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DENNIS ERIKSSON
Un isomorphisme de Deligne-Riemann-Roch, 2008

A Deligne-Riemann-Roch isomorphism

Dennis Eriksson

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CONTENTS

FOREWORD

Then he [Poirot] uttered an exclamation:
"But where is it that you are taking me? This is the seashore ahead of us!"
"Ah, I must explain our geography to you. You'll see for yourself in a minute. There's a creek, you see, Camel Creek, they call it, runs inland - looks almost like a river mouth, but it isn't - it's just a sea. To get to Alderbury by land you have to go right inland and round the creek, but the shortest way from one house to the other is to row across this narrow bit of the creek. Alderburry is just opposite - there, you can see the house through the trees".

"Five little pigs", Agatha Christie

Towards the end of this work, Thomason's ghost appeared in the author's dream. Encouraged by the story of Thomason and Trobaugh, he was hoping for some divine intervention. However, even before asking anything, Thomason started to explain, somewhat reluctantly, the definition of a local complete intersection morphism, and not even the definition of [P71] but rather that of [WF85]. To Thomason I could have explained why he shouldn't be listed as a coauthor.

Concerning working with stacks, I met José Burgos on a train (also in a dream) where I explained to him that the category of G-equivariant sheaves on a scheme X is naturally equivalent to that of ordinary (cartesian) sheaves on the quotient stack $[X/G]$. For him this was very natural, and he made an immediate reference to Thomason's work, where he observes that by Morita equivalence, the G-sheaves on $X \times_H G$ are equivalent to that of the H-sheaves on X. Of course, there is no real relationship between the two observations, albeit both being true, but this was one of the main motivations for introducing stacks in the mixture.

We finish with the following synopsis of "The Dreams in the Witch House" by H. P. Lovecraft:

"Plagued by insane nightmare visions, Walter Gilman seeks help in Miskatonic University's infamous library of forbidden books, where, in the pages of Abdul Alhazred's dreaded Necronomicon, he finds terrible hints that seem to connect his own studies in advanced mathematics with the fantastic legends of elder magic".

Notation and conventions

The category of schemes (resp. algebraic spaces) over a fixed scheme S is denoted by Sch/S (resp. Esp/S). Likewise, the category of smooth finite type schemes over S is denoted by Sm/S.

A regular scheme will always mean a separated, Nötherian regular scheme.

If X is a scheme, an "$\mathcal{O}_X$-module" will denote a sheaf of modules, as opposed to a general presheaf of modules.

Let S be an scheme, $Y \to X$ a morphism of algebraic S-spaces. Also, let G be a separated algebraic X-space group-object, flat and of finite presentation over X. Then the stack-quotient $[Y/G/X]$ is always considered to be the fppf-stack quotient (see [GL00] 2.4.2 for a definition and Cor. 10.8 for a proof of that it defines an Artin stack).

A vector bundle on an algebraic stack will always refer to a locally free sheaf of finite rank.

If $\mathcal{C}$ is any category, the nerve of the category, $N\mathcal{C}$ will be the simplicial set whose objects in degree n are the sequences

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n.$$

Let E be a vector bundle on an algebraic stack $\mathcal{X}$. A filtration of vector bundles $F_0 \subset \dots \subset F_i \subset F_{i+1} \subset \dots \subset E$ is said to be an admissible filtration whenever every quotient F_i/F_{i-1} is also a vector bundle.

INTRODUCTION

This thesis consists of two main parts.

Brauer-Manin obstruction for zero-cycles on curves

The first part (c.f. chapter 1) concerns a positive result on a Hasse-type principle for zero-cycles of degree one curves. The "Hasse principle", stated somewhat informally, says that whenever a variety over a number field k has points in every completion of k, it actually has a k-rational point. Formulated liked this the Hasse principle is well-known to be false in general, but there are however many instances where one can say something. We prove that under finitude of the Tate-Shafarevich group, if a curve over a number field has a zero-cycle of degree one in every completion orthogonal to a finite subquotient of the Brauer-group it has a global zero-cycle of degree one. We also introduce the concept of a "generic period" and relate it to the Brauer-Manin obstruction, as well as give another description of the generic period in terms of Suslin homology. The result on zero-cycles on curves was also independently found by Victor Scharaschkin and a joint article on this topic is being processed.

A Deligne-Riemann-Roch isomorphism

The second part is the principal part of this thesis. This work grew out of an attempt to answer equivariant analogues of Deligne's program in [Del87]. This on the other hand was inspired by the philosophy of S. Arakelov in [Ara74a], [Ara74b] and is an attempt to understand the works of [Fal84] and [HG84]. The program has two parts, one geometric and one analytic, and we will only treat the geometric. The context is the following for which Deligne's introduction in [Del87] is the best reference. If $f : X \to Y$ is a proper morphism of smooth quasi-projective varieties over a field k, one version of the Grothendieck-Riemann-Roch (c.f. [Ful98], chapter 15) states that there is a commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\operatorname{Td}(T_f)\operatorname{ch}} & \bigoplus_i \operatorname{CH}^i(X)_{\mathbb{Q}} . \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{\operatorname{ch}} & \bigoplus_i \operatorname{CH}^i(Y)_{\mathbb{Q}} \end{array}$$

As a special case, we get an identity

$$\mathrm{c}_1(Rf_*(E)) = f_*(\operatorname{Td}(T_f^\vee)\operatorname{ch}(E))^{(1)}$$

where $f_*(?)^{(1)}$ denotes the degree 1-part of $f_*(?)(E)$. For smooth varieties we have an identification $\operatorname{Pic}(Y) = \operatorname{CH}^1(Y)$ and the above can thus be understood as an equality in the rational Picard group, which has a categorial interpretation as the category of rational line bundles $\mathfrak{Pic}(X)_{\mathbb{Q}}$ modulo isomorphisms (c.f. Definition 6.5.0.2). The departure point is the following:

Question 1. Can we define two functors from the category of vector bundles to the category of line bundles (possibly up to torsion)

$$\mathbf{P}(X) \to \mathfrak{Pic}(Y)_{\mathbb{Q}}$$

and a natural isomorphism of the two, that naturally realizes the above identity?

One of the functors, the lifting of $\mathrm{c}_1(Rf_*(E))$ has a natural interpretation as the determinant (in the sense of [FK76]) of the perfect complex $Rf_*(E)$, which is a natural line bundle. We are led to the following question:

Question 2. Can we define categorial versions of all the groups involved, introduce the proper kind of functors, and define isomorphisms thereof, lifting the Grothendieck-Riemann-Roch identity?

In [Del87] these questions are completely solved for the case of a smooth curve $C \to S$ and the determinant of the cohomology. To answer this question in general one is led, among many other things, to consider the category of virtual vector bundles on a scheme X, defined as in [Del87], section 4. It amounts to the fundamental groupoid of Quillen's construction of the algebraic K-theory space of X. It arises in a purely categorical fashion as a universal Picard category with respect to an exact category in a way analogous to that of the relation of the category $\mathbf{P}(X)$ and $K_0(X)$. In chapter 2 we recall this theory and prove that the same construction is possible in the context of Waldhausen categories, i.e. replacing exact categories by categories with cofibrations and weak equivalences. This allows us to put ourself in the setting of [RT90]. As such, the dependence on the derived category is revealed and we can consider virtual categories of perfect complexes on a scheme and define pushforward of a proper perfect morphism $f : X \to Y$ of Nötherian schemes in quite large generality, as well as define virtual categories of complexes with support on a closed subset. This part is more or less independent of the rest of the text, but can be used to extend several of the results on functoriality from the regular case to the general case. As another corollary we find that the derived category of a (small) complicial biWaldhausen category $\mathcal{C}$ determine, via virtual categories, $K_0(\mathcal{C})$ and $K_1(\mathcal{C})$ and thus we have a derived category-categorization of $K_1(\mathcal{C})$. Not surprisingly it involves "distinguished triangles of distinguished triangles".

The correct equivariant base-setting for the above is, if one is to believe the philosophy of [Toe99a], that of algebraic stacks. In chapter 3 we give a treatment of this idea. We associate various virtual categories to an algebraic stack and record their main properties. This includes the new *cohomological virtual category*, $W(\mathcal{X})$, inspired by [Toe99a], which is a virtual version of the K-cohomology spectrum of *ibid.* For a regular scheme this is just the usual virtual category tensor $\mathbb{Q}$. Here we also prove a fundamental splitting principle, Theorem 3.2.1, inspired by [Fra90], which gives us a criterium for when we can descend an isomorphism in a Picard category from a flag variety. Following [Gra92], we also show that the virtual category of an algebraic stack admits canonical Adams-operations (c.f. Proposition 3.3.1).

The next chapter 4 is the most important background-material for applications to functoriality. We apply the philosophy of [Rio06] and establish a version of the following meta-Theorem (compare Theorem 4.0.6): Any operation on the presheaf $X \mapsto K_0(-)_{\mathbb{Q}}$ on the category of regular schemes has a canonical lifting to an operation on the cohomological virtual category of a regular algebraic stack, strictly stable under base-change of the same. Again applying the philosophy of [Rio06] we are able to construct a virtual version of the weight filtration of K-cohomology. For regular schemes this gives a weight filtration of the virtual category tensor $\mathbb{Q}$, and since there is much control of K_0 and K_1 of such a scheme it will allow us to make sense of a statement like: Given a regular scheme X, and a line bundle on L, for big enough n, there is a canonical isomorphism $(L-1)^{\otimes n} = 0$ valid in the virtual category of X tensor $\mathbb{Q}$. This whole machinery is made possible by a certain rigidity-statement that is based on the fact that $K_1(\mathbb{Z})$ is torsion and so certain homotopy-classes in question lift to homotopies up to unique homotopy and define operations on the virtual category. Now, the correct setting of 2 is without a doubt working with simplicial sets such as Quillen's construction of K-theory and some formalism of ∞-categories, and one might ask oneself if it there is a strictification-process (c.f. [WD89]) that allows us to lift homotopy-classes to canonical maps of spaces, up to some weak form of equivalence. The obstructions to doing so rationally are given by elements of a group of power-series with coefficients in $K_i(\mathbb{Z})_{\mathbb{Q}}$, and it is well-known from Borel's work on regulators (c.f. [Bor74]) that $K_i(\mathbb{Z})_{\mathbb{Q}}$ has free rank 1 for $i \equiv 1 \mod 4, i > 1$ and thus the above theory is insufficient to provide much finer information than for virtual categories. Thus any attempt to provide a full solution to the Deligne-problem necessarily needs new ideas. Results in this and the previous chapter are stated for somewhat general algebraic stacks, but in the actual applications of this thesis only the non-equivariant and the μ_n-equivariant case will be considered. It is the hope that we can prove functorial Lefschetz-type formulas for algebraic stacks in the future and that these results will be helpful.

In chapter 5 we establish a fundamental functorial base-change-property, the excess-base-change formula (c.f. Theorem 5.1.2). If $f : X \to Y$ is a proper morphism of smooth quasi-projective varieties over a field k and $q : Y' \to Y$ is a any morphism, $X' := X \times_Y Y'$ and we have morphisms induced by base-change $f' : X' \to Y', q' : X' \to X$, this states that

$$Rf'_* \left(\lambda_{-1}(E) \otimes Lq'^*(x)\right) = Lq^* Rf_*(x)$$

is an equality in $K_0(Y')$, for any $x \in K_0(X)$. Here E is the so called excess-bundle (c.f. [Ful98], 6.3) that arises whenever the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

is not transversal and $\lambda_{-1}(E) = \sum_{i=0}^{\infty}(-1)^i \wedge^i E$. Whenever f or q is flat $E = 0$ and this is just the usual base-change-formula. This is established as a functorial isomorphism, up to sign, valid for algebraic stacks satisfying the additional condition that any coherent sheaf on the algebraic stack is the quotient of a vector bundle. Particular focus has been paid to this isomorphism for several reasons. First of all, it gives an example of how to construct functorial isomorphisms of a special type in a relatively simple situation, and how to use the deformation to the normal cone to resolve these types of questions. Second of all, it is a formula that we will have to apply in many different contexts and in particular in the very statements of our functoriality Riemann-Roch-like theorems of the next chapter.

In the final chapter 6 we put together the obtained results and reap the fruits of our labor. The first of these is a functorial construction of multiplicative characteristic classes on the cohomological virtual category on a regular algebraic stack. These are conceived as certain multiplicative determinant functors $\mathbf{P}(X) \to W(X)^*$ determined by compatibility with base-change and what they are on the category of line bundles, just like in the classical situation (c.f. [WF85], chapter I, §4). The main example to have in mind is that of Bott's cannibalistic class arising in the Adams-Riemann-Roch theorem. Next we formulate a completely unique Lefschetz-isomorphism for cyclic diagonalizable groups T acting on regular schemes (c.f. section 6.2). This is a functorial version of Thomason's coherent Lefschetz trace-formula in [Tho92] and relates the trace of a T-representation on X with the trace of the representation restricted to the fixed-point set X^T of T acting on X. We also establish a completely unique functorial Adams-Riemann-Roch theorem, at least rationally, thus giving a positive response to Deligne's program for the determinant of the cohomology, for the category of regular schemes and projective morphisms between them. In the final section we relate this Adams-Riemann-Roch-isomorphism to the Deligne-Riemann-Roch-isomorphism for curves already alluded to and show they necessarily must coincide. We also apply this to study a conjecture of [Köc98].

The proof of all these "functoriality theorems" use the deformation to the normal cone and an adaption of the techniques that arise to the functorial context. To a great extent this has already been worked out by J. Franke in [Fra] who have also established a solution (unpublished) to Deligne's program in a less restrictive context. Loc. cit. is the culmination of a series of papers ([Fra90], [Fra91]) and is in a context different from ours. J. Franke focuses on a Grothendieck-Riemann-Roch formulation of the Riemann-Roch problem and obtains general results by considering Chern classes as intersection classes. However, in the non-equivariant setting we focus on the Adams-Riemann-Roch formulation for regular schemes. No attempt to compare the two theorems have been made, but since their constructions are very similar this should be possible with the formalism available in the respective papers. It should lastly be noted that the author has been very lucky to have access to the unpublished manuscript [Fra].

To summarize, this thesis includes the following main results:

- Proof of that, under finitude of the Tate-Shafarevich group, the Brauer-Manin obstruction to the existence of zero-cycles of degree one is the only one for curves (c.f. chapter 1).
- Study of determinants on triangulated categories coming from Waldhausen categories (c.f. section 2.3).
- Virtual categories for algebraic stacks and general results of the same. This includes functorial Adams operations and a splitting principle (c.f. chapter 3).
- Rigidity for virtual categories (c.f. chapter 4).
- Functorial excess-formula for certain types of algebraic stacks (c.f. chapter 5).
- Functorial Lefschetz theorem for $\mu_n \times \mathbb{G}_m^r$-actions and functorial Adams-Riemann-Roch. The latter case applied to curves allows us to recover (up to torsion) previous results of D. Mumford, P. Deligne and T. Saito, as well as confirm a version of a conjecture on equivariant Grothendieck-Riemann-Roch of B. Köck in the special case of the first Chern class (c.f. chapter 6).

Part I

1. BRAUER-MANIN OBSTRUCTION FOR ZERO-CYCLES ON CURVES

No need for doctors half the time. The French understand these things.

1.1 Brauer-Manin obstruction

For the purposes of this paper, a variety is a finite type, separated and geometrically integral scheme over a field k. We first recall the definition of the Brauer-Manin obstruction. Henceforth the symbols X and U will be used to denote geometrically integral varieties over a field k, and in case they are mentioned together U is a non-empty open subset of X.
Suppose k is perfect. Set $\mathrm{Br}(X) := \mathrm{H}^2_{et}(X, \mathbb{G}_m)$. By functoriality, an L-point (for L/k a finite field-extension) $\operatorname{spec} L \to X$ defines a homomorphism $\operatorname{Br} X \to \operatorname{Br} L$. Furthermore, since L/k is finite, we can take corestriction $\operatorname{Br} L \stackrel{\text{Cores}}{\to} \operatorname{Br} k$. Hence, by extending by linearity, we obtain a pairing between

$$Z_0(X) \times \mathrm{Br}(X) \to \operatorname{Br} k. \tag{1.1}$$

Here $Z_0(X)$ denotes the group of 0-cycles on X. Now, let k be a number field, and set k_v to be the completion of k at a place v. For a k-variety X, we denote $X \times_k k_v$ by X_v, and $X \times_k \overline{k}$ by $\overline{X}$ for a separable closure $\overline{k}$ of k. Denote by $Z_0^a(X)$ the group $\prod_v Z_0(X_v)$.
Suppose for the purposes of the introduction that X is smooth. In the case B is a subgroup of the unramified Brauer group $\mathrm{Br}_{nr}(X)$ (the Brauer group of any smooth compactification of X, which we will assume for now exists; it is a birational invariant, c.f. [4], Théorème 7.4), it is possible to show that we can define a pairing

$$Z_0^a(X) \times B \to \mathbb{Q}/\mathbb{Z}$$

as follows: Given a zero-cycle $(z_v)_v$ and an element $\alpha \in B$, one obtains elements $(\alpha(z_v)_v) \in \prod_v \operatorname{Br} k_v$ by evaluating as in (1.1). By local class field theory we have injections $i_v : \operatorname{Br} k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ and it is possible to show that by virtue of $\alpha \in \mathrm{Br}_{nr}(X)$, the element $\langle (z_v), \alpha \rangle = \sum_v i_v(\alpha(z_v))$ is a finite sum and thus well-defined.

Now, one puts, for $B \subseteq \mathrm{Br}_{nr} X$,

$$Z_0^a(X)^B = \{(z_v) \in Z_0^a(X) \mid \forall v, \deg z_v = 1, \langle (z_v), \alpha \rangle = 0, \forall \alpha \in B\}.$$

The corestriction-map is the identity on the level of $\mathbb{Q}/\mathbb{Z}$ for non-archimedean places (see [16], XI, Prop 2, ii) and XIII, Theorem 1). Using this together with the fundamental short exact sequence of class-field theory,

$$0 \to \mathrm{Br}\, k \to \oplus_v \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z} \to 0,$$

one shows that $Z_0(X)^1$, the zero-cycles of degree 1, are indeed included in the above defined set. This follows in a way completely analogous to the case of points (see Manin, loc.cit.). For the same reasons, the above pairing with $\mathrm{Br}_0(k)$, i.e. the image of $\mathrm{Br}\, k$ in $\mathrm{Br}\, X$, is seen to be zero. If the set $Z_0^a(X)^B$ is non-empty, we say there is no obstruction associated to B for existence of zero-cycles of degree 1. Now let $\mathcal{V}$ be a class of varieties with an assignment $B = B(X) \subseteq \mathrm{Br}_{nr}(X)$ for any $X \in \mathcal{V}$. If for all X in $\mathcal{V}$ we have

$$Z_0^a(X)^B \neq \emptyset \Longrightarrow Z_0(X)^1 \neq 0$$

then we say the Brauer-Manin obstruction is the only one to the existence of zero-cycles of degree 1 associated to B. Let $\mathrm{Br}_1(X) = \ker[\mathrm{Br}\, X \to \mathrm{Br}(\overline{X})]$. The group

$$\text{Б}(X) := \ker \left[\mathrm{Br}_1 X \to \prod_v \mathrm{Br}_1 X_v / \mathrm{Br}\, k_v \right]$$

is the group of (algebraic) locally constant elements of the Brauer group. Whenever X is a proper variety, the quotient of $\text{Б}(X)$ by the image $\mathrm{Br}_0(k)$ of $\mathrm{Br}\, k$ comes with a canonical isomorphism

$$\text{Б}(X)/\mathrm{Br}_0(k) \simeq \text{Ш}^1(\mathrm{Pic}(\overline{X})) = \ker \left(\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \to \prod_v \mathrm{H}^1(k_v, \mathrm{Pic}(\overline{X}_v)) \right)$$

via the Hochschild-Serre spectral sequence (see [19], Corollary 2.3.9, the case $M = \mathbb{Z}$. This is an isomorphism essentially because $H^3(k, \mathbb{G}_m) = 0$ for local and global fields). Here and henceforth $H^i(k, M)$ denotes étale cohomology, which reduces to Galois-cohomology of $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ with values in M. By Lemma 1.3.3 below, $\text{Б}(X) \subseteq \mathrm{Br}_1(X)$ is unramified in the sense that it does not depend on the choice of smooth compactification.

Let X be a smooth quasi-projective variety defined over a perfect field k. Denote the semi-Albanese variety of X by SAlb_X^0. There is a certain torsor SAlb_X^1 under SAlb_X^0, universal with respect to morphisms into torsors under semiabelian varieties (see definition 1.3.0.1). The *period*, $P = P_X$, of X is defined as the order of $[\mathrm{SAlb}^1_X]$ in $H^1(k, \mathrm{SAlb}^0_X)$. In Theorem 1.4.1 in section 1.4 we give another characterization of the period in terms of Suslin homology (see Appendix). The *index* $I = I_X$ of a variety X over a field k is defined to be least positive degree of a zero-cycle on X with respect to k. Define the generic period $\widetilde{P}$ as the supremum of all P_U over all open non-empty subsets U of X. Note that $P_U \mid I_U$ and and it is well known that the index of an open subset of X is the same as that of X (see [3], "Complément"). Thus we see that all P_U are bounded by I, so the supremum exists. Moreover $\widetilde{P}|I$. Our first result is a stronger version of a theorem originally due to S. Saito [14] (stronger, because here we only need the conjecturally finite group $Б(C)/\mathrm{Br}_0(k)$ as opposed to the whole Brauer group). S. Saito's theorem has also been reproved by Colliot-Thélène in [2].

Theorem 1.1.1. *Let C be a smooth projective curve over a number field k, let A be its Jacobian, and assume that $Ш^1(A)$ is finite. Then the obstruction associated to $Б(C)$ for zero-cycles of degree 1 is the only one:*

$$\text{if } Z_0^a(C)^{Б(C)} \neq \emptyset \text{ then } I = 1.$$

We have an obvious corollary which does not include the group Z_0^a (see introduction):

Corollary 1.1.2. *Let C be as above and keep the same assumptions. Then if C has no Brauer-Manin obstruction associated to $Б(C)$ for points, then C has a zero-cycle of degree 1, i.e.:*

$$\text{if } C(\mathbb{A}_k)^{Б(C)} \neq \emptyset \text{ then } I = 1.$$

For general projective, smooth, geometrically integral varieties k-varieties Colliot-Thélène (see [2]) has conjectured that the Brauer-Manin obstruction is the only obstruction to the existence of global 0-cycles of degree 1 on X. In §1.3, we shall prove a very weak version of this conjecture.

Theorem 1.1.3. *Let X be a projective, smooth geometrically integral variety over a number field k. Denote by A the Albanese variety of X and assume that*

$$Ш^1(A) = \ker[\mathrm{H}^1(k, A) \to \bigoplus_v \mathrm{H}^1(k_v, A)]$$

is finite. If $Z_0^a(X)^{Б(X)} \neq \emptyset$ then $\widetilde{P} = 1$.

Thus the Brauer-Manin obstruction is the only obstruction to the generic *period* being 1. The proof makes use of semi-Albanese torsors, and thus depends in an essential way on a result due to Harari and Szamuely Theorem 1.1 [5].

Remark 1.1.3.1. We note that P_U can indeed be larger than P_X for U open in X. For example, if X is a proper curve of genus 0, then via the anti-canonical embedding it can be written as a conic in $\mathbb{P}^2$:

$$X : aX^2 + bY^2 = cZ^2.$$

Hence the index is either 1 or 2, and it is 1 exactly when we have a rational point. Now, removing two points at infinity, we obtain

$$U : ax^2 + by^2 = c$$

which is a torsor under a torus. Because P_U divides I, it is either 1 or 2, and because the torsor is trivial exactly when P_U is 1, we see that $P_U = I$. Hence we have in this case that $\widetilde{P} = I$. However since the Albanese of X is trivial, P_X is certainly 1. In general one can show that for smooth curves over any field, the two invariants are the same.
To the author's knowledge, not very much is known about the quotient $I/\widetilde{P}$, but in general it is not always 1. For arbitrary varieties, even over a number field, they are not equal. Indeed, one can construct varieties with points exactly when they have a zero-cycle of degree 1, without Б-obstruction and without rational points. O. Wittenberg informs the author that the del Pezzo surface of degree 4

$$\begin{aligned} vw &= x^2 - 5y^2 \\ (v+w)(v+2w) &= x^2 - 5z^2 \end{aligned}$$

examined by Birch and Swinnerton-Dyer in [1] provides such an example.

Furthermore, in [21], O. Wittenberg relates the generic period to more classical invariants. He shows among other things that for a smooth proper geometrically integral variety X over a number field k, such that $X(\mathbb{A}_k) \neq \emptyset$ and the Tate-Shafarevich group of the Picard variety of X over k is finite, there is an equivalence between the following statements (see ibid., Theorem 3.3.1):

(a) $X(\mathbb{A}_k)^{Б} \neq \emptyset$;

(b) The elementary obstruction (see [19], Definition 2.3.5) of X vanishes ;

(c) The generic period is 1.

Remark 1.1.3.2. In [12] it is claimed that over a global field k and a principally polarized abelian variety A over k, there is the Cassels-Tate pairing (defined in loc.cit.)

$$Ш^1(A) \times Ш^1(A) \to \mathbb{Q}/\mathbb{Z},$$

which is perfect modulo maximal divisible subgroups. Having such a result in the function field case is exactly what is lacking to prove that the Brauer-Manin-obstruction associated to Б is the only one for abelian varieties over function fields. If we would have this one could extend Theorem 1.1.1 to any global field. In personal communication B. Poonen informs the author that to his knowledge there is no written proof of this fact in the function field case.

1.2 A Short Proof of S. Saito's Theorem

In this section we give our proof of Theorem 1.1.1. We let C be a geometrically integral, smooth projective curve over a number field k, without any Б(C)-obstruction to zero-cycles of degree 1. Also suppose that $Ш^1(A)$ is finite, where A is the Jacobian variety $\mathrm{Pic}^0_{C/k}$ of C. Recall that there is a uniquely defined morphism $p : C \to \mathrm{Pic}^1_{C/k}$, where $\mathrm{Pic}^1_{C/k}$ is a torsor under A, and that the morphism is universal with respect to morphisms into torsors under abelian varieties. See Theorem 1.3.1 for the statement of the existence over a perfect field (well known for curves) and a reference for proof, or the discussion in chapter V, paragraph 23, [17] for the case of a general field. By functoriality of the pairings, if C has no Б(C)-obstruction to the existence of zero-cycles of degree 1, then $\mathrm{Pic}^1_{C/k}$ does not have any obstruction associated to Б$(\mathrm{Pic}^1_{C/k})$. Using a result of Manin, by finitude of $Ш^1(A)$, $\mathrm{Pic}^1_{C/k}$ has a k-rational point (see Theorem 6.2.3 of [19] for a proof, or the original article of Manin [8]).
Denote by $\mathrm{CH}_0(X)$ the usual 0-th Chow group of a variety X, that is, the full group of zero-cycles modulo rational equivalence and recall that $\overline{k}$ denotes a separable closure of k. We now record the following general (well known) fact:

Lemma 1.2.1. *[[11], Prop 2.5] Let X be a smooth, proper and geometrically integral variety over a global field k and assume that X has a zero-cycle of degree one locally everywhere. Then* $\mathrm{Pic}(X) \simeq \mathrm{Pic}(\overline{X})^{\Gamma_k}$. *In the case X is a curve, we see that in particular,* $\mathrm{CH}_0(X) = \mathrm{Pic}(X)$ *surjects onto the k-points of the Picard-scheme* $\mathrm{Pic}_{X/k}(k) = \mathrm{Pic}(\overline{X})^{\Gamma_k}$.

Proof. We include a proof for completeness. The Hochschild-Serre spectral sequence provides us with the exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{\Gamma_k} \to \mathrm{Br}\,k \xrightarrow{j} \mathrm{Br}\,X.$$

If X has a k-point, this point splits the map j and so j is injective. By a restriction-corestriction argument the same stays true if X has a zero-cycle of degree 1. Global class-field theory tells us that the map $\mathrm{Br}\,k \to \oplus_v \mathrm{Br}\,k_v$ is injective. The condition that we have a zero-cycle locally everywhere gives us that $\oplus_v \mathrm{Br}\,k_v \to \oplus_v \mathrm{Br}\,X_v$ is injective, and one concludes that $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^{\Gamma_k}$ must be surjective, and hence bijective. □

We have a natural identification $\mathrm{Pic}_{X/k} = \coprod_{n\in\mathbb{Z}} \mathrm{Pic}^n_{X/k}$, where $\mathrm{Pic}^n_{X/k}$ is the n-th Baer sum of torsors. This comes with a natural map $\overline{\deg}$: $\mathrm{Pic}_{X/k}(k) \to \mathbb{Z}$, sending an element in $\mathrm{Pic}^n_{X/k}(k)$ to $n \in \mathbb{Z}$, and the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{CH}_0(X) & \longrightarrow & \mathrm{Pic}_{X/k}(k) \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle \overline{\deg}} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}.$$

Since the first map is a bijection, we see that $\overline{\deg}$ is surjective whenever deg is, and thus with the assumption that we have an adelic zero-cycle of degree 1 orthogonal to Б(C), we see that the index is one and the theorem is proved.

1.3 Brauer-Manin obstruction and Generic Periods

In this section we give the proof of the main theorem, roughly as follows. After recalling the definition of the semi-Albanese torsor we first show that under the right conditions the period is equal to 1 (Lemma 1.3.2). We then recall that the group of locally constant elements is invariant under restriction to Zariski opens (Lemma 1.3.3) and then put this together to prove Theorem 1.1.3. The main technical tool is a recent result on the Hasse principle for semiabelian varieties by Harari and Szamuely [5].

First, let U be a quasi-projective, smooth variety over a field k. Recall that a semi-abelian variety is a commutative group-variety which is an extension of an abelian variety by a torus. Suppose we are given a k-morphism $p : U \to S^1$ where S^1 is a torsor under a semi-abelian variety S^0 over k, with the following universal property: Given any k-morphism $m : U \to T^1$, where T^1 is a torsor under a semi-abelian variety T^0 over k, there is a unique k-morphism

$f^1 : S^1 \to T^1$ such that $h \circ p = f^1$ and there a unique k-morphism of algebraic groups $f^0 : S^0 \to T^0$ such that f^1 is f^0-equivariant. This clearly determines the quadruple (U, S^0, S^1, p), if it exists, up to unique isomorphism.

Definition 1.3.0.1. A quadruple (U, S^0, S^1, p) as above is a semi-Albanese torsor of U.

Suppose that k is perfect. The following is a formal consequence of its solution over an algebraically closed field [18] (see Theorem 7) and the descent theory of [17] (see p. 112, 4.22), which was already remarked in [13]:

Theorem 1.3.1. *Let U be a quasi-projective smooth variety over a perfect field k, then a semi-Albanese torsor exists.*

Remark 1.3.1.1. If X is also proper the semi-Albanese variety is the Albanese variety, and the semi-Albanese torsor is an "Albanese torsor", i.e. it is universal with respect to morphisms into torsors under abelian varieties. In this case it is desirable to write it as Alb^1_X instead. In general, if X_c denotes a smooth compactification of X, then if $H^1(X_c, \mathcal{O}_{X_c}) = 0$ the abelian-variety part of the semi-Albanese variety is trivial and the semi-Albanese torsor is a torsor under a torus and is universal with respect to morphisms to torsors under tori.

Lemma 1.3.2. *Let V be a torsor under a semi-abelian variety S which is an extension of an abelian variety A by a torus T, defined over a number field k and suppose that $Ш^1(A)$ is finite. Then the obstruction associated to $Б(V)$ for zero-cycles of degree one is the only one for rational points on V. That is, if $Z_0^a(V)^{Б(V)} \neq \emptyset$ then $V(k) \neq \emptyset$.*

Proof. Since V has a zero-cycle of degree one locally everywhere, it actually has a k_v-rational point for every v. For each place v, let Q_v be such a k_v-rational point, and suppose that $Q = (Q_v)_v$ is any point in $\prod_v V(k_v)$ which we can suppose is adelic. A restriction-corestriction argument shows that $i_v(\alpha(z_v)) = \deg(z_v) i_v(\alpha(Q_v))$ for any zero-cycle z_v on V_v, for α locally constant (i.e. in $Б(V)$). Hence we can replace all zero-cycles of degree one with this adelic point Q, which will be orthogonal to $Б(V)$. The statement we now want to prove is well known for rational points whenever S is an abelian variety (see Theorem 6.2.3 of [19] for a proof, or the original article of Manin [8]) or S to be a torus (see Theorem 6.2.1, [19]). The result for $Б(V)$ in the case of arbitrary S is a result of Harari and Szamuely, [5]. □

We record the following lemma of invariance of Б under restriction to open subsets.

Lemma 1.3.3. *[[15], Lemma 6.1 or [6], Theorem 2.1.1]Let k be a number field and let X be a smooth proper geometrically integral variety over k. Suppose U is a non-empty Zariski-open set in X. Then*

$$\text{Б}(U) = \text{Б}(X).$$

We now turn to:

Proof. (of Theorem 1.1.3). Let X be as in the theorem. Let U be an open set of X, and let $p\colon U \to \mathrm{SAlb}^1_U$ be its semi-Albanese torsor. Since by the above lemma, $\text{Б}(U) = \text{Б}(X)$, and these elements are locally constant, U has no $\text{Б}(U)$-obstruction. By functoriality of the Brauer-Manin pairing, the same holds true for SAlb^1_U. Because of the finiteness assumption on $\text{Ш}^1(A)$, Lemma 1.3.2 implies that the torsor is trivial and so P_U, the period for U, is 1. This is true for any open in X, and hence the generic period is 1. □

Remark 1.3.3.1. Suppose now that U is a torsor under a torus, and X a smooth compactification thereof. If X has no $\text{Б}(X)$-obstruction associated to zero-cycles of degree 1, by Theorem 1.1.3 we thus obtain that the period of U is 1 (this rests only on Theorem 6.2.1, [19] and doesn't utilise the full result of [5]). But then it has a point and we recover a result by Colliot-Thélène and Sansuc saying that the Brauer-Manin obstruction is the only one for smooth compactifications of k-torsors under tori (see [19], Theorem 6.3.1, and the remark afterwards saying that we only need to consider locally constant elements). In any case, the generic period contains more information than the period associated to only X. An interesting question (suggested by Colliot-Thélène) would be to calculate the generic period of (a compactification of) a non-abelian algebraic group and compare it to its index.

1.4 Alternative Description of the Period

In this section we give an additional description of the period as a cokernel of a map $\deg : \mathrm{h}_0(\overline{X})^{\Gamma_k} \to \mathbb{Z}$. Suppose k is perfect. The semi-Albanese scheme of $X \to \operatorname{spec} k$ is the k-group scheme

$$\underline{\mathrm{SAlb}}_{X/k} = \coprod_{n\in\mathbb{Z}} \mathrm{SAlb}^n_X$$

where SAlb^n_X is the n-fold Baer sum of torsors, and for $n = 0$ it is the semi-Albanese variety. In [13], 1.2, Ramachandran shows that this is a group-scheme with various functorial and universal properties. We mimic his approach: We have an obvious Γ_k-equivariant map $X(\overline{k}) \to \mathrm{SAlb}^1_X(\overline{k})$, and we define a map from the group of zero-cycles $Z_0(\overline{X})$ to the $\overline{k}$-points of the Albanese scheme to be the unique group-homomorphism whose restriction to $X(\overline{k})$ is the above map. Taking Galois-invariants gives a homomorphism of groups $Z_0(X) \to \underline{\mathrm{SAlb}}_{X/k}(k)$. By naturality and the fact that $S(\mathbb{A}^1_k) = S(k)$ for semi-abelian varieties S we see that this map factors over the Suslin homology group $\mathrm{h}_0(X)$ (see Appendix for a definition of Suslin homology and argue as in [20], Lemma 3.1).

Hence, there is a canonical homomorphism $\mathrm{h}_0(X) \to \underline{\mathrm{SAlb}}_{X/k}(k)$ such that the restriction to degree 0 is the generalized Albanese map (which we will refer to simply as the "Albanese map") of [20] and the structural morphism $X \to \operatorname{spec} k$ induces the following commutative diagram (where the exactness on the left of the first line is our definition of $\mathrm{h}_0(X)^0$)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{h}_0(X)^0 & \longrightarrow & \mathrm{h}_0(X) & \longrightarrow & \mathrm{h}_0(k) = \mathbb{Z} \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathrm{SAlb}^0_X(k) & \longrightarrow & \underline{\mathrm{SAlb}}_{X/k}(k) & \longrightarrow & \underline{\mathrm{SAlb}}_{k/k}(k) = \mathbb{Z} \end{array}$$

Definition 1.4.0.1. Let X be a quasi-projective smooth variety over a perfect field k. We define

$$S_X = \#\{\operatorname{coker} \deg : \mathrm{h}_0(\overline{X})^{\Gamma_k} \to \mathbb{Z}\}.$$

Theorem 1.4.1. *Let X be a quasi-projective, smooth geometrically integral variety over a field of characteristic 0. Then $S_X = P_X$.*

Proof. By the preceding remarks we have the following commutative diagram of Galois modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{h}_0(\overline{X})^0 & \longrightarrow & \mathrm{h}_0(\overline{X}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle p} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{SAlb}^0_X(\overline{k}) & \longrightarrow & \underline{\mathrm{SAlb}}_{X/k}(\overline{k}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Here p is the Albanese map (loc. cit.). Taking Galois cohomology gives us the following diagram

$$\begin{array}{ccccc} \mathrm{h}_0(\overline{X})^{\Gamma_k} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}^1(k, \mathrm{h}_0(\overline{X})^0) \\ \downarrow & & \| & & \downarrow{\scriptstyle p} \\ \underline{\mathrm{SAlb}}_{X/k}(k) & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}^1(k, \mathrm{SAlb}^0_X(\overline{k})). \end{array}$$

The image of 1 in $\mathrm{H}^1(k, \mathrm{SAlb}^0_X)$ is represented by the cocycle $\sigma \mapsto x_0^\sigma - x_0$, for $x_0 \in \mathrm{SAlb}^1_X(\overline{k})$, i.e. it is the class of the semi-Albanese torsor. Now, the generalized theorem of Roitman can be formulated as that under the conditions of the lemma, the Albanese map is surjective and the kernel of the same is uniquely divisible. This is the main theorem of Spiess and Szamuely [20]. It is established for a smooth connected variety X admitting a smooth projective connected compactification over an arbitrary algebraically closed field of char.p (for $p \geq 0$) and states that the Albanese map is an isomorphism on the prime-to-p torsion subgroups. Since we are in characteristic 0, the smooth projective compactficiation is provided by Hironaka. Finally, because for a uniquely divisible Galois-module M, $\mathrm{H}^i(k, M) = 0$ for $i > 0$, the rightmost homomorphism is an isomorphism which finishes the theorem. □

1.5 Appendix - Suslin homology, h_0

In this section we recall some basic properties of the Suslin homology-group h_0. Let X and Y be any separated schemes of finite type over a field k. If Y is connected, an elementary finite correspondence from Y to X over k is an integral closed subscheme Z of $X \times_k Y$, finite and surjective over Y. A finite correspondence between X and Y is a formal $\mathbb{Z}$-linear sum of elementary finite correspondences, and we denote the group of such as $\mathrm{Cor}(Y, X)$. Any closed subscheme Z of $X \times_k Y$ defines a finite correspondence by associating to it the correspondence $\sum n_i Z_{i,\mathrm{red}}$ where the sum is over irreducible components Z_i of Z, such that $Z_{i,\mathrm{red}}$ is finite and surjective over Y, and n_i is the geometric multiplicity of $Z_{i,\mathrm{red}}$ in Z (compare [9], Construction 1.3). If $Y = \coprod_\alpha Y_\alpha$ is the decomposition of Y into its connected components, one defines $\mathrm{Cor}(Y, X) = \oplus_\alpha \mathrm{Cor}(Y_\alpha, X)$. Note that the finite correspondences from $\mathrm{spec}\, k$ to X, $\mathrm{Cor}(\mathrm{spec}\, k, X)$, is just the group of zero-cycles on X.

If P is a k-point of Y and Z is a closed subscheme of $X \times_k Y$, denote by $Z(P)$ the scheme-theoretic fiber of Z over $X = X \times_k P$. Consider the points 0 and 1 of $\mathbb{A}^1_k$ and define a map $\mathrm{Cor}(\mathbb{A}^1_k, X) \to \mathrm{Cor}(\mathrm{spec}\, k, X)$ by

$$Z \mapsto Z(0) - Z(1).$$

We define $\mathrm{h}_0(X)$, the 0-th Suslin homology of X, to be the group of zero-cycles on X modulo the group generated by finite correspondences coming from $\mathbb{A}^1_k$ to X in the above sense. We note the following properties, which are not difficult to show.

Proposition 1.5.1. *Let X, Y be two separated schemes of finite type over a field k. Then the following holds:*

(a) *Let* $\mathrm{CH}_0(X)$ *be the 0-th Chow group. There is a map* $\mathrm{h}_0(X) \to \mathrm{CH}_0(X)$ *which is moreover an isomorphism if the structural morphism* $X \to \operatorname{spec} k$ *is proper.*

(b) h_0 *is covariantly functorial with respect to morphisms* $f : X \to Y$.

(c) *The degree map mapping a zero-cycle* $\sum n_i P_i \mapsto \sum n_i [k(P_i) : k]$ *factors over* $\mathrm{h}_0(X) \to \mathrm{h}_0(k) = \mathbb{Z}$ *where the map* $\mathrm{h}_0(X) \to \mathrm{h}_0(k)$ *is given by the structural morphism* $X \to \operatorname{spec} k$.

For lack of a specially tailored reference, we include the following proofs:

Proof. Let V be a dimension 1 integral closed subscheme of $X \times_k \mathbb{P}^1_k$ which is dominant on the second factor. Then $[V(0)] - [V(1)]$ is a zero-cycle on X and it follows as in [7], Proposition 1.6, that rational equivalence on zero-cycles is generated by the relation determined by all such V. Now, if $Z \subseteq X \times_k \mathbb{A}^1$ is an elementary finite correspondence its closure $\overline{Z}$ in $X \times_k \mathbb{P}^1_k$ defines such an object. This shows there is always a map $\mathrm{h}_0(X) \to \mathrm{CH}_0(X)$. Suppose $X \to \operatorname{spec} k$ is moreover proper. A closed integral subscheme $\overline{Z} \subseteq X \times_k \mathbb{P}^1_k$ which is dominant over $\mathbb{P}^1_k$ is proper over $\operatorname{spec} k$ by virtue of X being proper, and if $\overline{Z}$ is moreover of dimension one it is finite over $\mathbb{P}^1_k$. As such it determines a closed integral subscheme $Z \subseteq X \times_k \mathbb{A}^1_k$ which is a finite elementary correspondence from $\mathbb{A}^1_k$ to X. This is inverse to the above operation and proves (a).
Now, let $f : X \to Y$ be a morphism of separated schemes of finite type over a Noetherian base S. If $Z \subseteq X$ is a closed integral subscheme, finite and surjective over S, then the schematic image $f(Z)$ is a closed integral subscheme of Y, finite and surjective over S (see [9], Lemma 1.4). If Y is connected, the pushforward is defined as $f_*(Z) = df(Z)$ where $d = [k(Z) : k(f(Z))]$, which is finite, and the definition for general Y is similar. The above rule thus assigns a homomorphism $f_* : \mathrm{Cor}(V, X) \to \mathrm{Cor}(V, Y)$ to a separated scheme V of finite type over S. Putting $S = \mathbb{A}^1_k$ or $\operatorname{spec} k$ we obtain pushforwards on $\mathrm{Cor}(\mathbb{A}^1_k, -)$ respectively $\mathrm{Cor}(\operatorname{spec} k, -)$, and they clearly respect the obvious compatibility conditions for restricting to the points 0 and 1, so we obtain a homomorphism

$$f_* : \mathrm{h}_0(X) \to \mathrm{h}_0(Y).$$

The last point is now a consequence of the definition of the degree-map.

□

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Part II

2. SOME PRELIMINARIES

This little pig went to the market

2.1 The virtual category

Given a small exact category $\mathcal{C}$, we can consider its K-theory. The first case of K_0 can be defined explicitly in terms of the category $\mathcal{C}$, as the Grothendieck group of $\mathcal{C}$. This is the free abelian group on the objects of $\mathcal{C}$, modulo the relationship $B = A + C$ if

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathcal{C}$. A more sophisticated approach was taken by Quillen, [Qui73], where he constructs a certain topological space $BQ\mathcal{C}$ associated to a (small) exact category $\mathcal{C}$ such that

$$K_i(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C}).$$

This encompasses the old definition of Grothendieck groups and gives new groups satisfying certain functorial properties when we specialize $\mathcal{C}$ to the exact categories of coherent or locally free sheaves on schemes. We refer to loc.cit. for details.

Now, let $X \in \mathrm{ob}(\mathrm{Top}_\bullet)$ be a pointed topological space. One defines the fundamental groupoid of X to be the category whose objects are points of X, and morphisms are homotopy-classes of paths, i.e. it is associated to the diagram

$$[PX \rightrightarrows X]$$

where PX is the space of paths of X. Denote the corresponding functor by

$$\pi_f : \mathrm{Top}_\bullet \to \mathrm{Grp}\,.$$

Deligne, [Del87] defines a category of virtual objects of an exact category, which offers a type of generalization of the derived category and also the K_0 of the category.

