $(c_1, \ldots, c_k)$ des cycles de $C_0\Gamma$. On a l'égalité en loi suivante:*

$$(h_{c_1}, \ldots, h_{c_k}) \stackrel{\mathcal{L}}{=} \left(e^{i\left(W(u_{c_1})+\sigma_{\text{int}}(c_1)T\right)}, \ldots, e^{i\left(W(u_{c_k})+\sigma_{\text{int}}(c_k)T\right)}\right).$$

**Théorie continue abélienne.** La notion de potentiel de double couche permet en fait de définir une holonomie aléatoire le long de tous les cycles sur $M$, c'est-à-dire le long des combinaisons linéaires de lacets de $PM$. Soit $c$ un tel cycle. Nous avons déjà noté qu'il lui était associé un cycle homologue à zéro $c^\perp$, par:

$$c^\perp = c - (\lambda_1 \ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1}).$$

Soient $U_1, \ldots, U_{2g}$ des variables aléatoires uniformes sur $U(1)$, indépendantes de $T$ et de $W$. On pose alors

$$\Theta_c = U_1^{\lambda_1} \ldots U_{2g}^{\lambda_{2g}} x_1^{\nu_1} \ldots x_{p-1}^{\nu_{p-1}}.$$

Soit $(\Omega, P)$ un espace de probabilités qui supporte $T$, $W$ et les $U_i$.

**Définition 0.15 (3.1.1)** *Pour tout cycle $c \in CM$, on définit la variable aléatoire suivante sur $(\Omega, P)$:*

$$WH_c = \exp i(W_0(u_{c^\perp}) + \sigma_{\text{int}}(c^\perp)T)\ \Theta_c.$$

Cette famille de variables aléatoires a, sur toute famille de cycles qui peut s'inscrire dans un graphe, la loi de l'holonomie aléatoire sous la mesure de Yang-Mills. Un résultat d'unicité que nous énoncerons plus tard montre qu'il suffit de prouver une propriété de régularité de cette famille pour montrer que sa loi complète est celle de l'holonomie aléatoire de Yang-Mills.

Pour parler de régularité, il faut munir l'espace des chemins d'une topologie. La première qui vienne à l'esprit est la topologie induite par la distance uniforme pour une métrique riemannienne sur $M$, mais il s'avère qu'elle n'est pas assez fine. Une métrique étant fixée, on note $\ell(c)$ la longueur d'un chemin $c$. On pose alors

$$d_1(c, c') = d_\infty(c, c') + |\ell(c) - \ell(c')|.$$

La topologie que cette distance induit sur $PM$ ne dépend pas de la métrique et c'est elle que nous adopterons systématiquement. Cette distance s'étend à l'espace $CM$ des cycles sur $M$ et on a le résultat de régularité suivant:

**Proposition 0.16 (3.1.9)** *L'application $c \mapsto WH_c$ est continue de $(CM, d_1)$ dans l'espace des variables aléatoires de carré intégrable sur $(\Omega, P)$.*

**Passage à la limite continue.** Reprenons l'étude du cas général là où nous l'avions laissée. La première étape vers la limite continue consiste à considérer une famille de graphes de plus en plus fins et à prendre la limite projective des espaces de probabilités qui leurs sont associés. En fait, nous utilisons la famille des graphes dont les arêtes sont géodésiques par morceaux pour une certaine métrique sur $M$. Nous disposons alors d'une holonomie aléatoire le long de tous les chemins géodésiques par morceaux. La deuxième étape consiste à montrer qu'un procédé d'approximation naturel de chemins quelconques par des chemins géodésiques par morceaux permet de définir l'holonomie aléatoire le long de tous les chemins de $PM$, de façon indépendante de tous les choix effectués. Notons que les surfaces à bord requièrent, pour des raisons techniques, un traitement particulier mais qu'on obtient un résultat final identique pour des surfaces à bords et des surfaces fermées. La loi de la famille de variables aléatoires qui est alors construite est une mesure sur l'espace $\mathcal{F}(PM,G)$. On la transforme en une mesure sur le quotient $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$ et c'est cette mesure que nous appelons mesure de Yang-Mills.

Nous nous restreignons pour l'instant au cas où $M$ n'a pas de bord. Nous fixons $q$ lacets simples disjoints $L_1,\ldots,L_q$ sur $M$. Pour choisir une métrique sur $M$, nous utilisons le résultat suivant:

**Proposition 0.17 (2.2.1)** *Il existe une métrique riemannienne sur $M$ dont le volume riemannien est égal à $\sigma$ et telle que les $L_i$ soient géodésiques.*

On définit alors l'ensemble $\mathcal{G}$ des graphes $\Gamma$ dont les arêtes sont géodésiques par morceaux et qui sont tels que $L_1,\ldots,L_q \in \Gamma^*$. Cet ensemble est ordonné par la relation de plus grande finesse et satisfait la propriété suivante:

**Proposition 0.18 (2.2.5)** *Etant donné $\Gamma_1,\Gamma_2$ dans $\mathcal{G}$, il existe $\Gamma_3$ dans $\mathcal{G}$ tel que $\Gamma_1 < \Gamma_3$ et $\Gamma_2 < \Gamma_3$.*

Cette propriété découle des propriétés locales des géodésiques, qui ne peuvent se couper plusieurs fois sans être confondues. Elle peut sembler anodine mais c'est parce que l'ensemble de tous les graphes sur $M$ ne la vérifie pas qu'il est si compliqué de passer à la limite continue. En effet, deux chemins, même très réguliers, peuvent découper une infinité de composantes connexes dans $M$, auquel cas il n'existe aucun graphe qui les contienne tous les deux.

Cette propriété était le dernier maillon manquant pour pouvoir affirmer que la famille des espaces de probabilités $(G^\Gamma, P^\Gamma(x_1,\ldots,x_q)), \Gamma\in\mathcal{G}$ constitue, avec les fonctions $f_{\Gamma_1\Gamma_2}$, un système projectif. Le fait que tous ces espaces de probabilités soient boréliens compacts permet alors d'affirmer qu'ils possèdent une limite projective, qui est un espace $(\Omega, P(x_1,\ldots,x_q))$ sur lequel sont simultanément définies toutes les variables définies sur les $G^\Gamma$. Autrement dit, pour tout chemin géodésique par morceaux $\zeta$, il existe une variable $H_\zeta$ définie sur $\Omega$ qui est l'holonomie aléatoire le long de $\zeta$.

Il faut maintenant prouver un résultat d'approximation. Pour deux variables aléatoires $X$ et $Y$ définies sur le même espace et à valeurs dans $G$, nous notons $d_P(X,Y) = Ed(X,Y)$. La notion de convergence que nous utiliserons pour les variables aléatoires est celle induite par cette distance. Le résultat fondamental est le suivant.

**Proposition 0.19 (2.6.6)** *Soit $c$ un chemin de $PM$. Pour toute suite $(\alpha_n)_{n\geq 0}$ de chemins géodésiques par morceaux qui converge à extrémités fixées vers $c$, la suite $(H_{\alpha_n})_{n\geq 0}$ converge* *vers une variable aléatoire qui ne dépend que de c et que nous noterons $H_c$. De plus, pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que si $c'$ est un autre chemin de PM de mêmes extrémités que c et si $d_1(c, c') < \delta$, alors $d_P(H_c, H_{c'}) < \varepsilon$.*

Pour prouver ce résultat, on commence par s'intéresser aux chemins dont l'image est une sous-variété de $M$. Ces chemins ont l'avantage de posséder un voisinage tubulaire qu'on peut paramétrer en utilisant des coordonnées de Fermi. Ces coordonnées permettent un contrôle commode de l'approximation par des chemins géodésiques par morceaux. Pour estimer la distance entre les variables aléatoires, on utilise un résultat de combinatoire des chemins qui permet de décomposer n'importe quel chemin raisonnable en un produit de chemins élémentaires appelés lassos (voir figure 2.1). L'estimation fondamentale présentée dans le cadre de la théorie discrète (proposition 0.6) permet de contrôler l'holonomie le long de lassos dont la boucle n'est pas trop grosse. Une fois le résultat acquis pour les sous-variétés, on l'étend aisément aux chemins de $PM$ qui en sont des concaténations.

A ce stade, on a certes construit une holonomie aléatoire, mais il faut vérifier qu'elle ne dépend pas, au moins en loi, du choix de la métrique riemannienne qui a servi au passage à la limite. Dans ce but, on montre qu'on peut approcher un graphe quelconque sur $M$ par des graphes géodésiques par morceaux en préservant la structure de ce graphe (cf. 2.7.3). On en déduit l'indépendance souhaitée et on montre au passage que la fonction de partition conditionnelle ne dépend pas du graphe dans lequel on la calcule.

Il est alors temps de s'occuper des surfaces à bord. Si $M$ en est une, on la plonge dans une surface $M_1$ obtenue en bouchant les trous de $M$ avec des disques. L'idée est de définir l'holonomie aléatoire sur $M$ comme la restriction aux chemins de $M$ de celle qu'on sait déjà construire sur $M_1$. Cependant, et cette remarque anticipe les propriétés markoviennes de la mesure de Yang-Mills, si nous voulons que ce procédé donne un résultat qui ne dépende que de $M$ et pas de la façon dont on l'a fermée, il faut conditionner l'holonomie sur $M_1$ par rapport à sa valeur le long de chaque composante du bord de $M$. Le problème est qu'on ne souhaite pas forcément conditionner la mesure sur $M$. On commence donc par montrer qu'il existe une loi naturelle de l'holonomie le long du bord d'une surface, lorsqu'on ne la conditionne pas ou pas complètement: dans le résultat qui suit, on n'impose la valeur de l'holonomie que le long de $k$ composantes de $\partial M$ parmi $p$.

**Proposition 0.20 (2.8.1)** *Soit $\Gamma$ un graphe sur M tel que $L_1, \ldots, L_q \in \Gamma^*$. La loi de la variable aléatoire $(h_{N_1}, \ldots, h_{N_p}, h_{L_1}, \ldots, h_{L_q})$ définie sur l'espace de probabilités $(G^\Gamma, P(x_1, \ldots, x_k, y_1, \ldots, y_q))$ ne dépend pas de $\Gamma$.*

On construit ensuite l'holonomie sur $M$ conditionnellement à sa valeur sur toutes les composantes de $\partial M$ et on intègre les mesures obtenues pour restituer à l'holonomie le long du bord sa loi naturelle.

On aboutit au théorème central de la construction. Notons qu'il ne définit pas encore la mesure de Yang-Mills.

**Théorème 0.21 (2.9.1)** *Soit $(M, \sigma)$ une surface avec ou sans bord munie d'une mesure de surface. Soient $L_1, \ldots, L_q$ des lacets simples disjoints de PM, dont l'image est soit intérieure à M, soit égale à une des composantes de son bord. Etant donné des éléments $x_1, \ldots, x_q$ de G, il existe un espace de probabilités $(\Omega, \mathcal{A}, P(x_1, \ldots, x_q))$ et une famille de variables aléatoires $(H_c)_{c \in PM}$ à valeurs dans G définies sur cet espace, telle que:*

*1. Pour tout graphe $\Gamma = \{a_1, \ldots, a_r\}$ sur $M$ tel que $L_1, \ldots, L_q \in \Gamma^*$, la loi de $(H_{a_1}, \ldots, H_{a_r})$ est la mesure de Yang-Mills discrète $P^\Gamma(x_1, \ldots, x_q)$ sur $G^\Gamma$.*
*2. Pour tout chemin $c$ de $PM$ et toute suite $(c_n)_{n\geq 0}$ de chemins de $PM$ tels que $c_n \xrightarrow{d_1} c$ à extrémités fixées, on a $H_{c_n} \xrightarrow{d_P} H_c$.*
*La loi de cette famille de variables aléatoires est complètement déterminée par ces deux propriétés. De plus, elle a les propriétés supplémentaires suivantes:*
*3. Si $c_1$ et $c_2$ sont deux chemins qu'on peut concaténer pour former $c_1c_2$, alors on a presque sûrement l'égalité $H_{c_1c_2} = H_{c_2}H_{c_1}$.*
*4. Si $\varphi : M \longrightarrow M$ est un difféomorphisme tel que $\varphi_*\sigma = \sigma$, alors $\varphi$ induit une permutation de l'ensemble $PM$ et les familles $(H_c)_{c\in PM}$ et $(H_{\varphi(c)})_{c\in PM}$ ont la même loi.*

L'unicité établie par ce théorème assure que la famille de variables qu'on a construite en utilisant un bruit blanc dans le cas où $G = U(1)$ a bien la même loi que celle qu'on obtient par la méthode générale.

La loi dont ce théorème établit l'existence et l'unicité est une mesure $\mu_0(x_1, \ldots, x_q)$ sur l'espace $\mathcal{F}(PM, G)$ muni de la tribu engendrée par les cylindres. Cependant, cette mesure ne mérite pas encore le nom de mesure de Yang-Mills. Comme nous l'avons expliqué au début de cette introduction, la donnée d'une classe de connexions modulo transformations de jauge permet de déterminer une classe du quotient $\mathcal{F}(LM, G)/\mathcal{F}(M, G)$, mais pas mieux. Nous allons donc prendre successivement l'image de $\mu_0(x_1, \ldots, x_q)$ par l'application de restriction $\mathcal{F}(PM, G) \longrightarrow \mathcal{F}(LM, G)$ puis par la projection canonique $\mathcal{F}(LM, G) \longrightarrow \mathcal{F}(LM, G)/\mathcal{F}(M, G)$.

Pour définir une tribu sur $\mathcal{F}(LM, G)/\mathcal{F}(M, G)$, on commence par y définir une famille de fonctions. Notons $G^n/\operatorname{Ad}$ le quotient de $G^n$ par l'action diagonale de $G$ définie par $g.(g_1, \ldots, g_n) = (\operatorname{Ad}(g)g_1, \ldots, \operatorname{Ad}(g)g_n)$. Nous désignerons par $[g_1, \ldots, g_n]$ la classe d'un élément $(g_1, \ldots, g_n)$. Notons $H_l, l \in LM$ le processus canonique sur $\mathcal{F}(LM, G)$, qui est muni de la tribu engendrée par les cylindres $\mathcal{C}$. Etant donné $n$ lacets $l_1, \ldots, l_n$ basés au même point, la classe de conjugaison conjointe $[H_{l_1}, \ldots, H_{l_n}]$ est une fonction invariante par l'action de $\mathcal{F}(M, G)$ et nous notons $\mathcal{H}_{l_1,\ldots,l_n}$ la fonction qu'elle induit sur le quotient. La tribu $\mathcal{A}$ que nous considérons est la tribu engendrée par ces variables aléatoires. Un autre choix naturel aurait été celui de la tribu des éléments invariants de $\mathcal{C}$. Un argument utilisant le théorème de Blackwell permet au moins de montrer que ces deux tribus concurrentes ont la même complétion par rapport à la mesure $\mu_0(x_1, \ldots, x_q)$. Nous pouvons maintenant définir la mesure de Yang-Mills:

**Définition 0.22 (2.10.4)** *On appelle mesure de Yang-Mills sur $M$ et on note $\mu_M$ la projection sur l'espace $(\mathcal{F}(LM, G)/\mathcal{F}(M, G), \mathcal{A})$ de la mesure $\mu_0$. De même, étant donné des éléments $t_1, \ldots, t_q$ de $G/\operatorname{Ad}$, on appelle mesure de Yang-Mills conditionnelle par rapport à $L_1, \ldots, L_q$ et on note $\mu_M(t_1, \ldots, t_q)$ la projection sur $(\mathcal{F}(LM, G)/\mathcal{F}(M, G), \mathcal{A})$ de la mesure $\mu_0(x_1, \ldots, x_q)$, où les $x_i$ sont des représentants des $t_i$.*

Il est implicite dans cette définition que la projection sur $\mathcal{A}$ de $\mu_0(x_1, \ldots, x_q)$ ne dépend que de la classe de conjugaison des $x_i$ (cf. 2.10.3). Il est par ailleurs également possible de voir $\mathcal{A}$ comme une tribu sur $\mathcal{F}(LM, G)$ et de considérer la mesure de Yang-Mills comme une mesure sur $\mathcal{F}(LM, G)$ invariante par l'action de $\mathcal{F}(M, G)$. Après avoir étendu de façon naturelle la distance $d_P$ aux variables à valeurs dans $G^n/\operatorname{Ad}$, on montre la propriété de régularité suivante, qui est une conséquence de la régularité de la famille $(H_l)_{l\in LM}$.

**Proposition 0.23 (2.10.7)** *Soit* $((l_{1,k}, \dots, l_{n,k}))_{k\geq 0}$ *une suite de n-uplets de lacets telle que*
*1. pour tout* $k \geq 0$, *les lacets* $l_{1,k}, \dots, l_{n,k}$ *sont basés au même point,*
*2. pour tout* $i = 1, \dots, n$, *il existe un lacet* $l_i$ *tel que* $l_{i,k} \xrightarrow{d_1} l_i$.
*Alors*

$$\mathcal{H}_{l_{1,k},\dots,l_{n,k}} \xrightarrow{d_P} \mathcal{H}_{l_1,\dots,l_n}.$$

On remarque enfin que si l'on fixe un point $m$ dans $M$ et que l'on ne considère que les lacets basés en $m$, on a encore accès à toute l'information disponible dans la tribu $\mathcal{A}$. Ce résultat sera mis a profit pour l'étude du découpage et du recollement des surfaces.

**Structure à petite échelle de la mesure de Yang-Mills: le cas abélien.** La construction étant maintenant terminée, il est possible d'examiner plus en détail les liens entre les deux constructions dont nous disposons dans le cas où $G = U(1)$. Nous avons montré que la mesure de Yang-Mills pouvait être construite à partir d'un bruit blanc, nous allons montrer qu'on peu reconstruire un bruit blanc à partir de la mesure de Yang-Mills. Ceci revient à calculer la courbure de la connexion aléatoire sous-jacente à l'holonomie aléatoire, pour autant qu'on puisse parler de connexion aléatoire. En effet, comme toujours lorsqu'on prétend construire une mesure sur un espace de fonctions lisses, on la construit en fait sur un espace de fonctions beaucoup moins régulières, voire de distributions. Ainsi, la probabilité pour que l'holonomie aléatoire que nous avons construite soit celle d'une vraie connexion lisse est nulle.

Nous considérons sur $M$ une suite de graphes $(\Gamma_n)_{n\geq 0}$ telle que pour tout $n$, $\Gamma_n$ ait exactement $n$ faces $F_{j,n}$, $j = 1, \dots, n$. On suppose également que $\sigma(F_{j,n}) = \frac{\sigma(M)}{n}$ et que le diamètre de ces faces tend uniformément vers 0, c'est-à-dire que $\sup_j \operatorname{diam}(F_{j,n}) \longrightarrow 0$, une métrique quelconque étant choisie sur $M$. Nous choisissons une orientation de $M$ et orientons les bords des faces selon la convention usuelle. Pour tout couple $(j, n)$, $n \geq 0$, $1 \leq j \leq n$, on abrège par $\mathcal{H}_{j,n}$ la variable $\mathcal{H}_{\partial F_{j,n}}$ définie sur $(\Omega_M, \mu_M(x_1, \dots, x_p))$ et on identifie cette variable avec une variable à valeurs dans $\mathbf{C}$ en identifiant $U(1)$ avec $\{z \in \mathbf{C}, |z| = 1\}$.

Nous imitons alors la construction d'une intégrale de Wiener classique. Pour tout $n \geq 0$, soit $E_n$ l'espace des fonctions sur $M$ constantes sur chaque face de $\Gamma_n$. La réunion des $E_n$ est dense dans $L^2(M, \sigma)$. On définit sur chaque $E_n$ une forme linéaire $I_n$: si $f_n \in E_n$ et si $f_{j,n}$ est sa valeur sur $F_{j,n}$, on pose

$$I_n(f_n) = \frac{1}{i} \sum_{j=1}^{n} f_{j,n}(\mathcal{H}_{j,n} - 1).$$

**Théorème 0.24 (3.2.1)** *Soit* $f$ *une fonction de* $L^2(M)$ *et* $(f_n)_{n\geq 0}$ *une suite de fonctions convergeant vers* $f$ *dans* $L^2$ *et telle que* $f_n \in E_n$. *Alors la suite* $(I_n(f_n))_{n\geq 0}$ *converge dans* $L^2(\Omega_M, \mu_M(x_1, \dots, x_p))$ *vers une variable aléatoire* $I(f)$ *qui ne dépend que de* $f$. *On peut décrire la loi de cette variable comme suit. Soit* $W_f^0$ *une v.a. gaussienne réelle centrée de variance* $\| f \|_{L_0^2}^2 = \| f - \frac{1}{\sigma(M)} \int_M f d\sigma \|_{L^2}^2$. *Soit* $T$ *une variable* $\mathcal{N}(0, \sigma(M))$ *conditionnée à prendre ses valeurs dans* $\exp^{-1}(x)$, *indépendante de* $W_f^0$. *On a l'égalité en loi suivante:*

$$I(f) \stackrel{\mathcal{L}}{=} W_f^0 + \left(T + \frac{i}{2}\right) \frac{1}{\sigma(M)} \int_M f \, d\sigma.$$

Ceci montre que l'application $f \mapsto I(f)$ restreinte aux fonctions de moyenne nulle est un bruit blanc. Ceci permet aussi d'interpréter la variable $T$ comme la courbure totale de la connexion aléatoire, qui ne dépend dans le cas déterministe que de la topologie de $P$.

**Structure à petite échelle de la mesure de Yang-Mills: le cas semi-simple.** Dans le cas abélien, le fait qu'on puisse construire un bruit blanc à partir de l'holonomie le long de petits lacets montre que la tribu asymptotique engendrée par ces petits lacets contient presque toute l'information sur la mesure, plus précisément toute l'information qui concerne les lacets homologues à zéro. Nous montrons que dans le cas où $G$ est semi-simple, par exemple lorsque $G = SU(2)$, la situation est radicalement différente.

Soit $L$ un lacet simple sur $M$ qui est le bord d'un ouvert $D$ difféomorphe à un disque. Pour chaque $n \geq 0$, considérons un graphe sur $D$ qui a exactement $n$ faces $F_{1,n}, \ldots, F_{n,n}$ telles que $\sigma(F_{i,n}) = \frac{\sigma(D)}{n}$ pour $i = 1 \ldots n$.

Si $G$ était abélien, on aurait l'égalité entre cycles $L = \partial F_{1,n} + \ldots + \partial F_{n,n}$, pourvu que les orientations soient les bonnes. Ceci entraînerait $\mathcal{H}_L = \mathcal{H}_{\partial F_{1,n}} \ldots \mathcal{H}_{\partial F_{n,n}}$ et pour toute fonction $f$ continue sur $G/\operatorname{Ad} = G$, on aurait :

$$E[f(\mathcal{H}_L)|\mathcal{H}_{\partial F_{1,n}}, \ldots, \mathcal{H}_{\partial F_{n,n}}] = f(\mathcal{H}_L).$$

Lorsque $G$ est semi-simple, la comportement de la mesure est diamétralement opposé.

**Théorème 0.25 (4.1.1)** *Pour toute fonction $f$ continue sur $G/\operatorname{Ad}$, on a la convergence suivante:*

$$E[f(\mathcal{H}_L)|\mathcal{H}_{\partial F_{1,n}}, \ldots, \mathcal{H}_{\partial F_{n,n}}] \xrightarrow[n\to\infty]{L^2} Ef(\mathcal{H}_L).$$

Ainsi, il ne reste aucune information lorsqu'on regarde la mesure à l'échelle microscopique, en tout cas lorsqu'on la regarde de cette façon. Ceci indique que partir d'un bruit blanc sur $M$ pour construire la mesure dans ce cas risque fort de mener à une impasse.

La preuve de ce théorème repose principalement sur deux points techniques, qui sont l'utilisation de la théorie des caractères sur $G$ et l'étude du processus $\chi_\alpha(B_t)$ lorsque $(B_t)_{t>0}$ est un mouvement brownien sur $G$ et $\alpha$ une représentation irréductible de $G$.

**Découpage et recollement de surfaces.** Nous consacrons enfin un chapitre à l'étude de la chirurgie des surfaces du point de vue de la mesure de Yang-Mills. Deux idées principales s'en dégagent. La première concerne le caractère markovien de la mesure de Yang-Mills. En termes informels, il s'agit du fait que si l'on sépare une surface en deux morceaux en tracant un ou plusieurs lacets simples dessus, les holonomies aléatoires le long de lacets qui restent de part et d'autre de la frontière sont indépendantes conditionnellement à la valeur de l'holonomie le long de la frontière. Donnons un énoncé précis.

Soient $M_1$ et $M_2$ deux surfaces orientées dont les bords s'écrivent respectivement $\partial M_1 = N_1 \cup \ldots \cup N_{p_1} \cup B_1 \cup \ldots \cup B_p$ et $\partial M_2 = N'_1 \cup \ldots \cup N'_{p_2} \cup B'_1 \cup \ldots \cup B'_p$. On suppose $p > 0$. Soit $M$ la surface obtenue en identifiant chaque $B_i$ avec $B'_i$ par un difféomorphisme qui renverse l'orientation. Notons $L_1, \ldots, L_p$ des lacets de $M$ dont les images sont $B_1 = -B'_1, \ldots, B_p = -B'_p$. Considérons la mesure de Yang-Mills sur $M$ comme une mesure sur $(\mathcal{F}(LM,G), \mathcal{A})$. Il y a deux sous-tribus naturelles de $\mathcal{A}$ dans ce contexte, qui sont les $\widetilde{\mathcal{A}}_i = \sigma(\mathcal{H}_{l_1,\ldots,l_n}, l_k \in LM_i)$, $i = 1, 2$. De plus, toute fonction $f_i$ sur $\mathcal{F}(LM_i, G)$ permet de définir une fonction $\widetilde{f}_i$ sur $\mathcal{F}(LM, G)$ et il est équivalent de dire que $f_i$ est $\mathcal{A}_i$-mesurable ou de dire que $\widetilde{f_i}$ est $\widetilde{\mathcal{A}_i}$-mesurable. De ce fait, nous identifions $\mathcal{A}_i$ et $\widetilde{\mathcal{A}_i}$, ainsi que $f_i$ que $\widetilde{f_i}$.

**Théorème 0.26 (5.1.1)** *Les tribus $\mathcal{A}_1$ et $\mathcal{A}_2$ sont indépendantes sur $(\mathcal{F}(LM,G),\mathcal{A},\mu_M)$ conditionnellement à la variable aléatoire $(\mathcal{H}_{L_1},\ldots,\mathcal{H}_{L_p})$. Soient $f_1$ et $f_2$ deux fonctions mesurables sur $(\mathcal{F}(LM_1,G),\mathcal{A}_1)$ et $(\mathcal{F}(LM_2,G),\mathcal{A}_2)$ respectivement. Alors le produit $f_1f_2$ peut être vu comme une fonction $\mathcal{A}$-mesurable sur $\mathcal{F}(LM,G)$ et pour tous $t_1,\ldots,t_p \in G/\operatorname{Ad}$, on a l'égalité suivante:*

$$E_{\mu_M}[f_1f_2|\mathcal{H}_{L_1}=t_1,\ldots,\mathcal{H}_{L_p}=t_p] = \mu_M(t_1,\ldots,t_p)(f_1f_2) = \mu_{M_1}(t_1,\ldots,t_p)(f_1)\mu_{M_2}(t_1,\ldots,t_p)(f_2).$$

*Enfin, le théorème reste vrai si on remplace les mesures $\mu_M,\mu_{M_1},\mu_{M_2}$ par leurs versions conditionnelles par rapport à des variables choisies parmi $\mathcal{H}_{N_1},\ldots,\mathcal{H}_{N_{p_1}}$, $\mathcal{H}_{N'_1},\ldots,\mathcal{H}_{N'_{p_2}}$.*

En appliquant ce théorème à la situation la plus simple où une surface fermée $M$ est obtenue en recollant deux surfaces $M_1$ et $M_2$ dont le bord a une seule composante, on obtient la relation

$$Z_M\mu_M(f) = \int_{G/\operatorname{Ad}} Z_{M_1}(t)\mu_{M_1}(t)(f_1)\; Z_{M_2}(t^{-1})\mu_{M_2}(t^{-1})(f_2)\,dt$$

qui prouve que les objets qui se recollent le plus naturellement ne sont pas les mesures de probabilités mais les mesures munies de leurs masses naturelles.

L'examen de la situation où l'on coupe une anse d'une surface le long d'un cercle conduit à un théorème de découpage semblable au précédent (5.1.3).

La deuxième idée importante de ce chapitre est celle qui guide les deux théorèmes de recollement. Il ne semble pas qu'elle soit déjà apparue dans la littérature, aussi allons-nous la développer un peu. Le problème consiste à déterminer s'il est possible de reconstituer la mesure de Yang-Mills sur une surface $M$ obtenue par recollement de deux surfaces $M_1$ et $M_2$ le long d'un cercle à partir des mesures de Yang-Mills sur $M_1$ et $M_2$. Choisissons un point sur ce cercle. Il est commode de voir les mesures de Yang-Mills sur $M_1$, $M_2$ et $M$ comme des mesures sur les espaces de fonctions de lacets basés en $m$. La connaissance des mesures sur $M_1$ et $M_2$ nous donne accès à la classe de conjugaison conjointe des holonomies le long de tous les lacets de $L_mM_1$ d'une part et le long de tous les lacets de $L_mM_2$ d'autre part. La problème consiste à recoller ces deux classes conjointes. Prenons l'exemple du groupe $SO(3)$. La classe de conjugaison conjointe de plusieurs rotations est la donnée non seulement de leurs angles mais également des positions relatives de leurs axes dans $\mathbf{R}^3$. Ainsi, dans la situation décrite par la figure 5.2, nous connaissons les angles des trois rotations $H_L, H_{l_1}, H_{l_2}$, nous connaissons également d'une part la position relative des axes de $H_L$ et $H_{l_1}$ et d'autre part celle des axes de $H_L$ et $H_{l_2}$. Ce n'est pas suffisant pour retrouver la classe jointe du triplet $H_L, H_{l_1}, H_{l_2}$: il manque une information qui est une rotation autour de l'axe de $H_L$, c'est-à-dire une conjugaison par un élément du centralisateur de $H_L$. Ceci montre qu'il est nécessaire d'ajouter quelque chose aux mesures de Yang-Mills sur $M_1$ et $M_2$ pour pouvoir les recoller, en l'occurence une mesure sur le centralisateur de l'holonomie le long du bord qu'on recolle. Nous obtenons finalement le théorème suivant. On note $C(x)$ le centralisateur d'un élément $x$ et $dz$ la mesure de Haar sur $C(x)$ qui est un sous-groupe de $G$.

**Théorème 0.27 (5.2.6)** *Soient $\Omega_1$ et $\Omega_2$ les espaces sur lesquels sont définies les mesures de Yang-Mills sur $M_1$ et $M_2$. Soit $t$ un élément de $G/\operatorname{Ad}$ et $x$ un représentant de $t$. Soit $\Omega_R$ l'espace $\Omega_1\times C(x)\times\Omega_2$ muni de la mesure $\mu_{M_1}(t)\otimes dz\otimes\mu_{M_2}(t^{-1})$. Il existe sur $\Omega_R$ une famille de variables aléatoires dont la loi est la mesure de Yang-Mills $\mu_M(t)$ sur $M$.*

Nous examinons également le cas où l'on ajoute une anse à une surface en recollant deux composantes de son bord. Le problème se pose dans les mêmes termes et on montre un théorème tout à fait similaire (cf. 5.2.10).

**Fonctions de partition conditionnelles.** Au cours de la preuve des résultats de découpage, il apparaît que les fonctions de partition conditionnelles, qu'on a définies initialement comme les masses naturelles des désintégrations de la mesure de Yang-Mills discrète, vérifient des propriétés remarquables qui constituent un pendant algébrique à la combinatoire des surfaces. En se servant de l'invariance de la mesure de Yang-Mills par les difféomorphismes qui préservent la surface, on montre que la fonction de partition conditionnelle d'une surface $M$ par rapport aux composantes de $\partial M$ ne dépend en fait de $M$ que par son genre $g$, sa surface totale $T$ et par le nombre $p$ des composantes de $\partial M$. On prouve alors un théorème qui rassemble les propriétés principales de ces fonctions. Notons que ce théorème avait été déjà essentiellement démontré par Witten [Wi].

**Théorème 0.28 (5.3.1)** *Pour tout* $(p, g, T) \in \mathbf{N}^2 \times \mathbf{R}_+^*$, *la fonction* $Z_{g,p,T}$ *est continue et symétrique sur* $(G/\operatorname{Ad})^p$. *On peut en donner au moins deux expressions:*

$$Z_{p,g,T}(t_1, \ldots, t_n) = \int_{G^{2g+p}} p_{\sigma(M)}(y_1^{-1}x_1y_1 \ldots y_p^{-1}x_py_p[a_1, b_1] \ldots [a_g, b_g])\, da_1db_1 \ldots da_gdb_gdy_1 \ldots dy_p,$$

$$Z_{p,g,T}(t_1, \ldots, t_p) = \sum_{\alpha \in \widehat{G}} (\dim \alpha)^{2-2g} e^{-\frac{c_2(\alpha)}{2}T} \prod_{i=1}^{p} \frac{\chi_\alpha(t_i)}{\dim \alpha},$$

*cette dernière étant un développement en série de caractères de représentations irréductibles de* $G$. *De plus, pour tout* $(p', g', T')$ *et tous* $t_1, \ldots, t_p, t'_1, \ldots, t'_{p'} \in G/\operatorname{Ad}$, *on a les relations suivantes:*

$$\int_{G/\operatorname{Ad}} Z_{p+1,g,T}(t_1, \ldots, t_p, t) Z_{p'+1,g',T'}(t^{-1}, t'_1, \ldots, t'_{p'})\, dt = Z_{p+p',g+g',T+T'}(t_1, \ldots, t_p, t'_1, \ldots, t'_{p'}),$$

$$\int_{G/\operatorname{Ad}} Z_{p+2,g,T}(t_1, \ldots, t_p, t, t^{-1})\, dt = Z_{p,g+1,T}(t_1, \ldots, t_p).$$

On a dit que les propriétés algébriques de ces fonctions correspondaient aux propriétés combinatoires de la chirurgie des surfaces, or toute surface peut être obtenue à partir d'un petit nombre de surfaces élémentaires par des opérations de recollement: il suffit par exemple de disposer de disques et de pantalons, c'est-à-dire de sphères à trois trous. Comme on s'y attend, les fonctions de partition vérifient une propriété correspondante.

**Proposition 0.29 (5.3.2)** *La famille de fonctions* $Z_{p,g,T}$ *est complètement déterminée par les fonctions* $Z_{1,0,T}$ *et* $Z_{3,0,T}$, $T > 0$.

Reste à identifier ces fonctions élémentaires.

**Proposition 0.30 (5.3.3)** *La fonction* $Z_{1,0,T}$ *est la projection sur* $G/\operatorname{Ad}$ *de la solution fondamentale de l'équation de la chaleur* $p_T$ *sur* $G$.

La fonction $Z_{3,0,T}$ est d'interprétation moins immédiate. Tout d'abord, remarquons que toutes les fonctions $Z_{p,g,T}$, vues comme des fonctions centrales sur $G^p$, sont des solutions de l'équation de la chaleur par rapport à chacune de leurs variables (cf. 5.3.4). Ceci explicite la dépendance en $T$ de ces fonctions. Pour comprendre la nature de $Z_{3,0,T}$, on utilise le produit de convolution des fonctions centrales sur $G$. On démontre:

**Proposition 0.31 (5.3.5)** *Soient $f$ et $g$ deux fonctions centrales de $L^2(G)$. Alors on a l'égalité suivante dans $L^2(G)$:*

$$\int_{G^2} f_1(x_1)f_2(x_2)Z_{3,0,T}(x_1,x_2,x)\,dx_1dx_2 = \left[e^{-T\frac{\Delta}{2}}(f_1 * f_2)\right](x).$$

Autrement dit, à la limite lorsque $T$ tend vers 0, $Z_{3,0,T}$ est le noyau distributionnel de l'opérateur de convolution.

Enfin, on montre que les fonctions de partition conditionnelles peuvent être vues comme les fonctions de transition du champ aléatoire markovien que la mesure de Yang-Mills définit. Dans cet ordre d'idées, nous montrons qu'une partie de la loi de l'holonomie aléatoire peut être écrite explicitement en ne se servant que de ces fonctions de partition.

# Introduction

This thesis is devoted to the construction and to the study of the Yang-Mills measure in two dimensions. The quadridimensional and pseudo-Riemannian equivalent of this measure is used by physicists in gauge theories such as quantum electrodynamics and quantum chromodynamics, in order to describe the fundamental interactions. It appears in path integrals, which are known to be often ill defined. Physicists describe the Yang-Mills measure as a probability measure on the space of connections modulo gauge transformations on a principal bundle with compact structure group. We are interested in the case where the base space of this bundle is a compact orientable surface. The informal expression of the measure is the following:

$$d\mu(\omega) = \frac{1}{Z}\, e^{-\frac{1}{2}S(\omega)}\, D\omega, \tag{1}$$

where $S$ is the Yang-Mills action, that is, the $L^2$ norm of the curvature. The constant $Z$ is a normalization constant and the measure $D\omega$ should be a translation invariant measure on the space of connections.

The construction of the measure in a discrete setting, where the base space of the fiber bundle is replaced by a graph traced on a surface, is quite well understood thanks to the work of Witten [Wi]. In contrast, the continuum limit of this construction, which should allow to put a genuine manifold as base space, still remains problematic. Several works in this direction have led to substantial progress but not yet to an entirely satisfactory solution.

This work presents a complete and unified approach of the discrete theory and of its continuum limit. We give a geometrically consistent definition of the Yang-Mills measure, under the form of a random holonomy along a wide, intrinsic and natural class of loops. This definition allows us to study combinatorial properties of the measure, like its Markovian behaviour under the surgery of surfaces, as well as properties specific to the continuous setting, for example, some of its microscopic properties. In particular, we clarify the links between the Yang-Mills measure and the white noise and show that there is a major difference between the Abelian and semi-simple theories. We prove that it is possible to construct a white noise using the measure as a starting point and vice versa in the Abelian case but we show a result of asymptotic independence in the semi-simple case which suggests that it is impossible to extract a white noise from the measure.

**Statement of the problem.** We are given a surface $M$, a Lie group $G$ and a principal bundle $P$ over $M$. The surface $M$ is a differentiable two-dimensional compact orientable manifold, with or without boundary. It is endowed with a measure $\sigma$ which is equivalent to the Lebesgue measure in any chart, with a positive smooth density. The group $G$ is a compact connected

Lie group. In most examples, it will be either Abelian or semi-simple. The fiber bundle $P$ is a principal fiber bundle over $M$ with structure group $G$.

Recall that a connection on $P$ is a $G$-invariant choice of a horizontal distribution in $P$ and that this choice can be represented by a $\mathfrak{g}$-valued 1-form $\omega$ on $P$, where $\mathfrak{g}$ is the Lie algebra of $G$. The curvature of the connection $\omega$ is the $\mathfrak{g}$-valued 2-form $\Omega$ on $P$ defined by

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X),\omega(Y)]. \tag{2}$$

The curvature can be considered an $\operatorname{ad} P$-valued 2-form on $M$, where $\operatorname{ad} P$ is the fibre bundle associated with $P$ by the adjoint representation of $G$ on $\mathfrak{g}$. If we choose an orientation on $M$, $\Omega$ can be identified with a section of $\operatorname{ad} P$. An ad-invariant scalar product on $\mathfrak{g}$ allows to define a metric on $\operatorname{ad} P$ and hence the norm $\| \Omega \|$ of the curvature. This norm does not depend on the choice of the orientation of $M$ and the Yang-Mills action is defined on the space $\mathcal{A}$ of connections on $P$ by

$$\begin{aligned} S : \mathcal{A} &\longrightarrow \mathbf{R}_+ \\ \omega &\longmapsto S(\omega) = \int_M \| \Omega \|^2 \, d\sigma. \end{aligned}$$

Our aim is to give a sense to the informal expression (1). The first problem is of course that there is no translation invariant measure on the infinite dimensional affine space $\mathcal{A}$. Another one is the invariance of the action $S$ under the action of a huge group, that of gauge transformations of $P$. This group, denoted by $\mathcal{J}$, is the group of diffeomorphisms of $P$ over the identity of $M$ that commute with the action of $G$. It acts by pull-back on $\mathcal{A}$ and preserves $S$, since it acts on the curvature by pointwise conjugation, which does not change the norm in $\operatorname{ad} P$. Because of this invariance, the constant $Z$ should be proportional to the volume of $\mathcal{J}$, hence be infinite. To avoid this problem, we try to construct the measure on the quotient space $\mathcal{A}/\mathcal{J}$ instead of $\mathcal{A}$. This means that we will be able to integrate only gauge-invariant functions against the Yang-Mills measure, in agreement with the physical principle saying that observable quantities must be gauge-invariant. On the other hand, this quotient space has a much more complicated structure than an affine space. This is why one usually tries to avoid to work directly on it and prefer to work on a function space, as we explain below. The work of D. Fine [Fi1, Fi2] is an exception from this point of view, since the author analyzes the geometrical structure of $\mathcal{A}/\mathcal{J}$ in order to give sense to (1).

**The Yang-Mills measure as random holonomy.** A starting point may be to ask what functions we want to be able to integrate against the Yang-Mills measure. Physicists' answer this question is that we must be able to integrate Wilson loops.

A connection $\omega$ on $P$ defines a parallel transport along regular paths on $M$. The parallel transport along a given path $c : [0,1] \longrightarrow M$ is a $G$-equivariant diffeomorphism of the fiber $P_{c(0)}$ into the fiber $P_{c(1)}$, denoted by $\operatorname{hol}(\omega, c)$. If $c$ is a loop and if we fix a point $p$ in the fiber $P_{c(0)}$, this diffeomorphism can be represented by the element $g$ of $G$ such that $\operatorname{hol}(\omega, c)(p) = pg$. If we choose another point in $P_{c(0)}$, we find another element of $G$ conjugate to $g$. So, for any representation $\rho$ of $G$ and any loop $l$, one defines the Wilson loop $W_{l,\rho}$ by

$$W_{l,\rho}(\omega) = \operatorname{tr} \rho(\operatorname{hol}(\omega, l)).$$

The functions that we want to integrate are central functions of the holonomy along loops. We just noticed that the holonomy along a loop determines a conjugacy class in $G$. We must also take into account the action of the gauge group, which conjugates by the same element of $G$ the holonomies along all loops based at the same point. Let $LM$ denote the set of regular paths on $M$ and $\mathcal{F}(M,G)$, $\mathcal{F}(LM,G)$ the sets of $G$-valued functions on $M$, $LM$. An element $j$ of the group $\mathcal{F}(M,G)$ acts on an element $f$ of $\mathcal{F}(LM,G)$ by:

$$j.f(l) = j(l(0))^{-1}f(l)j(l(0)).$$

It is possible to define a map from $\mathcal{A}/\mathcal{J}$ into the quotient $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$, mapping a connection to the class of the holonomy that it determines along the elements of $LM$. An argument of Sengupta [Se1] proves that this map is injective. Thus, we change our point of view: we seek now a measure on the space $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$, viewing this space as a space of generalized connections modulo gauge transformations. In fact, we shall construct a measure on $\mathcal{F}(LM,G)$ and take the quotient of this measure by $\mathcal{F}(M,G)$. In other words, we really want to construct a random holonomy instead of a random connection. This will be easier because we can use the classical tools of probability to construct a measure on a function space.

At this point, it is necessary to characterize more precisely the Yang-Mills measure. Either one tries to extract more information from the informal expression of the measure or one looks for other description of this measure. The last option is the one that we choosed in this work, using the combinatorial description given by Migdal and Witten. The first one is based on the Gaussian character of the measure and was used by Driver and Sengupta [Dr1, Se1, Se2].

**Gaussian interpretation: curvature of the random connection.** Assume that $G$ is abelian, for example $G = U(1)$. The relation (2) between a connection and its curvature becomes linear. A formal change of variables in (1) gives

$$d\mu(\Omega) = \frac{1}{Z'}e^{-\frac{1}{2}\|\Omega\|^2}\, D\Omega. \tag{3}$$

Since $G$ is abelian, the fiber bundle $\mathrm{ad}(P)$ is trivial and may be identified with $M\times\mathfrak{g}$. We recognize in (3) the expression of a Gaussian measure on the Hilbert space of square integrable $\mathfrak{g}$-valued functions on $M$. This leads us to the main idea of the interpretation of (1): under the Yang-Mills measure, the random curvature of a connection has a Gaussian distribution, it is a $\mathfrak{g}$-valued white noise on $M$ with intensity $\sigma$.

This argument is of course specific to the abelian case, since in general, $\Omega$ is a quadratic function of $\omega$. Nevertheless, the fact that $M$ is two-dimensional allows to overcome this problem: it is always possible to get back to a situation similar to the abelian case by a gauge fixing procedure. This requires a word of explanation.

If we choose a local trivialization of $P$ on an open subset $U \subset M$, i.e. a local section $s : U \longrightarrow P$ of $P$, we can pull-back by $s$ all objects living on $P$, in particular the connection and curvature forms. One denotes usually $A = s^*\omega$ and $F = s^*\Omega$. These forms on $M$ satisfy a structure equation $F = dA + [A,A]$ identical to (2). Let $j$ be an element of the gauge group $\mathcal{J}$. This element can act in two different ways in this situation, either by transforming the section $s$ into $j \circ s$ or by transforming the forms $\omega$ and $\Omega$ into $j^*\omega$ and $j^*\Omega$. These two ways are indistinguishable from the base space, since $(j \circ s)^* = s^*j^*$. So, we denote without ambiguity $s^*j^*\omega = (j\circ s)^*\omega$ by $A^j$.

What is specific to the two-dimensional case is that given a connection, it is possible to choose $s$ in such a way that $[A, A] = 0$. Choose $U$ small enough to admit local coordinates $x, y$. Set $m = (0,0)$ in these coordinates and choose $p$ in $P_m$. Then define $s$ along the $y$-axis by lifting it horizontally, starting at $p$. Now, starting from each point $(0, y_0)$ in $U$, define $s$ on the line through $(0, y_0)$ parallel to the $x$-axis by lifting it horizontally, starting at $s(0, y_0)$. The section $s$ is smooth and horizontal along all lines parallel to the $x$-axis. Thus, $A = s^*\omega$ takes the form $A = A_y dy$ and $[A, A] = 0$. When one looks at $P$ through $s$, what one sees is similar to the abelian case, up to the fact that the section through which the relation between connection and curvature should be linear depends on the connection.

**From the curvature to the holonomy.** The next step is to define a random holonomy using the random curvature. The method is based on deterministic links between holonomy and curvature. Assume that $G = U(1)$ and take $\mathbf{R}^2$ as base space, although it is not a compact surface. Given a connection $\omega$ on the fiber bundle $\mathbf{R}^2 \times G$ and a simple loop $l$ which bounds a domain $D$, the Stokes formula gives

$$\mathrm{hol}(\omega, l) = \exp i \oint_l A = \exp i \int_D dA = \exp i \int_D F = \exp i (F, \mathbf{1}_D)_{L^2}.$$

This formulation is easily extended to the random case. Indeed, pick a white noise $W$ on $\mathbf{R}^2$, i.e. an isometry from $L^2(\mathbf{R}^2)$ into a vector space of Gaussian random variables. One can replace $F$ by $W$ in the last expression and define a random holonomy along $l$ by

$$H_l = \exp iW(\mathbf{1}_D).$$

The construction that we present in the abelian case in chapter 3 is an extension of this procedure to surfaces whose topology is non trivial and where the interior of a loop is not well defined.

It is possible here to understand better the difficulties of Driver and Sengupta. They tried to use this method in the case of a non-Abelian structure group. But in this case, the holonomy is not $\exp \int_l A$, but $P \exp \oint_l A$, which is a compact notation for the solution of the differential equation

$$\begin{cases} \dot{h}_t = A(\dot{l}(t))h_t \\ h_0 = 1. \end{cases}$$

The Stokes formula does not work in this frame. In some sense, one has to choose in which order one multiplies the small elements of $G$ obtained by integrating $F$ over small squares inside $D$. It is not surprising that Driver and Sengupta had to use the coordinate system that allows to define a section through which $[A, A] = 0$ in order to determine this order. The problem is that the class of loops along which they are able to define the random holonomy depends strongly on this choice of coordinates.

It should be noted that B. Driver [Dr2] and L. Gross [Gr] introduced a new local object in order to replace the white noise in this context and that this could lead to a way around the problem.

Although we do not treat this point in our work, we cannot conlude this conclusion without mentioning the semi-classical limit of the Yang-Mills measure. The remarkable fact is that, when the total surface of $M$ tends to 0, the measure concentrates on the set of flat connections over $M$ and tends to the volume measure associated with the natural symplectic structure on this space. There are a lot of references on this subject which is closely related to the geometry of some moduli spaces [Fo, BS, KS, Se3, Liu, AB].

**Combinatorial approach.** Our starting point is the combinatorial approach initiated by A.A. Migdal in 1975 [Mi] and improved by E.Witten in 1991 [Wi]. The idea is to replace the base space of the fiber bundle by a graph on a surface. This leads to a finite dimensional problem, where we define the random holonomy only along the paths of a graph. We also define conditional versions of the random holonomy. These conditional versions will play a technical role in the continuous construction on surfaces with boundary and lead also to the definition of very important objects, the conditional partition functions. The main property of the discrete theory is the invariance by subdivision. It explains that, up to some restrictions, the law of the random holonomy is independent of the graph in which one works.

The next step towards the continuum limit is to take the projective limit of the discrete measures associated with the graphs whose edges are piecewise geodesic for some Riemannian metric on $M$. This allows us to define a random holonomy along all piecewise geodesic paths on $M$. Then, we prove that this random holonomy can be extended by continuity to the set $PM$ of piecewise embedded paths on $M$, using a very natural approximation procedure. The law of this new random holonomy is a measure on $\mathcal{F}(PM, G)$ which is proved to be independent of all choices made during the construction. This measure is pushed forward on $\mathcal{F}(LM, G)/\mathcal{F}(M, G)$ and then becomes what we call the Yang-Mills measure. This measure is characterized by its consistence with the discrete theory and a continuity property. It is multiplicative, as a random holonomy is expected to be, and invariant by area-preserving diffeomorphisms.

All along the discrete construction, we study the special case $G = U(1)$. This analysis leads us to a second construction of a random holonomy, specific to the Abelian case, based on the Gaussian character of the measure in two dimensions. The characterization of the Yang-Mills measure given earlier allows us to prove that this random holonomy has the same law as that defined by the general procedure. Then we show that the holonomy along very small loops can be used to construct a white noise on $M$, by means of a Wiener-like integral.

It is then natural to try to adapt the extraction of the white noise to the general case. We prove a result in the semi-simple case that strongly suggests that this is impossible. Indeed, the $\sigma$-algebra generated by the holonomies along very small loops seems to satisfy a zero-one law.

