0. Introduction. In the whole paper, we will say that $i: M' \rightarrow M$ is an immersion of smooth manifolds if $M'$ is a submanifold of $M$, and if $i$ is the corresponding injection map. In particular the topology of $M'$ is the topology induced by the topology of $M$. In differential geometry, such maps $i: M' \rightarrow M$ are also called embeddings.

Let $i: M' \rightarrow M$ be an immersion of complex manifolds, let $\eta$ be a holomorphic vector bundle on $M'$, let $(\xi, v)$ be a holomorphic complex of vector bundles on $M$ which provides a resolution of the sheaf $i_* \mathcal{O}_{M'}(\eta)$.

We assume that the vector bundle $\eta$, the normal bundle $N$ to $M'$ in $M$ and the complex $\xi$ are equipped with Hermitian metrics $g^\eta$, $g^N$, $h^\xi$. The purpose of this paper is to construct a current $T(h^\xi)$ on $M$ which has three essential properties:
Under certain compatibility assumptions between the various metrics, then the following equation of currents holds

\[ \frac{\partial \bar{\partial}}{2i\pi} T(h^t) = Td^{-1}(g^N) \text{ch}(g^u)\delta_{M'} - \text{ch}(h^t) \]

In (0.1), the forms on the right hand side are obtained in Chern-Weil theory by using the holomorphic Hermitian connections associated with the given metrics. Also \( \delta_{M'} \) denotes the current on \( M \) given by integration on \( M' \).

- The wave front set of the current \( T(h^t) \) is included in the real conormal bundle \( N^*_R \) to \( M' \) in \( M \).
- The current \( T(h^t) \) pulls back naturally by holomorphic maps which are transversal to \( M' \).

In the case where \( M' = \phi \), i.e., if the complex \((\xi, v)\) is acyclic, currents similar to \( T(h^t) \) were already considered by Bott-Chern [BoC] and Donaldson [D]. In this case, the form \( T(h^t) \) was obtained in [BGSl] by using Quillen’s superconnections [Q1] and the number operator of the complex \((\xi, v)\), and was called a Bott-Chern form.

To construct our current \( T(h^t) \) in full generality, we essentially use the results of Bismut [B2]. In [B2], it was proved that the Quillen superconnection Chern character forms \( \omega_u \) (which here depend on a parameter \( u \) which scales the chain map \( v \)) converge as \( u \to +\infty \) to a current \( \omega_\infty \) which is explicitly determined. Also the speed of convergence of \( \omega_u \) towards \( \omega_\infty \) was studied in [B2], and certain key microlocal estimates were established. Our construction of \( T(h^t) \) then combines the double transgression formulas of [BGSl] for superconnections with the results of [B2]. The current \( T(h^t) \) is here called a Bott-Chern current.

The current \( T(h^t) \) is smooth on \( M \setminus M' \), and in general is not locally integrable. We study the singularity of \( T(h^t) \) near \( M' \), and we also calculate \( T(h^t) \) as a principal part (in the sense of distributions) of its restriction to \( M \setminus M' \).

In the whole paper, we assume the considered manifolds to be compact. Most of our results extend to the non compact case in the obvious way.

Our main motivation for introducing the current \( T(h^t) \) is related to our previous work on direct images of Hermitian vector bundles. In fact, in our earlier work [BGSl, 2, 3], we calculated the curvature of the holomorphic Hermitian connection on the determinant of a direct image equipped with the Quillen metric [Q2], and we obtained a differential form version of the Theorem of Riemann-Roch-Grothendieck. In our situation, the determinant lines \( \lambda(\eta) \) and \( \lambda(\xi) \) (which are the determinants of the cohomology of \( \eta \) and \( \xi \)) are canonically isomorphic. In view of the results of [BGSl, 2, 3] and of (0.1), one may suspect that the current \( T(h^t) \) will play a role in comparing the metrics on \( \lambda(\eta) \) and \( \lambda(\xi) \).

In our forthcoming work [BGSS], we give a direct verification that the currents \( T(h^t) \) verify certain functorial properties, which make them natural candidates to appear in a formula comparing the Quillen metrics of \( \lambda(\eta) \) and \( \lambda(\xi) \). The wave front set properties of \( T(h^t) \) play a key role in [BGSS], essentially because we can form
the product of such currents associated with submanifolds \( M' \) and \( M'' \) intersecting transversally. Also in [BGSS5], we relate our current \( T(h^2) \) to the arithmetic characteristic classes of Gillet and Soulé [GS].

Our paper is organised as follows. In Section 1, we recall the main results of [B2]. In Section 2, we construct our current \( T(h^2) \) and we prove it pulls back naturally under maps transversal to \( M' \). In Section 3, we describe the singularity of \( T(h^2) \) near \( M' \) and we calculate \( T(h^2) \) as a principal part.

The results contained in this paper were announced in [BGS4].

Acknowledgements. The authors are indebted to the referee of this paper for his comments and suggestions.

I. Superconnection currents and resolutions of vector bundles. Let \( i: M' \to M \) be an immersion of compact complex manifolds, let \( \eta \) be a holomorphic Hermitian vector bundle on \( M' \), and let \((\xi, v)\) be a holomorphic complex of Hermitian vector bundles on \( M \) which provides a resolution of the sheaf \( i_* \mathcal{O}_{M'}(\eta) \).

In this section, we recall the results of Bismut [B2] on the large parameter behavior of certain Quillen's superconnection forms associated with \((\xi, v)\). In particular, the microlocal estimates in [B2, Section 3] will play a key role in establishing the main properties of the Bott-Chern current, which we construct in Section 2.

This section is organized as follows. In a), we describe the main properties of the complex \((\xi, v)\). In b), we recall the assumption (A) of [B2] which is a compatibility condition between metrics on the complex \((\xi, v)\) and metrics on the normal bundle \( N \) to \( M' \) in \( M \) and on the vector bundle \( \eta \). In c), we introduce the wave front sets of currents on \( M \). In d), we give some properties of Quillen's superconnections [Q1]. In e), we recall the double transgression formulas of Bismut-Gillet-Soulé [BGSI] for superconnection forms. Finally in f), we describe the results established in Bismut [B2].

a) A holomorphic chain complex. Let \( M \) be a compact connected complex manifold of complex dimension \( d \). Let \( M' \) be a finite disjoint union of compact connected complex submanifolds of \( M' \). Let \( i \) be the immersion \( M' \to M \).