Let $\mathcal{C}$ be a small exact category. The category of virtual objects of $\mathcal{C}$, $V(\mathcal{C})$

is the following: Objects are loops in BQC around a fixed zero-point, and morphisms are homotopy-classes of homotopies of loops. Recall that BQC is the geometrical realization of the Quillen Q-construction of C. Addition is the usual addition of loops. This construction is the fundamental groupoid of the space ΩBQC. In case C is not small we will always consider an equivalent small category, and ignore any purely categorical issues this might cause. We record the following proposition:

Proposition 2.1.1. *$V(C)$ is a groupoid, i.e. any morphism is an isomorphism, and the set of equivalence-classes is in natural bijection with $K_0(C)$. For any object $c \in obV(C)$, we have $Aut_{V(C)}(c) = \pi_1(\Omega BQC) = K_1(C)$.*

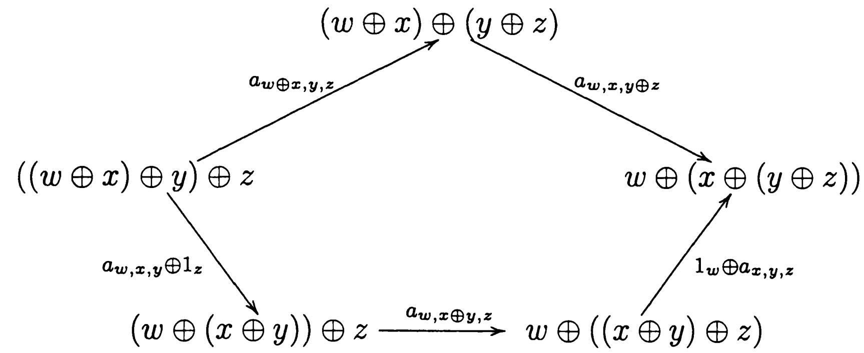
Deligne also provides a more algebraic and universal definition of $V(C)$. We will give an additional description.

2.2 Algebraic definition

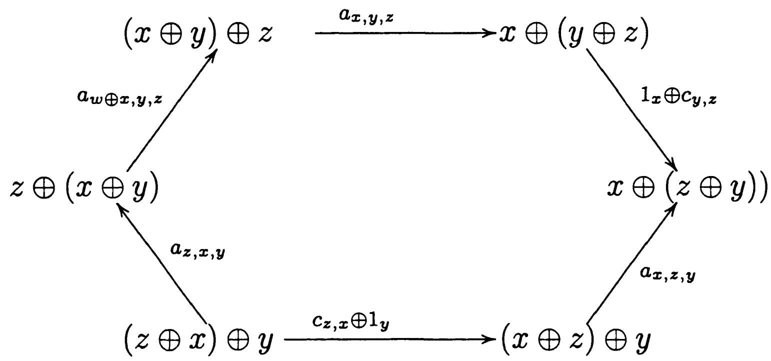
The above category is a so called universal Picard category with respect to C. A (commutative) Picard category is a groupoid C with an auto-equivalence $P \mapsto P \oplus Q$ for any object Q of C, satisfying certain compatibility-isomorphisms plus some commutativity and associativity-restraints (c.f. [Del77], XVIII, Définition 1.4.2 for the definition of a strictly commutative Picard category, or [GL00], 14.4, axiome du pentagone et de l'hexagone): There is an associativity-isomorphism

$$a_{x,y,z} : (x \oplus y) \oplus z \to x \oplus (y \oplus z)$$

such that



commutes. There is a commutativity-isomorphism $c_{x,y} : x \oplus y \to y \oplus x$ such that



commutes. It follows that a category such as this has a zero-object, has inverses etc. (see [Del77], XVIII, 1.4.4). In other words, a Picard category is a symmetric monoidal groupoid whose functor $- \oplus Q$ is an equivalence of categories for each object Q in $\mathcal{C}$. It is moreover said to be strictly commutative if $c_{x,x} : x \oplus x \to x \oplus x$ is the identity. In general we denote by $\epsilon(x) = c_{x,x}$. An additive functor between Picard categories is defined to be a monoidal functor between Picard categories.

Observe we merely have isomorphisms $B \oplus (-B) \to 0$, not equality. For any exact category $\mathcal{C}$, the universal Picard category $V(\mathcal{C})$ is a Picard category $\mathcal{C}$ with a functor $[] : (\mathcal{C}, is) \to V(\mathcal{C})$ which is universal with respect to morphisms $T : (\mathcal{C}, is) \to P$ into Picard categories P, satisfying the following compatibility conditions:

a) For any short exact sequence

$$\mathcal{A} : 0 \to A' \to A \to A'' \to 0,$$

there is an isomorphism, functorial with respect to isomorphisms of exact sequences,

$$T(\mathcal{A}) : T(A) \to T(A') \oplus T(A'').$$

b) For any 0-object of $\mathcal{C}$, there is an isomorphism $T(0) \simeq 0$.

c) If $\phi : A \to B$ is an isomorphism, with exact sequence $0 \to A \to B \to 0$, the induced map $T(\phi)$ is the composite

$$T(A) \to T(0) \oplus T(B) \to T(B).$$

d) T is compatible with filtrations, i.e. for an admissible filtration $A \subset B \subset C$, the diagram

$$\begin{array}{ccc} T(C) & \longrightarrow & T(A) + T(C/A) \\ \downarrow & & \downarrow \\ T(B) + T(C/B) & \longrightarrow & T(A) + T(B/A) + T(C/B) \end{array}$$

is commutative.

In [Del87] it is shown that the functor $(\mathcal{C}, is) \to V(\mathcal{C})$ factors as

$$(\mathcal{C}, is) \to (D^b(\mathcal{C}), q.i.) \to V(\mathcal{C}). \tag{2.1}$$

Here $D^b(\mathcal{C})$ is the derived category of $\mathcal{C}$ (supposed to be the full subcategory of a fixed abelian category), formed out of all complexes with bounded cohomology, where homotopic complexes are identified, and then localized at the thick subcategory of acyclic complexes. The extra suffix is to denote we consider the category where the objects are the same, but the morphisms are the quasi-isomorphisms, i.e. morphisms in the category of complexes that induces an isomorphism in the derived category.

2.3 Additional descriptions

In this section we give some additional descriptions of the virtual category associated to the category of vector bundles over a fixed scheme X. But first, given a small exact category $\mathcal{C}$, there are two major spaces (or more properly, S^1-spectra in the sense of topology) can be associated to this, namely that of the Waldhausen and Quillen K-theory. However, they are naturally homotopic (see [RT90] Theorem 1.11.2) hence have naturally equivalent fundamental groupoids and hence naturally equivalent virtual categories. If X is a scheme, the Quillen K-theory space, the naive one, is the Waldhausen K-theory of bounded complexes of vector bundles in the category of coherent $\mathcal{O}_X$-modules. The Thomason K-theory is the corresponding theory obtained by taking the Waldhausen K-theory of perfect complexes (see [P71], I.5.1. or [RT90], 2.1.1). K^Q provides the "correct" definition of higher K-theory, however, K^T provides the better definition in terms of functorial properties. We denote them by K^Q and K^T respectively. Also, denote by $\mathbf{P}(X)$ the category of (coherent) locally free sheaves on X.

Proposition 2.3.1 ([RT90], Corollary 3.9, Proposition 3.10). *We have natural maps*

$$\Omega BQ(\mathbf{P}(X)) \to K^Q(X) \to K^T(X).$$

For general X, the first map is an homotopy-equivalence. Whenever X has an ample family of line bundles ([P71], II. 2.3) or the resolution property (i.e. any coherent sheaf is the quotient of a coherent locally free sheaf) the last map is also an homotopy equivalence. In particular, whenever X has an ample family of line bundles we can define the virtual category as the fundamental groupoid of K^T.

Proof. The only non-obvious part is to show that the fundamental groupoid of K^T is a Picard category and that the natural map above is an equivalence of Picard categories. However, this follows from the description of the map and the fact that $K^T = \Omega BS$. □

X has in particular an ample family of line bundles whenever X is separated and regular, or is quasi-projective over an affine scheme. [RT90], Lemma 3.5 gives an additional list of spaces homotopy-equivalent to $K^T(X)$ for quasi-compact schemes X.

Definition 2.3.1.1. The fundamental groupoid of K^Q and K^T are denoted by $V(X)$ or $V^Q(X)$ and $V^T(X)$ respectively.

Both these definitions make $V^?$ into a contravariant functor from the category of schemes to the category of groupoids,

$$V^? : \mathit{Schemes} \to \mathrm{Grp},$$

via the pullback operation, $Lf^* = f^*$, and we have a natural transformation of functors $V^Q \to V^T$. It is not, in general, covariant with respect to even proper morphisms $f : X \to Y$. However, we have:

Proposition 2.3.2 ([RT90], 3.16.4-3.16.6). *V^T is a covariant functor from the category*

- *of Nötherian schemes and perfect (see [RT90], 2.5.2) proper morphisms. For example, any local complete intersection (as defined in [P71], VIII, Proposition 1.7) proper morphism between Nötherian schemes.*
- *of quasi-compact schemes and perfect projective morphisms.*
- *of quasi-compact schemes and flat proper morphisms.*

Definition 2.3.2.1. Let $\mathbf{A}$ be a Waldhausen category ([RT90], Example 1.3.6) with weak equivalences w. By $(\mathbf{A}, w)$ we denote the category having the same objects as $\mathbf{A}$ but the morphisms being weak equivalences. Given a Picard category P, a determinant functor from $\mathbf{A}$ to P is a functor

$$[-] : (\mathbf{A}, w) \to P$$

which satisfies the following constraints:

(a) For any cofibration exact sequence

$$\Sigma : A' \rightarrowtail A \twoheadrightarrow A''$$

an isomorphism $\{\Sigma\} : [A] \to [A'] \oplus [A'']$ functorial with respect to weak equivalences of cofibration sequences.

(b) For any object A, the cofibration sequence $\Sigma : A = A \twoheadrightarrow 0$ decomposes the identity-map as:

$$[A] \overset{\{\Sigma\}}{\to} [A] \oplus [0] \overset{\delta^R}{\to} [A]$$

where $\delta^R : [A] \oplus [0] \to [A]$ is given by the structure of $[0]$ as a right unit (see Lemma 2.3.3 for a unicity and existence statement).

(c) Suppose we have a commutative diagram

$$\begin{array}{lccccc}
\Sigma' : & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\
 & \downarrow & & \downarrow & & \downarrow \\
\Sigma : & A & \rightarrowtail & B & \twoheadrightarrow & C \\
 & \downarrow & & \downarrow & & \downarrow \\
\Sigma'' : & A'' & \rightarrowtail & B'' & \twoheadrightarrow & C'' \\
 & & & & & \\
 & \Sigma_A & & \Sigma_B & & \Sigma_C
\end{array}$$

were all the vertical and horizontal lines are cofibration sequences. Then the diagram

$$\begin{array}{ccccc}
[B'] \oplus [B''] & \overset{\{\Sigma_B\}}{\longleftarrow} & [B] & \overset{\{\Sigma\}}{\longrightarrow} & [A] \oplus [C] \\
\downarrow {\scriptstyle \{\Sigma'\} \oplus \{\Sigma''\}} & & & & \downarrow {\scriptstyle \{\Sigma_A\} \oplus \{\Sigma_C\}} \\
[A'] \oplus [C'] \oplus [A''] \oplus [C''] & & \longrightarrow & & [A'] \oplus [A''] \oplus [C'] \oplus [C'']
\end{array}$$

is commutative.
It is furthermore said to be commutative if P is commutative and the following holds:

(d) The triangle

$$\begin{array}{ccc}
[A'] \oplus [A''] & \longrightarrow & [A''] \oplus [A'] \\
 & \searrow \quad \swarrow & \\
 & [A' \bigcup A''] &
\end{array}$$

commutes.

We record the following lemma:

Lemma 2.3.3. *Suppose* $[]: (\boldsymbol{A}, w) \to P$ *is a determinant functor and* P *is commutative. Then for any 0-object of* $\boldsymbol{A}$, $[0]$ *has the structure of a unit in* P, *i.e. there are canonical isomorphisms* $\delta^L : [0] \oplus B \simeq B$ *and* $\delta^R : B \oplus [0] \simeq B$. *In particular, there is a canonical isomorphism* $[0] \simeq 0$ *with any unit object* 0 *of* P.

Proof. Applying $[]$ to the cofibration sequence

$$0 \rightarrowtail 0 \twoheadrightarrow 0$$

we obtain an isomorphism $[] : [0] \oplus [0] \simeq [0]$. By [Riv72], 2.2.5.1 $[0]$ has a unique structure of a unit such that $[0] = \delta^R([0]) = \delta^L([0])$. □

We note the following theorem which extends Deligne's categorical description of the virtual category:

Theorem 2.3.4. *Let* $\boldsymbol{A}$ *be a small Waldhausen category with weak equivalences* w. *Then there is a universal category for determinant functors:*

$$[-] : (\boldsymbol{A}, w) \to V(\boldsymbol{A}).$$

More precisely, for any Picard category P, *the category of determinant functors is equivalent to the category of additive functors* $V(\boldsymbol{A}) \to P$. *Moreover, this category is the fundamental groupoid of the Waldhausen* K*-theory space of* $\boldsymbol{A}$.

Proof. The proof is essentially by definition. Recall that the Waldhausen K-theory space is the loop space of the geometric realization of the bisimplicial set $N_\bullet w S_\bullet \mathbf{A}$ where $wS_p\mathbf{A}$ is the category whose objects are, for $0 \le i \le j \le p$, sequences $A_i \rightarrowtail A_j$ of cofibrations with $A_0 = 0$ and with choices of quotients A_j/A_i, and natural compatibility with composition so that $A_i \rightarrowtail A_j \rightarrowtail A_k$ coincides with $A_i \rightarrowtail A_k$ for $i \le j \le k$, and whose morphisms between two objects A and A' are given by weak equivalences $A_i \to A'_i$ making all the diagrams commute. $N_p w S_q \mathbf{A}$ is the p-nerve of the category $wS_q\mathbf{A}$. The categories $wS_0\mathbf{A}, wS_1\mathbf{A}, wS_2\mathbf{A}$ are, respectively, the trivial category, the category of objects of $\mathbf{A}$ and weak equivalences as morphisms, and the category of cofibration sequences with weak equivalences of cofibration sequences as morphisms.

The geometric realization in question is the (left-right) realization

$$|q \mapsto |p \mapsto N_p w S_q \mathbf{A}||.$$

Thus we obtain from the above description that the "0-simplices" are simply reduced to a point and the "1-simplices" in the $S_\bullet$-direction is obtained by adjoining the set

$$|p \mapsto N_p w S_1 \mathbf{A}| \times \Delta^1.$$

This defines a canonical map $|wS_1| \wedge S^1 \to |N_\bullet wS_\bullet \mathbf{A}|$, and by adjunction a map $|wS_1| \to \Omega|N_\bullet wS_\bullet \mathbf{A}| = K(\mathbf{A})$. By applying the fundamental groupoid-functor we obtain a functor $[] : \mathbf{A} \to (w^{-1}\mathbf{A}, w) = \pi_f(|wS_1\mathbf{A}|) \to \pi_f(K(\mathbf{A}))$, by sending an object to the loop represented by $A \in N_0 wS_1\mathbf{A}$. We verify that this is a determinant functor:
Axiom a: A cofibration sequence

$$\Sigma : A \rightarrowtail B \twoheadrightarrow C$$

defines an element in $N_0wS_2(\mathbf{A})$, and the face-maps to $N_0wS_1(\mathbf{A})$ are given by $\partial_0\Sigma = A, \partial_1\Sigma = B, \partial_2\Sigma = C$, thus providing a path from $[B]$ to $[A] + [C]$. A weak equivalence of cofibration sequences defines an element in

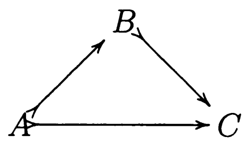
$$N_1wS_2\mathbf{A}$$

whose faces are in $N_1wS_1\mathbf{A}$, which provides the necessary path.
Axiom b: This is just a simplicial identity corresponding to the degeneracy $N_0wS_1(\mathbf{A}) \to N_0wS_2(\mathbf{A}) \to N_0wS_1(\mathbf{A}), A \mapsto [A \to A \to 0] \mapsto A$.

Axiom c: We first show that the commutativity can be rephrased as: if $A \rightarrowtail B \rightarrowtail C$ of cofibrations then

$$\begin{array}{ccc} [C] & \longrightarrow & [A] + [C/A] \\ \downarrow & & \downarrow \\ [B] + [C/B] & \longrightarrow & [A] + [B/A] + [C/A] \end{array}$$

commutes. This is clear since



is a 3-simplex (an object in $N_0wS_3\mathbf{A}$) and provides the necessary relationship between morphisms induced from the 2-simplices in $N_0wS_2\mathbf{A}$ (one also needs to use the commutativity in Axiom d, which is easy). For the full theorem we use that the two 3-simplices $A' \rightarrowtail B' \rightarrowtail B$ and $A' \rightarrowtail A \rightarrowtail B$ are glued together along the 2-simplex $A' \rightarrowtail B$.
We need to prove this construction is universal. Let $[] : (\mathcal{A}, w) \to P$ be a determinant functor. We construct maps $f_n : wS_n(\mathbf{A}) \to \prod_{i=1}^n P$, where the latter denotes the naive sum of Picard categories with indice-wise objects, homs and additions. We put

$$f_n(0 \rightarrowtail A_1 \rightarrowtail \ldots \rightarrowtail A_n) = ([A_1], [A_2] - [A_1], \ldots, [A_n] - [A_{n-1}]).$$

Equip $\prod^{\bullet} P$ with the structure of a simplicial category by the "bar simplicial resolution"-structure; the "n-simplices" are given by the product of categories $\prod_1^n P$ and that $d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$ and for $0 < i < n$, $d_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n)$ and $d_n(g_1, \ldots, g_{n-1}, g_n) = d_n(g_1, \ldots, g_{n-1})$. The face-maps are given by

$$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 0, g_{i+1}, \ldots, g_n).$$

By taking nerves, we obtain a map of bisimplicial sets $N_{\bullet} wS_{\bullet}(\mathbf{A}) \to N_{\bullet} \prod^{\bullet} P$. It is readily viewed that the simplicial category $\prod^{\bullet} P$ has a natural augmentation to the "constant" simplicial category P and as such a natural morphism of nerves $N_{\bullet} \prod^{\bullet} P \to N_{\bullet} P$. Upon applying geometric realizations and fundamental groupoids we obtain a canonical map $K(\mathcal{A}) \to |P|$ of topological spaces. Or rather a map $d(K(\mathcal{A})) \to S|P|$ of simplicial sets, where d denotes the diagonal of a bisimplicial set and S is the functor associating to a simplicial set its singular complex. Applying the functor π_f to this gives us the required canonical functor, in view of the fact that in general $\pi_f S|P| = \pi_f P = P$.

Remark 2.3.4.1. Let $\mathbf{A}$ be any (small) saturated Waldhausen category (see [RT90], Definition 1.2.5), so that in particular the localization $w^{-1}\mathbf{A}$ is well behaved. Then any functor from the groupoid $F : (w^{-1}\mathbf{A}, w) \to P$ to a (small) groupoid P corresponds in fact to a map of topological spaces $|F| : |wS_1\mathbf{A}| \to |P|$, and the functor F is recovered by applying the functor π_f. This follows from the fact that for two simplicial sets X and Y, there is a natural bijection $\hom_{Top}(|X|, |Y|) = \hom_S(X, S|Y|)$ and that the fundamental groupoid of a simplicial set Y and $S|Y|$ are in fact the same; see [PGG99], chapter I, Proposition 2.2 and section 8. It is also clear that the fundamental groupoid of the nerve of a groupoid is in fact the same groupoid. Also, any functor $F : \pi_f(K(\mathbf{A})) \to P$ of (small) Picard categories correspond to a map of topological spaces $K(\mathbf{A}) \to |P|$ by, for example, sending an n-simplex of the simplicial set $|wS_q A\mathbf{A}|$ of the form $A_1 \rightarrowtail A_2 \rightarrowtail \ldots \rightarrowtail A_{n+1}$ to the n-th nerve $F(A_{n+1}) \simeq F(A_{n+1}/A_n) \oplus F(A_n) \simeq \ldots \simeq F(A_{n+1}/A_n) \oplus F(A_n/A_{n-1}) \oplus \ldots \oplus F(A_1)$ of P. The functor $|F|$ can again be recovered by applying the functor π_f. Thus, the above problem of describing determinant functors can likewise be formulated as a lifting-problem of maps of topological spaces. We ask for which continuous functions $f : |wS_1\mathbf{A}| \to |P|$ there is a lift as in the diagram below

$$\begin{array}{ccc} |wS_1\mathbf{A}| & \longrightarrow & K(\mathbf{A}) \\ & \searrow & \Big\downarrow \exists \\ & & |P|. \end{array}$$

We leave the precise reformulation to the interested reader, as it will not be used in the rest of this article.

□

Remark 2.3.4.2. Since any exact category can be equipped with the structure of a biWaldhausen category, where the weak equivalences are the isomorphisms and the cofibrations are the admissible monomorphisms, it is clear that the above definition generalizes that of Deligne [Del87]. It is a simple exercise to verify that in this case the above axioms for determinant functors are equivalent to those given in loc.cit.. We also have the following stronger assertion

Proposition 2.3.5 ([Wal85] Theorem 1.9). *The Waldhausen K-theory spectrum of an exact category $\mathcal{E}$, $K(\mathcal{E})$, is naturally homotopy-equivalent to the K-theory spectrum of Quillen. A fortiori it induces an equivalence of fundamental groupoids and Picard categories.*

Proposition 2.3.6 ([RT90], 1.9.6). *Suppose in addition that $\boldsymbol{A}$ is complicial biWaldhausen so that it is a full subcategory of $C(\mathcal{A})$ for an abelian category $\mathcal{A}$. Furthermore suppose that it is closed under taking exact sequences in $C(\mathcal{A})$, is closed under finite degree shifts and co($\boldsymbol{A}$). Then $Ho(\boldsymbol{A}) = w^{-1}\boldsymbol{A}$ is a triangulated category and admits a calculus of fractions.*

Remark 2.3.6.1. By [RT90], Theorem 1.9.2, we can suppose that cofibrations are the degree-wise admissible monomorphisms with quotients in $\mathbf{A}$.

Proposition 2.3.7. *With the above notation, a determinant functor $[] : (\boldsymbol{A}, w) \to P$ admits the following equivalent description as a functor factoring via $D(\boldsymbol{A}) = Ho(\boldsymbol{A}) := w^{-1}\boldsymbol{A}$:*
(a'): For any distinguished triangle $\Sigma : A \to B \to C \to A[1]$ there is an isomorphism functorial with respect to isomorphisms (in $D(\boldsymbol{A})$):

$$\{\Sigma\} : [B] \simeq [A] \oplus [C].$$

(b'): For any object A, the distinguished triangle $\Sigma : A = A \to 0 \to A[1]$ decomposes the identity-map as:

$$[A] \xrightarrow{\{\Sigma\}} [A] \oplus [0] \xrightarrow{\delta^R} [A] .$$

(c'): For any distinguished triangle of distinguished triangles, i.e. a diagram of the form:

$$\begin{array}{lccccccc}
\Sigma_A : & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & A'[1] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma_B : & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & A'[1] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma_C : & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & A'[1] \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma_A[1] : & A'[1] & \longrightarrow & A[1] & \longrightarrow & A''[1] & \longrightarrow & A'[2] \\
& & & & & & & \\
& \Sigma' & & \Sigma & & \Sigma'' & & \Sigma'[1]
\end{array}$$

where all the rows and columns are distinguished triangles, the following diagram is commutative:

$$\begin{array}{ccccc}
[B'] \oplus [B''] & \xleftarrow[\{\Sigma_B\}]{} & [B] & \xrightarrow{\{\Sigma\}} & [A] \oplus [C] \\
\downarrow{\scriptstyle \{\Sigma'\}\oplus\{\Sigma''\}} & & & & \downarrow{\scriptstyle \{\Sigma_A\}\oplus\{\Sigma_C\}} \\
[A'] \oplus [C'] \oplus [A''] \oplus [C''] & & \longrightarrow & & [A'] \oplus [A''] \oplus [C'] \oplus [C'']
\end{array}$$

In case P is also commutative, we add the axiom
(d') The natural triangles

$$A \to A \oplus B \to A \to A[1]$$

and

$$B \to B \oplus B \to A \to B[1]$$

induce a commutative diagram

$$\begin{array}{ccc}
[A \oplus B] & \longrightarrow & [B \oplus A] \\
\downarrow & & \downarrow \\
[A] \oplus [B] & \longrightarrow & [B] \oplus [A].
\end{array}$$

Proof. We can suppose by the above remark that the cofibrations are given by degree-wise split monomorphisms, and these yield all distinguished triangles in $D(\mathbf{A})$. It is thus clear that the above data $(a') - (c')$ determine the data $(a)-(c)$, and that the data (d) and (d') are equivalent, so we show the converse statement. Since $D(\mathbf{A})$ admits a calculus of fractions it is immediate to verify that if a cofibration sequence $\Sigma : A' \rightarrowtail B' \twoheadrightarrow C'$ is isomorphic to a distinguished triangle $A \to B \to C \to A[1]$ we have an induced isomorphism $[B'] \simeq [A'] \oplus [C']$ such that the obvious diagram commutes. By (a) it does not depend on the choice of Σ. (b') is clearly equivalent to (b). To establish that the data of (c) determine that of (c'), we first notice that if $u : A \to B$ is any morphism in $D(\mathbf{A})$, and if we have two choices of cones of u, C and C', with an isomorphism $f : C \to C'$, by (a') applied to the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \| & & \| & & \downarrow{\scriptstyle f} & & \| \\ A & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A[1] \end{array}$$

there is an isomorphism $[C] = [C']$ which does not depend on the choice of f. Hence the object $[cone(u)]$ is determined up to unique isomorphism, as opposed to $cone(u)$, and for any distinguished triangle $A \xrightarrow{u} B \to C \to A[1]$ a canonical isomorphism $[C] = [cone(u)]$. Furthermore, any diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

can be completed into a diagram of the form in (c') by taking mapping cones and it follows that we can assume that our diagram is of that form. Lastly, if we have a map $u : A \to B$, the map $B \to cone(u)$ can be chosen to be represented by a cofibration and we are reduced to the case of cofibrations. □

Remark 2.3.7.1. Determinants on (small) triangulated categories were also studied in [Bre], where similar results were obtained.

Corollary 2.3.8 (Knudsen, [Knu]). *Let $i : \mathcal{E} \to \mathcal{A}$ be an exact fully faithful embedding of an exact category $\mathcal{E}$ in an abelian category $\mathcal{A}$, such that for any morphism in $\mathcal{E}$ which is an epimorphism in $\mathcal{A}$, is admissible in $\mathcal{E}$. Denote by $C(\mathcal{E})$ the full subcategory of bounded complexes of the category of complexes in $\mathcal{A}$. Then we have a natural equivalence of categories between the virtual category of $(\mathcal{E}, is)$ and the virtual category of $(C(\mathcal{E}), q.i.)$ of complexes in E with quasi-isomorphisms in $\mathcal{A}$.*

Proof. Equip the category $C(\mathcal{E})$ with the structure of a complicial biWaldhausen category where the weak equivalences are given by quasi-isomorphisms and the cofibrations are either of the two following: degree-wise admissible monomorphisms or degree-wise split monomorphisms whose quotients lie in $C(\mathcal{E})$. Denote the corresponding biWaldhausen categories by $\mathbf{E}$ and $\widetilde{\mathbf{E}}$. By [RT90], Theorem 1.11.7, we have natural homotopy-equivalences

$$K(\mathcal{E}) \simeq K(\mathbf{E}) \simeq K(\widetilde{\mathbf{E}})$$

and hence equivalent virtual categories. Moreover, this does not depend on the choice of $\mathcal{A}$. □

If $i : \mathcal{E} \to \mathcal{A}$ is the fully faithful Gabriel-Quillen embedding reflecting exactness (see [RT90], Appendix A), or if $\mathcal{E}$ is the category of coherent vector bundles and $\mathcal{A}$ is the category of coherent sheaves respectively on a scheme, i satisfies the above hypothesis.

Corollary 2.3.9. *Let $\boldsymbol{A}$ be a small complicial biWaldhausen category. Then K_0 and K_1 are functorially (with respect to triangulated functors, i.e. functors preserving the above structures) determined by the structure of a triangulated category of the homotopy category $Ho(\boldsymbol{A}) = D^b(\boldsymbol{A})$ and its isomorphisms.*

3. VIRTUAL CATEGORIES ASSOCIATED TO ALGEBRAIC STACKS

This little pig stayed at home

3.1 Various categories

We will freely use the language of Appendix C in this chapter where we expand slightly on the concept of a virtual category of an algebraic stack. We will always consider the a stack as a simplicial sheaf via the extended Yoneda functor C.0.18.1. Also, for the purposes of this section, all algebraic stacks are separated locally of finite type over some (non-fixed) Nötherian scheme S.

Definition 3.1.0.1. Given an algebraic stack $\mathcal{X}$, there are for our purposes four main candidates for virtual categories one might consider, namely any one of the following Picard categories

(a) the virtual category of locally free sheaves on $\mathcal{X}$, $V(\mathcal{X}) = V_{naive}(\mathcal{X})$.

(b) the virtual category of coherent $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$, $V_{coh}(\mathcal{X})$.

(c) the fundamental groupoid of $K^{sm}(\mathcal{X})$, the cohomological virtual category, $W(\mathcal{X})$.

(d) the fundamental groupoid of $G^{sm}(\mathcal{X})$, the coherent cohomological virtual category, $W_{coh}(\mathcal{X})$.

By the remarks concluding the Appendix C we have additive functors of fibered Picard categories, $V(-) \to W(-)$ and $V_{coh}(-) \to W_{coh}(-)$. Notice that since the automorphism-group of any object of $W(\mathcal{X})$ or $W_{coh}(\mathcal{X})$ is a $\mathbb{Q}$-vector space they are automatically strictly commutative.

Definition 3.1.0.2. Since K^{sm} is flabby, to give operations involving $W(\mathcal{X})$ it is sufficient to construct functorial homotopies on the K-theory spaces of the vertices of simplicial algebraic space $\mathcal{N}(X/\mathcal{X})$ for some presentation of $\mathcal{X}$. The same remark applies to $W_{coh}(\mathcal{X})$. We will say that any such constructed operations are given by *cohomological descent.*

Given a morphism $F : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks locally of finite type over a Nötherian scheme S, recall that for a coherent sheaf $\mathcal{F}$ we can define $R^iF_*\mathcal{F}$ by a Cech-cohomology argument (compare [Del74], Définition 5.2.2.). We know by [Ols05], Theorem 1.2, that whenever F is moreover proper, $R^iF_*\mathcal{F}$ is coherent whenever $\mathcal{F}$ is coherent. Suppose in addition that F is of finite cohomological dimension so that $R^iF_*(\mathcal{F}) = 0$ for large enough i. Then the usual formula

$$RF_*(\mathcal{F}) = \sum (-1)^i R^i F_* \mathcal{F}$$

defines a pushforward on $RF_* : V_{coh}(\mathcal{X}) \to V_{coh}(\mathcal{Y})$. It is more subtle to define the corresponding functor $W_{coh}(\mathcal{X}) \to W_{coh}(\mathcal{Y})$. If $F : \mathcal{X} \to \mathcal{Y}$ is a proper morphism, and given a proper surjective morphism $X \to \mathcal{X}$ with X a scheme, we obtain a diagram of

$$\begin{array}{ccc} \mathcal{N}(X/\mathcal{X}) & & \\ \downarrow{\scriptstyle p} & \searrow{\scriptstyle q} & \\ \mathcal{X} & \xrightarrow{F} & \mathcal{Y} \end{array}$$

with proper morphisms and applying the functor $G^{sm}()$ we obtain a diagram

$$\begin{array}{ccc} G^{sm}(\mathcal{N}(X/\mathcal{X})) = G(\mathcal{N}(X/\mathcal{X})) & & \\ \downarrow{\scriptstyle p_*} & \searrow{\scriptstyle q_*} & \\ G^{sm}(\mathcal{X}) & & G^{sm}(\mathcal{Y}) \end{array}$$

By [Toe99a], Théorème 2.9, given a proper surjective morphism $X \to \mathcal{X}$ with X a scheme and $\mathcal{X}$ is Deligne-Mumford, there is a weak equivalence $G(\mathcal{N}(X/\mathcal{X})) \to G^{sm}(\mathcal{X})$. Applying the fundamental groupoid-construction thus gives an equivalence of categories $\pi_f(G^{sm}(\mathcal{N}(X/\mathcal{X}))) \to W_{coh}(\mathcal{X})$ and we define $RF_* = q_*(p_*)^{-1} : W_{coh}(\mathcal{X}) \to W_{coh}(\mathcal{Y})$ (compare [Toe99b], Section 3.2.2). We have essentially proved:

Proposition 3.1.1. *Suppose $F : \mathcal{X} \to \mathcal{Y}$ is a proper of finite cohomological dimension morphism of separated Deligne-Mumford stacks of finite type over a Nötherian base-scheme S. It is possible to define a functor $RF_* : W_{coh}(\mathcal{X}) \to W_{coh}(\mathcal{Y})$ such that the diagram*

$$\begin{array}{ccc} V_{coh}(\mathcal{X}) & \longrightarrow & W_{coh}(\mathcal{X}) \\ \downarrow{\scriptstyle RF_*} & & \downarrow{\scriptstyle RF_*} \\ V_{coh}(\mathcal{Y}) & \longrightarrow & W_{coh}(\mathcal{Y}) \end{array}$$

is commutative up to canonical equivalence of functors.

Proof. The statement is clear as soon as we can show that there is always a choice of a proper surjective $X \to \mathcal{X}$ with X a scheme. It is clearly independent of such a choice. But this is [Ols05], Theorem 1.1, which moreover shows we can pick X to be quasi-projective over S. □

The following uses a standard argument factorizing a projective morphism as a closed immersion and a projective bundle projection:

Proposition 3.1.2. *Suppose $F : \mathcal{X} \to \mathcal{Y}$ is a (representable) projective local complete intersection morphism of algebraic stacks with $\mathcal{Y}$ quasi-compact and $\mathcal{Y}$ has the resolution property, i.e. any coherent sheaf is the quotient of a locally free sheaf. Then there is a natural functor*

$$RF_* : V(\mathcal{X}) \to V(\mathcal{Y})$$

compatible with the functor defined on V_{coh} under the additive functor $V(-) \to V_{coh}(-)$.

Proof. Suppose $\mathcal{F}$ is a locally free sheaf on $\mathcal{X}$.
In case $f = i$ is a regular closed immersion, by assumption the coherent sheaf $i_*\mathcal{F}$ has a finite locally free resolution and the inclusion of the category of vector bundles into the category of coherent sheaves with finite locally free resolutions is an equivalence of categories. In the case of a projective bundle-projection, $\mathbb{P}(\mathcal{E}) \to \mathcal{Y}$, by Theorem C.0.20 the functor

$$\bigoplus_{i=0}^{\operatorname{rk}\mathcal{E}-1} V(\mathcal{Y}) \to V(\mathbb{P}(\mathcal{E}))$$

sending $(\mathcal{F}_i)_{i=0}^{\operatorname{rk}\mathcal{E}-1}$ to $\sum_{i=0}^{\operatorname{rk}\mathcal{E}-1} \pi^*\mathcal{F}_i \otimes \mathcal{O}(-i)$ is an equivalence of categories, thus we can assume our vector bundle is of the latter form. In this case one simply defines $R\pi_*$ by taking cohomology and can be calculated to be

$$R\pi_*(\pi^*\mathcal{F}_i \otimes \mathcal{O}(-i)) = \mathcal{F}_i \otimes R\pi_*\mathcal{O}(-i) = \begin{matrix} 0, & \text{if } i < 0 \\ \mathcal{F}_0 & \text{if } i = 0 \end{matrix}$$

Given a factorization of F as $F : \mathcal{X} \to \mathbb{P}(\mathcal{E}) \to \mathcal{Y}$ we define the functor $RF_* : V(\mathcal{X}) \to V(\mathcal{Y})$ as the obvious composition. Arguing as in [WF85], chapter V, one sees that this does not depend on factorization up to canonical isomorphism. The extension to algebraic stacks does not cause any serious issues. The last point is obvious. □

Remark 3.1.2.1. Whenever we are working in a category of stacks where perfect complexes can be used to define algebraic K-theory the above is just a consequence of preservation of perfectness of a complex under proper local complete intersection morphisms. The compatibility under composition is given by Grothendieck's spectral sequence.

Similarly, if E is a vector bundle on $\mathcal{Y}$, and $F : \mathcal{X} \to \mathcal{Y}$ is any morphism, we define a functor $LF^* : V(\mathcal{Y}) \to V(\mathcal{X})$ via $LF^*[E] = [F^*E]$.

Let us just recall the usual definition of the base-change morphism, which always exists. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a Cartesian diagram of schemes. By adjointness, we have an equality of morphisms in the derived category of quasi-coherent complexes schemes;

$$Hom(Lf^*Rg_*E, Rg'_*Lf'^*E) = Hom(Rg_*E, Rf_*Rg'_*Lf'^*E)$$

and since $Rf_*Rg'_* \simeq Rg_*Rf'_*$ this is equal to

$$Hom(Rg_*E, Rg_*Rf'_*Lf'^*E).$$

By the adjunction morphism $E \to Rf'_*Lf'^*E$ we thus obtain a map

$$Hom(Rg_*E, Rg_*E) \to Hom(Lf^*Rg_*E, Rg'_*Lf'^*E).$$

The base-change morphism is the morphism which is the image under the identity-map on the left-hand-side.

Definition 3.1.2.1. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a commutative diagram of schemes. We say the diagram is transversal or Tor-independent or that X and Y' are transversal or Tor-independent over Y ([P71], III, Définition 1.5) if the diagram is a Cartesian diagram of schemes, with Y quasi-compact, f quasi-compact and quasi-separated and if for any $x \in X, y' \in Y'$ mapping to the same point $y \in Y$, we have

$$Tor_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0, \text{ for } i > 0,$$

and f is of finite Tor-dimension.

Lemma 3.1.3. *[SGA6, IV 3.1] Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a transversal diagram, and let $E \in D^b(X)$ be a complex with quasi-coherent cohomology. In this case the base-change morphism is an isomorphism

$$Lf^*Rg_*E \simeq Rg'_*Lf'^*E.$$

Since it is natural it also satisfies descent with respect to any smooth equivalence relationship and thus we have

Corollary 3.1.4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a transversal Cartesian diagram of quasi-compact algebraic stacks with the resolution property and representable morphisms, f and f' local complete intersection projective morphisms. Then there is a natural transformation

$$Lg^*Rf_* = Rf'_*Lg'^*$$

of functors $V(\mathcal{Y}') \to V(\mathcal{X})$.

Proof. From the above one readily obtains that if a vector bundle E is f_*-acyclic, f_*E is also g^*-acyclic and that g'^*E is f'_*-acyclic, inducing an isomorphism $g^*f_*E \to f'_*g'^*E$. If f is a projective bundle-projection we can, by Theorem C.0.20, assume that E is of the form $\sum f^*E_i \otimes \mathcal{O}(-i)$ which is a sum of f_*-acyclic objects. In the case f is a closed immersion f_* is automatically exact. The general case is obtained via the composition of the two which by standard techniques is seen to be independent of the choice of the factorization. □

The following will be used later

Lemma 3.1.5. *The following diagrams are commutative whenever all of the morphisms are defined:*

(a) Let

$$\begin{array}{ccc} X'' & \xrightarrow{g''} & Y'' \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be the composition of two transversal cartesian diagrams. Then the third diagram is also transversal and the diagram

$$\begin{array}{ccccc} Lg^*R(fe)_* & & \longrightarrow & & R(f'e')_*Lg''^* \\ \| & & & & \| \\ Lg^*Rf_*Re_* & \longrightarrow & Rf'_*Lg'^*Re_* & \longrightarrow & Rf'_*Re'_*Lg''^* \end{array}$$

is commutative.

(b) Let

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

be composition of two transversal cartesian diagrams. Then the third diagram is also transversal and the diagram

$$\begin{array}{ccccc} L(gh)^*Rf_* & & \longrightarrow & & Rf'_*L(g'h')^* \\ \downarrow & & & & \downarrow \\ Lh^*Lg^*Rf_* & \longrightarrow & Lh^*Rf'_*Lg'^* & \longrightarrow & Rf'_*Lh'^*Lg'^* \end{array}$$

Proof. Left to the reader (compare the unproved result of [Del77], XII, Proposition 4.4). □

The following is trivial:

Lemma 3.1.6 (Projection formula). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a local complete intersection projective morphism of algebraic stacks with the resolution property. Suppose F is a virtual bundle on $\mathcal{Y}$ and E is a virtual bundle on $\mathcal{X}$. Then there is a functorial isomorphism $Rf_*(E \otimes Lf^*F) \to Rf_*(E) \otimes Lf^*F$ compatible with transversal base-change, i.e. for a diagram as in Corollary 3.1.4, there is a commutative diagram*

$$\begin{array}{ccc} Lg^*Rf_*(E \otimes Lf^*F) & \longrightarrow & Lg^*(Rf_*(E) \otimes Lf^*F) \\ \downarrow & & \downarrow \\ Rf'_*(Lg'^*E \otimes Lgf'^*F) & \longrightarrow & Rf'_*(Lg'^*E) \otimes Lg^*Lf^*F) \end{array}$$

where the horizontal lines are given by the projection-formula and the vertical lines are given by base-change.

3.2 A splitting principle

Below we sketch a criterion for when we can descend a morphism on the level of the maximal flag-variety to the base [1]. First, let E be an vector-bundle of rank $e+1$ on a separated algebraic stack $\mathcal{X}$. The space $p^1 : Y_1 = \mathbb{P}(E) \to X$ is a projective bundle which on which we have a canonical sub-line bundle $\mathcal{O}(-1)$, and a canonical quotient-bundle of $p^{1*}E$. Repeating this construction with the quotient-bundle, we eventually obtain a map $p : \mathcal{Y} = \mathcal{Y}_e \to \mathcal{Y}_{e-1} \to \ldots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{X}$, where the top space is the maximal flag-variety of E on $\mathcal{X}$, which also comes equipped with a canonical maximal flag. Suppose P is a functor from the category of separated algebraic stacks to the category of Picard categories such that for any $\mathcal{X}$ there is a distributive additive functor $V(-) \times P(-) \stackrel{\otimes}{\to} P(-)$ moreover satisfying the projective bundle axiom; for any $\mathcal{X}$, the functor

$$\times_{i=0}^{e} P(\mathcal{X}) \to P(\mathbb{P}(E))$$

given by $(f_i)_{i=0}^{e} \mapsto \sum_{i=0}^{e} p^{1*} f_i \otimes \mathcal{O}(-i)$ is an equivalence of categories. Then the following is a version of an observation of Franke in terms of Chow categories of ordinary schemes (see the article by J. Franke, "Chern Functors" in [Fra91], 1.13.2):

Theorem 3.2.1. *[Splitting principle] Let $p_1, p_2 : \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ be the two projections, and $r = pp_1 = pp_2$. Then*
(a) $p^* : P(\mathcal{X}) \to P(\mathcal{Y})$ is faithful.
(b) Suppose we have two objects $A, B \in \operatorname{ob} P(X)$, and $f : p^*A \to p^*B$ a morphism in $\operatorname{Hom}_{P(\mathcal{Y})}(p^*A, p^*B)$, then f comes from a (unique) morphism $h : A \to B$ if and only if $p_1^*(f) = p_2^*(f)$ in $\operatorname{Hom}_{P(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y})}(r^*A, r^*B)$.

Proof. From the projective bundle axiom it follows each p^{i*} is injective on the level of automorphism-groups, i.e. for any object A in $P(\mathcal{Y}_i)$, $\operatorname{Aut}_{P(\mathcal{Y}_i)}(A) \to \operatorname{Aut}_{\mathcal{Y}_{i+1}}(p^{i*}A)$ is injective, so the functor is faithful. For (b), the condition is obviously necessary. To prove that the condition is sufficient we can assume $A = B$. Let $0 = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_e = p^*E$ be the universal flag on $\mathcal{Y}$, and $L_i = E_i/E_{i-1}$, then by the projective bundle axiom we have natural isomorphisms

$$\operatorname{Aut}_{P(Y)}(A) = \bigoplus_{j_1=0}^{e} \ldots \bigoplus_{j_e=0}^{1} L_1^{j_1} \otimes \ldots \otimes L_e^{j_e} \otimes p^* \operatorname{Aut}_{P(\mathcal{X})}(A)$$

[1] recall that a flag is a sequence of sub-vector bundles $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n$ whose successive quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are also vector bundles. It is furthermore maximal if each such quotient is a line bundle.

and

$$\mathrm{Aut}_{P(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y})}(A) = \bigoplus_{j_1,j_1'=0}^{e} \cdots \bigoplus_{j_e=0,j_e'=0}^{1} p_1^* L_1^{j_1} \otimes p_2^* L_1^{j_1'} \otimes \ldots \otimes p_1^* L_e^{j_e} \otimes p_2^* L_e^{j_e'} \otimes r^* \mathrm{Aut}_{P(\mathcal{X})}(A).$$

Representing f in the form suggested above, we see that $p_1^*(f) = p_2^*(f)$ exactly when all components of f are zero except for the one belonging to $(j_1, \ldots, j_e) = (0, \ldots, 0)$, which means exactly that f is equal to p^*h for some morphism $h : A \to A$. Moreover h is unique because of (a). □

By base-change to the flag-variety we can suppose we have nice enough flags. If we define an isomorphism dependent on this flag, the content of the proposition is that this descends to the base whenever this isomorphism isn't dependent on the flag.