In the last part of this work, we study combinatorial properties of the measure. We prove the Markov property of the random holonomy, extending to the continuous setting a result that was proved in the discrete setting by C. Becker and A. Sengupta [BS]. Then, we study how it is possible to glue together the Yang-Mills measures on two surfaces $M_1$ and $M_2$ in order to get the measure on $M$, the surface obtained by gluing $M_1$ and $M_2$ together. We show that the measures on $M_1$ and $M_2$ do not determine the measure on $M$. There is a lack of information that can be parametrized by the centralizer of the holonomy along the common boundary of $M_1$ and $M_2$. Finally, we summarize the algebraic properties of the conditional partition functions, whose importance had already been recognized by Witten [Wi]. We prove that a few of them generate all others by algebraic transformations and identify these elementary functions. We also show that the partition functions may be considered the transition functions of the random holonomy as a Markov field and discuss to what extent they determine the Yang-Mills measure.

# Chapter 1

# Discrete Yang-Mills measure

In this chapter we construct and we study the discrete Yang-Mills measure. It is both the basis of the construction of the continuous Yang-Mills measure and the frame in which computations are possible. The main results are the invariance by subdivision of the discrete measure and the estimation of the law of the random holonomy along small loops.

## 1.1 Notations

Throughout this work, $M$ will denote a surface, i.e. a real differentiable two-dimensional manifold, compact, connected, orientable, with or without boundary. It is endowed with a Lebesguian measure $\sigma$, i.e. a measure which has positive smooth density with respect to the Lebesgue measure in any chart.

The boundary of $M$, if it is non empty, is the disjoint union of circles $N_1, \ldots, N_p$. Let us make explicit what we call smooth objects on $M$ and introduce the very useful notion of closure.

**Definition 1.1.1** *A closure of $M$ is a triple $(i, M, M_1)$, where $M_1$ is a closed surface, i.e. a surface without boundary and $i : M \longrightarrow M_1$ is an embedding. If the complementary of $i(M)$ in $M_1$ is diffeomorphic to a disjoint union of disks, the closure is said to be minimal.*

Given two closures $(i_1, M, M_1)$ and $(i_2, M, M_2)$ of $M$, $i_1(M)$ and $i_2(M)$ have diffeomorphic neighbourhoods in $M_1$ and $M_2$. So it makes sense to say that an application (resp. a bundle, a section,...) is smooth on $M$ if it is the restriction of a smooth application (resp. bundle, section,...) defined on an open neighbourhood of $M$ in one of its closures.

The second basic object is $G$, a compact connected Lie group, that will be chosen to be Abelian or semi-simple in most examples.

Let $P$ be a principal $G$-bundle over $M$. If $M$ has a boundary, $M$ retracts on a bunch of circles and $P$ is trivial. But if $M$ is closed, the possible topological types for $P$ are classified by $\pi_1(G)$. A pleasant way to see this is to cut $M$ along the boundary of a small disk. We get two disjoint pieces. The restrictions of a bundle $P$ over $M$ to both pieces are trivial and the topology of $P$ is completely determined by the transition function along the boundary of the disk. This transition function is a map $S^1 \longrightarrow G$ and it is a fact that two homotopic maps give rise to two homeomorphic bundles.

If $G = U(1)$, the element of $\pi_1(U(1)) \simeq \mathbf{Z}$ determined by $P$ corresponds to the Chern class of the complex line bundle associated with $P$. Note that when $G$ is semi-simple, $\pi_1(G)$ is finite.

A connection $\omega$ on $P$ is a choice at each point $p$ of $P$ of a subspace in $T_pP$ supplementary to the vertical subspace of vectors tangent to the action of $G$. Moreover, this distribution, called the horizontal distribution, has to be invariant by the action of $G$.

Let $c : [0,1] \longrightarrow M$ be a regular path on $M$. A connection $\omega$ allows to lift $c$ to a horizontal path in $P$ starting at any prescribed point in $P_{c(0)}$. The function that maps a point $p$ of $P_{c(0)}$ to the end point of the horizontal lift of $c$ starting at $p$ is called the parallel transport or holonomy of $\omega$ along $c$. It is a $G$-equivariant map $\mathrm{hol}(\omega, c) : P_{c(0)} \longrightarrow P_{c(1)}$. If $c_1$ and $c_2$ are two paths such that $c_1(1) = c_2(0)$, then the path $c_1c_2$ exists and we have

$$\mathrm{hol}(\omega, c_1c_2) = \mathrm{hol}(\omega, c_2) \circ \mathrm{hol}(\omega, c_1).$$

A gauge transformation is a diffeomorphism $j$ of $P$ over the identity of $M$ that commutes with the right action of $G$. Let $\omega$ be a connection on $P$. Let $c : [0,1] \longrightarrow M$ be a piecewise $C^1$ path. A gauge transformation $j$ allows to define a new connection $j^*\omega$ whose holonomy is related to that of $\omega$ through the relation:

$$\mathrm{hol}(j^*\omega, c) = j(c(1))^{-1} \circ \mathrm{hol}(\omega, c) \circ j(c(0)).$$

Remark that these holonomies are conjugate if $c$ is a loop. More detailed presentations of the theory of fiber bundles and connections can be found for example in [KN, Bl].

## 1.2 Graphs on $M$

In order to reduce to a discrete setting, we will replace $M$ by a graph drawn on $M$ and adapt the notions of fibre bundle, connection and gauge transformation.

### 1.2.1 Pregraphs

We say that an application $c : [0,1] \longrightarrow M$ is smooth (resp. an embedding) if it is the restriction of a smooth application (resp. embedding) defined on an open interval containing $[0,1]$.

**Definition 1.2.1** *A parametrized path on $M$ is a mapping $c : [0,1] \longrightarrow M$ which is the concatenation of a finite number of smooth embeddings.*

Two parametrized paths are said to be equivalent if they differ by an increasing reparametrization.

**Lemma 1.2.2** *The equivalence of parametrized paths preserves their orientation, image, end points, injectivity, injectivity on $(0,1)$.*

Equivalence classes of parametrized paths are called simply paths. The set of paths on $M$ is denoted by $PM$.

A path whose end points are equal is called a loop and a loop which is injective on $(0,1)$ is said to be simple. Given a path $a$, we denote by $a^{-1}$ the path obtained by reversing the orientation of $a$. An edge is an injective path $a$ such that $a([0,1]) \cap \partial M$ is empty or a finite union of segments.

**Definition 1.2.3** *A pregraph on $M$ is a set $\Gamma = \{a_1, \dots, a_r\}$ of edges that meet each other only at their end points, i.e. such that for each distinct $i$ and $j$ between 1 and $r$, one has*

$$a_i([0,1]) \cap a_j([0,1]) = a_i(\{0,1\}) \cap a_j(\{0,1\}).$$

We call support of a pregraph $\Gamma$ the union of the images of its edges. A pregraph $\Gamma$ is said to be connected if its support $\mathrm{Supp}(\Gamma)$ is connected.

We call faces of a pregraph $\Gamma$ the connected components of $M \setminus \mathrm{Supp}(\Gamma)$. We denote by $\mathcal{F}(\Gamma)$ the set of these faces.

**Proposition 1.2.4** *Let $\Gamma$ be a connected pregraph on $M$. Suppose that every face of $\Gamma$ is diffeomorphic to a disk. Then the map $H_1(\mathrm{Supp}(\Gamma); \mathbf{Z}) \longrightarrow H_1(M; \mathbf{Z})$ induced by the inclusion is surjective.*

*Proof*. Let $c : [0,1] \longrightarrow M$ be a loop. There exists on each face of $\Gamma$ a point that is not in the image of $c$. Let us fix such a point in each face and remove it from $M$. The remaining open set $U$ retracts on the support of $\Gamma$ because each face with a point removed retracts on its boundary. This retraction induces a homotopy from $c$ to a loop whose image is included in $\mathrm{Supp}(\Gamma)$. So each loop of $M$ is homotopic, thus homologous to a loop of $\mathrm{Supp}(\Gamma)$. This proves the result. □

### 1.2.2 Graphs

Given a pregraph $\Gamma$, we call path in $\Gamma$ a concatenation of edges of $\Gamma$, with natural or reverse orientation. We denote by $\Gamma^*$ the set of these paths.

**Definition 1.2.5** *A graph on $M$ is a connected pregraph $\Gamma$ whose faces are diffeomorphic to disks and such that for each component $N_i$ of $\partial M$, there exists an element of $\Gamma^*$ whose image is equal to $N_i$.*

The reason for which we choose these properties is that we want a graph to take the whole topology of $M$ into account, including its boundary.

**Definition 1.2.6** *Let $\Gamma_1$ and $\Gamma_2$ be two pregraphs. We say that $\Gamma_2$ is finer than $\Gamma_1$ and write $\Gamma_1 < \Gamma_2$ if each edge of $\Gamma_1$ is a path of $\Gamma_2^*$.*

**Proposition 1.2.7** *Let $\Gamma_0$ be a pregraph on $M$. There exists a graph $\Gamma$ which is finer than $\Gamma_0$. Moreover, if $\mathrm{Supp}(\Gamma_0)$ is contained in an open set diffeomorphic to a disk, it is possible to construct $\Gamma$ in such a way that it has the same number of faces as $\Gamma_0$.*

**Lemma 1.2.8** *A pregraph whose faces are diffeomorphic to disks is necessarily connected.*

*Proof*. Let $\Gamma$ be a pregraph with faces diffeomorphic to disks. Suppose that $\Gamma = \Gamma' \cup \Gamma''$, where $\Gamma'$ and $\Gamma''$ have disjoint supports. Each face of $\Gamma$ is a disk, so it has a connected boundary, which is included either in the support of $\Gamma'$ or in that of $\Gamma''$. The closures of the unions of faces whose boundary lies in $\mathrm{Supp}(\Gamma')$ (resp. $\mathrm{Supp}(\Gamma'')$) form a partition of $M$ into two closed sets, which is in contradiction with connectedness of $M$. □

Before to prove proposition 1.2.7, let us recall some classical facts about the topology of $M$. If $M$ has no boundary and is not a sphere, its universal covering is diffeomorphic to a plane. In this plane, it is always possible to choose a polygonal fundamental domain for the covering map, namely a $4g$-gonal domain if $g$ is the genus of $M$. This means that it is possible to see topologically $M$ as the result of the identification of some edges of a polygon. If $M$ is a shpere, it can be seen as a disk whose upper and lower half of the boundary have been identified. If $M$ has a boundary, there are holes in the universal covering, one for each boundary component in a fundamental domain. It is possible to choose a fundamental domain such that the holes are in its interior. Thus, it is possible to represent a surface with boundary by a picture like picture 1.1.

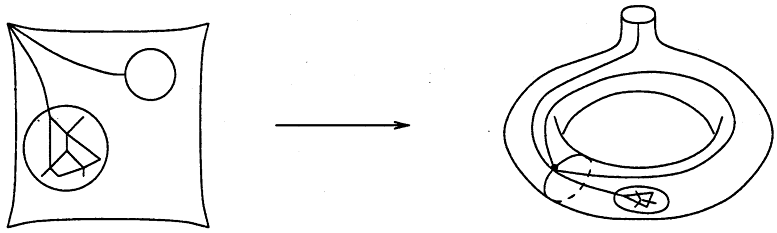

Figure 1.1: Fundamental domain in the universal covering for a torus with one hole.

*Proof of proposition* 1.2.7. By definition of an edge, the set $\mathrm{Supp}(\Gamma_0) \cap \partial M$ is a finite union of segments. Cutting some edges of $\Gamma_0$ in several pieces if necessary, we can assume that these segments are exactly images of edges. Then, it is possible to add to $\Gamma_0$ edges in such a way that $\mathrm{Supp}(\Gamma_0) \cap \partial M = \partial M$. If $\mathrm{Supp}(\Gamma_0)$ did not meet a component of $\partial M$ initially, it is necessary to add at least two edges on this component. So we can construct a pregraph $\Gamma_1$ which is finer than $\Gamma_0$, whose support contains $\partial M$.

Each face of $\Gamma_1$ is homeomorphic to the interior of a compact surface with boundary. On any such surface, there exists a graph, for example a triangulation. We add to $\Gamma_1$ the edges that are necessary to transform it into a graph on each face which is not diffeomorphic to a disk. All faces of the resulting pregraph $\Gamma_2$ are diffeomorphic to disks. By the preceding lemma, it is connected. Thus, it is a graph. The first part of the proposition is proved.

If $\mathrm{Supp}(\Gamma_0)$ is contained in a disk, it is possible to move this disk by a diffeomorphism of $M$ into any prescribed disk. It is possible to get the situation described by the picture, in the universal covering, where the disk is in the interior of a fundamental domain. Then it is easy to complete $\Gamma_0$ into a graph with the same number of faces and get back to the initial situation by the inverse diffeomorphism. □

## 1.3 Discrete holonomy and gauge transformations

Choose a graph $\Gamma = \{a_1, \ldots, a_r\}$ on $M$ and denote by $i : \mathrm{Supp}(\Gamma) \longrightarrow M$ the canonical injection. Consider the fiber bundle $i^*P$ on $\mathrm{Supp}(\Gamma)$. It is a trivial bundle, whatever the topology of $P$ was. We identify $i^*P$ with $\mathrm{Supp}(\Gamma) \times G$. Let $\mathcal{V}(\Gamma)$ denote the set of vertices of $\Gamma$.

**Lemma 1.3.1** *Let $\omega_1$ and $\omega_2$ be two connections on $i^*P$. Suppose that $\omega_1$ and $\omega_2$ have the same holonomy along each edge of $\Gamma$. Then there exists a gauge transformation $j$ of $i^*P$ that leaves* *the fibers over the points of $\mathcal{V}(\Gamma)$ invariant and such that $j^*\omega_1 = \omega_2$.*

*Proof.* Choose a parametrization of $a_1 \in \Gamma$. Set

$$j(a_1(t)) = \mathrm{hol}(\omega_1, a_{1|[0,t]}) \circ \mathrm{hol}(\omega_2, (a_{1|[0,t]})^{-1}).$$

Then

$$\mathrm{hol}(\omega_2, a_{1|[0,t]}) = j(a_1(t))^{-1} \circ \mathrm{hol}(\omega_1, a_{1|[0,t]}) = \mathrm{hol}(j^*\omega_1, a_{1|[0,t]}).$$

The assumption about $\omega_1$ and $\omega_2$ makes sure that it is possible to extend the construction of $j$ to the whole graph and that $j(m) = \mathrm{Id}_{P_m}$ for all vertex $m$ of $\Gamma$.

This implies that $\omega_2$ and $j^*\omega_1$ have the same holonomy, hence the same horizontal paths, so they are equal. □

In the discrete setting, we expect to be able to compute the holonomy only along edges of the graph. So we identify a connection with the holonomy that it determines along the edges of $\Gamma$ and, according to the preceding lemma, consider gauge transformations that act only on the fibers over the vertices of $\Gamma$. Finally, using the identification $i^*P = \mathrm{Supp}(\Gamma) \times G$, we can identify holonomies and gauge transformations with elements of $G$, defining for example $h_c(\omega)$ and $j(m)$ by $\mathrm{hol}(\omega, c)(c(0), 1) = (c(1), h_c(\omega))$ and $j(m, 1) = (m, j(m))$.

This leads us to the following definitions.

**Definition 1.3.2** *A discrete connection on $\Gamma$ is a map from $\Gamma$ into $G$. A discrete gauge transformation is a map from the set $\mathcal{V}(\Gamma)$ of vertices of $\Gamma$ into $G$.*

A discrete gauge transformation $j : \mathcal{V}(\Gamma) \longrightarrow G$ acts on a discrete connection $g = (g_1, \dots, g_r)$ by :

$$j \cdot g = (j(a_1(1))^{-1} g_1 j(a_1(0)), \dots, j(a_r(1))^{-1} g_r j(a_r(0))).$$

A discrete connection $g = (g_1, \dots, g_r)$ determines a multiplicative application from $\Gamma^*$ into $G$. Given a path $c = a_{i_1}^{\varepsilon_1} \dots a_{i_n}^{\varepsilon_n}$, $\varepsilon_i = \pm 1$, we can compute $g_{i_n}^{\varepsilon_n} \dots g_{i_1}^{\varepsilon_1}$ (with reversed order!). In other words, any path $c$ of $\Gamma^*$ gives rise to a map $h_c$ from $G^\Gamma$ to $G$ defined by

$$h_c(g_1, \dots, g_r) = g_{i_n}^{\varepsilon_n} \dots g_{i_1}^{\varepsilon_1}.$$

This map is well defined because there is only one way to decompose a path in product of edges. To see why, it is enough to consider the times at which a path crosses a vertex of $\Gamma$.

**Proposition 1.3.3** *Let $c_1$ and $c_2$ be two paths such that $c_1 c_2$ is also a path. Then*

$$h_{c_1 c_2} = h_{c_2} h_{c_1}.$$

This basic property of the discrete holonomy will be refered to as the multiplicativity of the holonomy.

A gauge transformation $j$ transforms $h_c$ in $h_c \circ j$, with

$$h_c \circ j = j(c(1))^{-1} h_c j(c(0)).$$

At this stage, it may be noticed that central functions of the holonomy along loops are invariant under gauge transformations. This is why they will play such a major role in the sequel.

## 1.4 Discrete Yang-Mills measure

We keep a graph $\Gamma = \{a_1, \ldots, a_r\}$ on $M$. In the discrete setting, a probability measure on the quotient space $\mathcal{A}/\mathcal{J}$ of connections modulo gauge transformations is represented by a probability measure on $G^\Gamma$ invariant under the action of $G^{\mathcal{V}(\Gamma)}$.

The basic example of such an invariant measure is the product of Haar measures. We shall construct the discrete Yang-Mills on $G^\Gamma$ as :

$$dP = \frac{dP}{dg}\, dg,$$

where $dg = dg_1 \otimes \ldots \otimes dg_r$. The density $\frac{dP}{dg}$ will be a product of central functions of holonomies along loops, a feature that makes $P$ invariant. Recall that a function $p$ is said to be central on $G$ if $p(xy) = p(yx)$, or equivalently $p(y^{-1}xy) = p(x)$ for all $x, y$ in $G$.

Each face $F$ of $\Gamma$ has a boundary which is the image of a path defined up to the choice of an origin and an orientation. So the function $h_{\partial F} : G^\Gamma \longrightarrow G$ is defined up to conjugation and inversion. For any central function $p$ invariant by inversion, the function $p \circ h_{\partial F}$ is well defined.

Let us denote by $(p_t)_{t>0}$ the fundamental solution of the heat equation on $G$ endowed with its biinvariant Riemannian metric, normalized to have total volume equal to 1. It satisfies

$$(\partial_t - \frac{1}{2}\Delta)p_t = 0 \quad \text{on} \quad \mathbf{R}_+^* \times G,$$

and for any function $f$ continuous on $G$,

$$\int_G f(g)p_t(g)\, dg \underset{t\to 0}{\longrightarrow} f(1).$$

For any positive $t$, $p_t$ is a positive central function, invariant by inversion, such that $\int_G p_t(g)\, dg = 1$. For the moment, the choice of the heat kernel may seem to be quite arbitrary. We shall discuss this point at the end of section 1.6.

For each face $F$ of $\Gamma$, the function $p_{\sigma(F)}(h_{\partial F}) : G^\Gamma \longrightarrow \mathbf{R}_+^*$ is well defined, with $\sigma$ denoting the surface measure on $M$. Set

$$D = \prod_{F \in \mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F}) : G^\Gamma \longrightarrow \mathbf{R}_+^*, \tag{1.1}$$

$$Z = \int_{G^\Gamma} D\, dg. \tag{1.2}$$

Now define $P$ on $(G^\Gamma, \mathrm{Bor}(G^\Gamma))$ by

$$dP = \frac{1}{Z}\, D\, dg. \tag{1.3}$$

Given $n$ paths $c_1, \ldots, c_n$ in $\Gamma^*$, we define the law of the discrete holonomy along $c_1, \ldots, c_n$ as the joint law of the $n$-uple $(h_{c_1}, \ldots, h_{c_n})$ under $P$.

## 1.5 Conditional Yang-Mills measure

When $M$ has a boundary, it is natural to want to impose the holonomy along the components of $\partial M$. It may also be useful to be able to impose holonomy along some other loops even if $M$ has no boundary.

### 1.5.1 Conditional Haar measure

**Proposition 1.5.1** *Let $n$ be a positive integer. Let $x$ be an element of $G$. There exists on $G^n$ a measure $\nu_x^n$ such that $g_n \ldots g_1 = x$ $\nu_x^n$-a.s. and such that for any function $f$ continuous on $G^n$ and any $i$ between 1 and $n$,*

$$\nu_x^n(f) = \int_{G^{n-1}} f(g_1, \ldots, g_{i-1}, (g_n \ldots g_{i+1})^{-1} x (g_{i-1} \ldots g_1)^{-1}, g_{i+1}, \ldots, g_n)\, dg_1 \ldots \widehat{dg_i} \ldots dg_n.$$

*Moreover, one has*

$$\nu_x^n(f) = \lim_{t \to 0} \int_{G^n} f(g_1, \ldots, g_n)\, p_t(g_n \ldots g_1 x^{-1})\, dg_1 \ldots dg_n.$$

*Finally, $\int_G \nu_x^n\, dx = dg$, as measures on $G^n$.*

*Proof*. Pick $i$ between 1 and $n$, and $t > 0$. By centrality of $p_t$ and then right invariance of $dg_i$, one has

$$\int_{G^n} f(g_1, \ldots, g_n) p_t(g_n \ldots g_1 x^{-1})\, dg = \int_{G^n} f(g_1, \ldots, g_n)\, p_t(g_i (g_{i-1} \ldots g_1) x^{-1} (g_n \ldots g_{i+1}))\, dg$$

$$= \int_{G^n} f(g_1, \ldots, g_{i-1}, g_i (g_n \ldots g_{i+1})^{-1} x (g_{i-1} \ldots g_1)^{-1}, g_{i+1}, \ldots, g_n)\, p_t(g_i)\, dg_i dg_1 \ldots \widehat{dg_i} \ldots dg_n$$

$$\underset{t \to 0}{\longrightarrow} \int_{G^{n-1}} f(g_1, \ldots, g_{i-1}, (g_n \ldots g_{i+1})^{-1} x (g_{i-1} \ldots g_1)^{-1}, g_{i+1}, \ldots, g_n)\, dg_1 \ldots \widehat{dg_i} \ldots dg_n.$$

Thus the limit exists and the last expression does not depend on $i$. It defines a probability measure on $G^n$.

If $f$ vanishes on the hypersurface $\{g_1 \ldots g_n = x\}$, then $\nu_x^n(f) = 0$. So we do have $g_n \ldots g_1 = x$ $\nu_x^n$-a.s.

Finally, since $\int_G p_t(g)\, dg = 1$,

$$\int_G \int_{G^n} f(g_1, \ldots, g_n)\, p_t(g_n \ldots g_1 x^{-1})\, dx dg = \int_{G^n} f(g_1, \ldots, g_n)\, dg,$$

which implies the last statement when $t$ tends to zero. □

### 1.5.2 Conditional Yang-Mills measure

Let $L_1, \ldots, L_q$ be disjoint simple loops of $\Gamma^*$ whose image is either a component of $\partial M$ or contained in the interior of $M$. We want to choose the law of $(h_{L_1}, \ldots, h_{L_q})$. For this, it is enough to be able to impose a deterministic value to each $h_{L_i}$. Let $(x_1, \ldots, x_q)$ be an element of $G^q$. Let $\Gamma'$ denote the set of edges of $\Gamma$ that do not appear in the decomposition of any $L_i$. We denote by $dg'$ the product of Haar measures on $G^{\Gamma'}$. The fact that the conditional Haar measure is not invariant by permutation of the factors on $G^n$ leads either to a very heavy or to an elliptic notation. We will choose the second option, except during a few lines. Suppose that $L_1 = a_{i_1^1}^{\varepsilon_1^1} \dots a_{i_{n_1}^1}^{\varepsilon_{n_1}^1}, \dots, L_q = a_{i_1^q}^{\varepsilon_1^q} \dots a_{i_{n_q}^q}^{\varepsilon_{n_q}^q}$, with $\varepsilon_j^i = \pm 1$. We denote by $d\nu_{x_1} \dots d\nu_{x_q} dg'$ the following measure on $G^\Gamma$:

$$d\nu_{x_1}^{n_1}(g_{i_1^1}^{\varepsilon_1^1}, \dots, g_{i_{n_1}^1}^{\varepsilon_{n_1}^1}) \dots d\nu_{x_q}^{n_q}(g_{i_1^q}^{\varepsilon_1^q}, \dots, g_{i_{n_q}^q}^{\varepsilon_{n_q}^q}) dg'.$$

With this notation, set:

$$Z(x_1, \dots, x_q) = \int_{G^\Gamma} D \; d\nu_{x_1} \dots d\nu_{x_q} dg', \tag{1.4}$$

$$dP(x_1, \dots, x_q) = \frac{1}{Z(x_1, \dots, x_q)} D \; d\nu_{x_1} \dots d\nu_{x_q} dg'. \tag{1.5}$$

The function $Z(x_1, \dots, x_q)$ is called conditional partition function on $M$ with respect to $L_1, \dots, L_q$.

**Proposition 1.5.2** *Choose $r \leq q$ and $x_{r+1}, \dots, x_q \in G$. The law of $(h_{L_1}, \dots, h_{L_r})$ under $P(x_{r+1}, \dots, x_q)$ is equal to:*

$$\frac{Z(x_1, \dots, x_q)}{Z(x_{r+1}, \dots, x_q)} \, dx_1 \dots dx_r,$$

*where each element $x_i$ corresponds to the loop $L_i$. In particular, the law of $(h_{L_1}, \dots, h_{L_q})$ under $P$ is $\frac{1}{Z} Z(x_1, \dots, x_q) \, dx_1 \dots dx_q$.*

*Proof*: The last part of the statement is just the case $r = q$. Let $f$ be a continuous function on $G^r$. We have:

$$\begin{aligned}
\int_{G^\Gamma} f(h_{L_1}, \dots, h_{L_r}) \, dP(x_{r+1}, \dots, x_q) &= \frac{1}{Z(x_{r+1}, \dots, x_q)} \int_{G^\Gamma} f(h_{L_1}, \dots, h_{L_r}) D \; d\nu_{x_{r+1}} \dots d\nu_{x_q} dg' \\
&= \frac{1}{Z(x_{r+1}, \dots, x_q)} \int_{G^r} \int_{G^\Gamma} f(h_{L_1}, \dots, h_{L_r}) D \; d\nu_{x_1} \dots d\nu_{x_q} dg' \, dx_1 \dots dx_r \\
&= \int_{G^r} f(x_1, \dots, x_r) \left[ \frac{1}{Z(x_{r+1}, \dots, x_q)} \int_{G^\Gamma} D \; d\nu_{x_1} \dots d\nu_{x_q} dg' \right] dx_1 \dots dx_r \\
&= \int_{G^r} f(x_1, \dots, x_r) \frac{Z(x_1, \dots, x_q)}{Z(x_{r+1}, \dots, x_q)} \, dx_1 \dots dx_r.
\end{aligned}$$

□

**Corollary 1.5.3** *The map $(x_1, \dots, x_q) \mapsto P(x_1, \dots, x_q)$ is a disintegration of the measure $P$ with respect to the random variable $(h_{L_1}, \dots, h_{L_q})$. This means that*
*1. $(h_{L_1}, \dots, h_{L_q}) = (x_1, \dots, x_q)$ $P(x_1, \dots, x_q)$-a.s.*
*2. denoting by $\eta$ the law of $(h_{L_1}, \dots, h_{L_q})$ under $P$, we have*

$$P = \int_{G^q} P(x_1, \dots, x_q) \, d\eta(x_1, \dots, x_q).$$

*Proof*. The first part is a direct consequence of the definition of $P(x_1, \dots, x_q)$. A simple computation proves the second one:

$$\begin{aligned}
\int_{G^q} P(x_1, \dots, x_q) \, d\eta(x_1, \dots, x_q) &= \frac{1}{Z} \int_{G^q} P(x_1, \dots, x_q) Z(x_1, \dots, x_q) \, dx_1 \dots dx_q \\
&= \frac{1}{Z} \int_{G^q} D \; \nu_{x_1} \dots \nu_{x_q} \, dg' \, dx_1 \dots dx_q \\
&= P.
\end{aligned}$$

□

This corollary says that the $P(x_1,\ldots,x_q)$ are really what we expected them to be. Now, given a measure $\beta$ on $G^q$, we can choose the law of $(h_{L_1},\ldots,h_{L_q})$ to be $\beta$ by putting the measure $P_\beta=\int_{G^q}P(x_1,\ldots,x_q)\,d\beta(x_1,\ldots,x_q)$ on $G^\Gamma$.

### 1.5.3 Gauge transformations

Let us compute how the measure $P(x_1,\ldots,x_q)$ is transformed by a gauge transformation.

**Lemma 1.5.4** *Let $j$ be a discrete gauge transformation. The following equality holds :*

$$j_*P(x_1,\ldots,x_q)=P(y_1x_1y_1^{-1},\ldots,y_qx_qy_q^{-1}),$$

*where $y_i=j(L_i(0))$.*

*Proof.* For sake of simplicity, let us write the proof in the case $q=1$, the general case being exactly similar, only with heavier notations. Suppose that $L_1=a_1\ldots a_m$. Let $f$ be a continuous function on $G^\Gamma$.

$$\begin{aligned}
j_*P(x_1)(f) &= \frac{1}{Z(x_1)}\int_{G^\Gamma} f\circ j \prod_{F\in\mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F})\,\nu^m_{x_1}(g_1,\ldots,g_m)dg_{m+1}\ldots dg_r\\
&= \frac{1}{Z(x_1)}\int_{G^\Gamma} f \prod_{F\in\mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F}\circ j^{-1})\,\nu^m_{x_1}(j(a_1(1))g_1j(a_1(0))^{-1},\ldots\\
&\qquad\qquad \ldots,j(a_m(1))g_mj(a_m(0))^{-1})\,dg_{m+1}\ldots dg_r\\
&= \frac{1}{Z(x_1)}\int_{G^\Gamma} f \prod_{F\in\mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F})\,\nu^m_{j(L_1(0))x_1j(L_1(0))^{-1}}(g_1,\ldots,g_m)dg_{m+1}\ldots dg_r\\
&= \frac{Z(y_1x_1y_1^{-1})}{Z(x_1)}P(y_1x_1y_1^{-1})(f),
\end{aligned}$$

with $y_1=j(L_1(0))$. Setting $f$ to be identically 1, we get $Z(y_1x_1y_1^{-1})=Z(x_1)$. □

Let us state the invariance property of the partition function that we just proved :

**Proposition 1.5.5** *For any $y_1,\ldots,y_q$ in $G$, one has*

$$Z(y_1^{-1}x_1y_1,\ldots,y_q^{-1}x_qy_q)=Z(x_1,\ldots,x_q).$$

According to this result, the conditional partition function can also be viewed as a function on $(G/\operatorname{Ad})^q$, where $Ad$ is the adjoint action on $G$ given by $\operatorname{Ad}(y)x=y^{-1}xy$. We will use this point of view in the next paragraph.

It is clear now that $P(x_1,\ldots,x_q)$ is not gauge invariant in general. We will explain how to overcome this problem.

Let $t$ be an element of the quotient $G/\operatorname{Ad}$, that is, a conjugacy class in $G$. Let $x$ be an element of this class. The measure $\int_G\delta_{yxy^{-1}}\,dy$ does not depend on the choice of $x$ in $t$. We shall denote it by $\delta_t$. Similarly, we denote $\delta_{t_1}\otimes\ldots\otimes\delta_{t_q}$ by $\delta_{t_1,\ldots,t_q}$. Set

$$P(t_1,\ldots,t_q)=\int_{G^q}P(x_1,\ldots,x_q)\,\delta_{t_1,\ldots,t_q}(x_1,\ldots,x_q).$$

**Proposition 1.5.6** *The measure $P(t_1,\ldots,t_q)$ is gauge invariant.*

*Proof*: Let $j$ be a discrete gauge transformation and set $y_i = j(L_i(0))$. According to the lemma 1.5.4, we have:

$$\begin{aligned}
j_*P(t_1,\ldots,t_q) &= \int_{G^q} j_*P(x_1,\ldots,x_q)\ \delta_{t_1,\ldots,t_q}(x_1,\ldots,x_q)\\
&= \int_{G^q} P(y_1x_1y_1^{-1},\ldots,y_qx_qy_q^{-1})\ \delta_{t_1,\ldots,t_q}(x_1,\ldots,x_q)\\
&= \int_{G^q} P(y_1z_1x_1^0z_1^{-1}y_1^{-1},\ldots,y_qz_qx_q^0z_q^{-1}y_q^{-1})\ dz_1\ldots dz_q\\
&= \int_{G^q} P(z_1^{-1}x_1^0z_1,\ldots,z_q^{-1}x_q^0z_q)\ dz_1\ldots dz_q\\
&= \int_{G^q} P(x_1,\ldots,x_q)\ \delta_{t_1,\ldots,t_q}(x_1,\ldots,x_q)\\
&= P(t_1,\ldots,t_q),
\end{aligned}$$

where each $x_i^0$ was an arbitrary element of $t_i$. □

It will emerge later that the measures $P(t_1,\ldots,t_q)$ are in fact more natural than the $P(x_1,\ldots,x_q)$.

## 1.6 Invariance by subdivision

The invariance by subdivision is the main feature of the discrete theory. It allows to prove that the law of the discrete holonomy along given loops does not depend on the graph in which one computes it.

The fact that the heat kernel $(p_t)_{t>0}$ is a convolution semi-group will play a central role in the proof. This means that for any $x \in G$ and any $s,t$ such that $0 < s < t$,

$$\int_G p_s(xy^{-1})p_t(y)\ dy = p_t(x).$$

Let $\Gamma_1$ and $\Gamma_2$ be two graphs on $M$. Suppose that $\Gamma_2$ is finer than $\Gamma_1$ and set $\Gamma_1 = \{a_1.\ldots,a_r\}$. By definition, each edge $a_i$ of $\Gamma_1$ is a path in $\Gamma_2^*$ and it gives rise to a function $h_{a_i} : G^{\Gamma_2} \longrightarrow G$. The $r$-uple of those functions constitutes a single function $(h_{a_1},\ldots,h_{a_r}) : G^{\Gamma_2} \longrightarrow G^r = G^{\Gamma_1}$ that we denote by $f_{\Gamma_1\Gamma_2}$. the invariance by subdivision is expressed by the following result :

**Theorem 1.6.1** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs on $M$ such that $\Gamma_2$ is finer than $\Gamma_1$. Let $L_1,\ldots,L_q$ be disjoint simple loops of $\Gamma_1^*$. Let $x_1,\ldots,x_q$ be elements of $G$. Then*
*1. The map $f_{\Gamma_1\Gamma_2} : G^{\Gamma_2} \longrightarrow G^{\Gamma_1}$ is surjective.*
*2. This map satisfies : $(f_{\Gamma_1\Gamma_2})_* P^{\Gamma_2}(x_1,\ldots,x_q) = P^{\Gamma_1}(x_1,\ldots,x_q)$.*

From now on, it will be sometimes necessary to write explicitely the graph in which we consider objects such as $P$, $Z$, $D$.

We begin by proving that it is always possible to go from one graph to a finer graph by a finite sequence of elementary transformations.

**Lemma 1.6.2** *Let $\Gamma$ and $\Gamma'$ be two graphs such that $\Gamma < \Gamma'$. There exist an increasing sequence of graphs $\Gamma_0 = \Gamma < \Gamma_1 < \ldots < \Gamma_n < \ldots$, stationnary of limit $\Gamma'$ and such that for any nonnegative $n$, one can transform $\Gamma_n$ into $\Gamma_{n+1}$ by one of the two following elementary operations:*
*(V) Add a vertex to $\Gamma_n$, i.e. replace an edge $a$ by two edges $b$ and $c$ such that $a = bc$,*
*(E) Add an edge to $\Gamma_n$, this new edge joining two vertices of $\Gamma_n$.*

*Proof.* We proceed by induction on $n$. $\Gamma_0$ is given, equal to $\Gamma$. Suppose $\Gamma_n$ given, with $\Gamma_n < \Gamma'$. Recall that $\mathcal{V}(\Gamma)$ denotes the set of vertices of $\Gamma$.

- We have $\mathcal{V}(\Gamma_n) \subset \mathcal{V}(\Gamma') \cap \mathrm{Supp}(\Gamma_n)$. If this inclusion is a strict one, pick an element of $(\mathcal{V}(\Gamma') \cap \mathrm{Supp}(\Gamma_n)) \backslash \mathcal{V}(\Gamma_n)$. It is a vertex of $\Gamma'$ which is on an edge of $\Gamma_n$ whithout being one of its end points. By an operation (V), we add this vertex to $\Gamma_n$ and get $\Gamma_{n+1}$ which is still finer than $\Gamma'$. Note that $\mathrm{Card}(\Gamma_{n+1}) = \mathrm{Card}(\Gamma_n) + 1$.
- If $\mathcal{V}(\Gamma_n) = \mathcal{V}(\Gamma') \cap \mathrm{Supp}(\Gamma_n)$, then each edge of $\Gamma_n$ is an edge of $\Gamma'$. In other words, $\Gamma_n \subset \Gamma'$. If this inclusion is a strict one, there exists an edge of $\Gamma'$ which is not an edge of $\Gamma_n$ and by connectedness of $\Gamma'$ we may assume that this edge has at least one of its end points on $\mathrm{Supp}(\Gamma_n)$. By an operation (E), we add this edge to $\Gamma_n$ and get $\Gamma_{n+1}$ which is still connected and finer than $\Gamma'$. The pregraph $\Gamma_{n+1}$ is a graph. Indeed, we just noticed that it is connected and it is finer than $\Gamma_0$, so that its support contains $\partial M$. It can happen that the operation (E) cuts a face in two pieces, but they are still diffeomorphic to disks. We also have $\mathrm{Card}(\Gamma_{n+1}) = \mathrm{Card}(\Gamma_n) + 1$.
- If $\Gamma_n = \Gamma'$, just set $\Gamma_{n+1} = \Gamma_n$.

At each step, the fact that $\Gamma_n$ is a graph implies that $\Gamma_{n+1}$ is also a graph : connectedness is preserved, as well as boundary properties. The faces of a graph are not modified by an operation (V) and it can happen that an operation (E) cuts a face into two pieces, which are still diffeomorphic to disks.

For each $n$, $\Gamma_n < \Gamma'$ implies $\mathrm{Card}(\Gamma_n) \leq \mathrm{Card}(\Gamma')$. On the other hand, elementary operations increase strictly the cardinal of the graph. Thus, there is necessarily only a finite number of such operations before the sequence becomes stationnary. □

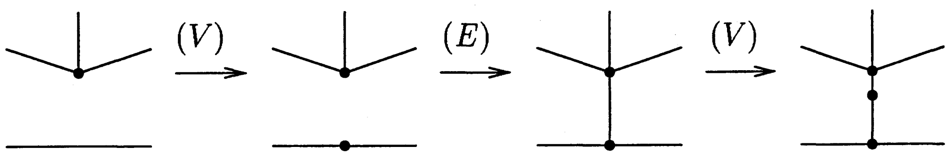


Figure 1.2: Examples of elementary transformations of a graph.

**Lemma 1.6.3** *Let $\Gamma_1 < \Gamma_2 < \Gamma_3$ be three graphs. Then*

$$f_{\Gamma_1\Gamma_3} = f_{\Gamma_1\Gamma_2} \circ f_{\Gamma_2\Gamma_3}.$$

*Proof.* It is the associativity of the product in $G$. □

This lemma shows that it is enough to prove the theorem 1.6.1 when $\Gamma_2$ can be deduced from $\Gamma_1$ by an elementary operation. One recovers the general case by composition.

During the proof, we set $\beta = \delta_{(x_1,\dots,x_q)}$ and adopt the notations $P_\beta = P(x_1,\dots,x_q)$ and $Z_\beta = Z(x_1,\dots,x_q)$ in order to make the expressions shorter.

*Proof of theorem* 1.6.1 : 1. We prove that $f_{\Gamma_1\Gamma_2}$ is surjective. If $\Gamma_2$ can be deduced from $\Gamma_1$ by an operation (E), $f_{\Gamma_1\Gamma_2}$ is just the projection that forgets the factor associated with the new edge. It is of course surjective. In the case of an operation (V), $f_{\Gamma_1\Gamma_2}$ preserves all factors except those associated with the two new edges, that are multiplied. It is also surjective.

2. Let us begin by the case of an operation of type (E). We fix some notations. Set $\Gamma_1 = \{a_1,\dots,a_r\}$ and $\Gamma_2 = \{a_1,\dots,a_r,b\}$. The new edge $b$ is located in a face $F_0$ of $\Gamma_1$. Two situations are possible : either $b$ has one end point on $\partial F_0$ or it has both. In the first case, $F_0$ is still a face of $\Gamma_2$, with a new factor $bb^{-1}$ in its boundary. In the second case, $F_0$ is cut into two faces $F_1$ and $F_2$ by $b$. Let us consider this second case. The boundaries of $F_0$, $F_1$ and $F_2$ can be written respectively $\partial F_0 = c_1c_2$, $\partial F_1 = c_1b^{-1}$ and $\partial F_2 = bc_2$. Let $f$ be a continuous function on $G^{\Gamma_1}$.

$$\int_{G^{\Gamma_1}} f\, d\left((f_{\Gamma_1\Gamma_2})_* P_\beta^{\Gamma_2}\right) = \frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_2}} f(g_1,\dots,g_r) p_{\sigma(F_1)}(g_{r+1}^{-1}h_{c_1}) p_{\sigma(F_2)}(h_{c_2}g_{r+1}) \prod_{F\in\mathcal{F}(\Gamma_1)\setminus F_0} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1}\dots d\nu_{x_q} dg',$$

where $g_{r+1}$ is the element associated with $b$. Since the $L_i$'s are paths in $\Gamma_1$, the new edge $b$ is not involved in their decomposition. Thus we can isolate $dg_{r+1}$ in $dg'$ and integrate against it. We use the fact that the heat kernel is a convolution semi-group. We get the following expression :

$$\begin{aligned} &= \frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f(g_1,\dots,g_r) p_{\sigma(F_1)+\sigma(F_2)}(h_{c_1}h_{c_2}) \prod_{F\in\mathcal{F}(\Gamma_1)\setminus F_0} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1}\dots d\nu_{x_q} dg' \\ &= \frac{Z_\beta^{\Gamma_1}}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f\, dP_\beta^{\Gamma_1}. \end{aligned}$$

Setting $f = 1$, we get $Z_\beta^{\Gamma_1} = Z_\beta^{\Gamma_2}$ and the result.

The case where the new edge does not cut $F_0$ into two faces is even simpler : the factor $bb^{-1}$ vanishes in all computations, because $f$ does not depend on the factor associated to $b$.

Now consider the case of an operation (V). Set $\Gamma_1 = \{a_1,\dots,a_r\}$ and $\Gamma_2 = \{b,c,a_2,\dots,a_r\}$, with $a_1 = bc$. The edge $a_1$ can be on the boundary of one or two faces, depending on the fact that it is on $\partial M$ or not.

$$\int_{G^{\Gamma_1}} f\, d\left((f_{\Gamma_1\Gamma_2})_* P_\beta^{\Gamma_2}\right) = \frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_2}} f(g_cg_b,g_2,\dots,g_r) D^{\Gamma_1}(g_cg_b,g_2,\dots,g_r)\, \nu_{x_1}\dots\nu_{x_q} dg',$$

where $(g_b,g_c,g_2,\dots,g_r)$ denotes the generic element of $G^{\Gamma_2}$. We have to discuss two cases : either $a_1$ is involved in the decomposition of one of the $L_i$'s, say $L_1$, or it is not. If it is not, we can isolate $dg_bdg_c$ in the $dg'$ term. By integrating against $dg_b$, the dependence in $g_c$ disappears by right invariance of $dg_b$ and we get

$$\dots = \frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f(g,g_2,\dots,g_r) D^{\Gamma_1}(g,g_2,\dots,g_r)\, \nu_{x_1}\dots\nu_{x_q} dg' = \frac{Z_\beta^{\Gamma_1}}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f\, dP_\beta^{\Gamma_1},$$

and we conclude as before. If $a_1$ is involved in the decomposition of $L_1$, we can suppose that $L_1 = a_1 \dots a_m$, with $m \geq 2$. We write $d\nu_{x_1}$ in a convenient way, putting the contioning on $g_m$, which is necessarily distinct from $g_1$. We get :

$$\frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f(g_c g_b, g_2, \dots, \widetilde{g_m}, \dots, g_r) D^{\Gamma_2}(g_c g_b, g_2, \dots, \widetilde{g_m}, \dots, g_r)\, dg_b dg_c dg_2 \dots dg_{m-1} \nu_{x_2} \dots \nu_{x_q} dg',$$

where $\widetilde{g_m} = x_1 (g_{m-1} \dots g_2 g_c g_b)^{-1}$. This is equal to

$$\frac{1}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f(g, g_2, \dots, \widetilde{g_m}, \dots, g_r) D^{\Gamma_2}(g, g_2, \dots, \widetilde{g_m}, \dots, g_r)\, dg dg_2 \dots dg_{m-1} \nu_{x_2} \dots \nu_{x_q} dg',$$

with $\widetilde{g_m}$ now equal to $x_1(g_{s_1} \dots g_2 g)^{-1}$. This is one more time equal to

$$\frac{Z_\beta^{\Gamma_1}}{Z_\beta^{\Gamma_2}} \int_{G^{\Gamma_1}} f\, dP_\beta^{\Gamma_1}$$

and we get the result. □

**Corollary 1.6.4** *Let $c_1, \dots, c_n$ be paths that are simultaneously elements of $\Gamma_1^*$ and $\Gamma_2^*$, where $\Gamma_1$ and $\Gamma_2$ are two graphs such that $\Gamma_1 < \Gamma_2$. Then the law of the discrete holonomy along $c_1, \dots, c_n$ is the same on $\Gamma_1$ and $\Gamma_2$. In other words, the law of $(h_{c_1}^{\Gamma_1}, \dots, h_{c_n}^{\Gamma_1})$ on $(G^{\Gamma_1}, P_\beta^{\Gamma_1})$ and the law of $(h_{c_1}^{\Gamma_2}, \dots, h_{c_n}^{\Gamma_2})$ on $(G^{\Gamma_2}, P_\beta^{\Gamma_2})$ are equal.*

*Proof*. It is enough to verify that $(h_{c_1}^{\Gamma_1}, \dots, h_{c_n}^{\Gamma_1}) \circ f_{\Gamma_1 \Gamma_2} = (h_{c_1}^{\Gamma_2}, \dots, h_{c_n}^{\Gamma_2})$. This is true if the $c_i$'s are edges of $\Gamma_1$, thus it is true in general by multiplicativity. □

During the proof of the theorem, we also proved an important result about the conditional partition function:

**Proposition 1.6.5** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs such that $\Gamma_1 < \Gamma_2$. Take $L_1, \dots, L_q$ and $x_1, \dots, x_q$ as usual. Then*

$$Z^{\Gamma_1}(x_1, \dots, x_q) = Z^{\Gamma_2}(x_1, \dots, x_q).$$

Let us discuss briefly the choice of the heat kernel in the definition of $P$. This choice is the key of the physical relevance of the theory. It is a physicist, A. Migdal [Mi], who suggested first to use the heat kernel in the mathematical formulation of the theory. Nevertheless, it is possible to construct a discrete theory using any other convolution semigroup. For example, Albeverio, Høegh Krohn and Holden investigated some properties of the random fields obtained this way [Al]. But it would probably be much more difficult to construct a continuous theory without the nice regularity properties that characterize the heat kernel among all other convolution semigroups.

## 1.7 Invariance by area-preserving diffeomorphisms

The manifold $M$ is given with its differentiable structure and the Lebesguian surface measure $\sigma$. Let $\Gamma$ be a graph on $M$ and $\phi : M \longrightarrow M$ a diffeomorphism such that $\phi_*\sigma = \sigma$. Then $\phi$ transforms $\Gamma$ into a graph $\phi(\Gamma)$ and induces a bijection between faces of $\Gamma$ and $\phi(\Gamma)$ that preserves the surface. Thus, the natural bijection induced between $G^\Gamma$ and $G^{\phi(\Gamma)}$ preserves the discrete Yang-Mills measure. Let us state in a slightly more general way this invariance property.

**Proposition 1.7.1** *Let $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ be two surfaces and $\phi : M_1 \longrightarrow M_2$ a diffeomorphism such that $\phi_*\sigma_1 = \sigma_2$. Let $\Gamma_1$ be a graph on $M_1$ and $\Gamma_2 = \phi(\Gamma_1)$ the corresponding graph on $M_2$. Still denoting by $\phi : G^{\Gamma_1} \longrightarrow G^{\Gamma_2}$ the induced bijection, one has*

$$\phi_* P^{\Gamma_1}(x_1, \ldots, x_q) = P^{\Gamma_2}(x_1, \ldots, x_q).$$

Thus, for each family $(c_1, \ldots, c_n)$ of paths in $\Gamma_1^*$, the law of the discrete holonomy along $(c_1, \ldots, c_n)$ equals the law of the discrete holonomy along $(\phi(c_1), \ldots, \phi(c_n))$.

## 1.8 Examples

In this section, we will compute the law of the discrete holonomy im two basic situations.

### 1.8.1 Holonomy along an open path

Let $\Gamma$ be a graph and $c \in \Gamma^*$ be an open path, i.e. a path such that $c(0) \neq c(1)$. Let $L_1, \ldots, L_q$ be disjoint simple loops of $\Gamma^*$ and $t_1, \ldots, t_q$ be elements of $G/\operatorname{Ad}$. Let us compute the law of $h_c$ under $P(t_1, \ldots, t_q)$. We will use the gauge invariance of $P(t_1, \ldots, t_q)$.

Let $f$ be a continuous function on $G$ and $j$ a discrete gauge transformation. Recall from the proposition 1.5.4 that $j_* P(t_1, \ldots, t_q) = P(t_1, \ldots, t_q)$, so that:

$$\begin{aligned} \int_{G^\Gamma} f(h_c)\, dP(t_1, \ldots, t_q) &= \int_{G^\Gamma} f(h_c \circ j)\, dP(t_1, \ldots, t_q) \\ &= \int_{G^\Gamma} f(j(c(1))^{-1} h_a j(c(0)))\, dP(t_1, \ldots, t_q). \end{aligned}$$

Thus the law of $h_c$ is right and left invariant on $G$ : it is the Haar measure.

### 1.8.2 Holonomy along the boundary of a small disk

Let $\Gamma$ be a graph on $M$, $L_1, \ldots, L_q$ be disjoint simple loops of $\Gamma^*$. Let $l$ be a loop of $\Gamma^*$ which is the boundary of a disk $D$ such that $L_i([0,1])$ is not constained in $\overline{D}$ for each $i$. We will estimate the law of $h_l$ on $(G^\Gamma, P_\beta)$.