Let \( N \) be the complex normal vector bundle to \( M' \) in \( M \), and let \( N^* \) be its dual. Let

\[
(\xi, v): 0 \to \xi_0 \xrightarrow{v} \xi_{m-1} \to \cdots \to \xi_0 \xrightarrow{v} 0
\]

be a chain complex of holomorphic vector bundles on \( M \).

Let \( \eta \) be a holomorphic vector bundle on \( M' \). We assume there is a holomorphic restriction map \( r: \xi_{0|M'} \to \eta \) which is such that we have an exact sequence of sheaves

\[
0 \to \mathcal{O}_M(\xi_0) \xrightarrow{v} \mathcal{O}_M(\xi_{m-1}) \xrightarrow{v} \cdots \xrightarrow{v} \mathcal{O}_M(\xi_0) \xrightarrow{v} i_* \mathcal{O}_{M'}(\eta) \to 0.
\]
In particular, the complex \((\xi, v)\) is acyclic on \(M \setminus M'\).

For \(x \in M', 0 \leq k \leq m\), let \(F_{k,x}\) be the \(k\)-th homology group of the complex \((\xi, v)_x\).

Set \(F_x = \bigoplus_{k=0}^m F_{k,x}\).

The following results are consequences of the local uniqueness of resolutions (see Serre [S, IV, Appendix 1] and Eilenberg [E, Theorem 8]) and are proved in [B2, Section 1].

- For \(k = 0, \ldots, m, x \in M'\), the dimension of \(F_{k,x}\) is locally constant on each \(M_j\), so that \(F_k\) is a holomorphic vector bundle on \(M'\).
- For \(x \in M', U \subset M\), let \(\partial_U v(x)\) be the derivative of the chain map \(v\) calculated in any given local holomorphic trivialization of \((\xi, v)\) near \(x\). Then \(\partial_U v(x)\) acts on \(F_x\). When acting on \(F_x\), \(\partial_U v(x)\) only depends on the image \(y \in N_x\) of \(x\). For \(x \in M', y \in N_x\), we write \(\partial_y v(x)\) instead of \(\partial_U v(x)\).
- For any \(x \in M', y \in N_x\), \((\partial_y v)^2(x) = 0\). If \(y \in N\), let \(i_y\) be the interior multiplication operator by \(y\) acting on the exterior algebra \(\Lambda(N^*)\). \(i_y\) acts like \(i_y \otimes 1\) on \(\Lambda N^* \otimes \eta\). Then, the graded holomorphic complex \((F, \partial_y v)\), on the total space of the vector bundle \(N\), is canonically isomorphic to the Koszul complex \((\Lambda N^* \otimes \eta, i_y)\).

b) Assumption (A) on the Hermitian metrics of a chain complex. We now assume that \(\xi_0, \ldots, \xi_m\) are equipped with smooth Hermitian metrics \(h^{\xi_0}, \ldots, h^{\xi_m}\). We equip \(\xi = \bigoplus_{k=0}^m \xi_k\) with the metric \(h^\xi\) which is the orthogonal sum of the metrics \(h^{\xi_0}, \ldots, h^{\xi_m}\). Let \(v^*\) be the adjoint of \(v\) with respect to the metric \(h^\xi\). Using finite dimensional Hodge theory, we get the identification of smooth vector bundles on \(M'\) for \(0 \leq k \leq m\)

\[
(1.3) \quad F_k \simeq \{ f \in \xi_k : v f = 0 ; v^* f = 0 \}.
\]

As a smooth vector subbundle of \(\xi_k\), the right hand side of (1.3) inherits a Hermitian metric from the metric \(h^{\xi_k}\). Using the identification (1.3), we find that for every \(k = 0, \ldots, m\), \(F_k\) is a holomorphic Hermitian vector bundle on \(M'\). Let \(h^{F_k}\) denote the Hermitian metric on \(F_k\). We equip \(F = \bigoplus_{k=0}^m F_k\) with the metric \(h^F\) which is the orthogonal sum of the metrics \(h^{F_0}, \ldots, h^{F_m}\).

Let \(g^N, g^n\) be Hermitian metrics on the vector bundles \(N, \eta\). We equip the vector bundle \(\Lambda N^* \otimes \eta\) with the tensor product of the metric induced by \(g^N\) on \(\Lambda(N^*)\) and of the metric \(g^n\).

**Definition 1.1.** Given metrics \(g^N, g^n\) on \(N, \eta\), we will say that the metrics \(h^{\xi_0}, \ldots, h^{\xi_m}\) on \(\xi_0, \ldots, \xi_m\) verify assumption (A) if the canonical identification of holomorphic chain complexes on \(N\)

\[
(1.4) \quad (F, \partial_y v) \cong (\Lambda N^* \otimes \eta, i_y)
\]

also identifies the metrics.

The following result is proved in [B2, Proposition 1.6].
Proposition 1.2. Given metrics $g^N, g^n$ on $N, \eta$, there exist metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on $\xi_0, \ldots, \xi_m$ which verify assumption (A).

c) Wave front sets. If $\gamma$ is a current on $M$, we note $WF(\gamma)$ the wave front set of $\gamma$. For the definition and properties of wave front sets, we refer to Hörmander [H, Chapter VIII].

Let us just recall that $WF(\xi)$ is a closed conic subset of $T^*_R M \setminus \{0\}$. Also if $p$ is the projection $T^*_R M \to M$, $p(WF(\gamma))$ is exactly the singular support of $\gamma$, whose complement in $M$ is the set of points $x$ such that $\gamma$ is $C^\infty$ on a neighborhood of $x$.

Let $N^*_R$ be the real conormal bundle to $M'$ in $M$. Let $\mathcal{D}'_{N^*_R}$ be the set of currents $\gamma$ on $M$ which are such that $WF(\gamma) \subset N^*_R$. In particular, currents in $\mathcal{D}'_{N^*_R}$ are smooth on $M \setminus M'$. By [H, Definition 8.2.2], $\mathcal{D}'_{N^*_R}$ has a natural topology which we now describe.