3.3 Adams and λ-operations on the virtual category

Let S be a scheme, and $\mathcal{X}$ an algebraic stack over S. Recall that we denote by $\mathbf{P}(\mathcal{X})$ the category of vector bundles on $\mathcal{X}$. Denote by $V(\mathcal{X})$ the virtual category thereof. We have the following result which is more or less contained in [Gra92];

Proposition 3.3.1. *There is a unique family of determinant functors $\Psi^k : \mathbf{P}(\mathcal{X}) \to V(\mathcal{X})$, and thus $\Psi^k : V(\mathcal{X}) \to V(\mathcal{X})$, stable under pullback, such that*

- *If L is a line bundle, $\Psi^k(L) = L^{\otimes k}$.*
- $\Psi^k \circ \Psi^{k'} \simeq \Psi^{kk'}$.

Proof. Unicity of the operations clearly follows from the characterizing properties and the splitting principle (Theorem 3.2.1). To prove existence, we apply the ideas of loc.cit.. Let N be a complex of vector bundles, and CN be the cone of the identity morphism $\mathrm{id} : N \to N$. Furthermore, let S^k be the k-th symmetric power, so that the p-th term of S^kCN is $S^{k-p}N \otimes \wedge^p N$, whenever N is reduced to a vector bundle in degree 0 (for details, see loc.cit., p. 4). Finally, for a bounded complex $N_\bullet = [\ldots \to N_{i-1} \to N_i \to N_{i+1} \to \ldots]$, define the secondary Euler characteristic $\chi'(N_\bullet) = \sum (-1)^{p+1} p[N_p] \in V(\mathcal{X})$. One of the key ideas of loc.cit. (formula (3.1)) is the formula in $K_0(\mathcal{X})$, for a vector bundle E,

$$\Psi^k(E) = \chi'(S^kCE).$$

We propose the same definition for Adams operations in the virtual category $V(\mathcal{X})$. Clearly $\Psi^k(L) = L^{\otimes k}$ for a line bundle L. Now, given a flag $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n$, define $E_1 \cdot E_2 \ldots \cdot E_n$ to be the image of $E_1 \otimes E_2 \otimes \ldots \otimes E_n$ in $S^n E_n$. Suppose that we have an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, and consider the filtration

$$\begin{aligned} S^k CE' &= CE'.CE'.\ldots.CE'.CE' \\ &\subseteq CE'.CE'.\ldots.CE'.CE \\ &\subseteq CE'.CE'.\ldots.CE.CE \subseteq \ldots \\ &\subseteq CE.CE.\ldots.CE.CE = S^k CE \end{aligned}$$

induces by a certain addivity of the secondary Euler characteristic, isomorphisms

$$\begin{aligned} \chi'(S^k CE) &= \chi'(S^k CE'') + \chi'(S^k CE') + \sum_{i=0}^{k-1} \chi'(S^i CE'' \otimes S^{k-i} CE') \\ &= \chi'(S^k CE'') + \chi'(S^k CE') \end{aligned}$$

since the secondary Euler-characteristic of a product of acyclic complexes is 0 and by the multilinearity-property of loc.cit (formula (2.1)). We need only verify that this operation respects filtrations. Let $F \subseteq H \subseteq E$ be an admissible filtration, and consider the double graded filtrations of $S^k CE$, $A_{\bullet,\bullet}$, where $A_{i,j} = S^{k-i-j} CE.S^j CF.S^i CH$. Applying secondary Euler characteristics in every direction, we obtain that the diagram of isomorphisms

$$\begin{array}{ccc} \chi'(S^k CE) & \longrightarrow & \chi'(S^k CH) + \chi'(S^k CE/H) \\ \downarrow & & \downarrow \\ \chi'(S^k CF) + \chi'(S^k CE/F) & \longrightarrow & \chi'(S^k CF) + \chi'(S^k CH/F) + \chi'(S^k CE/H) \end{array}$$

constructed above commutes. Condition "b)" of Definition 2.3.2.1 is trivial. Also everything is clearly stable under pullback. The last point now follows by unicity. □

Remark 3.3.1.1. In the next chapter we will show that whenever we restrict ourselves to regular schemes, the constructed Adams-operations are actually unique, at least whenever one inverts 2 or more primes in the virtual category.

We record the following corollary (of the splitting principle applied to the above case):

Corollary 3.3.2. *$\Psi^k : V(\mathcal{X}) \to V(\mathcal{X})$ is a ring-homomorphism in the sense that there are natural isomorphisms, for $A, B \in \mathrm{obV(X)}$,*

$$\Psi^k(A \otimes B) \simeq \Psi^k(A) \otimes \Psi^k(B)$$

compatible with the above sum-operation and compatible with base-change.

Proof. We only need to verify the multiplicative operation. It suffices to show that for any $A \in V(\mathcal{X}), B \in \mathbf{P}(\mathcal{X})$, $\Psi^k(A \otimes B) = \Psi^k(A) \otimes \Psi^k B)$ naturally. Or, by the splitting principle since Ψ^k is already an additive determinant functor, that if B is a line bundle, then $\Psi^k(A \otimes L) = \Psi^k(A) \otimes L^{\otimes k}$ naturally. For this we can assume that A is also a line-bundle M, in which case we have $\Psi^k(M \otimes L) = (M \otimes L)^{\otimes k} = M^{\otimes k} \otimes L^{\otimes k} = \Psi^k(M) \otimes \Psi^k(L)$. □

We also have a functor

$$\lambda_{-1} : \mathbf{P}(\mathcal{X}) \to V(\mathcal{X})$$

defined as follows. If E is a vector-bundle on $\mathcal{X}$, we define $\lambda_{-1}(E)$ as the alternating product of exterior powers $\sum_{i=0}^{\infty}(-1)^i \wedge^i E$. This is an object which is unique up to canonical isomorphism. Given a short exact sequence of vector-bundles

$$0 \to F \xrightarrow{\pi} E \to E/F \to 0$$

we can define an isomorphism

$$\lambda_{-1}(E) \to \lambda_{-1}(F) \otimes \lambda_{-1}(E/F) \tag{3.1}$$

as follows; for an integer k, such that $0 \leq k \leq n = \operatorname{rk} F$, we have a well known natural filtration (c.f. [P71], Ch. V, Lemme 2.2.1) of $\wedge^k F$ whose i-th instance is given by $F^i \wedge^k E = \operatorname{Im}[\wedge^i F \otimes \wedge^{k-i} E \to \wedge^k E]$, and with successive quotients $\wedge^i F \otimes \wedge^{k-i} F/E$, thus giving isomorphisms

$$\wedge^k E \simeq \sum_{i=0}^{k} \wedge^i F \otimes \wedge^{k-i} E/F. \tag{3.2}$$

Now, given two virtual vector bundles A and B, we have

$$A \otimes B + (-A) \otimes B = (A + (-A)) \otimes B = 0 \otimes B = 0$$

and thus an isomorphism $(-A) \otimes B) \simeq -(A \otimes B)$. Analogously we obtain $A \otimes (-B) \simeq -(A \otimes B)$. The diagram

$$\begin{array}{ccc} (-A) \otimes (-B) & \longrightarrow & -(A \otimes (-B)) \\ \downarrow & & \downarrow \\ -((-A) \otimes B) & \longrightarrow & --(A \otimes B) = A \otimes B \end{array}$$

is only commutative up to sign $\epsilon(A \otimes B)$ (c.f. [Del87], 4.11 a) + b)). We define the isomorphism $(-1)^k \wedge^k E = \sum_{i=0}^k [(-1)^i \wedge^i F] \otimes [(-1)^{k-i} \wedge^{k-i} E/F]$ via (3.2), the isomorphism

$$\begin{aligned}(-1)^k \wedge^i F \otimes \wedge^{k-i} E/F &= (-1)^{k-i}(((-1)^i \wedge^i F) \otimes \wedge^{k-i} E/F) \\ &= ((-1)^i \wedge^i F) \otimes (-1)^{k-i} \wedge^{k-i} E/F.\end{aligned}$$

The isomorphisms (3.1) and (3.2) are easily verified to be compatible (up to sign) with successive admissible filtrations $F' \subset F \subset E$ by considering the double filtration $F^{i,j}F = \mathrm{Im}[\wedge^i F' \otimes \wedge^j F \otimes \wedge^{k-i-j} E \to \wedge^k E]$.

These operations are however only commutative and associative up to a nightmare of signs, but become commutative and associative once one get rid of these.

3.4 Deformation to the normal cone

In this section we recall for the convenience of the reader some very well-known facts about the deformation to the normal cone slightly extended to the case of stacks. Some of these observations already appear in [Fra]. A reference for details of the below is [Ful98], chapter "Deformation to the Normal Cone".

First of all, given a section $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ of a rank n vector bundle $\mathcal{E}$ on an algebraic stack X, one has an induced dual section $s^{\vee} : \mathcal{E}^{\vee} \to \mathcal{O}_{\mathcal{X}}$ given by the composition of $\mathcal{O}_{\mathcal{X}} \to \mathcal{E}$ with $\mathcal{E} \otimes \mathcal{E}^{\vee} \to \mathcal{O}_{\mathcal{X}}$. This leads to a sequence

$$0 \to \Lambda^n \mathcal{E}^{\vee} \to \Lambda^{n-1} \mathcal{E}^{\vee} \to \ldots \to \Lambda^2 \mathcal{E}^{\vee} \to \mathcal{E}^{\vee} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{Z(s)} \to 0,$$

called the Koszul complex, which in local coordinates is given by

$$e_1 \wedge \ldots \wedge e_j \mapsto \sum_{i=0}^{j} (-1)^i s^{\vee}(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_j$$

where $Z(s)$ is the zero-locus of the section s. We say that s is a regular section if the above complex is exact.

Suppose that $i : \mathcal{X} \to \mathcal{Y}$ is a regular closed immersion of algebraic S-stacks, and let $N = N_{\mathcal{Y}/\mathcal{X}}$ be the normal bundle to i. We have a map $\mathbb{P}^1_{\mathcal{X}} \to \mathbb{P}^1_{\mathcal{Y}}$, and we define M as the blow-up of $\mathbb{P}^1_{\mathcal{Y}}$ in $\mathcal{X} \times \{\infty\}$ i.e. $Proj(\mathcal{I})$ where $\mathcal{I}$ is the ideal of the immersion I (see [GL00], 14.3). We have a natural 1-morphism from $\pi : M \to \mathbb{P}^1_{\mathcal{Y}} \to \mathbb{P}^1_S$ which is flat if Y is, and by the universal property of blowing-up the map $\mathbb{P}^1_{\mathcal{X}} \to \mathbb{P}^1_{\mathcal{Y}}$ lifts to M, so we have a diagram

$$\begin{array}{ccc} \mathbb{P}^1_X & \xrightarrow{\qquad} & M\,. \\ & \searrow \qquad \swarrow_{\pi} & \\ & \mathbb{P}^1_S & \end{array}$$

Moreover, $N_{\mathcal{X}\times\{\infty\}/\mathbb{P}^1_{\mathcal{Y}}} \simeq N_{\mathcal{X}/\mathcal{Y}} \oplus N_{\infty/\mathbb{P}^1} \simeq N \oplus 1$ (see [Ful98], Appendix B.6.3 and chapter IV, Proposition 3.6). The last isomorphism $N_{\infty/\mathbb{P}^1} \simeq 1$ is non-canonical, but we fix one once and for all to lax the notation. The exceptional divisor on M is a Cartier divisor isomorphic to $\mathbb{P}(N \oplus 1)$ and the fibre over ∞ is isomorphic to the union of this exceptional divisor and $Bl_{\mathcal{X}}\mathcal{Y}$, gluing together along a subscheme isomorphic to $\mathbb{P}(N)$. We have that for $s \in \mathbb{P}^1$, $\pi^{-1}(s) = \mathcal{Y}$ if $s \neq \infty$, and is equal to $\mathbb{P}(N \oplus 1) \cup Bl_{\mathcal{X}}\mathcal{Y}$, and the image of X does not meet $Bl_{\mathcal{X}}\mathcal{Y}$. In fact, it is embedded via the sub-bundle 1 in $N \oplus 1$ where we embed it as $x \mapsto (0, x)$. In other words, it embeds as $\mathcal{X} = \mathbb{P}(1) \to \mathbb{P}(N \oplus 1)$ or as the zero-section $X \to N$ followed by the open immersion $N \to \mathbb{P}(N \oplus 1)$. Let $p : P = \mathbb{P}(N \oplus 1) \to \mathcal{X}$ be the projection. On $\mathbb{P}(N \oplus 1)$ we have a universal exact sequence

$$0 \to \mathcal{O}_P(-1) \to p^*N \oplus 1 \to \xi \to 0$$

where $\mathcal{O}_P(-1)$ is the universal sub-bundle and $\xi = T_{P/\mathcal{X}}(-1)$ is the universal quotient-bundle. By [Ful98], Appendix B. 5.6., we know that the section determined by

$$\mathcal{O}_P(-1) \to p^*N \oplus 1 \to p^*N \tag{3.3}$$

is a regular section with zero-locus equal to $\mathcal{X} = \mathbb{P}(1) \subseteq \mathbb{P}(N \oplus 1)$. We also have another section defined by $1 \to p^*N \oplus 1 \to \xi$ which is also regular with the same zero-locus. The following is at worst an exercise:

Proposition 3.4.1. • *Suppose that we have two regular immersions $i : \mathcal{X} \hookrightarrow \mathcal{Y}, j : \mathcal{Y} \hookrightarrow \mathcal{Z}$ with normal bundles N_i, N_j and N_h, where $h = j \circ i$. We have an exact sequence, localized on $\mathcal{X}$:*

$$0 \to N_j \to N_h \to N_i \to 0.$$

• *We can simultaneously deform to the normal cone for i and $j \circ i$, to the natural embedding of the form*

$$\mathcal{X} \hookrightarrow \mathbb{P}(N_i \oplus 1) \hookrightarrow \mathbb{P}(N_h \oplus 1).$$

- *If we have a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \end{array}$$

where f, f' are closed regular embeddings, the two deformations associated to them are compatible in the sense that they restrict to a Cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad g' \quad} & \mathcal{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbb{P}(N_{f'} \oplus 1) & \xrightarrow{g} & \mathbb{P}(N_f \oplus 1) \end{array}$$

over the infinite fiber.

- *Let $i : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a closed regular embedding. Let $N^\vee$ be a vector-bundle on $\mathcal{Y}$, and suppose we are given a regular section on N which gives a Koszul resolution $s : N^\vee \to \mathcal{O}_\mathcal{Y}$ of $\mathcal{O}_\mathcal{X}$. Then under the deformation to the normal cone this restricts over the infinite fiber to the Koszul-resolution $\pi^* N_i^\vee(-1) \to \mathcal{O}_P$ given by (3.3).*

Proof. The first property is [WF85], chapter IV, Proposition 3.4, the second and third follows from [Ful98], Appendix B, B.6.9. Denote by M the deformation to the normal cone of i. Let $q : M \to \mathbb{P}^1_\mathcal{Y} \to \mathcal{Y}$ and $\pi : \mathbb{P}(N_i \oplus 1) \to \mathcal{X}$ be the natural projections, and $g : P = \mathbb{P}(N_i \oplus 1) \to M$ be the natural inclusion. $q^* s : q^* N^\vee \to \mathcal{O}_M$ is determined by a section which has a simple zero on the exceptional divisor, so it factors over $q^* N^\vee \otimes \mathcal{O}(P) \to \mathcal{O}_M$. This is a Koszul resolution of $\mathbb{P}^1_\mathcal{X}$. Restricting to P, we obtain a Koszul-resolution

$$g^* \left(q^* N^\vee \otimes \mathcal{O}(P) \right) = g^* q^* N^\vee \otimes \mathcal{O}(P)|_P \to \mathcal{O}_P$$

of $\mathcal{X}$. By [WF85], chapter IV, Proposition 3.2, $i^* N^\vee = N_i^\vee$, and $q \circ g = i \circ \pi$, so $g^* q^* N^\vee = \pi^* g^* N^\vee = \pi^* N_g^\vee$. We also have, by [Ful98], B.6.3, $\mathcal{O}(P)|_P = \mathcal{O}_P(-1)$, and that the section is the right one is clear. □

Lemma 3.4.2. *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g'} & \mathcal{Y} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{X}' & \xrightarrow{g} & \mathcal{Y}' \end{array}$$

be a Cartesian diagram. Let M' and M be the deformations to the normal cone of $\mathcal{X}' \subseteq \mathcal{Y}'$ and $\mathcal{X} \subseteq \mathcal{Y}$ respectively, and $G : M \to M'$ the induced map. Then we have

$$\widetilde{\mathcal{Y}}' = G^{-1}(\widetilde{\mathcal{Y}}), \mathbb{P}(N') = G^{-1}(\mathbb{P}(N))$$

and

$$\mathbb{P}_{\mathcal{X}'}(N' \oplus 1) = G^{-1}(\mathbb{P}_{\mathcal{X}}(N \oplus 1)).$$

Proof. By [Ful98], Appendix B, B.6.9, given the above kind of diagram, and exceptional divisors E (resp. E') of $Bl_{\mathcal{Y}}\mathcal{Y}'$ (resp. $Bl_{\mathcal{X}}\mathcal{X}'$), and $G : Bl_{\mathcal{Y}}\mathcal{Y}' \to Bl_{\mathcal{X}}\mathcal{X}'$ then $G^{-1}E = E'$. Applying this to the diagrams

$$\begin{array}{ccc}
\mathcal{X} \times \{\infty\} & \xrightarrow{g'} & \mathcal{Y} \times \{\infty\} \\
\downarrow & & \downarrow \\
\mathcal{X}' \times \{\infty\} & \xrightarrow{g} & \mathcal{Y}' \times \{\infty\} \\
\downarrow & & \downarrow \\
\mathbb{P}^1_{\mathcal{X}'} & \xrightarrow{G} & \mathbb{P}^1_{\mathcal{Y}'}
\end{array}$$

one obtains the lemma. □

4. RIGIDITY AND OPERATIONS ON VIRTUAL CATEGORIES

This little pig had roast beef

In this section we exhibit certain rigidity-properties of the virtual categories we are working on, and also the main technical results of this part of the thesis. As such, it rests heavily on the results obtained in [Mor99], [Rio06] and [VV99], and can perhaps in many instances be seen as reformulations of results therein obtained. The formulation in terms of K-cohomology was inspired from [Toe99a].

The main result of this section (Theorem 4.0.6) can be phrased, in a certain situation, that there is a certain commutative diagram

$$\begin{array}{ccc} hom_{\mathcal{H}(\mathfrak{R})}(\mathcal{K}_{\mathbb{Q}}, \mathcal{K}_{\mathbb{Q}}) & = & \hom_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}}) \\ & \searrow \quad \swarrow \nearrow & \\ & \hom(V_{\mathbb{Q}}, V_{\mathbb{Q}}) & \end{array}.$$

Here $\hom_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}})$ is a set of natural endo-transformations of the presheaf $K_0(-)$ on the category of regular schemes, and $\hom(V_{\mathbb{Q}}, V_{\mathbb{Q}})$ is the set of endo-functors of the virtual category of algebraic vector bundles strictly stable under pullback. Finally, $\mathcal{K}_{\mathbb{Q}}$ is a simplicial sheaf representing (rational) algebraic K-theory. This allows us to associate functorial operations on $V_{\mathbb{Q}}$ via the corresponding operations on K_0. We refer to the theorem for a precise formulation.

Let X be a separated regular Nötherian scheme of finite Krull dimension d. Then it is well known (see for example [WF85], chapter V, Corollary 3.10, [P71], chapter VI, Théorème 6.9 or use [RT90], Theorem 7.6 and (10.3.2)) then any element x of $K_0(X)$ of virtual rank 0 is nilpotent, and moreover we have $x^{d+1} = 0$. One can prove this in several ways, but one of the most natural ways is to construct a certain filtration on $K_0(X)$ which can be compared to other filtrations in terms of dimension of supports, a filtration that will terminate for natural reasons (see loc.cit.). One such filtration is the γ-filtration F_γ^p, built out of the λ-ring structure on $K_0(X)$ (see [WF85],

chapter III, p. 48 or [P71], chapter V, 3.10). We wish to incarnate this kind of nilpotence in the virtual category of X. Obviously, if x is a virtual vector-bundle of rank 0, then we know that a high enough power of it is isomorphic to a zero-object, but only non-canonically. A naive idea is to search for a decreasing filtration Fil^p of $V(X)$ which has the property that the functors $\mathrm{Fil}^p \to \mathrm{Fil}^{p-1}$ are faithful functors, and for big enough p, Fil^p is a category with exactly one morphism between any two objects.
The approach we have chosen to the problem is to construct the filtration already on the level of classifying spaces of the $\mathbb{P}^1$-spectrum representing rational algebraic K-theory in $\mathcal{SH}(S)$, and then use simplicial realizations to obtain a canonical filtration of $BQ\mathbf{P}(X)$ which eventually terminates or becomes trivial. For the notation used in this section we refer the reader to the Appendix.
Grayson (see [Gra95]) proposes that there should be a multiplicative filtration W^p of a space $K(X)$ representing the K-theory of X;

$$\dots W^2 \to W^1 \to W^0 = K(X)$$

such that the two following properties are satisfied:

(a) For any t, the quotient W^t/W^{t+1} is the simplicial realization of a simplicial abelian group.

(b) The Adams operations Ψ^k act by multiplication by k^t on W^t/W^{t+1}.

Such a filtration would immediately give an exact couple and thus give rise to an Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

relating "motivic cohomology" (that is, cohomology of $\mathbb{Z}(t) = \mathbb{Z}(t)^W := \Omega^{2t}(W^t/W^{t+1})$, in the sense of spectra with negative homotopygroups) on the left with algebraic K-theory on the right. In loc. cit. it is noted that the Postnikoff filtration satisfies the first but not the second property. For smooth schemes over a field [Lev05] constructed a coniveau-filtration which gives the correct spectral sequence for smooth varieties over a field.
The starting point of this section is the following theorem, which states that if we tensor with $\mathbb{Q}$ we can construct a Grayson-like filtration with various functorial properties. The author ignores if the filtration of [Lev05] coincides with the one considered in this section, both considered as objects of the appropriate homotopy category of schemes.

Theorem 4.0.3. *There are H-groups* $\{\mathrm{Fil}^{(i)}\}_{i=0}^{\infty}$ *(i.e. group-objects) and* $\{\mathbb{H}^{(i)}\}_{i=0}^{\infty}$ *of* $\mathcal{H}(\mathfrak{R}_S)_\bullet$*, determined up to unique isomorphism, satisfying the following properties:*

(a) $\mathrm{Fil}^0 = (\mathbb{Z}\times \mathrm{Gr})_{\mathbb{Q}}$ *and for any* $i \geq 0$*, there are morphisms* $\mathrm{Fil}^{(i+1)} \to \mathrm{Fil}^{(i)}$.

(b) For any i, j*, there are natural pairings* $\mathrm{Fil}^{(i)} \wedge \mathrm{Fil}^{(j)} \to \mathrm{Fil}^{(i+j)}$ *making, for* $i' \leq i, j' \leq j$*, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Fil}^{(i)} \wedge \mathrm{Fil}^{(j)} & \longrightarrow & \mathrm{Fil}^{(i+j)} \\ \downarrow & & \downarrow \\ \mathrm{Fil}^{(i')} \wedge \mathrm{Fil}^{(j')} & \longrightarrow & \mathrm{Fil}^{(i'+j')} \end{array}.$$

(c) For any i, j*, there are natural pairings* $\mathbb{H}^{(i)} \wedge \mathbb{H}^{(j)} \to \mathbb{H}^{(i+j)}$.

(d) There is a factorization $\mathrm{Fil}^{(i+1)} \to \mathrm{Fil}^{(i+1)} \times \mathbb{H}^{(i)} \overset{\Phi_i}{\approx} \mathrm{Fil}^{(i)}$ *which is compatible with the two above products. The pairings are also associative in the obvious sense.*

(e) The Adams operations Ψ^k *act on all the above objects and morphisms and acts purely by multiplication by* k^i *on* $\mathbb{H}^{(i)}$.

Proof. It follows from Theorem A.0.12 that we have a filtration of $\mathbf{BGL}_{\mathbb{Q}}$ in $\mathcal{SH}(\mathfrak{R}_S)$ given by $\mathrm{Fil}^p = \bigoplus_{n\geq p}\mathbb{H}^n$. By definition there is an evaluation-functor $\mathrm{ev}_n : \mathcal{SH}(\mathfrak{R}_S) \to \mathcal{H}(\mathfrak{R}_S)_\bullet$ sending a spectra (c.f. Appendix A) $\mathbf{E}$ to $\mathbf{E}_n$. Evaluating at 0 we obtain a canonical filtration of $\mathrm{ev}_0(\mathbf{BGL}_{\mathbb{Q}}) \simeq (\mathbb{Z}\times \mathrm{Gr})_{\mathbb{Q}}$ (see Definition A.0.7.3), a filtration $\{\mathrm{Fil}^{(i)}\}_{i=0}^{\infty}$ in $\mathcal{H}(\mathfrak{R}_S)_\bullet$. We similarly define $\mathbb{H}^{(i)} = \mathrm{ev}_0(\mathbb{H}^i)$ so that $\mathrm{Fil}^{(i)} = \mathbb{H}^{(i)} \oplus \mathrm{Fil}^{(i-1)}$. They are the 0-th space of a $\mathbb{P}^1$-spectrum and automatically H-groups. We similarly define Adams operations Ψ^k on the various objects via the same functor ev_0. We need to verify the other claimed properties.

Let $\mathcal{X}$ be a pointed simplicial sheaf, and define $\Omega^j X = \underline{Hom}_{\Delta^{op}\mathrm{Shv}_\bullet(\mathfrak{R}_S)}(S^j, \mathcal{X})$, the right adjoint to $S^j \wedge -$. Also denote by $R\Omega^j$ the total derived functor of Ω^j in $\mathcal{H}(\mathfrak{R}_S)_\bullet$.

Now, as before, denote by Φ the Yoneda-functor

$$\Phi : \mathfrak{R}_S \to \Delta^{op}\mathrm{Shv}(\mathfrak{R}_{S,\mathrm{sm}}) \to \mathcal{H}(\mathfrak{R}_S)$$

and for any object $G \in \mathcal{H}(\mathfrak{R}_S)$ we denote by ϕG the presheaf

$$\mathfrak{R}_S \ni U \mapsto \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\Phi U, G).$$

We then have:

Lemma 4.0.4. *Let $j \geq 0$. We have the following natural isomorphisms of presheaves on $\mathfrak{R}_S$ in the following cases:*

- $\phi(R\Omega^j(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}) = K_j(-)_{\mathbb{Q}}$.
- $\phi(R\Omega^j\mathbb{H}^{(i)}) = K_j(-)^{(i)}$, *the presheaf of sections of $K_j(-)$ with Ψ^k-eigenvalue k^i (which is independent of $k \geq 2$).*
- $\phi(\mathrm{Fil}^{(i)}) = F^iK_0(-)_{\mathbb{Q}} = \bigoplus_{p \geq i} K_0(-)^{(p)}$, *where*

  $$F^iK_0(X)_{\mathbb{Q}} = \bigcup_{Z \subset X} \mathrm{im}[K_0^Z(X)_{\mathbb{Q}} \to K_0(X)_{\mathbb{Q}}]$$

  and the union is over all closed subschemes $Z \subset X$ of codimension at least i, and $K_0^Z(X)$ is the Grothendieck group of complexes of vector bundles on X acyclic outside of Z, and the map is the natural one.
- *Let* $\mathbb{P}^\infty = \mathrm{colim}_n \mathbb{P}^n$, *then* $\phi(\mathbb{P}^\infty) = \mathrm{Pic}(-)$, $\phi(R\Omega\mathbb{P}^\infty) = \mathbb{G}_m$ *and* $\phi(R\Omega^j\mathbb{P}^\infty) = 0$ *otherwise.*

Proof. In view of how the Adams-operations act on the various objects involved, using Theorem A.0.8 the first non-trivial part is the equality $\bigoplus_{p \geq i} K_0(-)^{(p)} = F^iK_0(-)_{\mathbb{Q}}$ which is [Sou92], chapter I, Lemma 6 . The last point is established as in [VV99], Section 4, Proposition 3.8. □

Lemma 4.0.5. *[[Rio06], proof of Théorème III.29 + 31,] Suppose S is a regular scheme, and $\mathcal{X}$ and $\mathcal{Y}$ are objects of $\mathcal{H}(\mathfrak{R}_S)_\bullet$. Then the natural maps*

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\bullet, \mathfrak{R}_S^{op}\,\mathrm{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

and

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\mathfrak{R}_S^{op}\,\mathrm{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

are bijective in the case $\mathcal{Y}$ and $\mathcal{X}$ are products of any of the following:

- $(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$.
- $\mathrm{Fil}^{(i)}$.
- $\mathbb{H}^{(i)}$.
- $\mathbb{P}^\infty$.

Proof. The cited proof goes through with the following remarks. By ibid, Lemme III.19, for any objects X and E in $\mathcal{H}(\mathfrak{R}_S)_\bullet$ with E an H-group, there is an injection $\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(X, E) \to \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(X, E)$ whose image is that of morphisms $X \xrightarrow{f} E$ such that $f^*(\bullet) = \bullet \in \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(S, E)$. Thus one reduces to the non-pointed case. The objects in question are retracts of $(\mathbb{Z} \times \mathrm{Gr})_\mathbb{Q}$ or equal to $\mathbb{P}^\infty$, which are the cases treated in the reference and one concludes. □

We are now ready to complete the proof of Theorem 4.0.3. Using the above two lemmas we deduce morphisms $\mathrm{Fil}^{(i)} \times \mathrm{Fil}^{(j)} \to \mathrm{Fil}^{(i+j)}$ from the morphisms $F^iK_0(-) \times F^jK_0(-) \to F^{i+j}K_0(-)$ and similarly for $\mathbb{H}^{(i)} \times \mathbb{H}^{(j)} \to \mathbb{H}^{(i+j)}$. As in ibid, Lemma III.33 we have the following proposition: for an H-group E and objects A, B of $\mathcal{H}(\mathfrak{R}_S)_\bullet$, the map $\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(A \wedge B, E) \to \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(A \times B, E)$ is injective and its image consists of morphisms $A \times B \to E$ such that the restriction to $\bullet \times B$ and $A \times \bullet$ is zero. It follows that both of the two morphisms factor as $\mathrm{Fil}^{(i)} \wedge \mathrm{Fil}^{(j)} \to \mathrm{Fil}^{(i+j)}$ and $\mathbb{H}^{(i)} \wedge \mathbb{H}^{(j)} \to \mathbb{H}^{(i+j)}$. The same argument shows the necessary diagrams are commutative. The Adams-operations act appropriately for the same reason.

□

The main theorem of this section is now the following:

Theorem 4.0.6 (Rigidity). *[proof of [Rio06], Section III.10] Suppose $S = \operatorname{spec}\mathbb{Z}$. In the cases considered in the above lemma, except whenever $\mathcal{Y}$ involves a factor of $\mathbb{P}^\infty$, the morphisms*

$$\mathrm{Hom}_f(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

and

$$\mathrm{Hom}_{f,\bullet}(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\bullet,\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

have natural sections (and are thus surjective), which are canonical up to unique isomorphism (see A.0.13.1 for a definition of the functor V). Moreover, any isomorphism of two functors $V_\mathcal{X} \to V_\mathcal{Y}$ is unique up to unique isomorphism.

Proof. This follows directly from Lemma 4.0.5 and by considering the composition

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_f(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

obtained from pre-rigidity in Proposition A.0.14. The essential point is that the groups $\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X}, \Omega\mathcal{Y})$ disappear since they can all be related to $K_1(\mathbb{Z}) = \mathbb{Z}/2$-modules, and all the groups in question are 2-divisible by construction. Now, suppose we have $\phi \in \mathrm{Hom}_f(V_{\mathcal{X}}, V_{\mathcal{Y}})$ and an automorphism ϕ, i.e. a functorial isomorphism of functors $\phi \simeq \phi$. Suppose for simplicity that $\mathcal{X} = \mathcal{Y} = (\mathbb{Z} \times \mathrm{Gr}$. It is easy to see it determines an element in $\mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_1(-)_{\mathbb{Q}})$, and moreover that any such element determines an automorphism of ϕ. The latter group is zero by Theorem A.0.8 and an argument analogous to the proof in the previous lemma. □

Under the conclusion of the above theorem we say that the functor $V_{\mathcal{X}} \to V_{\mathcal{Y}}$ lifts that of $\phi\mathcal{X} \to \phi\mathcal{Y}$. We say the lifting given by the section of the theorem is given by "rigidity".

To state the next proposition, denote by $\mathfrak{Pic}_{\frac{1}{n}}(\mathcal{X})$ and $\mathfrak{Pic}_{\mathbb{Q}}(\mathcal{X})$ the Picard category of line bundles on $\mathcal{X}$ localized at an integer n or $\mathbb{Q}$ respectively (c.f. B.0.14.1).

Proposition 4.0.7. *Let $\mathfrak{RCh}/S$ be the category of regular algebraic stacks over S and let $\Phi' : \mathfrak{RCh}/S \to \mathcal{H}(\mathfrak{R})$ be the functor determined by the extended Yoneda-functor (see Definition C.0.18.1) and for an object $\mathcal{X}$ of $\mathcal{H}(\mathfrak{R})$, denote by $\phi'(\mathcal{X})$ the functor $\mathfrak{RCh}/S \to \mathrm{Grp}$ such that $\phi'(\mathcal{X})(\mathcal{Y}) = V_{\mathcal{X}}(\Phi'(\mathcal{Y}))$. Then we have the following equivalences of functors:*

- *$\phi'((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}})$ = the fibered Picard category over $\mathfrak{RCh}/S$ that is the fundamental groupoid of K-cohomology.*
- *Let n be an integer. Then $\phi'(\mathbb{P}^\infty[\frac{1}{n}]) = \mathfrak{Pic}[\frac{1}{n}]$ and $\phi'(\mathbb{P}^\infty_{\mathbb{Q}}) = \mathfrak{Pic}_{\mathbb{Q}}$, the fibered category of line bundles localized at n or $\mathbb{Q}$ over $\mathfrak{RCh}/S$, associating to any object of $\mathfrak{RCh}/S$ the category of localized linebundles thereupon.*

Proof. The first statement is essentially by definition. Consider the second statement. For a simplicial sheaf $\mathcal{X}$ and a sheaf of groups G, a G-torsor is a simplicial sheaf $\mathcal{Y} \to \mathcal{X}$ with a free action of G such that $\mathcal{Y}/G = \mathcal{X}$. In other words, a collection of G-torsors $\mathcal{Y}[n]$ on $\mathcal{X}[n]$ such that for a morphism $\phi : [n] \to [m]$ there are induced morphisms ϕ^* interchanging the data in the obvious manner. Now, it follows from [VV99], Section 4, Proposition 3.8 that for a simplicial sheaf $\mathcal{X}$, $\phi'(\mathbb{P}^\infty)(\mathcal{X})$ is the category of $\mathbb{G}_m$-torsors on $\mathcal{X}$. Thus, for a regular algebraic space U it is clear that this is the category of line bundles on U. Let U be an regular algebraic stack with smooth presentation $X \to U$ with X an algebraic space. Then U identifies with the simplicial sheaf whose n-simplices are given by $X \times_U X \times_U \ldots \times_U X$, n-time, and face and edge-maps by repeated diagonals and projections as face and edge-maps. Since a morphism of line bundles is necessarily injective one verifies that a

$\mathbb{G}_m$-torsor on U necessarily has isomorphisms as transition-morphisms, and we conclude by smooth descent.

□

Remark 4.0.7.1. By [AK04], Lemma 3.2, it follows that a Deligne-Mumford stack $\mathcal{M}$, separated and of finite type over a Nötherian base scheme with moduli space M, then $\mathfrak{Pic}(M)_{\mathbb{Q}} \to \mathfrak{Pic}(\mathcal{M})_{\mathbb{Q}}$ is an equivalence of categories.

A priori the operations given by rigidity are abstract and one might want to relate them to other operations. One standard way of doing so is as follows. Restricting any "virtual" operation given by a morphism $(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ along $\mathbb{P}^\infty \to (\{0\} \times \mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ gives us the behavior of the operation on an actual line bundle where we can often write down explicitly what it does. Then by the splitting principle one can often compare this to other operations.

Definition 4.0.7.1. We denote by $F^iW(-)$ (resp. $W^{(i)}$) the category fibered in groupoids $\phi'(\mathrm{Fil}^{(i)})$ (resp. $\phi'(\mathbb{H}^{(i)})$) over $\mathfrak{RCh}$. Notice that for an algebraic stack $\mathcal{X}$ that is not an algebraic space $F^0W(\mathcal{X}) = W(\mathcal{X})$ is in general not the virtual category $V(\mathcal{X})_{\mathbb{Q}}$ of $\mathcal{X}$.

We record the following.

Theorem 4.0.8. *The functors $F^iW(-)$ have the following properties.*

(a) The functors $F^{i-1}W(-) \to F^iW(-)$ are faithful.

(b) There are pairings, unique up to unique isomorphism,

$$F^iW(-) \times F^jW(-) \to F^{i+j}W(-)$$

lifting the pairings $F^iK_0(X)_{\mathbf{Q}} \times F^jK_0(X)_{\mathbf{Q}} \to F^{i+j}K_0(X)_{\mathbf{Q}}$ on regular schemes, such that for $i' \le i, j' \le j$, we have a commutative diagram

$$\begin{array}{ccc} F^iW(-) \times F^jW(-) & \longrightarrow & F^{i+j}W(-) \\ \downarrow & & \downarrow \\ F^{i'}W(-) \times F^{j'}W(-) & \longrightarrow & F^{i'+j'}W(-) \end{array}$$

(c) There are unique pairings $W^{(i)} \times W^{(j)} \to W^{(i+j)}$, extending the usual pairings $K_0(X)^{(i)} \times K_0(X)^{(j)} \to K_0(X)^{(i+j)}$ on regular schemes.

(d) The pairings are compatible with the isomorphism

$$F^iW(-) = F^{i-1}W(-) \times W^{(i)}$$

and they all satisfy the obvious associativity constraints.

(e) *The above is compatible with zero-objects in that a zero-object in one variable maps to a zero-object in the second.*

(f) *The Adams-operations act on all the objects and functors involved, and these operations are moreover, up to unique isomorphism, uniquely defined as liftings of the usual Adams operations.*

(g) *Let $\mathcal{X}$ be a regular algebraic stack of dimension d with finite affine.stabilizers. Then $F^{d+2}W(\mathcal{X})$ is equivalent to the trivial category with one object and the identity as only morphism and we have an equivalence of categories: $W(\mathcal{X}) = \bigoplus_{i=0}^{d+1} W^{(i)}(\mathcal{X})$.*

Proof. (b),(c),(d) and (f) are clear from rigidity. For (a), it is enough to show that for any $\mathcal{X}$ and object x of $F^{i-1}W(\mathcal{X})$, $\mathrm{Aut}_{F^{i-1}W(\mathcal{X})}(x) \to \mathrm{Aut}_{F^iW(\mathcal{X})}(x)$ is injective. But this is clear since this map identifies with the injection $F^{i-1}K_1^{sm}(\mathcal{X}) \to F^iK_1^{sm}(\mathcal{X})$. (e) follows from the description of the pairing in Theorem 4.0.3. Since we will only be concerned with (g) for a scheme we give the proof in this case.

Lemma 4.0.9. *Let X be a regular scheme with $d = \dim X$, and $i = 0, 1$. Recall that $F^jK_i(X)_{\mathbb{Q}}$ is the filtration on $K_i(X)_{\mathbb{Q}}$ determined by $F^jK_i(X)_{\mathbb{Q}} = \bigoplus_{p\geq j} K_i(X)^{(p)}$. Then $F^{d+i+1}K_i(X)_{\mathbb{Q}} = 0$.*

Proof. Consider the Quillen coniveau spectral sequence

$$E_1^{p,q}(X) = \bigoplus_{x\in X^{(p)}} K_{-p-q}(k(x)) \Longrightarrow K_{-p-q}(X)$$

where $X^{(p)}$ denotes the codimension p-points of X and $k(x)$ is the residue field of x. By [Sou85] Theoreme 4, iv), we have that, for $i = 0, 1$,

$$K_i(X)_{\mathbb{Q}} = \bigoplus_{p=0}^{d} E_2^{p,-p-i}(X)_{\mathbb{Q}}.$$

Furthermore, it is remarked in [HG87], proof of Theorem 8.2, that the Adams operations Ψ^k act on the spectral sequence and in particular on $E_r^{p,-p-i}(X)$ by k^{p+i} ($i = 0, 1$ and $r \geq 1$). Also, $E_r^{p,-p-i}(\Psi^k)$ converge towards the Adams operations on $K_i(X)$. Thus, for $i = 0, 1$, it follows that $K_i(X)^{(d+i+1+k)} = 0$ for $k \geq 0$ so that $F^{d+i+1}K_i(X)_{\mathbb{Q}} = 0$. Of course, for $i = 0$, this is well-known. □

We immediately deduce that the categories $F^pW(X)$ are trivial, i.e. all objects are uniquely isomorphic, for $p \geq \dim X + 2$. □

Remark 4.0.9.1. The proof of property (g) in the case of a regular algebraic space goes through verbatim. The general case is obtained in a similar way, but one has to work instead with the spectral sequence $E_1^{p,q}(\mathcal{X}) = \bigoplus_{\xi \in X^{(p)}} K^{sm}_{-p-q}(G_{\xi,\mathrm{red}}) \Longrightarrow K^{sm}_{-p-q}(\mathcal{X})$ which exists by a Brown-Gersten argument applied to the flabby S^1-spectrum representing cohomological K-theory and by virtue of $G_p^{sm} = K_p^{sm}$ by Poincaré duality for the cohomology of the $K_\bullet$-theory for regular algebraic stacks (see Theorem C.0.22). Then each $G_{\xi,\mathrm{red}}$ is a gerbe banded by some reduced algebraic group H, which is in fact necessarily an abstract finite group over the algebraic closure of the moduli space. To understand the Adams operations we can by étale descent moreover suppose that the moduli space $\operatorname{spec} k(x)$ of $G_{\xi,\mathrm{red}}$ is separably closed so that the gerbe is trivial and $G_{\xi,\mathrm{red}} = [\operatorname{spec} k(x)/H]$. By the arguments of [Tho86], 2.3 there is a spectral sequence

$$E_1^{p,q} = K_q(\prod^p H) \to K^{sm}_{q-p}(G_{\xi,\mathrm{red}}).$$

Then $K_i^{sm}(G_{\xi,\mathrm{red}})_{\mathbb{Q}} = K_i(\operatorname{spec} k(x))^H_{\mathbb{Q}}$ for all i. The Adams operations Ψ^k act on $K_i^{sm}(G_{\xi,\mathrm{red}})$ via the restriction of $K_i(\operatorname{spec} k(x))$ to the H-invariant part and thus by k^i. The rest is similiar but skipped.

Remark 4.0.9.2. From [Lev99], Theorem 11.5 it follows that if R is a Dedekind domain, and X is a regular finite type $\operatorname{spec} R$-scheme, then the γ-filtration on $K_n(X)$ for any integer n terminates after $d+n+1$ steps.

We harvest some obvious corollaries:

Corollary 4.0.10. *Let $\mathcal{X}$ be a regular algebraic stack. The Adams operations on $W(\mathcal{X})$ are compatible with the Adams operations constructed on $V(\mathcal{X})$ in Proposition 3.3.1 under the functor $V(\mathcal{X}) \to W(\mathcal{X})$. Moreover, there is a determinant functor* $\det : W(\mathcal{X}) \to \mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}}$ *such that the diagram*

$$\begin{array}{ccc} V(\mathcal{X}) & \longrightarrow & W(\mathcal{X}) \\ \downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \Psi^k} \\ V(\mathcal{X}) & \longrightarrow & W(\mathcal{X}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ \mathfrak{Pic}(\mathcal{X}) & \longrightarrow & \mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}} \end{array}$$

commutes up to canonical natural transformation.

Proof. By rigidity the two Adams-operations coincide on line bundles and we conclude by the splitting principle. Moreover, $\mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}}$ clearly satisfies coherent descent since it is a localization of the category $\mathfrak{Pic}(\mathcal{X})$ which does. The determinant functor then exists by cohomological descent and the diagram commutes again by rigidity and the splitting principle. □

Corollary 4.0.11. *There are unique λ-operations on $F^iW(-)$ λ-operations satisfying, for a regular algebraic stack $\mathcal{X}$,*

$$\lambda^k(x+y) = \sum_{j=0}^{k} \lambda^j(x) \otimes \lambda^{k-j}(y)$$

and for a vector bundle E one has $t_E : \lambda^k E \simeq \wedge^k E$ and for an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ a commutative diagram of isomorphisms

$$\begin{array}{ccc} \lambda^k(E) & = & \sum_{j=0}^{k} \lambda^j(E') \otimes \lambda^{k-j}(E'') \\ \Big\| t & & \Big\| t \\ \wedge^k(E) & = & \sum_{j=0}^{k} \wedge^j(E') \otimes \wedge^{k-j}(E'') \end{array}$$

with the lower row defined as in (3.1).

Proof. Unicity is clear by the splitting principle. Existence of the λ-operations are given by rigidity, and we suppose for simplicity that $i = 0$. For a line bundle rigidity provides us with an isomorphism $\lambda^k L \simeq \wedge^k L = L$ if $k = 1$ and an isomorphism with 0 if $k > 1$. Now recall the identity

$$\Psi^k - \lambda^1 \Psi^{k-1} + \ldots + (-1)^{k-1} \lambda^{k-1} \Psi^1 + (-1)^k k \lambda^k = 0$$

in on the level of K_0. It thus lifts by rigidity to W. When E is of rank 2, it is left as an exercise to the reader, using the exact sequences

$$0 \to S^{n-1}E \otimes \wedge^2 E \to S^n E \otimes E \to S^{n+1}E \to 0,$$

to use our construction of Ψ^k on the level on V to give the same identity, inductively. Here of course rigid λ operations are replaced by the usual $\wedge$ operations. One then uses the former corollary to deduce, again inductively, an isomorphism $\lambda^k E \simeq \wedge^k E$ valid for vector bundles of rank 2 satisfying the additivity property searched for.