Let $\rho$ be the function defined on $G$ by $\rho(x) = d(1, x)$, where $d$ is the biinvariant Riemannian distance. We want to estimate the size of $\rho(h_l)$.

$$\int_{G^\Gamma} \rho(h_l)\, dP_\beta = \frac{1}{Z_\beta} \int_{G^\Gamma} \rho(h_l) \prod_{F \in \mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1} \ldots d\nu_{x_q} dg'.$$

We need a result about graphs.

**Lemma 1.8.1** *Let $M$ be a surface and $\Gamma$ a graph on $M$. There exists a subgraph of $\Gamma$ which has only one face.*

*Proof.* If a graph has more than one face, there exists an edge which is on the boundary of two different faces. If we remove this edge, the resulting pregraph is still a graph. Indeed, this removed edge was necessarily in the interior of $M$, thus boundary properties are preserved. The faces of the new pregraph are those of the old one, except two faces that were glued along a segment. So the new face is diffeomorphic to a disk and by lemma 1.2.8, we know that the new pregraph is connected. In a finite number of such steps, one gets a subgraph of $\Gamma$ with only one face. □

The pregraph constituted by the $L_i$'s cuts $M$ into several pieces homeomorphic to surfaces with boundary $M_1, \ldots, M_k$. The graph $\Gamma$ induces a graph on each $M_i$. By the preceding lemma, there exists a subgraph $\Gamma'$ of $\Gamma$ that has exactly one face on each $M_i$ and such that $L_1, \ldots, L_q \in \Gamma'^*$. Now add to $\Gamma'$ the edges required to form $l$ and, if necessary, a simple path connecting $l$ with $\mathrm{Supp}(\Gamma')$. Finally, the assumption that the image of $L_i$ is never contained in $\overline{D}$ shows that it is possible, maybe by adding some vertices to $\Gamma'$, to be sure that each $L_i$ has an edge outside $D$. We get a graph $\Gamma''$ which is included in $\Gamma$ and in which we will compute, using the invariance by subdivision. We use the notation $P_\beta$ and $Z_\beta$.

$$\begin{aligned}\int_{G^\Gamma} \rho(h_l)\, dP_\beta^\Gamma &= \int_{G^{\Gamma''}} \rho(h_l)\, dP_\beta^{\Gamma''} \\ &= \frac{1}{Z_\beta}\int_{G^{\Gamma''}} \rho(h_l) \prod_{F\in\mathcal{F}(\Gamma'')} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1}\ldots d\nu_{x_q} dg'.\end{aligned}$$

**Lemma 1.8.2** *The function $t \longrightarrow \| p_t \|_\infty$ is decreasing on $(0,\infty)$.*

*Proof.* Let $0 < s < t$ be two positive times. Let $x$ be an element of $G$. We can estimate $p_t(x)$ in the following way, keeping in mind that $p_t$ is a positive function on $G$ :

$$p_t(x) = \int_G p_s(xy^{-1})p_{t-s}(y)\, dy \leq \| p_s \|_\infty \int_G p_{t-s}(y)\, dy \leq \| p_s \|_\infty .$$ □

Recall that each $L_i$ has at least an edge outside $\overline{D}$. For each $L_i$, we put the conditioning of $d\nu_{x_i}$ on one of these edges.

On the other hand, each face $F$ of $\Gamma''$ which is not included in $D$ is included in a face of $\Gamma'$, i.e. in a $M_i$: its surface is greater than $\sigma(M_i) - \sigma(D)$. We assume that $\sigma(D) \leq \frac{1}{2}\inf_i \sigma(M_i)$. Then

$$\int_{G^{\Gamma''}} \rho(h_l)\, dP_\beta \leq \frac{1}{Z_\beta} \prod_{F\not\subset D} \sup_{x\in G} |p_{\sigma(F)}(x)| \int_{G^{\Gamma''}} \rho(h_l) \prod_{F\subset D} p_{\sigma(F)}(h_{\partial F})\, dg'$$

The last integral is nothing but an integral against the discrete Yang-Mills measure on $D$, which is a surface with boundary. Using the invariance by subdivision inside $D$, we can replace the graph induced by $\Gamma''$ by a very simple graph whose support is just $\partial D = l([0,1])$. This leads to

$$\int_{G^\Gamma} \rho(h_l)\, dP_\beta^\Gamma \leq \frac{1}{Z_\beta} \prod_{i=1}^{k} \| p_{\frac{\sigma(M_i)}{2}} \|_\infty \int_G \rho(g) p_{\sigma(D)}(g)\, dg. \tag{1.6}$$

We are led to a problem of estimation of the heat kernel at small time.

**Lemma 1.8.3** *The following estimates hold :*

$$\int_G \rho(g)^4 p_t(g)\, dg = O(t^2), \quad \int_G \rho(g)^2 p_t(g)\, dg = O(t), \quad \int_G \rho(g) p_t(g)\, dg = O(\sqrt{t}).$$

Let $d$ denote the dimension of $G$. We use the following result proved in [Va](V.4.3):

**Proposition 1.8.4** *There exists a positive constant $C$ such that for all $t \in (0,1)$, all $g \in G$,*

$$\frac{1}{C} t^{-\frac{d}{2}} e^{-\frac{C\rho(g)^2}{t}} \leq p_t(g) \leq C t^{-\frac{d}{2}} e^{-\frac{\rho^2(g)}{Ct}}.$$

*Proof.* The first estimate implies both others. We use normal coordinates at the identity of $G$. Let $D_R$ be a geodesic disk of radius $R$ around 1, with $R$ such that exp is a diffeomorphism from $B(0,R) \subset T_1G$ onto $D_R$. We cut the integral according to $G = D_R \cup D_R^c$. One $D_R^c$, we have:

$$\int_{D_R^c} \rho(g)^4 p_t(g)\, dg \leq C t^{-\frac{d}{2}} \operatorname{diam}(G)^4 e^{-\frac{R^2}{Ct}} \leq C_1 t^{-\frac{d}{2}} e^{-\frac{R^2}{Ct}}.$$

For the part corresponding to $D_R$, we use spherical coordinates $(r,\theta)$ on $\exp^{-1}(D_R) = B(0,R)$. Note that on $B(0,R)$, the image of the Haar measure by $\exp^{-1}$ can be compared to the Lebesgue measure, so:

$$\begin{aligned} \int_{D_R} \rho(g)^4 p_t(g)\, dg &\leq C_2 \int_{[0,R]\times S^{d-1}} r^4 p_t(\exp(r,\theta)) r^{d-1}\, dr d\theta \\ &\leq C_3 t^{-\frac{d}{2}} \int_0^R r^{d+3} e^{-\frac{r^2}{Ct}}\, dr \\ &\leq C_3 t^{2+\frac{d}{2}} t^{-\frac{d}{2}} \int_0^\infty \frac{r^{d+3}}{t^{\frac{d+3}{2}}} e^{-\frac{r^2}{Ct}} \frac{dr}{t^{\frac{1}{2}}} \\ &\leq C_4 t^2. \end{aligned}$$

This estimation remains true if we replace $R$ by $R' < R$. Thus, for $t$ small enough, we have:

$$\begin{aligned} \int_G \rho(g)^4 p_t(g)\, dg &= \int_{D_{t^{1/4}}} \rho(g)^4 p_t(g)\, dg + \int_{D^c_{t^{1/4}}} \rho(g)^4 p_t(g)\, dg \\ &\leq C_4 t^2 + C_1 t^{-\frac{d}{2}} e^{-\frac{1}{Ct^{1/2}}} = O(t^2). \end{aligned}$$

□

Finally, we deduce from relation 1.6 and the preceding lemma the following proposition:

**Proposition 1.8.5** *Let $\Gamma$ be a graph on $M$. Let $L_1, \ldots, L_q$ be disjoint simple loops of $\Gamma^*$ and $x_1, \ldots, x_q$ be elements of $G$. Let $l$ be the boundary of a disk $D$ such that none of the $L_i$'s has its image contained in $\overline{D}$. There exist two positive constants $s$ and $C$ depending on the $L_i$'s but not on the $x_i$'s such that if $\sigma(D) \leq s$, then*

$$\int_{G^\Gamma} \rho(h_l)\, dP(x_1, \ldots, x_q) \leq C\sqrt{\sigma(D)}.$$

This regularity property will play an essential role in the construction of the continuous measure.

## 1.9 Discrete Abelian theory

### 1.9.1 Decomposition of cycles

Until now, we only used the compactness of $G$. We will finish this first chapter with a detailed study of the case $G = U(1)$. All results could be extended without conceptual problems to the case $G = U(1)^n$, i.e. the general compact Abelian case, but this would also make the notations much heavier.

We fix $M$, $\sigma$ as usual and a graph $\Gamma$ on $M$. Our aim is to analyze the law of the family $(h_c)_{c\in\Gamma^*}$. Set $\{a_1,\dots,a_r\} = \Gamma$. Since $G$ is Abelian, the function $h_c : G^\Gamma \longrightarrow G$ associated with a path $c$ depends only on the number of occurences of each $a_i$ in the decomposition of $c$, not on the order of the edges in this decomposition. In other words, the function $h_c$ depends only on the image of $c$ by the natural morphism of monoids $\Gamma^* \longrightarrow \mathbf{Z}^\Gamma$ which sends $a_i$ to $(0,\dots,1,\dots,0)$ with a 1 at the $i$-th place. Conversely, each element of $\mathbf{Z}^\Gamma$ determines without ambiguity a function from $U(1)^\Gamma$ into $U(1)$.

So, the natural index space in this context is $\mathbf{Z}^\Gamma$ instead of $\Gamma^*$ and this allows to consider linear combination of paths. Let us denote by $C\Gamma \subset \mathbf{Z}^\Gamma$ the set of linear combination of loops, also called cycles. We are especially interested in the law of $(h_c)_{c\in C\Gamma}$. The reason for which we consider only loops will become clear at the end of chapter 2. Basically, it is because for an arbitrary $G$, the holonomy along an open path is not a gauge-invariant function of a connection.

Let us recall a classical result about the homology of $M$.

**Theorem 1.9.1** *Let $g$ be the genus of $M$ and $p$ the number of connected components of $\partial M$. Then*

$$H_1(M;\mathbf{Z}) \simeq \begin{cases} \mathbf{Z}^{2g} & \text{if } p = 0 \\ \mathbf{Z}^{2g+p-1} & \text{if } p > 0. \end{cases}$$

If $p > 0$, one can construct a system of loops representing a basis of $H_1(M)$ by taking $p-1$ components of $\partial M$ and $2g$ loops of $M$ that generate the $H_1$ of a minimal closure of $M$, i.e. a surface obtained from $M$ by gluing a disk along each boundary component.

So, let us choose such a system composed by $\ell_1,\dots,\ell_{2g}$ in $\Gamma^*$ and $p-1$ loops $N_1,\dots,N_{p-1}$ that we denote just as the corresponding boundary components, with an abuse of notation. We can obtain the $\ell_i$'s by deforming an arbitrary system of generators using the same technique as in the proof of the proposition 1.2.4.

Now let $c$ be a cycle in $C\Gamma$. There is an unique decomposition

$$c = \lambda_1\ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1} + c^\perp,$$

with $\lambda_i, \nu_j \in \mathbf{Z}$ and $c^\perp \in C\Gamma$ a cycle homologous to zero. Let us denote by $C_0\Gamma$ the submodule of $C\Gamma$ spanned by the cycles homologous to zero.

**Proposition 1.9.2** *If $\partial M$ is empty (resp. non empty), the boundaries of all faces except one chosen arbitrarily (resp. of all faces) form a basis of the submodule $C_0\Gamma$ of $C\Gamma$.*

We will prove this proposition very soon. Set $\mathcal{F}(\Gamma) = \{F_1,\dots,F_n\}$ and choose for each $F_i$ a cycle $\partial F_i$ whose image is the boundary of $F_i$. We can write :

$$c = \lambda_1\ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1} + \mu_1\partial F_1 + \ldots + \mu_n\partial F_n, \tag{1.7}$$

the decomposition being non unique if $M$ is closed. The relation 1.7, together with the multiplicativity of the holonomy, shows that the law of the family $(h_c)_{c\in C\Gamma}$ is completely determined by the law of what we will call a fundamental system :

$$(h_{\ell_1},\dots,h_{\ell_{2g}},h_{N_1},\dots,h_{N_{p-1}},h_{\partial F_1},\dots,h_{\partial F_n}).$$

*Proof of proposition* 1.9.2 : To begin with, suppose that $M$ has no boundary. We proceed by induction on $n = \operatorname{Card}\mathcal{F}(\Gamma)$. If $n = 1$, the only loop in $C\Gamma$ is $\partial F$ and it is homologically trivial.

Now suppose that the result is true for a graph with $n-1$ faces. Let $\Gamma$ be a graph with $n$ faces. There is an edge of $\Gamma$, say $a_r$, which is on the boundary of two distinct faces, say $F_{n-1}$ and $F_n$. Let $\Gamma' = \{a_1,\dots,a_{r-1}\}$ be the graph obtained by removing $a_r$. It has $n-1$ faces $F_1,\dots,F_{n-2},F_{n-1}\cup F_n$. Let $c$ be a cycle of $C_0\Gamma$. We can decompose it uniquely in $c = c_0 + pa_r$ with $p \in \mathbf{Z}$ and $c_0 \in C\Gamma'$. We can also write $\partial F_{n-1} = a_r + b$ with $b \in C\Gamma'$. So , we have $c = (c_0 - pb) + p\partial F_{n-1}$. By induction, $c_0 - pb$, which is homologous to zero in $\Gamma'$, is a linear combination of $\partial F_1,\dots,\partial F_{n-2}$. Thus, $\partial F_1,\dots,\partial F_{n-1}$ generate the submodule of homologically trivial cycles in $C\Gamma$. On the other hand, $\partial F_1,\dots,\partial F_{n-2}$ are linearly indendent by induction and $\partial F_{n-1}$ is independent of the submodule that they generate, because it contains the edge $a_r$. This gives the result when $M$ is closed.

If $M$ has a boundary, consider a minimal closure $i_1 : M \longrightarrow M_1$ of $M$ and identify $M$ with $i_1(M)$. Let $c$ be a cycle homologous to zero in $M$. It is also homologous to zero in $M_1$ and can be decomposed using the result on $M_1$ into :

$$c = \sum_{F_i\in\mathcal{F}(\Gamma),F_i\subset M} \mu_i\partial F_i + \nu_1 N_1 + \dots + \nu_{p-1}N_{p-1},$$

because the $N_i$'s are the boundaries of the faces of $\Gamma$ on $M_1 - M$. This decomposition gives, in $H_1(M)$,

$$[c] = 0 = \nu_1[N_1] + \dots + \nu_{p-1}[N_{p-1}],$$

implying $\nu_1 = \dots = \nu_{p-1} = 0$ and $c = \mu_1\partial F_1 + \dots + \mu_n\partial F_n$. The independence of the $\partial F_i$'s on $M_1$ implies their independence on $M$. □

### 1.9.2 Study of a fundamental system

We want to study the discrete Yang-Mills measure conditioned by the holonomies along the boundary components of $M$. Let $x_1,\dots,x_p$ be elements of $U(1)$. Under $P_\beta = P(x_1,\dots,x_p)$, the law of $(h_{N_1},\dots,h_{N_{p-1}})$ is deterministic, equal to $\delta_{(x_1,\dots,x_{p-1})}$.

**Proposition 1.9.3** *Under the measure $\nu_{x_1}\otimes\dots\otimes\nu_{x_q}\otimes dg'$ on $G^\Gamma$, the variables $h_{\ell_1},\dots,h_{\ell_{2g}}$, $h_{\partial F_1},\dots,h_{\partial F_{n-1}}$ are uniform and independent on $U(1)$.*

*Proof.* We compute the characteristic function of $(h_{\ell_1},\dots,h_{\ell_{2g}},h_{\partial F_1},\dots,h_{\partial F_{n-1}})$, seen as a $\mathbf{C}^{2g+n-1}$-valued random variable. In order to simplify the notations, we choose an orientation of $M$ and assume that each $N_i \subset \partial M$ and each $\partial F_i$ is oriented according to the usual convention. Let $\lambda_1,\dots,\lambda_{2g},\mu_1,\dots,\mu_{n-1}$ be integers.

$$F(\lambda_1,\dots,\lambda_{2g},\mu_1,\dots,\mu_{n-1}) \;=\; \int_{U(1)^\Gamma} h_{\ell_1}^{\lambda_1}\dots h_{\ell_{2g}}^{\lambda_{2g}} h_{\partial F_1}^{\mu_1}\dots h_{\partial F_{n-1}}^{\mu_{n-1}}\, dP_\beta$$

$$= \int_{U(1)^\Gamma} h_{\lambda_1\ell_1+\ldots+\lambda_{2g}\ell_{2g}+\mu_1\partial F_1+\ldots+\mu_{n-1}\partial F_{n-1}}\, dP_\beta$$
$$= \int_{U(1)^\Gamma} h_{a_1}^{\alpha_1}\ldots h_{a_r}^{\alpha_r}\, dP_\beta,$$

where $\sum_i \lambda_i\ell_i + \sum_j \mu_j\partial F_j = \sum_k \alpha_k a_k$. Suppose that the $a_i$'s are labeled in such a way that $N_1 = a_1\ldots a_{i_1},\ldots, N_p = a_{i_{p-1}+1}\ldots a_{i_p}$ with $1 < a_1 < \ldots < a_{i_p}$.

$$F(\lambda_i,\mu_j) = \int_{U(1)^\Gamma} (h_{a_1}^{\alpha_1}\ldots h_{a_{i_1}}^{\alpha_{i_1}})\ldots(h_{a_{i_{p-1}+1}}^{\alpha_{i_{p-1}+1}}\ldots h_{a_{i_p}}^{\alpha_{i_p}})h_{a_{i_p+1}}^{\alpha_{i_p+1}}\ldots h_{a_r}^{\alpha_r}\, dP_\beta$$
$$= \int_{U(1)^\Gamma} g_1^{\alpha_1}\ldots g_{i_1}^{\alpha_{i_1}}\, d\nu_{x_1}\ldots\int_{U(1)^\Gamma} g_{i_{p-1}+1}^{\alpha_{i_{p-1}+1}}\ldots g_{i_p}^{\alpha_{i_p}}\, d\nu_{x_p}\int_{U(1)^\Gamma} g^{\alpha_{i_p+1}}\, dg\ldots\int_{U(1)^\Gamma} g^{\alpha_r}\, dg.$$

This product is zero if one of the $\alpha_k$'s with $k \geq i_p + 1$ is nonzero. Otherwise, $\partial M$ is non empty and the cycle $\sum\lambda_i\ell_i + \sum\mu_j\partial F_j$ has all its edges on $\partial M$. Thus, we have an equality

$$\sum_{i=1}^{2g}\lambda_i\ell_i + \sum_{j=1}^{n-1}\mu_j\partial F_j = \sum_{k=1}^{p}\nu_k N_k,$$

which, in $H_1(M)$, implies $\nu_p[N_p] = \sum\lambda_i[\ell_i] - \sum_{k<p}\nu_k[N_k]$. Since $[N_p] = -[N_1] - \ldots - [N_{p-1}]$, this implies $\lambda_i = 0$ for all $i$ and $\nu_k = \nu_p$ for all $k$. We get

$$\sum_{j=1}^{n-1}\mu_j\partial F_j = \nu\sum_{i=1}^{p} N_i = \nu\sum_{j=1}^{n}\partial F_j.$$

Since $\partial F_1,\ldots,\partial F_n$ are independent, the comparison of the coefficients of the $\partial F_n$'s gives $\nu = 0$ and then $\mu_j = 0$. Finally, the cycle $\sum\lambda_i\ell_i + \sum\mu_j\partial F_j$ is equal to zero and $F(\lambda_i,\mu_j) = 1$. Thus, we proved that $F(\lambda_i,\mu_j)$ is equal to zero, except if all $\lambda_i$'s and $\mu_j$'s are zero, in which case it is equal to 1. This proves the result. □

The last element to study in the fundamental system is $h_{\partial F_n}$. We have

$$\partial F_n = \sum_{i=1}^{p} N_i - \sum_{j=1}^{n-1}\partial F_j,$$

so that $h_{\partial F_n} = x_1\ldots x_p h_{\partial F_1}^{-1}\ldots h_{\partial F_{n-1}}^{-1}$ under $\nu_{x_1}\otimes\ldots\otimes\nu_{x_p}\otimes dg'$.

**Proposition 1.9.4** *Set $x = x_1\ldots x_p$ if $M$ has a boundary and $x = 1$ if $M$ is closed. For any function $f$ continuous on $G^{2g+n+p}$,*

$$\int_{G^\Gamma} f(h_{\ell_1},\ldots,h_{\ell_{2g}},h_{N_1},\ldots,h_{N_{p-1}},h_{\partial F_1},\ldots,h_{\partial F_n})\, dP(x_1,\ldots,x_p) =$$

$$\int_{G^{2g+n}} f(u_1,\ldots,u_{2g},x_1,\ldots,x_{p-1},v_1,\ldots,v_n)p_{\sigma(F_1)}(v_1)\ldots p_{\sigma(F_n)}(v_n)\, du_1\ldots du_{2g}\, d\nu_x^n(v_1,\ldots,v_n).$$

Note that, in the Abelian setting, the measure $\nu_x^n$ is invariant by permutations of the factors in $U(1)^n$.

### 1.9.3 Gaussian aspect of the Abelian theory

We proved that the law of the whole family $(h_c)_{c \in C\Gamma}$ is determined by the law of a fundamental system and we just described this law. So we could consider that the proposition 1.9.4 is the answer to our question. In fact, it is possible to be much more explicit by taking the gaussian character of the Abelian theory into account. The crucial part of the law of a fundamental system is of course that of $(h_{\partial F_1}, \ldots, h_{\partial F_n})$. We will concentrate on this law.

**Proposition 1.9.5** *Let $Y_1, \ldots, Y_n$ be independent centered real gaussian random variables with $Y_i \sim \mathcal{N}(0, \sigma(F_i))$. Let $S = Y_1 + \ldots + Y_n$ be their sum. For each $i = 1, \ldots, n$, set*

$$X_i = Y_i - \frac{\sigma(F_i)}{\sigma(M)} S.$$

*Let $T$ be a real random variable, independent of the $Y_i$'s, with the following discrete law :*

$$P(T = t) = \begin{cases} \left( \sum_{s, e^{is} = x} e^{-\frac{s^2}{2\sigma(M)}} \right)^{-1} e^{-\frac{t^2}{2\sigma(M)}} & \text{if } e^{it} = x \\ 0 \text{ otherwise,} & \end{cases}$$

*where, as before, $x = x_1 \ldots x_p$ if $M$ has a boundary and $x = 1$ if $M$ is closed. Then, for any function $f$ continuous on $G^n$,*

$$\int_{G^\Gamma} f(h_{\partial F_1}, \ldots, h_{\partial F_n}) \, dP_\beta = E \, f\left( e^{i\left(X_1 + \frac{\sigma(F_1)}{\sigma(M)} T\right)}, \ldots, e^{i\left(X_n + \frac{\sigma(F_n)}{\sigma(M)} T\right)} \right) \tag{1.8}$$

The law of $T$ described in this theorem is just that of a $\mathcal{N}(0, \sigma(M))$ random variable conditioned to take its values in $\exp^{-1}(x)$, where $\exp(t) = e^{it}$. We shall discuss the meaning of this variable in section 3.2.2.

*Proof.* In this proof, we set $\sigma_i = \sigma(F_i)$ and $\sigma_M = \sigma(M)$. One easily computes

$$EX_iX_j = \delta_{ij}\sigma_i - \frac{\sigma_i \sigma_j}{\sigma_M}$$

and $\sum X_i = 0$ a.s. The law of $(X_1, \ldots, X_n)$ has no density with respect to Lebesgue measure on $\mathbf{R}^n$, but that of $(X_1, \ldots, X_{n-1})$ does, on $\mathbf{R}^{n-1}$. Denote by $C$ the $(n-1) \times (n-1)$ covariance matrix of $(X_1, \ldots, X_{n-1})$. One easily checks that $C^{-1}$ is given by

$$(C^{-1})_{ij} = \frac{\delta_{ij}}{\sigma_i} + \frac{1}{\sigma_n}.$$

So the density of the law of $(X_1, \ldots, X_n)$ is :

$$d\eta(t_1, \ldots, t_{n-1}) = \frac{1}{Z} \exp -\frac{1}{2} \left( \sum_{i=1}^{n-1} \frac{t_i^2}{\sigma_i} + \sum_{i,j=1}^{n-1} \frac{t_i t_j}{\sigma_n} \right) dt_1 \ldots dt_n.$$

Let us fix a number $t_0$ such that $e^{it_0} = x$. We also set $t_n = -t_1 - \ldots - t_{n-1}$. We can compute the right term of (1.8): it is equal to

$$\frac{1}{Z}\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^{n-1}} f\left(e^{it_1+\frac{\sigma_1}{\sigma_M}(t_0+2k\pi)},\ldots,e^{it_{n-1}+\frac{\sigma_{n-1}}{\sigma_M}(t_0+2k\pi)},e^{it_n+\frac{\sigma_n}{\sigma_M}(t_0+2k\pi)}\right)$$

$$\exp-\frac{1}{2}\left(\sum_{i=1}^{n-1}\frac{t_i{}^2}{\sigma_i}+\sum_{i,j=1}^{n-1}\frac{t_it_j}{\sigma_n}\right)\exp\frac{-(t_0+2k\pi)^2}{2\sigma_M}\,dt_1\ldots dt_{n-1}$$

$$= \frac{1}{Z}\sum_{k,q_1,\ldots,q_{n-1}\in\mathbf{Z}}\int_{[0,2\pi]^{n-1}} f(e^{it_1},\ldots,e^{it_{n-1}},e^{i(t_n+t_0)})$$

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{\sigma_i}(t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi))^2+\right.$$

$$\left.+\frac{1}{2\sigma_n}\left(\sum_{i=1}^{n-1}t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi)\right)^2\right)\exp-\frac{(t_0+2k\pi)^2}{2\sigma_M}\,dt_1\ldots dt_{n-1}.$$

We do not care about normalization constants, since two probability measures with proportional densities are equal. Now we compute the left hand side of (1.8) using the following expression of the heat kernel :

$$p_s(e^{it}) = \frac{1}{Z}\sum_{p\in\mathbf{Z}} e^{-\frac{(t-2p\pi)^2}{2s}},$$

which is just the image by the exponential map of the heat kernel on $\mathbf{R}$. We get

$$\int_{G^\Gamma} f(h_{\partial F_1},\ldots,h_{\partial F_n})\,dP_\beta = \int_{G^n} f(v_1,\ldots,v_n)p_{\sigma_1}(v_1)\ldots p_{\sigma_n}(v_n)\nu_x^n(v_1,\ldots,v_n)$$

$$= \sum_{p_1,\ldots,p_n\in\mathbf{Z}}\int_{[0,2\pi]^{n-1}} f(e^{it_1},\ldots,e^{it_{n-1}},e^{i(t_n+t_0)})\exp\left(-\frac{1}{2}\sum_{i=1}^{n-1}\frac{(t_i-2p_i\pi)^2}{\sigma_i}\right)$$

$$\exp\left(-\frac{1}{2}\frac{(t_n+t_0-2p_n\pi)^2}{\sigma_n}\right)\,dt_1\ldots dt_{n-1}.$$

The result will be a consequence of the following equality:

$$\sum_{p_1,\ldots,p_n\in\mathbf{Z}}\exp\left(-\frac{1}{2}\sum_{i=1}^{n-1}\frac{(t_i-2p_i\pi)^2}{\sigma_i}-\frac{1}{2}\frac{(t_n+t_0-2p_n\pi)^2}{\sigma_n}\right)=$$

$$\sum_{k,q_1,\ldots,q_{n-1}\in\mathbf{Z}}\exp\frac{-(t_0+2k\pi)^2}{2\sigma_M}-\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{\sigma_i}(t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi))^2-$$

$$-\frac{1}{2\sigma_n}\left(\sum_{i=1}^{n-1}t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi)\right)^2$$

Setting $q_n = -q_1 - \ldots - q_{n-1}$, we have

$$\sum_{i=1}^{n-1}\frac{1}{\sigma_i}(t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi))^2+\frac{1}{\sigma_n}(\sum_{i=1}^{n-1}t_i+2\pi q_i-\frac{\sigma_i}{\sigma_M}(t_0+2k\pi))^2=$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} \frac{1}{\sigma_i}([t_i + 2\pi q_i] - \frac{\sigma_i}{\sigma_M}(t_0 + 2k\pi))^2 + \frac{1}{\sigma_n}\left([-t_n - 2\pi q_n - (t_0 + 2k\pi)] + \frac{\sigma_n}{\sigma_M}(t_0 + 2k\pi)\right)^2 \\
&= \sum_{i=1}^{n-1} \frac{1}{\sigma_i}\left[(t_i + 2\pi q_i)^2 - \frac{2\sigma_i}{\sigma_M}(t_i + 2\pi q_i)(t_0 + 2k\pi) + \frac{\sigma_i^2}{\sigma_M^2}(t_0 + 2k\pi)^2\right] + \\
&\qquad + \frac{1}{\sigma_n}\left[(-t_n - 2\pi q_n - (t_0 + 2k\pi))^2 + \frac{\sigma_n^2}{\sigma_M^2}(t_0 + 2k\pi)^2 + \right. \\
&\qquad\qquad \left. + \frac{2\sigma_n}{\sigma_M}(-t_n - 2\pi q_n - (t_0 + 2k\pi))(t_0 + 2k\pi)\right] \\
&= \sum_{i=1}^{n-1} \frac{1}{\sigma_i}(t_i + 2\pi q_i)^2 + \frac{1}{\sigma_M}(t_0 + 2k\pi)^2\left(\sum_{i=1}^{n-1}\frac{\sigma_i}{\sigma_M} + \frac{\sigma_n}{\sigma_M}\right) \\
&\qquad - \frac{2}{\sigma_M}\sum_{i=1}^{n-1}(t_i + 2\pi q_i)(t_0 + 2k\pi) + \frac{2}{\sigma_M}\sum_{i=1}^{n-1}(t_i + 2\pi q_i)(t_0 + 2k\pi) \\
&\qquad\qquad - \frac{2}{\sigma_M}(t_0 + 2k\pi)^2 + \frac{1}{\sigma_n}(-t_n - 2\pi q_n - (t_0 + 2k\pi))^2 \\
&= \sum_{i=1}^{n-1} \frac{1}{\sigma_i}(t_i + 2\pi q_i)^2 + \frac{1}{\sigma_n}(t_n + t_0 + 2\pi q_n + 2k\pi)^2 - \frac{1}{\sigma_M}(t_0 + 2k\pi)^2.
\end{aligned}
$$

Setting $p_1 = q_1, \ldots, p_{n-1} = q_{n-1}$ and $p_n = -q_n - k$, we get the result. □

### 1.9.4 The double layer potential

To go further, we would like to represent isometrically the vector $(X_1, \ldots, X_n)$ by a vector of functions of $L^2(M, \sigma)$ naturally associated with $F_1, \ldots, F_n$. To begin with, remark that the vector $(\mathbf{1}_{F_1}, \ldots, \mathbf{1}_{F_n})$ has the same covariance as $(Y_1, \ldots, Y_n)$. Now set

$$u_i = \mathbf{1}_{F_i} - \frac{\sigma(F_i)}{\sigma(M)}.$$

The vector $(u_1, \ldots, u_n)$ has the same covariance as $(X_1, \ldots, X_n)$. Each $u_i$ can be seen as the orthogonal projection of $\mathbf{1}_{F_i}$ on the hyperplane $L_0^2(M, \sigma)$ of functions whose mean is equal to zero. In fact, the $u_i$'s are the most natural generalizations on $M$ of the classical index of a loop around a point in the plane. We will give a more direct definition of $u_i$.

To do this, we endow $M$ with a Riemannian metric whose volume coincides with $\sigma$. There exist a lot of such metrics, because $\sigma$ is equivalent to the Lebesgue measure in any chart with a smooth density, as well as the Riemannian volume of any Riemannian metric on $M$.

The choice of a compatible metric on $M$ gives rise to a Laplace operator $\Delta$ and to a Hodge operator $*$ on $\bigwedge^1(T^*M)$. There exists on $M$ a Green function $G : M \times M \longrightarrow \mathbf{R}_+$ defined outside the diagonal which is symmetric, smooth and such that

$$
\begin{cases}
\Delta G(x, \cdot) = \delta_x - \frac{1}{\sigma(M)} \ \forall x \in M \\
\int_M G(x, y)\, d\sigma(y) = 0 \ \forall x \in M \ \text{ when } \partial M = \emptyset \\
*dG_x = 0 \ \text{ on } \partial M \ \forall x \in M \ \text{ when } \partial M \neq \emptyset,
\end{cases}
\tag{1.9}
$$

where $G_x$ denotes the function $G(x,\cdot)$. A proof of this fact can be found in [Au]. Note that when $M$ has a boundary, there exists a solution to $\Delta\widetilde{G}_x = \delta_x$. Nevertheless, this choice would be incompatible with the condition $*dG_x = 0$ on $\partial M$ which implies $\int_M \Delta G_x = 0$.

**Definition 1.9.6** *Let $c$ be a path on $M$. We call double layer potential of $c$ the function $u_c$ defined on $M$ outside the image of $c$ by :*

$$u_c(x) = \int_c *dG_x.$$

Note that the double layer potential is additive: if $c_1$ and $c_2$ are two cycles of $C\Gamma$, then $u_{c_1+c_2} = u_{c_1} + u_{c_2}$ $\sigma$-a.e. on $M$.

**Proposition 1.9.7** *Let $c$ be a simple loop which is the boundary of a subset $U$ of $M$. Set $V = U^c$. Then*

$$u_c(x) = \frac{\sigma(V)}{\sigma(M)}1_U(x) - \frac{\sigma(U)}{\sigma(M)}1_V(x) = 1_U - \frac{\sigma(U)}{\sigma(M)}.$$

*In particular, $u_c \in L^2(M,\sigma)$ and $\| u_c \|_2 = \left(\frac{\sigma(U)\sigma(V)}{\sigma(M)}\right)^{\frac{1}{2}}$.*

*Proof.* Let $x$ be in $U$. Since $*dG_x = 0$ on $\partial M$, we have:

$$u_c(x) = \int_{\partial U} *dG_x = -\int_{\partial V} *dG_x = -\int_V \delta_x - \frac{1}{\sigma(M)} = \frac{\sigma(V)}{\sigma(M)}.$$

Now let $x$ be in $V$.

$$u_c(x) = \int_{\partial U} *dG_x = \int_U \delta_x - \frac{1}{\sigma(M)} = -\frac{\sigma(U)}{\sigma(V)}.$$

The last part of the statement follows easily. □

**Corollary 1.9.8** *The vector $(u_1,\ldots,u_n)$ is equal to $(u_{\partial F_1},\ldots,u_{\partial F_n})$.*

To go from functions $u_{\partial F_i}$ to random variables $X_i$, we need an isometry of $L^2(M,\sigma)$ into a gaussian space, in other words a white noise on $(M,\sigma)$. Let us consider a white noise

$$\begin{array}{rcl} W : L^2(M,\sigma) & \longrightarrow & \mathcal{G} \\ u & \longmapsto & W(u) \end{array}$$

such that for any $u, v \in L^2(M)$, $W(u)$ and $W(v)$ are real centered gaussian random variables such that $E[W(u)W(v)] = (u,v)_{L^2}$. The proposition 1.9.5 can be rewritten in the following form:

**Proposition 1.9.9** *The following equality holds in law:*

$$(h_{\partial F_1},\ldots,h_{\partial F_n}) \overset{\text{law}}{=} \left(e^{i\left(W(u_{\partial F_1})+\frac{\sigma(F_1)}{\sigma(M)}T\right)},\ldots,e^{i\left(W(u_{\partial F_n})+\frac{\sigma(F_n)}{\sigma(M)}T\right)}\right).$$

We would like to extend this result to arbitrary cycles homologous to zero. Let $c_1, \ldots c_k$ be cycles of $C_0\Gamma$. For each $i$, $c_i$ is a linear combination of the $\partial F_i$ thus $u_{c_i}$ is well defined and is in $L^2(M)$. So, $W(u_{c_i})$ is well defined.

We have to generalize the term $\frac{\sigma(F_i)}{\sigma(M)}$. Since $c_i$ is homologous to zero, it is the boundary of a two-chain denoted by $\alpha$. If $M$ has a boundary, $H_2(M) = 0$ and $\alpha$ is well defined by $\partial\alpha = c$. We identify $\sigma$ with a 2-form on $M$ and set $\sigma(\alpha) = |\langle \sigma, \alpha \rangle|$, using the natural pairing between 2-forms and 2-chains. So the number

$$\sigma_{\text{int}}(c) = \frac{\sigma(\alpha)}{\sigma(M)}$$

is well defined. If $M$ is closed, $H_2(M) \simeq \mathbf{Z}$ and $\alpha$ is defined up to a multiple of $[M]$. So the number $\sigma_{\text{int}}(c)$ is only defined modulo 1. But in this case, $T$ takes its values in $\exp^{-1}(1) = 2\pi\mathbf{Z}$ so that $\exp i\sigma_{\text{int}}(c)T$ is well defined.

**Proposition 1.9.10** *Let $(c_1, \ldots, c_k)$ be cycles of $C_0\Gamma$. Then the following equality in law holds:*

$$(h_{c_1}, \ldots, h_{c_k}) \stackrel{\text{law}}{=} \left(e^{i\left(W(u_{c_1}) + \sigma_{\text{int}}(c_1)T\right)}, \ldots, e^{i\left(W(u_{c_k}) + \sigma_{\text{int}}(c_k)T\right)}\right).$$

*Proof*. By proposition 1.9.9, the result is true when $(c_1, \ldots, c_k) = (\partial F_1, \ldots, \partial F_n)$. Since the boundaries of the faces constitute a basis of $C_0\Gamma$, it is sufficient to show that the new set of variables defined using the white noise satisfy the same multiplicativity property as $(h_c)_{c \in C_0\Gamma}$.

On one hand, $W$ is linear and the double layer potential is additive, so that $\exp iW(u_{c_1+c_2}) = \exp iW(u_{c_1}) \exp iW(u_{c_2})$. On the other hand, $c_1 = \partial\alpha_1$ and $c_2 = \partial\alpha_2$ imply $c_1 + c_2 = \partial(\alpha_1 + \alpha_2)$, so $\sigma_{\text{int}}$ is also additive. This proves the result. □

The results that we proved in this section are the starting point of the more detailed investigation that will be done in chapter 3, after the continuous Yang-Mills measure has been constructed.

Some properties of the double layer potential will be proved in the next chapter, using a favorable technical context. Nevertheless, it is necessary to state here a fundamental property that will be proved at the end of the chapter 3.

**Theorem 1.9.11** *For any path $c$ of $PM$, the function $u_c$ is in $L^2(M, \sigma)$.*

# Chapter 2

# Continuous Yang-Mills measure

In chapter 1, we defined a random holonomy along the paths of a graph on $M$. Our aim in this chapter is to extend this definition to all paths on $M$. The problem is that there are families of paths that cannot be realized as subfamilies of any $\Gamma^*$, $\Gamma$ being a graph on $M$: even two smooth paths can cross each other an infinity of times an give rise to an infinity of connected components on $M$, a situation in which we are unable to write the joint law of their holonomies using the tools of the preceding chapter.

The two properties of the discrete theory that are essential to our purpose are the invariance by subdivision, expressed in theorem 1.6.1 and the regularity property of the proposition 1.8.5. Our basic idea is to cover $M$ with finer and finer graphs and to prove that the discrete measures on these graphs converge in some sense to a continuous object that will be called continuous Yang-Mills measure.

## 2.1 Projective systems

Let $(\Gamma_\lambda)_{\lambda\in\Lambda}$ be a family of graphs on $M$, which approximates correctly $M$, whatever this means exactly. In sections 1.4 and 1.5, we explained how to construct a family of probability spaces corresponding to this family of graphs. We will now consider the projective limit of this family of probability spaces.

**Definition 2.1.1** *Let $\Lambda$ be an ordered set such that for all $\lambda, \mu \in \Lambda$, there exists $\nu \in \Lambda$ such that $\lambda < \nu$ and $\mu < \nu$. A projective family of probability spaces indexed by $\Lambda$ is a family $(\Omega_\lambda, P_\lambda)$ of probability spaces together with a family of measurable maps $f_{\lambda\mu} : \Omega_\mu \longrightarrow \Omega_\lambda$ defined for all $\lambda < \mu$, such that*
*1.* $f_{\lambda\lambda} = \mathrm{Id}_{\Omega_\lambda}$,
*2.* $f_{\lambda\nu} = f_{\lambda\mu} \circ f_{\mu\nu}$ *if* $\lambda < \mu < \nu$,
*3.* $f_{\lambda\mu}(\Omega_\mu) = \Omega_\lambda$,
*4.* $(f_{\lambda\mu})_* P_\mu = P_\lambda$.

The projective limit of such a system is by definition the set

$$\varprojlim \Omega_\lambda = \Omega = \{(\omega_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \Omega_\lambda \mid \forall \lambda < \mu, f_{\lambda\mu}(\omega_\mu) = \omega_\lambda\}.$$

For each $\lambda \in \Lambda$, the projection on the $\lambda$-th coordinate defines a map $f_\lambda : \Omega \longrightarrow \Omega_\lambda$. The main result is that, under certain conditions, there exist a $\sigma$-algebra and a probability measure on $\Omega$ which are consistent with all $P_\lambda$ via the maps $f_\lambda$.

**Theorem 2.1.2 ([Ck] 2.2)** *If all $\Omega_\lambda$ are compact Borel probability spaces, then there exist a $\sigma$-algebra and a probability measure $P$ on $\Omega$ such that, for all $\lambda$ in $\Lambda$, $f_{\lambda *}P = P_\lambda$. The space $(\Omega, P)$ is called projective limit of the family $(\Omega_\lambda, P_\lambda; f_{\lambda\mu})$.*

It would be appealing to take the family of all graphs as index set $\Lambda$, with the order defined in 1.2.6. The problem is that given two graphs, it is not always true that there exists a third graph which is finer than both others, just because two edges belonging to two different graphs can intersect very badly. So, the first assumption about the ordering on $\Lambda$ would not be satisfied.

## 2.2 Piecewise geodesic graphs

We are led to consider a family of graphs small enough for that problem not to occur. A convenient family is that of graphs with edges piecewise geodesic for some Riemannian metric on $M$. Another possibility is to consider graphs with piecewise real analytic edges for some complex structure on $M$. This has been investigated by Ashtekar and Lewandowski [AL].

We fix a surface $(M, \sigma)$. For technical reasons, we suppose that $M$ is closed, until the section 2.8 where we shall derive the general case from the case without boundary.

Let us choose $q$ disjoint simple loops $L_1, \ldots, L_q$ on $M$ whose image is a smooth submanifold of $M$, in other words, $q$ disjoint embeddings of $S^1$ into $M$. We will sometimes think of these loops as the boundary of a submanifold of $M$ or just as loops along which we want to contition the holonomy.

**Proposition 2.2.1** *There exists a Riemannian metric on $M$ whose Riemannian volume coincides with $\sigma$ and such that $L_1, \ldots, L_q$ are geodesics.*

*Proof.* If $q$=0, let us choose an arbitrary metric on $M$. Its Riemannian volume is equivalent to $\sigma$, with a smooth density. Multiplying the metric by an appropriate smooth positive function, we get a new metric, conformal to the first one, whose Riemannian volume is exactly $\sigma$.

If $q > 0$, the proof is more complicated. Let us first construct a metric for which all $L_i$'s are geodesic. Each $L_i$ has a tubular neighbourhood in $M$ which is diffeomorphic to a cylinder $S^1 \times (-1, 1) \ni (\theta, t)$, with $L_i = \{t = 0\}$. In these coordinates, $L_i$ is certainly geodesic for the metric $d\theta^2 + dt^2$. If the tubular neighbourhoods were chosen small enough to be disjoint, this procedure defines a Riemannian metric on the reunion of these cylinders and we extend it arbitrarily to a metric $g_0$ on the whole surface $M$. The loops $L_1, \ldots, L_q$ cut $M$ into submanifolds with boundary $M_1, \ldots, M_k$. Multiplying $g_0$ by a good positive function which is identically equal to 1 in a neighbourhood of each $L_i$, we can obtain a new metric $g_1$ for which all $L_i$ are still geodesics and also such that

$$\mathrm{vol}_{g_1}(M_i) = \sigma(M_i) \quad \forall i = 1 \ldots k. \tag{2.1}$$

Now we must redistribute the surface inside each $M_i$. We adapt a proof of Moser's theorem, which is for example proved in [BG].

**Theorem 2.2.2 (Moser)** *Let $\alpha$ and $\beta$ be two 2-forms on a closed compact surface $M$ such that*

$$\int_M \alpha = \int_M \beta.$$

*There exists a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi^*\beta = \alpha$.*

We choose two forms $\alpha$ and $\beta$ representing respectively $\sigma$ and $\mathrm{vol}_{g_1}$. We know more about them than what is needed in Moser's theorem, but we also want to prove more: we would like to find a diffeomorphism of $M$ that sends $\alpha$ to $\beta$ and also that preserves the $L_i$'s, so that they remain geodesic after pulling back $g_1$ by the diffeomorphism.

For each $i = 1 \dots q$, let $j_i : L_i([0,1]) \longrightarrow M$ denote the canonical injection. The fact that $\int_M \beta - \alpha = 0$ implies that there exists a form $\gamma \in \Lambda^1(T^*M)$ such that $d\gamma = \beta - \alpha$. We show that $\gamma$ can be chosen such that $j_i^*\gamma = 0$ for all $i$, or in other words such that $\gamma(X) = 0$ for any vector $X$ tangent to a $L_i$. Pick $\gamma$ such that $d\gamma = \beta - \alpha$, consider the element $(\int_{L_1} \gamma, \dots, \int_{L_q} \gamma)$ of $\mathbf{R}^q$ and the $q$-uple $([L_1], \dots, [L_q])$ of vectors of $H_1(M; \mathbf{Z})$. Suppose that $n_1, \dots, n_q$ are integers such that $\sum_i n_i[L_i] = 0$. Then $\sum_i n_i[L_i]$ is the boundary of a 2-chain in $M$, which is necessarily a linear combination of $M_1, \dots, M_k$. Thanks to (2.1), this implies $\int_{\sum n_i L_i} \gamma = 0 = \sum_{i=1}^q n_i \int_{L_i} \gamma$. We proved that a relation $\sum n_i[L_i] = 0$ implies $\sum n_i \int_{L_i} \gamma = 0$. Thus there exists a linear form $\zeta$ on $H_1(M; \mathbf{Z})$ such that $(\zeta, [L_i]) = \int_{L_i} \gamma$. This linear form can be represented by an element of $H^1(M; \mathbf{R})$ and this element can be represented by a closed 1-form on $M$ that we still denote by $\zeta$. The form $\gamma - \zeta$ satisfies:

$$\begin{cases} d(\gamma - \zeta) = d\gamma = \beta - \alpha \\ \int_{L_i} \gamma - \zeta = 0 \quad \forall i = 1 \dots q. \end{cases}$$

This last relation proves that for each $i$, $j_i^*(\gamma - \zeta)$ is exact on $L_i$ and can be written $du_i$ with $u_i \in C^\infty(L_i)$. Let $u$ be a smooth function on $M$ such that $u_{|L_i} = u_i$ for each $i$. The form $\gamma - \zeta - du$ satisfies:

$$\begin{cases} d(\gamma - \zeta - du) = \beta - \alpha \\ j_i^*(\gamma - \zeta - du) = 0 \quad \forall i = 1 \dots q. \end{cases}$$

We proved that it is possible to choose $\gamma$ such that $\gamma(X) = 0$ for each vector $X$ tangent to a $L_i$ and we choose $\gamma$ in that way. The end of the proof is similar to that of Moser's theorem.

For each $t \in [0,1]$, set $\alpha_t = (1-t)\alpha + t\beta$ and define the vector field $X_t$ on $M$ by $i_{X_t}\alpha_t = -\gamma$. The field $X_t$ depends smoothly on $t$ and induces a flow $(\phi_t)_{t \in [0,1]}$. We compute the derivative of $\phi_t^*\alpha_t$. For any $t_0 \in (0,1)$,

$$\left.\frac{d}{dt}\right|_{t=t_0} (\phi_t^*\alpha_t) = \left.\frac{d}{dt}\right|_{t=t_0} (\phi_t^*\alpha_{t_0}) + \left.\frac{d}{dt}\right|_{t=t_0} (\phi_{t_0}^*\alpha_t).$$

The second term of the r.h.s. is equal to

$$\phi_{t_0}^* \left( \left.\frac{d}{dt}\right|_{t=t_0} \alpha_t \right) = \phi_{t_0}^*(\beta - \alpha).$$

We denote by $\mathcal{L}_{X_t}$ the Lie derivative with respect to the field $X_t$ and use Cartan's relation $\mathcal{L}_{X_t} = d \circ i_{X_t} + i_{X_t} \circ d$. We find that the first term is equal to

$$\mathcal{L}_{X_{t_0}} \phi_{t_0}^* \alpha_{t_0} = d\left(i_{X_{t_0}} \phi_{t_0}^* \alpha_{t_0}\right) = d(\phi_{t_0}^* i_{X_{t_0}} \alpha_{t_0}) = -d(\phi_{t_0}^*\gamma) = -\phi_{t_0}^*(d\gamma) = -\phi_{t_0}^*(\beta - \alpha).$$

Thus, $\frac{d}{dt}\phi_t^*\alpha_t = 0$, so that $\phi_1^*\beta = \phi_1^*\alpha_1 = \phi_0^*\alpha_0 = \alpha$. For any vector $X$ tangent to a $L_i$, the equality $\alpha_t(X_t, X) = i_{X_t}\alpha_t(X) = \gamma(X) = 0$ proves that the field $X_t$ is tangent to $L_i$. So the flow $\phi_t$ preserves the $L_i$'s.

The Riemannian volume of the metric $g = \phi_1^* g_1$ is $\phi^* \operatorname{vol}_{g_1} = \sigma$. Moreover, $\phi_1$ is an isometry from $(M, g)$ into $(M, g_1)$ that preserves the $L_i$'s. Since they are geodesics for $g_1$, they are also geodesics for $g$. □

From now on, we fix on $M$ a metric given by the last proposition. Let us recall a classical result that summarizes the main property of the geodesics that we will use. A proof of a local version of this theorem can be found for example in [dC] (proposition 3.4.2). The compactness of $M$ allows to globalize the result.