Let $U$ be a small open set in $M$, which we identify with an open ball in $R^{2\ell}$. Over $U$, we identify $T^*_R M$ with $U \times R^{2\ell}$. Let $\varphi$ be a smooth current on $R^{2\ell}$ with compact support included in $U$. Let $\Gamma$ be a closed conic subset of $R^{2\ell}$ such that $\Gamma \cap N^*_R = \emptyset$ on $M' \cup U$, and let $m$ be an integer. If $\gamma$ is a current, let $\hat{\gamma}(\xi)$ be the Fourier transform of $\varphi \gamma$ (which is here considered as a distribution on $R^{2\ell}$). If $\gamma \in \mathcal{D}'_{N^*_R}$, set

$$p_{U, \Gamma, \varphi, m}(\gamma) = \sup_{\xi \in \Gamma} |\xi|^m |\hat{\gamma}(\xi)|. \tag{1.5}$$

If $\gamma_n$ is a sequence of currents in $\mathcal{D}'_{N^*_R}$, we will say that $\gamma_n$ converge to $\gamma \in \mathcal{D}'_{N^*_R}$ if

- $\gamma_n \to \gamma$ in the sense of distributions;
- If $U, \Gamma, \varphi, m$ are taken as before

$$p_{U, \Gamma, \varphi, m}(\gamma_n - \gamma) \to 0. \tag{1.6}$$

Definition 1.3. $P^M_{M'}$ denotes the vector space of currents $\omega$ on $M$ which have the following two properties:
- $\omega$ is a sum of currents of type $(p, p)$;
- The wave front set $WF(\omega)$ of $\omega$ is included in $N^*_R$.

$P^M_{M'}$ is the vector space of currents $\omega \in P^M_{M'}$ which are such that there exist currents $\alpha, \beta$ for which

$$WF(\alpha) \subset N^*_R, \quad WF(\beta) \subset N^*_R, \quad \omega = \partial \alpha + \bar{\partial} \beta. \tag{1.7}$$

We equip $P^M_{M'}$ with the topology induced by $\mathcal{D}'_{N^*_R}(M)$. If $M' = \emptyset$, we will write $P^M_{M''}, P^M_{M'}$ instead of $P^M_{M''}, P^M_{M'}$.

d) Quillen's superconnections. We now assume that $\xi_0, \ldots, \xi_m$ are equipped with Hermitian metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. We otherwise use the notations of Section 1.a).
Set

\[ \xi_+ = \bigoplus_{k \text{ even}} \xi_k, \quad \xi_- = \bigoplus_{k \text{ odd}} \xi_k. \]

Then \( \xi = \xi_+ \oplus \xi_- \) is a \( \mathbb{Z}_2 \)-graded Hermitian vector bundle. \( \text{End} \xi \) is naturally \( \mathbb{Z}_2 \)-graded, the even (resp. odd) elements in \( \text{End} \xi \) commuting (resp. anticommuting) with the operator \( \tau = \pm 1 \) on \( \xi_{\pm} \) which defines the \( \mathbb{Z}_2 \) grading.

For \( 0 \leq k \leq m \), let \( \nabla_{\xi_k} \) be the holomorphic Hermitian connection on \( \xi_k \). Then \( \nabla_{\xi} = \bigoplus_{k=0}^m \nabla_{\xi_k} \) is the holomorphic Hermitian connection on the vector bundle \( \xi \).

We now briefly recall the definition of a superconnection in the sense of Quillen \([Q1]\). The bundle of algebras \( \Lambda(T^*_R M) \otimes \text{End} \xi \) is naturally \( \mathbb{Z}_2 \)-graded. Let \( S \) be a smooth section of \( \Lambda(T^*_R M) \otimes \text{End} \xi^{\text{odd}} \). Then, by definition, \( \nabla_{\xi} + S \) is a superconnection on the \( \mathbb{Z}_2 \)-graded vector bundle \( \xi \).

In the sequel, \( \nabla_{\xi} \) will be considered as a first-order differential operator acting on the set of smooth sections of \( \Lambda(T^*_R M) \otimes \xi \). The curvature \( (\nabla_{\xi} + S)^2 \) of the superconnection \( \nabla_{\xi} + S \) is then a smooth section of \( \Lambda(T^*_R M) \otimes \text{End} \xi^{\text{even}} \).

If \( A \in \text{End} \xi \), we define its supertrace \( \text{Tr}_s[A] \in \mathbb{C} \) by

\[ \text{Tr}_s[A] = \text{Tr}[\tau A]. \]

We extend \( \text{Tr}_s \) as a linear map from \( \Lambda(T^*_R M) \otimes \text{End} \xi \) into \( \Lambda(T^*_R M) \), with the convention that, if \( \omega \in \Lambda(T^*_R M) \), \( A \in \text{End} \xi \)

\[ \text{Tr}_s[\omega A] = \omega \text{Tr}_s[A]. \]

If \( B, B' \in \Lambda(T^*_R M) \otimes \text{End} \xi \), let \( [B, B'] \) be the supercommutator

\[ [B, B'] = BB' - (-1)^{\deg B \deg B'} B'B. \]

Then by \([Q1]\), \( \text{Tr}_s \) vanishes on supercommutators.

Let \( \phi \) be the homomorphism from \( \Lambda^{\text{even}}(T^*_R M) \) into itself which to \( \omega \in \Lambda^{2p}(T^*_R M) \) associates \( \phi(\omega) = (2i\pi)^{-p}\omega \).

Let \( S \) be an odd smooth section of \( \Lambda(T^*_R M) \otimes \text{End} \xi \). The basic result of Quillen \([Q1]\) asserts that the form \( \phi(\text{Tr}_s[\exp(-(\nabla_{\xi} + S)^2)]) \) is closed and represents in cohomology the Chern character of \( \xi_0 - \xi_1 + \cdots + (-1)^m \xi_m \).

e) **Double transgression formulas.** We make the same assumptions as in Section 1.d). Let \( v^* \) be the adjoint of \( v \). Set

\[ V = v + v^*. \]

\( V \) is a smooth section of \( \text{End}^{\text{odd}} \xi \). For \( u \geq 0 \), let \( A_u \) be the superconnection

\[ A_u = \nabla_{\xi} + \sqrt{u} V. \]
Then $A_u^2$ is the curvature of the superconnection $A_u$. $A_u^2$ is a smooth section of $(A(T^*_R M) \otimes \text{End } \xi)^{\text{even}}$.