One proceeds by induction on the rank of E. Given a filtration $F' \subset E$ one defines an isomorphism $t_{E,F'} : \lambda^k E \to \wedge^k E$ by requiring the diagram in the statement to commute. Given two filtrations $F'' \subset F' \subset E$ one verifies that by induction hypothesis that $t_{E,F''} = t_{E,F'}$ in a way stable by base-change of regular schemes. Given two unequal line bundles $L' \subset E$ and $L'' \subset E$ we consider the Grassmannian $p : \mathrm{Gr}_{L',L'',2}(E) \to X$ classifying bundles M of rank 2 such that $L', L'' \subset M \subset E$. Then $Rp_*1 = 1$ so that by the projection formula $A = Rp_*(Lp^*A)$ for any A and it follows that $t_{p^*E,p^*L'}$ determines $t_{E,L'}$. But by what we established $t_{p^*E,p^*L'} = t_{p^*E,\mathcal{M}} = t_{p^*E,p^*L''}$ for $\mathcal{M} \subset p^*E$ the universal rank 2 sub bundle on $\mathrm{Gr}_{L,L',2}(E)$. We conclude by the splitting principle. □

Corollary 4.0.12. *Let $\mathcal{X}$ be as in (g) in Theorem 4.0.8. There are γ-operations on the virtual category $W(\mathcal{X}) = W$, $j \geq 2$, γ^j, inducing the natural operations on $K_{0,\mathbb{Q}}$. For a virtual bundle of rank 0,*

$$\gamma^j(v) \in W^{(j)}.$$

Furthermore, for any two virtual objects x and y in F^iW, we have a family of isomorphisms in $F^{i+j-1}W$, functorial in x and y;

$$\gamma^k(x+y) \simeq \bigoplus \gamma^j(x)\gamma^{k-j}(y). \tag{4.1}$$

Since for a line bundle L, $1 - L$ identifies with an object of F^1W, we have that $(1-L)^i$ is an object of F^iW. We then have canonical isomorphisms in F^iW:

$$\gamma^i(1-L) \simeq (1-L)^i$$

and

$$\gamma^i(L-1) = 0 \text{ for } i \geq 2.$$

Given a vector bundle E of rank n, we have a canonical isomorphism $\gamma^n(E-n) = (-1)^n\lambda_{-1}(E)$ and for any $k > 0$ a trivialization $\gamma^{k+n}(E-n) = 0$ such that for a short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ of ranks n', n and n'' we have an isomorphism

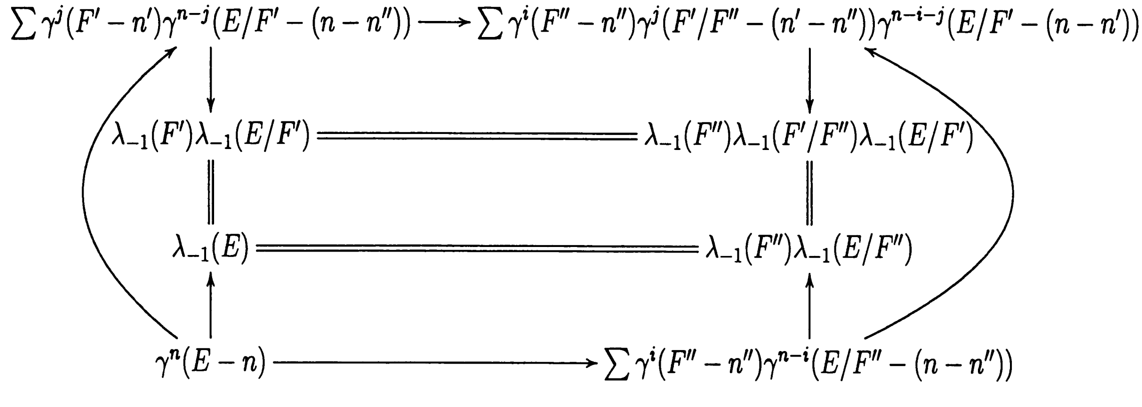
$$\begin{array}{ccc} \gamma^n(E-n) & = & \sum_{i=0}^n \gamma^i(E'-n') \otimes \gamma^{n-i}(E''-n'') . \\ \| & & \| \\ (-1)^n\lambda_{-1}(E) & = & (-1)^{n'}\lambda_{-1}(E') \otimes (-1)^{n''}\lambda_{-1}(E'') \end{array}$$

Thus the functor $E \mapsto \lambda_{-1}E$ defined on vector bundles has essential image in $F^{\mathrm{rk}\,E}W$. Moreover, the trivialization $\gamma^{k+n}(E-n) = 0$ is compatible with the trivializations given by (4.1) and admissible filtrations.

Proof. All statement except the last one are direct consequences of rigidity. By definition (c.f. [WF85], chapter III) and rigidity we are given a relationship of γ and λ-operations $\sum_{i=0} \gamma^i(u)t^i = \sum_i \lambda^i(u)\left(\frac{t}{1-t}\right)^i$ so that, in view of that $1/(1-t)^{r+1} = \sum \binom{j+r}{j} t^j$, the relationship, for $k > 0$, in $W(\mathcal{X})$

$$\gamma^k(u) = \sum_{i=0}^{k} \lambda^i(u)\binom{k-1}{k-i} = \lambda^k(u+k-1).$$

If u is a virtual bundle of rank 0 we deduce the equality in F^kW compatible with sums and the product on the filtration. Now, for a line bundle L one has $\lambda^n(-L) = (-L)^n$ and thus one has for a vector bundle of rank n a canonical isomorphism $\gamma^n(E-n) = \lambda^n(E-1) = (-1)^n\lambda_{-1}(E)$, where the latter isomorphism is an isomorphism in $V(\mathcal{X})$. We hereby identify $\lambda_{-1}(E)$ as an element in $F^nW(\mathcal{X})$. One also obtains a trivialization $\gamma^{k+n}(E-n) = \lambda^{k+n}(E+k-1) = \wedge^{k+n}(E+k-1) = 0$ in $W(\mathcal{X})$. We need to verify that the isomorphism lies in $F^{k+n}W(\mathcal{X})$. We proceed by induction. For a line bundle this is given by rigidity. Suppose we have verified the all the given statements for vector bundles of rank strictly less than n for all regular schemes. Suppose E is of rank n. Given an admissible filtration $F'' \subset F' \subset E$ of with F'' and F' of ranks n'' and n' respectively consider the following diagram



where the morphisms are given by the various trivializations $\gamma^i(\diamondsuit - \operatorname{rk}\diamondsuit) = 0$ in F^iW whenever $i > \operatorname{rk}\diamondsuit$ and the isomorphism $\gamma^{\operatorname{rk}\diamondsuit}(\diamondsuit - \operatorname{rk}\diamondsuit) = \lambda_{-1}(\diamondsuit)$ given by induction hypothesis. The outer contour commutes by definition of the γ-operations. The middle square commutes by compatibility with admissible filtrations of (3.3), the upper square commutes by induction hypothesis as does the right hand diagram. Thus the diagram determined by $F' \subset E$ commutes if and only if the diagram determined by the diagram determined by $F'' \subset E$ commutes. Arguing as in the previous corollary one sees that this morphism is independent of choice of F'' and F' and by the splitting principle one obtains that the diagram associated to $F' \subset E$ commutes. In the same way we obtain a canonical trivialization $\gamma^{n+k}(E-n) = 0$ in $F^{n+k}W(\mathcal{X})$ for

$k > 0$. We need to verify that its image in $W(\mathcal{X})$ coincides with the trivialization $t : \lambda^{k+n}(E+k-1) = \wedge^{k+n}(E+k-1) = 0$. This follows by additivity and induction on k.

□

Corollary 4.0.13. *Let X be a regular scheme of dimension d . Then for any virtual bundle v in $W(X)$ of rank 0, $k > 1$,*

$$\gamma^{d+k}(v) \simeq 0$$

canonically.

Proof. This follows from Theorem 4.0.8 and Corollary 4.0.12. □

Proposition 4.0.14. *Let X be a regular scheme of dimension d. Then for any k, and a virtual bundle v, an isomorphism $\Psi^k(v) = k^i v$ in $W(X)$ defines a projection of v into $W^{(k)}$. Thus it implies that there is a canonical isomorphism $\Psi^n(v) = k^n v$ for any n.*

Proof. By Theorem 4.0.8 there is an equivalence $W(X) = \bigoplus_{j=0}^{d+1} W^{(j)}(X)$ so that v is equivalent to an object of the form $\sum \pi_i v_i$ with $\pi_i : W^{(i)} \to W(X)$. Applying Ψ^j we obtain an isomorphism $(k^j - k^i)v_i = 0$ in $W^{(i)}$ and so $v_j = 0$ for $j \neq 0$. □

The following is trivial:

Corollary 4.0.15. *Let X be a regular scheme. Then the functor $R : W(X) \to W(X)$ associating to a virtual bundle u the virtual bundle*

$$R(u) = (v - \operatorname{rk} u) - (\det u - 1)$$

has essential image in F^2W. For u and v both virtual bundles on $W(X)$ there is a canonical isomorphism

$$R(u+v) = R(u) + R(v) - (\det u - 1)(\det(v - 1)$$

in $F^2W(X)$ where the product $(\det u - 1)(\det v - 1)$. These are stable by pullback of regular schemes. They moreover correspond to the isomorphisms defined by $\operatorname{rk}(u+v) = \operatorname{rk} u + \operatorname{rk} v$ and

$$\det(u+v) - 1 = (\det u - 1)(\det v - 1) + (\det u - 1) + (\det v - 1)$$

defined as in [Del87], (9.7.8). Thus, a virtual bundle of rank 0 and with trivialized determinant bundle defines an element in F^2W.

5. A FUNCTORIAL EXCESS FORMULA

This little pig had none

This section is the report of a failed attempt to construct a completely canonical simplicial description of the excess-formula of [P71] VII, Proposition 2.7. A word about application is in order. The argument is given for quite general stacks. However, by [Tot04] essentially all stacks in question are stacks of the form $[U/GL_n]$ with U quasi-affine. This is good to keep in mind since it reduces the arguments to equivariant geometry on quasi-affine schemes, but the arguments in the general case are however the same and we proceed to give this result.
Given a 2-cartesian diagram of algebraic stacks with representable morphisms it is innocuous to identify it with a Cartesian diagram of schemes or algebraic spaces with descent data. We will thus routinely apply the terminology "cartesian diagram" for a 2-cartesian diagram and moreover work with a strictly commutative version of the virtual category of vector bundles, defined as follows:

Definition 5.0.15.1. The objects of $V_{\pm}(X)$ and $V(X)$ are the same. We define $\mathrm{Hom}_{V_{\pm}(X)}(A, B)$ as the quotient of $\mathrm{Hom}_{V(X)}(A, B)$ by the relation that two morphisms $h, h' : A \to B$ are equal if $h \circ h'^{-1} = [-1] \in \mathrm{Aut}(A)$. Here $[-1]$ corresponds to the nontrivial element in the image of $\pm 1 = \mathrm{Aut}_{V(\mathbb{Z})}(0) \to \mathrm{Aut}_{V(X)}(0) = \mathrm{Aut}_{V(X)}(A)$ determined by the morphism $X \to \operatorname{spec} \mathbb{Z}$.

Lemma 5.0.16. *Let $\epsilon(X)$ be the automorphism of X determined by the symmetry $[X] + [Y] \simeq [Y] + [X]$ evaluated at $Y = X$. Then $\epsilon(X)$ is in the above image.*

Proof. (c.f. [Del87], 4.9) Let X be a vector bundle and consider the sequence $X \xrightarrow{u} X + X \xrightarrow{v} X$ where u is the diagonal and v is the map $(a, b) \mapsto a - b$. Then the symmetry automorphism induces 1 on the sub object and multiplication by (-1) on the quotient object. □

As an informal corollary of the above proposition is that most of the naive diagrams of isomorphisms we can write up commute, in particular the isomorphisms (3.3) and (3.1) become completely canonical and commutative.

5.1 A rough excess-isomorphism

Suppose we are given a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & Y' \end{array}$$

with f a closed immersion. Let x be a virtual vector-bundle on Y, and suppose we have two morphisms $\sigma : N^{\vee} \to \mathcal{O}_{Y'}, \sigma' : N'^{\vee} \to \mathcal{O}_{X'}$ defining Koszul-resolutions of $\mathcal{O}_Y$ and $\mathcal{O}_X$ respectively that are compatible in the sense that we have a morphism $\gamma : g^*N^{\vee} \to N'^{\vee}$ compatible with the resolutions $g^*\sigma$ and σ' in such a way that the natural diagram

$$\begin{array}{ccc} g^*N^{\vee} & \xrightarrow{\gamma} & N'^{\vee} \\ \downarrow & & \downarrow \\ g^*\mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_{X'} \end{array}$$

commutes. This implies that f and f' are both closed regular immersions. Writing $\mathcal{I}$ and (resp. $\mathcal{I}'$) for the ideals defining the immersions $f : X \to Y$ (resp. $f' : X' \to Y'$) and let $N_f^{\vee} = \mathcal{I}/\mathcal{I}^2$ and $N_{f'}^{\vee} = \mathcal{I}'/\mathcal{I}'^2$ be the conormal bundles of the immersions. By restricting to X via f' we obtain a commutative diagram

$$\begin{array}{ccc} f'^*g^*N^{\vee} & \xrightarrow{f'^*\gamma} & f'^*N'^{\vee} \\ \downarrow & & \downarrow \\ g'^*N_f^{\vee} & \longrightarrow & N_{f'}^{\vee} \end{array}$$

where the vertical morphisms necessarily are isomorphisms. Denote the kernel of γ by F. The bundle $f'^*F \simeq E$ is called the "excess bundle" (c.f. [Ful98], Section 6.3. Our definition is however dual that of *ibid*), so F provides an extension of E to X'. Also, suppose that the virtual vector-bundle x extends to a virtual vector-bundle $x_{Y'}$ on Y', i.e. there is an isomorphism

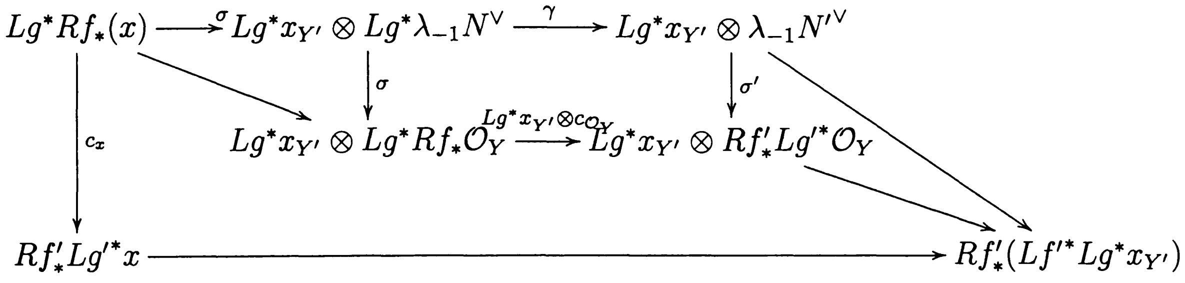
$r : f^*x_{Y'} \to x$. Then we define a rough excess-isomorphism by

$$\begin{aligned}
\Psi_{x_{Y'},\sigma,\sigma',r}(x) : Lg^*Rf_*(x) &\overset{\sigma,x_{Y'},r}{\simeq} \lambda_{-1}(Lg^*N^\vee) \otimes Lg^*(x_{Y'}) \\
&\overset{\gamma}{\simeq} \lambda_{-1}(F) \otimes \lambda_{-1}(N'^\vee) \otimes Lg^*(x_{Y'}) \\
&\overset{\sigma'}{\simeq} Rf'_*(\mathcal{O}_X) \otimes \lambda_{-1}(F) \otimes Lg^*(x_{Y'}) \\
&\simeq Rf'_*(Lf'^*(\lambda_{-1}(F) \otimes Lg^*(x_{Y'}))) \\
&\simeq Rf'_*(\lambda_{-1}(E) \otimes Lg'^*x).
\end{aligned}$$

The first isomorphism is given by the extension $x_{Y'}$ together with the resolution σ and then applying g^*. On the second to last line we use the projection formula once again, and then the definition of f'^* for the last line. We will later show that this isomorphism is independent of the subscripts $x_{Y'}, \sigma, \sigma', r$. The first result in this direction is the following which links it to the base-change-isomorphism:

Lemma 5.1.1. *Suppose that the above square is Tor-independent. Then the constructed isomorphism coincides with the image of the base-change isomorphism (see Lemma 3.1.3 of this thesis) in the virtual category.*

Proof. Denote by $Lg^*Rf_*x \overset{c_x}{\to} Rf'_*Lg'^*x$ the base-change isomorphism. The statement of the lemma is that for an extension $x_{Y'}$ of x to Y', the outer contour of following diagram



is commutative. Here the outer right contour connecting Lg^*Rf_*x and $Rf'_*Lg'^*x$ is the proposed rough excess-isomorphism. First of all, the triangles commute per definition. The square commutes since it is induced by the natural isomorphisms $Lg^*\lambda_{-1}N^\vee \to \lambda_{-1}N'^\vee$ and the commutative diagram

$$\begin{array}{ccccc} g^*\mathcal{I} & \longrightarrow & \mathcal{I}' & & \\ \downarrow & & \downarrow & & \\ g^*\mathcal{O}_{Y'} & \longrightarrow & \mathcal{O}_{X'} & & \\ \downarrow & & \downarrow & & \\ g^*f_*\mathcal{O}_Y & \longrightarrow & f'_*\mathcal{O}_X & = & f'_*g'^*\mathcal{O}_X \end{array}$$

with exact columns and vertical isomorphisms. By the same description, the middle horizontal morphism is clearly the base-change-isomorphism for the trivial bundle $\mathcal{O}_X$. For the inner contour, we can assume that $V = f^*V'$ for an actual vector bundle V' on Y'. A vector bundle V on Y is f_*-acyclic (recall that f is a closed immersion) as well as g'^*-acyclic, and thus the description of [Del77], Tome 3, XVII, 4.2.12 and Proposition 4.2.13 applies. The statement of the latter is a comparison of the base-change-isomorphism in the derived category and the one induced on actual complexes as well as compatibility with projection formula, and amounts to what we need. Details are left to the reader. □

Let S henceforth be a regular base scheme and let "virtual category" and "virtual objects" be substitutes for "strictly commutative virtual category" and "strictly commutative virtual objects" until the end of this section. The main theorem of this section is the following:

Theorem 5.1.2. *Suppose we have a Cartesian square $\mathcal{E}$ of algebraic S-stacks and representable morphisms*

$$\begin{array}{ccc} X & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & Y' \end{array}$$

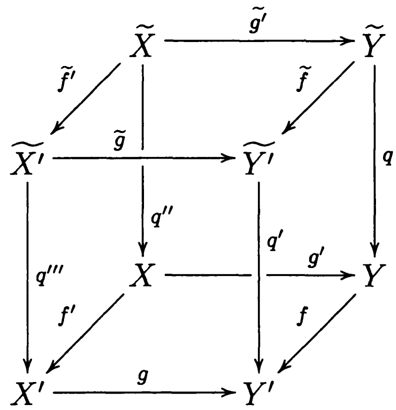
where g and g' are arbitrary morphisms, and f and f' are (representable) projective local complete intersection morphisms.

Then there is a unique family of isomorphisms of functors in the (strictly commutative) virtual category of vector bundles:

$$\Psi_{\mathcal{E}} : Rf'_*\left(\lambda_{-1}(E) \otimes Lg'^*(x)\right) \simeq Lg^*Rf_*(x)$$

where E is the excess bundle, with the following properties:

(a) *It is stable under transversal base-change. Consider the following cubical diagram*



where the right-hand and left-hand vertical squares are transversal and Cartesian. The upper and lower diagrams are denoted by $\tilde{\mathcal{E}}$ and $\mathcal{E}$ respectively and are supposed to be as in the introduction. Symbolically we summarize the cube by a morphism of diagrams $Q : \tilde{\mathcal{E}} \to \mathcal{E}$. There is then a commutative diagram

$$\begin{array}{ccc}
R\tilde{f}'_*(\lambda_{-1}(\tilde{E}) \otimes L\tilde{g}'^*(Lq^*x)) & \xrightarrow{\Psi_{\tilde{\mathcal{E}}}(q^*(x))} & L\tilde{i}^*R\tilde{f}_*(Lq^*(x)) \quad . \\
\downarrow{\scriptstyle q\tilde{g}'=g'q''',\, q''^*E=\tilde{E}} & & \downarrow{\scriptstyle base-change} \\
R\tilde{f}'_*(\lambda_{-1}(q''^*E) \otimes (Lq''^*Lg'^*x)) & & L\tilde{g}^*Lq'^*Rf_*(x) \\
\downarrow & & \\
R\tilde{f}'_*Lq''^*(\lambda_{-1}(E) \otimes Lg'^*x)) & & \downarrow{\scriptstyle iq'''=q'\tilde{g}} \\
\downarrow{\scriptstyle base-change} & & \\
Lq'''^*(Rf_*(\lambda_{-1}(E) \otimes Lg'^*x)) & \xrightarrow{q'''^*\Psi_{\mathcal{E}}(x)} & Lq'''^*Lg^*Rf_*(x)
\end{array}$$

We will sometimes write this as $\Psi_{\tilde{\mathcal{E}}} \circ Q^* \simeq Q^* \circ \Psi_{\mathcal{E}}$. In particular there is an isomorphism $\Psi_{\tilde{\mathcal{E}}}(q^*(x)) \simeq q'''^*\Psi_{\mathcal{E}}(x)$.

(b) *Suppose f is a regular closed immersion and suppose that we are given a (representable) regular closed immersion $h : Z \to Y'$ such that the fiber product $Z \times_{Y'} Y$ is empty. Denote by $h' : Z' \to X'$ the fiber product $Z \times_{Y'} X'$ and suppose it is also a regular closed immersion. Then*

$$Rh'_*(\mathcal{O}_{Z'}) \otimes Rf'_* \left(\lambda_{-1}(E) \otimes Lg'^*(x)\right)$$

and

$$Rh'_*(\mathcal{O}_{Z'}) \otimes Lg^*Rf_*(x)$$

are both canonically trivialized by the condition that Z' (resp. Z) doesn't intersect X (resp. Y). We demand that the excess-isomorphism interchanges these trivializations.

(c) *Normalization: Suppose that f is a closed embedding of a Cartier divisor Y in Y', and that $X = X'$. Let*

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_{Y'} \xrightarrow{\sigma} \mathcal{O}_Y \to 0$$

be the canonical Koszul resolution. Then, whenever x extends to a virtual bundle $x_{Y'}$ on Y', $\Psi_{\mathcal{E}}$ is given by the rough excess-isomorphism:

$$\begin{array}{rcl}\Psi_{\mathcal{F}_{Y'},\sigma} : Lg^*Rf_*(x) & \overset{\sigma}{\simeq} & \lambda_{-1}(Lg^*\mathcal{O}(-Y)) \otimes Lg^*(x_{Y'}) \\ & \overset{g=g'f,f'=\mathrm{id}}{\simeq} & \lambda_{-1}(Lg^*Lf'^*\mathcal{O}(-Y)) \otimes g'^*f'^*x_{Y'} \\ & \simeq & \lambda_{-1}(E) \otimes g'^*x.\end{array}$$

(d) *Given the composition of upper Cartesian diagram $\mathcal{E}$ and lower Cartesian diagram $\mathcal{E}'$ (giving $\mathcal{E}''$),*

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X'' & \xrightarrow{g''} & Y'' \end{array}$$

with associated excess-bundles E, E' and E'', the following diagram commutes

$$\begin{array}{ccc} R(f' \circ e')_*(\lambda_{-1}(E'') \otimes Lg^*x) & & \\ \downarrow{\scriptstyle f'\circ e' = f'e'} & \searrow{\scriptstyle \Psi_{\mathcal{E}''}} & \\ Rf'_*Re'_*(\lambda_{-1}(E'') \otimes Lg^*x) & & \\ \downarrow{\scriptstyle 0\to E\to E''\to e'^*E'\to 0} & & \\ Rf'_*(\lambda_{-1}(E') \otimes Re'_*(\lambda_{-1}(E) \otimes Lg^*x)) & & \\ \downarrow{\scriptstyle \Psi_{\mathcal{E}}} & & \\ Rf'_*(\lambda_{-1}(E') \otimes Lg'^*Re_*(x)) & & \\ \downarrow{\scriptstyle \Psi_{\mathcal{E}'}} & & \downarrow \\ Lg''^*Rf_*Re_*(x) & \xrightarrow{f\circ e = fe} & Lg''^*R(f \circ e)_*(x) \end{array}$$

where we use the projection formula on second left up-to-down arrow. This will be naively written as

$$\Psi_{\mathcal{E}'}\Psi_{\mathcal{E}} = \Psi_{\mathcal{E}''}$$

(e) *Stability under the projection formula. There is a commutative diagram*

$$\begin{array}{ccc} Rf'_*(\lambda_{-1}(E) \otimes Lg'^*(x \otimes Lf^*y)) & \longrightarrow & Lg^*Rf_*(x \otimes Lf^*y) \\ \downarrow & & \downarrow \\ Rf'_*(\lambda_{-1}(E) \otimes Lg'^*(x)) \otimes Lg^*y & \longrightarrow & Lg^*Rf_*(x) \otimes Lg^*y \end{array}$$

where the horizontal isomorphisms are given by excess and the vertical ones are given by the projection formula.

Corollary 5.1.3. *[Excess Self-Intersection Formula] Let $i : Y \to Y'$ be a regular closed embedding with Y' being an algebraic stack with the resolution property. Then we have a functorial isomorphism*

$$Li^*Ri_*(x) \simeq \lambda_{-1}(N^\vee_{Y/Y'}) \otimes x.$$

Proof. Indeed, take $X = X' = Y$, $f = i$, $f' = j = \mathrm{id}$ and use that the excess-bundle is just the conormal-bundle. □

The method of proof that we have chosen is that of deformation to the normal cone and goes roughly as follows: We first prove the theorem for closed immersions in the model situation of a zero-section in a projective bundle. After this we prove the theorem for projective bundle-projections using standard techniques. One then needs to show that for general projective morphisms, the excess-isomorphism does not depend on choice of factorization and thereby define the isomorphism.

5.2 Excess for projective bundle-morphisms, uniqueness

Proposition 5.2.1. *Suppose that f is a projective-bundle projection. Then the excess-isomorphism associated to f is uniquely determined and is necessarily given by flat base-change.*

Proof. Conditions (a) and (e) show that the excess-isomorphism is necessarily given by flat base-change. Indeed, first one notes that if $f : \mathbb{P}(N) \to Y$ any (strictly commutative) virtual object of $V(\mathbb{P}(N))_\pm$ is equivalent to an object of the form $\sum_{i=0}^{\mathrm{rk}\,N-1} f^*x_i \otimes \mathcal{O}(-i)$ for objects x_i in $V(Y)_\pm$. By condition (e) we are reduced to the case of an object of the form $\mathcal{O}(-i)$. Next one applies it to the cube of the theorem with all but one side being given by a trivial diagram, i.e. two of the morphisms are the identity. Moreover, the base-change isomorphism is compatible with base-change and composition itself by Lemma 3.1.5. The trivialization-condition is trivial. □

As a corollary, the property (a) and (e) are satisfied if we take the excess-isomorphism to be the base-change-isomorphism in the case of a projective bundle projection. This is Lemma 3.1.5 and Lemma 3.1.6.

5.3 Excess for closed immersions, uniqueness

The main object of this section is to prove the following theorem

Theorem 5.3.1. *If f is a closed immersion, then the excess-isomorphism, if it exists, is uniquely determined by the conditions in the theorem.*

Proof. For the case of closed immersions we will use the deformation to the normal cone to reduce ourselves to the "linear" situation. First of all, denote by $\mathcal{E}$ the Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}^1_X & \xrightarrow{G'} & \mathbb{P}^1_Y \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ M' & \xrightarrow{G} & M \end{array}$$

where M' and M are the deformation to the normal cone of f' and f respectively. Denote by $\mathcal{E}_0$ and $\mathcal{E}_\infty$ the two following diagrams:

$$\begin{array}{ccc} X & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle \tilde{f}'} & & \downarrow{\scriptstyle \tilde{f}} \\ \mathbb{P}(N'\oplus 1) & \xrightarrow{\tilde{g}} & \mathbb{P}(N\oplus 1) \end{array} \tag{5.1}$$

where $\tilde{f}, \tilde{f}'$ the canonical zero-sections with projections π' and π, $\tilde{g}$ being given by the morphisms induced by an inclusion of vector-bundles $N' \subseteq g'^*N$. In the following we abuse notation a bit to show unicity, but only in the sense that was used in the formulation of the theorem. Thus $\Psi_\mathcal{E}$ denotes the (rough) excess-isomorphism for the diagram $\mathcal{E}$ for example and for a functor F, $F\Psi_\mathcal{E}$

denotes the image of the excess-isomorphism under F. We have embeddings $i_0 : \mathcal{E}_0 \to \mathcal{E}$ and $i_\infty : \mathcal{E}_\infty \to \mathcal{E}$ and a projection $\Pi : \mathcal{E} \to \mathcal{E}_0$, satisfying $\Pi \circ i_0 = \mathrm{id}$. By property (1) of the theorem we see that $\Psi_{\mathcal{E}_0} i_0^* = i_0^* \Psi_{\mathcal{E}}$ and $\Psi_{\mathcal{E}_\infty} i_\infty^* = i_\infty^* \Psi_{\mathcal{E}}$. Also, applying the natural transformations $\mathrm{id} \simeq L i_0^* L\Pi^*$ and $\mathrm{id} \simeq R\Pi_* R i_{0,*}$ one sees that the functor $\Psi_{\mathcal{E}_0}$ is determined by the functor $R i_{0,*} L i_0^* \Psi_{\mathcal{E}} L\Pi^*$. Fixing a rational function λ on $\mathbb{P}^1_{\mathbb{Z}}$ with divisor $(0) - (\infty)$ defines an isomorphism

$$\mathcal{O}(X') \overset{\lambda}{\simeq} \mathcal{O}(P(N' \oplus 1) \cup D') \simeq \mathcal{O}(\mathbb{P}(N' \oplus 1)) \otimes \mathcal{O}(D') \qquad (5.2)$$

of line-bundles on M'. Here $\mathbb{P}(N' \oplus 1)$ and D' are the two components of the blow-up at infinity in the deformation to the normal cone, intersecting in $\mathbb{P}(N')$, and the image of X does not intersect that of D'. We have natural Koszul resolutions

$$\begin{aligned} 0 \to \mathcal{O}(-X') \to \mathcal{O} \to \mathcal{O}_{X'} \to 0 \\ 0 \to \mathcal{O}(-D') \to \mathcal{O} \to \mathcal{O}_{D'} \to 0 \\ 0 \to \mathcal{O}(-\mathbb{P}(N' \oplus 1)) \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}(N'\oplus 1)} \to 0 \end{aligned}$$

and a resolution

$$\begin{aligned} 0 \to \mathcal{O}(-D') \otimes \mathcal{O}(-\mathbb{P}(N' \oplus 1)) \to \mathcal{O}(-D') \oplus \mathcal{O}(-\mathbb{P}(N' \oplus 1)) \to \mathcal{O} \to \\ \mathcal{O}_{D' \cap \mathbb{P}(N'\oplus 1)} = \mathcal{O}_{\mathbb{P}(N')} \to 0 \end{aligned}$$

which together define an isomorphism, via (5.2) (see also [WF85], Proposition 4.4)

$$\begin{aligned} 0 &\simeq (1 - \mathcal{O}(-X')) - (1 - \mathcal{O}(-D')) - (1 - \mathcal{O}(-\mathbb{P}(N' \oplus 1)) \\ &\quad + (1 - \mathcal{O}(-\mathbb{P}(N' \oplus 1))(1 - \mathcal{O}(-D')) \\ &\simeq \mathcal{O}_{X'} - \mathcal{O}_{D'} - \mathcal{O}_{\mathbb{P}(N'\oplus 1)} + \mathcal{O}_{\mathbb{P}(N')} \end{aligned}$$

and hence an isomorphism

$$\mathcal{O}_{X'} \simeq \mathcal{O}_{D'} + \mathcal{O}_{\mathbb{P}(N'\oplus 1)} - \mathcal{O}_{\mathbb{P}(N')}.$$

Via the projection-formula this gives an equality

$$\begin{aligned} i_{0,*}\Psi_{\mathcal{E}_0} &\simeq i_{0,*} i_0^* \Psi_{\mathcal{E}} \Pi^* \simeq \mathcal{O}_{X'} \otimes \Psi_{\mathcal{E}} \Pi^* \\ &\simeq \mathcal{O}_{\mathbb{P}(N'\oplus 1)} \otimes \Psi_{\mathcal{E}} \Pi^* + \mathcal{O}_{D'} \otimes \Psi_{\mathcal{E}} \Pi^* - \mathcal{O}_{D' \otimes \mathbb{P}(N'\oplus 1)} \otimes \Psi_{\mathcal{E}} \Pi^* \end{aligned}$$

The condition (b) of the theorem canonically determine $\mathcal{O}_{D'} \otimes \Psi_{\mathcal{E}} \Pi^*$ and $\mathcal{O}_{D' \otimes \mathbb{P}(N'\oplus 1)} \otimes \Psi_{\mathcal{E}} \Pi^*$ and thus $i_{0,*}\Psi_{\mathcal{E}_0}$ is determined by $\mathcal{O}_{\mathbb{P}(N'\oplus 1)} \otimes \Psi_{\mathcal{E}} \Pi^* = i_{\infty,*} i_\infty^* \Psi_{\mathcal{E}} \Pi^* = i_{\infty,*} \Psi_{\mathcal{E}_\infty} L i_\infty^* \Pi^*$. We recall from Lemma 4.5, chapter V, [WF85] (see the proof for this slightly more refined statement):

Lemma 5.3.2. *Let $F : P \to M$ be a regular embedding, and let $\Phi : Y \to M$ be a morphism, and take the fibre square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Phi} \\ P & \xrightarrow{F} & M \end{array} .$$

If f is a regular embedding of the same codimension as F, then this square is transversal.

Note also that by symmetry, the same conclusion holds with Ψ and ϕ in place of F and f. Now, the diagrams that arise for application of condition (a) are the following:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^1_X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & M' \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \mathbb{P}^1_X \\ \downarrow & & \downarrow \\ \mathbb{P}(N' \oplus 1) & \longrightarrow & M' \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^1_Y \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & M \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^1_Y \\ \downarrow & & \downarrow \\ \mathbb{P}(N \oplus 1) & \longrightarrow & M \end{array}$$

The morphisms $\mathbb{P}^1_X \hookrightarrow M', \mathbb{P}^1_Y \hookrightarrow M$ are both regular by Theorem 4.5, chapter IV, [WF85], all diagrams Cartesian and codimension is preserved so we can apply the above lemma. Hence all diagrams are transversal and we can apply (a).

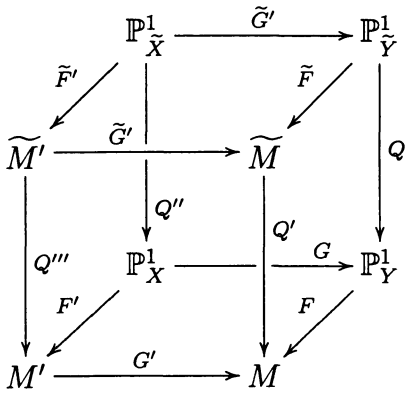
Thus we are reduced to showing uniqueness for the diagram $\mathcal{E}_\infty$. We proceed by induction on the dimension of E. In case the dimension of E is 0, then indeed the diagram is transversal and the isomorphism is fixed by transversal base-change. In the case the of $\operatorname{rk} E > 0$, consider the following Grassmannian $G = \mathrm{Gr}_{1,m,N} \to Y$ parameterizing flags $L \subseteq M \subseteq N$, with L, M of rank $1, m$ and $m = \operatorname{rk} N'$. Then $Lp^* : V(Y) \to V(G)$ is faithful, and an easy verification can be used to show that by the transversal base-change property we can assume our diagram is of the form

$$\begin{array}{ccc} \mathrm{Gr}_{1,m,g'^*N} & \xrightarrow{h'} & \mathrm{Gr}_{1,m,N} \\ \downarrow & & \downarrow \\ \mathbb{P}(h'^*\mathcal{M}) & \longrightarrow & \mathbb{P}(p^*N) \end{array}$$

where $\mathcal{L} \subset \mathcal{M} \subset p^*N$ is the universal flag on G. We can now filter p^*N/M by a maximal flag on G which is in particular a flag of p^*N including M. Now one easily sees that we can compose our big diagram as a composition of smaller diagrams which are either transversal or codimension 1-cases like in (c). By (d) this does not depend on the choice of flag and we conclude by the splitting principle.

5.4 Excess for closed immersions, rougher excess and existence

The previous section gave a recipe for the construction of the excess-isomorphism, which we spell out. Let M (resp. M') be the deformation to the normal cone of f (resp. f'). Denote by $i_D, i_{D\cap\mathbb{P}(N\oplus 1)}$ (resp. $i_{D'}, i_{D'\cap\mathbb{P}(N'\oplus 1)}$ the closed immersions of D and $D\cap\mathbb{P}(N\oplus 1)$ (resp. D' and $D'\cap\mathbb{P}(N'\oplus 1)$) in M (resp. M') and consider the diagram given by a cubical diagram as in (i). By the universal property of blow-ups ([Ful98], Appendix B.6.9) we obtain commutative diagram



where all the squares are Cartesian, except possibly the front and back vertical ones. We construct a sort of preliminary excess-isomorphism for the diagram $\mathcal{E}_\infty$ considered in (5.1). If ξ and ξ' denote universal quotient bundles on $P(N\oplus 1)$ respectively $P(N'\oplus 1)$, we have a short exact sequence

$$0 \to \pi'^* E \to \tilde{g}^*\xi^\vee \to (\xi')^\vee \to 0 \tag{5.3}$$

where E is the excess bundle of the diagram. Then we have a canonical Koszul resolution

$$0 \to \Lambda^n \xi^\vee \to \ldots \to \xi^\vee \to \mathcal{O}_{\mathbb{P}(N\oplus 1)} \to \tilde{f}_* \mathcal{O}_Y \to 0. \tag{5.4}$$

So, let $\mathcal{F}$ first be a vector bundle on X. Since tensoring with a vector-bundle preserves exactness,

$$\Lambda^\bullet \xi^\vee \otimes \pi^*\mathcal{F} \to \tilde{f}_*\mathcal{O}_Y \otimes \pi^*\mathcal{F} \tag{5.5}$$

Since $\tilde{f}_*\mathcal{O}_Y \otimes \pi^*\mathcal{F} = \tilde{f}_*(\mathcal{F})$ we have a canonical Koszul resolution:

$$\Lambda^\bullet \xi^\vee \otimes \pi^*\mathcal{F} \to \tilde{f}_*\mathcal{F}. \tag{5.6}$$

Applying $\tilde{g}^*$ to this we obtain

$$
\begin{aligned}
L\tilde{g}^* R\tilde{f}_* \mathcal{F} &= \tilde{g}^* \left(\sum (-1)^i \wedge^i \xi^\vee \otimes \pi^* \mathcal{F}\right) && (5.7)\\
&= \left(\sum (-1)^i \wedge^i \tilde{g}^* \xi^\vee \otimes \tilde{g}^* \pi^* \mathcal{F}\right) && (5.8)\\
&= \lambda_{-1}(\tilde{g}^* \xi^\vee) \otimes (\pi'^* g^*)\mathcal{F} && (5.9)\\
&= \lambda_{-1}((\xi')^\vee) \otimes \lambda_{-1}(\pi'^* E) \otimes \pi'^* g^* \mathcal{F} && (5.10)\\
&= R\tilde{f}'_* \left(\lambda_{-1}(E) \otimes Lg^* \mathcal{F}\right). && (5.11)
\end{aligned}
$$

The isomorphism from the third and fourth line comes from (5.3), the isomorphism between the fourth and the fifth come from a Koszul-resolution and the projection formula similar to that of (5.4). This is the rough excess-isomorphism already exhibited in Section 5.1.
In the case of a general diagram $\mathcal{E}$

$$
\begin{array}{ccc}
X & \xrightarrow{g'} & Y \\
\big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\
X' & \xrightarrow{g} & Y'
\end{array}
$$

the isomorphism $\Psi_{\mathcal{E}}$ is defined in the following fashion, using the notation of the previous section:

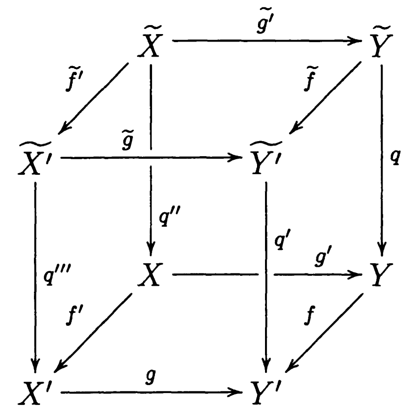
$$
\begin{aligned}
& Ri_{0,*} Lg^* Rf_*(\mathcal{F}) \\
\simeq\ & Ri_{0,*} Lg^* Rf_* Li_0^* L\Pi^*(\mathcal{F}) \\
\simeq\ & Ri_{0,*} Lg^* Li_0^* RF_* L\Pi^*(\mathcal{F}) \\
\simeq\ & Ri_{0,*} Li_0^* LG^* RF_* L\Pi^*(\mathcal{F}) \\
\overset{\lambda}{\simeq}\ & Ri_{\infty,*} Li_\infty^* LG^* RF_* L\Pi^*(\mathcal{F}) + \\
& Ri_{D',*} Li_{D'}^* LG^* RF_* L\Pi^*(\mathcal{F}) - \\
& Ri_{D'\cap P(N'\oplus 1),*} Li^*_{D'\cap \mathbb{P}(N'\oplus 1)} LG^* RF_* L\Pi^*(\mathcal{F}) \\
\simeq\ & Ri_{\infty,*} Li_\infty^* LG^* RF_* L\Pi^*(\mathcal{F}) \\
\simeq\ & Ri_{\infty,*} L\tilde{g}^* R\tilde{f}_*(\mathcal{F}) \\
\simeq\ & Ri_{\infty,*} R\tilde{f}'_*(\lambda_{-1}(E) \otimes Lg^*(\mathcal{F}))
\end{aligned}
$$

The term $Ri_{D',*} Li_{D'}^* LG^* RF_* L\Pi^*(\mathcal{F})$ is isomorphic by flat base-change to

$$Ri_{D',*} Li_{D'}^* L\Pi^* Lg^* Rf_*(\mathcal{F})$$

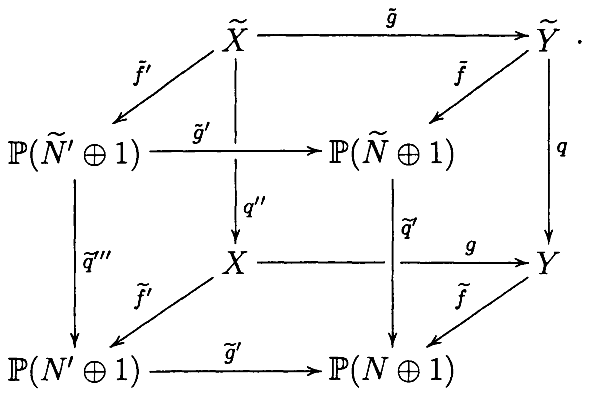
and is canonically trivialized since the intersection of D' and $D' \cap \mathbb{P}(N' \oplus 1)$ with Y is empty, and the second isomorphism is the rough excess-isomorphism of (5.7). Applying π_* on both sides gives the required isomorphism, since $\pi i_\infty \tilde{f}' = f'$ and $\pi i_0 = \mathrm{id}$. We claim this "rougher excess-isomorphism" satisfies the axioms of the theorem. It is immediate to verify condition (b). We need to verify the others.

Proposition 5.4.1. *Let*



be a commutative cube such as in condition (a) of Theorem 5.1.2. Then the rougher excess-isomorphism satisfies the conclusion of ibid.

Proof. This will essentially be by functoriality of the blow-up-construction. Keep the notation as introduced. In the cube in the introduction of this section the fibre over 0 is the diagram we start with, whereas the fiber at ∞ is, minus some unwanted factors:



The full factors over ∞ would include factors $D, D', \tilde D, \tilde D'$, which don't meet the images of $Y, X, \tilde Y$ and $\tilde X$ respectively. Let's denote by $i_0, i'_0, i'''_0, \tilde i_0, \tilde i'_0, \tilde i'''_0$ the inclusions of $Y, Y', X', \tilde Y, \tilde Y'$ and $\tilde X'$ in $\widetilde{M'}$ and $\widetilde M$ respectively. Also denote by $i_\infty, i'_\infty, i'''_\infty, \tilde i_\infty, \tilde i'_\infty, \tilde i'''_\infty$ (resp. $i_{\tilde D}$ and $i_{\tilde D'}$) the inclusions over ∞ of $Y, \mathbb{P}(N \oplus 1), \mathbb{P}(N' \oplus 1), \tilde Y, \mathbb{P}(\tilde N \oplus 1), \mathbb{P}(\widetilde{N'} \oplus 1)$ (resp. $\tilde D'$ and $\tilde D$) in

$\mathbb{P}^1_Y, M, M', \mathbb{P}^1_{\widetilde{Y}}, \widetilde{M}, \widetilde{M}'$ (resp. $\widetilde{M}$ and $\widetilde{M}'$). Also introduce the projections $\pi : \mathbb{P}^1_Y \to Y$ and $\overline{\pi} : \mathbb{P}^1_{\widetilde{Y}} \to \widetilde{Y}$.