**Theorem 2.2.3** *There exists a positive real number $R_M$, called convexity radius of $M$, such that if $x$ and $y$ are two points of $M$ contained in a ball of radius smaller than $R_M$, they are joined by a unique piece of minimizing geodesic and this piece of geodesic stays inside the ball.*

This theorem implies in particular the following result:

**Proposition 2.2.4** *Let $\zeta_1$ and $\zeta_2$ be two finite pieces of geodesics. The intersection of $\zeta_1$ and $\zeta_2$ is the union of a finite number of isolated points and at most two segments.*

*Proof.* If $\zeta_1$ and $\zeta_2$ meet at an infinity of points, it is easy to check that there exists a couple $(t_1, t_2)$ of times such that $\zeta_1(t_1) = \zeta_2(t_2)$ and $\dot{\zeta}_1(t_1) = \dot{\zeta}_2(t_2)$. So they are two pieces of the same infinite geodesic. If this geodesic is periodic, $\zeta_1$ and $\zeta_2$ can intersect along one or two segments. Otherwise, they have one segment in common plus a finite number of isolated points. □

We denote by $\mathcal{G}$ the set of graphs whose edges are piecewise geodesic and such that $L_1, \ldots, L_q$ are in $\Gamma^*$. The set $\mathcal{G}$ is ordered by the relation $<$.

**Proposition 2.2.5** *Given two graphs $\Gamma_1, \Gamma_2$ in $\mathcal{G}$, there exists $\Gamma_3$ in $\mathcal{G}$ such that $\Gamma_1 < \Gamma_3$ and $\Gamma_2 < \Gamma_3$.*

*Proof.* The idea is to superpose $\Gamma_1$ and $\Gamma_2$. Given an edge $a$ of $\Gamma_2$, we know that $a([0,1]) \cap \operatorname{Supp}(\Gamma_1)$ is a finite reunion of segments and points. So, it is possible to add a finite number of new vertices and new edges to $\Gamma_1$ in such a fashion that $a$ becomes a path in the new graph. Repeating this procedure for each edge of $\Gamma_2$ gives the result. □

Let us fix an element $(x_1, \ldots, x_q)$ of $G^q$. With each graph $\Gamma \in \mathcal{G}$, we associated a space $(G^\Gamma, P^\Gamma(x_1, \ldots, x_q))$. The last proposition states the last property that was missing for the family $(G^\Gamma, P^\Gamma(x_1, \ldots, x_q)), \Gamma \in \mathcal{G}$ to be a projective family of probability spaces. Each $G^\Gamma$ is compact, so theorem 2.1.2 asserts that this projetive family has a projective limit $(\Omega, \mathcal{A}, P(x_1, \ldots, x_q))$ which is a probability space endowed with functions $f_\Gamma : \Omega \longrightarrow G^\Gamma$ such that $f_{\Gamma*}P(x_1, \ldots, x_q) = P^\Gamma(x_1, \ldots, x_q)$. This space contains in itself the same information that is contained in all spaces $(G^\Gamma, P^\Gamma(x_1, \ldots, x_q))$: each random variable $h_\zeta : G^\Gamma \longrightarrow G$ gives rise to a random variable

$$H_\zeta : \Omega \xrightarrow{f_\Gamma} G^\Gamma \xrightarrow{h_\zeta} G$$

and the law of a $n$-uple $(h_{\zeta_1}, \ldots, h_{\zeta_n})$ computed in any graph of $\mathcal{G}$ is always equal to that $(H_{\zeta_1}, \ldots, H_{\zeta_n})$ under $P$. Moreover, since any piecewise geodesic path can be seen as a path in a graph, there is a well-defined random variable $H_\zeta$ on $\Omega$ associated with any such path $\zeta$. Remark that the multiplicativity property is preserved:

**Proposition 2.2.6** *Let $\zeta_1$ and $\zeta_2$ be two piecewise geodesic paths such that $\zeta_1(1) = \zeta_2(0)$. Then $H_{\zeta_1\zeta_2} = H_{\zeta_2}H_{\zeta_1}$ a.s.*

From now on, we use greek letters to denote the piecewise geodesic paths and denote by $PGM$ the set of these paths.

## 2.3 Preliminary results

### 2.3.1 Lassos

**Definition 2.3.1** *A lasso is a simple loop or a path of the form $l = cbc^{-1}$, where $c$ is an injective path and $b$ a simple loop which meets $c$ only at its base point. The loop $b$ is detetermined by $l$ and is called the buckle of the lasso $l$.*

A notion of lasso close to this one has already been used by Driver in [Dr2]. In [GKS], Gross, King and Sengupta also suggested that the use of lassos might be helpful in this construction.

Lassos are useful at least for two reasons: the first one is that it is easy to compute the law of their holonomy and the second one is that any reasonable loop can be decomposed in some sense into a product of lassos. Let us begin with this second point.

There is a natural equivalence relation between paths, which is the following:

**Definition 2.3.2** *Two paths are said to be basically equivalent if one of them can be written $c_1c_2c_2^{-1}c_3$ and the other one $c_1c_3$, where $c_1, c_2, c_3 \in PM$. Two paths $c$ and $c'$ are equivalent, and we denote $c \simeq c'$, if there exists a finite chain $c = c_0, \ldots, c_n = c'$ such that any two successive terms of this chain are basically equivalent.*

**Lemma 2.3.3** *Let $\zeta_1$ and $\zeta_2$ be two paths of $PGM$ and suppose that $\zeta_1 \simeq \zeta_2$. Then $H_{\zeta_1} = H_{\zeta_2}$ $P(x_1, \ldots, x_q)$-a.s.*

*Proof.* This is a consequence of the multiplicativity of the random holonomy. □

Let us define the class of paths that can be decomposed into a product of lassos.

**Definition 2.3.4** *A path $c \in PM$ is said to have finite self-intersection if there exists a graph $\Gamma$ such that $c \in \Gamma^*$.*

Remark that this definition is not the usual one of finite self-intersection. Indeed, our definition allows for a path a finite number of points and also a finite number of segments as auto-intersection set. In particular, the proposition 2.2.4 shows that a piecewise geodesic path, which is a concatenation of injective pieces of different geodesics, has finite self-intersection in the sense of 2.3.4.

The Riemannian metric chosen on $M$ allows us to compute the length of a path $c$, that we denote by $\ell(c)$.

**Proposition 2.3.5** *Let $c$ be a path with finite self-intersection.*
*1. If $c(0) \neq c(1)$ then $c$ is equivalent to a unique product $c \simeq l_1 \dots l_p c'$, where the $l_i$'s are lassos which are non equivalent to a constant loop and $c'$ is an injective path joining $c(0)$ to $c(1)$. Moreover, if $b_i$ denotes the buckle of the lasso $l_i$ for each $i$, the following inequality holds:*

$$\ell(c) \geq \sum_{i=1}^{p} \ell(b_i) + \ell(c').$$

*2. If $c(0) = c(1)$, the result remains true after removing $c'$ of all expressions.*

*Proof.* We proceed by induction on the number of edges in a decomposition of $c$ as a path in a graph. If $c$ is an edge, we are in the first case and the result is true. Suppose that $c = a_1 \dots a_r$. If $c$ is an injective path or a simple loop, the result is true. Otherwise, the idea is to trace $c$ out until the first time it intersects itself. Let $i$ be the smaller integer such that there exists $1 \leq j < i$ verifying $a_j(0) = a_i(1)$. Such an $i$ exists, and $(i,j) \neq (r,1)$. We have:

$$c \simeq a_1 \dots a_{j-1} \cdot a_j \dots a_i \cdot (a_1 \dots a_{j-1})^{-1} \cdot a_1 \dots a_{j-1} \cdot a_{i+1} \dots a_r.$$

It can happen that the first piece or the last piece of this decomposition are empty, respectively if $j = 1$ or $i = r$, but these two situations cannot coexist. If $j = i - 1$, it is possible that $a_i = a_j^{-1}$ so that $a_j a_i$ is equivalent to a constant path. This cannot happen if $j < i - 1$, in which case $a_j \dots a_i$ is a genuine simple path. Thus, the product of the three first terms is either equivalent to a constant path, or is a simple loop (if $j = 1$), or a lasso. The product of the two last terms is the product of a number of edges which is positive and strictly less than $p$. So by induction, this path $\widetilde{c}$ is equivalent to $l_1 \dots l_q c'$, or to $l_1 \dots l_q$ if $c$ is a loop. Note that $\ell(c) = \ell(a_1) + \dots + \ell(a_r) = \ell(\widetilde{c}) + \ell(a_j) + \dots + \ell(a_i)$. So, by induction, $\ell(c) \geq \ell(b_1) + \dots + \ell(b_q) + \ell(c') + \ell(a_j) + \dots + \ell(a_i)$. In the case $j = i - 1$ and $a_j = a_i^{-1}$, we have $c \simeq \widetilde{c}$ and the result is true with a strict inequality. Otherwise, there exists a lasso $l_0$ such that $c \simeq l_0 \widetilde{c}$ and the length of the buckle of this lasso $l_0$ is exactly $\ell(a_j) + \dots + \ell(a_i)$. □

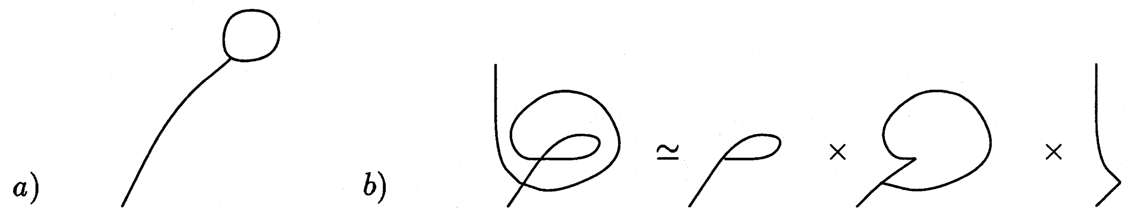


Figure 2.1: a) A lasso. b) An example of decomposition.

### 2.3.2 Holonomy along small piecewise geodesic loops

In order to estimate the holonomy along a small lasso, we need, according to the proposition 1.8.5, to know the area enclosed by its buckle. This area can be controlled by the length of the buckle using an isoperimetric inequality. We recall a classical fact about open covering of metric compact sets. A proof can be found in [Ma].

**Lemma 2.3.6** *Let $M$ be a metric compact set. Let $(O_i)_{i \in I}$ be an open covering of $M$. There exists a positive real number $R$ called Lebesgue number of this covering, such that for any ball $B$ of radius smaller than $R$ in $M$, there exists an $i \in I$ such that $B \subset O_i$.*

**Proposition 2.3.7** *There exist $R > 0$ and $K > 0$ such that any simple loop $l$ contained in a ball $B$ of radius smaller than $R$ is the boundary of an open set $U \subset B$ such that*

$$\sigma(U) \leq K\ell(l)^2.$$

*Proof.* Let $R_0$ be such that any closed geodesic ball of $M$ of radius smaller than $R_0$ is diffeomorphic to a disk. Let $B_1, \ldots, B_n$ be a covering of $M$ by open balls of radius $R_0$. Let us denote by $g$ the metric on $M$ and $g_0$ the euclidean metric on $\mathbf{R}^2$. For each $i$, there exists a diffeomorphism $\phi_i : \overline{B_i} \longrightarrow \overline{D}(0,1) \subset \mathbf{R}^2$. Since the metrics $g$ and $\phi^* g_0$ can be compared on $\overline{B_i}$, the usual isoperimetric inequality on $\overline{D}(0,1)$ gives rise to an inequality on $\overline{B_i}$, with some constant $K_i$. Let $K$ be the supremum of $K_1, \ldots, K_n$. Let $R$ be a Lebesgue number of the covering $B_1, \ldots, B_n$. Then the statement holds with this choice of $K$ and $R$. □

Now we can estimate the holonomy along a small lasso:

**Proposition 2.3.8** *There exist $L_0 > 0$ and $K > 0$ such that if $\lambda$ is a piecewise geodesic lasso whose buckle $\beta$ has a length smaller than $L_0$, then*

$$E\rho(H_\lambda) = Ed(H_\lambda, 1) \leq K\ell(\beta).$$

*Proof.* The lasso $\lambda$ can be written $\sigma\beta\sigma^{-1}$, so thanks to invariance by conjugation of the distance on $G$, we have $E\rho(H_\lambda) = E\rho(H_\beta)$. Let $L_0$ be shorter than the shortest length of a loop non homotopic to a point and also shorter than the radius $R$ given py the proposition 2.3.7. Then the hypothesis $\ell(\beta) \leq L_0$ implies that $\beta$ is the boundary of a small disk $D$. Using proposition 1.8.5, we get

$$E\rho(H_\beta) \leq C\sqrt{\sigma(D)} \leq K\ell(\beta).$$ □

This result suggests that it will be possible to prove regularity results for the random holonomy using the following distance between $G$-valued random variables:

**Definition 2.3.9** *Let $X$ and $Y$ be two $G$-valued random variables defined on the same probability space. The distance $d_P(X,Y)$ is defined by:*

$$d_P(X,Y) = Ed(X,Y),$$

*where $d$ is the biinvariant Riemannian distance on $G$.*

The first example of such regularity results is the following one:

**Proposition 2.3.10** *Let $\zeta$ be a piecewise geodesic loop of length smaller than $L_0$. Then*

$$d_P(H_\zeta, 1) \leq K\ell(\zeta).$$

*Proof.* Since $\zeta$ is piecewise geodesic, it has finite self-intersection. So it is equivalent to a product of piecewise geodesic lassos: $\zeta \simeq \lambda_1 \dots \lambda_p$. This gives:

$$\begin{aligned} d_P(H_\zeta, 1) &= d_P(H_{\lambda_1} \dots H_{\lambda_p}, 1) \\ &\leq d_P(H_{\lambda_1} \dots H_{\lambda_p}, H_{\lambda_2} \dots H_{\lambda_p}) + \dots + d_P(H_{\lambda_{p-1}} H_{\lambda_p}, H_{\lambda_p}) + d_P(H_{\lambda_p}, 1) \\ &\leq d_P(H_{\lambda_1}, 1) + \dots + d_P(H_{\lambda_p}, 1) \\ &\leq K(\ell(\beta_1) + \dots \ell(\beta_p)) \\ &\leq K\ell(\zeta). \end{aligned}$$

□

### 2.3.3 Double layer potential of small piecewise geodesic loops

Using the same techniques as in the preceding pararaph, we will estimate the double layer potential of a small loop. This is the first step in the proof of the proposition 2.6.8, that will play an important role in the study of the Abelian theory.

Recall that the definition of the potential (see 1.9.6) depends on a Riemannian metric on $M$ whose Riemannian volume is equal to $\sigma$. For the moment, we only know that the potential of any element of $PM$ is in $L^2(M, \sigma)$ (see theorem 1.9.11).

**Proposition 2.3.11** *Let $l$ be a lasso with buckle $b$. Suppose that $\ell(b) \leq L_0$, where $L_0$ is the length given by 2.3.8. Then*

$$\| u_l \|_{L^2} \leq K\ell(b).$$

*Proof.* The length $L_0$ is such that $b$ is necessarily the boundary of a disk $D$ whose area satisfies $\sigma(D) \leq K\ell(b)^2$. Thus, by proposition 1.9.7,

$$\| u_b \|_{L^2} = \left( \frac{\sigma(D)\sigma(D^c)}{\sigma(M)} \right)^{\frac{1}{2}} \leq \sigma(D)^{\frac{1}{2}} \leq K\ell(b).$$

Since $u_l = u_b$ a.e., we have the result. □

As in the preceding paragraph, this result can be extended to loops with finite self-intersection.

**Proposition 2.3.12** *Let $l$ be a loop with finite self-intersection and of length smaller than $L_0$. Then*

$$\| u_l \|_{L^2} \leq K\ell(l).$$

*Proof.* Let us write that $c$ is equivalent to a product of lassos: $c \simeq l_1 \dots l_p$. Two paths that are equivalent have the same double layer potential almost everywhere, so that

$$\begin{aligned} \| u_c \|_{L^2} &\leq \| u_{l_1} + \dots + u_{l_p} \|_{L^2} \\ &\leq \| u_{l_1} \|_{L^2} + \dots + \| u_{l_p} \|_{L^2} \\ &\leq K(\ell(b_1) + \dots + \ell(b_p)) \\ &\leq K\ell(c). \end{aligned}$$

□

The fact that the propositions 2.3.10 and 2.3.12 are very similar will allow us later, in proposition 2.5.3 for example, to transpose directly some regularity results about the random variables $H$ to the double layer potential.

### 2.3.4 Topology on the space of paths

According to the proposition 1.8.5, it seems to be necessary to control the surface left between two loops in order to control the distance between their holonomies. Given a Riemannian metric on $M$, the uniform distance defined as follows:

$$d_\infty(c_1, c_2) = \inf \sup_{t\in[0,1]} d(c_1(t), c_2(t))$$

allows to control this surface, where the infimum is taken over all reparametrizations of $c_1$ and $c_2$.

In the paper [Be], C. Becker says that the double layer potential of a loop depends continuously of this loop in norm $L^2$ when the set of loops on $M$ is endowed with the topology induced by the uniform norm. His proposition 3.2 depends on the validity of this assertion, which is probably true if one restricts to simple loops, but not if one allows loops to have a self-intersection, even a finite one. Let us describe a counterexample. Becker stated his result on $\mathbf{R}^2$, but this does not change the situation very much. Let $M$ be the sphere $S^2$ embedded as usual in $\mathbf{R}^3$, endowed with the standard metric. Let us consider the pencil of planes $(P_t)_{t\in[0,4\pi)}$ containing the horizontal line $z = 0, y = -1$, indexed in the following way: denoting by $C_t$ the intersection of $S^2$ with the lower half-space bounded by $P_t$, we have $\sigma(C_t) = t$. For any $t \in [0, 4\pi)$, denote by $c_t$ a loop based at $(0, -1, 0)$ whose image is the intersection of $P_t$ with $S^2$, oriented negatively with respect to the $z$ axis. Let $0 < t_1 < \ldots < t_n < 4\pi$ be $n$ distinct times and set $c = c_{t_1} \ldots c_{t_n}$. For each $i$, $u_{c_i} = \mathbf{1}_{C_{t_i}} - \frac{t_i}{4\pi}$. Thus,

$$\begin{aligned} \| u_c \|_{L^2}^2 &= \| \mathbf{1}_{C_{t_1}} + \ldots + \mathbf{1}_{C_{t_n}} \|_{L^2}^2 + \frac{(t_1 + \ldots + t_n)^2}{16\pi^2} - 2\left(\frac{t_1 + \ldots + t_n}{4\pi}, \mathbf{1}_{C_{t_1}} + \ldots + \mathbf{1}_{C_{t_n}}\right)_{L^2} \\ &\geq n^2 t_1 + \frac{n^2 t_1^2}{16\pi^2} - \frac{n^2 t_n^2}{2\pi}. \end{aligned}$$

Suppose that $t_1 = \frac{1}{n}$ and $t_n = \frac{2}{n}$. Then $\| u_c \|_{L^2}^2$ is of the order of $n$, so it grows to infinity when $n$ tends to infinity. But at the same time, the loop $c$ tends to the constant loop equal to $(0, -1, 0)$ in the topology induced by the distance $d_\infty$. The potential of this constant loop being equal to zero, this contradicts the continuity.

Therefore, it is necessary to endow the space of paths with a topology finer than that induced by $d_\infty$ if we expect some kind of continuity of the double layer potential and of the random holonomy. It has emerged in the last paragraphs that the length plays a role in the continuity results.

**Definition 2.3.13** *On the set of paths PM, we define the distance $d_1$ by*

$$d_1(c_1, c_2) = d_\infty(c_1, c_2) + |\ell(c_1) - \ell(c_2)|.$$

**Proposition 2.3.14** *The topology induced on PM by the distance $d_1$ does not depend on the Riemannian metric chosen on M.*

*Proof*. By compactness of $M$, two different metrics induce two equivalent Riemannian distances on $M$ and thus two equivalent distances $d_1$ on $PM$. □

## 2.4 Approximation of embedded paths

We want to extend the definition of the random holonomy to all paths in $PM$, by approximation. Since any path of $PM$ is, by definition, a concatenation of embedded submanifolds of $M$, it is natural to begin with those paths who are embedded submanifolds themselves.

### 2.4.1 Tubular neighbourhoods and Fermi coordinates

These paths have the following nice property: they possess a tubular neighbourhood that can be described using Fermi coordinates. Let us fix a path $c$ which is an embedded submanifold. The proof of the following result can be found in [Gy]. Let us fix a parametrization of $c$ and a vector field $N$ along $c$, unitary and normal to $c$.

**Proposition 2.4.1** *There exists a positive real number $r$ such that the mapping*

$$\begin{array}{rcl}\psi : [0,1]\times[-r,r] & \longrightarrow & M \\ (t,s) & \longmapsto & \exp_{c(t)}(sN_{c(t)})\end{array}$$

*is a diffeomorphism onto its image, which is called tubular neighbourhood of $c$ or tube around $c$. The coordinates $(t,s)$ are called Fermi coordinates. They satisfy:*
*1. For any fixed $t_0$, the curve $s \mapsto \psi(t_0,s)$ is a piece of geodesic normal to $c$.*
*2. For any couple $(t,s)\in[0,1]\times[-r,r]$, $d(\psi(t,s),c([0,1]))=s$.*

We shall always assume that the radius of the tubular neighbourhoods that we consider are smaller than the convexity radius $R_M$ of $M$, defined in 2.2.3.

### 2.4.2 Piecewise geodesic approximation

The path $c$ is fixed until the end of the next section, together with a tubular neighbourhood of radius $r$.

**Proposition 2.4.2** *Let $x,y,z$ be three real numbers such that $0\le x<y<z\le r$. There exists a piecewise geodesic path $\sigma$ such that*
*1. $\sigma(0)=\psi(0,y)$ and $\sigma(1)=\psi(1,y)$,*
*2. $\sigma((0,1))\subset\psi((0,1)\times(x,z))$,*
*3. $\sigma$ is injective.*

We construct $\sigma$ as an approximation of the path $c_y : t\mapsto\psi(t,y)$, in the same way as one would approximate a curve in $\mathbf{R}^2$ by piecewise linear paths.

**Lemma 2.4.3** *Set* $\delta_n(c_y)=\sup_{0\le k\le n-1} d\left(c_y\left(\frac{k}{n}\right),c_y\left(\frac{k+1}{n}\right)\right)$. *Then* $\delta_n(c_y)\underset{n\to\infty}{\longrightarrow}0$.

*Proof*: The norm of the velocity of $c$ is bounded. □

For $n$ large enough and for each $k=1\dots n$, the points $c_y(\frac{k-1}{n})$ and $c_y(\frac{k}{n})$ are close enough to be joined by a unique minimizing geodesic $\zeta_{n,k}$ that stays at a distance smaller than $\delta_n$ of $c_y(\frac{k}{n})$. We will always assume that $n$ is large enough for this property to be true and set $\zeta_n=\zeta_{n,1}\cdots\zeta_{n,n}$.

**Lemma 2.4.4** *For n large enough, $\zeta_n$ is the graph of a continuous function in Fermi coordinates. More precisely, there exists a continuous function $\varphi : [0,1] \longrightarrow (-r,r)$ such that for each $t \in [0,1]$, $\zeta_n(t) = \psi(t, \varphi(t))$.*

*Proof.* It is sufficient to prove that each $\zeta_{n,k}$ is the graph of a continuous function defined on $[\frac{k-1}{n}, \frac{k}{n}]$ and that these functions can be put together to form $\varphi$. Let $n$ be such that $\delta_n < \frac{r-y}{2}$. We show that $\zeta_n$ stays inside the tubular neighbourhood of $c$.

The first point is that $\zeta_{n,k}$ cannot meet the horizontal boundary $\psi([0,1] \times \{-r, r\})$, because this boundary is at distance $r$ of $c$ and $\zeta_{n,k}$ stays at distance smaller than $\delta_n + y < r$.

The vertical part of the boundary $\psi(\{0,1\} \times [-r,r])$ is made of two pieces of minimizing geodesics, so that $\zeta_{n,k}$, which is also minimizing, cannot meet twice one of these pieces without belonging to the same infinite geodesic. This is impossible because the geodesics supporting the vertical boundary meet $c_y$ only once.

The only way $\zeta_{n,k}$ could exit the tube around $c$ would be to exit through one piece of the vertical boundary and get back through the other. Suppose that $n$ is large enough for $\delta_n$ being smaller than $\frac{1}{2}d(\psi(\{0\} \times [-r,r]), \psi(\{1\} \times [-r,r]))$. Then the situation described above cannot happen, since any two points of the image of $\zeta_{n,k}$ are at distance smaller than $\delta_n$. So, $\zeta_{n,k}$ stays inside the tube.

Each vertical slice $t = t_0$ of the tube is a minimizing piece of a geodesic that meets $c$ only once inside the tube, so that it meets $\zeta_{n,k}$ at most one time. Thus, $\zeta_{n,k}$ is the graph of a smooth function defined on the segment $[\frac{k-1}{n}, \frac{k}{n}]$, equal to $y$ at both end points of this segment. All these functions can be put together to make $\varphi$, which is continuous. □

*Proof of proposition 2.4.2.* Choose $n$ be large enough for $\zeta_n$ to be the graph of a function in Fermi coordinates and such that $\delta_n < \inf(z-y, y-x)$. As a graph, $\zeta_n$ is necessary injective, which is statement 3. The inequality $|\varphi(t) - y| = d(\zeta_n(t), c_y) \leq \delta_n$ shows that $\zeta_n$ stays in $\psi([0,1] \times (x,z))$. Together with the fact that $\zeta_n$ meets at most once each vertical boundary, this gives statement 2. Statement 1 is a direct consequence of the definition of $\zeta_n$. So $\sigma = \zeta_n$ has all the properties required. □

## 2.5 Random holonomy along embedded paths

We suppose that the surface of the tube is smaller than the constant $s$ given by the proposition 1.8.5. We prove that the random holonomy along a piecewise geodesic approximation of $c$ converges in probability to a random variable and that this limit does not depend on the particular choice of the approximation.

### 2.5.1 Existence of a limit random holonomy

For $n \geq 0$, set $x_n = \frac{r}{2^{n+1}}$, $y_n = \frac{3}{2}\frac{r}{2^{n+1}}$, $z_n = \frac{r}{2^n}$ and let $\sigma_n$ be a path given by the proposition 2.4.2. For each $n \geq 0$, let $\lambda_n$ denote the vertical segment joining $(0,0)$ to $(0, y_n)$ and $\rho_n$ the vertical segment joining $(1, y_n)$ to $(1, 0)$. Finally, set $\alpha_n = \lambda_n \sigma_n \rho_n$.

**Proposition 2.5.1** *The sequence of random variables $(H_{\alpha_n})_{n \geq 0}$ is a Cauchy sequence with respect to the distance $d_P$.*

*Proof.* Let $m \geq n$ be two integers. We want to estimate $d_P(H_{\alpha_m}, H_{\alpha_n}) = d_P(H_{\alpha_n^{-1}\alpha_m}, 1)$. But $\alpha_n^{-1}\alpha_m$ is equivalent to a simple loop which is the boundary of an open set contained in $\psi([0,1]\times[0,\frac{r}{2^n}])$. Thus, the assumption on the surface of the tube allows us to apply proposition 1.8.5. We get:

$$d_P(H_{\alpha_m}, H_{\alpha_n}) \leq C\sigma\left(\psi\left([0,1]\times\left[0,\frac{r}{2^n}\right]\right)\right) \leq \frac{C'}{2^n}.$$

This proves the result. □

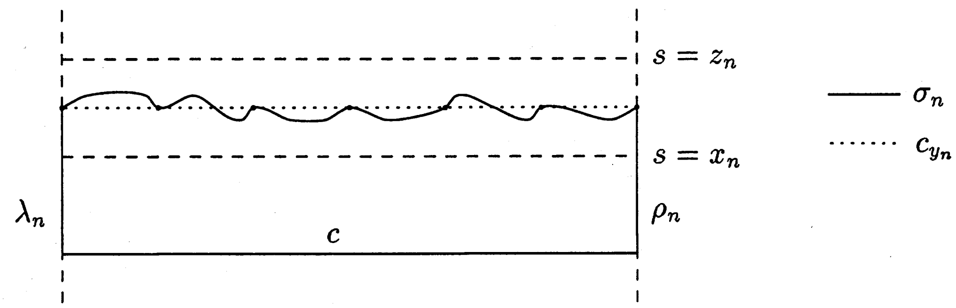


Figure 2.2: Definition of the sequence $(\alpha_n)$.

The space of $G$-valued random variables on $(\Omega, \mathcal{A}, P_\beta)$ endowed with the distance $d_P$ is complete: it can be isometrically embedded in a $L^1$ space by embedding $G$ isometrically in some $\mathbf{R}^n$. So the sequence $(H_{\alpha_n})_{n\geq 0}$ has a limit that we denote by $H_c$, anticipating the fact that this limit does not depend on the choice of the sequence $(\alpha_n)$.

### 2.5.2 Unicity of the limit random holonomy

**Lemma 2.5.2** *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha$ is an* injective *piecewise geodesic path with the same end points as $c$, such that $\alpha((0,1)) \subset \psi((0,1)\times(-r,r))$ and such that $d_\infty(c,\alpha) < \delta$, then $d_P(H_c, H_\alpha) < \varepsilon$.*

In this statement, it is not necessary to control $|\ell(\alpha) - \ell(c)|$ because $\alpha$ is assumed to be injective.

*Proof.* Let $C$ be the constant given by the proposition 1.8.5. Let $n$ be such that $d_P(H_c, H_{\alpha_n}) < \varepsilon/2$ and $C\sigma(\psi([0,1]\times[-\frac{r}{2^n},\frac{r}{2^n}])) < \varepsilon/2$. Set $\delta = \frac{1}{2^{n+1}}$ and suppose that $d_\infty(c,\alpha) < \delta$. Then $\alpha$ meets $\alpha_n$ only at its end points. Thus $\alpha_n\alpha^{-1}$ is the boundary of an open set included in $\psi([0,1]\times[-\frac{r}{2^n},\frac{r}{2^n}])$, so that

$$\begin{aligned} d_P(H_c, H_\alpha) &\leq d_P(H_c, H_{\alpha_n}) + d_P(H_{\alpha_n}, H_\alpha) \\ &\leq \frac{\varepsilon}{2} + C\sigma\left(\psi\left([0,1]\times\left[-\frac{r}{2^n},\frac{r}{2^n}\right]\right)\right) \\ &\leq \varepsilon. \end{aligned}$$

□

The control of the length of $\alpha$ allows to drop all restrictive conditions on $\alpha$, unless those concerning end points. The main result of this section is the following:

**Proposition 2.5.3** *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha$ is a piecewise geodesic path with the same end points as $c$ and such that $d_1(c,\alpha) < \delta$, then $d_P(H_c, H_\alpha) < \varepsilon$.*

**Lemma 2.5.4** *For all $\delta > 0$, there exists $\eta > 0$ such that if $c'$ is another path such that $d_\infty(c,c') < \eta$, then $\ell(c') > \ell(c) - \delta$.*

Note that this result is not symmetric in $c$ and $c'$. Indeed, it is true that there exists $\eta'$ such that $d_\infty(c,c') < \eta'$ implies $\ell(c) > \ell(c') - \delta$, but $\eta'$ may be much smaller than $\eta$ (consider for $c'$ a zigzag approximating a straight line for example). One could reformulate this result by saying that for any sequence $c_n$ converging uniformly to $c$, $\liminf \ell(c_n) \geq \ell(c)$.

*Proof*: Let $n$ be large enough for the following inequality to hold:

$$\ell(c) - \sum_{k=0}^{n-1} d(c(\tfrac{k}{n}), c(\tfrac{k+1}{n})) < \frac{\delta}{2}.$$

Let $c'$ be a path with fixed parametrization such that $d_\infty(c,c') < \frac{\delta}{4n}$. Then, on one hand, $\ell(c') \geq \sum d(c'(\frac{k}{n}), c'(\frac{k+1}{n}))$. On the other hand,

$$\begin{aligned} d(c(\tfrac{k}{n}), c(((k+1))/n)) &\leq d(c(\tfrac{k}{n}), c'(\tfrac{k}{n})) + d(c'(\tfrac{k}{n}), c'(\tfrac{k+1}{n})) + d(c'(\tfrac{k+1}{n}), c(\tfrac{k+1}{n})) \\ &\leq \frac{\delta}{2n} + d(c'(\tfrac{k}{n}), c'(\tfrac{k+1}{n})). \end{aligned}$$

Thus, $\ell(c') \geq \sum_{k=0}^{n-1} d(c(\frac{k}{n}), c(\frac{k+1}{n})) - \frac{\delta}{2} \geq \ell(c) - \delta$. We see that $\eta = \frac{\delta}{4n}$ is a possible choice. □

*Proof of proposition 2.5.3.* Denote by $\delta_0$ the distance between $c(0)$ and $c(1)$. Assume that $d_\infty(\sigma, c)$ is smaller than $\inf(r, \delta_0/5)$. Recall that $r$ is assumed to be smaller than the convexity radius of $M$ (see 2.2.3).

The points $\alpha(0)$ and $\alpha(1)$ are respectively in the balls $B_0 = B(c(0), 2d_\infty(\alpha, c))$ and $B_1 = B(c(1), 2d_\infty(\alpha, c))$. These balls are disjoint, hence there exists a last time $\tau_0$ at which $\alpha$ exits $B_0$ and a first time $\tau_1$ at which it enters $B_1$. The points $\alpha(\tau_0)$ and $\alpha(\tau_1)$ are necessarily inside the tube, for the points of $M$ at distance smaller than $r$ of $c$ are inside the tube or in $B_0 \cup B_1$. In Fermi coordinates, we can write $\alpha(\tau_0) = (t_0, s_0)$ and $\alpha(\tau_1) = (t_1, s_1)$. Note that $t_0 > 0$ and $t_1 < 1$: otherwise, we would have $|s_0|$ or $|s_1|$ equal to $2d_\infty(\alpha, c)$.

Let $\gamma_0$ be the path that follows $c$ from time 0 to $t_0$ and then the geodesic normal to $c$ from $(t_0, 0)$ to $(t_0, s_0)$. Similarly, let $\gamma_1$ be the path that follows the normal geodesic from $(t_1, s_1)$ to $(t_1, 0)$ and then $c$ from time $t_1$ to 1. We write $\alpha$ in the following way:

$$\alpha \simeq \alpha_{|[0,\tau_0]}\gamma_0^{-1} \cdot \gamma_0 \alpha_{|[\tau_0,\tau_1]}\gamma_1 \cdot \gamma_1^{-1}\alpha_{|[\tau_1,1]}.$$

The first and the third terms are small loops that we shall study later. Let us consider the central term $\gamma_0\alpha_{|[\tau_0,\tau_1]}\gamma_1$. It is contained in the tube around $c$ and has the same end points as $c$. Let us decompose it according to 2.3.5 into a product $\lambda_1 \ldots \lambda_p \xi$, where the $\lambda_i$'s are lassos based at $c(0)$ and $\xi$ is an injective path between $c(0)$ and $c(1)$. It is obvious that $d_\infty(c,\xi) \leq d_\infty(c,\alpha)$. This tells us, by proposition 2.5.2, that $H_\xi$ can be made arbitrarily close to $H_c$ by taking $d_\infty(c,\alpha)$ sufficiently small.

Let us fix a positive $\varepsilon$ and $\delta_1$ such that $d_\infty(c,\alpha) < \delta_1$ implies $d_P(H_c, H_\xi) < \varepsilon/2$. It is enough now to control $d_P(H_\xi, H_\alpha)$.

$$\begin{aligned} d_P(H_\xi, H_\alpha) &= d_P(H_\xi, H_{\alpha_{|[0,\tau_0]}\gamma_0^{-1}} H_{\lambda_1} \ldots H_{\lambda_p} H_\xi H_{\gamma_1^{-1}\alpha_{|[\tau_1,1]}}) \\ &\leq \sum_{i=1}^{p} d_P(H_{\lambda_i}, 1) + d_P(H_{\alpha_{|[0,\tau_0]}\gamma_0^{-1}}, 1) + d_P(H_{\gamma_1^{-1}\alpha_{|[\tau_1,1]}}, 1). \end{aligned}$$

We are led to consider the random variables associated with loops with finite self-intersection. According to 2.3.8, it it is sufficient to control their lengths. We already know by 2.5.4 that we can have $\ell(\xi) \geq \ell(c) - \varepsilon/8$ provided $\delta_1$ and so $d_\infty(c,\xi)$ is small enough. If we impose now that $d_1(c,\alpha) < \delta_1$, instead of $d_\infty(c,\alpha) < \delta_1$, then we also get $\ell(\alpha) < \ell(c) + \varepsilon/8$.

Then $0 < \ell(\alpha) - \ell(\xi) < \varepsilon/4$. Let us denote by $\beta_1, \ldots, \beta_p$ the buckles of the lassos $\lambda_1, \ldots, \lambda_p$. By 2.3.5,

$$\ell(\xi) + \sum_i \ell(\beta_i) \leq \ell(\gamma_0) + \ell(\alpha_{|[\tau_0,\tau_1]}) + \ell(\gamma_1),$$

and so

$$\sum_i \ell(\beta_i) + \ell(\alpha_{|[0,\tau_0]}\gamma_0^{-1}) + \ell(\gamma_1^{-1}\alpha_{|[\tau_1,1]}) \leq \frac{\varepsilon}{4} + 2\ell(\gamma_0) + 2\ell(\gamma_1).$$

Since $\ell(\gamma_i) \leq 2d_\infty(c,\alpha) + \ell(c([0,1]) \cap B_i)$, the lengths appearing in the right hand side can be made small by taking $d_\infty(c,\alpha)$ small enough. This is exactly what was needed to control $d_P(H_\xi, H_\alpha)$. This gives us a $\delta_2$ such that $d_1(c,\alpha) < \delta_2$ implies $d_P(H_c, H_\alpha) < \varepsilon$. □

**Corollary 2.5.5** *Let $(\beta_n)_{n\geq 0}$ be any sequence of piecewise geodesic paths with the same end points as $c$ that converges to $c$. Then the sequence $(H_{\beta_n})$ converges to $H_c$.*

This proves that the variable $H_c$ does not depend on the particular choice of the sequence of paths approximating $c$.

### 2.5.3 Continuity of the double layer potential (1)

Following step by step the proofs of propositions 2.5.2 and 2.5.3 and replacing statements about random variables by statements about the double layer potential, according to the remark made at the end of paragraph 2.3.3, we get the following results:

**Lemma 2.5.6** *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha$ is an injective piecewise geodesic path with the same end points as $c$, such that $\alpha((0,1)) \subset \psi((0,1) \times (-r,r))$ and such that $d_\infty(c,\alpha) < \delta$, then $\| u_\alpha - u_c \|_{L^2} < \varepsilon$.*

**Proposition 2.5.7** *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha$ is a piecewise geodesic path with the same end points as $c$ and such that $d_1(c,\alpha) < \delta$, then $\| u_\alpha - u_c \|_{L^2} < \varepsilon$.*

## 2.6 Random holonomy along arbitrary paths

### 2.6.1 Construction of the random holonomy

Let $c$ be an element of $PM$. By definition, it can be written $c = c_1 \dots c_p$, where the $c_i$'s are embedded paths, but this decomposition is far to be unique. Nevertheless, we prove that the random variable $H_{c_p} \dots H_{c_1}$ depends only on $c$.

**Lemma 2.6.1** *Let $c$ be a path. There exists a sequence of piecewise geodesic paths with the same end points as $c$ that converges to $c$.*

*Proof.* Given a decomposition $c = c_1 \dots c_p$ of $c$ into a product of embedded paths, we concatenate sequences of paths that converge to each $c_i$ with fixed end points and get the required sequence. □

**Proposition 2.6.2** *Let $(\alpha_n)$ be a sequence of piecewise geodesic paths that converges with fixed end points to $c$. The sequence $(H_{\alpha_n})$ converges to the product $H_{c_p} \dots H_{c_1}$.*

The following corollary is in fact the main result of this paragraph.

**Corollary 2.6.3** *The product $H_{c_p} \dots H_{c_1}$ is independent of the choice of the decomposition of $c$ and it is equal to the common limit of all sequences $(H_{\alpha_n})$ associated with sequences $(\alpha_n)$ of piecewise geodesic paths converging to $c$ with fixed end points. We shall denote it by $H_c$.*

*Proof of proposition 2.6.2.* We cut $\alpha_n$ in a way that corresonds to the decomposition of $c$. Let us fix a parametrization of $c$ such that $c_i = c_{|[\frac{i-1}{p},\frac{i}{p}]}$. Let us also fix a parametrization of each $\alpha_n$ such that the uniform convergence $d(c, \alpha_n) \to 0$ holds with these parametrizations. Set $\alpha_{i,n} = \alpha_{n|[\frac{i-1}{p},\frac{i}{p}]}$. Let us show that for each $i$, $\alpha_{i,n} \underset{n\to\infty}{\longrightarrow} c_i$.

The first point is that $d_\infty(c_i, \alpha_{i,n}) \leq d_\infty(c, \alpha_n) \underset{n\to\infty}{\longrightarrow} 0$, parametrizations being fixed. Now let us choose $\varepsilon > 0$ and $n$ large enough such that for all $i = 1, \dots, p$, $\ell(\alpha_{i,n}) \geq \ell(c_i) - \frac{\varepsilon}{p}$ (using lemma 2.5.4) and $\ell(\alpha_n) \leq \ell(c) + \varepsilon$. Then

$$\begin{aligned} -\frac{\varepsilon}{p} \leq \ell(\alpha_{i,n}) - \ell(c_i) &= \ell(\alpha_n) - \ell(c) - \sum_{j \neq i} (\ell(\alpha_{j;n}) - \ell(c_j)) \\ &\leq 2\varepsilon. \end{aligned}$$

So we also have $\ell(\alpha_{i,n}) \underset{n\to\infty}{\longrightarrow} \ell(c_i)$ for all $i$. Consider now for each $i$ and each $n$ the path $\widetilde{\alpha}_{i,n}$ which is $\alpha_{i,n}$ concatenated at both end points with a minimizing piece of geodesic in order to have the same end points as $c_i$. If $n$ is large enough, $\alpha_n$ is close enough to $c$ for these pieces of minimizing geodesic to be uniquely defined. So, these geodesic segments cancel out in the product $\widetilde{\alpha}_{1,n} \dots \widetilde{\alpha}_{p,n}$ which is equivalent to $\alpha_{1,n} \dots \alpha_{p,n}$. On the other hand, we have $\widetilde{\alpha}_{i,n} \xrightarrow{d_1} c_i$. Indeed, the small geodesic pieces stay close to each $c_i$ and their length converges to zero. Thus, the corollary 2.5.5 implies $H_{\widetilde{\alpha}_{i,n}} \underset{n\to\infty}{\longrightarrow} H_{c_i}$ which gives the result:

$$H_{\alpha_n} = H_{\widetilde{\alpha}_{p;n}} \dots H_{\widetilde{\alpha}_{1;n}} \underset{n\to\infty}{\longrightarrow} H_{c_p} \dots H_{c_1}.$$

□

### 2.6.2 Continuity of the random holonomy

At this point, we constructed a random holonomy along each path $c$ of $PM$. This random holonomy is a $G$-valued random variable on the probability space $(\Omega, \mathcal{A}, P(x_1, \ldots, x_q))$. Let us state some of its basic properties.

**Proposition 2.6.4** *Let $c, c_1, c_2$ be elements of $PM$.*
*1. $H_{c^{-1}} = H_c^{-1}$ a.s.*
*2. The variable $H_c$ depends only on the equivalence class of $c$ for the relation $\simeq$.*
*3. If $c_1$ and $c_2$ satisfy $c_1(1) = c_2(0)$ then $H_{c_1 c_2} = H_{c_2} H_{c_1}$ a.s.*

*Proof*: (3) is obvious by putting together two decompositions of $c_1$ and $c_2$ and (2) is a direct consequence of (1). To prove (1), just note that this is true for piecewise geodesic paths by construction, and that if $\alpha_n \xrightarrow[n\to\infty]{d_1} c$, then $\alpha_n^{-1} \xrightarrow[n\to\infty]{d_1} c^{-1}$. □

We still have to prove that the law of this random holonomy does not depend on choice of the Riemannian metric used in the construction. For this, we need a regularity property which is the object of the next proposition.

**Proposition 2.6.5** *Let $c$ be a path of $PM$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $c'$ is another path of $PM$ with the same end points as $c$ and if $d_1(c, c') < \delta$, then $d_P(H_c, H_{c'}) < \varepsilon$.*

*Proof.* Let $\delta_0 > 0$ be given by the proposition 2.5.3 such that for any piecewise geodesic path $\alpha$ with the same end points as $c$, $d_1(c, \alpha) < \delta_0$, implies $d_P(H_c, H_\alpha) < \varepsilon$. Let $\delta = \frac{\delta_0}{2}$. Suppose that $c'$ is a path of $PM$ with the same end points as $c$ such that $d_1(c, c') < \delta$. Let $\alpha$ be a piecewise geodesic path such that, simultaneously, $d_1(\alpha, c') < \delta$ and $d_P(H_{c'}, H_\alpha) < \frac{\varepsilon}{2}$. Then $d_1(c, \alpha) < \delta_0$, so that

$$d_P(H_c, H_{c'}) \leq d_P(H_c, H_\alpha) + d_P(H_\alpha, H_{c'}) < \varepsilon.$$ □

Let us state a result that summarizes the results of the procedure of piecewise geodesic approximation. It is in fact the center of the continuum limit procedure. We put together the propositions 2.6.2, 2.6.3 and 2.6.5.

**Proposition 2.6.6** *Let $c$ be a path of $PM$. For any sequence $(\alpha_n)_{n\geq 0}$ of piecewise geodesic paths converging to $c$ with fixed end points, the sequence $(H_{\alpha_n})_{n\geq 0}$ converges to a random variable that depends only on $c$ and that we denote by $H_c$. Moreover, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $c'$ is another path of $PM$ with the same end points as $c$ and if $d_1(c, c') < \delta$ then $d_P(H_c, H_{c'}) < \varepsilon$.*

### 2.6.3 Continuity of the double layer potential (2)

One more time, we transpose directly the preceding arguments to the double layer potential and get the following result:

**Proposition 2.6.7** *Let $c$ be a path of $PM$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $c'$ is another path of $PM$ with the same end points as $c$ and if $d_1(c, c') < \delta$, then $\| u_c - u_{c'} \|_{L^2} < \varepsilon$.*

**Corollary 2.6.8** *Let $l$ be a loop of $LM$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $l'$ is another loop of $LM$ and if $d_1(l, l') < \delta$, then $\| u_l - u_{l'} \|_{L^2} < \varepsilon$.*

*Proof.* Let $\delta_0$ be given by the preceding proposition and set $\delta = \frac{\delta_0}{3}$. Let $l' \in LM$ be such that $d_1(l,l') < \delta$. We have in particular $d(l(0),l'(0)) < \delta$. Let $\sigma$ be a minimizing geodesic from $l(0)$ to $l'(0)$. Then $\widetilde{l'} = \sigma l' \sigma^{-1}$ satisfies $u_{\widetilde{l'}} = u_{l'}$ a.e. and $d_1(\widetilde{l'},l) < \delta_0$. Moreover, $\widetilde{l'}$ has the same end points as $l$. Thus,

$$\| u_l - u_{l'} \|_{L^2} = \| u_l - u_{\widetilde{l'}} \|_{L^2} < \varepsilon.$$

□

## 2.7 Law of the random holonomy

For the moment, we are only able to write down the law of the holonomy along piecewise geodesic paths. We want to show that the law of the holonomy along arbitrary families of paths with finite self-intersection is what we expect it to be, namely that given by the discrete theory. The goal of this section is to prove the following proposition:

**Proposition 2.7.1** *Let $\Gamma = \{a_1,\ldots,a_r\}$ be a graph on $M$ such that $L_1,\ldots L_q \in \Gamma^*$. For any function $f$ continuous on $G^\Gamma$, we have:*

$$Ef(H_{a_1},\ldots,H_{a_r}) = \int_{G^\Gamma} f \, dP^\Gamma_\beta.$$

A very important consequence of this result is the independence of the construction with respect to the Riemannian metric:

**Corollary 2.7.2** *The law of the family $(H_c)_{c\in PM}$ does not depend on the choice of the Riemannian metric that was used throughout the construction.*

*Proof.* Consider two families of variables obtained with two different choices of metric. By the preceding proposition, these families have the same law on the set of paths that are piecewise geodesic for, say, the first metric. By proposition 2.6.5, both families are continuous in a sense that is strong enough to guarantee that their laws coincide on the whole set $PM$. □

In order to prove the proposition 2.7.1, we need a technical result about the approximation of graphs by piecewise geodesic graphs. Before to state this result, let us make some remarks about the edges and faces in a graph in $M$.

Recall that a path and hence an edge must by definition have non-zero derivatives at its end points. This avoids pathological behaviours. For example, consider all edges that share a given vertex of a graph and a small geodesic circle centered at this vertex. If the radius of this circle is small enough, each edge cuts it only once, and the order of the intersection points, which does not depend on the radius of the circle, defines a cyclic order on the set of these edges.

Now, consider two edges that are adjacent for this order. They bound at least one common face. Thus, if $M$ is oriented, a couple of adjacent edges determines a face of the graph (see fig. 2.3). Conversely, given a face, any two consecutive edges of the boundary of this face are adjacent at the vertex that they share, or eventually at both vertices if they share two.

**Proposition 2.7.3** *Let $\Gamma = \{a_1,\ldots,a_r\}$ be a graph such that $L_1,\ldots,L_q \in \Gamma^*$. For any $\varepsilon > 0$, there exists a graph $\Gamma_\varepsilon = \{\alpha_1,\ldots,\alpha_r\}$ with piecewise geodesic edges such that:*
*1. $\Gamma_\varepsilon$ and $\Gamma$ have the same vertices,*

*2. For each $i = 1, \ldots, r$, $d_1(\alpha_i, a_i) < \varepsilon$,*
*3. $L_1, \ldots, L_q \in \Gamma^*_\varepsilon$,*
*4. For each $i = 1, \ldots, r$, $a_i$ and $\alpha_i$ are in the same connected component of the complementary of the unions of the images of the $L_i$'s.*
*Let us denote by $\alpha : \Gamma^* \longrightarrow \Gamma^*_\varepsilon$ the multiplicative map that sends $a_i$ to $\alpha_i$. It is possible to construct $\Gamma_\varepsilon$ in such a way that this map induces a one-to-one correspondence still denoted by $\alpha : \mathcal{F}(\Gamma) \longrightarrow \mathcal{F}(\Gamma_\varepsilon)$ such that $\partial\alpha(F) = \alpha(\partial F)$ and $\sigma(F - \partial F) < \varepsilon$, where $-$ denotes the symmetric difference.*

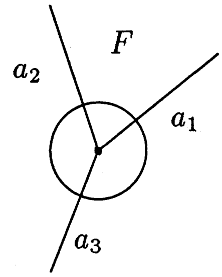


Figure 2.3: The face determined by two adjacent edges.