Let $\mathcal{N}_n$ be the number operator of the complex $\xi$. Namely for $0 \leq k \leq m$, if $h \in \xi_k$

$$\mathcal{N}_n h = k h.$$ 

We now recall a result of [BGSI-I].

**Theorem 1.4.** The forms $\text{Tr}_s[\exp(-A_u^2)]$ and $\text{Tr}_s[\mathcal{N}_n \exp(-A_u^2)]$ lie in $P^M$ and depend smoothly on $u \geq 0$. Moreover, for $u > 0$, the following identities hold:

$$\frac{\partial}{\partial u} \text{Tr}_s[\exp(-A_u^2)] = -d \text{Tr}_s \left[ \frac{V}{2\sqrt{u}} \exp(-A_u^2) \right]$$

(1.11)

$$\text{Tr}_s \left[ \frac{V}{\sqrt{u}} \exp(-A_u^2) \right] = \frac{1}{u} (\bar{\partial} - \partial) \text{Tr}_s[\mathcal{N}_n \exp(-A_u^2)].$$

In particular,

$$\frac{\partial}{\partial u} \text{Tr}_s[\exp(-A_u^2)] = \frac{1}{u} \bar{\partial} \partial \text{Tr}_s[\mathcal{N}_n \exp(-A_u^2)].$$

(1.12)

**Proof.** These results are proved in [BGSI, Theorem 1.15]. Note the difference of signs with respect to [BGSI] since here $v$ decreases the degree in $\xi$ by one, while in [BGSI], $v$ increases the degree in $\xi$ by one. \qed

**f) Convergence of superconnection currents.** We make the same assumptions as in Sections 1.a) and 1.d). Set

$$F_+ = \bigoplus_{k \text{ even}} F_k, \quad F_- = \bigoplus_{k \text{ odd}} F_k.$$ 

$F = F_+ \oplus F_-$ is a Hermitian $\mathbb{Z}_2$-graded vector bundle. If $y \in N$, let $\bar{y}$ be the conjugate element of $y$ in $\bar{N}$. Then $y \in N$ represents $Y = y + \bar{y} \in N_R$. In particular, if $N$ is equipped with a metric $g^N$, $|Y|^2 = 2|y|^2$.

The superconnection formalism of Quillen can also be applied to the $\mathbb{Z}_2$-graded vector bundle $F = F_+ \oplus F_-$. Let $(\partial_y v)^*$ be the adjoint of $\partial_y v$ with respect to the metric $h^F$ on $F$. Then $(\partial_y v)^*$ is an antiholomorphic function of $y$. Set

$$\partial_Y V = \partial_y v + (\partial_y v)^*.$$ 

$\partial_Y V$ is an odd section of $\text{End } F$. If we use the canonical identification (1.4), then

$$\partial_Y V = i_y + i_y^*.$$
For $0 < k < m$, let $\nabla_{F_k}$ be the holomorphic Hermitian connection on the vector bundle $F_k$. Then $\nabla_F = \bigoplus_{k=0}^{m} \nabla_{F_k}$ is the holomorphic Hermitian connection on $F$.

Let $B$ be the superconnection on $F$

\begin{equation}
B = \nabla_F + \partial_Y V.
\end{equation}

The curvature $B^2$ of $B$ is a smooth section of $\Lambda(T_R^* N) \otimes \text{End} F$. The operator $N_H$ acts naturally on $F$. $N_H$ is simply multiplication by $k$ on $F_k$.

By [B2, Proposition 3.1], we know that the form on $N$, $\text{Tr}_e[N_H \exp(-B^2)]$ decays as $|y| \to +\infty$ faster than $\exp(-C|y|^2)$ (for one $C > 0$).

$\delta_M$ denotes the current of integration on the orientable manifold $M'$.

Let $C^1(M)$ be the set of continuous differential forms on $M$ which have continuous first derivatives. Let $\| \cdot \|_{C^1(M)}$ be a norm on $C^1(M)$ such that $\|\mu_n\|_{C^1(M)} \to 0$ if and only if $\mu_n$ tends to 0 uniformly on $M$ together with its first derivatives.

We now recall the result of Bismut announced in [B1] and proved in [B2, Theorems 3.2, 4.1 and 4.3].

**Theorem 1.5.** As $u \to \infty$, we have the following convergence of currents on $M$

\begin{equation}
\text{Tr}_e[\exp(-A_u^2)] \to \int_N \text{Tr}_e[\exp(-B^2)] \delta_M'. \text{ in } P^M_M.
\end{equation}

\begin{equation}
\text{Tr}_e[\sqrt{u} V \exp(-A_u^2)] \to 0 \text{ in } P^M_M.
\end{equation}

\begin{equation}
\text{Tr}_e[N_H \exp(-A_u^2)] \to \int_N \text{Tr}_e[N_H \exp(-B^2)] \delta_M'. \text{ in } P^M_M.
\end{equation}

There exists $C > 0$ such that if $\mu$ is a smooth differential form on $M$, then for $u \geq 1$

\begin{equation}
\left| \int_M \mu \left\{ \text{Tr}_e[\exp(-A_u^2)] - \int_N \text{Tr}_e[\exp(-B^2)] \right\} \delta_M' \right| \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}
\end{equation}

\begin{equation}
\left| \int_M \mu \text{Tr}_e[\sqrt{u} V \exp(-A_u^2)] \right| \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}
\end{equation}

\begin{equation}
\left| \int_M \mu \left\{ \text{Tr}_e[N_H \exp(-A_u^2)] - \int_N \text{Tr}_e[N_H \exp(-B^2)] \right\} \delta_M' \right| \leq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}.
\end{equation}