Then we find that the following diagram is commutative

$$
\begin{array}{ccccc}
\overline{g}^*\overline{f}_*q^* & \xrightarrow{base-change} & \overline{g}^*q'^*f_* & \xrightarrow{q'\overline{g}=gq'''} & q'''^*g^*f_* \\
\big\downarrow{\scriptstyle \overline{\pi}\overline{i}_0=\mathrm{id}} & & & & \big\downarrow{\scriptstyle \pi i_0=\mathrm{id}} \\
\overline{g}^*\overline{f}_*\overline{i}_0^*\overline{\pi}^*q^* & & & & q'''^*g^*f_*i_0^*\pi^* \\
\big\downarrow{\scriptstyle base-change} & & & & \big\downarrow{\scriptstyle base-change} \\
\overline{g}^*\overline{i_0'}^*\overline{F}_*\overline{\pi}^*q^* & \xrightarrow{\pi Q=q\overline{\pi}} & \overline{g}^*\overline{i_0'}^*\overline{F}_*Q^*\pi^* & & q'''^*i^*i_0'^*F_*\pi^* \\
\big\downarrow{\scriptstyle \overline{Gi_0''''}=\overline{i_0'}g} & & \big\downarrow{\scriptstyle base-change} & \nearrow{\scriptstyle Q'i_0'\overline{g}=i_0'gq'''} & \big\downarrow{\scriptstyle Gi_0'''=i_0'g} \\
\overline{i_0''''}^*\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & & \overline{g}^*\overline{i_0'}^*Q'^*F_*\pi^* & & \\
\big\downarrow{\scriptstyle \pi Q=q\overline{\pi}} & & & & \big\downarrow \\
\overline{i_0''''}^*\overline{G}^*\overline{F}_*Q^*\pi^* & \xrightarrow{base-change} & \overline{i_0''''}^*\overline{G}^*Q'^*F_*\pi^* & \xrightarrow{Q'\overline{Gi_0''''}=Gi_0'''q'''} & q'''^*i_0''''^*G^*F_*\pi^*
\end{array}
$$

This is just applying Lemma 3.1.5 to the definition of base-change.

Apply $\overline{i}'''_{0,*}$ to the above diagram. We continue

$$
\begin{array}{ccc}
\overline{i}'''_{0,*}\overline{i_0''''}^*\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & \xrightarrow{\text{see above}} & \overline{i}'''_{0,*}q'''^*i'''_{0,*}G^*F_*\pi^* \\
\big\downarrow{\scriptstyle projection-formula} & & \big\downarrow{\scriptstyle base-change} \\
\mathcal{O}_{\overline{X}'}\otimes\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & & Q''''^*i'''_{0,*}i_0''''^*G^*F_*\pi^* \\
\big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \lambda+projection-formula} \\
\left(\mathcal{O}_{\mathbb{P}(\overline{N}'\oplus 1))}+\mathcal{O}_{\overline{D}}-\mathcal{O}_{\mathbb{P}(\overline{N}'\oplus 1))\cap\overline{D}}\right)\otimes\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & \longrightarrow & Q''''^*\left(\left(\mathcal{O}_{\mathbb{P}(N'\oplus 1))}+\mathcal{O}_D-\mathcal{O}_{\mathbb{P}(N'\oplus 1))\cap D}\right)\otimes G^*F_*\pi^*\right) \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}_{\mathbb{P}(\overline{N}'\oplus 1))}\otimes\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & \longrightarrow & Q''''^*\left(\mathcal{O}_{\mathbb{P}(N'\oplus 1))}\otimes G^*F_*\pi^*\right) \\
\big\downarrow{\scriptstyle projection-formula} & & \big\downarrow{\scriptstyle projection-formula} \\
\overline{i''''_{\infty *}}\,\overline{i''''_{\infty}}^*\overline{G}^*\overline{F}_*\overline{\pi}^*q^* & \longrightarrow & Q''''^*i''''_{\infty *}i''''^*_{\infty}G^*F_*\pi^* \\
\big\downarrow & & \big\downarrow \\
\overline{i''''_{\infty *}}g^*f_*q^* & \longrightarrow & \overline{i''''_{\infty *}}q'''^*g^*f_*.
\end{array}
$$

The upper square in this diagram is commutative by our choice of $\lambda \in k(\mathbb{P}^1_{\mathbb{Z}})$

and by verifying immediately (c.f. Lemma 3.1.6) that the base-changes involved are stable under the projection formula.

In the next square, the induced isomorphism induces a commutative diagram of isomorphisms:

$$\begin{array}{ccc} \mathcal{O}_{\overline{X}'} & \longrightarrow & Q'''^{*}\mathcal{O}_{X'} \\ \downarrow \lambda & & \downarrow \lambda \\ \mathcal{O}_{\mathbb{P}(\overline{N}'+1))} + \mathcal{O}_{\overline{D}} - \mathcal{O}_{\mathbb{P}(\overline{N}'+1))\cap\overline{D}} & \longrightarrow & Q'''^{*}(\mathcal{O}_{\mathbb{P}(N'+1))} + \mathcal{O}_{D} - \mathcal{O}_{\mathbb{P}(N'+1))\otimes D}) \end{array}$$

Moreover, in the second line there is an induced isomorphism

$$\mathcal{O}_{\mathbb{P}(\overline{N}'+1))} \simeq Q'''^{*}\left(\mathcal{O}_{\mathbb{P}(N'+1))}\right)$$

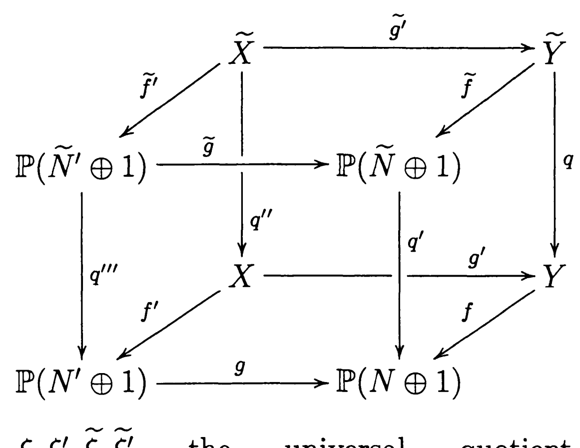
and thus also an induced isomorphism of superfluous terms (see Lemma 3.4.2):

$$\mathcal{O}_{\overline{D}} - \mathcal{O}_{\mathbb{P}(\overline{N}'+1))\cap\overline{D}} \simeq Q'''^{*}\left(\mathcal{O}_{D} - \mathcal{O}_{\mathbb{P}(N'+1))\cap D}\right).$$

Hence calculating the Euler-characteristic to 0 is compatible with this base-change, and the next square is also commutative.

Finally, the lower square is commutative for the same reason as the other first square above, i.e. by Lemma 3.1.5. Applying $\overline{\pi}_*$ as in the definition of our morphism we obtain the commutativity, modulo showing commutativity in the model situation.

With the notation of the left diagram of (5.1), we are left to consider the diagram



Denote by $\xi, \xi', \widetilde{\xi}, \widetilde{\xi}'$ the universal quotient bundles on $\mathbb{P}(N\oplus 1), \mathbb{P}(N'\oplus 1), \mathbb{P}(\widetilde{N}\oplus 1)$ and $\mathbb{P}(\widetilde{N}'\oplus 1)$ respectively. The proposition is that

$$\begin{array}{ccc}
R\tilde{f}'_*(\lambda_{-1}(\tilde{E}) \otimes L\tilde{g}'^*(Lq^*x)) & \longrightarrow & L\tilde{g}^* R\tilde{f}_*(Lq^*(x)) \\
\downarrow & & \downarrow \\
R\tilde{f}'_*(\lambda_{-1}(q''^*E) \otimes (Lq''^* Lg'^* x)) & & L\tilde{g}^* Lq'^* Rf_*(x) \\
\downarrow & & \downarrow \\
R\tilde{f}'_* Lq''^*(\lambda_{-1}(E) \otimes Lg'^* x)) & & Lq'''^* Lg^* Rf_*(x) \\
\downarrow & & \downarrow{=} \\
Lq'''^*(Rf_*(\lambda_{-1}(E) \otimes Lg'^* x)) & \longrightarrow & Lq'''^* Lg^* Rf_*(x)
\end{array}$$

is a commutative diagram, where the horizontal morphisms are the already constructed candidates for the excess-formula isomorphisms in (5.7). We will show that the above diagram is commutative by chopping it up into smaller pieces.

$$\begin{array}{ccccc}
\tilde{g}^*\tilde{f}_*q^* & \xrightarrow{\text{base-change}} & \tilde{g}^*q'^*f_* & \xrightarrow{q'\tilde{g}=gq'''} & q'''^*g^*f_* \\
\downarrow{\scriptstyle\text{resolution}} & & & & \downarrow{\scriptstyle\text{resolution}} \\
\tilde{g}^*(\lambda_{-1}(\tilde{\xi}^\vee) \otimes \tilde{\pi}^*q^*) & \longrightarrow & \lambda_{-1}(\tilde{g}^*\tilde{\xi}^\vee) \otimes \tilde{g}^*\tilde{\pi}^*q^* & \longrightarrow & q'''^*g^*(\lambda_{-1}(\xi^\vee) \otimes \pi^*) \\
\downarrow & \swarrow\!\!= & & & \downarrow \\
\lambda_{-1}(\tilde{g}^*\tilde{\xi}^\vee) \otimes \tilde{g}^*\tilde{\pi}^*q^* & & \longrightarrow & & \lambda_{-1}(q'''^*g^*\xi^\vee) \otimes q'''^*g^*\pi^* \\
\downarrow{\scriptstyle 0\to\tilde{\pi}'^*\tilde{E}\to\tilde{g}^*\tilde{\xi}^\vee\to\tilde{\xi}'^\vee\to 0} & & & & \downarrow{\scriptstyle 0\to\pi'^*E\to g^*\xi^\vee\to\xi'^\vee\to 0} \\
\lambda_{-1}(\tilde{\xi}'^\vee) \otimes \lambda_{-1}(\tilde{\pi}'^*\tilde{E}) \otimes \tilde{g}^*\tilde{\pi}^*q^* & & \longrightarrow & & q'''^*\left(\lambda_{-1}(\xi'^\vee) \otimes \lambda_{-1}(\pi^*E) \otimes g^*\pi^*\right) \\
\downarrow{\scriptstyle\text{projection-formula}} & & & & \downarrow{\scriptstyle\text{projection-formula}} \\
\tilde{f}_*\left(\lambda_{-1}(\tilde{E}) \otimes \tilde{g}'^*q^*\right) & \xrightarrow{q''^*E=\tilde{E}} & \tilde{f}_*(\lambda_{-1}(q''^*E) \otimes q''^*g'^*) & \xrightarrow{\text{base-change}} & q'''^*(f_*(\lambda_{-1}(E) \otimes g'^*))\,.
\end{array}$$

The upper middle diagram commutes by general nonsense; the isomorphisms are just given by certain natural transformations.
To see why the lower middle diagram commutes, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & q'''^*\pi'^*E & \longrightarrow & q'''^*g^*\xi^\vee & \longrightarrow & q'''^*(\xi')^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\pi}'^*\tilde{E} & \longrightarrow & \tilde{g}^*\tilde{\xi}^\vee & \longrightarrow & (\tilde{\xi}')^\vee & \longrightarrow & 0
\end{array}$$

where all the vertical maps are isomorphisms. Indeed, all the maps exist by functoriality, and they are isomorphisms by the assumption on transversality.

The upper and lower diagrams are commutative again by applying Lemma 3.1.5. The details are left to the reader. □

Proposition 5.4.2. *Let $e : Y \to Y', f : Y' \to Y''$ be two regular closed immersions, and $g : X'' \to Y''$ another regular closed immersion, with associated diagrams $\mathcal{E}, \mathcal{E}'$ and big diagram $\mathcal{E}''$,*

$$\begin{array}{ccc} X & \xrightarrow{g''} & Y \\ \downarrow{\scriptstyle e'} & \mathcal{E} & \downarrow{\scriptstyle e} \\ X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & \mathcal{E}' & \downarrow{\scriptstyle f} \\ X'' & \xrightarrow{g} & Y''. \end{array}$$

Then $\Psi_{\mathcal{E}''}$ is the composition of $\Psi_{\mathcal{E}'}$ and $\Psi_{\mathcal{E}}$ in the sense of Theorem 5.1.2, condition (d).

Proof. By a deformation to the normal cone argument we can suppose that our immersions are of the form

$$\begin{array}{ccc} X & \xrightarrow{g''} & Y \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ \mathbb{P}_X(N' \oplus 1) & \xrightarrow{g'} & \mathbb{P}_Y(N \oplus 1) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathbb{P}_X(M' \oplus 1) & \xrightarrow{g} & \mathbb{P}_Y(M \oplus 1). \end{array}$$

Denote by $p : \mathbb{P}_Y(M \oplus 1) \to Y, p' : \mathbb{P}_X(M' \oplus 1) \to X, \pi : \mathbb{P}_Y(N \oplus 1) \to Y$ and $\pi' : \mathbb{P}_X(N' \oplus 1) \to X$ the various projections. Define

$$N^{\perp} := \ker[M^{\vee} \to N^{\vee}] = (M/N)^{\vee}, N'^{\perp} := \ker[M'^{\vee} \to N'^{\vee}]) = (M'/N')^{\vee}.$$

On $\mathbb{P}_Y(M \oplus 1)$ we have a canonical exact sequence

$$0 \to \mathcal{O}(-1) \to p^*(M \oplus 1) \to T_{P/Y}(-1) \to 0.$$

The map $\mathcal{O}(-1) \to p^*M \oplus 1 \to p^*(M)$ defines a regular section of $p^*M(1)$ which vanishes exactly at Y, and in the same way we get a regular section of the vector bundle $p^*(M/N)(1)$ which vanishes exactly at $\mathbb{P}_Y(N \oplus 1)$ (see [Ful98], Appendix B. 5.6). We have the following commutative diagram with exact columns and lines

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{E}' & \longrightarrow & g^*p^*N^{\perp}(-1) & \longrightarrow & p'^*N'^{\perp}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{E}'' & \longrightarrow & g^*p^*M^{\vee}(-1) & \longrightarrow & p'^*M'^{\vee}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{E} & \longrightarrow & g^*p^*N^{\vee}(-1) & \longrightarrow & p'^*N'^{\vee}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{5.12}$$

Here $\mathbb{E}, \mathbb{E}', \mathbb{E}''$ denote vector-bundles on $\mathbb{P}(M' \oplus 1)$ which extend the excess-bundles E, E' and E'' respectively. Using the canonical Koszul resolutions, the proposition is that the following diagram is commutative:

$$\begin{array}{ccc}
R(f'\circ e')_*(\lambda_{-1}(E'')\otimes Lg''^*(\mathcal{F})) & \longrightarrow & \lambda_{-1}(\mathbb{E}'')\otimes\lambda_{-1}(Lp'^*M'^{\vee}(-1))\otimes L(g''\circ p')^*(\mathcal{F}) \\
\downarrow & & \downarrow \\
Rf'_*(Re'_*(\lambda_{-1}(Le'^*E')\otimes\lambda_{-1}(E)\otimes Lg''^*(\mathcal{F}))) & \textbf{(A)} & \lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(\mathbb{E})\otimes\lambda_{-1}(Lp'^*M'^{\vee}(-1))\otimes L(g''\circ p')^*(\mathcal{F}) \\
\downarrow & & \downarrow \\
Rf'_*(\lambda_{-1}(E')\otimes Re'_*(\lambda_{-1}(E)\otimes Lg''^*(\mathcal{F}))) & \longrightarrow & \lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(Lp'^*N'^{\perp}(-1))\otimes\lambda_{-1}(Lp'^*N'^{\vee}(-1))\otimes\lambda_{-1}(\mathbb{E})\otimes(Lp'^*Lg''^*\mathcal{F}) \\
\downarrow & \textbf{(B)} & \| \\
Rf'_*\big(\lambda_{-1}(E')\otimes\lambda_{-1}(L\pi'^*N'^{\vee}(-1))\otimes\lambda_{-1}(\pi'^*E)\otimes L\pi'^*Lg''^*(\mathcal{F})\big) & \longrightarrow & \lambda_{-1}(Lp'^*N'^{\perp}(-1))\otimes\lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(p'^*N'^{\vee}(-1))\otimes\lambda_{-1}(\mathbb{E})\otimes(Lp'^*Lg''^*\mathcal{F}) \\
\downarrow & \textbf{(C)} & \downarrow \\
Rf'_*\big(\lambda_{-1}(E')\otimes\lambda_{-1}(Lg'^*L\pi^*N^{\vee}(-1))\otimes Lg'^*L\pi^*(\mathcal{F})\big) & \longrightarrow & \lambda_{-1}(Lp'^*N'^{\perp}(-1))\otimes\lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(Lg''^*Lp^*N^{\vee}(-1))\otimes(Lp'^*Lg''^*\mathcal{F}) \\
\downarrow & & \| \\
Rf'_*\big(\lambda_{-1}(E')\otimes Lg'^*Re_*(\mathcal{F})\big) & \textbf{(D)} \searrow & \| \\
\downarrow & \textbf{(E)} & \lambda_{-1}(Lp'^*N'^{\perp}(-1))\otimes\lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(Lg^*Lp^*N^{\vee}(-1))\otimes(Lp'^*Lg''^*\mathcal{F}) \\
\downarrow & \nearrow & \downarrow \\
\big(\lambda_{-1}(Lp'^*N'^{\perp}(-1))\otimes\lambda_{-1}(\mathbb{E}')\otimes\lambda_{-1}(Lg^*Lp^*N^{\vee}(-1))\otimes Lg^*Lp^*\mathcal{F}\big) & \textbf{(F)} & \big(\lambda_{-1}(Lg^*Lp^*N^{\perp}(-1))\otimes\lambda_{-1}(Lg^*Lp^*N^{\vee}(-1))\otimes Lg^*Lp^*\mathcal{F}\big) \\
\downarrow & \swarrow \ \textbf{(G)} & \downarrow \\
(\lambda_{-1}(Lg^*Lp^*M^{\vee}(-1))\otimes Lg^*Lp^*(\mathcal{F})) & \longrightarrow & Lg^*R(f\circ e)_*(\mathcal{F})
\end{array}$$

A few comments are in order. Since there is a Koszul resolution $N^{\vee}(-1) \to \mathcal{O}_{\mathbb{P}_Y(N\oplus 1)}$ of Y on $\mathbb{P}_Y(N \oplus 1)$ the virtual bundle $Re_*(\mathcal{F})$ has

an extension to $\mathbb{P}_Y(M \oplus 1)$ given by $\lambda_{-1}(Lp^*N^\vee(-1)) \otimes Lp^*\mathcal{F}$ and the left arrow of diagram (**E**) is defined using this. The composition of the left arrows of the diagrams (**B**), (**C**) and (**D**) constitute the isomorphism determined by $\mathcal{E}$ and finally the composition of the leftmost arrows of (**E**), (**F**) and (**G**) give the isomorphism determined by $\mathcal{E}'$.
Moreover (**F**) commutes since the isomorphism (3.3) is compatible with filtrations and the diagram (5.12) and the composition of all the rightmost downwards arrows is the isomorphism determined by $\mathcal{E}''$ for the same reason. We refrain from giving all the details of the fact that the other diagrams commute. The interested reader can however easily verify that they do using that all the Koszul-resolutions involved are compatible and using the following lemma which we give without proof, which after an inspection takes care of (**A**) and (**B**). The other diagrams commute for similar, albeit slightly more involved, reasons.

Lemma 5.4.3. *Suppose $X \to Y$ is a closed regular immersion and let $\sigma : M^\vee \to \mathcal{O}_Y$ be a Koszul resolution of X (i.e. dual to a regular section $\mathcal{O}_Y \to M$) and let N' be a subbundle of M. Let $N^\perp = \ker : M^\vee \to N^\vee$, and let $g : Z \to Y$ be the substack of Y defined by $\sigma(N^\perp)$. Then restricting σ to $N^\perp$ we obtain a Koszul resolution for $\mathcal{O}_Z$, and $g^*\sigma : g^*N^\vee \to \mathcal{O}_Z$ defines a Koszul-resolution of $\mathcal{O}_Y$ on Z. Per definition we have an exact sequence*

$$0 \to N^\perp \to M^\vee \to N^\vee \to 0. \tag{5.13}$$

Suppose furthermore that we are given a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & Y \\ {\scriptstyle f}\nwarrow\searrow{\scriptstyle \pi} & & \swarrow{\scriptstyle p} \\ & X & \end{array}$$

Let F be a virtual vector-bundle on X, and let F' be a virtual vector bundle on Z which extends to a virtual vector bundle F'_Y on Y. Then the diagram

$$\begin{array}{ccc} \lambda_{-1}(N^\vee) \otimes \lambda_{-1}(N^\perp) \otimes F'_Y \otimes p^*F & \xrightarrow{(5.13)} & \lambda_{-1}(M^\vee) \otimes F'_Y \otimes p^*F \\ \downarrow{\scriptstyle \Phi_{\sigma'}} & & \downarrow{\scriptstyle \Phi_\sigma} \\ g_*(F' \otimes \lambda_{-1}(g^*N^\vee) \otimes \pi^*F) & \xrightarrow{\Phi_{g^*\sigma}} & g_*(F' \otimes f_*(F)) \end{array}$$

commutes. Here Φ_* denotes the isomorphism determined by the Koszul resolution "$*$".

□

Proposition 5.4.4. *Consider the isomorphism considered in condition (c). Then this coincides with the above constructed isomorphism.*

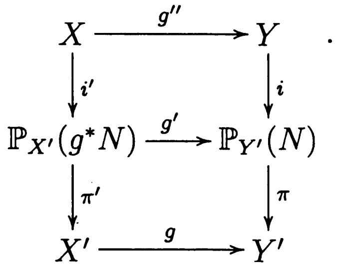
Proof. This is easy. By the argument of Proposition 3.4.1 the Koszul resolution determined by the Cartier divisor deforms to the Koszul resolution in the model situation. Since the constructed excess-isomorphism has been verified to be stable under such transformations we are done. □

We thus conclude the demonstration in the case of a regular closed immersion. □

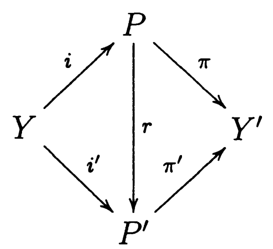
5.5 General excess isomorphism

In this section we will tie together the isomorphisms constructed in the preceding sections and finally construct the total excess isomorphism.

Suppose first that we have a diagram $\mathcal{E}$ and a decomposition of a proper morphism $f : Y \to Y'$ as $f_\tau : Y \xrightarrow{i} \mathbb{P}_{Y'}(N) \xrightarrow{\pi} Y'$ where i is a regular closed immersion and π is a projective bundle-projection. By base-change we obtain the composition of two Cartesian diagrams



We define the excess-isomorphism $\Psi_{\mathcal{E},\tau}$ via the naive composition of the excess-isomorphisms of the two diagrams. Given a factorization τ, denote by E_τ the associated excess bundle. In general, if we have two factorizations with a morphism r,



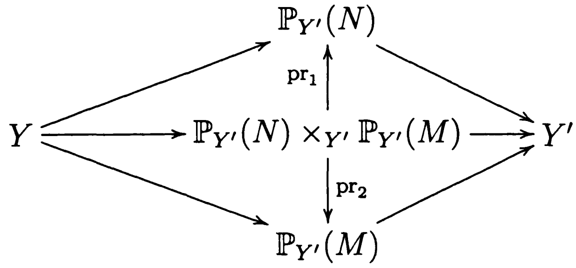
with π and π' smooth, we get an isomorphism

$$\phi_{\tau,\tau',r} : E_\tau \to E_{\tau'}$$

with the property that

$$\phi_{\tau',\tau'',r}\phi_{\tau,\tau',s} = \phi_{\tau,\tau'',sr}.$$

Now, given two arbitrary factorizations τ, τ', we compare them with the diagonal



and put

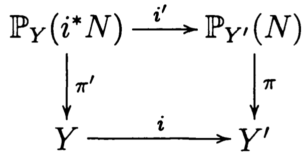
$$\phi_{\tau,\tau'} = \phi_{\tau,\tau\times\tau',\mathrm{pr}_1}\left(\phi_{\tau',\tau\times\tau',\mathrm{pr}_2}\right)^{-1} : E_{\tau'} \to E_{\tau}.$$

This defines an isomorphism $\phi_{\tau,\tau'} : E_{\tau'} \to E_{\tau}$ which satisfies the cocycle condition $\phi_{\tau,\tau'}\phi_{\tau',\tau''} = \phi_{\tau,\tau''}$ so they glue together an virtual excess-bundle E, determined up to canonical isomorphism. Of course, one could directly define it as the kernel of "le complexe cotangent" of Illusie as defined in [GL00], Théorème-définition 17.3 (3) (notice there is however a famous error in the definition of the lisse-étale site which causes it to not be functorial. Since all our morphisms in question are representable this doesn't cause any issues) Given any such factorization τ, we define an excess-morphism $\Psi_{\mathcal{E},\tau}$ as the naive composition of the two, using the above definition of the excess-bundle. We give a standard list of basic compatibilities needed:

Lemma 5.5.1. *We suppose the diagrams $\mathcal{E}$ implicit.*

- *Let i be a section to a projection bundle-projection $\pi : \mathbb{P}_Y(N) \to Y$. Then $\Psi_\pi\Psi_i = \mathrm{Id}$.*

- *Suppose that we have a Cartesian diagram*



with i a regular immersion. Then $\Psi_i\Psi_{\pi'} = \Psi_\pi\Psi_{i'}$.

- *Suppose we have a diagram*

$$\begin{array}{ccc} & & \mathbb{P}_{Y'}(N) \\ & \overset{j}{\nearrow} & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\;i\;} & Y' \end{array}$$

with i and j being regular closed immersions and π the projective bundle-projection. Then $\Psi_i = \Psi_\pi \Psi_j$.

Proof. In the first case there is no excess and the statement becomes that the composition of two base-change isomorphisms is the base-change of the composition, which is the identity.
For the second case, we can suppose by additivity and standard reduction that our bundles are of the form $\pi^* F \otimes \mathcal{O}(-k)$ for $0 \leq k < n = \operatorname{rk} N$. The argument is now a lengthy but elementary application of the relationship $R\pi_* \pi^* = \mathrm{Id}$ together with stability under base-change already established for closed immersions, which we leave to the interested reader.
For the third point, one uses the Lichtenbaum-trick to reduce to the case of a morphism with section:

$$\begin{array}{ccc} \mathbb{P}_Y(i^*N) & \xrightarrow{\;i'\;} & \mathbb{P}_{Y'}(N)\,. \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \pi'} & \overset{j}{\nearrow} & \downarrow{\scriptstyle \pi'} \\ Y & \xrightarrow{\;i\;} & Y' \end{array}$$

Here the diagram is Cartesian and s is the section determined by j and base-change: Then $\Psi_\pi \Psi_j = \Psi_\pi \Psi_{i'} \Psi_s = \Psi_i \Psi_{\pi'} \Psi_s = \Psi_i$ by the preceding statements. □

Proposition 5.5.2. *With the above virtual excess bundle the $\Psi_{\mathcal{E},\tau}$ does not depend on choice of τ.*

Proof. Let τ and τ' be two different factorizations as above. Considering again the diagram

$$\begin{array}{ccccc} & \overset{i}{\nearrow} & \mathbb{P}_{Y'}(N) & \overset{\pi}{\searrow} & \\ & & \uparrow{\scriptstyle q'} & & \\ Y & \xrightarrow{\;i''\;} & \mathbb{P}_{Y'}(N) \times_{Y'} \mathbb{P}_{Y'}(M) & \xrightarrow{\;\pi''\;} & Y' \\ & & \downarrow{\scriptstyle q} & & \\ & \underset{i'}{\searrow} & \mathbb{P}_{Y'}(M) & \underset{\pi'}{\nearrow} & \end{array}$$

Put $\Psi_{\mathcal{E},\tau}$ equal to the naive composition of the isomorphisms induced by i and π. By abuse of notation, we obtain equalities of isomorphisms

$$\Psi_{f,\sigma} = \Psi_\pi \Psi_i = \Psi_\pi \Psi_{q'} \Psi_{i''} = \Psi_{\pi''} \Psi_{i''} = \Psi_{\pi'} \Psi_q \Psi_{i''} = \Psi_{\pi'} \Psi_{i'} = \Psi_{f,\tau}.$$

Hence all isomorphisms $\Psi_{f,?}$ are in fact one single morphism, defining $\Psi_{\mathcal{E}}$.

□

It remains to prove:

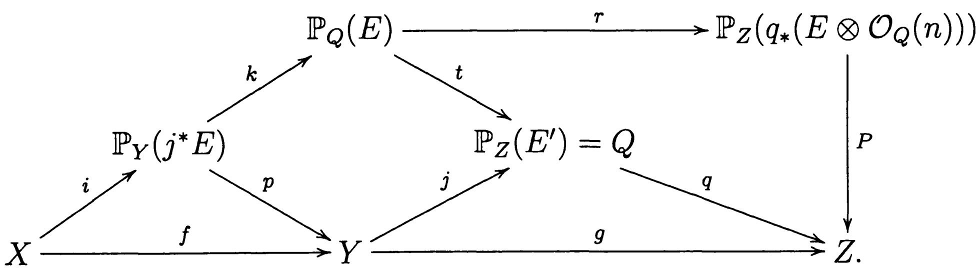
Theorem 5.5.3. *Suppose that we have morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with f and g projective local complete intersection morphisms. Then

$$\Psi_f \circ \Psi_g = \Psi_{fg}.$$

We can factor as follows, for big enough n;



The first lower triangle can be written in this form because Y has the resolution property, $q_*(E \otimes \mathcal{O}_Q(n))$ is locally free for large enough n and both f and g are supposed to be projective (see the argument in [WF85], chapter IV, Proposition 3.12). Now, we have by Lemma 5.5.1 and Proposition 5.5.2:

$$\Psi_g \Psi_f = \Psi_q \Psi_j \Psi_p \Psi_i = \Psi_q \Psi_t \Psi_k \Psi_i = \Psi_P \Psi_r \Psi_k \Psi_i = \Psi_P \Psi_{rki} = \Psi_{gf}.$$

6. APPLICATIONS TO FUNCTORIALITY

This little pig cried 'wee wee wee' all the way home

6.1 Explicit construction of characteristic classes

Let $\mathcal{X}$ be an algebraic stack and consider the virtual category $V(\mathcal{X})$. Consider the full subcategory $V(\mathcal{X})^*$ of $V(\mathcal{X})$ consisting of the elements whose image $[-]$ in $K_0(\mathcal{X})$ is invertible. It is clear that tensor-product on $V(\mathcal{X})$ satisfies the pentagonal and hexagonal axioms for a Picard category (c.f. 2.2). Also, by construction, for a fixed object B in $V(\mathcal{X})^*$, $A \mapsto A \otimes B$ is essentially surjective, and fully faithful since it acts on the automorphism-group $K_1(\mathcal{X})$ of an object by $[B]$ which is an automorphism in view of the fact that $K_1(\mathcal{X})$ is a $K_0(\mathcal{X})$-module. It follows that the category $V(\mathcal{X})^*$ together with the tensor product is a Picard category and thus for any object B in $V(\mathcal{X})^*$ there is an element, B^{-1}, unique up to unique isomorphism such that $B \otimes B^{-1} = 1$. As with the category $V(\mathcal{X})$, $V(\mathcal{X})^*$ comes equipped with a plethora of sign-anomalies associated with the fact that they are not strictly commutative. We start by showing that certain characteristic classes are constructible in a quite general context whenever we ignore these signs.

Definition 6.1.0.1. Suppose $(P, \oplus)$ is a Picard category with a distributive functor $\otimes : P \times P \to P$ with associativity and commutativity-constraints satisfying the hexagonal and pentagonal axioms (c.f. 2.2) so that $\otimes$ makes P into a (non-unital) monoidal category. We call $(P, \oplus, \otimes)$ a Picard ring and often omit reference to $\oplus$ and $\otimes$. It is said to be strictly commutative if the operations $\oplus$ and $\otimes$ are strictly commutative, i.e. the symmetry-isomorphism $X \oplus X \to X \oplus X$ and $Y \otimes Y \to Y \otimes Y$ is the identity. A ring functor of Picard rings is a functor of Picard rings which is monoidal for both operations $\oplus$ and $\otimes$.
We say that a category P fibered over a category $\mathcal{C}$ is a category fibered in Picard rings over a category $\mathcal{C}$ if for any object X of $\mathcal{C}$, $P(X)$ is a Picard ring, and such that for any morphism $f : X \to Y$ in $\mathcal{C}$, there is a ring functor $f^* : P(Y) \to P(X)$ satisfying the obvious associativity constraints.

Clearly the virtual category $V(\mathcal{X})$ is a Picard ring and V defines a category fibered in Picard rings over the category of algebraic stacks. In general, for a Picard ring P, we can consider the full subcategory of elements P^* whose isomorphism-class in $\pi_0(P)$ is invertible under the operation $\pi_0(\otimes)$. By the same argument as above, $(P^*, \otimes)$ forms a Picard category, and similarily for a category fibered in Picard rings P one obtains a category fibered in Picard categories P^*. Similarly for the cohomological virtual category $W(\mathcal{X})$.

For the next proposition, recall that by [Tot04], Theorem 1.1, Proposition 1.3, a normal separated Nötherian algebraic stack (over $\operatorname{spec}\mathbb{Z}$) with affine geometric stabilizers has the resolution property if and only if it is of the form $[U/GL_d]$ for quasi-affine U. In particular, a regular algebraic stack with affine stabilizers is of the form $[U/GL_d]$ for regular quasi-affine U if and only if it has the resolution property. This is in the same spirit as the following result which we shall also quote often:

Theorem 6.1.1 (Jouanolou-Thomason, [Wei89], Proposition 4.4). *Let X be a scheme admitting an ample family of line bundles. Then there is a vector bundle $\xi \to X$ and a ξ-torsor $f : T \to X$ such that T is affine.*

Proposition 6.1.2 (Multiplicative characteristic classes in cohomological virtual categories on quotient stacks). *Consider the cohomological virtual category W considered as a Picard ring fibered over the category of regular algebraic stacks with the resolution property and finite affine stabilizers (by the above, necessarily of the form $[U/GL_d]$ for quasi-affine U). Suppose we are given a powerseries*

$$F = 1 + \sum_{i=1}^{\infty} a_i x^i \in 1 + x\mathbb{Q}[[x]].$$

There is then a unique functor $\Theta : V \to W$, up to unique isomorphism, such that:

(a) Θ is a determinant functor $V \to W^*$.

(b) For a line bundle L on $\mathcal{X}$ there exists an isomorphism $\Theta(L) = F(L-1)$, which is well-defined by virtue of point (7) of Theorem 4.0.8

Proof. As for existence, we first recall the formalism of Hirzebruch polynomials. Let R be a λ-ring and denote by γ the corresponding γ-structure (c.f. [WF85], chapter III). Suppose that $\phi(x) \in 1 + xR[[x]]$. We can associate multiplicative maps $M_\phi(x) : R \to R$ as follows. First, for u a line element, we simply define

$$M_\phi(u) = \phi(u-1).$$

If e is a sum of line elements u_i, we set

$$M_\phi(e) = \prod_i M_\phi(u_i).$$

If W_i are independent variables, we consider the power-series

$$M_\phi(W_i t) = \sum H_j^\phi(s_1, \ldots, s_j) t^j$$

for some degree j-homogenous polynomial H_j^ϕ in the elementary symmetric functions s_k in the W_i. Here the H_j^ϕ are the associated (multiplicative) Hirzebruch polynomials. Now, regular schemes have the resolution property so by [Tot04], Theorem 1.1 they are of the form prescribed. Let X be a regular scheme, $R = K_0(X)_\mathbb{Q}$ and $\phi = F$. The associated H_j^F and M_F define homomorphisms $K_0(X)_\mathbb{Q} \to K_0(X)_\mathbb{Q}$ functorial on the category of regular schemes. By rigidity they define functors, which we denote by $H_j : W \to W^{(j)}$ and $M : W \to W$, such that for a line bundle L on a regular algebraic stack, $H_j(L) = a_j(L-1)^j$ and $M(L) = 1 + \sum_{i=1}^\infty a_i(L-1)^k$. This sum is again well-defined by point (7) of Theorem 4.0.8. Rigidity also implies that for a sum of virtual bundles $u + v$ on an algebraic stack $\mathcal{X}$, there is a canonical isomorphism $H_j(u+v) \to \sum H_i(u) \otimes H_{i-j}(v)$ in $W^{(j)}$. It follows that there is an isomorphism $M(u+v) \to M(u) \otimes M(v)$ in $W(\mathcal{X})^*$ and thus Θ defines a determinant functor by the composition $V(\mathcal{X}) \to W(\mathcal{X}) \to W(\mathcal{X})^*$ and it is functorial by construction and satisfies the conditions of the theorem.
We are left to establish unicity. Suppose $\mathcal{X} = [U/GL_d]$ is a regular algebraic stack with quasi-affine U. By the splitting principle it is sufficient to verify that the object $\Theta(L)$ in $W(\mathcal{X})$ is uniquely determined. The trivial bundle on U is then GL_d-equivariantly ample and there exists a GL_d-equivariant locally split monomorphism $L \subset \mathcal{O}^r$ for big enough r. This defines a section $i : \mathcal{X} \to \mathbb{P}^n_\mathcal{X}$ to the natural projection $p : \mathbb{P}^n_\mathcal{X} \to \mathcal{X}$ and $L = i^*\mathcal{O}(1) = Li^*\mathcal{O}(1)$. Since the algebraic stacks have the resolution property we can apply the functor Ri_* and we then have an isomorphism

$$\begin{array}{ccccc}
Ri_*\Theta(L) & \longrightarrow & Ri_*Li^*\Theta(\mathcal{O}(1)) & \longrightarrow & Ri_*(\mathcal{O}_\mathcal{X}) \otimes \Theta(\mathcal{O}(1)) \\
\downarrow & & \downarrow & & \downarrow \\
Ri_*F(L-1) & \longrightarrow & Ri_*Li^*F(\mathcal{O}(1)-1) & \longrightarrow & Ri_*(\mathcal{O}_\mathcal{X}) \otimes F(\mathcal{O}(1)-1)
\end{array}$$

Since $Rp_*Ri_* = \mathrm{id}$ the isomorphism $\Theta(L) \to F(L-1)$ is determined by the isomorphism $\Theta(\mathcal{O}(1)) \to F(\mathcal{O}(1)-1)$ on $\mathbb{P}^n_\mathcal{X}$. However, $\mathcal{O}(1)$ on $\mathbb{P}^n_\mathcal{X}$ is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^n_\mathbb{Z}$ via the unique (in general non-representable) morphism $\mathcal{X} \to \operatorname{spec}\mathbb{Z}$ and this isomorphism is tautologically rigidified since $W(\mathbb{P}^n_\mathbb{Z}) = V(\mathbb{P}^n_\mathbb{Z})_\mathbb{Q}$ doesn't have any non-trivial automorphisms. □

Any class constructed in the above fashion will be called the associated multiplicative class to the given data. We now harvest the following corollary of the previous proposition.

Corollary 6.1.3 (Bott's cannibalistic class). *Let $\mathcal{X}$ be a regular algebraic stack with finite affine stabilizers of the form $[U/GL_d]$ for a quasi-affine U.*

- *For $k \geq 1$, there is then a unique Bott-element $\theta_k : V(\mathcal{X}) \to W(\mathcal{X})$ which is k times the associated multiplicative class of the polynomial $F(x) = k^{-1}(1 + (x+1) + (x+1)^2 + \ldots + (x+1)^{k-1}) \in 1 + \mathbb{Q}[x]$.*
- *Suppose that v is a virtual vector bundle of rank r. Then $\theta_2(-v)$ is equal to*

$$\frac{1}{2^r}\left(1 - \frac{v-r}{2} + \frac{(v-r)^2 - \gamma^2(v-r)}{4} - \frac{(v-r)^3 - 2\gamma^2(v-r)(v-r) + \gamma^3(v-r)}{8}\right)$$

modulo $F^{(4)}W(\mathcal{X})$ [1].

Proof. This is all contained in the preceding proposition, except for the last point which is a direct calculation of the relevant Hirzebruch polynomial. □

6.2 An explicit functorial Lefschetz formula for cyclic diagonal actions

In this section we recall and formulate the Lefschetz-Riemann-Roch theorem of [Tho92] (in particular, Théorème 3.5) for regular schemes with the action of cyclic diagonalizable group (see below), and make it functorial. Recall that a regular scheme is to be understood as Nötherian, separated regular scheme.

Let S be a connected separated Nötherian scheme, and $T = \operatorname{spec} S[M]$ a diagonalizable group of finite type determined by an abelian group M. By [AG70a], I.4.7.3, a T-representation E on S is equivalent to a grading of weights $\bigoplus_{\lambda \in M} E_\lambda$ and so the K-groups of T-equivariant locally free sheaves are given by $K_*(S,T) = K_*(S) \otimes \mathbb{Z}[M]$, and to any prime ideal ρ of $\mathbb{Z}[M]$ consider

$$K_\rho = \{\lambda \in M | 1 - [\lambda] \in \rho \subseteq \mathbb{Z}[M]\}$$

[1] Here taking a Picard category P modulo a full sub Picard category P', is to be understood as the category P'' defined as follows: objects are those of P with the equivalence relation that $a \sim b$ if $a - b$ is in the essential image of P' in P. Morphisms are described by removing automorphisms coming from P'. It is clearly also a Picard category.

and associate to it the sub group-scheme $D_S(M/K_\rho) = T_\rho \subseteq T$. T_ρ is called the support of ρ and has the property that for any closed diagonalizable sub group-scheme $T' = D_S(M/K) \subseteq T$, ρ is an inverse image of $\mathbb{Z}[M] \to \mathbb{Z}[M/K]$ if and only if $T_\rho \subseteq T'$ (see loc.cit. Proposition 1.2). Given a T-equivariant S-scheme X, we denote by $i : X^\rho \to X$ the fixed-point scheme of X under T_ρ.

Theorem 6.2.1 ([Tho92]). *Keep the above assumptions and assume in addition that X is a regular scheme.*

(a) X^T is also a regular scheme.

(b) For any prime ideal ρ of $\mathbb{Z}[M]$, we have an isomorphism of localizations at ρ, $i_* : K_*(X^\rho, T)_{(\rho)} \simeq K_*(X, T)_{(\rho)}$.

(c) If N_i is the normal bundle to $i : X^\rho \to X$, the inverse to i_* is given by $(\lambda_{-1}N_i^\vee)^{-1} \otimes i^*$ (part of the statement is that $\lambda_{-1}(N_i^\vee)$ is invertible in $K_0(X^\rho, T)_{(\rho)}$).

(d) Suppose that Y is also a regular and T-equivariant S-scheme with $j : Y^\rho \to Y$ and that $f : X \to Y$ is a proper T-equivariant morphism with induced morphism $f' : X^\rho \to Y^\rho$, then we have the formula

$$Rf_*(\mathcal{F}) = Rf'_*\left((\lambda_{-1}f'^*N_j^\vee) \otimes (\lambda_{-1}N_i^\vee)^{-1} \otimes Li^*\mathcal{F}\right)$$

in $K_*(Y^\rho, T)_{(\rho)}$.

We digress for a short moment on the following case. Suppose $M = \mathbb{Z}^r \oplus \mathbb{Z}/n$ and $T = D_S(M) = \mathbb{G}_m^r \times \mu_n$, a "cyclic diagonalizable group" (compare [Tho92], Remarque 1.5 and [Seg68]) and let X be a finite-dimensional connected regular S-scheme with trivial T-action, and L a line-bundle on X with no trivial eigenvalues by the action of T. That is, L is given by a line bundle L_0 and a grading $\lambda \in M\backslash 0$. Let Φ_n be the n-th cyclotomic polynomial and $\rho = \rho_T : \ker[\mathbb{Z}[M] = \mathbb{Z}[T_0, T_1, \ldots, T_r]/(T_0^n - 1) \to \mathbb{Z}[T, T_1, \ldots, T_r]/(\Phi_n)]$ where the homomorphism is the canonical one. Then $K_\rho = \emptyset$ so $X^\rho = X^T$ and we can verify directly that the element $1 - L$ is invertible in $K_0(T, X)_{(\rho)}$. Indeed, first we see that when $L_0 = 1$ is the trivial bundle, $1 - \lambda$ is invertible since it is not zero in $\mathbb{Z}[M]_{(\rho)}/\rho$ which is just the field $\mathbb{Q}(\mu_n)(x_1, \ldots, x_r)$, the function field of the n-th cyclotomic field with r independent variables. Then, we calculate, $K_0(X^T, T)_{(\rho)}$:

$$\begin{aligned}\frac{1}{1-\lambda L_0} &= \frac{1}{1-\lambda+\lambda-\lambda L_0} = \frac{1}{1-\lambda}\frac{1}{1-\lambda/(1-\lambda)L_0} \\ &= \frac{1}{1-\lambda}\sum\left(\frac{\lambda}{1-\lambda}\right)^k(1-L_0)^k.\end{aligned}$$

Since the rank 0-part of K-theory of a regular scheme is nilpotent, more precisely $(1-L_0)^k = 0$ for $k > \dim X$, this sum is well-defined [2].