*Proof.* The property 4 is a consequence of 2 and 3. Indeed, if $a_i$ is in a given connected component, $\alpha_i$ meets this component if $\varepsilon$ is small enough, by 2. But $\alpha_i$ could only exit this component by crossing $L_i$ at a point which is not an end point of $\alpha_i$, which is impossible by 3 and by the definition of graphs.

Let $\mathcal{V}(\Gamma) = \{s_1, \ldots, s_p\}$ denote the set of vertices of $\Gamma$. Let $r$ be a positive real number that "localizes the vertices of $\Gamma$", i.e. small enough to satisfy the following properties:
1. The balls $B(s_i, r)$ are pairwise disjoint.
2. For every pair $(a_i, s_j)$ with $a_i \in \Gamma$ and $s_j$ an end point of $a_i$, $a_i$ meets only once and transversally any circle centered at $s_j$ and of radius smaller than $r$. Moreover, the length of the portion of $a_i$ in the corresponding ball is smaller than $\varepsilon/16$.
3. For any pair $(a_i, s_j)$ where $s_j$ is not an end point of $a_i$, $a_i$ does not meet the ball $B(s_j, r)$.
4. The sum of the surfaces of the ball $B(s_i, r)$ is smaller than $\varepsilon/2$.
5. $r < \varepsilon/16$ and $r < R_M$, where $R_M$ is the convexity radius of $M$.
All properties remain true for $r' < r$ once they are true for $r$, so that it is not a problem to get them simultaneously.

Let $t$ be a positive real number such that $\sigma(\{d(\cdot, \Gamma) < t\}) < \frac{\varepsilon}{2}$. Let $\widehat{a}_i$ denote the portion of $a_i$ outside the disks of radius $r$ around its end points. Let $\delta$ be the smallest distance between the images of two distinct $\widehat{a}_i$. For each $i$, let $\gamma_i$ be an injective piecewise geodesic path with the same end points as $\widehat{a}_i$, such that $d_1(\gamma_i, \widehat{a}_i) < \inf(\varepsilon/4, \delta/2, t)$ and that never meets the balls $B(s_j, r)$, except at its ends. This last condition can be obtained because $a_i$ cuts $B(s_j, r)$ transversally: in a neighbourhood of each end point of $\widehat{a}_i$, there is a half-tube around $\widehat{a}_i$ that does not meet $B(s_j, r)$. It is possible around each end point of $\widehat{a}_i$ to construct $\gamma_i$ inside this half-tube. By definition of $\delta$, the $\gamma_i$'s are disjoint.

Now define $\alpha_i$ for each $i$ such that $a_i$ is not piecewise geodesic as the concatenation of the minimizing geodesic from $a_i(0)$ to $\widehat{a}_i(0)$, of $\gamma_i$ and of the minimizing geodesic from $\widehat{a}_i(1)$ to

$a_i(1)$ (see fig. 2.4). Assumption 5 ensures that these minimizing geodesics are well defined. Assumptions 2 and 5 imply that $d_1(\alpha_i, a_i) < \varepsilon$. The edges of the decompositions of the $L_i$'s are already piecewise geodesic. Hence we only rename them, setting $\alpha_i = a_i$.

The $\alpha_i$'s are edges. Moreover, they were constructed in such a way that they meet only at their ends: we already noticed that they do not meet outside the balls around the vertices of $\Gamma$, and they cannot meet more than once inside these balls according to the local properties of geodesics. Thus, the graph $\Gamma_\varepsilon = \{\alpha_1, \ldots, \alpha_r\}$ exists and has the same vertices as $\Gamma$.

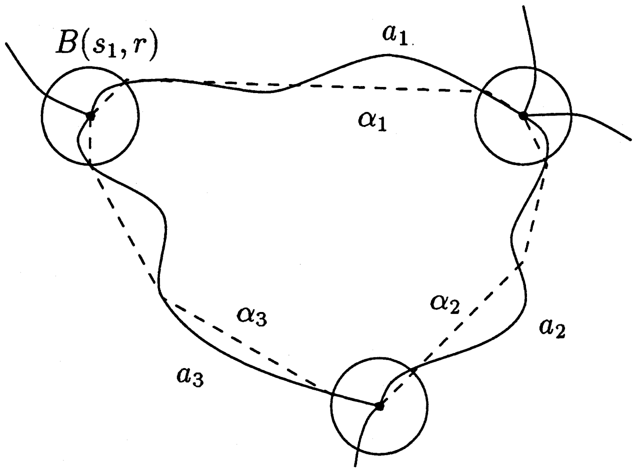


Figure 2.4: Definition of the edges of the graph $\Gamma_\varepsilon$.

We just proved that properties 1 and 2. Property 3 is true because we kept the edges corresponding to the $L_i$'s and property 4 follows, according to the remark made at the beginning of the proof. It remains to prove the last part of the statement.

Consider edges of $\Gamma$ that share a given vertex. They are given a cyclic order. By definition, the corresponding $\alpha_i$'s cut the circle of radius $r$ around this vertex in the same order, so that the multiplicative application $\alpha : \Gamma^* \to \Gamma_g^*$ defined by $\alpha(a_i) = \alpha_i$ preserves the cyclic order at each vertex.

Given a pair of edges of $\Gamma$ that determine the face $F$, the pair of corresponding edges of $\Gamma_\varepsilon$ is a pair of adjacent edges that determine a face of $\Gamma_\varepsilon$. This face does not depend on the particular choice of the edges that represent $F$ and we denote it by $\alpha(F)$. By construction, we have the relation $\partial(\alpha(F)) = \alpha(\partial F)$.

The symmetric difference of $F$ and $\alpha(F)$ is contained in the reunion of the balls $B(s_j, r)$ and the sets $\{d(\cdot, \widehat{a}_i) < t\}$. By assumption 4) and by definition of $t$, we know that the total volume of these sets is smaller than $\varepsilon$. Thus, $\mathrm{vol}(F - \partial F) < \varepsilon$. Moreover, this inequality characterizes $\alpha(F)$ among the faces of $\Gamma_\varepsilon$ that have $\alpha(\partial F)$ as boundary, if there is more than one, provided $\sigma(M)$ is greater than $2\varepsilon$. □

*Proof of proposition 2.7.1.* For each integer $n$, the preceding proposition gives a graph $\Gamma_{\frac{1}{n}} = \{\alpha_{1,n}, \ldots, \alpha_{r,n}\}$. For each $i = 1, \ldots, r$, the sequence $(\alpha_{i,n})$ converges to $a_i$ with fixed end points, so that $H_{\alpha_{i,n}} \longrightarrow H_{a_i}$. In particular, we have the convergence in law:

$$(H_{\alpha_{1,n}}, \ldots, H_{\alpha_{r,n}}) \xrightarrow[n\to\infty]{\text{law}} (H_{a_1}, \ldots, H_{a_r}).$$

Thus, for any function $f$ continuous on $G^\Gamma$,

$$\begin{aligned} Ef(H_{a_1},\ldots,H_{a_r}) &= \lim_{n\to\infty} \frac{1}{Z^{\Gamma_{\frac{1}{n}}}} \int_{G^\Gamma} f(g_1,\ldots,g_r) \prod_{F\in\mathcal{F}(\Gamma_{\frac{1}{n}})} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1}\ldots d\nu_{x_q}\, dg' \\ &= \left(\lim_{n\to\infty} \frac{1}{Z^{\Gamma_{\frac{1}{n}}}}\right) \int_{G^\Gamma} f(g_1,\ldots,g_r) \prod_{F\in\mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F})\, d\nu_{x_1}\ldots d\nu_{x_q}\, dg', \end{aligned}$$

using the fact that $\sigma(\alpha(F))$ tends to $\sigma(F)$ when $n$ tends to infinifty.

Recall from proposition 1.6.5 that the conditional partition functions computed in two graphs, one being finer than the other, are equal. But given two piecewise geodesic graphs, there exists a third one which is finer than both others, as was proved in 2.2.5. Thus the partition function is the same for all piecewise geodesic graphs, and the sequence $\left(Z^{\Gamma_{\frac{1}{n}}}\right)$ is constant. Its value can be computed by setting $f$ identically equal to 1: we find that it is equal to $Z^\Gamma$. This proves the result. □

By the way, we proved the following important result:

**Proposition 2.7.4** *Let $\Gamma$ be a graph such that $L_1,\ldots,L_q \in \Gamma^*$. Then the value of the conditional partition function $Z^\Gamma(x_1,\ldots,x_q)$ does not depend on $\Gamma$.*

## 2.8 Surfaces with boundary

At the beginning of this chapter, we restricted ourselves for technical reasons to surfaces without boundary. In this section, we will extend the construction of the random holonomy to the case of surfaces with boundary.

### 2.8.1 Natural law of the holonomy along the boundary

Let $(M,\sigma)$ be a surface with a boundary $\partial M = N_1\cup\ldots\cup N_p$. In order to construct the holonomy along the paths of $M$, we shall embed $M$ in a minimal closure and use the construction described in the preceding sections. But if we want this procedure to give a result independent of the closure of $M$, and we do, it is necessary to condition the holonomy along every component of $\partial M$. If this was not our first intention, say if we expected only to impose the holonomy along $N_1$, to be equal to $x$ for example, we need to know the natural law of the holonomy along the whole boundary under $P(x)$. Then, we will artificially impose this natural law when working on the closure of $M$. We begin by defining this natural law.

Let $L_1,\ldots,L_q$ be disjoint simple loops on $M$ whose image is included in the interior of $M$. Let $N_1,\ldots,N_k$ be the components of $\partial M$ along which we want to impose the holonomy. Let $x_1,\ldots,x_k,y_1,\ldots,y_q$ be elements of $G$. The $k$ first elements correspond to the components of $\partial M$, the $q$ others to the interior loops.

**Proposition 2.8.1** *Let $\Gamma$ be a graph on $M$ such that $L_1,\ldots,L_q\in\Gamma^*$. The law of the random variable $(h_{N_1},\ldots,h_{N_p},h_{L_1},\ldots,h_{L_q})$ defined on the probability space $(G^\Gamma,P(x_1,\ldots,x_k,y_1,\ldots,y_q))$ does not depend on $\Gamma$. We will denote it by $\beta(x_1,\ldots,x_k,y_1,\ldots,y_q)$.*

**Lemma 2.8.2** *The propositions 2.7.3 and 2.7.4 hold on surfaces with boundary.*

*Proof*. Note that 2.7.3 implies 2.7.4, using the computation done at the end of its proof. Thus, it is sufficient to prove that 2.7.3 holds. For this, embed $M$ in a minimal closure $M_1$. A graph $\Gamma$ on $M$ induces a graph on $M_1$, which can be approximated by piecewise geodesic graphs. If each component of $\partial M$ is the image of one of the $L_i$'s, then the property (4) says exactly that the approximating graphs stay inside $M$. □

*Proof of proposition 2.8.1*. Let us endow $M$ with a Riemannian metric for which $N_1, \dots, N_p$, $L_1, \dots, L_q$ are geodesics. The law of $(h_{N_1}, \dots, h_{N_p}, h_{L_1}, \dots, h_{L_q})$ does not depend on $\Gamma$ provided it is piecewise geodesic, by invariance by subdivision.

Now consider an arbitrary graph $\Gamma$. According to the preceding lemma, we can approximate it by piecewise geodesic graphs, for which the law we are interested in is always the same. The convergence of the joint law of the holonomy along all edges proves the result. □

The lemma 1.5.2 gives us an expression of $\beta(x_1, \dots, x_k, y_1, \dots, y_q)$. We state it here again.

**Lemma 2.8.3** *The following equality between measures on $G^{p+q}$ holds:*

$$\beta(x_1, \dots, x_k, y_1, \dots, y_q) =$$
$$\delta_{(x_1,\dots,x_k)} \otimes \frac{Z(x_1, \dots, x_k, x'_{k+1}, \dots, x'_p, y_1, \dots, y_q)}{Z(x_1, \dots, x_k, y_1, \dots, y_q)} \, dx'_{k+1} \dots dx'_p \otimes \delta_{(y_1,\dots,y_q)}.$$

In the particular case where we do not want to condition the measure at all, the last expression is still true, with the convention that a conditional partition function without parameters is equal to 1.

### 2.8.2 Definition of the random holonomy

Let $M_1$ be a minimal closure of $M$ endowed with a surface measure $\sigma_1$ that extends $\sigma$. We see $N_1, \dots, N_p$, $L_1, \dots, L_q$ as loops on $M_1$. There is a measurable space $(\Omega_1, \mathcal{A}_1)$ on which we constructed a family of measurable functions $(H_c)_{c \in PM_1}$. On this measurable space, we put the following probability:

$$P_1 = \int_{G^{p+q}} P(x'_1, \dots, x'_p, y'_1, \dots, y'_q) \, d(\beta(x_1, \dots, x_k, y_1, \dots, y_q))(x'_1, \dots, x'_p, y'_1, \dots, y'_q).$$

In other words, we insist on the law of $(H_{N_1}, \dots, H_{N_p}, H_{L_1}, \dots, H_{L_q})$ being the natural one under $P(x_1, \dots, x_k, y_1, \dots, y_q)$.

We consider the restriction of the family $(H_c)_{c \in CM_1}$ to $M$, i.e. we restrict the index set to $PM$.

**Proposition 2.8.4** *The law of the restriction $(H_c)_{c \in PM}$ does not depend on $M_1$. If $\Gamma = \{a_1, \dots, a_r\}$ is a graph on $M$ such that $L_1, \dots, L_q \in \Gamma^*$, then the law of $(H_{a_1}, \dots, H_{a_r})$ under $P_1$ is the discrete Yang-Mills measure $P_M(x_1, \dots, x_k, y_1, \dots, y_q)$.*

*Proof*. The regularity property 2.6.5 of the random holonomy on $M_1$ is still true for its restriction to $M$. Thus, the second assertion implies the first one, using the fact that any family of paths on $M_1$ can be approximated by piecewise geodesic families.

Let $\Gamma$ be a graph as in the statement and $f$ be a continuous function on $G^\Gamma$.

$$E_{P_1} f(H_{a_1},\ldots,H_{a_r}) = \int_{G^{p+q}} \frac{d(\beta(x_1,\ldots,x_k,y_1,\ldots,y_q))}{Z_{M_1}(x'_1,\ldots,x'_p,y'_1,\ldots,y'_q)} \int_{G^\Gamma} f(g_1,\ldots,g_r) \prod_{F\in\mathcal{F}(\Gamma),F\subset M_1} p_{\sigma_1(F)}(h_{\partial F})\, d\nu_{x'_1}\ldots d\nu_{x'_p} d\nu_{y'_1}\ldots d\nu_{y'_q}\, dg'.$$

In $M_1$, the loops $N_1,\ldots,N_p$ bound $p$ disks $D_1,\ldots,D_p$ which are the only faces of $\Gamma$ that are not inside $M$. Thus,

$$\begin{aligned} Z_{M_1}(x'_1,\ldots,x'_p,y'_1,\ldots,y'_q) &= \int_{G^\Gamma} \prod_{F\in\mathcal{F}(\Gamma)\, F\subset M} p_{\sigma(F)}(h_{\partial F}) \prod_{i=1}^{p} p_{\sigma_1(D_i)}(h_{N_i}) \\ &\qquad d\nu_{x'_1}\ldots d\nu_{x'_p} d\nu_{y'_1}\ldots d\nu_{y'_q}\, dg' \\ &= \prod_{i=1}^{p} p_{\sigma_1(D_i)}(x'_i) Z_M(x'_1,\ldots,x'_p,y'_1,\ldots,y'_q). \end{aligned}$$

Using this last relation, we get:

$$\begin{aligned} E_{P_1} f(H_{a_1},\ldots,H_{a_r}) &= \int_{G^{p+q}} \frac{d(\beta(x_1,\ldots,x_k,y_1,\ldots,y_q))}{Z_M(x'_1,\ldots,x'_p,y'_1,\ldots,y'_q)} \int_{G^\Gamma} f(g_1,\ldots,g_r) D^\Gamma_M \\ &\qquad d\nu_{x'_1}\ldots d\nu_{x'_p} d\nu_{y'_1}\ldots d\nu_{y'_q}\, dg' \\ = \int_{G^{p-k}} &\frac{Z(x_1,\ldots,x_k,x'_{k+1},\ldots,x'_p,y_1,\ldots,y_q)}{Z(x_1,\ldots,x_k,y_1,\ldots,y_q)} \frac{dx'_{k+1}\ldots dx'_p}{Z(x_1,\ldots,x_k,x'_{k+1},\ldots,x'_p,y_1,\ldots,y_q)} \\ &\int_{G^\Gamma} f(g_1,\ldots,g_r) D^\Gamma_M\, d\nu_{x_1}\ldots d\nu_{x_k} d\nu_{x'_{k+1}}\ldots d\nu_{x'_p} d\nu_{y_1}\ldots d\nu_{y_q}\, dg' \\ &= \frac{1}{Z(x_1,\ldots,x_k,y_1,\ldots,y_q)} \int_{G^\Gamma} f(g_1,\ldots,g_r) D^\Gamma_M\, d\nu_{x_1}\ldots d\nu_{x_k} d\nu_{y_1}\ldots d\nu_{y_q}\, dg' \\ &= P(x_1,\ldots,x_k,y_1,\ldots,y_q)(f). \end{aligned}$$

□

## 2.9 Summary of the properties of the random holonomy

### 2.9.1 Existence, unicity in law and main properties

Let us summarize what has been done in this chapter. We started with a surface $(M,\sigma)$, with or without boundary. We choosed on $M$ disjoint simple loops $L_1,\ldots,L_q$, whose image is either a boundary component of $M$ or contained in the interior of $M$. We picked $q$ elements $x_1,\ldots,x_q$ in $G$. We almost proved the following theorem:

**Theorem 2.9.1** *There exists a probabilty space $(\Omega,\mathcal{A},P(x_1,\ldots,x_q))$ and a family of $G$-valued random variables $(H_c)_{c\in PM}$ on this space, such that:*
*1. For any graph $\Gamma=\{a_1,\ldots,a_r\}$ on $M$ such that $L_1,\ldots,L_q\in\Gamma^*$, the law of $(H_{a_1},\ldots,H_{a_r})$ is the discrete Yang-Mills measure $P^\Gamma(x_1,\ldots,x_q)$ on $G^\Gamma$.*
*2. For any path $c$ of $PM$ and any sequence $(c_n)_{n\geq 0}$ of paths of $PM$ such that $c_n \xrightarrow[n\to\infty]{d_1} c$ with*

*fixed end points, we have* $H_{c_n} \xrightarrow[n\to\infty]{d_P} H_c$.
*The law of this family of random variables is uniquely defined by these two properties. Moreover, it has the following properties:*
*3. If* $c_1$ *and* $c_2$ *are paths that can be concatenated to form* $c_1c_2$*, then* $H_{c_1c_2} = H_{c_2}H_{c_1}$ *a.s.*
*4. If* $\varphi : M \longrightarrow M$ *is a diffeomorphism such that* $\varphi_*\sigma = \sigma$*, then* $\varphi$ *induces a permutation of the set of paths PM and the families* $(H_c)_{c\in PM}$ *and* $(H_{\varphi(c)})_{c\in PM}$ *have the same law.*

*Proof*. We already proved the existence of the family. When $M$ has a boundary, the probability space is that associated with a minimal closure of $M$. Let us prove the uniqueness in law. This law is a probability measure on the set $\mathcal{F}(PM,G)$ endowed with the $\sigma$-algebra generated by cylinder sets. So it is characterized by its finite-dimensional marginals. Since any family of paths can be approximated by families of paths in graphs, for example piecewise geodesic paths for some metric, the law of the random holonomy along an arbitrary finite family of paths is determined by properties (1) and (2).

Property (3) was already proved in proposition 2.6.4 for closed surfaces. For surfaces with boundary, the construction by restriction of the random holonomy on a minimal closure obviously preserves the multiplicativity.

Property (4) was proved at the discrete level in proposition 1.7.1. Since the law of the whole family is determined by discrete laws, it is also true is the continuous setting. □

Given $(M,\sigma)$, $L_1,\ldots,L_q$, $x_1,\ldots,x_q$, the law whose existence and uniqueness is stated by this theorem is a measure on $(\mathcal{F}(LM,G),\mathcal{C})$, where $\mathcal{C}$ is the $\sigma$-algebra generated by the cylinder sets. We shall denote this measure by $\mu_0(x_1,\ldots,x_q)$, or just $\mu_0$ if $q=0$. We keep the notation $(H_c)_{c\in PM}$ for the canonical process on the space $(\mathcal{F}(LM,G),\mathcal{C})$.

### 2.9.2 Disintegration formula

Consider a surface $(M,\sigma)$. Recall from proposition 1.5.3 that the conditional discrete measures constitute a disintegration of the free discrete Yang-Mills measure. We want to extend this result to the continuous setting. As usual, $L_1,\ldots,L_q$ are loops on $M$.

**Proposition 2.9.2** *The map* $(x_1,\ldots,x_q)\mapsto\mu_0(x_1,\ldots,x_q)$ *provides a disintegration of the measure* $\mu_0$ *on* $(\mathcal{F}(PM,G),\mathcal{C})$ *with respect to the random variable* $(H_{L_1},\ldots,H_{L_q})$.

*Proof*. By construction, $(H_{L_1},\ldots,H_{L_q}) = (x_1,\ldots,x_q)$ $\mu_0(x_1,\ldots,x_q)$-a.s. Let $(c_1,\ldots,c_n)$ be a family of paths of $PM$. We need to prove that, for any function $f$ continuous on $G^q$,

$$E_{\mu_0} f(H_{c_1},\ldots,H_{c_q}) = \int_{G^q} E_{\mu_0(x_1,\ldots,x_q)} f(H_{c_1},\ldots,H_{c_q})\, d\eta(x_1,\ldots,x_q),$$

where $\eta$ is the law of $(H_{L_1},\ldots,H_{L_q})$ under $\mu_0$. We already know that this result is true if $c_1,\ldots,c_n$ are paths in a graph. If they are not, we can approximate them in the $d_1$-topology by paths in graphs so that both expectations appearing in the formula converge. Since $G$ is compact, $f$ is bounded and the dominated convergence theorem applies. □

## 2.10 Yang-Mills measure

### 2.10.1 Definition of the Yang-Mills measure

In this paragraph, we will explain why and how the measure $\mu_0$ defined in the preceding section still has to be transformed in order to become something that might be called Yang-Mills measure.

According to the formal description, Yang-Mills measure should be a measure on the quotient space $\mathcal{A}/\mathcal{J}$ of connections modulo gauge transformations. But an element of this space does not determine a holonomy along each path on $M$ that could be intrinsically represented by an element of $G$. Indeed, the holonomy along an open path $c$, i.e. such that $c(0) \neq c(1)$, can be transformed into any other $G$-equivariant diffeomorphism of the fiber over $c(0)$ into the fiber over $c(1)$ by an appropriate gauge transformation. The fact that the law of the random holonomy along an edge is always uniform on $G$ could be thought of as a reflect of this geometric property. This is why we will restrict to the set $LM$ of loops on $M$ instead of $PM$. Thus we will consider the family $(H_l)_{l\in LM}$ whose law is a probability measure on $(\mathcal{F}(LM,G),\mathcal{C})$, where we keep the notation $\mathcal{C}$ for the $\sigma$-algebra generated by the cylinders.

But it is still not true that an element of $\mathcal{A}/\mathcal{J}$ determines an element of $G$ as holonomy along each loop. Gauge transformations act by conjugation on the holonomy along loops. More precisely, they conjugate in the same way the holonomies along loops based at the same point. Let us denote by Ad the diagonal action of $G$ on $G^n$ defined by:

$$\mathrm{Ad}(g)(g_1,\ldots,g_n) = (\mathrm{Ad}(g)g_1,\ldots,\mathrm{Ad}(g)g_n).$$

Orbits of this action will be called joint conjugacy classes and the joint class of $(g_1,\ldots,g_n)$ will be denoted by $[g_1,\ldots,g_n]$. We can reformulate our observation by saying that an element of $\mathcal{A}/\mathcal{J}$ determines the the joint conjugacy class of the holonomy along all loops based at the same point. Sengupta proved the converse of this statement (prop. 2.1.2 in [Se1] ):

**Proposition 2.10.1 ([Se1])** *Let $\omega_1$ and $\omega_2$ be two connections on $M$. Let $m_0$ be a point on $M$. Suppose that along any finite family of loops $l_1,\ldots,l_n$ based at $m_0$, the joint conjugacy classes of the holonomies defined by $\omega_1$ and $\omega_2$ are equal. Then $\omega_1$ and $\omega_2$ belong to the same class in $\mathcal{A}/\mathcal{J}$.*

Let the group $\mathcal{F}(M,G)$ act on $\mathcal{F}(LM,G)$ in the following way: if $j \in \mathcal{F}(M,G)$, $f \in \mathcal{F}(LM,G)$ and $l \in LM$, set

$$(j\cdot f)(l) = j(l(0))^{-1}f(l)j(l(0)).$$

This action extends the action of a discrete gauge transformation. We can summarize our observations as follows:

**Proposition 2.10.2** *The holonomy allows to define an injective map*

$$\mathcal{A}/\mathcal{J} \longrightarrow \mathcal{F}(LM,G)/\mathcal{F}(M,G).$$

This result says that the quotient space $\mathcal{F}(LM,G)/\mathcal{F}(M,G)$ can be viewed as an extension of the space of connections modulo gauge transformations. We want to define the Yang-Mills measure on this space. To begin with, we must define a convenient $\sigma$-algebra.

There is a set of natural functions on the quotient space: given $n$ loops $l_1, \ldots, l_n$ based at the same point, the joint class $[H_{l_1}, \ldots, H_{l_n}]$ is a well-defined function that we denote by $\mathcal{H}_{l_1,\ldots,l_n}$. We will consider the $\sigma$-algebra $\mathcal{A}$ generated by the set of these functions. Of course, we want to be able to consider random variables associated with families of loops that are not based at the same point. We claim that the $\sigma$-algebra $\mathcal{A}$ allows to do this. Indeed, let $l_1, \ldots, l_n$ be a family of loops that we rewrite $(l_1, \ldots, l_{i_1}), \ldots, (l_{i_p+1}, \ldots, l_n)$, putting together the loops based at the same points. Then we can define the variable $\mathcal{H}_{l_1,\ldots,l_n}$ by

$$\mathcal{H}_{l_1,\ldots,l_n} = (\mathcal{H}_{l_1,\ldots,l_{i_1}}, \ldots, \mathcal{H}_{l_{i_p+1},\ldots,l_n})$$

and this random variable is measurable with respect to $\mathcal{A}$. Remark that $\mathcal{A}$ may also be seen as a $\sigma$-algebra on $\mathcal{F}(LM, G)$, invariant by the action of $\mathcal{F}(M, G)$, since the functions $\mathcal{H}_{l_1,\ldots,l_q}$ are also naturally defined on this space. Another natural choice for $\mathcal{A}$ would have been to consider the $\mathcal{F}(M, G)$-invariant sets of the cylinder $\sigma$-algebra $\mathcal{C}$. We shall discuss this point at the end of this section.

**Proposition 2.10.3** *Let $(M, \sigma)$ be a surface. Let $L_1, \ldots, L_q$ be disjoint simple loops on $M$ whose image is either a component of the boundary of $M$ or contained in the interior of $M$. Let $(x_1, \ldots, x_q)$ be an element of $G^q$. The restriction of $\mu_0(x_1, \ldots, x_q)$ to $\mathcal{A}$ depends on each $x_i$ only through its conjugacy class.*

*Proof.* The point is to understand how $\mu_0(x_1, \ldots, x_q)$ is transformed under the action of $\mathcal{F}(M, G)$. Similarly to what we proved in 1.5.4, if $j$ is an element of $\mathcal{F}(M, G)$, then, setting $y_i = j(L_i(0))$, we have

$$j_* P(x_1, \ldots, x_q) = P(y_1^{-1} x_1 y_1, \ldots, y_q^{-1} x_q y_q).$$

Indeed, we already know that this equality holds when we evaluate these measures against functions of the holonomy along paths in a graph, and we extend it to general measurable functions by the usual approximation scheme.

Thus, the $\mu_0(x_1, \ldots, x_q)$-measure of sets invariant under the action of $\mathcal{F}(M, G)$ depends only on the conjugacy classes $[x_1], \ldots, [x_q]$. □

We denote by $t_i$ the conjugacy class of each $x_i$.

**Definition 2.10.4** *We call Yang-Mills measure on $M$ and denote by $\mu$ the image measure of $\mu_0$ on the quotient space $(\mathcal{F}(LM, G)/\mathcal{F}(M, G), \mathcal{A})$, or equivalently the restriction of $\mu_0$ to $(\mathcal{F}(LM, G), \mathcal{A})$.*

*Similarly, we call conditional Yang-Mills measure with respect to $L_1, \ldots, L_q$ and we denote by $\mu(t_1, \ldots, t_q)$ the image measure of $\mu_0(x_1, \ldots, x_q)$ on the quotient space $(\mathcal{F}(LM, G)/\mathcal{F}(M, G), \mathcal{A})$, or equivalently the restriction of $\mu_0(x_1, \ldots, x_q)$ to $(\mathcal{F}(LM, G), \mathcal{A})$.*

The first point of view keeps track of the quotient structure of the space $\mathcal{A}/\mathcal{J}$. Nevertheless, the second will often be technically more convenient.

**Proposition 2.10.5** *The map $(t_1, \ldots, t_q) \mapsto \mu(t_1, \ldots, t_q)$ defined on $(G/\operatorname{Ad})^q$ provides a disintegration of the measure $\mu$ with respect to the random variable $\mathcal{H}_{L_1}, \ldots, \mathcal{H}_{L_q}$.*

Note that since the $L_i$'s are not based at the same point, the variables $\mathcal{H}_{L_1,\ldots,L_q}$ and $(\mathcal{H}_{L_1},\ldots,\mathcal{H}_{L_q})$ are equal.

*Proof.* Let $(t_1,\ldots,t_q)$ be an element of $(G/\operatorname{Ad})^q$ and $(x_1,\ldots,x_q)\in G^n$ be such that $[x_i]=t_i$ for each $i$. Then $([H_{L_1}],\ldots,[H_{L_q}])=([x_1],\ldots,[x_q])$ $\mu_0(x_1,\ldots,x_q)$-a.s., so that

$$(\mathcal{H}_{L_1},\ldots,\mathcal{H}_{L_q})=(t_1,\ldots,t_q)\;\;\mu(t_1,\ldots,t_q)-\text{a.s.}$$

By 1.5.2, we know that the law of $(H_{L_1},\ldots,H_{L_q})$ under $\mu_0$ is $Z^{-1}Z(x_1,\ldots,x_q)\,dx_1\ldots dx_q$. We also proved in 2.9.2 that $\mu_0$ is disintegrated by the $\mu_0(x_1,\ldots,x_q)$, so that

$$\mu_0=\frac{1}{Z}\int_{G^q}Z(x_1,\ldots,x_q)\mu_0(x_1,\ldots,x_q)\;dx_1\ldots dx_q.$$

If we evaluate these measures on sets of $\mathcal{A}$ and use the invariance by conjugation of the conditional partition function stated in 1.5.5, we find:

$$\begin{aligned}\mu_0 &= \frac{1}{Z}\int_{G^q}Z([x_1],\ldots,[x_q])\mu_0([x_1],\ldots,[x_q])\;dx_1\ldots dx_q\\ &= \frac{1}{Z}\int_{(G/\operatorname{Ad})^q}Z(t_1,\ldots,t_q)\mu_0(t_1,\ldots,t_q)\;dt_1\ldots dt_q,\end{aligned}$$

where $dt$ is the image measure on $G/\operatorname{Ad}$ of the Haar measure. The last equality restricted to $\mathcal{A}$-measurable sets proves the result. □

### 2.10.2 Regularity properties

In order to study Yang-Mills measure, we need to say more about the set of joint conjugacy classes $G^n/\operatorname{Ad}$. We regard it as a set of compact subsets of $G^n$ and endow it with the Hausdorff distance, defined in general between two compact sets by

$$d(K_1,K_2)=\sup\big(\sup_{k_1\in K_1}\inf_{k_2\in K_2}d(k_1,k_2),\;\sup_{k_2\in K_2}\inf_{k_1\in K_1}d(k_1,k_2)\big).$$

**Lemma 2.10.6** *The canonical projection $G^n\longrightarrow G^n/\operatorname{Ad}$ is 1-Lipchitz.*

*Proof.* Let $(g_1,\ldots,g_n)$ and $(h_1,\ldots,h_n)$ be two elements of $G^n$.

$$\begin{aligned}d([g_1,\ldots,g_n],[h_1,\ldots,h_n]) &= \sup_{g\in G}\inf_{h\in G}d(\operatorname{Ad}(g)(g_1,\ldots g_n),\operatorname{Ad}(h)(h_1,\ldots,h_n))\\ &= \inf_{h\in G}d((g_1,\ldots,g_n),\operatorname{Ad}(h)(h_1,\ldots,h_n))\\ &\leq d((g_1,\ldots,g_n),(h_1,\ldots,h_n)).\end{aligned}$$

□

As before, this distance on $G^n/\operatorname{Ad}$ allows to define the distance

$$d_P(X,Y)=Ed(X,Y)$$

between $G^n/\operatorname{Ad}$-valued random variables defined on the same probability space. The regularity property 2.6.5 of the random holonomy becomes the following regularity property for the Yang-Mills measure:

**Proposition 2.10.7** *Let $((l_{1,k},\dots,l_{n,k}))_{k\geq 0}$ be a sequence of n-uples of loops such that*
*1. for each $k \geq 0$, the loops $l_{1,k},\dots,l_{n,k}$ are based at the same point,*
*2. for each $i = 1,\dots,n$, there exists a loop $l_i$ such that $l_{i,k} \xrightarrow[k\to\infty]{d_1} l_i$.*
*Then*

$$\mathcal{H}_{l_{1,k},\dots,l_{n,k}} \xrightarrow[k\to\infty]{d_P} \mathcal{H}_{l_1,\dots,l_n}.$$

*Proof*. The loops $l_i$ are necessarily based at the same point, denoted by $m$. Denoting by $m_k$ the base point of the $l_{i,k}$'s, we have $m_k \longrightarrow m$. For each $k$, let $z_k$ denote an arbitrary path joining $m$ to $m_k$. Then for each $i$ $z_k l_{i,k} z_k^{-1} \xrightarrow{d_1} l_i$ with fixed basepoint, so that

$$H_{z_k l_{i,k} z_k^{-1}} \xrightarrow{d_P} H_{l_i}.$$

Since the projection on $G^n/\operatorname{Ad}$ reduces the distances, this implies

$$[H_{z_k l_{1,k} z_k^{-1}},\dots,H_{z_k l_{n,k} z_k^{-1}}] \xrightarrow[k\to\infty]{d_P} [H_{l_1},\dots,H_{l_n}].$$

The left hand side term is equal to

$$[H_{z_k}^{-1} H_{l_{1,k}} H_{z_k},\dots,H_{z_k}^{-1} H_{l_{n,k}} H_{z_k}] = [H_{l_{1,k}},\dots,H_{l_{n,k}}],$$

so that the result is proved. □

### 2.10.3 Remarkable subfamilies of random variables

We study two special subfamilies of random variables defined on $(\mathcal{F}(LM,G)/\mathcal{F}(M,G),\mathcal{A},\mu)$, using the results proved in the preceding paragraphs.

We begin by the family $(\mathcal{H}_l)_{l\in LM}$. Each variable is $G/\operatorname{Ad}$-valued and this family satisfies a very nice regularity property:

**Proposition 2.10.8** *Let $l$ be in $LM$ and $(l_n)_{n\geq 0}$ be a sequence of loops that converges to $l$. Then $\mathcal{H}_{l_n} \xrightarrow{d_P} \mathcal{H}_l$.*

This is the only situation where we can forget about end points. Unfortunately, this family does not generate $\mathcal{A}$, since it does not contain any information about joint conjugacy classes.

Now fix a point $m \in M$ and consider the set $L_m M$ of loops based at $m$. We are interested in the family $(\mathcal{H}_{l_1,\dots,l_n})_{l_i\in L_m M}$. It has the same property as that stated in 2.10.7, the condition on end points being always satisfied. What is interesting here is the following fact:

**Proposition 2.10.9** *The family $(\mathcal{H}_{l_1,\dots,l_n})_{l_i\in L_m M}$ generates the $\sigma$-algebra $\mathcal{A}$.*

*Proof*. Let $l_1,\dots,l_n$ be $n$ loops on $M$ based at a point $m_1$. Let $c$ be a path joining $m$ to $m_1$. Then the equality

$$\mathcal{H}_{cl_1c^{-1},\dots,cl_nc^{-1}} = \mathcal{H}_{l_1,\dots,l_n}$$

proves that it is always possible to get back to loops based at $m$. □

This subfamily satisfies also a multiplicativity property. Indeed, the joint conjugacy class of some elements of $G$ determines the joint class of all products of these elements. For example, there is a well defined map from $G^n/\operatorname{Ad}$ to $G^{n-1}/\operatorname{Ad}$ that sends $[g_1,\ldots,g_{n-1},g_n]$ to $[g_1,\ldots,g_{n-2},g_ng_{n-1}]$. The multiplicativity can be expressed by saying that for any $l_1,\ldots,l_n \in L_mM$, this maps sends almost surely $\mathcal{H}_{l_1,\ldots,l_n}$ to $\mathcal{H}_{l_1,\ldots,l_{n-1}l_n}$.

**Remark.** Let us discuss the definition of the $\sigma$-algebra $\mathcal{A}$. For this, we consider the Yang-Mills measure an invariant measure on the space $\mathcal{F}(L_mM,G)$, because in this setting, the action of the gauge group is that of the finite dimensional group $G$. We use this fact below in order to integrate functions over the orbits of this action.

We could have made another natural choice of an invariant $\sigma$-algebra on $\mathcal{F}(L_mM,G)$, namely that of invariant sets of the cylinder $\sigma$-algebra $\mathcal{C}$ on $\mathcal{F}(L_mM,G)$. Let us denote by $\mathcal{C}_\mathcal{I}$ this $\sigma$-algebra. It is clear that $\mathcal{A} \subset \mathcal{C}_\mathcal{I}$ and it is very likely that this inclusion is in fact an equality. We prove that the completions $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}_\mathcal{I}}$ with respect to $\mu_M(x)$ are equal.

We use the separability of $L_mM$ proved in the next lemma.

**Lemma 2.10.10** *Let $M$ be a surface and $m$ a point of $M$. The loop space $L_mM$ endowed with the $d_1$-topology is separable.*

*Proof.* We construct a countable dense subset of $L_mM$. The first point is that $M$ itself is separable. Choose a countable dense subset $\Pi \subset M$ containing $m$. Endow $M$ with a Riemannian metric such that $\partial M$ is geodesic if it is non empty. Let $R_M$ be the convexity radius of $M$: two points at distance smaller than $R_M$ are joined by a unique minimizing geodesic. Define $\Lambda$ to be the set of loops obtained by concatenation of a finite number of geodesic segments joining two points of $\Pi$ at distance smaller than $R_M$. The set $\Lambda$ is countable because it is equipotent to a subset of finite sequences of $\Pi$. We claim that it is dense in $L_mM$. Indeed, any geodesic segment of length smaller than $R_M$ can be approximated by segments joining points of $\Pi$, since a small piece of geodesic depends continuously on its end points. Thus, the $d_1$-closure of $\Lambda$ contains the set of piecewise geodesic loops and we already know that this set is dense in $L_mM$.□

Let $\Lambda \subset L_mM$ be a countable dense subset. Let $\mathcal{A}^\Lambda$ denote $\sigma(\mathcal{H}_{\lambda_1,\ldots,\lambda_n} \in \Lambda)$. Let $\mathcal{C}^\Lambda$ denote $\sigma(H_\lambda, \lambda \in \Lambda)$ and $(\mathcal{C}^\Lambda)_\mathcal{I}$ denote the invariant sets of this $\sigma$-algebra. It is clear that $\mathcal{A}^\Lambda \subset (\mathcal{C}^\Lambda)_\mathcal{I}$. It is also clear that $\mathcal{A}^\Lambda$ is a separable $\sigma$-algebra. Finally, one easily checks that any atom of $\mathcal{A}^\Lambda$ is contained in an atom of $(\mathcal{C}^\Lambda)_\mathcal{I}$. Thus, Blackwell's theorem implies (see [DM]) that $\mathcal{A}^\Lambda = (\mathcal{C}^\Lambda)_\mathcal{I}$.

We use this equality to prove the inclusion $\mathcal{C}_\mathcal{I} \subset \widetilde{\mathcal{A}}$, which implies the result. The point is that the continuity in probability of the map $l \mapsto \mathcal{H}_l$ and the density of $\Lambda$ in $L_mM$ imply that $\widetilde{\mathcal{C}^\Lambda}$ contains $\mathcal{C}$ and that $\widetilde{\mathcal{A}^\Lambda}$ contains $\mathcal{A}$. This last inclusion implies the equality $\widetilde{\mathcal{A}^\Lambda} = \widetilde{\mathcal{A}}$. Thus, it is sufficient to prove that $\mathcal{C}_\mathcal{I} \subset \widetilde{\mathcal{A}^\Lambda}$. Let $f$ be a $\mathcal{C}_\mathcal{I}$-measurable function. As a $\mathcal{C}$-measurable function, it is $\mu_M(x)$-almost surely the limit of a sequence $(f_n)$ of $\mathcal{C}^\Lambda$-measurable functions. Let us integrate these functions $f_n$ over the orbits of the action of $G$ on $\mathcal{F}(L_mM,G)$, using the Haar measure on $G$. We get a sequence of $(\mathcal{C}^\Lambda)_\mathcal{I}$-measurable functions still converging to $f$. Thus, $f$ is measurable with respect to the completion $\widetilde{(\mathcal{C}^\Lambda)_\mathcal{I}} = \widetilde{\mathcal{A}^\Lambda}$ and we get the result.

# Chapter 3

# Abelian theory

In this chapter, we continue the investigation of the case $G = U(1)$ started in section 1.9. Recall that we had reconstructed the random holonomy along loops homologous to zero in a graph, using a white noise on $M$ (see proposition 1.9.10). We extend now this reconstruction to all cycles of $M$, using the unicity properties of the Yang-Mills measure proved in chapter 2.

Then we show that it is possible to proceed backwards, namely to extract a white noise on $M$ from the Yang-Mills measure on $M$, more precisely, using the random holonomy along very small loops. This makes clear the relationship between the random holonomy and the white noise in this Abelian case.

## 3.1 The random holonomy as a white noise functional

As usual, $M$ may have a boundary $\partial M = N_1 \cup \ldots \cup N_p$. Choose elements $x_1, \ldots, x_p$ in $U(1)$ and set $x = x_1 \ldots x_p$ or $x = 1$ if $M$ has no boundary. We denote by $CM$ the set of cycles on $M$, i.e. the set of linear combination of loops with integer coefficients and by $C_0\Gamma$ the set of cycles homologous to zero. The family of random variables $(H_l)_{l \in LM}$, that we used in 2.10 to construct the Yang-Mills measure $\mu_M(x_1, \ldots, x_p)$, extends by multiplicativity to the cycles of $CM$ and gives rise to a measure on $\mathcal{F}(CM, U(1))$. In this Abelian setting, the action of $\mathcal{F}(M, U(1))$ is trivial.

We seek a result similar to 1.9.9, valid for all cycles on $M$. We begin by defining a family of random variables using a white noise on $M$ and prove later that it has the law of a Yang-Mills random holonomy.

Recall that we proved earlier that the holonomies along a system of loops representing a basis of $H_1(M; \mathbf{Z})$ are independent uniform variables on $U(1)$, independent of the holonomies along loops homologous to zero (see proposition 1.9.4).

### 3.1.1 Definition of the white noise functional

Recall that $g$ denotes the genus of $M$. In order to define the double layer potential (see 1.9.6), we need a Riemannian metric on $M$, that we choose such that the boundary of $M$ is geodesic. Let $\ell_1, \ldots, \ell_{2g}$ be piecewise geodesic loops such that

$$\mathcal{B} = ([\ell_1], \ldots, [\ell_{2g}], [N_1], \ldots, [N_{p-1}])$$

is a basis of $H_1(M; \mathbf{Z})$.

Let $c$ be a cycle of $CM$. We can decompose it in $H_1(M;\mathbf{Z})$:

$$[c] = \lambda_1[\ell_1] + \ldots + \lambda_{2g}[\ell_{2g}] + \nu_1[N_1] + \ldots + \nu_{p-1}[N_{p-1}].$$

Let $U_1, \ldots, U_{2g}$ be $2g$ independent uniform random variables on $U(1)$. Set

$$\Theta_c = U_1^{\lambda_1} \ldots U_{2g}^{\lambda_{2g}} x_1^{\nu_1} \ldots x_{p-1}^{\nu_{p-1}}.$$

There is a cycle $c^\perp$ of $C_0M$, i.e. a cycle homologous to zero, associated with $c$, defined by

$$c^\perp = c - (\lambda_1\ell_1 + \ldots + \lambda_{2g}\ell_{2g} + \nu_1 N_1 + \ldots + \nu_{p-1}N_{p-1}).$$

Recall that $u_{c^\perp}$ denotes the doule layer potential of $c^\perp$. The cycle $c^\perp$ is the boundary of a 2-chain $\alpha$. We defined an element $\sigma_{\text{int}}(c^\perp)$ of $\mathbf{R}$ (resp. $\mathbf{R}/\mathbf{Z}$) when $M$ has a boundary (resp. no boundary) by $\sigma_{\text{int}}(c^\perp) = \frac{|\langle\sigma,\alpha\rangle|}{\sigma(M)}$.

Let $W$ be a white noise on $M$, independent of the $U_i$'s. Let $T$ be a variable independent of $W$ and the $U_i$'s, whose law is that described in proposition 1.9.5. Finally, denote by $W_0$ the projection of $W$ on the hyperplane of zero-mean functions: for any function $u \in L^2(M,\sigma)$,

$$W_0(u) = W\left(u - \frac{1}{\sigma(M)}\int_M u\, d\sigma\right).$$

We are able to define what will be proved to be a second realization of the random holonomy along cycles. Denote by $(\Omega, P)$ a probability space that supports $W$, $T$ and the $U_i$'s.

**Definition 3.1.1** *For each cycle $c \in CM$, define the following random variable on $(\Omega, P)$:*

$$WH_c = \exp i(W_0(u_{c^\perp}) + \sigma_{\text{int}}(c^\perp)T)\, \Theta_c.$$

### 3.1.2 Regularity of the new random holonomy

In order to prove that this family has the law of the random holonomy, we will check that this is true for a restricted class of paths, namely piecewise geodesic paths, and extend this partial result by continuity. This is why we are interested in the regularity of this new family.

We begin by extending the distance $d_1$ to the space $CM$ of cycles.

**Definition 3.1.2** *Let $c = n_1l_1 + \ldots + n_kl_k$ and $c' = n'_1l'_1 + \ldots + n'_{k'}l'_{k'}$ be two cycles on $M$, written as combinations of loops. If $k \neq k'$, set $d_1(c,c') = 1$. If $k = k'$, let $\pi(c,c')$ be the set of permutations $\tau \in S_k$ such that $n'_{\tau(i)} = n_i$ for all $i = 1, \ldots, k$. If $\pi(c,c') = \emptyset$, set $d_1(c,c') = 1$. Otherwise, set*

$$d_1(c,c') = \inf_{\tau\in\pi(c,c')} \sum_{i=1}^{k} n_i d_1(l_i, l'_{\tau(i)}).$$

We will use proposition 2.6.8 about the continuity of the double layer potential of loops to prove the next proposition. Recall that 2.6.8 was proved only on surfaces without boundary.

**Proposition 3.1.3** *Suppose that $M$ has no boundary. Let $c$ be a cycle and $(c_n)_{n\geq 0}$ be a sequence of cycles such that $c_n \xrightarrow{d_1} c$. Then $u_{c_n} \xrightarrow{L^2} u_c$.*

*Proof*: Decompose $c$ as $n_1 l_1 + \ldots + n_k l_k$. Let $N = n_1 + \ldots + n_k$. Fix $\varepsilon > 0$. For each $l_i$, there exists $\delta_i < 1$ such that $d_1(l'_i, l_i) < \delta_i$ implies $\| u_{l'_i} - u_{l_i} \|_{L^2} < \frac{\varepsilon}{kN}$ for any loop $l'_i$. Suppose that $n$ is such that $d_1(c_n, c) < \inf \delta_i$. Then $c_n$ can be written $n_1 l'_1 + \ldots + n_k l'_k$, with $\sum n_i d_1(l_i, l'_i) < \inf \delta_i$. For each $i$, we have in particular $d_1(l_i, l'_i) < \delta_i$. Thus,

$$\| u_{c_n} - u_c \|_{L^2} \leq \sum_{i=1}^{k} n_i \| u_{l'_i} - u_{l_i} \|_{L^2} < \frac{kN\varepsilon}{kN} = \varepsilon. \qquad \square$$

The homology class of a cycle $c$ depends continuously on $c$, even for the distance $d_\infty$. Thus, $c^\perp$ and hence $u_{c^\perp}$ depends continuously on $c$, and, by continuity of the white noise which is an isometry, the map $c \mapsto W_0(u_{c^\perp})$ is continuous for the $L^2$ norm when $M$ is closed. We want to extend this result to surfaces with boundary. For this, we study $u_c$ when $c$ is homologous to zero.

Let $M$ be a surface, with or without boundary and fix $c \in C_0 M$. Let $x$ and $y$ be two points of $M$ outside the image of $c$. If necessary, we modify locally the $\ell_i$'s in a neighbourhood of $x$ in order to make sure that $x$ meets none of them. Let $\ell_x$ be the boundary of a small disk $D_x$ around $x$, small enough not to meet $c$ and not to contain $y$. The module $H_1(M - \{x, y\})$ is generated by $[\ell_1], \ldots, [\ell_{2g}], [N_1], \ldots, [N_{p-1}], [\ell_x], [\ell_y]$. In $M - \{x, y\}$, we have the equality

$$[c] = \sum_{i=1}^{2g} \lambda_i [\ell_i] + \sum_{j=1}^{p-1} \nu_j [N_j] + p[\ell_x] + q[\ell_y]$$

for some $p \in \mathbf{Z}$. This equality also holds in $M$, where $[c] = [\ell_x] = [\ell_y] = 0$, and this proves that $\lambda_i = \nu_j = 0$. Thus, $[c] = p[\ell_x] + q[\ell_y]$ in $M - \{x, y\}$.