If $U, \Gamma, \varphi, m$ are taken as in Section 1.c), there exists $C > 0$ such that for $u \geq 1$

\begin{equation}
p_{U, \Gamma, \varphi, m} \left( \text{Tr}_e[\exp(-A_u^2)] - \int_N \text{Tr}_e[\exp(-B^2)] \delta_M' \right) \leq \frac{C}{\sqrt{u}}
\end{equation}
\[
p_{\nu, \Gamma, \varphi, m}(\text{Tr}_s[\sqrt{u} V \exp(-A^2_u)]) \leq \frac{C}{\sqrt{u}}
\]

(1.16) \[
p_{\nu, \Gamma, \varphi, m}\left( \text{Tr}_s[N_H \exp(-A^2_u)] - \int_N \text{Tr}_s[N_H \exp(-B^2)] \delta_{M'} \right) \leq \frac{C}{\sqrt{u}}.
\]

Observe that if the manifolds \( M \) and \( M' \) are not assumed to be compact, there is an obvious analogue of Theorem 1.5. The forms \( \mu \) must be taken to have compact support, and the various constants \( C \) depend explicitly on the compact set containing the support of the considered forms \( \mu \).

Recall that the Todd polynomial is an ad-invariant polynomial on matrices which is such that if \( C \) is a diagonal matrix with diagonal entries \( x_1, \ldots, x_p \), then

(1.17) \[
Td(C) = \prod_{i=1}^p \frac{x_i}{1 - e^{-x_i}}.
\]

Let \( (Td^{-1})' \) be the add-invariant polynomial which is such that if \( C \) is taken as before, then

(1.18) \[
(Td^{-1})'(C) = \frac{\partial}{\partial b} \left\{ \prod_{i=1}^p \frac{1 - e^{-(x_i+b)}}{x_i + b} \right\}_{b=0}
\]

If \( N, \eta \) are equipped with metrics \( g^N, g^n \), we denote by \( \nabla^N, \nabla^n \) the corresponding holomorphic Hermitian connections, and by \( (\nabla^N)^2, (\nabla^n)^2 \) their curvatures.

**THEOREM 1.6.** The form \( \text{Tr}_s[N_H \exp(-A^2_u)] \) on \( M \) is closed. The form \( \int_N \text{Tr}_s[N_H \exp(-B^2)] \) on \( M' \) is closed. If the metrics \( h^5, \ldots, h^m \) verify assumption (A) with respect to metrics \( g^N, g^n \) on \( N, \eta \), then

(1.19) \[
\int_N \text{Tr}_s[\exp(-B^2)] = (2in)^{\dim N} \left( - (\nabla^n)^2 \right) \left( Td^{-1} (\nabla^n)^2 \right) \text{Tr}[\exp(- (\nabla^n)^2)]
\]

(1.20) \[
\text{Tr}_s[N_H \exp(-A^2_u)] = \sum_{k=0}^m (-1)^k \text{Tr}_s[(-(\nabla^N)^2)^k]
\]

and so the form (1.20) is closed on \( M \). The form \( \int_N \text{Tr}_s[N_H \exp(-B^2)] \) is closed by [B2, Theorem 4.3]. The first line of (1.19) is a result of Mathai-Quillen [MQ, Theorem 4.5]. The second line of (1.19) is proved in [B2, Theorem 4.3].
Remark 1.7. Let $\tilde{M}$ be a complex submanifold of $M$. Assume that $\tilde{M}' = \tilde{M} \cap M'$ is a complex submanifold of $M$ and that $TM' = TM' \cap T\tilde{M}$.

Let $\tilde{N}$ be the normal bundle to $\tilde{M}'$ in $\tilde{M}$. Then $\tilde{N} \subset N$ and $\tilde{N} = N$ if and only if $\tilde{M}$ and $M'$ are transversal. Also $(\xi, v)|_{\tilde{M}}$ provides a resolution of $\eta|_{\tilde{M}}$ if and only if $\tilde{M}$ and $M'$ are transversal.

An analogue of the results stated in Theorems 1.5 and 1.6 is proved in [B2, Theorem 5.1] when replacing $M$ by $\tilde{M}$. The explicit formulas (1.19) have to be adequately modified.

II. A Bott-Chern current. In this Section, we construct our Bott-Chern singular current $T(h^2)$ associated with the chain complex $(\xi, v)$ equipped with the metric $h^2$. Our construction of $T(h^2)$ is a straightforward application of results in [BGS1] and in [B2]. We also study the behavior of the current $T(h^2)$ by pull back, and by integration along the fiber.

This Section is organized as follows. In a), we construct the current $T(h^2)$, and we show that its wave front set is included in the real conormal bundle to $M'$ in $M$.

In b), we study the pull back of $T(h^2)$ by a map $f$ transversal to $M'$. Finally in c), we consider the integral along the fiber of a submersion of our current $T(h^2)$.

Our assumptions and notations are the same as in Section 1.

a) A generalized Bott-Chern current. Let $E$ be a holomorphic vector bundle of dimension $k$ on the manifold $M$, let $g^E$ be a Hermitian metric on $E$. Let $\nabla^E$ be the corresponding holomorphic Hermitian connection on $E$, and let $\Omega^E = (\nabla^E)^2$ be its curvature. If $Q$ is an ad-invariant polynomial on $(k, k)$ matrices, we use the notation

$$Q(g^E) = Q\left(\frac{-\Omega^E}{2i\pi}\right).$$

In the sequel, $\text{ch}$ denotes the function $A \in \text{End}(C^k) \rightarrow \text{Tr}[\exp(A)]$.

We fix once and for all one square root of $2i\pi$ which we note $(2i\pi)^{1/2}$. The formulas which we will write in the sequel do not depend on this choice.

Let $\varphi$ be the homomorphism of $\Lambda(T^*_k M)$ into itself which to $\alpha \in \Lambda^i(T^*_k M)$ associates $(2i\pi)^{-i/2} \alpha$. $\varphi$ also acts on currents in the same way. More generally, $\varphi$ will also act on forms or currents on the manifolds $M'$ and $N$.

We now make the same assumptions as in Section 1.f), and we use the notations of Section 1.