Definition 6.2.1.1. Fix a cyclic diagonalizable group-scheme T over $\operatorname{spec}\mathbb{Z}$ and denote by $\mathfrak{R}_T$ the category whose objects are regular T-schemes and morphisms are T-equivariant morphisms of T-schemes. For a T-scheme, denote by $|X| = X^T$ and $V(X,T)$ the virtual category of T-equivariant vector bundles on X denote by $V(X,T)_{(\rho)}$ the localization (c.f. B.0.14.1) of $V(X,T)$ at the prime ideal $\rho = \rho_T$ exhibited above and then at $\mathbb{Q}$. Also, denote by α_X the virtual bundle $\lambda_{-1}(N^\vee_{|X|/X})$ in $V(|X|,T)$.

The following lemma gives an explicit construction of the class $\lambda_{-1}(N^\vee_{|X|/X})^{-1}$ appearing in Thomason's result in a special case.

Lemma 6.2.2 (Inverting λ_{-1}). *Let T be a cyclic diagonalizable group-scheme corresponding to a finitely generated abelian group $M = \mathbb{Z}^r \times \mathbb{Z}/n$. Let X be a regular scheme with a trivial T-action and E a vector bundle on X with no trivial eigenvalues for the action of T. There is then a unique way of expressing the inverse bundle $\lambda_{-1}(E)^{-1}$ as a power-series in $V(X,T)_{(\rho)}$ such that it stable under base-change and compatible with exact sequences.*

Proof. First notice that there is an equivalence of categories $V(X,T)_{(\rho)} = (V(X)_{\mathbb{Q}} \otimes \mathbb{Q}[M])_{(\rho)}$ [3]. Let E be such a vector bundle. Since $|X| = X$ it is given by a grading $E = \oplus_{\lambda\in M,\lambda\neq 0} E_\lambda$. Then we propose that for a virtual bundle u_λ on X with pure weight $\lambda \neq 0$

$$\Lambda_{-1}(u_\lambda) = (1-\lambda)^{\operatorname{rk} u} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda - 1}\right)^k \gamma^k(u - \operatorname{rk} u).$$

If $k > \dim X + 1$, by Corollary 4.0.13, there is a completely canonical trivialization $\gamma^k(u) = 0$ in $V(X)_{\mathbb{Q}}$ and by truncating the powerseries at such a k these isomorphisms glue together to an object. By the same corollary, for $u = L$ a line bundle, $\gamma^k(L-1) = 0$ for $k > 1$ so that $\Lambda_{-1}(u_\lambda) = (1-\lambda)(1 + \frac{\lambda}{1-\lambda}(L-1)) = 1 - \lambda L = \lambda_{-1}(\lambda L)$. By Corollary 4.0.12, $\Lambda_{-1}((u+v)_\lambda) = \Lambda_{-1}(u_\lambda)\Lambda_{-1}(v_\lambda)$ so that $u \mapsto \lambda_{-1}(u_\lambda)$ defines

[2] In [GV02], results similar to those of [Tho92] were obtained, but with a different choice of localization (c.f. [GV02], Section 2). Clearly the results in this text can be reformulated with respect to such localizations.

[3] Here the tensor product $V \otimes M$ for a Picard category and a vector space M refers to the Picard category whose objects are formal finite sums $\sum v_i \otimes m_i$ with v_i an object of V and $m_i \in M$ and $\otimes$ is bilinear. Morphisms are determined by the condition $\operatorname{Hom}_{V\otimes M}((v \otimes m, v' \otimes m')) = \operatorname{Hom}(v, v')_V \otimes M$.

an additive functor from $V(X)$ to the Picard category $V(X,T)^*_{(\rho)}$ of invertible elements and for a vector bundle $E = \oplus_{\lambda\in M,\lambda\neq 0}E_\lambda$ with no trivial eigenvalues for the action of T, $\Lambda_{-1}(E) = \otimes_{\lambda\in M,\lambda\neq 0}\Lambda_{-1}(E_\lambda) = \lambda_{-1}(E)$. Now, returning to the case of a virtual bundle $u = \oplus_{\lambda\in M,\lambda\neq 0}u_\lambda$, we put $\lambda_{-1}(u) = \otimes_{\lambda\in M,\lambda\neq 0}\lambda_{-1}(u_\lambda)$. And thus for the same vector bundle we propose the element $\lambda_{-1}(E)^{-1} = \Lambda_{-1}(-E)$.
Next, we show that this constructed class is unique. By the splitting principle we can suppose that E is a line bundle. The scheme X is regular and thus has an ample family of line bundles (c.f. [P71], II 2.2.4). The argument of [Wei89], Proposition 4.4 then provides us with a T-equivariant torsor $\operatorname{spec} R \to X$ under a T-equivariant vector bundle, which can be chosen with trivial T-action. Then $\operatorname{spec} R$ is regular and $V(T,X)_{(\rho)} \to V(T,\operatorname{spec} R)_{(\rho)}$ is an equivalence of categories so we can suppose X is affine regular. But then the trivial bundle $\mathcal{O}_X$ is equivariantly ample and choosing a surjection $\mathcal{O}^n \twoheadrightarrow L^\vee$ we obtain that $L = i^*(\mathcal{O}(1))$ for a section $i : X \to \mathbb{P}^n_X$ to the natural projection $p : \mathbb{P}^n_X \to X$. Then $Ri_*(\Lambda_{-1}(L_\lambda)) = Ri_*Li^*\Lambda_{-1}(L_\lambda) = Ri_*\mathcal{O}_X \otimes \Lambda_{-1}(\mathcal{O}(1)_\lambda)$ and Ri_* is faithful by virtue of it having a right inverse Rp_*. But $\Lambda_{-1}(\mathcal{O}(1)_\lambda)$ is the unique pullback of $\Lambda_{-1}(\mathcal{O}(1)_\lambda)$ on $\mathbb{P}^n_{\mathbb{Z}}$ and in this case $V(\mathbb{P}^n_{\mathbb{Z}},T)_{(\rho)}$ has no nontrivial morphisms and the objects $\Lambda_{-1}(\pm\mathcal{O}(1)_\lambda)$ are uniquely determined. □

Let X be a regular T-scheme and denote by $X^T = |X|$ and suppose that $|X| \to X$ is a closed regular immersion. Assume in addition that the square

$$\begin{array}{ccc} |X| & \xrightarrow{i_X} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ |Y| & \xrightarrow{i_Y} & Y \end{array}$$

is Cartesian. First, also suppose that f is a closed regular immersion. We obtain a surjection $N^\vee_{X/Y} \twoheadrightarrow N^\vee_{|X|/|Y|}$ and the kernel is the excess bundle E. By [Ful98], Example 6.3.2, we also have a surjection $N^\vee_{|Y|/Y} \twoheadrightarrow N^\vee_{|X|/X}$ whose kernel is also E. Hence we obtain the formula

$$\lambda_{-1}(N^\vee_{|Y|/Y}) = \lambda_{-1}N^\vee_{|X|/X} \otimes \lambda_{-1}(E)$$

and since $N^\vee_{|Y|/Y}$ and $N^\vee_{|X|/X}$ have no non-trivial eigenvalues for the action of T so that

$$\lambda_{-1}(E) = (\lambda_{-1}N^\vee_{|X|/X})^{-1} \otimes \lambda_{-1}(N^\vee_{|Y|/Y})$$

where the $(\lambda_{-1}N^\vee_{|X|/X})^{-1}$ is defined as above. Via the projection formula we immediately see that the Lefschetz-formula above takes the form of an excess intersection-formula, valid without any localization. Note that since X is regular it has the resolution-property, and by [Tho87b] it also has the T-equivariant resolution property so the excess-formula of Theorem 5.1.2 can be applied to stacks of the form $[X/T]$ and it is clearly valid after localization.

In the rest of this section we put together the already constructed isomorphisms to obtain a functorial Lefschetz-formula. Fix T a cyclic diagonalizable group of finite type (i.e. of the form $D_{\mathrm{spec}\,\mathbb{Z}}(\mathbb{Z}^r \oplus \mathbb{Z}/n)$. Let X be a T-equivariant regular scheme, and denote by $|X|$ the fixed point set $i_X : |X| \to X$ of the action of T, and write α_X for the class $\lambda_{-1}(N_{|X|/X})$ in $V(X,T)_{(\rho)}$. Denote by $L_X : V(X,T)_{(\rho)} \to V(|X|,T)_{(\rho)}$ the functor $x \mapsto \alpha_X^{-1} \otimes Li_X^* x$ where α_X^{-1} is the class constructed above. Then

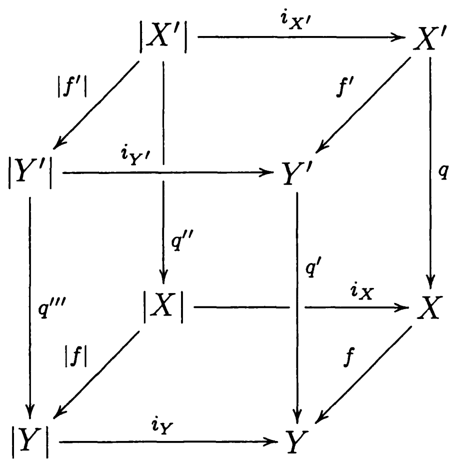
Lemma 6.2.3. *Let X be a regular T-scheme for a cyclic diagonalizable group. Then there are natural equivalences of functors $L_X Ri_* = \mathrm{id}$ and $Ri_* L_X = \mathrm{id}$. Moreover, for $q : X' \to X$ with induced morphism $|q| : |X'| \to |X|$ there is a natural isomorphism $L_{X'} Lq^* = \alpha_{X'/X} L_X |q|^*$ for $\alpha_{X'/X} = \lambda_{-1}(\ker[N^\vee_{|X|/X} \twoheadrightarrow N^\vee_{|X'|/X'}])$.*

Proof. By Corollary 5.1.3, there is a self-intersection formula $Li_X^* Ri_{X,*} = \alpha_X$ and thus naturally $\alpha_X^{-1} Li_X^* Ri_{X,*} = \mathrm{id}$. By Theorem 6.2.1, Ri_* induces a bijection on automorphism-groups and surjection on objects and is thus an equivalence of categories and to exhibit a natural isomorphism $Ri_{X,*} L_X = \mathrm{id}$ it suffices to establish $Ri_{X,*} L_X Ri_{X,*} = Ri_{X,*}$. We can construct such an isomorphism using the isomorphism $Li_X^* Ri_{X,*} = \alpha_X$ already established. Given $q : X' \to X$, then $\alpha_{X'/X} \alpha_X^{-1} = \alpha_{X'}^{-1}$ and we thus define the isomorphism in the second part of the lemma. □

Corollary 6.2.4. *Given a proper morphism $f : X \to Y$, there is a canonical isomorphism of functors $\Upsilon_f : R|f|_* L_X \to L_Y Rf_*$.*

Proof. Apply the above explicit equivalence of categories to the composition of functors $R|f|_* Ri_{Y,*} = Ri_{X,*} Rf_*$. □

Given a projective morphism $f : X \to Y$ and any morphism $q : Y' \to Y$, both in $\mathfrak{R}_T$, there is the question of how the just established isomorphism transforms under base-change. Consider the cube



with commutative squares. Since the cube is not transversal in the sense of Theorem 5.1.2 we cannot directly apply the functorial excess-formula to calculate this. We proceed as follows. For a projective morphism $f : X \to Y$ in $\mathfrak{R}_T$ we can define the cotangent complex which is a two-term complex of equivariant vector bundles canonically determined up to canonical quasi-isomorphism, $L_{X/Y} = L_f$. If f is a closed immersion

$$L_{X/Y} = [N \to 0]$$

for N the conormal bundle and if f is smooth

$$L_{X/Y} = [0 \to \Omega_{X/Y}].$$

For a composition of projective local complete intersection morphism $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is an exact triangle

$$L_{X/Y} \to L_{Y/Z} \to L_{X/Z} \to L_{X/Y}[1]$$

(c.f. [P71], VIII, 2, the arguments are easily made equivariant). In the above setting, we obtain exact triangles

$$L_{|X|/X} \to L_{X/Y} \to L_{|X|/Y} \to L_{|X|/X}[1]$$

and

$$L_{|X|/|Y|} \to L_{|Y|/Y} \to L_{|X|/Y} \to L_{|X|/|Y|}[1].$$

Define E (resp. E') to be the homology of $[L_{X/Y} \to L_{X'/Y'}]$ (resp. $[L_{|X|/|Y|} \to L_{|X'|/|Y'|}]$. Then E is the excess-bundle of the Cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

whereas E' is represented by a complex whose $\lambda_{-1}(E') = \alpha_{E'}$ can be defined by $\Lambda_{-1}(L_{|X|/|Y|} - L_{|X'|/|Y'|})$ which is seen to be well-defined. Moreover, using excess and the isomorphism $i_{X,*}L_X = \text{id}$ one constructs an isomorphism

$$R|f'|_*(\alpha_{E'} \otimes Lq'''^*) = Lq''^* R|f|_* \tag{6.1}$$

and by functoriality we have an isomorphism $\alpha_{Y'/Y} \otimes \lambda_{-1}(E) = \alpha_{X'/X} \otimes \lambda_{-1}(E')$. We then have two isomorphisms

$$Lq'''^* L_Y Rf_* = \alpha_{Y'/Y} L_{Y'} Lq'^* Rf_* = \alpha_{Y'/Y} L_{Y'} Rf'_*(\lambda_{-1}(E) \otimes Lq^*) \tag{6.2}$$

$$\begin{aligned} Lq'''^* R|f|_* L_X &= R|f'|_*(\alpha_{E'} \otimes Lq''^* L_X) \\ &= R|f'|_*(\alpha_{E'} \otimes \alpha_{X'/X} \otimes L_{X'} Lq^*) \\ &= R|f'|_*(\lambda_{-1}(E) \otimes \alpha_{Y'/Y} \otimes L_{X'} Lq^*) \\ &= \alpha_{Y'/Y} R|f'|_* L_{X'}(\lambda_{-1}(E) \otimes Lq^*)) \end{aligned} \tag{6.3}$$

From the definition it is not difficult to verify that these two isomorphisms are compatible with the isomorphisms $L_Y Rf_* = R|f|_* L_X$ and $L_{Y'} Rf'_* = R|f'|_* L_{X'}$. The theorem is that these properties essentially characterize the Lefschetz-isomorphism:

Theorem 6.2.5 (Functorial Lefschetz-Riemann-Roch for cyclic diagonalizable groups). *Fix positive integers r and n and let $M = \mathbb{Z}^r \oplus \mathbb{Z}/n$ and let $T = \operatorname{spec} \mathbb{Z}[M]$ be the associated diagonalizable group. Consider the category $\mathfrak{R}_T^p$ of T-equivariant regular schemes and morphisms given by $f : X \to Y$ a T-equivariant morphism of regular schemes which is equivariantly projective, i.e. factors equivariantly into a projective bundle $X \hookrightarrow \mathbb{P}(E) \to Y$ for a closed immersion $X \hookrightarrow \mathbb{P}(E)$ and an equivariant vectorbundle E on Y. Denote the induced morphism on fixed points $|f| : |X| \to |Y|$. Then there is a family, unique up to unique isomorphism, of functor-isomorphisms for f a morphism in $\mathfrak{R}_T^p$,*

$$\Upsilon_f : R|f|_* L_X \to L_Y Rf_*$$

satisfying the following compatibilities:

(a) Stability under composition: Given $X \xrightarrow{f} Y, Y \xrightarrow{g} Z$ in $\mathfrak{R}_T^p$, the composition

$$R(|g||f|)_* L_X \xrightarrow{\Upsilon_f} R|g|_* L_Y Rf_* \xrightarrow{\Upsilon_g} L_Z R(gf)_*$$

is Υ_{gf}.

(b) *Stability under base-change in $\mathfrak{R}_T$; if $q' : Y' \to Y$ is an equivariant morphism such that $q : X' \to Y'$ is also in $\mathfrak{R}_T$, the isomorphisms (6.2) and (6.3) intertangle to give a commutative diagram:*

$$\begin{array}{ccc} Lq'''^* L_Y Rf_* & \xrightarrow{(6.2)} & \alpha_{Y'/Y} L_{Y'} Rf'_*(\lambda_{-1}(E) \otimes Lq^*) \\ \downarrow{\scriptstyle \Upsilon_f} & & \downarrow{\scriptstyle \Upsilon_{f'}} \\ Lq'''^* R|f|_* L_X & \xrightarrow{(6.3)} & \alpha_{Y'/Y} R|f'|_* L_{X'}(\lambda_{-1}(E) \otimes Lq^*) \end{array}$$

(c) *Suppose Z is also in $\mathfrak{R}_T$, and $h : Z \to Y$ is closed regular immersion T-equivariant immersion and $f : X \to Y$ is a morphism $\mathfrak{R}_T$ whose image is disjoint with that of Z. Then both sides of Υ_f are canonically trivialized and we require that Υ respects these trivializations.*

(d) *The isomorphism is compatible with the projection-formula, i.e. the diagram*

$$\begin{array}{ccc} R|f|_* L_X(u \otimes Lf^* v) & \longrightarrow & L_Y Rf_*(u \otimes Lf^* v) \\ \downarrow & & \downarrow \\ R|f|_* L_X u \otimes L|f|^* Li_Y^* v & & L_Y(Rf_* u \otimes v) \\ \downarrow & & \downarrow \\ R|f|_* L_X u \otimes Li_Y^* v & \longrightarrow & L_Y Rf_* u \otimes Li_Y^* v \end{array}$$

commutes.

Remark 6.2.5.1. The proof proceeds as in the case of the functorial excess-formula and also follows the corresponding proof for Grothendieck-Riemann-Roch in the unpublished manuscript [Fra], which uses a reduction to the arithmetic case. This in turn is an adaption of the usual proof of [P71] to the functorial situation. Clearly the isomorphism exists in greater generality by Corollary 6.2.4, the stronger statement is the uniqueness-property. It is possible to establish a similar isomorphism for more general diagonalizable groups but one has to introduce a normalization-condition analogous to that of the rough excess-isomorphism in 5.1.

Proof. The isomorphism has already been constructed and the properties follow either from construction or from the discussion. We review its construction in the case of a closed immersion and a projective bundle projection to possibly clarify the situation.
Suppose $i : X \to Y$ is a closed immersion of regular T-schemes. Thus we can

apply the excess-formula to algebraic stacks of the form $[X/T]$ which gives an isomorphism, by the arguments preceding the theorem,

$$\alpha_Y^{-1} \otimes Li_Y^* Rf_* = R|f|_* \left(\alpha_X^{-1} \otimes Li_X^*\right).$$

It moreover satisfies the given conditions by virtue of them being satisfied for the excess-isomorphism. This also gives a description of Lefschetz for closed immersions via a rough excess-argument.
Now, suppose $f : \mathbb{P}(N) \to Y$ is a projective bundle projection for N a T-equivariant vector bundle on Y, whose restriction $N|_{|Y|}$ is diagonalized to $\oplus_{\lambda \in M} N_\lambda$. Then $|\mathbb{P}_{|Y|}(\oplus N_\lambda)| = \coprod_{\lambda \in M} \mathbb{P}_{|Y|}(N_\lambda)$ (c.f. [KK01], Proposition 2.9). Thus we are given a diagram

$$\begin{array}{ccccc} \coprod_{\lambda \in M} \mathbb{P}_{|Y|}(N_\lambda) & \longrightarrow & \mathbb{P}_{|Y|}(N|_{|Y|}) & \longrightarrow & \mathbb{P}_Y(N) \\ & \searrow & \downarrow & & \downarrow f \\ & & |Y| & \longrightarrow & Y \end{array}$$

with Cartesian square. We treat first the left triangle and suppose that $Y = |Y|$. Denote by $i_\lambda : \mathbb{P}_Y(N_\lambda) \to \mathbb{P}_Y(N)$ the closed immersion, $i = \coprod i_\lambda$: $\coprod \mathbb{P}_Y(N_\lambda) \to \mathbb{P}_Y(N)$, $|f|_\lambda = f i_\lambda$ and $|f| = fi$. For any virtual bundle x, we need to construct a functorial isomorphism

$$Rf_*(x) = \sum R|f|_{\lambda,*}(\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda{}^* x).$$

We can assume x is of the form $\sum_{k=0}^{n-1} Lf^* a_k \otimes \mathcal{O}(-k)$ for $n = \operatorname{rk} N$. The right hand side is thus isomorphic to, via the projection formula,

$$\sum_{k,\lambda} R|f|_{\lambda,*}(\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* \mathcal{O}(-k)) \otimes a_k). \tag{6.4}$$

By Corollary 5.1.3, there is a canonical isomorphism of functors $Li_\lambda^* Ri_{\lambda,*}(-) = \lambda_{-1} N_\lambda \otimes (-)$ and thus $\lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* Ri_{\lambda,*} = \mathrm{id}$. Moreover, it is known that $Ri_{\lambda,*}$ is an equivalence of categories. Thus any x is of the form $Ri_{\lambda,*} y$ and we deduce the isomorphism

$$Ri_{\lambda,*} \lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* x = \lambda_{-1}(N_\lambda)^{-1} \otimes Li_\lambda^* Ri_{\lambda,*} y = Ri_{\lambda,*} y = x.$$

Applying this isomorphism to (6.4) we obtain $\sum_k Rf_*(\mathcal{O}(-k)) \otimes a_k$. For $k > 0$, $Rf_*(\mathcal{O}(-k)) = 0$ and for $k = 0$, $Rf_*(\mathcal{O}) = \mathcal{O}$. In general we compose this with the excess-isomorphism for the Cartesian square. For a projective bundle-projection it is also true that $E = E' = 0$ for base-changes. That this respects composition is done exactly as in the case of the excess-isomorphism.

We are left to show that the morphism is unique. We can clearly suppose Y is connected and treat the cases of a closed immersion and projective bundle projections separately. In the case of a closed immersion it is immediate to verify that all the schemes that arise in the case of a deformation to the normal cone are regular T-schemes so thus we stay in the correct category. The essential point is the trivialization condition (c) to exclude unwanted factors. Then a deformation to the normal cone-argument analogous to that of argument related to the excess-formula shows that we are reduced to the case of an embedding $i : X \to \mathbb{P}(N)$ for some equivariant vector bundle N of rank n defined by an inclusion $L \subset N$ for some line bundle N. Let $p : \mathbb{P}_X(N) \to X$ be the projection. Then $|\mathbb{P}_X(N)| = \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda})$ (loc.cit.). For a virtual bundle u the isomorphism $u = Li^*Lp^*u$ and compatibility with the projection formula shows that we are reduced to showing that $\Upsilon_i(\mathcal{O})$ is uniquely determined.
We apply the base-change-property to the Cartesian diagram

$$\begin{array}{ccc} |X| & \longrightarrow & \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda}) \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_{P_X(N)}} \\ X & \longrightarrow & P_X(N) \end{array}$$

and see that $\Upsilon_i(\mathcal{O})$ is determined by the functor $\Upsilon_{|i|}$ and hence by $\Upsilon_{|i|}(\mathcal{O})$. Since $|\mathbb{P}_X(N)| = \coprod_{\lambda \in M} \mathbb{P}_{|X|}(N|_{|X|,\lambda})$ and we can assume Y to be connected we can assume furthermore that $Y = \mathbb{P}_{|X|}(N_\lambda)$ for some fixed $\lambda \in M$ and vector bundle $N = N_\lambda$ on $|X| = X$ with single grading λ. We need to verify that the Lefschetz-isomorphism $Rf_*\mathcal{O} \to Rf_*\mathcal{O}$ in this case necessarily is the identity. By 6.1.1 there exists a torsor $t : \operatorname{spec} R \to X$ under some vector bundle on X which we endow with the trivial action. Then by the affine bundle theorem of [Tho87a] there are equivalences of categories Lt^* : $V(X,T) \to V(\operatorname{spec} R, T)$ and thus $V(X,T)_{(\rho)} \to V(\operatorname{spec} R,T)_{(\rho)}$ as well. Similarly there is an equivalence $V(\mathbb{P}_X(N),T)_{(\rho)} \to V(\mathbb{P}_{\operatorname{spec} R}(t^*N),T)_{(\rho)}$ and by the base-change-property we can assume that X is in fact affine. Then we can choose a surjection $\mathcal{O}^r \to N^\vee$ so that we have a flag $L \subset N \subset \mathcal{O}^r$ which is concentrated on the single grading λ. Consider the Grassmannian $\tau : G = \mathrm{Gr}_{1,n,r} \to X$ of flags $L' \subset N' \subset \mathcal{O}^r$ on X with L' (resp. N') have rank 1 (resp. n). Then G is regular, has trivial T-action and the flag $L \subset N \subset \mathcal{O}^r$ defines a section $j : X \to G$. If $\mathcal{L} \subset \mathcal{N} \subset \mathcal{O}^r$ is the universal flag on G, $p^*j^*\mathcal{L} \subset p^*j^*\mathcal{N}$ similarly define a section $j' : \mathbb{P}(N) \to \mathbb{P}(\mathcal{N})$ so that we have a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{P}(N) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ G & \xrightarrow{i'} & \mathbb{P}(\mathcal{N}) \end{array}.$$

Then compatibility with the projection-formula shows that

$$Rj'_*\Upsilon_i = Rj'_*(\mathcal{O}_X) \otimes \Upsilon_{i'}.$$

However, Rj'_* is faithful since $R\tau_* Rj'_* = \mathrm{id}$ so Υ_i is determined by $\Upsilon_{i'}$. The varieties G and $\mathbb{P}(\mathcal{N})$ are equivariantly defined over $\operatorname{spec}\mathbb{Z}$. This shows that $\Upsilon_{i'}$ is obtained by base-change from $X \to \operatorname{spec}\mathbb{Z}$ so we can assume that $X = \mathbb{Z}$. In this case uniqueness is tautological since G is cellular so $K_1(G,T)_{(\rho),\mathbb{Q}}$ is a free $K_1(\mathbb{Z},T)_{(\rho),\mathbb{Q}}$-module and (c.f. beginning of this section)

$$K_1(\mathbb{Z},T)_{(\rho),\mathbb{Q}} = K_1(\mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}[M]_{(\rho)} = 0$$

since $K_1(\mathbb{Z}) = \pm 1$.

Now, consider the case of a projective bundle projection $p : \mathbb{P}_Y(N) \to Y$ for some T-equivariant vector bundle N on Y. Arguing as above, we reduce to the case of $Y = |Y|$ being an affine scheme and the case of a virtual bundle of the form $\mathcal{O}(-i)$ on $\mathbb{P}_Y(N)$. If $N = \oplus_{\lambda\in M} N_\lambda$ with N_λ of rank n_λ, choose a locally split injection $N \subset \mathcal{O}^r$ which restricts to $N_\lambda \subset \mathcal{O}_\lambda^{r_\lambda}$ on each grading and consider the Grassmannian $G = \mathrm{Gr}_{n,r}$ of flags $N' \subset \oplus_{\lambda\in M}\mathcal{O}_\lambda^{r_\lambda}$ with N' of rank n. In a way similar to the case of closed immersions we reduce to the case of the diagram

$$\begin{array}{ccc} \coprod_{\lambda\in M} \mathrm{Gr}_{n_\lambda,r_\lambda}(\mathcal{N}_\lambda) & \longrightarrow & \mathbb{P}_G(\mathcal{N}) \\ \downarrow & & \downarrow \\ \coprod_{\lambda\in M} \mathrm{Gr}_{n_\lambda,r_\lambda} & \longrightarrow & \mathrm{Gr}_{n,r} \end{array}$$

where $\mathcal{N}$ is the universal rank n subbundle of $\oplus_{\lambda\in M}\mathcal{O}_\lambda^{r_\lambda}$ on G. Again this diagram is equivariantly defined over $\operatorname{spec}\mathbb{Z}$ and we conclude as before. □

6.3 An Adams-Deligne-Riemann-Roch formula

In this section we state and prove a functorial Adams-Riemann-Roch formula. We will continuously work in the category $V(X)_{\mathbb{Q}}$ of virtual vector bundles

on a scheme X, which comes equipped with Adams operations and various other operations (c.f 3.3.1). This coincides with the cohomological virtual category $W(X)$, and whenever X is regular it also comes equipped with various additional operations considered in 4 which we shall use freely. Also recall that a regular scheme is a separated, Nötherian regular scheme.
We recall to the reader that one formulation of the Adams-Riemann-Roch formula is as follows (c.f. [WF85], V, Theorem 7.6). Suppose that $f : X \to Y$ is a projective local complete intersection morphism of schemes and that Y has an ample family of line bundles so that any coherent sheaf is the quotient of a coherent locally free sheaf. Also, define Ω_f to be the class of the cotangent-bundle of f, and $\theta_{k,f} = \theta_k(\Omega_f)$ where θ_k is the unique multiplicative characteristic class in $K_0(X)$ such that for a line bundle L, $\theta_k(L) = 1 + L + \ldots + L^{k-1}$. Then for any $k \geq 1$, we have a commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\theta_{k,f}^{-1} \otimes \Psi^k} & K_0(X)_{\mathbb{Q}} \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ K_0(Y) & \xrightarrow{\Psi^k} & K_0(Y)_{\mathbb{Q}} \end{array}$$

To formulate the functorial version of the Adams-Riemann-Roch formula, recall the following Lemma which is a corollary of Corollary 6.1.3 while noticing that for an regular scheme, $W(X) = V(X)_{\mathbb{Q}}$:

Lemma 6.3.1. *There is a unique family of functors, determined up to unique isomorphism, on the category of regular schemes, stable under base-change, such that for a regular scheme X*

$$\theta_k : V(X) \to V(X)_{\mathbb{Q}}$$

such that θ_k is a determinant functor $\mathbf{P}(X) \to V(X)^*_{\mathbb{Q}}$ and for a line bundle L on X there exists an isomorphism

$$\theta_k(L) = 1 + L + \ldots + L^{k-1} = 1 + L + \ldots + L^{k-1} = \sum_{j=0}^{k-1} a_{j,k}(L-1)^j$$

where $a_{j,k} = \sum_{i=j}^{k-1} \binom{i}{j}$.

Now, given a projective morphism $f : X \to Y$ of regular schemes factoring as $X \overset{i}{\hookrightarrow} P \overset{p}{\to} Y$ for a closed immersion i and smooth morphism p, define $\theta_k^{-1}(\Omega_f)_{i,p}$ to be the virtual bundle $\theta_k(N_i^\vee - i^*\Omega_{P/Y})$. We analyze its properties before stating the functorial Adams-Riemann-Roch-theorem. The usual proof in [P71], VIII, Proposition 2.2, shows that the virtual bundle $N_i^\vee - i^*\Omega_{P/Y}$ glues together to a virtual bundle Ω_f which is independent of factorization . We define $\theta_{k,f}^{-1} := \theta_k^{-1}(\Omega_f)$ which is an object dependent only on f determined up to unique isomorphism. Moreover, suppose $q : Y' \to Y$ is any morphism such that $f' : X' = Y' \times_Y X \to Y'$ is also a projective morphism of regular schemes. Then it is clear from the definition that there is a canonical isomorphism $Lq^*\theta_{k,f}^{-1} = \theta_{k,f'}^{-1} \otimes \theta_k E$ with E the excess bundle of the diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}.$$

One deduces from the splitting principle an isomorphism

$$\theta_k E \otimes \lambda_{-1}(E) = \Psi^k(\lambda_{-1}(E)) \tag{6.5}$$

and thus an isomorphism

$$Lq^*\theta_{k,f}^{-1} \otimes \lambda_{-1}(E) = \theta_{k,f'}^{-1} \otimes \Psi^k(\lambda_{-1}(E)). \tag{6.6}$$

Finally, for a composition of projective morphisms of regular schemes, $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is also a canonical isomorphism

$$\theta_{k,gf}^{-1} = \theta_{k,g}^{-1} \otimes Lf^*\theta_{k,g}^{-1} \tag{6.7}$$

(c.f. [P71], VIII, Proposition 2.6).

Theorem 6.3.2 (Functorial Adams-Riemann-Roch). *Suppose $f : X \to Y$ is a projective morphism of regular schemes (automatically a local complete intersection), and $k \geq 1$. There is a unique family of functorial isomorphisms*

$$\psi_{k,f} : \Psi^k Rf_* \to Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k) \tag{6.8}$$

characterized further by the following properties:

(a) Stability under composition of projective local complete intersection morphisms: That is, for a composition of projective morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the isomorphism

$$\begin{aligned} R(gf)_*(\theta_{k,gf}^{-1} \otimes \Psi^k(u)) &\overset{(6.7)}{=} R(gf)_*(\theta_{k,g}^{-1} \otimes Lf^*\theta_{k,g}^{-1} \otimes \Psi^k(u)) \\ &= Rg_*(\theta_{k,g}^{-1} \otimes Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u))) \\ &\overset{\psi_{k,f}}{=} Rg_*(\theta_{k,g}^{-1}\Psi^k Rf_*(u) \\ &\overset{\psi_{k,g}}{=} \Psi^k Rg_* Rf_*(u) \\ &= \Psi^k R(gf)_* u \end{aligned}$$

is $\psi_{k,gf}$.

(b) *Stability under the projection-formula: That is, the diagram*

$$\begin{array}{ccc}
\Psi^k Rf_*(u \otimes Lf^*v) & \longrightarrow & Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u \otimes Lf^*v)) \\
\downarrow & & \downarrow \\
\Psi^k Rf_*(u) \otimes \Psi^k(v) & \longrightarrow & Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \otimes \Psi^k(v)
\end{array}$$

commutes where the horizontal isomorphisms are given by $\Psi_{k,f}$ *and the vertical isomorphisms are given by the projection-formula.*

(c) *Compatibility with base-change and excess: Suppose* $q : Y' \to Y$ *is a morphism such that the induced morphism* $f' : X' = Y' \times_X Y \to X$ *is also a projective morphism of regular schemes, and denote by* $q' : X' \to X$ *the morphism obtained by base-change, and denote by* E *the associated excess bundle. Then the diagram*

$$\begin{array}{ccc}
Lq^*\Psi^k Rf_*(u) & \xrightarrow{Lq^*\psi_{k,f}} & Lq^*Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \\
\downarrow{\scriptstyle \text{Theorem } 5.1.2} & & \downarrow{\scriptstyle \text{Theorem } 5.1.2} \\
\Psi^k Rf'_*(\lambda_{-1}(E) \otimes Lq'^*u) & & Rf'_*(\lambda_{-1}(E) \otimes Lq'^*(\theta_{k,f}^{-1} \otimes \Psi^k(u)) \\
\downarrow{\scriptstyle \psi_{k,f'}} & & \downarrow \\
Rf'_*(\theta_{f',k}^{-1} \otimes \Psi^k(\lambda_{-1}(E) \otimes Lq'^*u)) & & Rf'_*(\lambda_{-1}(E) \otimes Lq'^*\theta_{k,f}^{-1} \otimes \Psi^k Lq'^*u) \\
\downarrow & \nearrow{\scriptstyle (6.6)} & \\
Rf'_*(\theta_{f',k}^{-1} \otimes \Psi^k(\lambda_{-1}(E)) \otimes \Psi^k Lq'^*u) & &
\end{array}$$

commutes, where the diagonal morphism is deduced from the isomorphism (6.6). In particular, for $k = 1$ *this reduces to the excess-isomorphism of Theorem 5.1.2 and if* q *is flat the isomorphism strictly commutes with pullback.*

(d) *Suppose we are given a closed immersion* $h : Z \to Y$ *whose image in* Y *doesn't intersect that of* X. *Then both* $Rh_*(\mathcal{O}_Z) \otimes \Psi^k Rf_*(u)$ *and* $Rh_*(\mathcal{O}_Z) \otimes Rf_*(\theta_{k,f}^{-1} \otimes \Psi^k(u))$ *are canonically trivialized. We demand that the isomorphism* $\psi_{k,f}$ *interchanges these trivializations. We don't require* Z *to be regular.*

Proof. The proof proceeds as in the case of the functorial excess-formula and also closely follows the corresponding proof for Grothendieck-Riemann-Roch in the unpublished article [Fra], which uses a reduction to the arithmetic case. We indicate the necessary changes from the case of the excess-isomorphism. Suppose $i : X \to Y$ is a regular closed immersion of regular schemes. Given a Koszul resolution built out of $s : N^\vee \to \mathcal{O}_Y$ of $\mathcal{O}_X$, one first defines a rough Adams-Riemann-Roch-isomorphism for closed immersions as follows: Let L be a line bundle on X and $\mathcal{L}$ an extension of X to a line bundle on Y. As in (6.5) we have a natural isomorphism $\theta_k(N^\vee) \otimes \lambda_{-1}(N^\vee) \simeq \Psi^k(\lambda_{-1}(N^\vee))$. Then we have an isomorphism

$$\Psi^k i_*(L) \simeq \Psi^k(\lambda_{-1}N^\vee \otimes \mathcal{L}) \simeq \theta_k(N^\vee) \otimes \lambda_{-1}N^\vee \otimes \mathcal{L}^{\otimes k}.$$

This is, by another projection-formula-argument, isomorphic to $i_*(\theta_{i,k}(N_i^\vee) \otimes \Psi^k L)$.

One needs to verify that the deformation to the normal cone-argument can be used to reduce to this case. The same proof goes through with the remark that if $X \to Y$ is a closed regular immersion of regular schemes, then the blow-up of Y in X is a regular scheme with regular exceptional divisor. Indeed, the exceptional divisor is simply $\mathbb{P}_X(N)$ so thus regular. It is a regular Cartier divisor in $Bl_X Y$ and so forces $Bl_X Y$ to be regular (see [AG67], 19.1.1). Thus we stay in the correct category of regular schemes while deforming.

We are left to show that the morphism is unique. A deformation to the normal cone-argument (which is justified by the above reasoning) shows that we are reduced to the case of an embedding $i : X \to \mathbb{P}(N)$ for some vector bundle N of rank d defined by an inclusion $L \subset N$ for some line bundle L. Let $p : \mathbb{P}(N) \to X$ be the projection. By stability under the projection formula we have a commutative diagram

$$\begin{array}{ccccc} Ri_*(\Psi^k(E) \otimes \theta_{i,k}) & \longrightarrow & Ri_*(\Psi^k(Li^*Lp^*E) \otimes \theta_{i,k}) & \longrightarrow & \Psi^k(Lp^*E) \otimes Ri_*(\theta_{i,k}) \\ \downarrow & & \downarrow & & \downarrow \\ \Psi^k(Ri_*E) & \longrightarrow & \Psi^k(Ri_*Li^*Lp^*E) & \longrightarrow & \Psi^k(Lp^*E) \otimes \Psi^k(Ri_*\mathcal{O}_X) \end{array}$$

and thus we can assume that $E = \mathcal{O}_X$. By Theorem 6.1.1, there exists an affine torsor $T \to Y$ under some vector bundle E on Y, and base-change to this variety is an equivalence of virtual categories so we can assume X and Y are affine. We loose the assumption that $Y = \mathbb{P}(N)$ but gain that X is affine. By another deformation to the normal cone argument we can again assume $Y = \mathbb{P}(N)$. Since X is affine $N^\vee$ is generated by global sections $\mathcal{O}_X^n \to N^\vee$ for some n and we have an injection $N \subset \mathcal{O}_X^n$. Consider the flag variety $G = \mathrm{Gr}_{\mathrm{n,d,1,X}}$ of locally split flags $L' \subset N' \subset \mathcal{O}^d$, with L' and N' of rank 1 and d respectively. The flag $L \subset N \subset \mathcal{O}^d$ defines a section $s : X \to G$ to

$r : G \to X$. If $\mathcal{L} \to \mathcal{N} \to \mathcal{O}^d$ denotes the universal flag on G we have the following commutative diagram

$$\begin{array}{ccc} G = \mathrm{Gr}_{n,d,1} & \xrightarrow{i'} & \mathbb{P}_G(\mathcal{N}) \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle r} & & {\scriptstyle s'}\uparrow\downarrow{\scriptstyle r'} \\ X & \xrightarrow{i} & \mathbb{P}(N) = \mathbb{P}(s^*\mathcal{N}) \end{array}$$

with the section s' is defined by the flag $r^*L \subset r^*N$ on G. Then $Rr_*\mathcal{O}_G = \mathcal{O}_X$ and $Rr'_*\mathcal{O}_{\mathbb{P}_G(\mathcal{N})} = \mathcal{O}_{\mathbb{P}(N)}$. Then there is no excess for the Cartesian diagram of closed immersions and so $Ls^*\theta_{k,i'}^{-1} = \theta_{k,i}^{-1}$ and $Ls^*Ri_* = Ls'^*Ri'_*$. We have the commutative diagrams

$$\begin{array}{ccc} Rs'_*(Ri_*(\theta_{k,f}^{-1})) & \xrightarrow{Rs'_*(\psi_{k,i})} & Rs'_*(\Psi^k Ri_* 1) \\ \downarrow & & \downarrow \\ Rs'_* Ls'^* Ri'_*(\theta_{k,i'}^{-1}) & \longrightarrow & Rs'_* Ls'^*(\Psi^k Ri'_* 1) \\ \downarrow & & \downarrow \\ Ri'_*(\theta_{k,i'}^{-1}) \otimes Rs'_*(1) & \xrightarrow{\psi_{k,i'} \otimes Rs'_*(1)} & \Psi^k Ri'_*(1) \otimes Rs'_*(1) \end{array}$$

where the upper square is the base-change-property and the lower square is the natural transformation associated to the projection-formula. Since $Rr'_*Rs'_* = \mathrm{id}$, Rs'_* is faithful and thus $\psi_{k,i}$ is determined by $\psi_{k,i'} \otimes Rs'_*(1)$ which is determined by $\psi_{k,i'}$. However, X is regular and we can suppose without loss of generality that it is connected and so integral. As such it has a morphism to $\operatorname{spec}\mathbb{Z}$ and we may assume that X is $\operatorname{spec}\mathbb{Z}$. In this case the virtual category under consideration doesn't have any non-trivial automorphisms, since $K_1(\mathbb{Z}) \pm 1$ and we tensor with $\mathbb{Q}$. Thus the isomorphism in question is uniquely rigidified in the case of a closed immersion.

Suppose now that $f : \mathbb{P}(N) = X \to Y$ is a projective bundle projection for some vector bundle N on Y of rank d. By Theorem C.0.20 we can assume $u = \sum_{i=0}^{d-1} Lf^*u_i \otimes \mathcal{O}(-i)$ for virtual bundles u_i on Y. By the projection-formula and the multiplicative property of the Adams operations of Corollary 3.3.2, we only need to define the isomorphism for bundles of the type $u = \mathcal{O}(-i)$. We calculate both sides. First, $\Psi^k Rf_*(\mathcal{O}(-i)) = 0$ if $i > 0$ and isomorphic to 1 if $i = 0$. On the other hand, there is a universal exact sequence

$$0 \to \Omega_{X/Y} \to f^*N^\vee \otimes \mathcal{O}(1) \to \mathcal{O}_X \to 0$$

on X. Then we obtain the isomorphism $\theta_k(f^*N^\vee \otimes \mathcal{O}(1)) = \theta_k(\mathcal{O}_X) \otimes \theta_k(\Omega_{X/Y}) = k\theta_k(\Omega_{X/Y})$. Thus we want to construct isomorphisms

$$Rf_*(\theta_k(f^*N^\vee \otimes \mathcal{O}(1)) \otimes \mathcal{O}(-ik)) = \begin{matrix} 1/k & \text{if } i = 0 \\ 0 & \text{if } i = 1, \ldots, d-1 \end{matrix}.$$

We need the following lemma:

Lemma 6.3.3 ([WF85], II, Lemma 3.3). *Let R be a commutative ring in which k is invertible. For $a, b \in R$, define $a \oplus b = (1+a)(1+b) - 1$ and for an integer j define $[j]a = a \oplus a \oplus \ldots \oplus a$ taken j times. Let $a_1, \ldots, a_d, Z$ be independent variables and define*

$$R[[a_1, \ldots, a_d, Z]] \ni F_{n,k}(Z) = (1+Z)^{nk} \prod_{j=1}^{d} \frac{Z \oplus a_j}{[k](Z \oplus a_j)}.$$

There exists unique elements $b_0^{i,k}, \ldots, b_d^{i,k} \in R[[s_1, \ldots, s_d]]$ (where s_j are the elementary symmetric functions in the a_j) such that

$$F_{i,k}(Z) \equiv b_0^{i,k} + b_1^{i,k} Z + \ldots + b_{d-1}^{i,k} Z^{d-1} \text{ mod } \prod_{j=1}^{d} Z \oplus a_j$$

and we have

$$\sum_{v=0}^{d-1} (-1)^v b_v^{i,k} = \begin{matrix} 1/k & \text{if } n = 0 \\ 0 & \text{if } i = 1, \ldots, d-1 \end{matrix}.$$

In particular, this result holds as an identity on $R = K_0(\mathbb{P}(N)_X)_\mathbb{Q}$ whenever inserting $Z = (\mathcal{O}(-1) - 1)$ and $s_j = \gamma^j(N - d)$ and by ridigity this lifts to the virtual category. Also, by rigidity there is a canonical isomorphism $F_{i,k}(Z)$ and $\theta_k(f^*N^\vee \otimes \mathcal{O}(1)) \otimes \mathcal{O}(-ik)$ and we define the Adams-Riemann-Roch-isomorphism as the isomorphism interchanging the two calculations we have done above. By functoriality of the rigidity-construction this isomorphism clearly satisfies all the proposed properties, except possibly the one concerning compatibility of composition of morphisms.