**Lemma 3.1.4** *With the preceding notations,* $u_c(x) - u_c(y) = p - q$.

*Proof*: Recall the definition of the Green function $G(\cdot, \cdot)$ from equation (1.9), and the value of the double layer potential of small loops from proposition 1.9.7. The 1-form $*dG_x - *dG_y$ is closed on $M - \{x, y\}$. So,

$$\begin{aligned} u_c(x) - u_c(y) &= \int_c *dG_x - *dG_y \\ &= p \int_{\ell_x} *dG_x - *dG_y + q \int_{\ell_y} *dG_x - *dG_y \\ &= p(u_{\ell_x}(x) - u_{\ell_x}(y)) + q(u_{\ell_y}(x) - u_{\ell_y}(y)) \\ &= p\left(\mathbf{1}_{D_x}(x) - \frac{\sigma(D_x)}{\sigma(M)} - \mathbf{1}_{D_x}(y) + \frac{\sigma(D_x)}{\sigma(M)}\right) + q\left(\mathbf{1}_{D_y}(x) - \mathbf{1}_{D_y}(y)\right) \\ &= p - q. \end{aligned} \qquad \square$$

**Remark.** If $M$ is closed, then $[\ell_x] + [\ell_y] = 0$ in $H_1(M - \{x, y\})$. In this case, $p$ and $q$ are defined up to an additive constant but the difference $p - q$ is well defined.

**Corollary 3.1.5** *1. When $M$ has no boundary, the double layer potential of cycles of $C_0 M$ does not depend on the choice of the metric.*

*2. The potential $u_c$ is constant on each connected component of the complementary of the image of c.*
*3. If $M_1$ is a minimal closure of $M$ and if we identify $M$ with a submanifold of $M_1$, then for any $c \in C_0M$, the potentials $u_c^M$ and $u_c^{M_1}{}_{|M}$ computed respectively in $M$ and $M_1$ differ only by an additive constant.*

*Proof*: 1. The lemma determines the potential of any cycle of $C_0M$ up to a constant. When $M$ has no boundary this constant is determined by the condition $\int_M u_c \, d\sigma = 0$.

2. Given a fixed point $y_0$, $u_c(x) - u_c(y_0)$ depends only on the homology class of $c$ in $M - \{x, y\}$, which does not change if $x$ stays in a given connected component.

3. Both functions $u_c^M$ and $u_c^{M_1}{}_{|M}$ satisfy the property shown in the lemma, with the same values of $p$ and $q$. Indeed, if $[c] = p[\ell_x] + q[\ell_y]$ in $H_1(M)$, then the same equality holds in $H_1(M - \{x, y\})$. Thus, they cannot differ by more than an additive constant. □

**Proposition 3.1.6** *Even if $M$ has a boundary, the map $c \mapsto W_0(u_{c^\perp})$ is continuous.*

*Proof*: By property (3) of the preceding corollary,

$$u_{c^\perp}^M - \frac{1}{\sigma(M)} \int_M u_{c^\perp}^M \, d\sigma = u_{c^\perp}^{M_1}{}_{|M} - \frac{1}{\sigma(M)} \int_M u_{c^\perp}^{M_1} \, d\sigma.$$

Together with 3.1.3, this shows that $u_{c^\perp}$ depends continuously on $c_0$. This was the only missing point. □

Now we study the term $\sigma_{\text{int}}(c)$, when $c \in C_0M$. We will show that it can be extracted from the double layer potential of $c$.

**Lemma 3.1.7** *Let $c$ be a cycle of $C_0M$.*
*1. If $M$ has no boundary, $\sigma_{\text{int}}(c)$ is the element $t \in \mathbf{R}/\mathbf{Z}$ such that $u_c$ takes its values in $\mathbf{Z} - t$.*
*2. If $M$ has a boundary, consider $M_1$ a minimal closure of $M$. Then $\sigma_{\text{int}}(c)$ is equal to $-\frac{\sigma(M_1)}{\sigma(M)}$ times the value of $u_c^{M_1}$ at any point of $M_1 - M$.*

*Proof*: 1. Let $x$ be a point of $M$ outside the image of $c$. Denote by $\alpha$ a 2-chain such that $c = \partial\alpha$. Then

$$\begin{aligned} u_c(x) = \int_c *dG_x &= \int_\alpha \delta_x - \frac{d\sigma}{\sigma(M)} \\ &\equiv -\frac{|\langle \sigma, \alpha \rangle|}{\sigma(M)} \pmod 1. \end{aligned}$$

2. Let $x$ be a point of $M_1 - M$. We have

$$u_c^{M_1}(x) = \int_\alpha \delta_x - \frac{d\sigma}{\sigma(M_1)} = -\frac{|\langle \sigma, \alpha \rangle|}{\sigma(M_1)} = -\frac{\sigma(M)}{\sigma(M_1)} \sigma_{\text{int}}(c).$$ □

Let $(c_n)_{n \geq 0}$ be a sequence of cycles of $C_0M$ such that $c_n \longrightarrow c$. Since $u_{c_n} \xrightarrow{L^2} u_c$ and since all these functions are locally constant, there is pointwise convergence outside the image of $c$. This implies:

**Proposition 3.1.8** *The map $c \mapsto \sigma_{\text{int}}(c)$ defined on $C_0M$ is continuous.*

Finally, the map $c \mapsto \Theta_c$ is locally constant on $C_0M$. We proved:

**Proposition 3.1.9** *The map $c \mapsto WH_c$ is continuous from $(CM, d_1)$ into the space of square integrable random variables on $(\Omega, P)$.*

The last property that we need is the multiplicativity:

**Proposition 3.1.10** *For any cycles $c_1$ and $c_2$ in $CM$,*

$$WH_{c_1+c_2} = WH_{c_1} WH_{c_2} \quad P - \text{a.s.}$$

*Proof*: This follows immediately from the following facts: $c^\perp$ depends linearly on $c$, the double layer potential is additive, the map $c \mapsto \sigma(c)$ is also additive and the map $c \mapsto \Theta_c$ is multiplicative. □

### 3.1.3 Identification of the random holonomies

We are now able to prove the main theorem of this section:

**Theorem 3.1.11** *The family of random variables $(WH_c)_{c \in CM}$ has the same law as the family $(\mathcal{H}_c)_{c \in CM}$ under $\mu_M(x_1, \ldots, x_p)$.*

*Proof*: According to the unicity statement of the theorem 2.9.1 and to the regularity property 3.1.9, it is enough to prove the equality of the laws for piecewise geodesic cycles. Let $\Gamma$ be a piecewise geodesic graph such that $\ell_1, \ldots, \ell_{2g} \in \Gamma^*$. Recall that these loops generate the first homology group of a minimal closure of $M$. Denote by $F_1, \ldots, F_n$ the faces of $\Gamma$. The arguments developped in 1.9.1 explain why it is enough to prove the equality of the laws for the fundamental system $(\ell_1, \ldots, \ell_{2g}, N_1, \ldots, N_p, \partial F_1, \ldots, \partial F_n)$.

On one hand, $\ell_1^\perp = \ldots = \ell_{2g}^\perp = 0$, so that $WH_{\ell_i} = U_i$ for all $i = 1, \ldots, 2g$. On the other hand, $N_1^\perp = \ldots = N_{p-1}^\perp = 0$ and $N_p^\perp = N_1 + \ldots + N_p$. Thus, $\sigma_{\text{int}}(N_p^\perp) = 1$ and $u_{N_p^\perp}$ is equal to zero so that $W_0(u_{N_p^\perp}) = 0$. This implies $WH_{N_j} = x_j$ for all $j = 1, \ldots, p-1$ and also for $j = p$. Finally, we already proved in 1.9.9 that $(WH_{\partial F_1}, \ldots, WH_{\partial F_n})$ and $(\mathcal{H}_{\partial F_1}, \ldots, \mathcal{H}_{\partial F_n})$ have the same law. This terminates the proof, since we know that $(\mathcal{H}_{\partial F_1}, \ldots, \mathcal{H}_{\partial F_n})$ and $(\mathcal{H}_{\ell_1}, \ldots, \mathcal{H}_{\ell_{2g}})$ are independent under the Yang-Mills measure. □

## 3.2 Small scale structure of the Yang-Mills field

### 3.2.1 Extraction of a white noise

In the first part of this chapter, we explained how the data of a white noise on $M$ and a bit more alea allows to reconstruct the Yang-Mills measure. We proceed now backwards: we try to extract a white noise from the Yang-Mills measure on a surface. In some sense, this amounts to compute the curvature of a Yang-Mills random connection.

As usual, $(M, \sigma)$ is given, as well as elements $x_1, \ldots, x_p$ of $G$, associated with the components of $\partial M$ and $x = x_1 \ldots x_p$ or $x = 1$ if $M$ has no boundary. We denote by $(\Omega_M, \mu_M(x_1, \ldots, x_p))$ the space of the Yang-Mills measure on $M$.

In order to study the measure at small scale, we construct on $M$ a sequence of partitions in the following way. Let $(\Gamma_n)_{n\geq 0}$ be a sequence of graphs on $M$ such that $\Gamma_n$ has exactly $n$ faces denoted by $F_{j,n}$, $j = 1, \ldots, n$. We assume that $\sigma(F_{j,n}) = \frac{\sigma(M)}{n}$ and also that the diameter of the faces decreases uniformly to 0, i.e. that for any metric on $M$, $\sup_j \text{diam}(F_{j,n}) \longrightarrow 0$. We fix an orientation of $M$ and assume that the boundaries of the $F_{j,n}$'s are oriented with the usual convention. For each couple $(j, n)$ with $n \geq 0$, $1 \leq j \leq n$, we denote the random variable $\mathcal{H}_{\partial F_{j,n}}$ defined on $(\Omega_M, \mu_M(x_1, \ldots, x_p))$ by $\mathcal{H}_{j,n}$ and see it as a $\mathbf{C}$-valued random variable, identifying $U(1)$ with $\{z \in \mathbf{C}, |z| = 1\}$.

For each $n \geq 0$, let $E_n$ denote the space of functions on $M$ constant on each face of $\Gamma_n$. Set $E_\infty = \cup_n E_n$. The assumption on the diameter of the faces $F_{j,n}$ imply that any continuous function on $M$ can be uniformly approximated by functions of $E_\infty$. Thus, $E_\infty$ is dense in $L^1(M, \sigma)$ and $L^2(M, \sigma)$ with their respective usual norms.

In order to define a kind of white noise, we will proceed as for the construction of the standard Wiener integral. We define a linear form $I_n$ on each $E_n$. Let $f_n$ be a function of $E_n$ and let $f_{j,n}$ be its value on $F_{j,n}$. We set

$$I_n(f_n) = \frac{1}{i} \sum_{j=1}^{n} f_{j,n}(\mathcal{H}_{j,n} - 1).$$

**Theorem 3.2.1** *Let $f$ be a square-integrable function on $M$ and $(f_n)_{n\geq 0}$ a sequence of functions converging to $f$ in $L^2$ norm and such that $f_n \in E_n$. Then the sequence $(I_n(f_n))_{n\geq 0}$ converges in $L^2(\Omega_M, \mu_M(x_1, \ldots, x_p))$ to a random variable $I(f)$ that does not depend on the choice of the sequence $(f_n)$. The law of this random variable can be described in the following way. Let $W_f^0$ be a centered gaussian random variable with variance $\| f \|_{L_0^2}^2 = \| f - \frac{1}{\sigma(M)} \int_M f d\sigma \|_{L^2}^2$. Let $T$ be a $\mathcal{N}(0, \sigma(M))$ random variable conditioned to take its values in $\exp^{-1}(x)$, independent of $W_f^0$. Then, the following identity holds in distribution:*

$$I(f) \stackrel{\text{law}}{=} W_f^0 + \left(T + \frac{i}{2}\right) \frac{1}{\sigma(M)} \int_M f \, d\sigma. \tag{3.1}$$

This proves in particular that the law of $I(f)$ does not depend on the choice of the orientation of $M$.

*Proof*: To prove this theorem, it is convenient to use the white noise realization of the Yang-Mills measure. Let $(\Omega, P)$ be a probability space on which a pair $(W, T)$ is defined, consisting in a white noise $W$ and a random variable $T$ independent of $W$, whose law is that described in the theorem. We do not need the variables $U_i$, because we are only computing the holonomy along loops that are homologous to zero. Set

$$Y_{j,n} = W(\mathbf{1}_{F_{j,n}}) \ , \ S_n = \sum_{j=1}^{n} Y_{j,n} \ , \ X_{j,n} = Y_{j,n} - \frac{1}{n} S_n.$$

We know by the theorem 3.1.11 that the law of the sequence $(I_n(f_n))$ can be represented on

$(\Omega, P)$ by

$$\left(\frac{1}{i}\sum_{j=1}^{n} f_{j,n}\left(e^{i(X_{j,n}+\frac{T}{n})}-1\right)\right)_{n\geq 0}.$$

We will prove the theorem for this sequence. For this, we study the following Lagrange inequality:

$$\left|\sum_{j=1}^{n} f_{j,n}\left(e^{i(X_{j,n}+\frac{T}{n})}-1\right) - i\sum_{j=1}^{n} f_{j,n}\left(X_{j,n}+\frac{T}{n}\right) + \frac{1}{2}\sum_{j=1}^{n} f_{j,n}\left(X_{j,n}+\frac{T}{n}\right)^2\right| \leq \sum_{j=1}^{n} |f_{j,n}|\left|X_{j,n}+\frac{T}{n}\right|^3 \quad (3.2)$$

We will often use of the following lemma:

**Lemma 3.2.2** *For each positive integer $p$, there exists a constant $C_p$ such that*

$$E|X_{j,n}|^p \leq \frac{C_p}{n^{\frac{p}{2}}}.$$

*Proof*: This is just a consequence of the fact that a centered gaussian random variable $X$ of variance $t$ satisfies $E|X|^p = C_p t^{\frac{p}{2}}$ for come constant $C_p$ independent of $t$ and that $X_{j,n}$ has variance $\frac{\sigma(M)}{n} - \frac{\sigma(M)}{n^2}$. □

We begin by showing that the right hand side term converges to zero in $L^2(\Omega, P)$.y

$$E\left|X_{j,n}+\frac{T}{n}\right|^6 \leq \sum_{k=0}^{6}\binom{6}{k} E|X_{j,n}|^{6-k} E\frac{|T|^k}{n^k} \leq \frac{K}{n^3},$$

so that

$$\begin{aligned} E\left|\sum_{j=1}^{n} |f_{j,n}|\left|X_{j,n}+\frac{T}{n}\right|^3\right|^2 &= \sum_{j,k=1}^{n} |f_{j,n}||f_{k,n}| E\left|X_{j,n}+\frac{T}{n}\right|^3 \left|X_{k,n}+\frac{T}{n}\right|^3 \\ &\leq \left(\sup_{j=1}^{n} n \parallel (X_{j,n}+\frac{T}{n})^3 \parallel_{L^2}\right)^2 \left(\sum_{j=1}^{n}\frac{|f_{j,n}|}{n}\right)^2 \\ &\leq \frac{1}{n}\parallel f_n \parallel_{L^1}^2 \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now look at the second order term of the left hand side of (3.2). Let $m_n = \frac{1}{\sigma(M)}\int_M f_n \, d\sigma$ denote the mean of $f_n$ and $f_n^0 = f_n - m_n$ denote its zero-mean part. We will use several times the fact that $\sum_j f_{j,n}^0 = 0$. We have

$$\sum_{j=1}^{n} f_{j,n}(X_{j,n}+\frac{T}{n})^2 = \sum_{j=1}^{n} f_{j,n}^0(X_{j,n}+\frac{T}{n})^2 + m_n\sum_{j=1}^{n}(X_{j,n}+\frac{T}{n})^2. \quad (3.3)$$

Let us study the first term of this decomposition. In all estimations, $C$ denotes a constant, i.e. a number that depends neither on $j$ nor on $n$. It may denote different constants at different lines.

$$\sum_{j=1}^{n} f_{j,n}^0 (X_{j,n} + \frac{T}{n})^2 = \sum_{j=1}^{n} f_{j,n}^0 X_{j,n}^2 + 2\sum_{j=1}^{n} \frac{f_{j,n}^0}{n} X_{j,n} T. \tag{3.4}$$

The first term of the right hand side term can be written:

$$\sum_{j=1}^{n} f_{j,n}^0 X_{j,n}^2 = \sum_{j=1}^{n} f_{j,n}^0 (Y_{j,n} - \frac{1}{n} S_n)^2 = \sum_{j=1}^{n} f_{j,n}^0 Y_{j,n}^2 - 2\sum_{j=1}^{n} \frac{f_{j,n}^0}{n} Y_{j,n} S_n.$$

On one hand,

$$E\left|\sum_{j=1}^{n} f_{j,n}^0 Y_{j,n}^2\right|^2 = E\left|\sum_{j=1}^{n} \frac{f_{j,n}^0}{n} n Y_{j,n}^2\right|^2 = \sum_{j=1}^{n} \frac{|f_{j,n}^0|^2}{2^{2n}} E|nY_{j,n}^2|^2 \leq \frac{C}{n} \parallel f_n^0 \parallel_{L^2} \underset{n\to\infty}{\longrightarrow} 0,$$

since $E|nY_{j,n}|^2$ depends neither on $j$ nor on $n$.

On the other hand,

$$\parallel Y_{j,n} S_n \parallel_{L^2}^2 = E(Y_{j,n}^2 S_n^2) \leq n(EY_{j,n}^2)^2 + EY_{j,n}^4 \leq \frac{C}{n}$$

implies

$$\parallel \sum_{j=1}^{n} \frac{f_{j,n}^0}{n} Y_{j,n} S_n \parallel_{L^2} \leq \sum_{j=1}^{n} \frac{|f_{j,n}^0|}{n} \parallel Y_{j,n} S_n \parallel_{L^2} \leq \frac{C}{2^{\frac{n}{2}}} \parallel f_n^0 \parallel_{L^1} \underset{n\to\infty}{\longrightarrow} 0.$$

We proved that the first term of the r.h.s. of (3.4) tends to 0. To study the second one, note that

$$\parallel X_{j,n} T \parallel_{L^2}^2 = EX_{j,n}^2 ET^2 \leq \frac{C}{n},$$

so that

$$\parallel \sum_{j=1}^{n} \frac{f_{j,n}^0}{n} X_{j,n} T \parallel_{L^2} \leq \sum_{j=1}^{n} \frac{|f_{j,n}^0|}{n} \parallel X_{j,n} T \parallel_{L^2} \leq \frac{C}{\sqrt{n}} \parallel f_{j,n}^0 \parallel_{L^1} \underset{n\to\infty}{\longrightarrow} 0.$$

We proved that the zero-mean part of $f_n$ does not contribute to the second order term. Let us study the last term of (3.3).

$$m_n \sum_{j=1}^{n} \left(X_{j,n} + \frac{T}{n}\right)^2 = m_n \sum_{j=1}^{n} X_{j,n}^2 + \frac{m_n T}{2^{n-1}} \sum_{j=1}^{n} X_{j,n} + \frac{m_n}{n} T^2.$$

We have $\sum_j X_{j,n} = 0$ a.s. and $\frac{m_n}{n} T^2 \longrightarrow 0$ a.s. . It remains

$$m_n \sum_{j=1}^{n} X_{j,n}^2 = m_n \sum_{j=1}^{n} Y_{j,n}^2 + \frac{m_n}{n} S_n^2 - \frac{m_n}{2^{n-1}} S_n^2.$$

Since the law of $S_n^2$ does not depend on $n$, the two last terms tend to zero. In order to determine the limit of the first one, we compute

$$E\left|\sum_{j=1}^{n} Y_{j,n}^2 - \frac{1}{n}\right|^2 = \sum_{j=1}^{n} E\left|Y_{j,n}^2 - \frac{1}{n}\right|^2 = \sum_{j=1}^{n} E\left(Y_{j,n}^4 + \frac{1}{n^2} - \frac{2Y_{j,n}^2}{n}\right) = \sum_{j=1}^{n} \frac{C}{n^2} + \frac{1}{n^2} - \frac{2}{n^2} \leq \frac{C}{n^2}.$$

Thus,

$$m_n \sum_{j=1}^{n} Y_{j,n}^2 = \sum_{j=1}^{n} (Y_{j,n}^2 - \frac{1}{n}) + m_n \xrightarrow[n\to\infty]{L^2} \lim_{n\to\infty} m_n = \frac{1}{\sigma(M)} \int_M f \, d\sigma.$$

We are done with the second order term. We finish the proof by studying the first order one.

$$\begin{aligned}
\sum_{j=1}^{n} f_{j,n} \left( X_{j,n} + \frac{T}{n} \right) &= \sum_{j=1}^{n} f^0 j, n X_{j,n} + m_n T \\
&= m_n T + \sum_{j=1}^{n} f_{j,n}^0 Y_{j,n} \\
&= m_n T + W(\sum_{j=1}^{n} f_{j,n}^0 \mathbf{1}_{F_{j,n}}) \\
&= W(f_n^0) + m_n T \xrightarrow[n\to\infty]{L^2} W(f^0) + \frac{T}{\sigma(M)} \int_M f \, d\sigma.
\end{aligned}$$

We have proved that

$$\frac{1}{i} \sum_{j=1}^{n} f_{j,n} (e^{i(X_{j,n} + \frac{T}{n})} - 1) \xrightarrow[n\to\infty]{L^2} W(f^0) + \left( T + \frac{i}{2} \right) \frac{1}{\sigma(M)} \int_M f \, d\sigma.$$

This limit does not depend on the choice of the sequence $(f_n)$. Thus the sequence $(I_n(f_n))$ converges also to a limit $I(f)$ that does not depend on the choice of $(f_n)$ and whose law is the law announced in the theorem. □

### 3.2.2 Meaning of the variable $T$

As a conclusion for this chapter, we will spend a few lines to suggest a geometric interpretation for the variable $T$, whose meaning could seem to be quite mysterious.

In a deterministic setting with a smooth connection $\omega$, a construction similar to that of the map $f \mapsto I(f)$ would have given the map:

$$f \mapsto \int_M f F(\omega),$$

where $F(\omega)$ is the curvature 2-form of $\omega$. As long as we consider zero-mean functions, the comparison between this formula and (3.2) is in agreement with the heuristic principle saying that the curvature of a Yang-Mills random connection is a white noise, as explained in the introduction.

If we take the function $f$ identically equal to 1 in the deterministic setting, we get the total curvature $\int_M F(\omega)$ of the fiber bundle $P$ on which $\omega$ lives. This quantity is well known to be independent of $\omega$ and to be a topological invariant of $P$, namely its first Chern class. The probabilistic counterpart of this total curvature seems then to be $I(1) = T$, droping out the imaginary part. This discussion becomes really meaningful when $M$ is closed, because $P$ is not necessarily trivial. We mentioned at the beginning of the discretization procedure in section 1.3 that we had lost any topological information about the structure of $P$. If we compute the "random Chern class" of $P$ at the end of the construction, we find a weighted sum of all possible Chern classes, with the smallest weights for the most complicated types of bundles. This was already suggested by Witten [Wi].

On the other hand, we can change our point of view in the following way: we have an expression of the random holonomy which depends explicitely on the Chern class of $P$. So if we replace $T$ by a deterministic multiple of $2\pi$ in the definition 3.1.1, we are able to construct a random holonomy consistent with any prescribed type of bundle $P$.

## 3.3 Square-integrability of the double-layer potential

In this section, we prove the theorem 1.9.11. We claim that it is enough to prove the theorem on closed surfaces. Indeed, we did not use the square-integrability of the double-layer potential to prove the results 3.1.4 and 3.1.5, which show that the result on a surface with boundary can be deduced from the result on a minimal closure of this surface. Thus, we assume that $M$ is closed.

**Proposition 3.3.1** *There exists $R_F > 0$ such that for all embedded path $c \in PM$ such that $\ell(c) < \frac{1}{4}R_F$, the double layer potential $u_c$ of $c$ is in $L^\infty(M)$.*

This proposition implies obviously the theorem. It implies even more, namely that the double layer potential of any path is in $L^\infty$.

*Proof*. The proof relies on three facts. The first one is that we know the divergence near the diagonal of the Green function in an open subset of $\mathbf{R}^2$. The second one is that, according to a classical theorem due to Gauss [Ch], any metric on $M$ is locally conformally flat. The third point is that the Green function is conformally invariant.

Since $M$ is compact, the second remark implies that there exists a radius $R_F$ such that any geodesic ball of $M$ of radius smaller than $R_F$ is conformally flat. Let us choose an embedded path $c$ of length smaller than $R_F/4$. For each $r > 0$, we denote by $B_r$ the ball $B(c(0), r)$. Since $\ell(c)$ is smaller than $R_F/4$, $c$ is contained in $B_{\frac{1}{2}R_F}$. Since the Green function $G$ is smooth outside the diagonal, $u_c$ is smooth outside $B_{\frac{1}{2}R_F}$. It is enough now to prove that it is bounded on $B_{\frac{3}{4}R_F}$ for example. Set $r = \frac{3}{4}R_F$.

The values of $u_c$ inside $\overline{B}_r$ depend only on the restriction of $G$ to $\overline{B}_r \times \overline{B}_r$. On this set, $G$ satisfies $\Delta G_x = \delta_x - \frac{1}{\sigma(M)}$. Our idea is to substract smooth functions to $G$ until we get something easier to compute than $G$ itself. Denote by $G^0$ the solution of:

$$\begin{cases} \Delta G^0_x = -\frac{1}{\sigma(M)} \\ G^0_x(y) = G_x(y) \ \ \forall y \in \partial B_r. \end{cases}$$

It is a smooth function inside $B_r$. The function $G^1$ defined by $G^1 = G - G^0$ satisfies

$$\begin{cases} \Delta G^1_x = \delta_x \\ G^1_x(y) = 0 \ \ \forall y \in \partial B_r. \end{cases}$$

So, $G^1$ is the fundamental solution of $\Delta$ inside $B_r$ with Dirichlet boundary conditions. We can decompose $u_c(x)$ for any $x$ in $B_r$ according to:

$$u_c(x) = \int_c *dG^0_x + \int_c *dG^1_x = u^0_c(x) + u^1_c(x).$$

The first term is smooth and we are led to study the second. This is where conformally flat coordinates are useful: we choose a local chart $\varphi : U \longrightarrow B_r$, where $U$ is an open subset of $\mathbf{R}^2$, such that the pull-back of the metric $g$ of $M$ by $\varphi$ is conformally equivalent to $dx^2 + dy^2$ on $U$. The point is that $\varphi^* G^1$ is not only the fundamental solution of $\Delta_{\varphi^* g}$ with respect to the measure induced by $\varphi^* g$, but also the fundamental solution of $\Delta_0 = \partial_x^2 + \partial_y^2$ with respect to the flat metric on $U$, by conformal invariance of the Green function. This tells us that $\varphi^* G^1$ diverges like $\frac{1}{4\pi} \log d(\cdot, \cdot)$ on the diagonal. In other words, there exists a smooth function $G^2$ on $B_r \times B_r$ such that $\varphi^* G^1 = \frac{1}{4\pi} \log d(\cdot, \cdot) + \varphi^* G^2$.

What we want is to prove that $u_c^1(x)$ is bounded inside $B_r$. It is equivalent to prove that $u_c^1 \circ \varphi$ is bounded on $U$. But for any $y \in U$,

$$u_c^1 \circ \varphi(y) = \int_c *dG^1_{\varphi(y)} = \int_{\varphi^{-1}(c)} *d(\varphi^* G^1_y).$$

Note that in this last term, the Hodge operator $*$ is that of the metric $\varphi^* g$. It is not the same operator as that of the flat metric on $U$. Fortunately, the fact that these two metrics are conformally equivalent implies that their Hodge operators are pointwise proportionnal, i.e. one is deduced from the other by the multiplication by a positive bounded smooth function. Thus, it is sufficient to prove that $y \mapsto \int_{\varphi^{-1}(c)} *d\varphi^* G^1_y$ is bounded, the Hodge operator being now that of the flat metric. As already noticed, we can remove a smooth part of $G^1$ and keep only the part

$$\frac{1}{4\pi} \int_{\varphi^{-1}(c)} *d \log d(y, \cdot).$$

A short computation shows that $1/4\pi * d \log d(y, \cdot)$ is nothing but $1/2\pi$ times the angle form $d\theta$ of the polar coordinates centered at $y$. This allows us to estimate very easily the integral of this form along a path.

For example, we know that the integral of this form along a simple loop is bounded by 1. On the other hand, it is obvious and easy to prove by a direct computation that the integral of this form along a straight segment is bounded by 1/2.

Consider the path $\varphi^{-1}(c)$. It is injective, hence it is possible to transform it into a simple loop by concatenating it with a finite number of segments. So, we can make the function that we want to estimate to be bounded by adding to it a finite number of bounded functions. This gives the result. □

# Chapter 4

# Small scale structure in the semi-simple case

The theorem 3.1.11 shows that it is possible to construct the Yang-Mills measure in a short and quite pleasant way when $G = U(1)$, using a white noise on $M$ as main ingredient. Is it possible to do something similar in general? The works of Sengupta and Driver [Dr2, Se1, Se2] lead to an ambiguous answer to that question. Indeed, in these works, the authors have constructed random holonomies, starting from a Lie algebra-valued white noise on $M$. Nevertheless, the family of loops along which they are able to define the holonomy is strongly dependent of a particular choice of coordinates on $M$, as we explained in the introduction. We think that this is more than a simple technical problem. Although there might exist some generalization of the construction made in section 3.1, we will show that a white noise is probably not the right object to start with.

Our idea is the following. Was it possible to realize the random holonomy using a white noise, it would be possible to find a lot of information by looking at the random holonomy at small scale, i.e. along very small loops. For example, the theorem 3.2.1 basically says that when $G = U(1)$, almost all the information about the holonomy along homologically trivial loops is available at infinitesimally small scale. We prove that, when $G$ is semi-simple, there is no information at all available at infinitesimally small scale, at least when one looks at it in the same way that we did in the Abelian case.

## 4.1 Statement of a zero-one law

We begin by stating the main result. The surface is $(M, \sigma)$ as usual. We assume that $G$ is a compact connected semi-simple Lie group, for example $SU(2)$. We choose $x_1, \ldots, x_p$ in $G$ and consider the probability space $(\Omega_M, \mu_M(x_1, \ldots, x_p))$.

Let $L$ be a simple loop on $M$ which is the boundary of an open set $D$ diffeomorphic to a disk. For each $n \geq 0$, consider a graph on $D$ which has exactly $n$ faces $F_{1,n}, \ldots, F_{n,n}$ such that $\sigma(F_{i,n}) = \frac{\sigma(D)}{n}$ for each $i$. This is very similar to the situation described in the section 3.2.

Was $G$ Abelian, we would have the equality of cycles $L = \partial F_{1,n} + \ldots + \partial F_{n,n}$, provided orientations are well chosen. This would imply $\mathcal{H}_L = \mathcal{H}_{\partial F_{1,n}} \ldots \mathcal{H}_{\partial F_{n,n}}$ and for any function $f$ continuous on $G/\operatorname{Ad} = G$,

$$E[f(\mathcal{H}_L) | \mathcal{H}_{\partial F_{1,n}}, \ldots, \mathcal{H}_{\partial F_{n,n}}] = f(\mathcal{H}_L).$$

When $G$ is semi-simple, the situtation is the opposite.

**Theorem 4.1.1** *For any function $f$ continuous on $G/\mathrm{Ad}$, the following convergence holds:*

$$E[f(\mathcal{H}_L)|\mathcal{H}_{\partial F_{1,n}},\ldots,\mathcal{H}_{\partial F_{n,n}}] \xrightarrow[n\to\infty]{L^2} Ef(\mathcal{H}_L).$$

## 4.2 Proof of the zero-one law

### 4.2.1 Computation of the conditional expectation

In this section, we will compute the conditional expectation appearing in the statement of the theorem, keeping $n$ fixed. We abbreviate $F_{i,n}$ in $F_i$.

For each $F_i$, consider a sequence $(L_{i,k})_{k\geq 0}$ of simple loops whose image is inside the interior of $F_i$ and such that $L_{i,k} \xrightarrow[k\to\infty]{d_1} \partial F_i$. The proposition 2.10.8 shows that the following convergence holds in probability:

$$(\mathcal{H}_{L_{1,k}},\ldots,\mathcal{H}_{L_{n,k}}) \xrightarrow[k\to\infty]{P} (\mathcal{H}_{\partial F_1},\ldots,\mathcal{H}_{\partial F_n}).$$

Let $f$ and $f_1$ be continuous functions on $G/\mathrm{Ad}$ and $(G/\mathrm{Ad})^n$ respectively. This convergence implies the following one:

$$E[f(\mathcal{H}_L)f_1(\mathcal{H}_{\partial F_1},\ldots,\mathcal{H}_{\partial F_n})] = \lim_{k\to\infty} E[f(\mathcal{H}_L)f_1(\mathcal{H}_{L_{1,k}},\ldots,\mathcal{H}_{L_{n,k}})].$$

We are led to the computation of the second expectation, keeping $k$ fixed. We abreviate temporarily $L_{i,k}$ by $L_i$.

We construct a particular graph on $M$ such that $L, L_1,\ldots,L_n \in \Gamma^*$ (see fig. 4.1). Outside $D$, it has only one face. Its support contains the components $N_1,\ldots,N_p$ of $\partial M$, paths $c_1,\ldots,c_p$ joining $L(0)$ to the $N_i$'s and simple loops $a_1,\ldots,a_g,b_1,\ldots,b_g$ that represent a basis of the $H_1$ of a minimal closure of $M$.

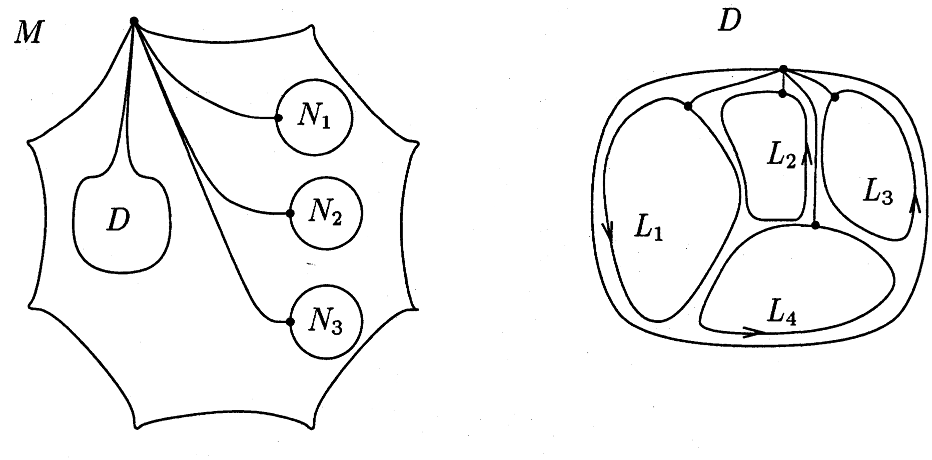


Figure 4.1: Aspect of $\Gamma$ outside and inside $D$.

The boundary of the unique face outside $D$ is $L^{-1}c_1N_1c_1^{-1}\ldots c_pN_pc_p^{-1}[a_1,b_1]\ldots[a_g,b_g]$, where $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$ of $a$ and $b$. This notation is the same as that of joint conjugacy classes, but the context will always make our meaning clear. Inside $D$, the support of $\Gamma$ contains $n$ paths $d_1,\ldots,d_n$ joining $L(0)$ to the $L_i(0)$. These paths meet pairwise only at $L(0)$. Inside $D$, this graph has $n+1$ faces: $n$ whose boundaries are the $L_i$'s and the last with boundary $Ld_nL_n^{-1}d_n^{-1}\ldots d_1L_1^{-1}d_1^{-1}$. We denote by $t$ the surface of this last face and, for each $i=1,\ldots,n$, by $t_i$ the surface of the face bounded by $L_i$.

We compute $E[f(\mathcal{H}_L)f_1(\mathcal{H}_{L_1},\ldots,\mathcal{H}_{L_n})]$ in this graph. It is equal to

$$\frac{1}{Z(x_1,\ldots,x_p)}\int_{G^\Gamma} f([h_L])f_1([h_{L_1}],\ldots,[h_{L_n}])p_{t_1}(h_{L_1})\ldots p_{t_n}(h_{L_n})$$
$$p_t(h_{d_1}^{-1}h_{L_1}^{-1}h_{d_1}\ldots h_{d_n}^{-1}h_{L_n}^{-1}h_{d_n}h_L)p_{\sigma(D^c)}([h_{b_g}^{-1},h_{a_g}^{-1}]\ldots[h_{b_1}^{-1},h_{a_1}^{-1}]h_{c_p}^{-1}h_{N_p}h_{c_p}\ldots h_{c_1}^{-1}h_{N_1}h_{c_1}h_L^{-1})$$
$$d\nu_{x_1}\ldots d\nu_{x_p}dg'$$
$$= \frac{1}{Z(x_1,\ldots,x_p)}\int_{G^{2(g+n)+p+1}} f([g])f_1([g_1],\ldots,[g_n])p_{t_1}(g_1)\ldots p_{t_n}(g_n)$$
$$p_t(y_1^{-1}g_1^{-1}y_1\ldots y_n^{-1}g_n^{-1}y_ng)p_{\sigma(D^c)}([a_g,b_g]\ldots[a_1,b_1]z_p^{-1}x_pz_p\ldots z_1^{-1}x_1z_1g^{-1})$$
$$dg\,dg_1\ldots dg_n\,dy_1\ldots dy_n\,dz_1\ldots dz_p\,da_1\ldots da_g\,db_1\ldots db_g.$$

We used the fact that under $\nu_{x_1}\ldots\nu_{x_p}dg'$, $h_{N_i}=x_i$ a.s. and $h_L$ and all $h_{L_i}$, $h_{a_i}$, $h_{b_i}$, $h_{c_i}$,$h_{d_i}$ are uniform and independent. When $k$ tends to infinity, each $t_i$ tends to $\frac{\sigma(D)}{n}$ and $t$ tends to zero. So, according to 1.5.1, we can drop $g$ and replace it by $y_n^{-1}g_ny_n\ldots y_1^{-1}g_1y_1$. This terminates the proof of the following proposition:

**Proposition 4.2.1** *The following equality holds:*

$$E[f(\mathcal{H}_L)|\mathcal{H}_{\partial F_{1,n}},\ldots,\mathcal{H}_{\partial F_{n,n}}]=$$
$$\left(\int p_{\sigma(D^c)}([a_g,b_g]\ldots[a_1,b_1]z_p^{-1}x_pz_p\ldots z_1^{-1}x_1z_1y_1^{-1}K_1y_1\ldots y_n^{-1}K_ny_n)\;dy_idz_ida_idb_i\right)^{-1}\cdot$$
$$\int f(y_n^{-1}K_ny_n\ldots y_1^{-1}K_1y_1)p_{\sigma(D^c)}([a_g,b_g]\ldots[a_1,b_1]z_p^{-1}x_pz_p\ldots z_1^{-1}x_1z_1y_1^{-1}K_1^{-1}y_1\ldots y_n^{-1}K_n^{-1}y_n)$$
$$dy_idz_ida_idb_i$$

*where $K_1,\ldots,K_n$ are arbitrary representatives of $\mathcal{H}_{\partial F_{1,n}},\ldots,\mathcal{H}_{\partial F_{n,n}}$.*

Since we are dealing with functions on $G/\operatorname{Ad}$, it is natural to use the theory of characters on $G$. We give a very short account of the results that we will use. For a detailed presentation of the subject, see for example [Br, Si].

### 4.2.2 Characters of a semi-simple Lie group

A representation of $G$ is a smooth morphism of groups $\rho$ from $G$ into the linear group of some $\mathbf{C}^n$. The integer $n$ is the dimension of $\rho$. Since $G$ is compact, we may always assume that $\rho(G)$ is included in the unitary group. A representation is said to be irreducible if there are no subspaces of $\mathbf{C}^n$ invariant by all $\rho(g)$ except $\mathbf{C}^n$ and 0. Two representations $\rho_1$ and $\rho_2$ of same dimension $n$ are said to be equivalent if there exists a linear isomorphism $\phi$ of $\mathbf{C}^n$ such that $\rho_1\circ\phi=\phi\circ\rho_2$. . The character of $\rho$ is the $\mathbf{C}$-valued function $\chi_\rho$ defined on $G$ by $\chi_\rho(g)=\operatorname{tr}\rho(g)$. Two equivalent representations have the same character. In fact, this is also a sufficient condition of equivalence. The usual properties of the trace imply that it is a central function on $G$, that is, $\chi_\rho(g)$ depends only on the conjugacy class of $g$. Since all representations are unitary, the relation $\chi_\rho(g^{-1}) = \chi_\rho(g)^*$ holds, the star denoting the complex conjugation. Note that $\chi_\rho(1) = \dim \rho$. The main theorem is the following:

**Theorem 4.2.2 (Peter-Weyl theorem)** *The set of characters of equivalence classes of irreducible representations of $G$ is an orthonormal basis of the space of central square-integrable functions on $(G, dg)$. Moreover, the algebra generated by this set is dense in the set of continuous $\mathbf{C}$-valued central functions on $G$ endowed with the uniform norm.*

According to this theorem, any continuous function can be approximated by linear combinations of products of characters. But it is a fact that such combinations can always be written as linear combinations of characters of irreducible representations. Thus it is sufficient to prove our theorem when $f$ is the character of an irreducible representation.

Characters satisfy orthogonality relations that give rise to useful formulas. We will mainly use two of them.

**Proposition 4.2.3** *For any $x, y, z \in G$ and for any irreducible representation $\alpha$,*

$$\int_G \chi_\alpha(xyx^{-1}z)\, dx = \frac{1}{\dim \alpha}\chi_\alpha(x)\chi_\alpha(y), \tag{4.1}$$

$$\int_{G^2} \chi_\alpha([a,b]x)\, da db = \frac{1}{(\dim \alpha)^2}\chi_\alpha(x). \tag{4.2}$$

Let us endow $G$ with its biinvariant metric normalized to have total volume equal to 1. This metric gives rise to a Laplace operator on $G$. A remarkable property of the characters is that they are eigenfunctions for this operator. More precisely, for any irreducible representation $\alpha$, there exists a positive real number $c_2(\alpha)$ such that

$$\Delta\chi_\alpha = -c_2(\alpha)\chi_\alpha.$$

A nice application of these properties is the computation of the character expansion of the heat kernel on $G$. Let us denote by $\widehat{G}$ the set of classes of irreducible representations of $G$.

**Proposition 4.2.4** *The following equality holds in $L^2(G, dg)$ and also pointwise on $G$:*

$$p_t = \sum_{\beta\in\widehat{G}} \dim \beta\, e^{-\frac{c_2(\beta)}{2}t}\chi_\beta.$$

*Proof*: We first prove the $L^2$ convergence. For any $t > 0$, the function $p_t$ is a central $L^2$ function on $G$. Thus it admits a decomposition

$$p_t = \sum_{\beta\in\widehat{G}} C_\beta(t)\chi_\beta.$$

The differential equation $(\frac{1}{2}\Delta - \partial_t)p_t = 0$ implies that $(\frac{1}{2}c_2(\beta) + \partial_t)C_\beta(t) = 0$. Thus, $C_\beta(t) = C_\beta(0)e^{-c_2(\beta)\frac{t}{2}}$. The constants $C_\beta(0)$ are determined using the fact that $p_t$ tends to $\delta_1$ as $t$ tends to 0. It is easily checked that $C_\beta(0) = \dim \beta$ is a convenient choice. Formally, it amounts to check that $\sum_\beta \dim \beta\chi_\beta = \delta_1$.

To see that the convergence holds pointwise, note that the expansion of $p_t$ is a series of continuous functions that converges normally. Since $p_t$ is continuous, it is not only the sum of this series in the $L^2$ sense, but also in the sense of the uniform convergence. □

### 4.2.3 Character computations

We go back to the big expression obtained in proposition 4.2.1. From now on, we fix an irreducible representation $\alpha$ and put $f = \chi_\alpha$. We compute the numerator $N_n$ of the conditional expectation. The computations using characters presented in this section and the next one are very close to those done by Witten in [Wi], when he expands explicitely partition functions in order to compute the symplectic volume of the moduli space of flat connections.

We begin by developing the heat kernel $p_{\sigma(D^c)}$ using proposition 4.2.4. We get a sum over $\beta$ of integrals of $\chi_\alpha$ of something times $\chi_\beta$ of something else. We integrate over the variables that appear only as arguments of $\chi_\beta$, first $a_1$ and $b_1$, and so on until $a_g$ and $b_g$, using (4.2). Each integration against $a_i$ and $b_i$ produces a factor $\frac{1}{(\dim\beta)^2}$. Then we integrate against $z_1, \ldots, z_n$ using (4.1). Each integration gives a factor $\chi_\beta(x_i)/\dim\beta$. At this stage, the arguments of $\chi_\alpha$ and $\chi_\beta$ are inverse of each other. Using the relation $\chi_\beta(g^{-1}) = \chi_\beta^*(g)$, we obtain:

$$N_n = \sum_{\beta\in\widehat{G}} (\dim\beta)^{1-2g-p}\, e^{-\frac{c_2(\beta)}{2}\sigma(D^c)} \prod_{i=1}^{p} \chi_\beta(x_i) \int_{G^n} \chi_\alpha\chi_\beta^*(y_n^{-1}K_n y_n \ldots y_1^{-1}K_1 y_1)\, dy_1\ldots dy_n.$$

We use the formal developement of the central function $\chi_\alpha\chi_\beta^* = \sum_{\gamma\in\widehat{G}}(\chi_\alpha\chi_\beta^*, \chi_\gamma)_{L^2}\,\chi_\gamma$. We have now a sum over $\beta$ and $\gamma$ but only $\chi_\gamma$ under the integral. We integrate against $y_1, \ldots, y_n$ using (4.1). Note that the factor $\frac{1}{\dim\gamma}$ produced by the last integration cancels out with the remaining $\chi_\gamma(1)$ . We find

$$N_n = \sum_{\beta,\gamma\in\widehat{G}} (\dim\beta)^{1-2g-p}(\dim\gamma)^{-(n-1)}\, e^{-\frac{c_2(\beta)}{2}\sigma(D^c)}(\chi_\alpha\chi_\beta^*, \chi_\gamma)_{L^2} \prod_{i=1}^{p}\chi_\beta(x_i)\prod_{i=1}^{n}\chi_\gamma(K_n), \quad (4.3)$$

with $(\chi_\alpha\chi_\beta^*, \chi_\gamma)_{L^2} = \int_G \chi_\alpha(g)\chi_\beta^*(g)\chi_\gamma^*(g)\, dg = (\chi_\alpha, \chi_\beta\chi_\gamma)_{L^2}$.

A similar and simpler computation leads to the following expression for the denominator $D_n$ of 4.2.1:

$$D_n = \sum_{\beta\in\widehat{G}} (\dim\beta)^{2-2g-p-n} e^{-\frac{c_2(\beta)}{2}\sigma(D^c)} \prod_{i=1}^{p}\chi_\beta(x_p)\prod_{i=1}^{n}\chi_\beta(K_i). \quad (4.4)$$

Remark that it is equivalent to evaluate a character at $K_i$ or on $\mathcal{H}_{\partial F_i}$. In order to prove the convergence, we need to know the asymptotic behaviour of an expression like

$$\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^{n}\chi_\beta(\mathcal{H}_{\partial F_i}).$$

The following result will be proved in the next section:

**Proposition 4.2.5** *For any $\beta \in \widehat{G}$, the following convergence holds:*

$$\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^{n}\chi_\beta(\mathcal{H}_{\partial F_i}) \xrightarrow[n\to\infty]{L^2} \dim\beta\, e^{-\frac{c_2(\beta)}{2}\sigma(D)}.$$

The inequality $|\chi_\beta(g)| \leq \dim\beta$ shows that the sequence is uniformly bounded by $\dim\beta$. This allows us to permute this convergence with the summation over $\beta$. We get the following $L^2$-limit for the numerator:

$$\sum_{\beta,\gamma\in\widehat{G}} (\dim\beta)^{1-2g-p} e^{-\frac{c_2(\beta)}{2}\sigma(D^c)} \prod_{i=1}^{p} \chi_\beta(x_p)\ (\chi_\alpha, \chi_\beta\chi_\gamma)_{L^2} \dim\gamma e^{-\frac{c_2(\gamma)}{2}\sigma(D)}.$$

By the same kind of arguments that we used to derive the expressions of $D_n$ and $N_n$, it is easy to check that this expression is equal to the following:

$$\int_{G^{p+2g+1}} \chi_\alpha(g) p_{\sigma(D)}(g^{-1}) p_{\sigma(D^c)}(g^{-1}z_1x_1z_1^{-1}\dots z_px_pz_p^{-1}[a_1,b_1]\dots[a_g,b_g])\, dg dz_i da_i db_i. \quad (4.5)$$

For the denominator, we find the following $L^2$-limit:

$$\sum_{\beta\in\widehat{G}} (\dim\beta)^{2-2g-p} e^{-\frac{c_2(\beta)}{2}(\sigma(D^c)+\sigma(D))} \prod_{i=1}^{p} \chi_\beta(x_p),$$

which is equal to:

$$\int_{G^{p+2g}} p_{\sigma(M)}(z_1x_1z_1^{-1}\dots z_px_pz_p^{-1}[a_1,b_1]\dots[a_g,b_g])\, dz_i da_i db_i \;=\; Z_M(x_1,\dots,x_p). \quad (4.6)$$

Using the fact that $p_{\sigma(D)}(g^{-1}) = p_{\sigma(D)}(g)$, we see that the quotient of (4.5) by (4.6) is equal to $E\chi_\alpha(\mathcal{H}_L)$. This proves the theorem, up to the proposition 4.2.5 that was admitted.

### 4.2.4 Zero-one law on the plane

In order to prove proposition 4.2.5, we begin with the following result, which can be seen as a reformulation of the zero-one law when the manifold $M$ is the plane $\mathbf{R}^2$.