Definition 2.1. For $s \in \mathbb{C}, 0 < \text{Re}(s) < \frac{1}{2}$, let $\zeta(s)$ be the current on $M$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} u^{s-1} \left\{ \text{Tr}_s[N \exp(-A_u^2)] - \left( \int_{N} \text{Tr}_s[N \exp(-B^2)] \right) \delta_{M'} \right\} du. \tag{2.1}$$

By the inequality (1.15) in Theorem 1.5, it is clear that the current $\zeta(s)$ is well-defined. More precisely, if $\mu$ is a smooth form on $M$, one easily verifies that the
function of \( s \in \mathbb{C}, 0 < \text{Re}(s) < \frac{1}{2} \), \( \int_M \mu'_{\xi}(s) \) extends into a meromorphic function of \( s \in \mathbb{C} \), which is holomorphic at 0.

**Definition 2.2.** \( \zeta_{\xi}'(0) \) denotes the current on \( M \) which is such that if \( \mu \) is any smooth differential form on \( M \), then

\[
\int_M \mu'_{\xi}(0) = \frac{\partial}{\partial s} \left[ \int_M \mu'_{\xi}(s) \right]_{s=0}.
\]

More explicitly, we have the obvious formula

\[
\zeta_{\xi}'(0) = \int_0^1 \left\{ \text{Tr}_s[N_H(\exp(-A_2^s)) - \exp(-A_0^2)] \right\} \frac{du}{u}
\]

\[+ \int_1^{+\infty} \left\{ \text{Tr}_s[N_H \exp(-A_2^s)] - \left( \int_N \text{Tr}_s[N_H \exp(-B^2)] \right) \delta_{M'} \right\} \frac{du}{u}
\]

\[- \Gamma'(1) \left\{ \text{Tr}_s[N_H \exp(-A_0^2)] - \left( \int_N \text{Tr}_s[N_H \exp(-B^2)] \right) \delta_{M'} \right\}.
\]

By Theorem 1.6, the form \( \text{Tr}_s[N_H \exp(-A_0^2)] \) is closed, and the current

\[\int_N (\text{Tr}_s[N_H \exp(-B^2)]) \delta_{M'} \]

is also closed.

**Remark 2.3.** Observe that if the complex (\( \xi, \nu \)) is acyclic, i.e., if \( M' = \phi \), then the current \( \zeta_{\xi}'(0) \) is smooth and coincides with the current which was already defined in [BGSl, Section 1.c)].

Recall that \( h^{s_0}, \ldots, h^{s_m} \) are the given Hermitian metrics on \( \xi_0, \ldots, \xi_m \) and that \( h^{s} \) is the metric on \( \xi \) which is the orthogonal sum of the metrics \( h^{s_0}, \ldots, h^{s_m} \).

**Definition 2.4.** Set

\[
\text{ch}(h^{s}) = \sum_0^m (-1)^k \text{ch}(h^{s_k})
\]

\[
T(h^{s}) = \varphi(\zeta_{\xi}'(0)).
\]

Let \( \gamma(h^{s}) \) be the current

\[
\gamma(h^{s}) = (2\pi i)^{-1/2} \int_0^{+\infty} \varphi(\text{Tr}_s[\sqrt{u} V \exp(-A_0^2)]) \frac{du}{2u}.
\]
Note that by the inequality (1.15) in Theorem 1.5, the current $\gamma(h^\delta)$ is also well-defined.

**Theorem 2.5.** The current $T(h^\delta)$ lies in $P^\delta_M$. The following identities of current hold

$$\frac{1}{2i\pi}(\overline{\partial} - \partial)T(h^\delta) = 2\gamma(h^\delta)$$

(2.6)

$$d\gamma(h^\delta) = ch(h^\delta) - \left(\int_N \varphi(Tr_s[exp(-B^2)])\right)\delta_{M'}.$$  

In particular,

$$\frac{\overline{\partial} \partial}{2i\pi} T(h^\delta) = \left(\int_N \varphi(Tr_s[exp(-B^2)])\right)\delta_{M'} - ch(h^\delta).$$

(2.7)

The wave front sets of the currents $T(h^\delta)$ and $\gamma(h^\delta)$ are included in $N^*_k$. If the metrics $h^\delta = (h^{\delta_1}, \ldots, h^{\delta_m})$ verify assumption (A) with respect to metrics $g^N, g^n$ on $N, \eta$, then

$$\frac{\overline{\partial} \partial}{2i\pi} T(h^\delta) = Td^{-1}(g^n) ch(g^n)\delta_{M'} - ch(h^\delta).$$

(2.8)

**Proof.** By Theorem 1.4, and by formula (2.3), it is clear that the current $\zeta^\delta(0)$ is a sum of currents of type $(p, p)$. Let us now prove that the wave front set of $\zeta^\delta(0)$ is included in $N^*_k$. Clearly the form $\int_0^1 Tr_s[N_H(exp(-A^2) - exp(-A^2_0))]\frac{du}{u}$ is smooth on $M$. Also by [H, Example 8.2.5], the wave front set of the current

$$\int_N Tr_s[N_H \exp(-B^2)]\delta_{M'}$$

is included in $N^*_k$. Let $\rho_\xi$ be the current

$$\rho_\xi = \int_1^{+\infty} \left\{Tr_s[N_H \exp(-A^2)] - \left(\int_N Tr_s[N_H \exp(-B^2)]\right)\delta_{M'}\right\}\frac{du}{u}.$$

We will prove that the wave front set of $\rho_\xi$ is still included in $N^*_k$. By using Duhamel’s formula, and by proceeding as in [BGS1, Section 1.c)], it is quite easy to see that $\rho_\xi$ is smooth on $M \setminus M'$. Take now $U, \Gamma, \varphi, M$ as in Section 1.c). By (1.16), we find that

$$P_{U, \Gamma, \varphi, M}(\rho_\xi) \leq C \int_1^{+\infty} \frac{du}{u^{3/2}} < +\infty.$$  

(2.9)
(2.10) \[ \int_M \mu d\gamma = -\int_M d\mu \gamma. \]