We now show uniqueness for $f : X = \mathbb{P}(N) \to Y$ a projective bundle projection for some vector bundle N on Y of rank d. By Theorem C.0.20 we can assume $u = \sum_{i=0}^{d-1} f^*u_i \otimes \mathcal{O}(-i)$ for virtual bundles u_i on Y. By additivity and compatibility with the projection formula we can assume that $u = \mathcal{O}(-i)$ for some $i, 0 \leq -i < d$. Again as in the case of a closed immersion we can assume that Y is affine and that we have an injection $N \subset \mathcal{O}_X^n$ and consider the Grassmannian $\mathrm{Gr}_{d,n,Y}$ of locally split flags $N' \subset \mathcal{O}^n$ with N' a rank d vector bundle with universal flag $\mathcal{N} \subset \mathcal{O}^n$. Again, arguing as above

one reduces to the case of $\mathbb{P}(\mathcal{N}) \to \mathrm{Gr}_{d,n,Y}$ and then $Y = \operatorname{spec}\mathbb{Z}$ where the statement is tautological.
Now, given a factorization $f : X \xrightarrow{i} \mathbb{P}(N) \xrightarrow{p} Y$ one defines $\psi_{k,f,i,p}$ via $\psi_{k,p}\psi_{k,i}$ which is defined by requiring that condition 1. holds. One needs to go over the same steps as in the case of the excess formula to establish that it is independent of factorization and satisfies the conditions of the theorem. They are proved similarly, with one exception:

Lemma 6.3.4. *Suppose we are given a Cartesian square*

$$\begin{array}{ccc} \mathbb{P}_X(N_X) & \xrightarrow{i'} & \mathbb{P}_Y(N) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{i} & Y \end{array}$$

of morphisms in of regular schemes with i, i' closed immersions, N a vector bundle on Y, p the natural projection and $N_X = N|_X$. Then $\psi_{k,p}\psi_{k,i'} = \psi_{k,i}\psi_{k,p'}$.

Proof. This can be done by direct calculation, but we show how to reduce to the arithmetic situation. All functors are compatible with the projection formula so we only need to show that $\psi_{k,p}\psi_{k,i'}(u) = \psi_{k,i}\psi_{k,p'}(u)$ for virtual bundles $u = \mathcal{O}(-i), i = 0, 1, \ldots, d-1$ with d being the rank of N. By Theorem 6.1.1 and the properties already established we may assume that Y, and thus X, is affine. Also, a deformation to the normal cone-argument shows that we can suppose $Y = \mathbb{P}_X(M)$ for a vector bundle M on X and a Grassmannian argument as in the proof of the main result of section 6.2 shows we can reduce to the case of a diagram of Grassmannians and reduces to the case of Grassmannians over $\operatorname{spec}\mathbb{Z}$ in which case the isomorphisms are rigidified. □

The rest of the proof is just like in the proof of the excess formula. We conclude the proof of the Adams-Riemann-Roch theorem. □

Recall that the relative dimension of a local complete intersection morphism $f : X \to Y$ is the rank of the virtual bundle Ω_f defined at the beginning of this section. This is locally constant on X. Also, denote by $F^iV(X) = F^iW(X)$ and $V(X)^{(i)} = W(X)^{(i)}$ where $F^iW(X)$ is part of the filtration of the cohomological virtual category exhibited in 4. This is motivated by the equivalence of categories $V(X)_{\mathbb{Q}} = W(X)$.

Corollary 6.3.5. *Let $f : X \to Y$ be a projective local complete intersection morphism of regular schemes of constant relative dimension n. Then the* *morphism $Rf_*: V(X) \to V(Y)$ restricts canonically to a morphism $Rf_*: F^iV(X) \to F^{i-n}V(Y)$. In other words, the essential image of $F^iV(X)$ in $V(Y)$ lies in (the essential image of) $F^{i-n}V(Y)$.*

Proof. Denote by p_j the composition of $V \to V^{(j)} \to V$. Then, by Proposition 4.0.14, any object $v \in F^iV(X)$ is equivalent to a sum of the form $\sum_{j\geq i} v_j$ for $v_j \in V(X)^{(j)}$. Let $P(t)$ be the multiplicative characteristic class associated to $(t-1)/\log(t)$ which exists and is unique by Proposition 6.1.2 and the arguments of Lemma 6.3.1. An application of the splitting principle establishes that for a virtual vector bundle v of rank r, there is a canonical isomorphism $\Psi^k P(v)\theta_k^{-1}(v) = k^{-r}P(v)$ stable under arbitrary base-change of regular schemes. An application of the above Riemann-Roch theorem to the virtual bundle $v = P(\Omega_f) \otimes P(\Omega_f)^{-1} \otimes v \in F^iV$ establishes an isomorphism, putting $v_j = p_j[P(\Omega_f)^{-1} \otimes v]$

$$\begin{aligned}\Psi^k Rf_*(P(\Omega_f) \otimes v_j) &= Rf_*(\theta_k(\Omega_f)^{-1} \otimes \Psi^k P(\Omega_f) \otimes v_j) \\ &= k^{j-n} Rf_*(P(\Omega_f) \otimes v_j).\end{aligned}$$

Because the Adams-Riemann-Roch isomorphism is functorial we obtain, by Proposition 4.0.14, a functorial projection of $Rf_*(v)$ onto $F^{i-n}V(Y)$. □

Definition 6.3.5.1. Given a projective local complete intersection morphism $f: X \to Y$ of regular schemes of constant relative dimension n, define $f_*: V^{(i)}(X) \to V^{(i-n)}(Y)$ to be the functor induced by the preceding corollary.

Remark 6.3.5.1. Using this, it should be possible to establish an analogous Grothendieck-Riemann-Roch formula for the "Chow categories" $V^{(i)}(X)$, at least for regular schemes. The author has not compared this construction to that of [Fra90] or such a result to the results of [Fra]. This approach should however be analogous to that of [HG87].

6.4 *Application to Adams-Riemann-Roch transformations*

In this section we propose an Adams-Riemann-Roch transformation a la Baum-Fulton-MacPherson-Gillet (c.f. [PB75], [WF83]).
For a scheme Z, denote by $G(Z) = G_0(Z)_{\mathbb{Q}}$, the Grothendieck group of coherent sheaves on Z, tensor $\mathbb{Q}$. Suppose that Z is a closed subscheme of a scheme X which is smooth of finite type over a regular scheme S. Denote by $K_0^Z(X)$ the Grothendieck group of complexes of coherent locally free sheaves acyclic away from Z and $K^Z(X) = K_0^Z(X)_{\mathbb{Q}}$. By Poincaré duality, $K_0^Z(X) = G_0(Z)$ and $K^Z(X) = G(Z)$ via the map sending a complex α representing an element in $K_0^Z(X)$ to its homology in $G_0(Z)$. We will use

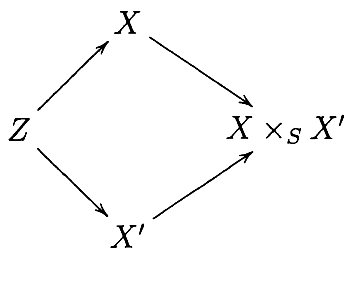
this implicitly in some instances. If we set $U = X \setminus Z$, we can evaluate the quotient-space X/U [4] on the object $\mathbb{Z}\times \mathrm{Gr}$ in Voevodsky's $\mathbb{A}^1$-homotopy category. The result identifies with the fundamental group of the homotopy-fiber $BQP(X) \to BQP(U)$ of Quillen's Q-construction on the category of coherent locally free sheaves on X and U respectively which is also $K_0^Z(X)$. Using this observation we can also construct Bott-classes and Adams-filtration for the group $K^Z(X)$. The cotangent-complex $L_{X/S}$ is a perfect complex on X and so quasi-isomorphic to a strictly perfect complex, i.e. a finite complex of locally free sheaves, and as such defines an element in $K(X)$. In particular there is an element $\theta_k(L_{X/S})^{-1}$ in $K(X)$. Moreover the Adams operations act on $K^Z(X)$ via a unique extension from the case $Z = X$, by evaluating the Adams-operations $\Psi^k : \mathbb{Z} \times \mathrm{Gr} \to \mathbb{Z} \times \mathrm{Gr}$ on X/U, which coincides with the Dold-Puppe construction (for the latter construction, if not comparison, c.f. [AD61] or [HG87]). We denote it by $\Psi^k_{Z,X}$.

Proposition 6.4.1. *Let Z be a closed subscheme of a scheme X, with X of smooth of finite type over a regular basescheme S. For a positive integer k, and a coherent sheaf $\mathcal{F}$ on Z, and a quasi-isomorphism $i_*(\mathcal{F}) \to E^\bullet$ for $E^\bullet$ a complex of locally free sheaves on X. We define the Adams-Riemann-Roch transformation*

$$\tau^k_{Z,X}(\mathcal{F}) : G(Z) \to G(Z)$$

via $\tau^k_{Z,X}(\mathcal{F}) = Li^*\theta_k(L_{X/S})^{-1} \otimes \Psi^k_{Z,X}(E^\bullet)$. This is well-defined and does not depend on the choice of X.

Proof. Suppose X and X' are two schemes smooth over S, with closed subscheme Z. We need to confirm $\tau^k_{Z,X}(\mathcal{F}) = \tau^k_{Z,X'}(\mathcal{F})$. Clearly $\tau^k_{Z,X}(\mathcal{F})$ does not depend on $E^\bullet$ and the choice of quasi-isomorphism. The result easily follows from the Adams-Riemann-Roch theorem for closed immersions of regular schemes while considering the square



□

[4] considered in Voevodsky's $\mathbb{A}^1$-homotopy category.

Proposition 6.4.2. *Fix a regular scheme S, and let Z be a scheme of finite type over S. Then, for any positive integer k, there are natural Adams-Riemann-Roch-transformations*

$$\tau_Z^k : G(Z)_{\mathbb{Q}} \to G(Z)_{\mathbb{Q}}$$

such that:

- *If $f : Y \to X$ is a proper morphism, $Rf_* \tau_Y^k = \tau_X^k Rf_*$.*
- *If $\alpha \in K_0(X)$ and $\beta \in G(X)$ then $\tau_X^k(\alpha \otimes \beta) = \Psi^k \alpha \otimes \tau_X^k(\beta)$.*
- *If Z is a closed subscheme of a regular scheme X with inclusion $i : Z \to X$, then $\tau_Z^k(\alpha) = \theta_k(Li^* L_{X/S})^{-1} \otimes \Psi^k(\alpha)$.*
- *If $f : Y \to X$ is a projective local complete intersection morphism, then*

$$\begin{array}{ccc} K(X) & \xrightarrow{\tau_Y^k} & K(X) \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle \theta^k(\Omega_f) f^*} \\ K(Y) & \xrightarrow{\tau_X^k} & K(Y) \end{array}$$

  commutes.
- *If Z is a closed subvariety of X, then*

  $\tau_X^k(\mathcal{O}_V) = [V] +$ *terms lying in the$(n+1)$st weight-filtration of$G(X)$.*

Proof. The extra details not already provided are given in the same way as in the Riemann-Roch theorem of [PB75], [WF83] or [Ful98], chapter 18. The essential contribution of the functorial Adams-Riemann-Roch theorem is the formulation of the above theorem in the case of supported K-theory. We leave the precise wording to the reader. □

6.5 Mumford's isomorphism and comparison with Deligne's isomorphism

Let $f : C \to S$ be an flat local complete intersection generically smooth proper morphism with geometrically connected fibers of dimension 1, with S any connected normal Nötherian locally factorial scheme. Given a virtual bundle v on C, denote by $\lambda_f(v) = \lambda(v) = \det Rf_* v$ for $\omega = \omega_{C/S}$ being the relative dualizing sheaf, also write $\lambda_n = \det Rf_* \omega_{C/S}^{\otimes n}$. Let $f : \overline{\mathcal{C}_g} \to \overline{\mathcal{M}_g}$

(resp. $\mathcal{C}_g \to \mathcal{M}_g$) be the universal stable curve of genus g (resp. universal smooth curve of genus g) and let $\Delta_g = \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ the discriminant locus of singular curves and write $\delta = \mathcal{O}_S(\Delta_g)$ and $\mu = \lambda_1^{\otimes 12} \otimes \delta^{-1}$. Then we have an isomorphism

$$\lambda_n = \mu^{n(n-1)/2} \otimes \lambda_1 \tag{6.9}$$

on $\overline{\mathcal{M}_g}$ which is unique up to sign (see [Mum77], Theorem 5.10). One deduces the same formula over a general base for a stable curve. In the case C is regular the corresponding factor δ^{-1} is related to the conductor of the curve (see [Sai88a]). In the case C is non-regular (with the same hypothesis on S) an unpublished result of J. Franke as a consequence of his functorial Riemann-Roch in [Fra] establish a formula of the "discriminant" as a localized Chern class.

The classical proof (see [Mum77], loc.cit.) in the stable case is a simple calculation using Grothendieck-Riemann-Roch and the facts for any $g \in \mathbb{N}$, we have that

(a) the Picard-group of the moduli-functor of smooth curves is torsion-free.

(b) $H^0(\mathcal{M}_{g,\mathbb{Z}}, \mathbb{G}_m) = \pm 1$ (c.f. [MB89], Lemme 2.2.3).

We show that our formalism and Adams-Riemann-Roch theorem restricted to dimension 1 implies a version of these results. It should also be noted that the context is greatly simplified by the assumption that we tensor with $\mathbb{Q}$. In particular, inverting 2 eliminates sign-considerations which are without a doubt the greatest obstacle to obtaining integral functorial isomorphisms.

Definition 6.5.0.1. Henceforth "$f : C \to S$ is a curve" is to be as above, with the additional hypothesis that C and S are regular.

Definition 6.5.0.2. (Compare with [Fal84]) Given a scheme X, define $\mathfrak{Pic}(X)_{\mathbb{Q}}$ to be the Picard category of line bundles on X with isomorphisms, $\mathfrak{Pic}(X)$, localized at $\mathbb{Q}$ (c.f. B.0.14.1). This category can be described as follows. The objects are (L, l) with L a line bundle and l a positive integer. Moreover, we have

$$\mathrm{Hom}_{\mathfrak{Pic}(X)_{\mathbb{Q}}}((L,l),(M,m)) = \lim_{n \to \infty} \mathrm{Hom}_{\mathfrak{Pic}(X)}(L^{\otimes nm}, M^{\otimes nl})$$

where the limit is taken over integers ordered by divisibility. Given two line bundles L, M on a scheme X, a $\mathbb{Q}$-morphism $f : L \to M$ i.e. for big enough n, there exists a morphism $L^{\otimes n} \to M^{\otimes n}$, up to obvious equivalence.

First a preliminary calculation showing that we obtain a version of (6.9).

Lemma 6.5.1. *Let $f : C \to S$ be a smooth curve. Then there is a unique canonical $\mathbb{Q}$-isomorphism*

$$\Delta_n : \lambda_n = \det Rf_* \omega_{C/S}^{\otimes n} \simeq \lambda_1^{\otimes(6n^2-6n+1)}$$

stable under base-change $S' \to S$.

Proof. Uniqueness follows from descent and the preceding remarks adding that $\mathcal{M}_g$ is smooth over $\operatorname{spec} \mathbb{Z}$ and so regular. We can also assume that S is the spectrum of a discrete valuation-ring since isomorphisms would then glue together to a global one by virtue of them being canonical. In this case f is automatically projective by [Lic68], Section 23. Let $\omega_f = \Omega_{C/S}$ be the relative dualizing bundle of f. Applying the Adams-Riemann-Roch theorem to the case $(1-\omega_f)$, we obtain the "Grothendieck-Serre duality"-isomorphism

$$(\lambda_0 \otimes \lambda_1^{-1})^{\otimes(k-1)} \simeq 1$$

and in particular for $k = 2$ one has a canonical $\mathbb{Q}$-isomorphism $\lambda_0 \simeq \lambda_1$. Consider the cannibalistic Bott-class

$$\theta_{f,2}^{-1} := \frac{1-\omega_f}{1-\omega_f^2} = \frac{1}{2}\left(1 + \frac{1-\omega_f}{2} + \left(\frac{1-\omega_f}{2}\right)^{\otimes 2}\right) + F^3V(X).$$

The truncation is sufficient for our purposes since the relative dimension $C \to S$ is 1 and Corollary 6.3.5, so that $F^3V(C)$ has image in $F^2V(S)$ and the determinant functor is trivial on this category. It is moreover stable under base-change by functoriality of the Adams-Riemann-Roch theorem. For $k = 2$, inserting this into the Adams-Riemann-Roch-theorem for the trivial line bundle and applying Grothendieck-Serre-duality this reduces to the expression

$$\lambda_1^{16} = \lambda_0^{16} = \lambda_0^7 \otimes \lambda_1^{-4} \otimes \lambda_2 = \lambda_1^3 \otimes \lambda_2$$

so that $\lambda_1^{13} = \lambda_2$. Repeatedly applying the theorem to the case of $1 - \omega_f^2$ one proceeds by induction on n to establish the general formula for λ_n. □

Thus for any curve $f : C \to S$ we obtain a canonical rational $\mathbb{Q}$-morphism $\Delta : \lambda(\omega^{\otimes 2}) \to \lambda(\omega)^{\otimes 13}$ which restricts to the above one over the smooth locus. This is the usual discriminant morphism considered in [Sai88a], for example. We intend to compare our Adams-Riemann-Roch-isomorphism with that of Deligne (see [Del87], Théorème 9.9). Let's just first recall the main ingredients. Let $C \to S$ be a local complete intersection projective morphism of schemes with geometrically connected fibers of dimension 1. Given two line bundles L and M on C, by [Del77], XVIII, 1.3.11, [Elk89], one can form the line bundle $\langle L, M\rangle$ on S. The symbol $\langle\cdot,\cdot\rangle$ satisfies bimultiplicativity with respect to tensor product and has a cohomological description: if u and v are virtual vector bundles of rank 0 on C, then (see [Del87], 7.3.1);

$$\langle \det u, \det v\rangle = \lambda(u \otimes v) \tag{6.10}$$

and so in particular

$$\langle L, M\rangle = \lambda(L\otimes M)\lambda(L)^{-1}\lambda(M)^{-1}\lambda(\mathcal{O}_C).$$

Using this the discriminant section Δ is equivalent to a rational section $\langle\omega,\omega\rangle \to \lambda(\omega)^{\otimes 12}$ which we shall also call the discriminant section. Given any virtual bundle v on C, define $R(v) = (v - \mathcal{O}_C^{\operatorname{rk} v}) - (\det v - \mathcal{O}_C)$, one defines $I_{C/S}C_2(v) = \lambda(-R(v))$.

Proposition 6.5.2. *Let $f : C \to S$ be a curve. Then there exists a unique canonical $\mathbb{Q}$-isomorphism*

$$\Xi : \lambda(\gamma^2(u - \operatorname{rk} u)) = I_{C/S}C_2(u)$$

functorial on virtual bundles u on C, such that the isomorphism is compatible with the trivializations for u a line bundle and such that the following condition Λ holds: for an isomorphism $u = v + w$ the isomorphism

$$\lambda(\gamma^2(u-\operatorname{rk} u)) \to \lambda(\gamma^2(v-\operatorname{rk} v))\otimes\lambda((v-\operatorname{rk} v))\otimes(w-\operatorname{rk} w)))\otimes\lambda(\gamma^2(w-\operatorname{rk} w))$$

is compatible with the isomorphism

$$I_{C/S}C_2(u) \to I_{C/S}C_2(v) \otimes \langle \det v, \det w\rangle \otimes I_{C/S}C_2(w)$$

via Ξ and (6.10).

Proof. The proof is inspired by a reduction to the line bundle technique in [Fra]. Suppose that E is a vector bundle on C. In case E is a line bundle both sides are canonically trivialized and we thereby define the isomorphism. Given a filtration $I : (0) \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E$ with line bundle quotients $E_i/E_{i-1} = L_i$ and $E = \sum_{i=1}^n L_i$. We have an induced isomorphism

$$\gamma^2(E - n) \simeq \sum_{i<j} (L_i - 1) \otimes (L_j - 1)$$

and thus an isomorphism $\lambda(\gamma^2(E - n)) = \bigotimes_{i<j}\langle L_i, L_j\rangle$ and the same relationship holds for $I_{C/S}C_2(E)$ (see [Del87] (9.7.9)). We thus obtain unicity and to construct an isomorphism we only need to show it that is independent of the filtration I. We proceed by induction on the length of

I. For E of rank 2, by Corollary 4.0.12 there is a canonical isomorphism $\gamma^2(E-2) = 1 - E + \det E = -R(E-2)$ [5] which gives an isomorphism which is compatible with arbitrary admissible rank 1 subbundles of E. We proceed by induction. For any admissible subbundle M of rank 1 or 2 of E we obtain an isomorphism $\Xi_M : \lambda(\gamma^2(E-n)) \to I_{C/S}C_2(E)$ by requiring the condition Λ. Moreover, if $L \subset M$ then one easily verifies that $\Xi_M = \Xi_L$ because of the induction hypothesis. In general, suppose we are given two different admissible rank 1 subbundles L and L' of E. Let $p : G = G_{2,L,L'}(E) \to C$ be the Grassmannian variety of admissible rank 2 subbundles M of E with two admissible sublinebundles $L \subset M$ and $L' \subset M$. Then $p : G \to C$ is birational and proper with $Rp_*1 = 1$. The above discussion carries over to the composition $G \to C \to S$ and one reduces to comparing the isomorphism Ξ_{p^*L} and $\Xi_{p^*L'}$ for the bundle p^*E. By the above they are equal to Ξ_M for the universal rank 2-bundle M on G. □

Proposition 6.5.3. *Suppose $C \to \operatorname{spec} R$ is a local complete intersection curve (c.f. 6.5.0.1, it is automatically projective by [Lic68], Section 23) where R be the spectrum of discrete valuation ring with special point s and generic point η. Let $\Omega_{C/S}$ be the coherent sheaf of relative differentials and $\omega_{C/S} = \omega$ the relative dualizing sheaf. The bundle $IC_2(\Omega_{C/S})$ is canonically trivialized over the generic point and the order of the trivialization at η is equal to the order of the discriminant.*

Proof. By the general theory $\Omega_{C/S}$ comes equipped with a natural morphism $\Omega_{C/S} \to \omega$ inducing an isomorphism $\det \Omega_{C/S} = \omega$. The Adams-Riemann-Roch theorem provides us with a canonical $\mathbb{Q}$-isomorphism

$$\lambda(1)^{\otimes 16} = \lambda(1 - 2(\Omega_{C/S} - 1) + (\Omega_{C/S} - 1)^{\otimes 2} - \gamma^2(\Omega_{C/S} - 1)).$$

By the cohomological description of the Deligne-pairing in (6.10), we have canonical isomorphisms $\lambda((\Omega_{C/S}-1)^{\otimes 2}) = \langle \omega, \omega \rangle$ and $-(\Omega_{C/S}-1) = -R(\Omega_{C/S}) - (\omega - 1)$ so that $\lambda(-(\Omega_{C/S} - 1)) = I_{C/S}C_2(\Omega_{C/S}) \otimes \lambda(\omega - 1) = I_{C/S}C_2(\Omega_{C/S})$ where the last isomorphism is by Lemma 6.5.1. Thus we obtain a canonical isomorphism

$$\lambda(\omega)^{\otimes 12} = \langle \omega, \omega \rangle \otimes I_{C/S}C_2(\Omega_{C/S})$$

which restricts to $\lambda(\omega)^{\otimes 12} = \langle \omega, \omega \rangle$ over the generic fiber via the trivialization $I_{C/S}C_2(\Omega_{C/S})$ over the generic fiber defined by the trivialization $I_{C/S}C_2(\omega_{C_\eta/\operatorname{spec} \eta}) = 1$. Thus the order of the generic trivialization $1 \to IC_2(\Omega_{C/S})$ is the discriminant. □

[5] This also follows from $\gamma^2(E-2) = \lambda^2(E-2+2-1) = \lambda^2(E-1)$.

Definition 6.5.3.1. Let R be a discrete valuation ring, and $C \to \operatorname{spec} R$ a curve with special point s and generic point η as above. Let (u,t) be a couple with u a virtual bundle on C with a trivialization $t : \det u_{|C_\eta} \to 1$ on the generic fiber. Then the bundle $\langle u, v\rangle$ has a canonical trivialization by t on R over the generic point via the isomorphism $\langle u, v\rangle = \langle \det u, \det v\rangle = \lambda((\det u - 1)\otimes(\det v - 1))$. Then for another virtual bundle v on C, we define $c_1^D(u,t).c_1(v)$ to be the order of this trivialization. In a similar vein, suppose that (u,s) is a couple with u a virtual bundle on C with an isomorphism $s : u|_{|C_\eta} \to L$ on the generic fiber with L a line bundle. Then $IC_2(u)$ has a canonical trivialization by s on R over the generic point, c.f. [Del87], Proposition 9.4 (ii) or above definition, and we define $c_2^D(u,s)$ to be the order of this trivialization.

The following is a slight extension of Lemma 2 in [Sai88a], to which we refer the reader for an idea of the proof.

Lemma 6.5.4. *Let X be a regular scheme and Z an effective divisor of X with complement U. Suppose we two strict perfect complexes E, F on X, and a have a quasi-isomorphism $t : E|_U \to F|_U$ over U. Denote by $\det t$ the corresponding rational section of the line bundle $\operatorname{Hom}_{\mathcal{O}_X}(\det E, \det F)$, and $\operatorname{div} t$ its divisor. Then the bivariant class $c_{1,Z}^X(E \to F)\cap$ acts as the intersection class $\mathrm{CH}_i(X) \to \mathrm{CH}_{i-1}(Z)$ given by simply restricting along $\operatorname{div} t$.*

Corollary 6.5.5. *The class $c_1^D(u,t).c_1(v)$ defined above coincides with $c_{1,C_s}^C(\det u, \det t).c_1(\det v) \cap [X]$.*

Proof. This follows from the above description and an application of Riemann-Roch for singular curves (on the special fiber) as in [Ful98], Example 18.3.4. □

Lemma 6.5.6. *Keep the assumptions and notations of the above definition. The cotangent sheaf $\Omega_{C/S}$ is a line bundle on the generic fiber and thus $IC_2(\Omega_{C/S})$ is canonically trivialized over the generic fiber. With this trivialization we have $c_2^D(\Omega_{C/S}) = c_{2C_s}^C(\Omega_{C/S})$, the associated localized Chern class (c.f. [Blo87], section 1).*

Proof. First of all, $\gamma^2(\Omega_{C/S} - 1) = \lambda^2\Omega_{C/S}$, where λ^2 denotes the λ-operation on the virtual category, so that $\lambda^2\Omega_{C/S}$ is trivialized over the generic fiber. By [Sai88b], Proposition 2.3 the alternating lengths of the cohomology of $\lambda^2(\Omega_{C/S})$ is $c_{2C_s}^C(\Omega_{C/S})$. It follows that the order of the above induced trivialization of $\det Rf_*(\gamma^2(\Omega_{C/S} - 1)) = \det Rf_*(\lambda^2\Omega_{C/S})$ is $c_{2C_s}^C(\Omega_{C/S})$. We conclude by Proposition 6.5.2. □

Corollary 6.5.7 (Conductor-Discriminant formula by T. Saito). *With the above assumptions, the order of the discriminant rational section* Δ *of the line bundle* $\mathrm{Hom}_{\mathcal{O}_S}(\langle\omega,\omega\rangle,\lambda(\omega)^{\otimes 12})$ *is equal to minus the Artin conductor of* $C \to S$ *(c.f. [Sai88a]).*

Proof. Combine Proposition 6.5.2 and Corollary 6.5.6 and the main result of [Blo87] which identifies the localized Chern class with the Artin conductor. □

Remark 6.5.7.1. The proof of Corollary 6.5.7 is essentially different from that of [Sai88a] in that it does not use any kind of semi-stable reduction techniques. It does however share similarity to that of [Fra].

Remark 6.5.7.2. Given a rich enough theory of integrals over Chern classes the reasoning above would imply that for $f : X \to S$ a projective local complete intersection morphism of regular schemes of pure dimension n and $\sum k_i = n$, $\lambda(\prod \gamma^{k_i}(E_i - \mathrm{rk}\, E_i)) = \det \int_{X/S} \prod c_{k_i}(E_i)$. It seems reasonable that it is possible to deduce these relations if one takes the theory of Chern classes to be that of [Elk89], at least rationally. Also, if f is flat generically smooth of relative dimension d and S a discrete valuation ring and a complex of vector bundles $E^\bullet$ on X acyclic outside a closed subscheme Z over the special point, the above suggests that the order of vanishing of $\lambda(\gamma^{d+1}(E^\bullet - d)))$ at the generic point is the localized Chern class $c_{d+1}^Z(E^\bullet)$. This is predicted by the analogous situation in Chow categories, [Fra].

6.6 A Deligne-Riemann-Roch formula for the Determinant of the cohomology

As before, given a virtual bundle v and a proper perfect morphism $f : X \to Y$ then $\lambda(v) = \det Rf_*(v)$.

Theorem 6.6.1. *Let* $f : C \to S$ *be a curve over a regular stack* S*, i.e. a curve over any smooth presentation of* S*. Then there is a unique canonical* $\mathbb{Q}$*-isomorphism*

$$\lambda(\theta_f^{-2} \otimes \Psi^2 u) \simeq \langle\omega,\omega\rangle^{\mathrm{rk}\, u} \otimes IC_2(\Omega_{C/S})^{\mathrm{rk}\, u} \otimes IC_2(u)^{-12} \langle \det u, \det u \otimes \omega^{-1}\rangle^6$$

which is compatible with the Deligne-isomorphism up to torsion.

Remark 6.6.1.1. A word of caution is in place. We haven't actually constructed classes θ_f^{-2} etc for the virtual category of vector bundles on any stack. The bundles $\lambda(\theta_f^{-2} \otimes \Psi^2 u)$ etc. in question refer to the bundles one obtains by smooth descent from the case of a regular scheme, in which case it does make sense.

Proof. By descent we can suppose that S is a regular scheme. Rigidity provides us with, for a virtual bundle u of rank m, a canonical and functorial isomorphism

$$\Psi^2(u) - m = (u - m)^{\otimes 2} + 2(u - m) - 2\gamma^2(u - m).$$

We can multiply this out and use that there is a functorial product on the filtration $F^i V(C)$ to cancel out all the terms that are in $F^3 V(C)$. This provides us with a choice of canonical isomorphism with the right-hand side of the Deligne-Riemann-Roch theorem, compatible with base-change and sums. We verify that it is unique. This is an argument given in [Fra] which we reconsider here (more or less verbatim). Given any functorial isomorphism as above the lack of compatibility with the Deligne-isomorphism for a virtual bundle u is given by an element $c_{X/Y}(u) \in H^0(S, \mathbb{G}_m)_{\mathbb{Q}}$ which is stable under pullback of smooth curves, as well as isomorphisms and sums of virtual bundles. Locally on the base S the virtual bundle u is a sum of line bundles, so we can assume u is a line bundle. To show that $c_{X/Y}(L) = 1$ we can assume S is the spectrum of an algebraically closed field. Given a line bundle of degree d it is the pullback of the universal degree d-bundle $\mathcal{P}_d$ of $\mathcal{C}_{g,d} \to P_{g,d}$, moduli of genus g-curves with a given degree d bundle, thus constant. We remark that $P_{g,d}$ is smooth over $\mathcal{M}_g$ and thus regular since $\mathcal{M}_g$ is smooth over $\operatorname{spec} \mathbb{Z}$.
Thus we are given universal constants $(c_{d,g}) \in H^0(\mathcal{M}_g, \mathbb{G}_m)_{\mathbb{Q}}$ on $\mathcal{M}_g$ which are 1 by virtue of the fact that $H^0(\mathcal{M}_g, \mathbb{G}_m) = \pm 1$.

□

6.7 A conjecture of Köck for the determinant of the cohomology

We recall the setting of [Köc98] in the special case of K_0. Let S be an separated Nötherian scheme and G a flat separated finite type group-scheme over S. Suppose we are given a G-projective local complete intersection morphism $f : X \to Y$ of G-equivariant schemes such that on Y any G-coherent module is the quotient of a locally free G-module. Denote by $K(X, G)$ the group $K_0(X, G)_{\mathbb{Q}}$ of G-equivariant K-theory of vector bundles of X tensor $\mathbb{Q}$. Then the method of Proposition 3.1.2 furnishes a pushforward $Rf_* : K(X, G) \to K(Y, G)$.

Definition 6.7.0.1. Fix a G-equivariant factorization $f : X \xhookrightarrow{i} \mathbb{P}_Y(\mathcal{E}) \xrightarrow{\pi} Y$ for some vector bundle $\mathcal{E}$ on Y and denote by Ω the sheaf of relative differentials of $\mathbb{P}(\mathcal{E}) \to Y$. Let d be the rank of $\mathcal{E}$ and denote by $\hat{K}(Y,G)$ the ring $K(Y,G)_{\mathbb{Q}}$ completed at the ideal generated by elements of the form $\lambda^1(\mathcal{E}) - d, \lambda^2(\mathcal{E}) - \binom{d}{2}, \ldots, \lambda^d(\mathcal{E}) - 1$.

Then the following is proved:

Proposition 6.7.1 ([Köc98], Theorem 4.5). *The Bott element $\theta'^{-1}_k := \theta_k(N_i - i^*\Omega)$ (defined as before) defines an element in $K(X,G) \otimes_{K(Y,G)} \hat{K}(Y,G)$ and for any $k \geq 2$ there is a commutative diagram*

$$\begin{array}{ccc} K(X,G) & \xrightarrow{\theta'^{-1}_k \otimes \Psi^k} & K(X,G) \otimes_{K(Y,G)} \hat{K}(Y,G) \\ \Big\downarrow{\scriptstyle Rf_*} & & \Big\downarrow{\scriptstyle \hat{R}f_*} \\ K(Y,G) & \xrightarrow{\Psi^k} & \hat{K}(Y,G) \end{array}$$

We recast it in the following way to provide a formula for the first Chern-class of the cohomology, or equivalently, the determinant of the cohomology. Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable proper local complete intersection morphism of separated Nötherian algebraic stacks with the resolution property and with smooth groupoid representations $[p, q : R \rightrightarrows X]$ and $[p', q' : R' \rightrightarrows Y]$ respectively, with induced morphisms $g : X \to Y, h : R \to R'$, and consider the following two "determinant of cohomology"-functors. First of all, given a vector bundle E on $\mathcal{X}$, one pushforwards to obtain a perfect complex Rf_*E on $\mathcal{Y}$, and then apply the determinant to obtain the determinant of the cohomology, $\lambda_1(E)$, considered as a linebundle on $\mathcal{Y}$. On the level of K-groups this corresponds to the homomorphism $K(\mathcal{X}) \stackrel{Rf_*}{\to} K(\mathcal{Y}) \stackrel{\det}{\to} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$, where the pushforward is the one exhibited in [Köc98] for quotient-stacks and Proposition 3.1.2 in this text, and the last homomorphism is the determinant homomorphism. In a different vein, consider the same vector bundle E, and consider the perfect complex Rg_*E on Y. Since one has the relation $\det Lq^* = q^* \det$, the base-change-isomorphism equips the determinant $\det Rg_*E$ with descent-data with respect to $R' \rightrightarrows Y$, thus we obtain another linebundle $\lambda_2(E)$ on $\mathcal{Y}$. The main observation of this section is that in the above setting, descent commutes with pushforward:

Lemma 6.7.2. *There is a natural equivalence of determinant functors $\lambda_1 \simeq \lambda_2$, and we denote both by $\lambda := \det Rf_*$.*

Proof. The proof is just unwinding the definitions. The definition of λ_1 is obtained by choosing an R-equivariant g_*-acyclic resolution $E \to F$, and then applying g_*, and finally the determinant functor. The descent-data for g_*F for acyclic F is given by the base-change-isomorphism, so that the following diagram is commutative

$$\begin{array}{ccccccc} Lp'^*Rg_*E & \longrightarrow & Rh_*Lp^*E & \longrightarrow & Rh_*Lq^*E & \longrightarrow & Lq'^*Rg_*E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ p'^*g_*F & \longrightarrow & h_*p^*F & \longrightarrow & h_*q^*F & \longrightarrow & q'^*g_*F \end{array}$$

in the derived category of perfect complexes on R'. The upper line is given by a quasi-isomorphism composed by smooth base-change and descent-data, whereas the lower one is an isomorphism given by ordinary smooth base-change and the vertical maps are the natural quasi-isomorphisms. Applying the determinant functor to the perfect complex Rg_*E transforms quasi-isomorphisms to isomorphisms and derived pullbacks to pullbacks and thus provides us with descent-data of Rg_*E. This is the definition of λ_2 and provides us with the requested equivalence of functors. □

Notice that λ_1 defines a determinant functor from the category of virtual vector bundles on $\mathcal{X}$ admitting a f_*-acyclic resolution, whereas λ_2 is defined on the category of virtual vector bundles on $\mathcal{X}$ admitting g_*-acyclic resolutions on X.

Theorem 6.7.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable projective local complete intersection morphism of regular stacks. Then we have equalities*

$$\det(Rf_*E)^{\otimes k} = \det(Rf_*(\theta_{f,k}^{-1} \otimes \psi^k E))$$

in $\mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$.

Proof. By Theorem 6.3.2, for a R-equivariant vector bundle E the Adams-Riemann-Roch isomorphism $\Psi^k Rg_*E = Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E)$ associated to $g : X \to Y$ is stable under smooth base-change and thus defines descent-data of the isomorphism of $\mathbb{Q}$-line bundles

$$\det(Rg_*E)^{\otimes k} = \det(\Psi^k Rg_*E) = \det Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E).$$

By Lemma 6.7.2 the class in Pic of left hand side coincides with the map $K(\mathcal{X}) \xrightarrow{Rf_*} K(\mathcal{Y}) \xrightarrow{\det} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$ and the right hand side $\det Rf_*(\theta_{g,k}^{-1} \otimes \psi^k E)$ coincides with $K(\mathcal{X}) \xrightarrow{\theta_k^{-1} \otimes \Psi^k} K(\mathcal{Y}) \xrightarrow{\det} \mathrm{Pic}(\mathcal{Y})_{\mathbb{Q}}$. □

Thus we obtain a non-completed version of Köck's Adams-Riemann-Roch. Define an action of $a \in K(Y, G)$ on $(r, L) \in \mathbb{Z} \oplus \mathrm{Pic}(Y, G)$ by

$$a.(r, L) = (r \cdot \mathrm{rk}\, a, \det(a)^r \otimes b^{\mathrm{rk}\, a}).$$

Then $\mathbb{Z} \oplus \mathrm{Pic}(Y, G)$ is a $K(Y, G)$-module and we put $\hat{\mathrm{Pic}}(Y, G)$ to be the quotient of $(\mathbb{Z} \oplus \mathrm{Pic}(Y, G)) \otimes_{K(Y,G)} \hat{K}(Y, G)$ by $\mathbb{Z} \otimes_{K(Y,G)} \hat{K}(Y, G)$. It follows from Lemma 6.7.2 that the image of the right side in $\hat{\mathrm{Pic}}(Y, G)$ of the above theorem necessarily coincides with the image under $Rf_*(\theta'^{-1}_k \otimes \Psi^k E)$ in $\hat{K}(Y, G) \xrightarrow{\det} \hat{\mathrm{Pic}}(Y, G)$. Also, by the usual equivalence of the Adams-Riemann-Roch and Grothendieck-Riemann-Roch theorem one obtains expressions for the Chern-classes and the corresponding equivariant Grothendieck-Riemann-Roch theorem for the first Chern-class. This is example 5.11 of loc.cit. which is only known under the condition that f is continuous with respect to the γ-filtration on the K-groups, i.e. if $F^n K$ denotes the γ-filtration on K, then for any n we require that there is an m such that $Rf_* F^m K(X, G) \subset F^n K(Y, G)$ (c.f. [Köc98], section 5).

APPENDIX

A. $\mathbb{A}^1$-HOMOTOPY THEORY OF SCHEMES

The chauffeur held open the door of the car. Lady Dittisham got in and the chauffeur wrapped the fur rug around her knees.

This section is to recall some necessary results and and to fix some notation. In what follows we have but slight extensions of the theorems in the reference-list, and we hope the reader agrees that not spelling out the proofs does not cause any harm. One word of warning though, we have almost completely ignored issues related to smallness of categories. This can be amended by inserting the word "universe" at the appropriate places.

Denote by Δ the category of totally ordered finite sets and monotonic maps. Hence, the objects are the finite sets $[n] = \{0 < 1 < 2 < \ldots < n\}$ and the morphisms of Δ are generated by the maps

$$\delta_i : [n-1] \to [n], \text{ defined by } \delta_i(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases}$$

and

$$\sigma_i : [n] \to [n-1], \text{ defined by } \sigma_i(j) = \begin{cases} j, & \text{if } j \leq i \\ j-1, & \text{if } j > i \end{cases}$$

These maps are the face resp. the degeneracy-maps, and satisfy the usual simplicial relationships ([PGG99], chapter 1). If $\mathcal{C}$ is any category, we denote by $s\mathcal{C}$ or $\Delta^{\text{op}}\mathcal{C}$ the category of simplicial objects of $\mathcal{C}$, i.e. the category whose objects are functors $\Delta^{\text{op}} \to \mathcal{C}$, and morphisms are natural transformations of functors.
Let T be a site, and denote by $\mathbf{Shv}(T)$ the category of sheaves of sets on T, and $\Delta^{\text{op}}\,\mathbf{Shv}(T)$ the category of simplicial sheaves. Note that if we are given a simplicial set E, we can associate to it the constant simplicial sheaf, which we also denote by E, and thus we obtain a functor

$$\Delta^{\text{op}}\,\text{Set} \overset{\text{constant}}{\longrightarrow} \Delta^{\text{op}}\mathbf{Shv}(T).$$

The standard n-simplices Δ^n define thus by the Yoneda lemma a cosimplicial object

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet} & \Delta^{\mathrm{op}}\,\mathbf{Shv}(T) \\ n & \mapsto & \Delta^n \end{array}$$

and we give the category $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ the structure of a simplicial category with a simplicial function object $\mathbf{hom}(-,-)$ given by

$$\mathbf{hom}(\mathcal{X},\mathcal{Y}) := \mathrm{Hom}_{\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)}(\mathcal{X}\times\Delta^\bullet,\mathcal{Y}).$$

Before continuing, we recall the fundamental lemma of homotopical algebra

Theorem A.0.4. *[[PGG99], II.3.10] Let $\mathcal{C}$ be a closed simplicial model category with associated homotopy-category $\mathcal{H}$, and $\mathcal{X},\mathcal{Y}\in\mathrm{ob}(\mathcal{C})$. Suppose furthermore that $\mathcal{X}'\to X$ is a trivial fibration with $\mathcal{X}'$ cofibrant and $\mathcal{Y}\to\mathcal{Y}'$ is a trivial cofibration with $\mathcal{Y}'$ fibrant. Then we have a natural identification*

$$\mathrm{hom}_{\mathcal{H}}(\mathcal{X},\mathcal{Y}) = \pi_0(\mathbf{hom}(\mathcal{X}',\mathcal{Y}')).$$

An adjoint to the functor $\Delta^{\mathrm{op}}\,\mathrm{Set}\to\Delta^{\mathrm{op}}\mathbf{Shv}(T)$ given by $X\mapsto\mathbf{hom}(*,X)$, which we sometimes write as $X\mapsto|X|$.

Definition A.0.4.1. Let $f:\mathcal{X}\to\mathcal{Y}$ be a morphism of simplicial pre-sheaves. Then

(a) f is said to be a (simplicial) weak equivalence if, for any conservative family $\{x:\mathbf{Shv}(T)\to\mathrm{Set}\}$ of points of T, $x(f):x(\mathcal{X})\to x(\mathcal{Y})$ is a homotopy-equivalence of simplicial sets.

(b) f is called a cofibration if it is a monomorphism.

(c) f is called a fibration if it has the right lifting property with respect to trivial cofibrations, i.e. cofibrations which are also weak equivalences.

Theorem A.0.5. *[[VV99], Theorem 2.1.4] For any (small) site with enough points T, the above equips $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with the structure of a closed model category.*

We denote by $\mathcal{H}_s(T)$ the corresponding homotopy-category obtained by inverting the weak equivalences in $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$. To fix ideas, unless explicitly mentioned, from here on S will denote a regular scheme and T a full subsite with enough points of Sch/S_{sm} the category of S-schemes equipped with the smooth topology [1], and denote the corresponding homotopy-category by $\mathcal{H}_s(T)$. Most often, we will be concerned with the category $\mathfrak{R}_S$ of regular S-schemes with the smooth topology. When $S = \operatorname{spec}\mathbb{Z}$, we write $\mathfrak{R}_\mathbb{Z} = \mathfrak{R}$. Since any smooth morphism locally for the étale topology has a section we can identify the various topoi of sheaves of regular S-schemes with étale or smooth topology or of affine regular S-schemes with the étale or smooth topology with a "big regular étale S"-topoi. They are given a conservative set of points by regular local strict henselian rings.

Definition A.0.5.1. Suppose T is such that for any $X \in ob(T)$, $\mathbb{A}^1_X$ is also an object in T. We say that $\mathcal{X} \in \mathcal{H}_s(T)$ is $\mathbb{A}^1$-local with respect to T, if for any $\mathcal{Y} \in \mathbf{Shv}(T)$ the map

$$\mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{Y}, \mathcal{X})$$

is bijective. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ is $\mathbb{A}^1$-local if for any $\mathbb{A}^1$-local object $\mathcal{Z}$, the natural map

$$\mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{X}, \mathcal{Z})$$

is bijective. Now equip $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with $\mathbb{A}^1$-local weak equivalences, cofibrations and $\mathbb{A}^1$-local fibrations. Then we have:

Theorem A.0.6 ([VV99], Theorem 2.3.2). *This equips $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with the structure of a closed model-category.*

Definition A.0.6.1. We denote the corresponding homotopy category by $\mathcal{H}(T)$. Whenever $T = Sm/S_{Nis}$, the corresponding homotopy-category is the $\mathbb{A}^1$-homotopy category of schemes over S defined by loc.cit., but it will not directly play a role in what we do. When the site is $T = \mathfrak{R}_{sm}$, the category of regular schemes with the smooth topology, the corresponding homotopy category is denoted by $\mathcal{H}(\mathfrak{R})$. We also have natural pointed analogues. Replacing in all previous definitions pointed versions, we obtain the $\mathbb{A}^1$-homotopy category of pointed simplicial sheaves $\mathcal{H}_\bullet(T)$ as a localization of the category of pointed simplicial sheaves; $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)_\bullet$. For two objects $(\mathcal{X}, x), (\mathcal{Y}, y) \in \mathcal{H}_\bullet(T)$ we define $X \wedge Y$ in the usual way as the coequalizer of

$$\mathcal{X} \times y, x \times \mathcal{Y} \rightrightarrows \mathcal{X} \times \mathcal{Y}.$$

For a simplicial sheaf $\mathcal{X}$, we denote by $\mathcal{X}_+$ the simplicial presheaf with a disjoint point. The functor $\mathcal{X} \to \mathcal{X}_+$ is left adjoint to the forgetful functor $\mathcal{H}_\bullet(T) \to \mathcal{H}(T)$.