**Proposition 4.2.6** *Let $(B_t^n)_{t\in\mathbf{R}_+,n\geq 0}$ be a sequence of independent Brownian motions on $G$. For any irreducible representation $\beta$ of $G$ and any positive real number $T$, the following convergence holds in distribution:*

$$\frac{1}{(\dim\beta)^{n-1}} \prod_{i=1}^{n} \chi_\beta(B^i_{\frac{T}{n}}) \xrightarrow[n\to\infty]{\text{law}} \dim\beta\, e^{-\frac{c_2(\beta)}{2}T}.$$

This result is really the center of the whole proof of the theorem. It is the place where the fact that $G$ is semi-simple will be used, in the following way. Given a representation $\rho$ of $G$, the differential at 1 of $\rho$ is a linear map from $T_1G$, the Lie algebra of $G$, into $\mathcal{L}(\mathbf{C}^n)$. The following statement is proved in Bourbaki (*Lie*, chap. I, § 6, N°2, corollary of the th. 1) [B2]:

**Proposition 4.2.7** *Let $G$ be a semi-simple group and $\rho$ a representation of $G$ of dimension $n$. Then*

$$d_1\rho(T_1G) \subset \mathfrak{sl}_n(\mathbf{C}),$$

*where $\mathfrak{sl}_n(\mathbf{C})$ denotes the set of endomorphisms of $\mathbf{C}^n$ whose trace is equal to zero.*

*Proof of proposition 4.2.6*: We consider a Brownian motion $(B_t)$ on $G$ and study the process $\chi_\alpha(B_t)$. There is a convenient way to represent $(B_t)$, as a solution of a Stratonovich stochastic differential equation [IW]. Recall that the data of a biinvariant metric on $G$ is equivalent to that of a scalar product on the Lie algebra of $G$, invariant by adjunction. Let $(X_1, \ldots, X_{\dim G})$ be a basis of $T_1G$ orthonormal for this scalar product. Each $X_i$ is seen as a left-invariant vector field on $G$. Let $W^1, \ldots, W^{\dim G}$ be independent real Brownian motions. Then the Brownian motion on $G$ satisfies:

$$\begin{cases} dB_t = \sum_{i=1}^{\dim G} X_i \circ dW_t^i \\ B_0 = 1 \end{cases} \tag{4.7}$$

The meaning of this notation is that, for any continuous function $f$ on $G$,

$$f(B_t) = f(1) + \sum_{i=1}^{\dim G} \int_0^t X_i f(B_s)\, dW_s^i + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds.$$

We apply this relation to $f = \chi_\alpha$. Using $\chi_\alpha(1) = \dim \alpha$ and $\Delta\chi_\alpha = -c_2(\alpha)\chi_\alpha$, it becomes:

$$\frac{\chi_\alpha(B_t)}{\dim \alpha} = 1 - \frac{c_2(\alpha)t}{2} + \frac{1}{\dim \alpha}\sum_{i=1}^{\dim G}\int_0^t X_i\chi_\alpha(B_s)\, dW_s^i - \frac{c_2(\alpha)}{2\dim \alpha}\int_0^t (\chi_\alpha(B_s) - \dim \alpha)\, ds.$$

Set

$$Y_t = -\frac{c_2(\alpha)}{2\dim \alpha}\int_0^t (\chi_\alpha(B_s) - \dim \alpha)\, ds,$$

$$Z_t = \frac{1}{\dim \alpha}\sum_{i=1}^{\dim G}\int_0^t X_i\chi_\alpha(B_s)\, dW_s^i.$$

We keep the notation $\rho(g)$ for the biinvariant distance $d(1, g)$ when $g \in G$. This $\rho$ has nothing to do with a representation of $G$ !

For any $X \in T_1G$, we have:

$$d_1\chi_\alpha(X) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{tr} \alpha(\exp tX) = \operatorname{tr} d_1\alpha(X) = 0,$$

according to 4.2.7. Thus the differential of $\chi_\alpha$ at 1 is zero. This implies that $|\chi_\alpha(g) - \dim \alpha| = O(\rho(g)^2)$ in a neighbourhood of 1. Using the lemma 1.8.3,we get:

$$\begin{aligned} E|Y_t|^2 &= CE\left|\int_0^t (\chi_\alpha(B_s) - \dim \alpha)\, ds\right|^2 \\ &\leq Ct\, E\int_0^t |\chi_\alpha(B_s) - \dim \alpha|^2\, ds \\ &\leq Ct\, E\int_0^t \rho(B_s)^4\, ds \\ &\leq Ct\int_0^t s^2\, ds \\ &\leq Ct^4. \end{aligned} \tag{4.8}$$

For each $i = 1, \ldots, \dim G$, the function $X_i\chi_\alpha$ is smooth and $X_i\chi_\alpha(1) = 0$. Thus, $|X_i\chi_\alpha(g)| = O(\rho(g))$ in a neighbourhood of 1. Thus,

$$\begin{aligned} E|Z_t^2| &= CE\left|\sum_{i=1}^{\dim G}\int_0^t X_i\chi_\alpha(B_s)\, dW_s^i\right|^2 \\ &= C\sum_{i=1}^{\dim G} E\left|\int_0^t X_i\chi_\alpha(B_s)\, dW_s^i\right|^2 \\ &= C\sum_{i=1}^{\dim G} E\int_0^t |X_i\chi_\alpha(B_s)|^2\, ds \\ &\leq C\dim G\int_0^t E\rho(B_s)^2\, ds \\ &\leq Ct^2. \end{aligned} \tag{4.9}$$

The preliminary study of the process $\chi_\alpha(B_t)$ is now finished. We consider a sequence $(B_t^n)$ as in the statement of 4.2.6, defined on a probability space $(\Omega, P)$. We look at the product

$$X_n = \prod_{i=1}^n (1 - \frac{c_2(\alpha)T}{2n} + Z_{T/n}^i + Y_{T/n}^i),$$

where the random variables with different exponents are independent. We would like to take the logarithm of this product. This requires some precautions. Set

$$\Omega_n = \left\{\left|Z_{T/n}^i + Y_{T/n}^i\right| < \frac{1}{3}\ \forall i = 1\ldots n\right\}.$$

A Chebishev inequality gives

$$P\left(|Z_t + Y_t| < \frac{1}{3}\right) \geq 1 - 9E|Z_t + Y_t|^2 \geq 1 - Ct^2,$$

implying

$$P(\Omega_n) \geq \left(1 - \frac{CT^2}{n^2}\right)^n \underset{n\to\infty}{\longrightarrow} 1.$$

We do not change any convergence in distribution on $\Omega$ if we replace $X_n$ by 1 outside $\Omega_n$. So we set

$$\widetilde{X}_n = X_n \mathbf{1}_{\Omega_n} + \mathbf{1}_{\Omega_n^c}.$$

Then $\operatorname{Log}\widetilde{X}_n$ is well defined, Log being the principal determination of the complex logarithm. In fact, we have more than that. If $n$ is such that $\frac{c_2(\alpha)T}{2n}$ is smaller than $\frac{1}{6}$, then each factor of $\widetilde{X}_n$ is of the form $(1-z)$ with $|z| < \frac{1}{2}$. For such a $z$,we have $|\log(1-z) + z| \leq |z|^2$. Thus, the equality $\operatorname{Log}(\widetilde{X}_n) = \mathbf{1}_{\Omega_n}\sum_{i=1}^n \log(1 - \frac{c_2(\alpha)T}{2n} + Z_{T/n}^i + Y_{T/n}^i)$ implies

$$\begin{aligned} \left|\operatorname{Log}\widetilde{X}_n + \sum_{i=1}^n\left(\frac{c_2(\alpha)T}{2n} - Z_{T/n}^i - Y_{T/n}^i\right)\right| \leq\ & \mathbf{1}_{\Omega_n}\sum_{i=1}^n\left|\frac{c_2(\alpha)T}{2n} - Z_{T/n}^i - Y_{T/n}^i\right|^2 + \\ & \mathbf{1}_{\Omega_n^c}\left|\sum_{i=1}^n \frac{c_2(\alpha)T}{2n} - Z_{T/n}^i - Y_{T/n}^i\right|. \end{aligned} \tag{4.10}$$

The last term tends to 0 in probability because $P(\Omega_n^c)$ tends to 0. Using (4.8) and (4.9), we find

$$E\left|\frac{c_2(\alpha)T}{2n} - Z^i_{T/n} - Y^i_{T/n}\right|^2 \le \frac{CT^2}{n^2},$$

so that the first term of the right hand side of (4.10) tends to 0 in $L^1$ norm.

Now let us study the left hand side. On one hand, (4.8) implies $\sum_i Y^i_{T/n} \xrightarrow{L^1} 0$, because

$$E\left|\sum_{i=1}^n Y^i_{T/n}\right| \le \sum_{i=1}^n E|Y^i_{T/n}| \le \frac{C}{n}.$$

On the other hand, using 4.9 and the fact that $EZ^i_{T/n} = 0$, we also have $\sum_i Z^i_{T/n} \xrightarrow{L^1} 0$, because

$$E\left|\sum_{i=1}^n Z^i_{T/n}\right|^2 = \sum_{i=1}^n E|Z^i_{T/n}|^2 \le \frac{C}{n}.$$

We deduce that $\operatorname{Log} \widetilde{X}_n$ converges in probability to $-\frac{c_2(\alpha)T}{2}$. This implies that $\widetilde{X}_n$, and so $X_n$, converge in probability to $e^{-\frac{c_2(\alpha)T}{2}}$. Finally, we get

$$\dim\alpha\, X_n \xrightarrow[n\to\infty]{\text{law}} \dim\alpha\, e^{-\frac{c_2(\alpha)T}{2}},$$

proving the proposition. □

The proof of the theorem is almost finished. It remains to prove that the proposition 4.2.6 implies the proposition 4.2.5.

*Proof of proposition 4.2.5*: We set $T = \sigma(D)$. Let $F$ be a continuous function on $\mathbf{R}$.

$$E\left[F\left(\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^n \chi_\beta(\mathcal{H}_{\partial F_i})\right)\right] = \frac{1}{Z(x_1,\ldots,x_p)}\int F\left(\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^n \chi_\beta(g_i)\right)$$
$$p_{T/n}(g_1)\ldots p_{T/n}(g_n)p_{\sigma(D^c)}([a_g,b_g]\ldots[a_1,b_1]z_p^{-1}x_pz_p\ldots z_1^{-1}x_1z_1y_1^{-1}g_1^{-1}y_1\ldots y_n^{-1}g_n^{-1}y_n)$$
$$dg_i da_i db_i dy_i dz_i.$$

We develop the heat kernel $p_{\sigma(D^c)}$ and integrate against all variables except $g_1,\ldots,g_n$. We find

$$\frac{1}{Z(x_1,\ldots,x_p)}\sum_{\gamma\in\widehat{G}}(\dim\gamma)^{1-2g-p}e^{-\frac{c_2(\gamma)}{2}\sigma(D^c)}\prod_{i=1}^p \chi_\gamma(x_p)$$
$$\int_{G^n} F\left(\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^n \chi_\beta(g_i)\right)\frac{1}{(\dim\gamma)^{n-1}}\prod_{i=1}^n \chi_\gamma(g_i)\, p_{T/n}(g_1)\ldots p_{T/n}(g_n)\, dg_1\ldots dg_n.$$

The proposition 4.2.6 says exactly that this last integral converges to

$$F(\dim\beta e^{-\frac{c_2(\beta)}{2}\sigma(D)})\dim\gamma e^{-\frac{c_2(\gamma)}{2}\sigma(D)}.$$

Thus,

$$E\left[F\left(\frac{1}{(\dim\beta)^{n-1}}\prod_{i=1}^{n}\chi_\beta(\mathcal{H}_{\partial F_i})\right)\right]\underset{n\to\infty}{\longrightarrow}$$

$$\frac{1}{Z(x_1,\ldots,x_p)}\sum_{\gamma\in\widehat{G}}(\dim\gamma)^{2-2g-p}e^{-\frac{c_2(\gamma)}{2}\sigma(M)}\frac{1}{(\dim\gamma)^{p-1}}\prod_{i=1}^{p}\chi_\gamma(x_i)\ F(\dim\beta e^{-\frac{c_2(\beta)}{2}\sigma(D)})$$

$$= \frac{Z(x_1,\ldots,x_p)}{Z(x_1,\ldots,x_p)}F(\dim\beta e^{-\frac{c_2(\beta)}{2}\sigma(D)}).$$

This proves that the announced convergence holds in distribution. We already noted that the function $\chi_\beta$ is bounded on $G$ by $\dim\beta$. So, the sequence that we study is uniformly bounded by $\dim\beta$. The result follows, since the convergence in distribution of a uniformly bounded sequence to a deterministic limit implies its $L^2$ convergence. □

# Chapter 5

# Surgery of surfaces

In this chapter, we study the effect of the surgery of surfaces on the Yang-Mills measure. This means that we consider a surface $M$ obtained by gluing two surfaces $M_1$ and $M_2$ together and that we study the relationships between the Yang-Mills measures on $M$, $M_1$ and $M_2$.

The first result is the Markov property of the Yang-Mills measure, which explains how the random holonomy on a piece of $M$ is embedded in the whole probability space supporting $\mu_M$. This corresponds to the operation of cutting a surface into smaller pieces and had already been partially studied by Becker and Sengupta (see below).

The second type of results corresponds to the operation of gluing surfaces together: we explain why the random holonomy on $M$ is not generated by its restrictions to smaller pieces of $M$ and show how it is possible to reconstruct the holonomy on $M$ using these restrictions and some extra information.

Then, we study the conditional partition functions, which arose naturally in the first part of the chapter. We show that they play the role of the transition functions of a Markov random field and discuss to what extent they determine the law of the random holonomy.

## 5.1 Cutting surfaces

### 5.1.1 Markov property of the Yang-Mills field

The first problem is the following: given a surface $M$ cut into two pieces $M_1$ and $M_2$, how is it possible to deduce the Yang-Mills measure on $M_1$ and $M_2$ from that on $M$? We already met this problem in the section 2.8 when we constructed the random holonomy on a surface with boundary starting from that on a minimal closure of this surface. In fact, we already proved a part of the theorem 5.1.1 in a special case.

This problem has been solved by Becker and Sengupta in the discrete setting, in [BS] (th. 3.1). The proof of the theorem 5.1.1 is inspired by this work.

It should be noted that the notion of Markov stochastic cosurface, introduced by Albeverio et al. in [Al], is a general point of view on random fields like the Yang-Mills measure which have the same kind of Markov property.

Let $M_1$ and $M_2$ be two oriented surfaces with boundary and denote $\partial M_1 = N_1 \cup \ldots \cup N_{p_1} \cup B_1 \cup \ldots \cup B_p$, $\partial M_2 = N'_1 \cup \ldots \cup N'_{p_2} \cup B'_1 \cup \ldots \cup B'_p$. We assume only that $p > 0$. In this chapter, we always orient the boundary of oriented surfaces according to the usual convention. For each $i = 1 \ldots p$, let $\psi_i : B_i \longrightarrow B'_i$ be an orientation-reversing diffeomorphism. Let $M$ be obtained by gluing $M_1$ and $M_2$ along $\psi_1, \ldots, \psi_p$. Finally, denote by $L_1, \ldots, L_p$ $p$ loops of $LM$ whose images are $B_1 = -B'_1, \ldots, B_p = -B'_p$. We identify $M_1$ and $M_2$ with submanifolds of $M$.

The probabilty space that supports the Yang-Mills measure on $M$ is $(\mathcal{F}(LM, G), \mathcal{A})$. Recall that $\mathcal{A} = \sigma(\mathcal{H}_{l_1,\ldots,l_n}, l_i \in LM)$. There are two special sub-$\sigma$-algebras on this space, namely $\widetilde{\mathcal{A}}_i = \sigma(\mathcal{H}_{l_1,\ldots,l_n}, l_k \in LM_i)$, $i = 1, 2$. Note that a function $f_i$ on $\mathcal{F}(LM_i, G)$ gives rise to a function $\widetilde{f}_i$ on $\mathcal{F}(LM, G)$ and that it is equivalent to say that $f_i$ is $\mathcal{A}_i$-measurable or to say that $\widetilde{f}_i$ is $\widetilde{\mathcal{A}}_i$- measurable. Thus, we identify $\mathcal{A}_i$ and $\widetilde{\mathcal{A}}_i$, as well as $f_i$ and $\widetilde{f}_i$.

**Theorem 5.1.1** *The $\sigma$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent on $(\mathcal{F}(LM, G), \mathcal{A}, \mu_M)$ conditionally to the random variable $(\mathcal{H}_{L_1}, \ldots, \mathcal{H}_{L_p})$. Moreover, let $f_1$ and $f_2$ be two measurable functions on $(\mathcal{F}(LM_1, G), \mathcal{A}_1)$ and $(\mathcal{F}(LM_2, G), \mathcal{A}_2)$ respectively. Then the product $f_1 f_2$ can be seen as a $\mathcal{A}$-measurable function on $\mathcal{F}(LM, G)$ and for any $t_1, \ldots, t_p \in G/\mathrm{Ad}$, the following equality holds:*

$$E_{\mu_M}[f_1 f_2 | \mathcal{H}_{L_1} = t_1, \ldots, \mathcal{H}_{L_p} = t_p] = \mu_M(t_1, \ldots, t_p)(f_1 f_2) = \mu_{M_1}(t_1, \ldots, t_p)(f_1)\mu_{M_2}(t_1, \ldots, t_p)(f_2).$$

*Finally, one can replace the measures $\mu_M, \mu_{M_1}, \mu_{M_2}$ by their conditional versions with respect to some variables among $\mathcal{H}_{N_1}, \ldots, \mathcal{H}_{N_{p_1}}, \mathcal{H}_{N'_1}, \ldots, \mathcal{H}_{N'_{p_2}}$ and all statements remain true.*

This theorem says two things. It says that the random holonomy on $M_1$ is independent of that on $M_2$ conditionally to the holonomy along the common boundary of $M_1$ and $M_2$ and it also says that the restriction to $\mathcal{A}_i$ of the measure $\mu_M(t_1, \ldots, t_p)$ is equal to the measure $\mu_{M_i}(t_1, \ldots, t_p)$, that is, the Yang-Mills measure on $M_i$.

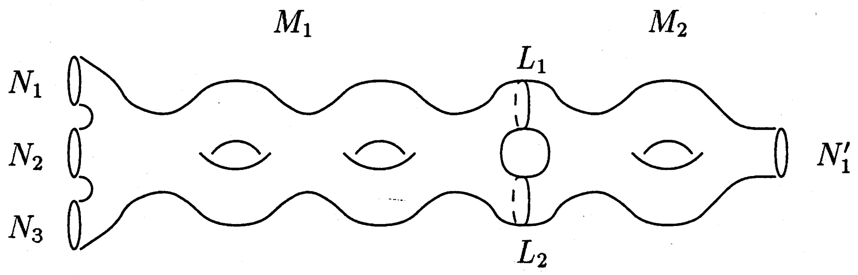


Figure 5.1: Two surfaces glued together.

*Proof.* Let $\Gamma$ be a graph on $M$ such that $L_1, \ldots, L_p \in \Gamma^*$. Denote by $\Gamma_i$ the graph induced by $\Gamma$ on $M_i$, $i = 1, 2$ and by $\Gamma_\partial$ the graph induced on $M_1 \cap M_2$. Let $f_1, f_2, f_\partial$ be three continuous gauge-invariant functions defined respectively on $G^{\Gamma_1}, G^{\Gamma_2}, G^{\Gamma_\partial}$. We choose $y = (y_1, \ldots, y_{p_1}) \in G^{p_1}$, $y' = (y'_1, \ldots, y'_{p_2}) \in G^{p_2}$ and compute the conditional expectation $E_{P(y,y')}[f_1 f_2 | f_\partial]$ under the conditional discrete Yang-Mills measure $P(y, y')$ on $G^\Gamma$, where each $y_i$ corresponds to $N_i$ and each $y'_j$ to $N'_j$. In order to avoid too long expressions, we use sometimes the following natural abbreviations: $x = (x_1, \ldots, x_p)$, $x^{-1} = (x_1^{-1}, \ldots, x_p^{-1})$, $dx = dx_1 \ldots dx_p$, $d\nu_x = d\nu_{x_1} \ldots d\nu_{x_p}$, $d\nu_y = d\nu_{y_1} \ldots d\nu_{y_{p_1}}$, $d\nu_{y'} = d\nu_{y'_1} \ldots d\nu_{y'_{p_2}}$.

$$\int_{G^\Gamma} f_1 f_2 f_\partial \, dP(y, y') = \frac{1}{Z_M(y, y')} \int_{G^\Gamma} f_1 f_2 f_\partial \prod_{F \in \mathcal{F}(\Gamma)} p_{\sigma(F)}(h_{\partial F}) \, d\nu_y d\nu_{y'} dg'$$

$$= \frac{1}{Z_M(y,y')}\int_{G^p} Z(y,x,y')\,dx\,\frac{1}{Z(y,x,y')}\int_{G^\Gamma} f_1 f_2 f_\partial D\,d\nu_x d\nu_y d\nu_{y'} dg'$$

$$= \frac{1}{Z_M(y,y')}\int_{G^p} f_\partial(x) Z(y,x,y')\,dx$$

$$\frac{1}{Z(y,x,y')}\int_{G^{\Gamma_\partial}}\left[\int_{G^{\Gamma_1\setminus\Gamma_\partial}} f_1 D^{\Gamma_1}\,d\nu_y dg'^{\Gamma_1\setminus\Gamma_\partial}\right]\left[\int_{G^{\Gamma_2\setminus\Gamma_\partial}} f_2 D^{\Gamma_2} d\nu_{y'} dg'^{\Gamma_2\setminus\Gamma_\partial}\right] d\nu_x$$

The two last integrals are gauge-invariant functions on $G^{\Gamma_\partial}$. So, they depend only on the values of $h_{L_1},\ldots,h_{L_p}$ and we can drop the integration against $d\nu_x$ over $G^{\Gamma_\partial}$. Setting $f_1$ and $f_2$ identically equal to 1, we get:

$$\int_{G^\Gamma} f_\partial\,dP(y,y') = \frac{1}{Z_M(y,y')}\int_{G^p} f_\partial(x) Z_{M_1}(y,x) Z_{M_2}(x^{-1},y')\,dx$$

But according to 1.5.2, the left hand side is equal to $Z_M(y,y')^{-1}\int_{G^p} f_\partial(x) Z_M(y,x,y^{-1})\,dx$. Since the equality holds for any continuous function $f_\partial$, the measures on both sides are equal. We deduce the following important relation:

$$Z_M(y,x,y') = Z_{M_1}(y,x) Z_{M_2}(x^{-1},y'),$$

using which we see that $\int f_1 f_2 f_\partial\,dP(y,y')$ is equal to:

$$\frac{1}{Z_M(y,y')}\int_{G^p} f_\partial(x) Z(y,x,y')\,\frac{1}{Z_{M_1}(y,x)}\left[\int_{G^{\Gamma_1\setminus\Gamma_\partial}} f_1 D^{\Gamma_1}\,d\nu_y dg'^{\Gamma_1\setminus\Gamma_\partial}\right](x)$$
$$\frac{1}{Z_{M_2}(x^{-1},y')}\left[\int_{G^{\Gamma_2\setminus\Gamma_\partial}} f_2 D^{\Gamma_2}\,d\nu_{y'} dg'^{\Gamma_2\setminus\Gamma_\partial}\right](x)\,dx$$
$$= \frac{1}{Z_M(y,y')}\int_{G^p} f_\partial(x)\left[\int_{G^{\Gamma_1}} f_1\,dP_{M_1}(y,x)\right]\left[\int_{G^{\Gamma_2}} f_2\,dP_{M_2}(x^{-1},y')\right] Z(y,x,y')\,dx.$$

This proves the theorem for the functions $f_1$ and $f_2$. If we choose a Riemannian metric on $M$ and use piecewise geodesic approximations of arbitrary paths, we can extend the last relation to arbitrary functions measurable with respect to $\mathcal{A}_1$, $\mathcal{A}_2$ and $\sigma(\mathcal{H}_{L_1},\ldots,\mathcal{H}_{L_q})$, which terminates the proof. □

It is worth stating separately the relation between conditional partition functions that was established during the proof.

**Proposition 5.1.2** *For all $x = (x_1,\ldots,x_p)$, $y = (y_1,\ldots,y_{p_1})$ and $y' = (y'_1,\ldots,y'_{p_2})$, the following relation holds:*

$$Z_M(y,x,y') = Z_{M_1}(y,x) Z_{M_2}(x^{-1},y').$$

### 5.1.2 An example

We give an example of this situation. Consider a closed surface $M$ of genus two realized as connected sum of two toruses. Let $M_1$ and $M_2$ denote the two halves of $M$. Set $L = \partial M_1 = -\partial M_2$. Let $f_1$ be a function on $(\mathcal{F}(LM_1,G),\mathcal{A}_1)$ and $f_2$ a function on $(\mathcal{F}(LM_2,G),\mathcal{A}_2)$. Then $f_1 f_2$ can be seen as a function on $(\mathcal{F}(LM,G),\mathcal{A})$. We just proved that, for all $t\in G/\operatorname{Ad}$, $\mu_M(t)(f_1f_2)=\mu_{M_1}(t)(f_1)\mu_{M_2}(t^{-1})(f_2)$. Since $\mu_M=\int_{G/\operatorname{Ad}}\mu_M(t)Z_M(t)\,dt$ (see proposition 2.10.5), we get:

$$\begin{aligned}\mu_M(f_1f_2) &= \frac{1}{Z_M}\int_{G/\operatorname{Ad}} Z_M(t)\mu_M(t)(f_1f_2)\,dt\\ &= \frac{1}{Z_M}\int_{G/\operatorname{Ad}} Z_{M_1}(t)\mu_{M_1}(t)(f_1)\,Z_{M_2}(t^{-1})\mu_{M_2}(t^{-1})(f_2)\,dt.\end{aligned}$$

We rewrite this last equality in a more symmetric form:

$$Z_M\mu_M(f)=\int_{G/\operatorname{Ad}} Z_{M_1}(t)\mu_{M_1}(t)(f_1)\,Z_{M_2}(t^{-1})\mu_{M_2}(t^{-1})(f_2)\,dt.$$

The point here is that the analytic objects that glue together in a simple way are not the probability measures, but the measures with their natural weights.

### 5.1.3 Cutting a handle

The other situation that can arise when one cuts a surface along a circle is that the surface remains connected. This is the second problem we want to investigate. As in the first section, it is easier to describe the geometrical setting starting from the end.

Let $M_1$ be a surface with boundary $\partial M=N_1\cup\ldots\cup N_p\cup B_1\cup B_2$. Let $\psi: B_1\longrightarrow B_2$ be an orientation-reversing diffeomorphism and let $M$ be obtained by gluing $M_1$ along $\psi$. Let $L$ be a loop whose image is $B_1=-B_2$. Note that, in contrast to the preceding situation, $M_1$ is not embedded in $M$, it is only immersed. Nevertheless, this immersion is enough to map $LM_1$ into $LM$. So, a function $f$ on $\mathcal{F}(LM_1,G)$ can be seen as a function $\widetilde{f}$ on $\mathcal{F}(LM,G)$ and $f$ is measurable with respect to $\mathcal{A}_1$ if and only if $\widetilde{f}$ is measurable with respect to $\widetilde{\mathcal{A}}_1=\sigma(\mathcal{H}_{l_1,\ldots,l_n}, l\in LM_1)$. As before, we identify $f$ and $\mathcal{A}_1$ with $\widetilde{f}$ and $\widetilde{\mathcal{A}}_1$.

There is no conditional independence in this case, but only a statement very similar to the second part of the theorem 5.1.1.

**Theorem 5.1.3** *Let $f$ be a measurable function on $(\mathcal{F}(LM_1,G),\mathcal{A}_1)$. Then for any $u_1,\ldots,u_p,t\in G/\operatorname{Ad}$, the following equality holds:*

$$\mu_M(u_1,\ldots,u_p,t)(f)=\mu_{M_1}(u_1,\ldots,u_p,t,t^{-1})(f).$$

*Proof.* The proof is similar to that of theorem 5.1.1. Let $\Gamma$ be a graph on $M$ such that $L\in\Gamma^*$. Let $\Gamma_1$ be the graph on $M_1$ that is mapped onto $\Gamma$ by the immersion of $M_1$ into $M$ and such that the diffeomorphism $\psi$ sends an edge of $\Gamma_1$ to another edge of $\Gamma_1$. Let $f$ be a continuous gauge-invariant function on $G^{\Gamma_1}$. Let $y_1,\ldots,y_p,x$ be elements of $G$. We abbreviate $(y_1,\ldots,y_p)$ by $y$, $d\nu_{y_1}\ldots d\nu_{y_p}$ by $d\nu_y$. We compute the expectation $E_{P(y,x,x^{-1})}[f]$ under the conditional discrete measure $P(y,x,x^{-1})$ on $G^{\Gamma_1}$.

$$\begin{aligned}\int_{G^{\Gamma_1}} f\,dP_{M_1}(y,x,x^{-1}) &= \frac{1}{Z_{M_1}(y,x,x^{-1})}\int_{G^{\Gamma_1}} fD^{\Gamma_1}\,d\nu_y d\nu_x d\nu_{x^{-1}}\,dg'\\ &= \int_{G^{\Gamma_\partial}}\frac{d\nu_x d\nu_{x^{-1}}}{Z_{M_1}(y,x,x^{-1})}\int_{G^{\Gamma_1\setminus\Gamma_\partial}} fD^{\Gamma_1}\,d\nu_y dg'.\end{aligned}$$

The last integral is a gauge-invariant function on $G^{\Gamma_\theta}$. So, it depends only on the value of $h_{B_1}$ and $h_{B_2}$. Since these values are equal, with the right orientations, this integral has the same value as the corresponding integral over $G^\Gamma$. Thus,

$$\begin{aligned}
\int_{G^{\Gamma_1}} f\, dP_{M_1}(y,x,x^{-1}) &= \frac{1}{Z_{M_1}(y,x,x^{-1})}\int_{G^\Gamma} fD^\Gamma\, d\nu_x d\nu_y dg' \\
&= \frac{Z_M(y,x)}{Z_{M_1}(y,x,x^{-1})}\int_{G^\Gamma} f\, dP_M(y,x) \\
&= \int_{G^\Gamma} f\, dP_M(y,x).
\end{aligned}$$

To see that the partition functions are equal, we just set $f$ identically equal to 1. The theorem is proved for $f$ and we extend this partial result to the general case by the usual approximation procedure, described in the proof of 5.1.1. □

One more time, a very interesting relation between the partition functions arised, namely:

**Proposition 5.1.4** *For any $t_1,\ldots,t_p,t \in G/\operatorname{Ad}$, the following equality holds:*

$$Z_M(t_1,\ldots,t_p,t,t^{-1}) = Z_{M_1}(t_1,\ldots,t_p,t).$$

## 5.2 Gluing surfaces

### 5.2.1 Gluing two surfaces together

In this section, we go back to the situation described at the beginning of this chapter: two surfaces $M_1$ and $M_2$ are glued together along one circle to form another surface $M$. The circle along which $M_1$ and $M_2$ are glued is the image of a loop $L$ in $M$. We choose a conjugacy class $t \in G/\operatorname{Ad}$ and consider the Yang-Mills measures $\mu_{M_1}(t)$, $\mu_{M_2}(t^{-1})$ and $\mu_M(t)$. According to the proposition 2.10.9, we can use the spaces of functions on based loop spaces to represent these Yang-Mills measures: we fix the point $m = L(0)$ and set $\Omega_{M_i} = \mathcal{F}(L_m M_i, G)/\operatorname{Ad}$, $\Omega_M = \mathcal{F}(L_m M, G)/\operatorname{Ad}$. The $\sigma$-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$ on these sets are generated by the random variables $\mathcal{H}_{l_1,\ldots,l_n}$ with $l_1,\ldots,l_n$ in $L_mM_1$, $L_mM_2$ and $L_mM$ respectively.

The theorem 5.1.1 says that the probability spaces $(\Omega_{M_i}, \mathcal{A}_i, \mu_{M_i}(t^{\pm 1}))$ are naturally isomorphic to two independent subspaces of $(\Omega_M, \mathcal{A}_M, \mu_M(t))$. It is natural to ask whether these two subspaces generate $\Omega_M$ or not, in other words whether the inclusion $\mathcal{A}_1 \vee \mathcal{A}_2 \subset \mathcal{A}$ is an equality or not. Let us start with an example.

#### 5.2.1.1 Study of an example

Consider two disks $D_1$ and $D_2$ glued together to form a sphere $S$ with equator $L$. In order to compare $\mathcal{A}_1 \vee \mathcal{A}_2$ and $\mathcal{A}$, choose $l_1 \in L_mD_1$ and $l_2 \in L_mD_2$. The random variable $\mathcal{H}_{l_1,L,l_2}$ is $\mathcal{A}$-measurable, but is it $\mathcal{A}_1 \vee \mathcal{A}_2$-measurable? The following informal argument answers negatively to this question.

The random variables $\mathcal{H}_{l_1,L}$ and $\mathcal{H}_{L,l_2}$ are respectively $\mathcal{A}_1$- and $\mathcal{A}_2$-measurable. It seems reasonable to believe that they provide the whole information about $\mathcal{H}_{l_1,L,l_2}$ available in $\mathcal{A}_1 \vee \mathcal{A}_2$. Let us explain why they do not determine $\mathcal{H}_{l_1,L,l_2}$. For this, we take the example $G = SO(3)$ and describe more carefully conjugacy classes and joint conjugacy classes in $G$.

If $r \in SO(3)$ is neither the identity nor a symmetry, it has an angle and an axis, which can be oriented in such a way that the angle is an element of $(0, \pi)$. Thus, $r$ has a half-axis and an angle. This angle characterizes the conjugacy class of $r$ (this is still true for the identity and the symmetries). Now, consider $(r_1, \ldots, r_n)$ and $(r'_1, \ldots, r'_n)$ two $n$-uples of rotations that have half-axes and angles $(u_i, \theta_i)$, $(u'_j, \theta'_j)$. They belong to the same joint conjugacy class if and only if $\theta_i = \theta'_i$ for all $i = 1 \ldots n$ and if there exists a rotation $R \in SO(3)$ such that $u'_i = R(u_i)$.

The random variables $\mathcal{H}_{l_1,L}$ and $\mathcal{H}_{L,l_2}$ determine the angles of the three rotations and the relative position of the half-axes of $H_{l_1}$ and $H_L$ on one hand and $H_L$ and $H_{l_2}$ on the other hand.

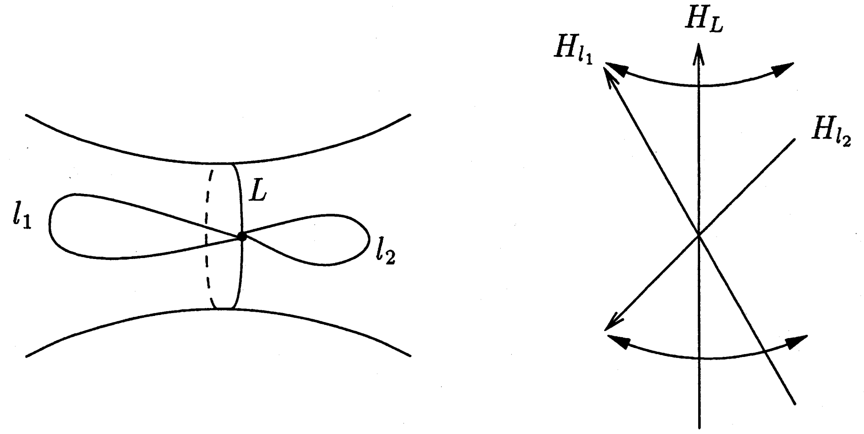


Figure 5.2: Lack of information about the joint class $[H_L, H_{l_1}, H_{l_2}]$ when $G = SO(3)$.

But this is not enough to determine the relative position of the three half-axes (see figure 5.2). There remains an undetermined rotation around the axis of $H_L$. Algebraically, we know the joint class $[H_{l_1}, H_L, H_{l_2}]$ up to the conjugation of $H_{l_2}$ by an element $z$ such that $z^{-1}H_L z = H_L$, that is, an element of the centralizer of $H_L$.

Now we claim that, under mild conditions on $l_1$ and $l_2$, the variable $\mathcal{H}_{l_1,L,l_2}$ is essentially what has to be added to $\mathcal{A}_1 \vee \mathcal{A}_2$ to get $\mathcal{A}$. More precisely, the completion of the $\sigma$-algebra $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \sigma(\mathcal{H}_{l_1,L,l_2})$ with respect to the measure $\mu_M(t)$ contains $\mathcal{A}$. The point is that, provided $H_{l_1}$, $H_L$ and $H_{l_1}$ have different axes, the variable $\mathcal{H}_{l_1,L,l_2}$ determines the rotation around the axis of $H_L$ that was missing. An algebraic proof of this fact will be given in the next paragraph. The mild conditions on $l_1$ and $l_2$ are the following: we need to be sure that the axes of $H_{l_1}$, $H_L$ and $H_{l_2}$ are almost surely different. This happens if the law of $\mathcal{H}_{l_1,l_2}$ has a density with respect to the natural measure on $G^2/\mathrm{Ad}$. This holds for example if $l_1$ and $l_2$ are small disjoint simple loops, according to the computations of laws done in the first chapter.

#### 5.2.1.2 The general case

In the general case, we follow the scheme of this example. The first important result is the following:

**Theorem 5.2.1** *Set $N = \dim G - \mathrm{rank}\, G + 1$. Let $L^1_1, \ldots, L^1_N$ and $L^2_1, \ldots, L^2_N$ be disjoint simple loops of $L_m M_1$ and $L_m M_2$ respectively, whose images do not meet the boundaries of $M_1$ and $M_2$. Then the completion of the $\sigma$-algebra $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \sigma(\mathcal{H}_{L^1_1,\ldots,L^1_N,L,L^2_1,\ldots,L^2_N})$ with respect to $\mu_M(t)$ contains $\mathcal{A}$.*

Note that the value of $N$ given in this theorem is an upper bound: it may happen, as in the case of $SO(3)$, that the result is true with a smaller number of loops.

The proof of this theorem requires some preliminary results. We choose on $M$ a Riemannian metric such that $L$ is geodesic. This is not essential for the proof of 5.2.1 but it will be useful at the end of this paragraph. The set of piecewise geodesic loops on $M$ based at $m$ is denoted by $PGL_mM$.

**Lemma 5.2.2** *On the space $\Omega_M$, the $\sigma$-algebra $\mathcal{A}$ is contained in $\mathcal{A}_{12}$, which is the completion of $\sigma(\mathcal{H}_{\lambda_1,\ldots,\lambda_n}, \lambda_1, \ldots, \lambda_n \in PGL_mM_1 \cup PGL_mM_2)$ with respect to $\mu_M(t)$.*

*Proof*. By the multiplicativity of the random holonomy described at the end of section 2.10.3, the holonomies along loops that are finite products of loops of $L_mM_1$ and $L_mM_2$ are $\mathcal{A}_{12}$-measurable. According to the continuity property stated in 2.10.7 and to the fact that the holonomy depends only on the equivalence class of the loops, the result depends on the fact that any loop of $L_mM$ can be approximated by loops that are equivalent to finite products of loops of $PGL_mM_1$ and $PGL_mM_2$. Consider a piecewise geodesic loop of $L_mM$. This loop cuts $L$ transversally at most a finite number of times, hence it is equivalent to a finite product of loops of $PGL_mM_1$ and $PGL_mM_2$. Since any loop of $L_mM$ can be approximated by piecewise geodesic loops, the result is proved. □

**Lemma 5.2.3** *Let $G$ be a compact connected Lie group of dimension $n$ and rank $k$. Set $N = n-k+1$. There exists a Borel subset $S_N \subset G^N$ of full Haar measure such that for all $(g_1, \ldots, g_N) \in S_N$, the closed subgroup of $G$ generated by $g_1, \ldots, g_N$ is $G$ itself.*

*Proof*. The key of this result is that almost every element of $G$ is regular, i.e. generates a maximal torus [Si]. Let $g_1$ be such a regular element. It generates a subgroup $G_1 = T_1$ of $G$ of dimension $k$. If $G_1$ is a proper subgroup of $G$, then $\dim G_1 < \dim G$ since $G$ is connected. Thus, the complementary of $G_1$ has full measure as well as the set of regular elements outside $G_1$. Let $g_2$ be such an element and $T_2$ the torus that it generates. Denote by $G_2$ the subgroup generated by $\{g_1, g_2\}$. Denote by $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{t}_2$ the Lie algebras of $G_1$, $G_2$ and $T_2$ respectively. The fact that $\mathfrak{t}_2 \not\subseteq \mathfrak{g}_1$ and $\mathfrak{g}_2 \supset \mathfrak{g}_1 + \mathfrak{t}_2$ shows that $\dim G_2 > \dim G_1$. Repeating this procedure $N = n-k+1$ times, we get a subgroup $G_N$ which is equal to $G$. It is clear from this construction that the set $S_N$ of convenient $N$-uples has full Haar measure in $G^N$. □

Before to prove the theorem, let us recall that, by a classical result of Kuratowski [Ku], a one-to-one measurable map between two Polish spaces sends Borel subsets to Borel subsets. We will use this result several times in the sequel to prove measurability results.

*Proof of the theorem 5.2.1*. The assumptions on $L_1^1, \ldots, L_N^1, L_1^2, \ldots, L_N^2$ ensure that the laws of the random variables $\mathcal{H}_{L_1^1,\ldots,L_N^1}$ and $\mathcal{H}_{L_1^2,\ldots,L_N^2}$ have densities with respect to the natural measure on $G^N/\operatorname{Ad}$ and hence are almost surely in $S_N/\operatorname{Ad}$. Pick $p$ loops $l_1, \ldots, l_p$ in $L_mM_1$ and $q$ loops $l'_1, \ldots, l'_q$ in $L_mM_2$. We abbreviate by $l, l', L^1$ and $L^2$ the corresponding families of loops.

We show that the random variables $\mathcal{H}_{l,L^1}$, $\mathcal{H}_{L^2,l'}$ and $\mathcal{H}_{L^1,L^2}$ determine $\mathcal{H}_{l,l'}$ if the values of $\mathcal{H}_{L^1}$ and $\mathcal{H}_{L^2}$ are in $S_N/\operatorname{Ad}$. By lemma 5.2.2, this is enough to prove the theorem. We do not even restrict ourselves to piecewise geodesic loops. Since $\mathcal{H}_{l,l'}$ is just a continuous projection of

$\mathcal{H}_{l,L^1,L^2,l'}$, it is sufficient to write this last variable as a function of the three given variables. For this, we construct a map:

$$(G^p \times S_N)/\operatorname{Ad} \times_{S_N} (S_N \times S_N)/\operatorname{Ad} \times_{S_N} (S_N \times G^q)/\operatorname{Ad} \xrightarrow{\kappa} (G^p \times S_N \times S_N \times G^q)/\operatorname{Ad}.$$

The symbols $\times_{S_N}$ above mean that the map $\kappa$ is only defined on the set of elements of the form $([u_1,\ldots,u_p,g_1,\ldots,g_N],[g'_1,\ldots,g'_N,h'_1,\ldots,h'_N],[h_1,\ldots,h_N,v_1,\ldots,v_q])$ such that $[g_1,\ldots,g_N] = [g'_1,\ldots,g'_N]$ and $[h_1,\ldots,h_N] = [h'_1,\ldots,h'_N]$.

We claim that it makes sense to construct $\kappa$ such that the image of such a triple is the unique element $[x_1,\ldots,x_p,r_1,\ldots,r_N,s_1,\ldots,s_N,y_1,\ldots,y_q]$ such that, with compact notations, $[x,r] = [u,g]$, $[r,s] = [g',h']$ and $[s,y] = [h,v]$. It is not difficult to see that such an element exists: if $z_g$ and $z_h$ are two elements of $G$ such that $\operatorname{Ad}(z_g)g = g'$ and $\operatorname{Ad}(z_h)h = h'$, then $[\operatorname{Ad}(z_g)u, g', h', \operatorname{Ad}(z_h)v]$ is a possible choice.

Suppose that $[x,r,s,y]$ and $[x',r',s',y']$ are two candidates. Since $[x,r] = [x',r']$, there exists $z_r \in G$ such that $\operatorname{Ad}(z_r)x = x'$ and $\operatorname{Ad}(z_r)r = r'$. Similarly, there exists $z_s$ such that $\operatorname{Ad}(z_s)y = y'$ and $\operatorname{Ad}(z_s)s = s'$. Now, $[r,s] = [r',s']$ implies $[r,s] = [\operatorname{Ad}(z_r)r, \operatorname{Ad}(z_s)s] = [\operatorname{Ad}(z_r z_s^{-1})r, s]$. This forces $z_r z_s^{-1}$ to be an element of $Z(G)$, so that $\operatorname{Ad}(z_r) = \operatorname{Ad}(z_s)$ and the two candidates are equal.

We have $\mu_M(t)$-almost surely $\mathcal{H}_{l,L^1,L^2,l'} = \kappa(\mathcal{H}_{l,L^1}, \mathcal{H}_{L^1,L^2}, \mathcal{H}_{L^2,l'})$, so that it remains only to prove that $\kappa$ is measurable. To see this, remark that $\kappa^{-1}$ is easier to define than $\kappa$: it is a restriction of three continuous projections defined on $(G^p \times G^N \times G^N \times G^q)/\operatorname{Ad}$. Moreover, since $\kappa$ is well defined, $\kappa^{-1}$ is injective. Thus, by the result of Kuratowski mentionned above, $\kappa^{-1}$ sends Borel subsets to Borel subsets, which means exactly that $\kappa$ is measurable. □

Now we will take a slightly different point of view. When we studied the case of $G = SO(3)$, we said that the difference of information between $\mathcal{A}_1 \vee \mathcal{A}_2$ and $\mathcal{A}$ seemed to be parametrized by the centralizer of $H_L$. This is what we want to develop now. This will lead us to construct a version of the Yang-Mills measure on $M$, using those on $M_1$, $M_2$ and a uniform random variable on $C(H_L)$. Since the centralizer of an element of $G$ does not depend only on its conjugacy class, we must choose a fictive value of $H_L$ inside the class $\mathcal{H}_L = t$ in order to speak about the centralizer of $H_L$. So, we choose $x \in t$ and compute as if we knew that $H_L = x$.

**Lemma 5.2.4** *There exists a measurable section $\tau : (t \times G^N)/\operatorname{Ad} \longrightarrow \{x\} \times G^N$.*

*Proof*. We use a theorem of Bourbaki (*Topologie*, chap. IX, § 6, N°9, th. 5) [B1], which says that there exists a Borel subset $R$ of $t \times G^N$ that meets once and only once each orbit of the action of $G$. This subset $R$ allows to define a section $\tau$. To prove that $\tau$ is measurable, consider a Borel subset $B \subset t \times G^N$. The fact that $\tau^{-1}(B) = p(B \cap R)$, where $p : t \times G^N \longrightarrow (t \times G^N)/G$ is the natural projection, together with the result of Kuratowski mentionned above, shows that $\tau^{-1}(B)$ is a Borel subset. Thus, $\tau$ is measurable. □

From now on, we fix such a section $\tau$. In order to justify the next construction, we spend a few lines to explain how the random variable $\mathcal{H}_{L^1,L,L^2}$ and the section $\tau$ determine almost surely a random variable with values in $C(x)/Z(G)$. Indeed, suppose that $\mathcal{H}_{L^1,L,L^2} = [u_1,\ldots,u_N,x,v_1,\ldots,v_N]$. Set $(x,g_1,\ldots,g_N) = \tau([x,u_1,\ldots,u_N])$ and $(x,h_1,\ldots,h_N) = \tau([x,v_1,\ldots,v_N])$. Then there exists $z \in G$ such that $[\operatorname{Ad}(z)g, x, h] = [u,x,v]$. It is easily seen that $z \in C(x)$ and that the class of $x$ modulo $Z(G)$ is well defined, provided $u, v \in S_N$. The conditional independence and gauge invariance of the measure $\mu_M(t)$ imply that the law of this variable is left and right invariant: it is uniform.

The centralizer $C(x)$ of $x$ is a subgroup of $G$ whose generic element will be denoted by $z$ and on which we put the Haar measure $dz$. Choose elements $u = (u_1, \ldots, u_{p_1}) \in (G/\operatorname{Ad})^{p_1}$, $u' = (u'_1, \ldots, u'_{p_2}) \in (G/\operatorname{Ad})^{p_2}$, corresponding to the boundary components of $M_1$ and $M_2$ and consider the space

$$(\Omega_R, \mathcal{A}_R, \mu_R(u,t,u')) = (\Omega_{M_1} \times C(x) \times \Omega_{M_2}, \mathcal{A}_1 \otimes \operatorname{Bor}(C(x)) \otimes \mathcal{A}_2, \mu_{M_1}(u,t) \otimes dz \otimes \mu_{M_2}(t^{-1}, u')).$$

Our goal is to construct a family of random variables on this space whose law is the Yang-Mills measure $\mu_M(u,t,u')$. According to the lemma 5.2.2, it is sufficient to construct a variable $\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n}$ on $\Omega_R$ for any $\lambda_1, \ldots, \lambda_n \in PGL_m M_1 \cup PGL_m M_2$. In what follows, we choose $2N$ piecewise geodesic loops $L^1_1, \ldots, L^1_N, L^2_1, \ldots, L^2_N$ as in the theorem 5.2.1 and keep them fixed.

Let $i$ be fixed, equal to 1 or 2. For any subset $\mathcal{L}$ of $L_m M_i$ containing $L, L^i_1, \ldots, L^i_N$, the spaces $\mathcal{F}(\mathcal{L}, G)/\operatorname{Ad}$ and $\mathcal{F}(\mathcal{L}, G)$ are endowed respectively with the $\sigma$-algebra generated by the function $\mathcal{H}_{l_1,\ldots,l_n}$, $l_1, \ldots, l_n \in \mathcal{L}$ and with the cylinder $\sigma$-algebra. The two cases of main interest are the case of finite $\mathcal{L}$ and the case $\mathcal{L} = L_m M_i$.

**Proposition 5.2.5** *For any subset $\mathcal{L}$ of $L_m M_i$ containing $L, L^1_1, \ldots, L^1_N$, the section $\tau$ defined in lemma 5.2.4 determines a measurable section $\sigma_{\mathcal{L}} : \mathcal{F}(\mathcal{L}, G)/\operatorname{Ad} \longrightarrow \mathcal{F}(\mathcal{L}, G)$ defined $\mu_R(u,t,u')$-a.s. such that for all $\omega \in \mathcal{F}(\mathcal{L}, G)/\operatorname{Ad}$, $\sigma_{\mathcal{L}}(\omega)(L) = x$.*

*Proof.* Set $\mathcal{L}' = \mathcal{L} - \{L, L^i_1, \ldots, L^i_N\}$. Let $\omega = [h, h_1, \ldots, h_N, (h_l)_{l \in \mathcal{L}'}]$ be an element of $(t \times S_N \times G^{\mathcal{L}'})/\operatorname{Ad}$. Set $(x, g_1, \ldots, g_N) = \tau([h, h_1, \ldots, h_N])$. Let $(x, g_1, \ldots, g_N, (g_l))$ be an element of $\{x\} \times S_N \times G^{\mathcal{L}'}$ such that $[x, g_1, \ldots] = [h, h_1, \ldots]$. If $(x, g_1, \ldots, g_N, g'_{N+1}, \ldots)$ is another such element, then for each $l \in \mathcal{L}'$, we have in particular $[g_1, \ldots, g_N, g_l] = [g_1, \ldots, g_N, g'_l]$. This implies that $g'_l = z g_l z^{-1}$ with $z \in C(g_1) \cap \ldots \cap C(g_N) = Z(G)$ the center of $G$, so that $g_l = g'_l$. This proves that the section $\tau$ extends uniquely to a section $\sigma_{\mathcal{L}} : (t \times S_N \times G^{\mathcal{L}'})/\operatorname{Ad} \longrightarrow \{x\} \times S_N \times G^{\mathcal{L}'}$. Since the random holonomy along $L^i_1, \ldots, L^i_N$ has a density with respect to the Haar measure, the set $(t \times S_N \times G^{\mathcal{L}'})$ has full $\mu_{M_1}(t)$-measure in $\mathcal{F}(L_m M_1, G)/\operatorname{Ad}$.