Clearly

\[ \int_M d\mu \gamma = \lim_{a \to +\infty} \int_0^a \left\{ \int_M \mu \operatorname{Tr}_s[\sqrt{u} V \exp(-A^2)] \right\} \frac{du}{2u} = -\lim_{a \to +\infty} \int_0^a \left\{ \int_M \mu d\operatorname{Tr}_s[\sqrt{u} V \exp(-A^2)] \right\} \frac{du}{2u}. \]

Using Theorems 1.4 and 1.5, we get

\[ \int_M \mu d\gamma = \int_M \mu \operatorname{Tr}_s[\exp(-A^2)] - \int_{M'} \iota^* \mu \int_N \operatorname{Tr}_s[\exp(-B^2)]. \]

We have thus proved the second line in (2.6). Similarly, if \( \mu \) is a smooth odd form on \( M \), by definition

\[ \int_M \mu (\bar{\partial} - \partial) \gamma(0) = \int_M (\bar{\partial} - \partial) \mu \gamma(0). \]

By Theorem 1.6, the currents \( \operatorname{Tr}_s[N_H \exp(-A^2)] \) and \( (\int_N \operatorname{Tr}_s[N_H \exp(-B^2)]) \delta_{M'} \) are closed. By Theorem 1.4, the forms \( \operatorname{Tr}_s[N_H \exp(-A^2)] \) lie in \( P^M \). Similarly, by [BGS1, Theorem 1.9], the form \( \operatorname{Tr}_s[N_H \exp(-B^2)] \) lies in \( P^N \). Therefore the currents \( \operatorname{Tr}_s[N_H \exp(-A^2)] \) and \( (\int_N \operatorname{Tr}_s[N_H \exp(-B^2)]) \delta_{M'} \) lie in \( P^M \) and are \( \bar{\partial} \) and \( \partial \) closed. We then get

\[ \int_M (\bar{\partial} - \partial) \mu \operatorname{Tr}_s[N_H \exp(-A^2)] = 0 \]

(2.14) \[ \int_{M'} \iota^* (\bar{\partial} - \partial) \mu \left( \int_N \operatorname{Tr}_s[N_H \exp(-B^2)] \right) = 0. \]

By proceeding as in (2.11) and by using Theorem 1.4 and (2.3), (2.13), (2.14), we immediately obtain the first line in (2.6). (2.7) follows from (2.6). (2.8) is a consequence
of the Mathai-Quillen formula [MQ, Theorem 4.5] stated in the first line of (1.19)
and of (2.7). Theorem 2.5 is proved.

b) The pull-back of the current $T(h^5)$ by a transversal map. Let $\tilde{M}$ be a compact
connected complex manifold. Let $f$ be a holomorphic map $\tilde{M} \to M$.

Definition 2.6. The map $f$ will be said to be transversal to $M'$ if for any $\tilde{x} \in \tilde{M}$
such that $f(\tilde{x}) = x \in M'$, then

\begin{equation}
\{ \eta \in T_x^* M ; f^* \eta = 0 \} \cap N_x^* = \{0\}.
\end{equation}

(2.15) is equivalent to

\begin{equation}
f^* T_{\tilde{x}} \tilde{M} + T_x M' = T_x M.
\end{equation}

Set $\tilde{M}' = f^{-1}(M')$. Then, if $f$ is transversal to $M'$, $\tilde{M}'$ is a complex submanifold
of $\tilde{M}$. Moreover as a subbundle of $T^* \tilde{M}|_{M'}$, the vector bundle $N^*$ on $M'$ pulls back
naturally into a subvector bundle $f^* N^*$ of $T^* \tilde{M}|_{\tilde{M}}$. If $f$ is transversal to $M'$, $f^* N^*$
is exactly the dual $N^*$ of the normal bundle $N$ to $M'$ in $\tilde{M}$.

From now on, we assume that $f$ is transversal to $M'$. Let $\tilde{i}$ be the embedding
$\tilde{M}' \hookrightarrow \tilde{M}$. Let

\[(f^* \xi, f^* v) : 0 \longrightarrow f^* \xi_m \stackrel{f^* v}{\longrightarrow} \cdots \longleftarrow f^* \xi_0 \longrightarrow 0\]

be the holomorphic chain complex of Hermitian vector bundles on $\tilde{M}$ which is
the pull-back by $f$ of the holomorphic chain complex of Hermitian vector bundles
$(\xi, v)$.

Let $f^*(\eta)$ be the holomorphic vector bundle of $\tilde{M}'$ which is the pull-back by $f$ of
the holomorphic vector bundle $\eta$ on $M'$.

Since $f$ is transversal to $M'$, using the local uniqueness of resolutions ([S, Chapter
IV, Appendix 1], [E, Theorem 8]), we find that $(f^* \xi, f^* v)$ provides a projective
resolution of $f^* \eta$, i.e., we have the exact sequence of sheaves

\begin{equation}
0 \longrightarrow \mathcal{O}_{\tilde{M}}(f^* \xi_m) \stackrel{f^* v}{\longrightarrow} \cdots \longrightarrow \mathcal{O}_{\tilde{M}}(f^* \xi_0) \longrightarrow \tilde{i}_* \mathcal{O}_{\tilde{M}}(f^* \eta) \longrightarrow 0.
\end{equation}

Also, by [H, Theorem 8.2.4], since $WF(\gamma(h^5)) \subset N^*_R$, $WF(T(h^5)) \subset N^*_R$, the current
$\gamma(h^5)$, $T(h^5)$ can be unambiguously pulled back into currents $f^*(\gamma(h^5))$, $f^*(T(h^5))$
on $\tilde{M}$.

Of course on $\tilde{M}$, we can also define currents $\gamma(f^* h^5)$, $T(f^* h^5)$.