[1] i.e. a full subcategory such that any open cover of T is an open cover of Sch/S_{Sm}.

The stable homotopy-category of schemes is stabilized out of the "unstable" one with the proper notion of a circle. As before, let T denote a (small) site with enough points.

Definition A.0.6.2. Let $\mathbf{T} \in \Delta^{op}\,\mathbf{Shv}(T)_\bullet$. A $\mathbf{T}$-spectra is a set $\mathbf{E} = (d_n, \mathbf{E}_n)_{n\in\mathbb{N}}$ of objects in $\Delta^{op}\,\mathbf{Shv}(T)_\bullet$ with morphisms

$$d_n : \mathbf{T} \wedge \mathbf{E}_n \to \mathbf{E}_{n+1}.$$

A morphism of $\mathbf{T}$-spectra $f : \mathbf{E} \to \mathbf{F}$ is a set of morphisms $f_n : \mathbf{E}_n \to \mathbf{F}_n$ such that the diagram commutes

$$\begin{array}{ccc} \mathbf{T} \wedge E_n & \longrightarrow & E_{n+1} \\ \downarrow & & \downarrow \\ \mathbf{T} \wedge F_n & \longrightarrow & F_{n+1} \end{array}.$$

Definition A.0.6.3. Let $\mathbf{E}$ be a $\mathbf{T}$-spectra, and denote by $\Omega_{\mathbf{T}}(-) = \Omega(-) = R\mathrm{Hom}(\mathbf{T}, -)$ the total derived functor (in $\mathcal{H}_\bullet(S)$) of the right adjoint to $\mathbf{T} \wedge -$. We say that $\mathbf{E}$ is a Ω-spectra if for any n the induced morphism

$$\mathbf{E}_n \to \Omega(\mathbf{E}_{n+1})$$

is in fact an isomorphism. We can naively construct "a" stable homotopy-theory by taking the category of Ω-spectras with respect to $\mathbf{T} = (\mathbb{P}^1, \infty)$, and denote it by $\mathcal{SH}_{naive}(T)$, and giving morphisms $E \to F$ by morphisms $E_n \to F_n$ in $\mathcal{H}_\bullet(S)$ for any n such that the obvious diagram commutes (c.f. [Rio06], Définition I.124).

Definition A.0.6.4. Let be a morphism $f : \mathbf{E} \to \mathbf{F}$ of $\mathbf{T}$-spectras. Then f is a projective cofibration if f_0 is a monomorphism and for any $n > 0$,

$$\mathbf{T} \wedge \mathbf{F}_n \bigvee_{\mathbf{T}\wedge\mathbf{E}_n} \mathbf{E}_{n+1} \to \mathbf{F}_{n+1}$$

is also a monomorphism. Its an $\mathbb{A}^1$-projective fibration (resp. $\mathbb{A}^1$-projective equivalence) if every map f_n is a $\mathbb{A}^1$-fibration (resp. $\mathbb{A}^1$-weak equivalence).

Theorem A.0.7 ([Rio06], Première partie). *Let $\mathbf{T} = (\mathbb{P}^1, \infty)$. The category of $\mathbf{T}$-spectras equipped with projective cofibrations as cofibrations, $\mathbb{A}^1$-projective fibrations as fibrations and $\mathbb{A}^1$-projective equivalences as weak equivalences is a closed model-category.*

We define the following as the stable homotopy-category of T.

Definition A.0.7.1. Let $\mathbf{T} = (\mathbb{P}^1, \infty)$. Then the stable homotopy-category $\mathcal{SH}(T)$ is the full subcategory, of the corresponding homotopy-category, of Ω-spectras.

Definition A.0.7.2. For a fixed scheme S, let $\mathrm{Gr}_{d,r}$ be the Grassmannian of locally free quotients of rank r of $\mathcal{O}_S^{d+r}$ viewed as an object of $\mathrm{Shv}(T)$. Notice that $\mathrm{Gr}_{d,r} \simeq \mathrm{Gr}_{r,d}$. Let $\mathcal{F}$ be a locally free sheaf of rank r. We have natural morphisms $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d+1,r}$ and $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d,r+1}$ by sending $\phi : \mathcal{O}^{d+r} \twoheadrightarrow \mathcal{F}$ to $\mathcal{O}^{d+r+1} \overset{(\phi,0)}{\twoheadrightarrow} \mathcal{F}$ and $\mathcal{O}^{d+r+1} \overset{\phi,\mathrm{id}}{\twoheadrightarrow} \mathcal{F} \oplus \mathcal{O}$ respectively. We denote by $\mathrm{Gr}_d = \lim_{\to} \mathrm{Gr}_{d,r}$ and $\mathrm{Gr} = \lim_{\to} \mathrm{Gr}_d$ for these maps. Here the direct limits are taken in $\mathbf{Shv}(T)$. Since all things here naturally pointed (by $\mathrm{Gr}_{\mathrm{d},0}$ for any d), we also obtain a pointed element $\mathrm{Gr} \in \mathcal{H}_\bullet(T)$. Notice that $\mathbb{P}^d = \mathrm{Gr}_{d,1} \simeq \mathrm{Gr}_{1,d}$ and denote by $\mathbb{P}^\infty = \mathrm{Gr}_1$.

By the method of [Rio06], Définition III.101, it is possible to define a sheaf $(\mathbb{Z} \times \mathrm{Gr})[\frac{1}{n}]$ and $(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ with a natural morphism $\mathbb{Z} \times \mathrm{Gr} \to (\mathbb{Z} \times \mathrm{Gr})[\frac{1}{n}]$ and $\mathbb{Z} \times \mathrm{Gr} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$. In a similar fashion to loc. cit., to lax notation first put $\mathrm{Gr}_{d,r} = \mathrm{Gr}^{d+r,r}$ so that $\mathbb{P}^d = \mathrm{Gr}^{d+1,1}$ and define a morphism $m_{a,d} : \mathrm{Gr}^{d,1} \to \mathrm{Gr}^{d^a,1}$ by sending a surjection $p : \mathcal{O}^d \twoheadrightarrow \mathcal{L}$ to $p^{\otimes a} : (\mathcal{O}^d)^{\otimes a} \twoheadrightarrow \mathcal{L}^{\otimes a}$. One verifies the relation

$$\begin{array}{ccc} \mathrm{Gr}^{d,1} & \xrightarrow{\quad m_{a,d} \quad} & \mathrm{Gr}^{d^a,1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}^{d+1,1} & \xrightarrow{m_{a,d+1}} \mathrm{Gr}^{(d+1)^a,1} = & \mathrm{Gr}^{(d^a+[(d+1)^a-d^a],1} \end{array}$$

and define $m_a : \mathbb{P}^\infty \to \mathbb{P}^\infty$ to be the induced morphism. The relation $m_{ab} = m_a m_b$ is easy. [2]

Definition A.0.7.3. One defines $\mathbb{P}^\infty[\frac{1}{n}]$ (resp. $\mathbb{P}^\infty_{\mathbb{Q}}$) as the inductive limit over m_a's ordered by division for $a = n^k, k \in \mathbb{N}$ (resp. m_a's ordered by division for all $a \in \mathbb{N}$).

---

[2] To make the above a proper definition and make the diagram commute on the nose, one needs to define a natural isomorphism $\delta_{a,d} : (\mathcal{O}^d)^{\otimes a} \to \mathcal{O}^{d^a}$. It can be done as follows. We define a strict total order on $\{e_1^1, e_2^1, \dots e_d^1\} \times \dots \times \{e_1^a, e_2^a, \dots e_d^a\}$, i.e. the structure of the category $[n^a - 1]$ inductively as follows. First $e_{i_1}^1 \times e_{i_2}^2 \times \dots \times e_{i_a}^a < e_{j_1}^1 \times e_{j_2}^2 \times \dots \times e_{j_a}^a$: If $\max i_k < \max j_l$. If there is equality $i_m = \max i_k = \max j_l = j_n$, then if $\max i_k \setminus i_n < \max j_l \setminus j_m$. Repeatedly removing such m, n's we obtain an order on all objects except when the i_k are a permutation of the j_l's. With these, pick the lexicographic order. We then define an isomorphism $\delta_{a,d} : (\mathcal{O}^d)^{\otimes a} \to \mathcal{O}^{d^a}$ by sending a basis-element $e_{i_1}^1 \times e_{i_2}^2 \times \dots \times e_{i_a}^a$ to the basis f_i with $i \in [n^a - 1]$ via the ordering just constructed. As such, the element $e_1^1 \times e_1^2 \times \dots \times e_1^a$ is the smallest element, $e_1^1 \times e_2^2 \times e_1^3 \times \dots \times e_1^a < e_2^1 \times e_1^2 \times e_1^1 \times \dots \times e_1^a$ and $e_d^1 \times e_1^2 \times \dots \times e_1^a$ is generally a big element.

One of the main observations of [VV99] is the following theorem, which states that algebraic K-theory is represented by an infinite Grassmannian. The version presented below is proven in exactly the same way as in the article in question, with the exception of using smooth descent for rational K-theory instead of Nisnevich descent. Note that since any smooth morphism locally for the étale topology has a section we have étale descent whenever we have smooth descent, and in the former case the statement we are looking for is [RT90], Theorem 11.11;

Theorem A.0.8 ([VV99], Theorem 4.3.13). *Let S be a regular scheme. Then we have canonical functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}_\bullet(\mathfrak{R}_{S,sm})}(S^n \wedge X_+, (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}) = \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}(X, \Omega^n(\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}) \simeq K_n(X)_{\mathbb{Q}}$$

for X a regular S-scheme, where K_n refers to Quillen's K-theory defined as above. In particular, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}(X, (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}) \simeq K_0(X)_{\mathbb{Q}}.$$

Proceeding as in [Rio06], Chapitre III, one constructs a product

$$(\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \wedge (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$$

in $\mathcal{H}_\bullet(\mathfrak{R}_{S,sm})$.

Proposition A.0.9. *Consider the natural map $t : \mathbb{P}^1 \to \{0\}\times\mathrm{Gr} \to (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$. Then the data $\mathbf{E} = (\mathbf{E}_i, d_i)$ defined by $\mathbf{E}_i = \mathbb{Z}\times\mathrm{Gr}$ and the product*

$$d_i : \mathbb{P}^1 \wedge (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \stackrel{t\wedge\mathrm{id}}{\to} (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \wedge (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$$

is a naive spectrum, which we denote by K_{naive}.

Proof. We need to show that the natural map

$$(\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \to R\mathrm{Hom}_\bullet((\mathbb{P}^1,\infty), (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}})$$

is an isomorphism. However, this follows from the fact that for any S-scheme X, the map

$$K_n(X) \to \{y \in K_n(\mathbb{P}^1_X), \infty^* y = 0 \in K_n(X)\}$$

given by $x \mapsto x \boxtimes u$, where $u = \mathcal{O}(1) - 1$, is bijective, which in turn is a consequence of the projective-bundle-formula for K-theory. □

Notice there is an obvious forgetful functor

$$\mathcal{SH}(T) \to \mathcal{SH}_{naive}(T).$$

There are a priori several liftings of the naive spectrum constructed above representing algebraic K-theory, and we make the following definition:

Definition A.0.9.1. A stable model for rational algebraic K-theory is an object $\mathcal{K} \in \mathcal{SH}(T)$, together with an isomorphism

$$\omega : forget(\mathcal{K}) \simeq (K_{naive})_{\mathbb{Q}}.$$

Theorem A.0.10 ([Rio06], Chapitre V). *When $S = \operatorname{spec} \mathbb{Z}$ and $T = \mathfrak{R}_{sm}$ there is a unique, up to unique isomorphism, stable model for rational algebraic K-theory, denoted by $\mathbf{BGL}_{\mathbb{Q}}$, thus defining a canonical rational stable model $\mathcal{S}H(\mathfrak{R}_{S,sm})$ for any regular scheme S via $\mathbf{BGL}_{S,\mathbb{Q}} := f^*\mathbf{BGL}_{\mathbb{Q}}$ where $f : S \to \operatorname{spec} \mathbb{Z}$ is the natural morphism.*

Let S be a Nötherian, regular scheme. By the Yoneda lemma, we have a functor

$$\Phi : T \to \mathbf{Shv}(T) \to \Delta^{\mathrm{op}}\,\mathbf{Shv}(T) \to \mathcal{H}(T).$$

If G is any object of $\mathcal{H}(T)$, we denote by ϕG the presheaf on T defined by

$$T \ni U \mapsto \operatorname{Hom}_{\mathcal{H}(T)}(\Phi U, G).$$

In particular, we have an isomorphism

$$\phi(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \simeq K_0(-)_{\mathbb{Q}}.$$

Theorem A.0.11 (Théorème III.29 in [Rio06]). *Let S be a regular scheme. Given two (pointed) presheaves $\mathcal{F}, \mathcal{G}$ on $\mathfrak{R}_S$ denote by $\operatorname{Hom}_{\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(\mathcal{F}, \mathcal{G})$ (resp. $\operatorname{Hom}_{\bullet,\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(\mathcal{F}, \mathcal{G})$) the set of (pointed) natural transformations from $\mathcal{F} \to \mathcal{G}$. Then the natural morphism*

$$\operatorname{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}, (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}) \to \operatorname{Hom}_{\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}})$$

(resp.

$$\operatorname{Hom}_{\mathcal{H}_\bullet(\mathfrak{R}_{S,sm})}((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}, (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}) \to \operatorname{Hom}_{\bullet,\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}}))$$

is bijective.

Theorem A.0.12. *[Théorème IV.72 in [Rio06]] We have a natural decomposition in terms of "Adams eigenspaces",*

$$\mathbf{BGL}_{\mathbb{Q}} \simeq \bigoplus_{i\in\mathbb{Z}} \mathbb{H}^{(i)}$$

In $SH(\mathfrak{R}_S)$.

Since the above theorem will play a crucial role, we will mention the highlights of the proof.
First, put

$$p_n = \frac{1}{n!}\log^n(1+U) \in \mathbb{Q}[[U]],$$

and

$$\sum k^n p_n = (1+U)^k = \Psi^k.$$

This element will play the role of the k-th Adams operator. Let A be an abelian group and define the operator Ω on $A[[U]]$ by the formula

$$\Omega(f) = (1+U)\frac{df}{dU}$$

and by A^{Ω} the following inverse limit:

$$\ldots \xrightarrow{\Omega} A[[U]] \xrightarrow{\Omega} A[[U]] \xrightarrow{\Omega} A[[U]].$$

Now, by [Rio06], Theorem IV. 55, then we have bijections

$$\mathrm{Hom}_{\mathcal{SH}(S)}(\mathbf{BGL}, \mathbf{BGL}_{\mathbb{Q}}) \simeq \lim(K_0(S)_{\mathbb{Q}})^{\Omega},$$

and there is a natural map

$$\mathbb{Q}^{\mathbb{Z}} \to \lim(K_0(S)_{\mathbb{Q}})^{\Omega}$$

induced by a commutative diagram (whenever A is a $\mathbb{Q}$-vector space, see loc.cit. Lemme IV. 66):

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\sigma} & A[[U]] \\ \downarrow s & & \downarrow \Omega \\ A^{\mathbb{N}} & \xrightarrow{\sigma} & A[[U]] \end{array}$$

where $s((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{\mathbb{N}}$ and $\sigma((a_n)_{n\in\mathbb{N}})) = \sum_{n\in\mathbb{N}} a_n p_n$. Here this induces an isomorphism $\Sigma : A^{\mathbb{Z}} \simeq \lim A^{\Omega}$ defined by $[(a_i)_{i\in\mathbb{Z}}]_{i\in\mathbb{Z}} \mapsto \sum a_{i+n}p_n \in$

$A[[U]]$(loc.cit. Corollaire IV.67), and this is the map alluded to above. Composing the above two maps, we obtain a map

$$\mathbb{Q}^{\mathbb{Z}} \to \mathrm{End}_{\mathcal{SH}(\mathfrak{R}_S)}(\mathbf{BGL}_{\mathbb{Q}}).$$

Consider the characteristic function π_i associated to $\{i\}$ viewed as an element $\mathbb{Q}^{\mathbb{Z}}$. It corresponds naturally to the element $(p_{i+n})_{n\in\mathbb{N}}$ in $\lim \mathbb{Q}^{\Omega}$ (where we put $p_k = 0$ for $k < 0$). They naturally define an orthogonal family of idempotents in $\mathrm{End}_{\mathcal{SH}(S)}(\mathbf{BGL}_{\mathbb{Q}})$. $\mathcal{SH}(\mathfrak{R}_S)$ being an pseudo-abelian category, we can consider the image of π_i, and we denote it by $\mathbb{H}^{(i)}$. It is formal that Ψ^k above acts as multiplication by k^i on $\mathbb{H}^{(i)}$.

Corollary A.0.13 ([Rio06], Corollaire IV.75). *We have a decomposition*

$$K_i(X)_{\mathbb{Q}} = \bigoplus_{j\in\mathbb{N}} K_i(X)^{(j)}.$$

Definition A.0.13.1. Let $\mathcal{C}$ be a closed simplicial model category, and suppose that $\mathcal{X}$ is an object of $\mathcal{C}$. Given a fibrant replacement $\mathcal{X} \to \mathcal{X}'$, consider the functor $V_{\mathcal{X}}$ taking an object X of $\mathcal{C}$ to the fundamental groupoid of $\mathbf{hom}(X, \mathcal{X}')$. This is independent up to unique isomorphism of the choice of fibrant replacement by abstract nonsense. We call $V_{\mathcal{X}}$ the associated category fibered in groupoids over $\mathcal{C}$. A 1- and 2-morphism of categories fibered in groupoids over $\mathcal{C}$ is the standard one and we denote by $\mathrm{Hom}_f(V_{\mathcal{X}}, V_{\mathcal{Y}})$ the set of 1-morphisms $V_{\mathcal{X}} \to V_{\mathcal{Y}}$ strictly functorial with respect to pullback.

Very often, these groupoids have the structure of Picard categories and they form Picard categories fibered in $\mathcal{C}$. Recall from [Fra91], 3.6 that a Picard category fibered over $\mathcal{C}$, P, is, for every object X of $\mathcal{C}$, a Picard category P_X and for every morphism $X \to Y$ an additive functor $P_Y \to P_X$ compatible with composition in the obvious sense.
The following proposition is formal, and is surely known in more generality:

Proposition A.0.14. *[Pre-rigidity, proof of [Rio06], Chapitre III, section 10] Let T be as above and consider the category of pointed or unpointed simplicial (pre-)sheaves on T. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are objects thereof, with $\mathcal{X}$ cofibrant and $\mathcal{Y}$ fibrant, with associated fibered categories in groupoids $V_{\mathcal{X}}, V_{\mathcal{Y}}$, and suppose that*

- $\mathrm{Hom}_{\mathcal{H}(C)}(\mathcal{X}, \Omega\mathcal{Y}) = 0.$
- $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$ *is an H-group.*

Then we have a canonical map

$$\mathrm{Hom}_{\mathcal{H}(C)}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_f(V_{\mathcal{X}}, V_{\mathcal{Y}})$$

which associates to an element of the left a functor of fibered categories ϕ : $V_{\mathcal{X}} \to V_{\mathcal{Y}}$, canonical up to unique isomorphism.

Proof. If Φ is in $\mathrm{Hom}_{\mathcal{H}(C)}(\mathcal{X}, \mathcal{Y}) = \pi_0(\mathbf{hom}(\mathcal{X}, \mathcal{Y}))$, it induces for any $X \in C$ a map $\phi_X : \mathcal{V}_{\mathcal{X}}(X) \to \mathcal{V}_{\mathcal{Y}}(X)$, functorial in X, by choice of a representative ϕ of Φ in $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$. If ϕ and ϕ' induce the same homotopy-class, there is a homotopy $h : \Delta^1 \times \mathcal{X} \to \mathcal{Y}$ from ϕ to ϕ' which gives an isomorphism $\mathrm{iso}_{h,X} : \phi_X \to \phi'_X$. Moreover, it is easy to see that if there are two homotopies h and h' which are homotopic, they induce the same isomorphism of functors. The obstruction for $\mathrm{iso}_{h,X}$ to be canonical lies in the fundamental group of $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$ which can be identified with $\mathrm{Hom}_{\mathcal{H}(C)}(\mathcal{X}, \Omega\mathcal{Y})$ which is 0 by assumption. □

B. LOCALIZATION OF PICARD CATEGORIES AND THE CASE OF QUOTIENTS OF SPLIT REDUCTIVE GROUPS.

In this chapter we define the localization of a Picard category and explain the main examples.

Definition B.0.14.1. Let P be a commutative Picard category, and let S be a multiplicative totally ordered set acting on P. Then we define the localized Picard category $S^{-1}P$ in the follow way: Elements are objects of the form $\frac{1}{s}p := (s,p)$ where $s \in S$, and $p \in P$, and we equip it with the structure of a commutative Picard category via

$$\frac{1}{s}p + \frac{1}{s'}p' = \frac{1}{ss'}(s'p + sp')$$

and

$$\mathrm{Hom}_{S^{-1}P}(\frac{1}{s}p, \frac{1}{s'}p') = \varinjlim_{d \in S} \mathrm{Hom}_P(ds'p, dsp').$$

The natural localization functor $\pi_S : P \to S^{-1}P$ is clearly universal with respect to additive S-equivariant functors of P to commutative Picard-categories which are S-divisible in the sense that multiplication by any $s \in S$ gives an autoequivalence.
The first example of localizations of Picard categories are given by localization at the multiplicative set $\{n^i, i \in \mathbb{Z}\}$ for some integer n. We can also localize with respect to $\mathbb{N} \setminus \{0\}$. The associated categories are denoted by $P[\frac{1}{n}]$ and $P_{\mathbb{Q}}$ respectively. In particular we can localize the virtual category of an algebraic stack $V(\mathcal{X})$.
When dealing with functorial Lefschetz later it will be necessary to localize with respect to other multiplicative sets. We treat the case of diagonalizable groups but to keep true to [Tho92] and later developments by for example [GV02] we also mention the case of a split reductive group [1]
First, recall that a diagonalizable group over a scheme S (c.f. [AG70a], I. 4.4, [AG70b]), §1 is a group scheme determined by a constant character-group (supposed to be of finite type). M, it will be denoted by $D_S(M)$. When

[1] The author hopes to treat this case in a near future.

$M = \mathbb{Z}^r$, $D_S(M) = \mathbb{G}_{m,S}^r$ and when $M = \mathbb{Z}/m$, $D_S(M) = \mu_{m,S}$. Recall that a split reductive group-scheme over a basescheme S is a smooth affine S-groupscheme with geometrically connected fibers that are reductive groups and with a split (i.e. diagonalizable) maximal torus (c.f. [AG70c], XXII 1.13, XIX 1.6, 2.7). These groups are uniquely defined over $\mathbb{Z}$ (c.f. [AG70c], XXV 1.1, 1.2, XXIII 1.1).

Lemma B.0.15 ([AG70a], Proposition 4.7.3). *Let D be a diagonalizable group corresponding to a finitely generated abelian group M, and let X be an S-scheme with trivial D-action. Then there is a natural equivalence between D-equivariant vector bundles on X and the category of M-graded vector bundles.*

Corollary B.0.16. *Let D, X, S be as in the previous lemma. Then if $[X/D]$ denotes the quotient stack, there is a natural equivalence of virtual categories*

$$V([X/D]) = V(X, D) = \times_{m \in M} V(X).$$

In other words, a D-equivariant virtual bundle on X is diagonalizable.

Definition B.0.16.1. Let G be a split reductive or diagonalizable group over $\operatorname{spec}\mathbb{Z}$ and denote by $R(G) = K_0(\mathbb{Z}, G)$ the representation-ring of G, i.e. the group of representations of G over $\operatorname{spec}\mathbb{Z}$. Suppose G acts on an algebraic space X and denote by $[X/G]$ the quotient stack. If ρ is a prime ideal of $R(G)$, let $S = R \setminus \rho$. Then S acts on $V([X/G]) = V(X, G)$ and we define the localization $V(X, G)_{(\rho)}$ to be the localization with respect to this set.

C. ALGEBRAIC STACKS

In this section we recall the theory of algebraic stacks. It is neither self-contained nor complete, and we refer the reader to for example [GL00] or [DM82] for more exhaustive treatments. Recall the following notion of an algebraic space over a base-scheme S ([GL00], définition 1.1). This is a sheaf on the category of affine schemes over S equipped with the etale topology coming with some additional algebraicity conditions.

We recall the following definitions (see Définition 3.1 and Définition 4.1, [GL00]):

Definition C.0.16.2. A pre-stack over S, or S-prestack, is an category $\mathcal{X}$ fibered in groupoids over $(Aff/S)_{et}$ such that

(a) The set of isomorphisms is a sheaf.
It is furthermore a stack, if

(b) For any covering $(\varphi_i : V_i \to U))$ in $(Aff/S)_{et}$, any descent datum (x_i, f_{ij}) relative to this cover is effective, i.e. $\mathcal{X}$ is a sheaf of groupoids. More generally, for any site $\mathcal{C}$, a stack on $\mathcal{C}$ is to be understood as a sheaf of groupoids.

(c) ([GL00], Définition 3.10.1) A morphism of stacks $F : \mathcal{X} \to \mathcal{Y}$ is said to be representable (schematic), if for every $U \in (Aff/S)$ and $y : U \to \mathcal{Y}$, the morphism $F_U : \mathcal{X} \times_{F,\mathcal{Y},y} U \to U$ is a morphism of S-algebraic spaces (S-schemes). Furthermore, given a property P of morphisms of algebraic spaces $X \to Y$, stable by base-change $Y' \to Y$ and is local for the etale topology on Y, we say that a representable morphism $F : \mathcal{X} \to \mathcal{Y}$ possess the property P if F_U does for every $U \in (Aff/S), y : U \to \mathcal{Y}$. For example, separated, quasi-compact, locally of finite type, locally of finite presentation, of finite type, an open immersion, a closed

immersion, affine, quasi-affine, finite, quasi-finite, proper, flat, non-ramified, smooth, étale and closed regular immersion. See loc.cit. 3.10 for additional properties.

We say that a stack is algebraic (or Artin) if

(d) The diagonal 1-morphism

$$\mathcal{X} \xrightarrow{\Delta} \mathcal{X} \times_S \mathcal{X}$$

is representable, separated and quasi-compact.

(e-i) There exists an algebraic S-space X (a presentation of $\mathcal{X}$), and a surjective, smooth 1-morphism

$$X \xrightarrow{P} \mathcal{X}.$$

(e-ii) It is said to be Deligne-Mumford if the above presentation can be chosen to be étale.

(f) Let P be a property of morphisms between algebraic spaces such that if for any smooth $X' \to X$, $X \to Y$ has property P if and only if $X' \to Y$ does. We then say that a 1-morphism (not necessarily representable) $F : \mathcal{X} \to \mathcal{Y}$ has property P if for any smooth presentation $y : Y \to \mathcal{Y}$ and smooth presentation $x : X' \to \mathcal{X} \times_{x,\mathcal{Y},F} Y$, $X' \xrightarrow{x} \mathcal{X} \times_{x,\mathcal{Y},F} Y \xrightarrow{F_y} Y$ has property P.

Furthermore, let T be a full subsite of $(Aff/S)_{et}$ (i.e. a full subcategory with a Grothendieck topology such that a cover in the former is one in the latter). By abuse of language, we say that that a category fibered in groupoids over T is resp. a stack, an algebraic stack or Deligne-Mumford stack if it is the restriction of a stack, algebraic stack or Deligne-Mumford stack.

We have the following examples;

- Any scheme or algebraic space is an algebraic stack. Algebraic spaces are exactly the algebraic stacks with trivial automorphism-group.
- ([GL00], Cor. 10.8) Let S be an scheme, $Y \to X$ a morphism of algebraic S-spaces. Also, let G be a separated algebraic X-space group-object, flat and of finite presentation over X. Then the fppf stack-quotient $[Y/G/X]$ is an algebraic stack.

- ([Knu83], Theorem 2.7) Let $2g - 2 + n > 0$. Then the moduli space of n-pointed stable curves of genus g, $\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n,\operatorname{spec}\mathbb{Z}}$ is a proper, smooth Deligne-Mumford stack over $\operatorname{spec}\mathbb{Z}$. The moduli space of n-point smooth curves of genus g is denoted by $\mathcal{M}_{g,n}$. To be explicit, in the latter case, for a scheme T, the objects of $\mathcal{M}_{g,n}(T)$ are smooth morphisms $p : C \to T$ with curves of genus g-fibers plus n distinct sections. A morphism from $p : C \to T$ and $p' : C' \to T$ is an isomorphism $f : C \to C'$ such that $p = p'f$ and preserving the sections.

- ([GL00], 2.4.6 and 4.6.3) The moduli space of principally polarized abelian schemes of dimension g, $\mathcal{A}_{g,1} = \mathcal{A}_{g,1,\operatorname{spec}\mathbb{Z}}$, is a Deligne-Mumford stack over $\operatorname{spec}\mathbb{Z}$.

The following is a version of the definition [GL00], Application 14.3.4

Definition C.0.16.3. We say that a representable 1-morphism $F : X \to Y$ of algebraic stacks is projective (or quasi-projective) if there is a coherent locally free sheaf $\mathcal{E}$ on Y and 2-commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{I} & \mathbb{P}(\mathcal{E}) \\ & \searrow & \downarrow{\scriptstyle P} \\ & & Y \end{array}$$

with I a (representable) closed immersion (resp. quasi-compact immersion) and P is the canonical projection.[1]

We have the following examples of projective morphisms.

- ([Tho87b], Corollary 3.4) Suppose Y is a normal scheme, S separated and Nötherian, $G \to S$ a separated group scheme of finite type. Suppose that we have a G-equivariant morphism $f : Y \to S$, i.e. a morphism $[Y/G/S] \to [S/G/S]$ which is a projective morphism of schemes. The morphism $[Y/G/S] \to [S/G/S]$ is projective in the following cases:
  - S is regular, $\dim_{Krull} X \leq 2$, G is affine and smooth with connected fibers.
  - G is semi-simple or split reductive, S has an ample family of line bundles, for example S affine or regular.
  - S regular, and G reductive.

[1] This is what many authors call a "strongly projective" (or "strongly quasi-projective") 1-morphism.

It follows as in [AG67], Proposition 5.5.5 (v) that if $f : X \to Y$ is an equivariant morphism of normal S-schemes with $Y \to S$ separated, and X is projective over the base S, then $f : X \to Y$ is actually equivariantly projective.

- ([PD69], [Knu83]) Consider the universal n-pointed (by $s_1, \ldots, s_n$) stable curve of genus g, $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, together with the relative dualizing sheaf $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$. For $k \geq 3$, the line bundle $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(s_1 + s_2 + \ldots + s_n)^{\otimes k}$ is very ample with respect to π and embeds $\overline{\mathcal{C}}_{g,n}$ into $\mathbb{P}(\pi_*(\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(s_1+s_2+\ldots+s_n)^{\otimes k}))$ and $\pi_*\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(s_1+s_2+\ldots+s_n)^{\otimes k}$ is locally free.
- The universal principally polarized abelian scheme of dimension g, $\pi : \mathcal{V}_{g,1} \to \mathcal{A}_{g,1}$ has a principal polarization λ and 4λ defines an embedding into projective space $\mathbb{P}^N_{\mathcal{A}_{g,1}}$ for $N = 4^g - 1$.

Given a presentation $X \to \mathcal{X}$, we can associate to it the following groupoid: $[X_1 \rightrightarrows X_0]$, where $X_0 = X$ and $X_1 = X \times_{\mathcal{X}} X$, the two arrows being the natural projections. More generally, and conversely, we have the following result:

Proposition C.0.17 ([GL00], Corollaire 10.6). *Let $\mathcal{X} = [s, t : X_1 \rightrightarrows X_0]$ be a groupoid of algebraic spaces X_0, X_1, and s, t both be faithfully flat and of finite presentation. Furthermore suppose that $\delta = (s, t) : X_1 \to X_0 \times_S X_0$ is separated and quasi-compact. Then the associated fppf S-stack (see [GL00], 9.3), $\mathcal{X}_{fppf}$, is an algebraic S-stack.*

Given an algebraic stack $\mathcal{X}$, an algebraic space X and a fppf-morphism $X \to \mathcal{X}$, the associated groupoid $[X_1 \rightrightarrows X_0]$ is an fppf-presentation of $\mathcal{X}$.

We now recast the above in a setting which will make it more natural to apply various auxiliary results, which is that of a simplicial setting.

Definition C.0.17.1. Let T be a site. The category of presheaves and sheaves on this site is denoted by $pShv(T)$ and $Shv(T)$ respectively.

The category of simplicial objects of a category $\mathcal{C}$, i.e. functors $\Delta^{op} \to \mathcal{C}$, is denoted by $\Delta^{op}\mathcal{C}$ or $s\mathcal{C}$.

Recall that whenever T has enough points a morphism of simplicial presheaves in T is said to be a local equivalence if it induces weak equivalences of simplicial sets on all stalks.
Let $U \to X$ be a morphism of an object X in T. The nerve of this morphism is the simplicial object $\mathcal{N}(U/X)$ whose n-simplices are given by the product

$U \times_X U \times_X U \ldots \times_X U$ (n times). Given a presheaf of simplicial sets on T we have an associated cosimplicial functor $\Delta \to \text{Set}$, $[n] \mapsto \mathcal{F}(\mathcal{N}(U/X)_n)$. The Cech cohomology with respect to the covering $U \to X$ is the simplicial set

$$\mathbf{H}(U/X, \mathcal{F}) := \text{holim}_\Delta\, \mathcal{F}(\mathcal{N}(U/X)_n).$$

We say that $\mathcal{F}$ satisfies descent if for any X and any covering $U \to X$ in T, the map

$$\mathcal{F}(X) \to \mathbf{H}(U/X, \mathcal{F})$$

is a weak equivalence.

Definition C.0.17.2. A presheaf $\mathcal{F}$ of simplicial sets on a site T is said to be flabby, if for any (and thus each) simplicially fibrant replacement $\mathcal{F} \to \mathcal{F}'$, and any $X \in T$, the map

$$\mathcal{F}(X) \to \mathcal{F}'(X)$$

is a weak equivalence of simplicial sets.

Theorem C.0.18 ([Toe99a], Theoreme 1.2). *$\mathcal{F}$ is flabby if and only if it satisfies descent.*

Thus any simplicially fibrant simplicial presheaf satisfies descent. It follows from the definition that a groupoid is flabby if and only if it is a stack. If $\mathcal{X}$ is an S-stack, there is sheaf of simplicial sets defined as follows: Let U be an object in (Aff/S), and let $\overline{\mathcal{X}}$ be the associated fibered category over (Aff/S). The category $F_{\mathcal{X}}(U) := Hom_{Cat/S}(\overline{U}, \mathcal{X})$ is a groupoid, and its nerve is a simplicial set $BF_{\mathcal{X}}$.

Definition C.0.18.1. Let T be a site, and consider the category of simplicial presheaves on T, $\Delta^{op} p\,\mathbf{Shv}(T)$. The full subcategory of simplicial sheaves is denoted by $\Delta^{op}\,\mathbf{Shv}(T)$. If $\mathfrak{Ch}$ is the category of stacks on T, we call the functor $B : \mathfrak{Ch}(T) \to \Delta^{op}\,\mathbf{Shv}(T)$ constructed above the extended Yoneda functor.

Proposition C.0.19 ([Hol], Corollary 4.5, Theorem 5.4). *Suppose that T is a (small) site with enough points with associated simplicial homotopy category $\mathcal{H}_s(T)$ (see Theorem A.0.5). Then the functor $\mathbf{Shv}(T) \to \mathcal{H}_s(T)$ determined by the Yoneda embedding is a full embedding. The essential image of the extended Yoneda functor from $\mathfrak{Ch}$ is equivalent to the category whose objects are stacks and morphisms are 1-morphisms up to 2-morphism.*

Furthermore, a (cartesian) quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module on an algebraic stack $\mathcal{X}$ viewed as a simplicial set is an assignment of a quasi-coherent (resp.

coherent, locally free, etc) $\mathcal{F}_n$ on each $\mathcal{X}_n$ such that for any $\phi : [n] \to [m]$ we have an isomorphism $\phi^* : \phi^*\mathcal{F}_n \to \mathcal{F}_m$ compatible with compositions $[n] \to [n'] \to [n'']$. Coherent and locally free sheaves are defined analogously. As an example (c.f. [Del74], 6.1.2), let G be a group scheme, finitely presented, separated and faithfully flat over a scheme S. Let X be an algebraic space over S. We say that G acts on X if there is a morphism $\mu : G \times_S X \to X$ satisfying the usual associativity and unit-constraints. If $\mathcal{F}$ is a $\mathcal{O}_X$-module, we say that G acts on $\mathcal{F}$, or that $\mathcal{F}$ is G-equivariant, if there is an isomorphism of $\mathcal{O}_{G\times_S X}$-modules

$$\phi : \mu^*\mathcal{F} = p_2^*\mathcal{F}$$

satisfying the associativity constraint, on $G \times_S G \times_S X$:

$$p_{23}^*(1 \times \mu)^*\phi = (\mu \times 1)^*\phi.$$

We employ the analogous definition for complexes of quasi-coherent $\mathcal{O}_X$-modules. To an algebraic space X with a group action G, we can form the following simplicial algebraic space:

$$[X/G/S] := X \leftleftarrows\!\!\!\to G \times_S X \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow}\; G \times_S G \times_S X \ldots$$

Here the maps are either projection or multiplication-maps, and the non-written arrows in the other directions are given by repeated applications of the unit-map e. The above condition that $\mathcal{F}$ is G-equivariant can equivalently be rephrased as that $\mathcal{F}$ is the degree 0-part of a cartesian $\mathcal{O}_{[X/G/S]}$-module on $[X/G/S]$ with descent-data.

Yet another way of defining a quasi-coherent $\mathcal{O}_\mathcal{X}$ on an algebraic stack $\mathcal{X}$, is in the following way: Given an algebraic space U and a 1-morphism with U an algebraic space, $s : U \to \mathcal{X}$, we have an quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_s$ on U. Given two 1-morphisms of algebraic spaces $s : U \to \mathcal{X}, t : V \to \mathcal{X}$, a morphism $f : U \to V$, and a 2-isomorphism $h : t \circ f \Rrightarrow s$, an isomorphism

$$\phi_{f,t,h} : f^*\mathcal{F}_t \simeq \mathcal{F}_s.$$

Given morphisms of algebraic spaces $U \xrightarrow{f} V \xrightarrow{g} W$, and 1-morphisms $s : U \to \mathcal{X}, t : V \to \mathcal{X}, w : W \to \mathcal{X}$, and 2-isomorphisms $h : t \circ f \Rrightarrow s$ and $j : w \circ g \Rrightarrow t$ an equality

$$\phi_{f,t,h} \circ f^*\phi_{g,w,j} = \phi_{f\circ g,w,h\circ j}.$$

Given two quasi-coherent $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ and $\mathcal{E}$, a morphism between them is morphism $\mathcal{F}_s \to \mathcal{E}_s$ for every morphism $s : U \to \mathcal{X}$ with U an algebraic space compatible with the isomorphism ϕ in the obvious way.

Definition C.0.19.1. The Quillen K-theory space of an algebraic stack $\mathcal{X}$, $K(\mathcal{X})$ is defined to be the space ΩBQC, with C being the exact category of (coherent) vector bundles on X. The K-theory groups $K_i(\mathcal{X})$ are defined to be π_i of the corresponding loops-space ΩBQC. Similarly, one defines the G-theory space and G-theory of an algebraic stack $\mathcal{X}$, $G_i(\mathcal{X})$, as the corresponding object considering the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules instead.

The main standard properties of $K-$ and G-theory are summarized in the following theorem (compare with [Toe99a], Proposition 2.2, note however that it does not seem to be true that most of the results in this proposition automatically generalize from the case of schemes. Indeed, this is the main point of the article [Tho87a] where the equivariant versions of non-cohomological K and G-theory are studied):

Theorem C.0.20. *Fix a separated algebraic stack $\mathcal{X}$. Then we have*

- *$K(-)$ is contravariantly functorial with respect to 1-morphisms of algebraic stacks, and is covariantly functorial with respect to representable projective morphisms between algebraic stacks with the resolution property.*
- *$G(-)$ is covariantly functorial with respect to proper representable 1-morphisms.*
- *Let $\mathcal{E}$ be a vector bundle of rank n on X, and consider the canonical bundle $\mathcal{O}(1)$ on $\pi : Proj_X(Sym^\bullet\mathcal{E}) = \mathbb{P}(\mathcal{E}) \to \mathcal{X}$. Then we have a homotopy equivalence*

$$\bigvee_{j=0}^{n-1} K(X) \to K(\mathbb{P}(\mathcal{E}))$$

  induced by $(f_j)_{j=0}^{n-1} \mapsto \sum_{j=0}^{n-1} \pi^* f_j \otimes \mathcal{O}(-j)$. Same formula holds for G.
- *Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$, and T a torsor of $\mathcal{E}$ over $\mathcal{X}$. Then $G(\mathcal{X}) \to G(T)$ is a homotopy equivalence.*

Proof. The first result is proven as in [Tho87a], Theorem 3.1. and most of the results are proven using the classical techniques or modifying the same using loc.cit. As we shall only need the above theorems in the special cases of their associated virtual categories we will contend ourselves with the above statements without proofs. □

An additional object will enter into our stage, K-cohomology, which in this form is borrowed from [Toe99a].

Definition C.0.20.1. Let $T = Aff/S_{sm}$, the category of affine S-schemes with the smooth topology. Denote by $K_{\mathbb{Q}}^{TT}$ a T-simplicially fibrant model of the simplicial presheaf on T that represents rational Thomason algebraic K-theory and let X be a simplicial T-sheaf. The K-cohomology K^{sm} is the simplicial presheaf (automatically flabby) $X \mapsto K^{sm}(X) := \mathbf{hom}(X, K_{\mathbb{Q}}^{TT})$. We define the K-cohomology groups $K_i^{sm}(X)$ to be $\pi_i(\mathbf{hom}(X, K_{\mathbb{Q}}^{TT}))$. Also define $G_{\mathbb{Q}}^{sm}$ to be the G-cohomology of [Toe99a].

The definition of $K^{sm}(X)$ of [Toe99a] is different, and exhibits $K^{sm}(\mathcal{X})$ more properly as a S^1-spectrum. But by *ibid* Proposition 2.2, the given spectrum is flabby when restricted to the small smooth site on the algebraic stack (i.e. a smooth presentation is a cover) and equal to ordinary (rational) Thomason K-theory for a regular Nötherian finite dimensional algebraic space or scheme. Because holim preserves weak equivalences, for a regular stack with smooth presentation $X \to \mathcal{X}$, we have weak equivalences $K^{sm}(\mathcal{X}) = \mathbf{H}(X/\mathcal{X}), K^{sm}) = \mathbf{H}(X/\mathcal{X}, K_{\mathbb{Q}}^{TT}) = \mathbf{hom}(\mathcal{X}, K_{\mathbb{Q}}^{TT})$ so Toen's K-cohomology necessarily coincides with our K-cohomology in this case. Also recall that for a scheme in addition to being finite dimensional Nötherian admit an ample family of line bundles $K_{\mathbb{Q}}^{TT}(X)$ represents rational Quillen K-theory. By [Tho85], Theorem 2.15 rational G-theory has étale descent for separated Nötherian schemes of finite Krull-dimension and thus rational G-theory has descent for algebraic spaces. It should be noted that Toen's corresponding G-cohomology theory does not have smooth descent in general so cannot be defined as values of an algebraic stack in some simplicial sheaf representing G-theory in $\mathbb{A}^1$-homotopy theory.
By [Toe99a], Proposition 1.6, there is a natural transformation $K \to K_{\mathbb{Q}}^{TT}$ that can be realized as, for a smooth presentation $X \to \mathcal{X}$, the augmentations $K(\mathcal{X}) \to \mathbf{H}(X/\mathcal{X}, K_{\mathbb{Q}})$.
With these remarks it follows from Theorem A.0.8 that we have the following proposition:

Proposition C.0.21. *Let $T = \mathfrak{R}_{S,sm}$ be the category of regular S-schemes with the smooth topology. Then for any regular algebraic S-stack $\mathcal{X}$ there is an $\mathbb{A}^1$-weak equivalence $K_{\mathbb{Q}}^{TT} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ so that*

$$K_i^{sm}(\mathcal{X}) = \mathrm{Hom}_{\mathcal{H}(T)}(\mathcal{X}, R\Omega^i(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}).$$

Proposition C.0.22 ([Toe99a], Proposition 2.2)**.** *The conclusions of Theorem C.0.20 hold with K (resp. G) replaced by K^{sm} (resp. G^{sm}), at least whenever restricted to the category of regular stacks. Moreover, for a regular algebraic stack there is Poincaré duality; the natural map $K^{sm}(\mathcal{X}) \to G^{sm}(\mathcal{X})$ is a weak equivalence.*

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