It remains to prove that $\sigma_{\mathcal{L}}$ is measurable. For this, we show that each coordinate of $\sigma_{\mathcal{L}}$ is measurable. For the $N+1$ separate coordinates, this follows from the measurability of $\tau$. For $l_0 \in \mathcal{L}'$, we write the $l_0$-th coordinate $\sigma_{l_0}$ as a composition of mappings:

$$(t \times S_N \times G^{\mathcal{L}'})/\operatorname{Ad} \xrightarrow{u} S_N \times (S_N \times G)/\operatorname{Ad} \xrightarrow{v} S_N \times G \xrightarrow{w} G,$$

with $u([h, h_1, \ldots, h_N, (h_l)_{l \in \mathcal{L}'}]) = (g_1, \ldots, g_N, [h_1, \ldots, h_N, h_{l_0}])$, where we set $(x, g_1, \ldots, g_N) = \tau([h, h_1, \ldots, h_N])$, $v(g_1, \ldots, g_N, [h_1, \ldots, h_N, h_{l_0}]) = (g_1, \ldots, g_N, g_{l_0})$ and $w(g_1, \ldots, g_N, g_l) = g_l$. Since $u$ and $w$ are obviously measurable, it is sufficient to show that $v$ is measurable. Note that $v$ is bijective. We consider $v^{-1}$ and prove that it maps a Borel set to a Borel set, using for the third time the same result of Kuratowski. The first point is that $v^{-1}$ is continuous. The second point is that $S_N \times G$ is a Borel subset of the Polish space $G^{N+1}$, so that the theorem applies and proves that $v$ is measurable, finishing the proof. □

By construction, the sections $\sigma_{\mathcal{L}}$ satisfy the following interesting property: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two subsets of $L_m M_i$ such that $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\sigma_{\mathcal{L}_1}$ and $\sigma_{\mathcal{L}_2}$ coincide on $\mathcal{L}_1$. This means that any element $\omega \in \mathcal{F}(\mathcal{L}_2, G)/\operatorname{Ad}$ determines by restriction an element $\widetilde{\omega} \in \mathcal{F}(\mathcal{L}_1, G)/\operatorname{Ad}$ and that $\sigma_{\mathcal{L}_2}(\omega) = \sigma_{\mathcal{L}_1}(\widetilde{\omega})$.

We are now able to define variables $\widetilde{\mathcal{H}}$ on $\Omega_R$. We use the notations $\sigma_1 = \sigma_{L_m M_1}$ and $\sigma_2 = \sigma_{L_m M_2}$. Choose $\lambda_1, \ldots, \lambda_n \in L_m M_1$ and $\lambda'_1, \ldots, \lambda'_{n'} \in L_m M_2$. Choose $\omega_R = (\omega_1, z, \omega_2)$ in $\Omega_R$. We set

$$\mathcal{H}_{\lambda_1,\ldots,\lambda_n,\lambda'_1,\ldots,\lambda'_{n'}}(\omega_R) = [\sigma_1(\omega_1)(\lambda_1), \ldots, \sigma_1(\omega_1)(\lambda_n), z^{-1}\sigma_2(\omega_2)(\lambda'_1)z, \ldots, z^{-1}\sigma_2(\omega_2)(\lambda'_{n'})z].$$

The fact that we choosed piecewise geodesic loops will only allow us to compute the laws of the variables $\widetilde{\mathcal{H}}$ during the proof of the following theorem, which is the second important statement of this section:

**Theorem 5.2.6** *The family of random variables $\widetilde{\mathcal{H}}$ has the same law as the family $\mathcal{H}$ under the Yang-Mills measure $\mu_M(u, t, u')$ on $M$.*

*Proof*. The first point is that the law of the whole family of variables $\widetilde{\mathcal{H}}$ is characterized by the individual law of each variable. Indeed, the law of a finite family like $(\widetilde{\mathcal{H}}_{l^1_1,\ldots,l^1_{n_1}}, \ldots, \widetilde{\mathcal{H}}_{l^p_1,\ldots,l^p_{n_p}})$ is just the projection on $G^{n_1}/\operatorname{Ad} \times \ldots \times G^{n_p}/\operatorname{Ad}$ of the law on $G^{n_1+\cdots+n_p}/\operatorname{Ad}$ of $\widetilde{\mathcal{H}}_{l^1_1,\ldots\ldots,l^p_{n_p}}$.

Let $\lambda_1, \ldots, \lambda_n$ be loops of $PGL_m M_1$ and $\lambda'_1, \ldots, \lambda'_{n'}$ be loops of $PGL_m M_2$. Let $f$ be a continuous function on $G^{n+n'}/\operatorname{Ad}$.

$$E[f(\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,\lambda'_1,\ldots,\lambda'_{n'}})] = \int_{\Omega_R} f\,(\sigma_1(\omega_1)(\lambda_1), \ldots, \sigma_1(\omega_1)(\lambda_n), \\ z^{-1}\sigma_2(\omega_2)(\lambda'_1)z, \ldots, z^{-1}\sigma_2(\omega_2)(\lambda'_{n'})z)\; d\mu_R(\omega_1, z, \omega_2).$$

By the preceding remark, $\sigma_1(\omega_1)(\lambda_i)$ and $\sigma_2(\omega_2)(\lambda'_i)$ depend only on the restrictions of $\omega_1$ to $\mathcal{L}_1 = \{L, L^1_1, \ldots, L^1_N, \lambda_1, \ldots, \lambda_n\}$ and of $\omega_2$ to $\mathcal{L}_2 = \{L, L^2_1, \ldots, L^2_N, \lambda'_1, \ldots, \lambda'_{n'}\}$. In other words, the expectation $E[f(\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,\lambda'_1,\ldots,\lambda'_{n'}})]$ appears to be a function of the variables $\mathcal{H}_{L,L^1_1,\ldots,L^1_N,\lambda_1,\ldots,\lambda_n}$ and $\mathcal{H}_{L,L^2_1,\ldots,L^2_N,\lambda'_1,\ldots,\lambda'_{n'}}$: it is equal to

$$\int_{\Omega_R} f(\sigma_{\mathcal{L}_1}(\mathcal{H}_{L,L^1_1,\ldots,L^1_N,\lambda_1,\ldots,\lambda_n})(\lambda_1, \ldots, \lambda_n), z^{-1}\sigma_{\mathcal{L}_2}(\mathcal{H}_{L,L^2_1,\ldots,L^2_N,\lambda'_1,\ldots,\lambda'_{n'}})(\lambda'_1, \ldots, \lambda'_{n'})z)\; d\mu_R(\omega_1, z, \omega_2).$$

Let $\Gamma_1$ and $\Gamma_2$ be graphs on $M_1$ and $M_2$ such that $L_1, \ldots, L_N, \lambda_1, \ldots, \lambda_n \in \Gamma_1^*$ and $L_1, \ldots, L_N$, $\lambda'_1, \ldots, \lambda'_{n'} \in \Gamma_2^*$. Such graphs exist because we chosed piecewise geodesic loops. Then the last expression is equal to

$$\int_{G^{\Gamma_1}\times C(x)\times G^{\Gamma_2}} f(\sigma_{\mathcal{L}_1}(h^1_L, h^1_{L^1_1}, \ldots, h^1_{L^1_N}, h^1_{\lambda_1}, \ldots, h^1_{\lambda_n})(\lambda_1, \ldots, \lambda_n), \\ z^{-1}\sigma_{\mathcal{L}_2}(h^2_L, h^2_{L^2_1}, \ldots, h^2_{L^2_N}, h^2_{\lambda'_1}, \ldots, h^2_{\lambda'_{n'}})(\lambda'_1, \ldots, \lambda'_{n'})z)\; dP_{M_1}(y, x)dzdP_{M_2}(y', x^{-1}).$$

Since $h^1_L = x$ $P_{M_1}(y, x)$-a.s., there exists a $C(x)$-valued random variable $z_1$ on $G^{\Gamma_1}$ such that

$$\sigma_{\mathcal{L}_1}(h^1_L, h^1_{L^1_1}, \ldots, h^1_{L^1_N}, h^1_{\lambda_1}, \ldots, h^1_{\lambda_n})(\lambda_1, \ldots, \lambda_n) = z_1^{-1}(h^1_{\lambda_1}, \ldots, h^1_{\lambda_n})z_1.$$

Similarly, there exists a $C(x)$-valued random variable $z_2$ on $G^{\Gamma_2}$ such that

$$\sigma_{\mathcal{L}_2}(h^2_L, h^2_{L^2_1}, \ldots, h^2_{L^2_N}, h^2_{\lambda'_1}, \ldots, h^2_{\lambda'_{n'}})(\lambda'_1, \ldots, \lambda'_{n'}) = z_2^{-1}(h^2_{\lambda'_1}, \ldots, h^2_{\lambda'_{n'}})z_2.$$

The integration against $dz$ allows to drop $z_1$ and $z_2$, using left and right invariance. Thus, the expectation that we compute is just equal to

$$\int_{G^{\Gamma_1}\times C(x)\times G^{\Gamma_2}} f(h^1_{\lambda_1},\ldots,h^1_{\lambda_n},z^{-1}h^2_{\lambda'_1}z,\ldots,z^{-1}h^2_{\lambda'_{n'}}z)\, dP_{M_1}(y,x)dzdP_{M_2}(y',x^{-1}).$$

Let us fix the variable $z$. Let $j$ be the discrete gauge transformation on $\Gamma_2$ equal to $z^{-1}$ at $L(0)$ and identically equal to 1 elsewhere. This transformation preserves $P_{M_2}(y',x^{-1})$ because $z\in C(x)$ and it transforms $z^{-1}h^2_{\lambda'_i}z$ into $h^2_{\lambda'_i}$. We find

$$\begin{aligned} &= \int_{C(x)} dz \int_{G^{\Gamma_1}\times G^{\Gamma_2}} f(h^1_{\lambda_1},\ldots,h^1_{\lambda_n},h^2_{\lambda'_1},\ldots,h^2_{\lambda'_{n'}})\, dP_{M_1}(y,x)dP_{M_2}(y',x^{-1}) \\ &= E_{\mu_M(y,y',x)} f(\mathcal{H}_{\lambda_1,\ldots,\lambda_n,\lambda'_1,\ldots,\lambda'_{n'}}) \end{aligned}$$

according to the property of conditional independence. □

### 5.2.2 Making a handle

To complete the picture of the properties of the Yang-Mills measure with respect to the surgery of surfaces, we consider now the situation decribed in section 5.1.3 and look for theorems similar to 5.2.1 and 5.2.6.

We keep the notations of 5.1.3. Recall that we glue together two components of the boundary of a surface $M_1$ to get the surface $M$. We denote by $\imath$ the natural immersion of $M_1$ into $M$ and by $L$ a loop on $M$ whose image is $\imath(B_1) = -\imath(B_2)$. Set $m = L(0)$ and define $m_1$, $m_2$ to be the preimages by $\imath$ of $m$ on $B_1$ and $B_2$ respectively.

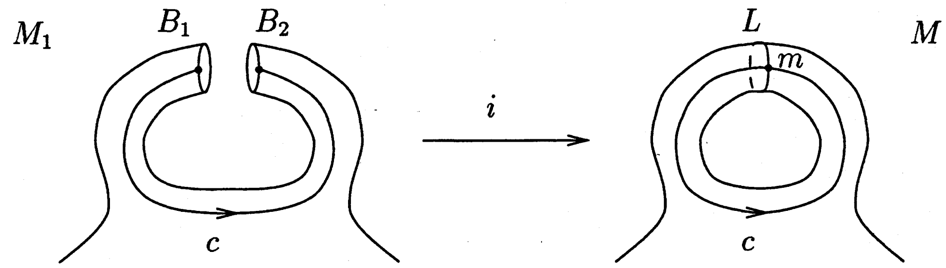


Figure 5.3: The new handle of $M$.

As in the preceding section, it is convenient to use based loops spaces: we set $\Omega_M = \mathcal{F}(L_mM,G)/\operatorname{Ad}$ and $\Omega_{M_1} = \mathcal{F}(L_{m_1}M_1,G)/\operatorname{Ad}$. The $\sigma$-algebra $\mathcal{A}_1$ on $\Omega_{M_1}$ can be seen as a sub-$\sigma$-algebra of $\mathcal{A}$ and we want to study the inclusion $\mathcal{A}_1\subset\mathcal{A}$.

Let us fix a path $c$ from $m_1$ to $m_2$ on $M_1$. Although $c$ is an open path in $M_1$, it becomes a loop on $M$, so that the holonomy along $c$ is well defined on $\Omega_M$ but certainly not $\mathcal{A}_1$-measurable. Once again, there is some information about $\mathcal{H}_c$ in $\mathcal{A}_1$ but there is also something missing, which is closely related to the centralizer of $H_L$. Indeed, denote by $L$ and $L'$ two loops based at $m_1$ and $m_2$ and whose images are $B_1$ and $-B_2$ respectively. Since the holonomies along $L$ and $L'$ must be equal after the gluing procedure, we choose arbitrarily an element $x\in t$ and compute as if $H_L = H_{L'} = x$. The variable $\mathcal{H}_{L,cL'c^{-1}}$ is $\mathcal{A}_1$-measurable and determines to some extent the holonomy along $c$: we interpret its value as that of $[x, H_c^{-1}xH_c]$ and this determines the value of $H_c$ up to left and right multiplications by elements of the centralizer $C(x)$.

The result corresponding to the theorem 5.2.1 is the following:

**Theorem 5.2.7** *Set $N = \dim G - \operatorname{rank} G + 1$. Let $L_1, \dots, L_N$ be disjoint simple loops of $L_m M_1$ that do not meet $\partial M_1$, except at their base point. Then the completion of the $\sigma$-algebra $\mathcal{A}_1 \vee \sigma(\mathcal{H}_{c,L_1,\dots,L_N})$ with respect to the measure $\mu_M(t)$ contains $\mathcal{A}$.*

We endow $M$ with a Riemannian metric such that $L$ is geodesic and choose $L_1, \dots, L_N$ piecewise geodesic for this metric. This metric on $M$ induces by pull-back by $\imath$ a metric on $M_1$. We need a result similar to the lemma 5.2.2.

**Lemma 5.2.8** *On the space $\Omega_M$, the $\sigma$-algebra $\mathcal{A}$ is contained in the completion of the $\sigma$-algebra $\sigma(\mathcal{H}_{\lambda_1,\dots,\lambda_n}, \lambda_1, \dots, \lambda_n \in \imath(PGL_{m_1}M_1 \cup \{c\}))$ with respect to the measure $\mu_M(t)$.*

*Proof.* Just as in the proof of 5.2.2, the point is to prove that any loop of $L_m M$ can be approximated by loops that are equivalent to finite products of loops of $\imath(PGL_{m_1}M_1 \cup \{c\})$. Consider a piecewise geodesic loop. Since it cuts at most a finite number of times $L$ transversally, it is equivalent to a finite product of loops of $PGL_m M$ that are the images by $\imath$ of loops based at $m_1$ or at $m_2$ or of paths with endpoints $m_1$ and $m_2$. Conjugation by $c$ or left or right multiplication by $c$ transform all these paths on $M_1$ into loops based at $m_1$, hence the holonomy along our piecewise geodesic loop can be expressed in terms of the holonomies along the path of $\imath(PGL_{m_1}M_1 \cup \{c\})$. Since any loop of $L_m M$ is a limit of piecewise geodesic loops, the result is proved. □

*Proof of the theorem 5.2.7.* The hypothesis on $L_1, \dots, L_N$ ensure that the law of their holonomy has a density with respect to the natural measure on $G^N/\operatorname{Ad}$. Thus, $\mathcal{H}_{L_1,\dots,L_N}$ takes almost surely its values in $S_N$. Let $l_1, \dots, l_p$ be $p$ loops of $\imath(PGL_{m_1}M_1 \cup \{c\})$. Since this proof is very close to that of 5.2.1, we do not write all details. The variables $\mathcal{H}_{c,L_1,\dots,L_N}$ and $\mathcal{H}_{L_1,\dots,L_N,l_1,\dots,l_p}$ determine $\mathcal{H}_{c,l_1,\dots,l_p}$ because they determine $\mathcal{H}_{c,L_1,\dots,L_N,l_1,\dots,l_p}$. This is proved using a map

$$(G \times S_N)/\operatorname{Ad} \times_{S_N} (S_N \times G^p)/\operatorname{Ad} \xrightarrow{\kappa'} (G \times S_N \times G^p)/\operatorname{Ad}.$$

It is easily checked that this map can be defined in such a way that $\mathcal{H}_{c,L,l} = \kappa'(\mathcal{H}_{c,L}, \mathcal{H}_{L,l})$. The fact that $\kappa'$ is measurable terminates the proof. □

We want also a theorem like 5.2.6 describing more explicitely how to construct a version of the random holonomy on $M$ using that on $M_1$ and a random variable on $C(x)$. So we consider the probability space

$$(\Omega_R, \mathcal{A}_R, \mu_R) = (\Omega_{M_1} \times C(x), \mathcal{A}_1 \otimes \operatorname{Bor}(C(x)), \mu_{M_1}(u, t, t^{-1}) \otimes dz).$$

We want to construct on this space a family of random variables $\widetilde{\mathcal{H}}_{l_1,\dots,l_n}$, with $l_1, \dots, l_n \in \imath(PGL_{m_1}M_1 \cup \{c\})$. We need a result similar to 5.2.5. We fix a section $\tau$ given by 5.2.4 as well as a new measurable section $\tau' : C(x)\backslash G \longrightarrow G$ of the natural projection. The existence $\tau'$ is given by an argument very similar to the proof of the lemma 5.2.4.

**Lemma 5.2.9** *For any subset $\mathcal{L}$ of $L_{m_1}M_1$ containing $L_1,\ldots,L_N,L,cL'c^{-1}$, the sections $\tau$ and $\tau'$ determine a measurable map $\sigma_{\mathcal{L}}:\mathcal{F}(\mathcal{L},G)/\operatorname{Ad}\longrightarrow\mathcal{F}(\mathcal{L}\cup\{c\},G)$, defined $\mu_{M_1}(u,t,t^{-1})$-a.s., such that for any $\omega\in\mathcal{F}(\mathcal{L},G)/\operatorname{Ad}$,*

$$\begin{cases}\sigma_{\mathcal{L}}(\omega)(L)=x\\ \sigma_{\mathcal{L}}(\omega)(c)^{-1}x\sigma_{\mathcal{L}}(\omega)(c)=\sigma_{\mathcal{L}}(\omega)(cL'c^{-1}).\end{cases}$$

*Proof.* Take $\omega$ in $\mathcal{F}(\mathcal{L},G)/\operatorname{Ad}$. By the lemma 5.2.5, we know that $\tau$ and $\omega$ determine a map of $\mathcal{F}(\mathcal{L},G)$ that satisfies the first property, namely that takes the value $x$ on the loop $L$. The point is to choose the value $\sigma_{\mathcal{L}}(c)$ in a convenient way. For this, we use the value $y=\sigma_{\mathcal{L}}(\omega)(cL'c^{-1})$ which is already defined. This $y$ is $\mu_{M_1}(u,t,t^{-1})$-a.s. in the class $t\in G/\operatorname{Ad}$, so that there exists $u\in G$ such that $y=u^{-1}xu$. This $u$ is not uniquely defined, but its class in the quotient $C(x)\backslash G$ is defined. We take for $\sigma_{\mathcal{L}}(\omega)(c)$ the image by $\tau'$ of this class. The map $\sigma_{\mathcal{L}}$ defined in this way is measurable and has the properties wanted. □

As in the preceding section, the maps $\sigma_{\mathcal{L}_1}$ and $\sigma_{\mathcal{L}_2}$ defined for two subsets $\mathcal{L}_1\subset\mathcal{L}_2$ coincide on $\mathcal{L}_1$. We also denote by $\sigma$ the map $\sigma_{L_{m_1}M_1}$.

We are now able to define the variables. Choose $\lambda_1,\ldots,\lambda_n$ in $\imath(PGL_{m_1}M_1)$ and $\omega_R=(\omega_1,z)$ in $\Omega_R$. We set

$$\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,c}(\omega_R)=[\sigma(\omega_1)(\lambda_1),\ldots,\sigma(\omega_1)(\lambda_n),\sigma(\omega_1)(c)z].$$

**Theorem 5.2.10** *The family of random variables $\widetilde{\mathcal{H}}$ has the same law as the family $\mathcal{H}$ under the Yang-Mills measure $\mu_M(u,t)$ on $M$.*

*Proof*: The proof is once again similar to that of the theorem 5.2.6. The argument invoked at the beginning of the proof of the theorem 5.2.6 shows that it is sufficient to prove the equality of the individual laws of the variables $\widetilde{\mathcal{H}}$ and $\mathcal{H}$.

Let $f$ be a continuous function on $G^{n+1}/\operatorname{Ad}$. Let $\lambda_1,\ldots,\lambda_n$ be elements of $\imath(PGL_{m_1}M_1)$. We have

$$Ef(\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,c})=\int_{\Omega_1\times C(x)}f(\sigma(\omega_1)(\lambda_1),\ldots,\sigma(\omega_1)(\lambda_n),\sigma(\omega_1)(c)z)\,d\mu_{M_1}(u,t,t^{-1})(\omega_1)dz.$$

By the remark preceding this theorem, $\sigma(\omega_1)(\lambda_i)$ and $\sigma(\omega_1)(c)$ depend only on the restriction of $\omega_1$ to $\mathcal{L}=\{L_1,\ldots,L_N,L,cL'c^{-1},\lambda_1,\ldots,\lambda_n\}$. Thus, the expectation is a function of $\mathcal{H}_{L_1,\ldots,L_N,L,cL'c^{-1},\lambda_1,\ldots,\lambda_n}$, namely:

$$\int_{\Omega_1\times C(x)}f(\sigma_{\mathcal{L}}(\mathcal{H}_{L_1,\ldots,L_N,L,cL'c^{-1},\lambda_1,\ldots,\lambda_n})(\lambda_1,\ldots,\lambda_n),\\ \sigma_{\mathcal{L}}(\mathcal{H}_{L_1,\ldots,L_N,L,cL'c^{-1},\lambda_1,\ldots,\lambda_n})(c)z)\,d\mu_{M_1}(u,t,t^{-1})(\omega_1)dz.$$

Let $\Gamma$ be a graph on $M$ such that $L_1,\ldots,L_N,\imath(\lambda_1),\ldots,\imath(\lambda_n),\imath(c)\in\Gamma^*$. Let $\Gamma_1$ be the graph on $M_1$ deduced by $\imath^{-1}$. Then the expectation $Ef(\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,c})$ is equal to

$$\int_{G^{\Gamma_1}\times C(x)}f(\sigma_{\mathcal{L}}([h_{L_1},\ldots,h_{L_N},h_L,h_{cL'c^{-1}},h_{\lambda_1},\ldots,h_{\lambda_n}])(\lambda_1,\ldots,\lambda_n),\\ \sigma_{\mathcal{L}}([h_{L_1},\ldots,h_{L_N},h_L,h_{cL'c^{-1}},h_{\lambda_1},\ldots,h_{\lambda_n}])(c)z)\,dP(y,x,x^{-1})dz.$$

where $y = (y_1, \ldots, y_p) \in G^p$ represents $u \in (G/\mathrm{Ad})^p$. There exist two $C(x)$-valued random variables on $G^{\Gamma_1}$ $z_3$ and $z_4$, such that

$$\begin{aligned} Ef(\widetilde{\mathcal{H}}_{\lambda_1,\ldots,\lambda_n,c}) &= \int_{G^{\Gamma_1}\times C(x)} f(z_3 h_{\lambda_1} z_3^{-1}, \ldots, z_3 h_{\lambda_n} z_3^{-1}, z_3 h_c z_4 z)\, dP(y,x,x^{-1}) dz \\ &= \int_{G^{\Gamma_1}\times C(x)} f(h_{\lambda_1}, \ldots, h_{\lambda_n}, h_c z)\, dP(y,x,x^{-1}) dz \\ &= \int_{C(x)} dz \int_{G^{\Gamma_1}} f(h_{\lambda_1}, \ldots, h_{\lambda_n}, h_c)\, dP(y,x,x^{-1}) \\ &= E_{\mu_M(t)} f(\mathcal{H}_{\imath(\lambda_1),\ldots,\imath(\lambda_n),c}). \end{aligned}$$

Between the first line and the second one, we used the biinvariance of $dz$ together with the invariance of $f$ by the diagonal action of $G$ on its arguments. Then we used for each $z$ the discrete gauge transformation $j$ identically equal to 1 except at $m_2$, with $j(m_2) = z^{-1}$. This gauge transformation preserves $dP(y,x,x^{-1})$,because $z \in C(x) = C(x^{-1})$. The last equality follows from the theorem 5.1.3. □

## 5.3 Conditional partition functions

The propositions 5.1.2 and 5.1.4 show that the conditional partition functions deserve to be studied separately. We are interested in the conditional partition functions with respect to the boundary components of a surface. The importance of these functions had been pointed out by Witten [Wi]. He already proved their algebraic properties using character expansions.

### 5.3.1 Algebraic properties of the partition functions

Let us summarize the properties of the conditional partition function that were proved at different points in the preceding chapters.

Let $(M, \sigma)$ be a surface, with a boundary $\partial M = N_1 \cup \ldots \cup N_p$ or without boundary. For any graph $\Gamma$ on $M$ and any $x_1, \ldots, x_p \in G$, the number $\int_{G^\Gamma} D^\Gamma\, d\nu_{x_1} \ldots d\nu_{x_p}\, dg'$ is well defined (see section 1.5). By the lemma 1.5.5, it depends only on the conjugacy classes of the $x_i$'s: it is a central function of the $x_i$'s. By the proposition 2.7.4, which is true on a surface with boundary by the lemma 2.8.2, this number does not depend on $\Gamma$. If $t_1 = [x_1], \ldots, t_p = [x_p]$, we denote this number by $Z_M(t_1, \ldots, t_p)$, or just $Z_M$ if $M$ is closed.

Consider now an area-preserving diffeomorphism between $(M, \sigma)$ and another surface $(M', \sigma')$, i.e. a diffeomorphism that sends $\sigma$ to $\sigma'$. Then an expression like $\int_{G^\Gamma} D^\Gamma\, d\nu_{x_1} \ldots d\nu_{x_p}\, dg'$ is obviously invariant by this diffeomorphism. Thus, the function $Z_M$ depends on $M$ only through its class modulo area-preserving diffeomorphisms. As a consequence of Moser's theorem, that we extended to the case of surfaces with boundary in the proof of 2.2.1, this class is easily parametrized by a triple $(p, g, T) \in \mathbf{N}^2 \times \mathbf{R}_+^*$, where $p$ is the number of components of $\partial M$, $g$ the genus of $M$ and $T$ the total surface of $M$. Another consequence of this invariance and of Moser's theorem is the symmetry of $Z_M$. Indeed, given any two components of $\partial M$, there exists a diffeomorphism of $M$ that permutes these components, hence an area-preserving diffeomorphism. Thus, for any $i$ and $j$ such that $1 \le i < j \le p$, $Z_M(t_1, \ldots, t_i, \ldots, t_j, \ldots, t_p) = Z_M(t_1, \ldots, t_j, \ldots, t_i, \ldots, t_p)$.

Let us give an expression of the function $Z_M$ that makes clear that it depends on $M$ only through $p, g, T$. We consider a graph with only one face on $M$, such that the boundary of this face is $[a_1, b_1] \dots [a_g, b_g]c_1^{-1}N_1c_1 \dots c_p^{-1}N_pc_p$, where $a_i, b_i$ are the edges of a polygonal fundamental domain in the universal covering of $M$ and each $c_i$ joins $N_i$ to a point on the boundary of this fundamental domain. We find

$$Z_{p,g,T}(t_1, \dots, t_n) = \int_{G^{2g+p}} p_{\sigma(M)}(y_1^{-1}x_1y_1 \dots y_p^{-1}x_py_p[a_1, b_1] \dots [a_g, b_g]) \, da_1db_1 \dots da_gdb_gdy_1 \dots dy_p, \tag{5.1}$$

where the $x_i$'s are arbitrary representatives of the $t_i$'s, and

$$Z_{0,g,T} = \int_{G^{2g+p}} p_{\sigma(M)}([a_1, b_1] \dots [a_g, b_g]) \, da_1db_1 \dots da_gdb_g \tag{5.2}$$

when $p = 0$, i.e. when $M$ is closed. From now on, we index the function $Z$ by the triple $(p, g, T)$ instead of the surface $M$.

The expression 5.1 shows also that $Z_{p,g,T}$ is a smooth central function on $G^p$ and a continuous function on $(G/\operatorname{Ad})^p$. On the other hand, the symmetry of $Z_{p,g,T}$ is less obvious in this form. Using character expansions, it is possible to give a manifestly symmetric expression of $Z_{p,g,T}$. The reader which is not familiar with the characters of a compact Lie group should read the beginning of the section 4.2.2 before to go further. Using the expansion of the heat kernel proved in proposition 4.2.4, we transform (5.1) and (5.2) into:

$$Z_{p,g,T}(t_1, \dots, t_p) = \sum_{\alpha \in \widehat{G}} (\dim \alpha)^{2-2g} e^{-\frac{c_2(\alpha)}{2}T} \prod_{i=1}^{p} \frac{\chi_\alpha(t_i)}{\dim \alpha}, \tag{5.3}$$

$$Z_{0,g,T} = \sum_{\alpha \in \widehat{G}} (\dim \alpha)^{2-2g} e^{-\frac{c_2(\alpha)}{2}T}. \tag{5.4}$$

Before to state these results in a theorem, recall that $G/\operatorname{Ad}$ is endowed with the image measure of the Haar measure by the canonical projection $G \longrightarrow G/\operatorname{Ad}$. As we said at the beginning of this section, this theorem was essentially already proved by Witten.

**Theorem 5.3.1** *For each $(p, g, T) \in \mathbf{N}^2 \times \mathbf{R}_+^*$, the function $Z_{g,p,T}$ is a continuous symmetric function on $(G/\operatorname{Ad})^p$. Moreover, for any $(p', g', T')$ and any $t_1, \dots, t_p, t'_1, \dots, t'_{p'} \in G/\operatorname{Ad}$, the following relations hold:*

$$\int_{G/\operatorname{Ad}} Z_{p+1,g,T}(t_1, \dots, t_p, t) Z_{p'+1,g',T'}(t^{-1}, t'_1, \dots, t'_{p'}) \, dt = Z_{p+p',g+g',T+T'}(t_1, \dots, t_p, t'_1, \dots, t'_{p'}), \tag{5.5}$$

$$\int_{G/\operatorname{Ad}} Z_{p+2,g,T}(t_1, \dots, t_p, t, t^{-1}) \, dt = Z_{p,g+1,T}(t_1, \dots, t_p). \tag{5.6}$$

*Proof*. The symmetry and continuity of $Z_{p,g,T}$ were already discussed. The relation (5.5) is a consequence of the proposition 5.1.2. Indeed, in this proposition, the number of components of the boundary of $M$ is $p_1 + p_2 - 2$, where $p_1$ and $p_2$ are the number of components of $\partial M_1$ and $\partial M_2$ and its genus and total surface are the sums of those of $M_1$ and $M_2$. This gives:

$$\int_{G/\operatorname{Ad}} Z_{p+1,g,T}(t_1, \dots, t_p, t) Z_{p'+1,g',T'}(t^{-1}, t'_1, \dots, t'_{p'}) \, dt = \int_{G/\operatorname{Ad}} Z_M(t_1, \dots, t_p, t, t'_1, \dots, t'_{p'}) \, dt.$$

In this last partition function, the variable $t$ corresponds to an interior loop of $M$, not to a component of the boundary. If we compute this function using an expression like $\int_{G^\Gamma} D^\Gamma \, d\nu_x \ldots$, where $[x] = t$, we see that the conditioning with respect to this interior loop disappears if we integrate over $x$, so that the last integral is exactly equal to $Z_{p+p',g+g',T+T'}(t_1,\ldots,t_p,t'_1,\ldots,t'_{p'})$.

Similarly, the relation (5.6) is a consequence of the proposition 5.1.4. In this gluing operation, the surface $M_1$ had lost two components of its boundary and gained one handle. Thus,

$$\int_{G/\,\mathrm{Ad}} Z_{p+2,g,T}(t_1,\ldots,t_p,t,t^{-1})\,dt = \int_{G/\,\mathrm{Ad}} Z_M(t_1,\ldots,t_p,t)\,dt.$$

Just as above, $t$ disappears when we integrate against $dt$ and we find $Z_{p,g+1,T}(t_1,\ldots,t_p)$. □

### 5.3.2 Building bricks of the theory

The two relations (5.5) and (5.6) are the analytic counterparts of the behaviour of the Yang-Mills measure under the two basic surgery operations. It is well known that a few elementary surfaces are enough to build any surface by a sequence of these basic operations, namely a disk and a three-holed sphere (see fig. 5.4). It is not surprising that a corresponding result holds for the conditional partition functions.

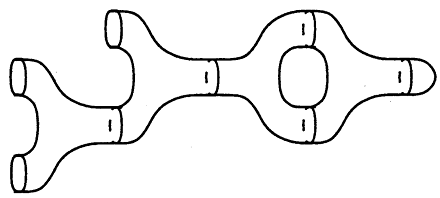

Figure 5.4: An example of decomposition in three-holed spheres and disks.

**Proposition 5.3.2** *The family of functions $Z_{p,g,T}$ is completely determined by the functions $Z_{1,0,T}$ and $Z_{3,0,T}$, $T > 0$, and the relations (5.5) and (5.6).*

*Proof.* We choose $g, p$ and construct the functions $Z_{g,p,T}$, $T > 0$, starting with the functions $Z_{1,0,T}$ and $Z_{3,0,T}$.

Suppose first that $p + 2g \geq 3$. In this case, repeated applications of (5.5) to the function $Z_{3,0,T}$ allow to compute $Z_{p+2g,0,T}$ for any $T > 0$. Now, $g$ applications of (5.6) to $Z_{p+2g,0,T}$ give the function $Z_{p,g,T}$.

The case $p + 2g = 2$ happens when $(p, g) = (0, 1)$ or $(2, 0)$. The first case is that of a closed torus. Start with a three-holed sphere and glue two components of its boundary. We get a torus with one hole. This corresponds to (5.6) applied to $Z_{3,0,T}$ to get $Z_{1,1,T}$. Now, it remains to glue a disk on the hole of the torus. In other words, the relation (5.5) applied to $Z_{1,1,T}$ and $Z_{1,0,T}$ gives $Z_{0,1,T}$. The case $(p, g) = (2, 0)$ is that of a cylinder, which is obtained by gluing a disk on a three-holed sphere. So, $Z_{2,0,T}$ is obtained by applying (5.5) to $Z_{3,0,T}$ and $Z_{1,0,T}$.

The case $p + 2g = 1$ happens only when $(p, g) = (1, 0)$ and the corresponding function is one of our building bricks.

Finally, $p = g = 0$ is a closed sphere, which can be obtained by gluing two disks together. So, (5.5) applied to $Z_{1,0,T}$ gives $Z_{0,0,T}$. □

The natural question arising from this result is to identify the elementary functions $Z_{1,0,T}$ and $Z_{3,0,T}$.

**Proposition 5.3.3** *The function $Z_{1,0,T}$ is the projection on $G/\operatorname{Ad}$ of the heat kernel $p_T$ on $G$.*

*Proof*. Any expression of $Z_{1,0,T}$, for example (5.1), proves this assertion. □

The meaning of $Z_{3,0,T}$ is less obvious. Let us consider it a central function on $G$.

**Lemma 5.3.4** *For any $(p, g) \in \mathbb{N}^2$, any $T, T' > 0$, the following relation holds between central functions on $G$:*

$$e^{T\frac{\Delta}{2}} Z_{p,g,T'}(x_1, \dots, x_{p-1}, \cdot)(x) = Z_{p,g,T+T'}(x_1, \dots, x_{p-1}, x).$$

In other words, $Z_{p,g,T}$ is a solution of the heat equation in each of its variables.

*Proof*. Given the fact that $\Delta\chi_\alpha = -c_2(\alpha)\chi_\alpha$ for any irreducible representation $\alpha$, this assertion is a simple consequence of 5.3. □

This lemma shows that the algebraic meaning of $Z_{3,0,T}$, if there is one, is contained in the formal limit $\lim_{T\to 0} Z_{3,0,T}$. Let us look at a three-holed sphere with a very small surface (see fig. 5.5).

At the $T \to 0$ limit, there remains only two adjacent circles that form a graph. If we remind that the conditional partition function is the density of the natural law of the holonomy along the boundary of a surface (see 2.8.1), we see in this case that $Z_{3,0,T}$ is closely related to the multiplication in $G$.

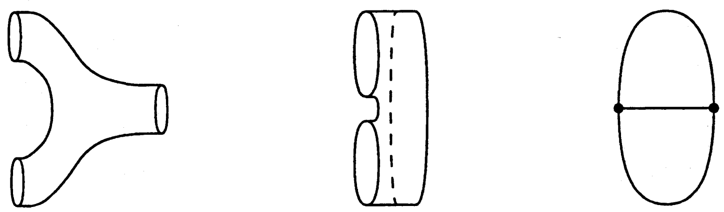

Figure 5.5: A thiner and thiner three-holed sphere.

Recall that the convolution product of two function $f, g \in L^2(G, dx)$ is defined by $f * g(x) = \int_G f(y)g(y^{-1}x)\, dx$. It is also a square-integrable function. Let us denote by $L^2(G)^G$ the space of central $L^2$ functions on $G$. It is easily checked that the convolution product is a commutative operation in $L^2(G)^G$. Indeed, let $f$ and $g$ be two central functions. Using the left invariance of the Haar measure and its invariance by inversion, we get:

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dy = \int_G f(xy)g(y^{-1})\, dy = \int_G g(y)f(y^{-1}x)\, dy = g * f(x).$$

It remains to check that $f * g$ is central.

$$f * g(xy) = \int_G f(z)g(z^{-1}xy)\,dz = \int_G f(z)g(yz^{-1}x)\,dz = \int_G f(xzy)g(z^{-1})\,dz = g * f(yx)$$

and the result follows by commutativity of $*$.

This product on $L^2(G)^G$ is what remains from the product on $G$ when one considers conjugacy classes. The following result relates $Z_{3,0,T}$ to this product.

**Proposition 5.3.5** *Let $f$ and $g$ be two functions of $L^2(G)^G$. Then the following equality holds in $L^2(G)^G$:*

$$\int_{G^2} f_1(x_1)f_2(x_2)Z_{3,0,T}(x_1,x_2,x)\,dx_1dx_2 = \left[e^{-T\frac{\Delta}{2}}(f_1 * f_2)\right](x).$$

This gives the interpretation that we were looking for: formally, $Z_{3,0,0}$ is the distributional kernel of the operator $* : L^2(G)^G \otimes L^2(G)^G \longrightarrow L^2(G)^G$. From this point of view, the commutativity of $*$ finds its geometric counterpart in the fact that two holes of a three-holed sphere are indistinguishable under area-preserving diffeomorphisms.

*Proof.* We use the fact that any central square-integrable function can be expanded into a series of characters. Thus, it is sufficient to prove the theorem when $f_1$ and $f_2$ are the characters of two irreducible representations $\alpha$ and $\beta$. We use the expansion of $Z_{3,0,T}$ given by (5.3).

$$\begin{aligned}\int_{G^2} \chi_\alpha(x_1)\chi_\beta(x_2)Z_{3,0,T}(x_1,x_2,x)\,dx_1dx_2 &= \sum_{\gamma\in\widehat{G}}(\dim\gamma)^{-1}e^{-\frac{c_2(\gamma)}{2}T}\chi_\gamma(x) \\ &\qquad \int_G \chi_\alpha(x_1)\chi_\gamma(x_1)\,dx_1 \int_G \chi_\beta(x_2)\chi_\gamma(x_2)\,dx_2 \\ &= \sum_{\gamma\in\widehat{G}}(\dim\gamma)^{-1}e^{-\frac{c_2(\gamma)}{2}T}\chi_\gamma(x)\delta_{\alpha,\gamma}\delta_{\beta,\gamma} \\ &= \delta_{\alpha,\beta}e^{-\frac{c_2(\alpha)}{2}T}\frac{\chi_\alpha(x)}{\dim\alpha}. \qquad (5.7)\end{aligned}$$

On the other hand, the orthogonality relations between characters imply:

$$\chi_\alpha * \chi_\beta(x) = \int_G \chi_\alpha(y)\chi_\beta(y^{-1}x)\,dy = \delta_{\alpha,\beta}\frac{\chi_\alpha(x)}{\dim\alpha}.$$

Finally, the fact that $\Delta\chi_\alpha = -c_2(\alpha)\chi_\alpha$ shows that the expression (5.7) is exactly equal to $\exp(T\frac{\Delta}{2})(\chi_\alpha * \chi_\beta)$. □

### 5.3.3 Transition fonctions of the Markov field

Consider the following very simple example. Take $M$ to be a cylinder $S^1 \times [0,1]$ endowed with the Riemannian volume of the standard product metric, with total volume equal to 1. Pick two elements $t_0$ and $t_1$ in $G/\mathrm{Ad}$ and consider the Yang-Mills measure $\mu_M(t_0^{-1}, t_1)$.

For each $s \in [0,1]$, let $l_s$ be a loop whose image is the slice $S^1 \times \{s\}$ in $M$. The family of random variables $\mathcal{H}_{l_s}, s \in [0,1]$ is a $G/\mathrm{Ad}$-valued process with index set $[0,1]$. The Markov property of the Yang-Mills field (theorem 5.1.1) implies that this process is a Markov process. Let us compute its transition functions. Choose $0 < s_1 < s_2 < 1$. Let $f_1$ and $f_2$ be two continuous functions on $G/\mathrm{Ad}$. We know by the proposition 1.5.2 that

$$E_{\mu_M(t_0^{-1},t_1)}[f_1(\mathcal{H}_{l_{s_1}})f_2(\mathcal{H}_{l_{s_2}})] = \frac{1}{Z_{2,0,1}(t_0^{-1},t_1)}\int_{(G/\mathrm{Ad})^2} f_1(u_1)f_2(u_2)Z_M(t_0^{-1},u_1,u_2,t_1)\,du_1du_2,$$

where the partition function in the integral is taken with respect to $l_0, l_{s_1}, l_{s_2}, l_1$. Using the proposition 5.1.2, we find that the expectation $E_{\mu_M(t_0,t_1)}[f_1(\mathcal{H}_{l_{s_1}})f_2(\mathcal{H}_{l_{s_2}})]$ is equal to

$$\frac{1}{Z_{2,0,1}(t_0^{-1},t_1)}\int_{(G/\mathrm{Ad})^2} f_1(u_1)f_2(u_2)Z_{2,0,s_1}(t_0^{-1},u_1)Z_{2,0,s_2-s_1}(u_1^{-1},u_2)Z_{2,0,1-s_2}(u_2^{-1},t_1)\,du_1du_2.$$

Thus, the conditional expectation $E[f_1(\mathcal{H}_{l_{s_1}})|\mathcal{H}_{l_{s_2}}]$ is equal to

$$\frac{1}{Z_{2,0,1}(t_0^{-1},t_1)}\int_{G/\mathrm{Ad}} f_1(u)Z_{2,0,s_1}(t_0^{-1},u)Z_{2,0,s_2-s_1}(u^{-1},\mathcal{H}_{l_{s_2}})Z_{2,0,1-s_2}(\mathcal{H}_{l_{s_2}}^{-1},t_1)\,du.$$

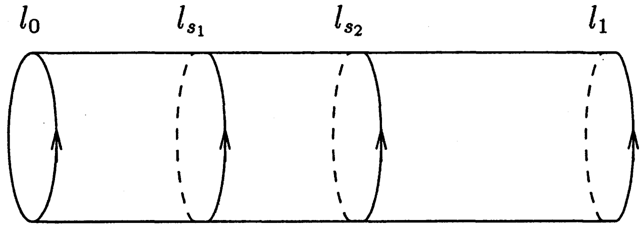


Figure 5.6: Transition functions on a cylinder.

The transition functions of the process are exactly the functions $Z_{2,0,T}$. This suggests that the functions $Z_{p,g,T}$ determine to some extent the law of the random holonomy. This was essentially the content of the proposition 1.5.2. More precisely, this proposition shows that it is possible to write down the law of the holonomy along a family of disjoint simple loops using only the partition functions. Using the continuity of the random holonomy, we can extend this statement a little bit. Indeed, let $l_1,\ldots,l_n$ be simple loops on $M$ that can be approximated by families of disjoint simple loops in such a way that none of the components of $M$ delimited by these families has a surface tending to 0. Then the density of the law of the variable $(\mathcal{H}_{l_1},\ldots,\mathcal{H}_{l_n})$ is the limit of the densities of the holonomies along the approximating families. Since $Z_{p,g,T}$ depends continuously on its parameters and also on $T$ provided $T > 0$, this limit density is also a combination of the functions $Z_{p,g,T}$.

Nevertheless, one cannot hope to express the law on $G^2/\mathrm{Ad}$ of a variable like $\mathcal{H}_{l_1,l_2}$ using partition functions when $l_1$ and $l_2$ are based at the same point. Indeed, partition functions are functions on the space $(G/\mathrm{Ad})^p$ which is much smaller than the space $G^p/\mathrm{Ad}$.

# Bibliography


[Al] ALBEVERIO, S., HØEGH-KROHN, R., AND HOLDEN, H. Stochastic Lie group-valued measures and their relations to stochastic curve integrals, gauge fields and Markov cosurfaces. In *Stochastic processes—mathematics and physics (Bielefeld, 1984)*. Springer, Berlin, 1986, pp. 1–24.

[AL] ASHTEKAR, A., AND LEWANDOWSKI, J. Projective techniques and functional integration for gauge theories. *J. Math. Phys. 36*, 5 (1995), 2170–2191.

[AB] ATIYAH, M. F., AND BOTT, R. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A 308*, 1505 (1983), 523–615.

[Au] AUBIN, T. *Some nonlinear problems in Riemannian geometry.* Springer-Verlag, Berlin, 1998.

[Be] BECKER, C. Wilson loops in two-dimensional space-time regarded as white noise. *J. Funct. Anal. 134*, 2 (1995), 321–349.

[BS] BECKER, C., AND SENGUPTA, A. Sewing Yang-Mills measures and moduli spaces over compact surfaces. *J. Funct. Anal. 152*, 1 (1998), 74–99.

[BG] BERGER, M., AND GOSTIAUX, B. *Géométrie différentielle.* Librairie Armand Colin, Paris, 1972. Maîtrise de mathématiques, Collection U/Série "Mathématiques".

[Bl] BLEECKER, D. *Gauge theory and variational principles.* Addison-Wesley Publishing Co., Reading, Mass., 1981.

[B1] BOURBAKI, N. *Éléments de mathématique. I: Les structures fondamentales de l'analyse. Fascicule VIII. Livre III: Topologie générale. Chapitre 9: Utilisation des nombres réels en topologie générale.* Hermann, Paris, 1958. Deuxième édition revue et augmentée. Actualités Scientifiques et Industrielles, No. 1045.

[B2] BOURBAKI, N. *Éléments de mathématique. Fasc. XXVI. Groupes et algèbres de Lie. Chapitre I: Algèbres de Lie.* Hermann, Paris, 1971. Seconde édition. Actualités Scientifiques et Industrielles, No. 1285.

[Br] BRÖCKER, T., AND TOM DIECK, T. *Representations of compact Lie groups.* Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[Ch] CHERN, S.-S. An elementary proof of the existence of isothermal parameters on a surface. *Proc. Amer. Math. Soc. 6* (1955), 771–782.

[Ck] CHOKSI, J. R. Inverse limits of measure spaces. *Proc. London Math. Soc. (3) 8* (1958), 321–342.

[DM] DELLACHERIE, C., AND MEYER, P.-A. *Probabilities and potential. C.* North-Holland Publishing Co., Amsterdam, 1988.

[dC] DO CARMO, M. P. A. *Riemannian geometry.* Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[Dr1] DRIVER, B. K. Two-dimensional Euclidean quantized Yang-Mills fields. In *Probability models in mathematical physics (Colorado Springs, CO, 1990).* World Sci. Publishing, Teaneck, NJ, 1991, pp. 21–36.

[Dr2] DRIVER, B. K. $YM_2$: continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys. 123*, 4 (1989), 575–616.

[Fi1] FINE, D. S. Quantum Yang-Mills on the two-sphere. *Comm. Math. Phys. 134*, 2 (1990), 273–292.

[Fi2] FINE, D. S. Quantum Yang-Mills on a Riemann surface. *Comm. Math. Phys. 140*, 2 (1991), 321–338.

[Fo] FORMAN, R. Small volume limits of 2-d Yang-Mills. *Comm. Math. Phys. 151*, 1 (1993), 39–52.

[Gy] GRAY, A. *Tubes.* Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.

[Gr] GROSS, L. A Poincaré lemma for connection forms. *J. Funct. Anal. 63*, 1 (1985), 1–46.

[GKS] GROSS, L., KING, C., AND SENGUPTA, A. Two-dimensional Yang-Mills theory via stochastic differential equations. *Ann. Physics 194*, 1 (1989), 65–112.

[IW] IKEDA, N., AND WATANABE, S. *Stochastic differential equations and diffusion processes*, second ed. North-Holland Publishing Co., Amsterdam, 1989.

[KS] KING, C., AND SENGUPTA, A. The semiclassical limit of the two-dimensional quantum Yang-Mills model. *J. Math. Phys. 35*, 10 (1994), 5354–5361. Topology and physics.

[KN] KOBAYASHI, S., AND NOMIZU, K. *Foundations of differential geometry. Vol. I.* John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[Ku] KURATOWSKI, K. *Topology. Vol. I.* Academic Press, New York, 1966. New edition, revised and augmented. Translated from the French by J. Jaworowski.

[Liu] LIU, K. Heat kernel and moduli space. *Math. Res. Lett. 3*, 6 (1996), 743–762.

[Ma] MASSEY, W. S. *A basic course in algebraic topology.* Springer-Verlag, New York, 1991.

[Mi] MIGDAL, A. A. Recursion equations in gauge field theories. *Sov. Phys. JETP 42*, 3 (1975), 413–418.

[Se1] SENGUPTA, A. The Yang-Mills measure for $S^2$. *J. Funct. Anal. 108*, 2 (1992), 231–273.

[Se2] SENGUPTA, A. Gauge theory on compact surfaces. *Mem. Amer. Math. Soc. 126*, 600 (1997), viii+85.

[Se3] SENGUPTA, A. Yang-Mills on surfaces with boundary: quantum theory and symplectic limit. *Comm. Math. Phys. 183*, 3 (1997), 661–705.

[Si] SIMON, B. *Representations of finite and compact groups.* American Mathematical Society, Providence, RI, 1996.

[Va] VAROPOULOS, N. T., SALOFF-COSTE, L., AND COULHON, T. *Analysis and geometry on groups.* Cambridge University Press, Cambridge, 1992.

[Wi] WITTEN, E. On quantum gauge theories in two dimensions. *Comm. Math. Phys. 141*, 1 (1991), 153–209.