Theorem 2.7. The following identities of currents hold on $M'$

\begin{equation}
\gamma(f^* h^5) = f^* \gamma(h^5)
\end{equation}

(2.18)

\begin{equation}
T(f^* h^5) = f^* T(h^5).
\end{equation}
Proof. We will only prove the second identity in (2.18). For $1 \leq a < +\infty$, let $\zeta^a(0)$ be the current on $M$

\begin{equation}
\zeta^a(0) = \int_0^1 \frac{\text{Tr}_s[N_H(\exp(-A^2_a)) - \exp(-A^2_0)]}{u} du
+ \int_1^a \left\{ \text{Tr}_s[N_H \exp(-A^2_a)] - \left( \int_N \text{Tr}_s[N_H \exp(-B^2)] \right) \delta_{M'} \right\} \frac{du}{u}
- \Gamma'(1) \left\{ \text{Tr}_s[N_H \exp(-A^2_0)] - \left( \int_N \text{Tr}_s[N_H \exp(-B^2)] \right) \delta_{M'} \right\} .
\end{equation}

By Theorems 1.5 and 2.5, we know that as $a \to +\infty$

\begin{equation}
\zeta^a(0) \to \zeta(0) \quad \text{in} \quad D_{N\bar{H}}(M).
\end{equation}

Similarly, we define in the same way currents $\zeta^a_{\gamma}(0)$ on $M'$ so that, as $a \to +\infty$

\begin{equation}
\zeta^a_{\gamma}(0) \to \zeta_{\gamma}(0) \quad \text{in} \quad D_{\bar{N}\gamma}(\tilde{M}).
\end{equation}

We claim that for any $a \geq 1$

\begin{equation}
\zeta^a_{\gamma}(0) = f^* \zeta^a(0).
\end{equation}

In fact, $\zeta^a_{\gamma}(0)$ is a sum of smooth currents carried by $M$ and smooth currents carried by $M'$. Then

- By definition, the smooth forms $\text{Tr}_s[N_H \exp(-A^2_a)]$ on $M$ pull back into the corresponding forms on $M'$ associated with the Hermitian chain complex $(f^*\xi, f^*v)$.

- Since $N = f_* \bar{N}$, we have the equality of forms on $\tilde{M}'$

\begin{equation}
\int_{\bar{N}} \text{Tr}_s[N_H \exp(-(f^*B)^2)] = f^* \int_N \text{Tr}_s[N_H \exp(-B^2)].
\end{equation}

- In view of (2.19)–(2.23), to obtain (2.22), it remains to prove that

\begin{equation}
f^* \delta_{M'} = \delta_{\tilde{M}'}.
\end{equation}

(2.24) can easily be proved by the methods of [H, Example 8.2.8], i.e., by approximating locally $\delta_{M'}$ by a sequence of smooth currents, which converge “transversally” to the current $\delta_{M'}$ of integration over $M'$, by pulling back these currents and using the transversality of the map $f$. So we have proved (2.22).
By [H, Theorem 8.2.4], and by (2.20), then as $a \to \infty$

\begin{equation}
(2.25) \quad f^* \zeta_2^a(0) \to f^* \zeta_2(0) \quad \text{in } \mathcal{D}'(\widetilde{M}).
\end{equation}

The second equality in (2.11) now follows from (2.21), (2.22), (2.25). \hfill \Box

Remark 2.8. Let $\widetilde{M}$ be a complex submanifold of $M$, and let $f$ be the embedding $M \hookrightarrow \widetilde{M}$. Then $\widetilde{M}$ is said to intersect $M'$ transversally if for any $x \in \widetilde{M} \cap M'$, $T_x \widetilde{M} + T_x M' = T_x M$, or equivalently if $f: \widetilde{M} \to M$ is transversal to $M'$. If $M$ and $\widetilde{M}$ intersect transversally, the current $T(h^z)$ has a well-defined restriction $f^* T(h^z)$ to $\widetilde{M}$, which coincides with $T(f^* h^z)$.

c) Integration along the fiber of the current $T(h^z)$. Let $B$ be a compact complex connected manifold. Let $\pi$ be a holomorphic submersion from $M$ into $B$, with connected fiber $Z$.

We assume that the restriction of $\pi$ to $M'$ is also a holomorphic submersion from $M'$ on $B$, whose fibers are denoted $Y$. Of course $Y = Z \cap M'$. More precisely, the fibers $Z$ intersect $M'$ transversally.

For $k \in \mathbb{N}$, let $C^k(M)$ be the set of differential forms on $M$ which are $k$ times continuously differentiable. Let $\| \cdot \|_{C^k(M)}$ be a norm on $C^k(M)$ such that if $(\mu_n)_n$ is a sequence of elements of $C^k(M)$, then $\|\mu_n\|_{C^k(M)} \to 0$ if and only if $\mu_n$ and its first $k$ derivatives converge to 0 uniformly on $M$ as $n \to +\infty$. We define $C^k(B)$ and $\| \cdot \|_{C^k(B)}$ in the same way.

Let $\mu$ be a smooth differential form on $M$. In [B2, Theorem 3.2], it was proved that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $u \geq 1$

\begin{equation}
(2.26) \quad \left\| \int_Z \text{Tr}_s[N_H \exp(-A^z_n)] - \int_Y i^* \mu \left( \int_N \text{Tr}_s[N_H \exp(-B^2)] \right) \right\|_{C^k(B)} \leq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}.
\end{equation}

For $0 < \Re(s) < \frac{1}{2}$, let $\eta(h^z)(\mu)(s)$ be the smooth differential form on $B$

\begin{equation}
(2.27) \quad \eta(h^z)(\mu)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{-1} \left[ \int_Z \mu \varphi(\text{Tr}_s[N_H \exp(-A^z_n)]) \right. \\
- \int_Y i^* \mu \left( \int_N \varphi(\text{Tr}_s[N_H \exp(-B^2)]) \right) \right] du.
\end{equation}

The fact that for $0 < \Re(s) < \frac{1}{2}, \eta(h^z)(\mu)(s)$ is a smooth form on $B$ of course follows from (2.26). Also $\eta(h^z)(\mu)(s)$ extends into a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$. In particular $\eta(h^z)(\mu)'(0)$ is a smooth form on $B$.

On the other hand, let $\pi_* (\mu T(h^z))$ be the current on $B$ which is the image of the current $\mu T(h^z)$ by $\pi_*$ (i.e., the integral along the fiber $Z$ of the current $\mu T(h^z)$). Since
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\( WF(T(h^2)) \subset N^*_R \), then \( WF(\mu T(h^2)) \subset N^*_R \). Also since \( \pi_{|M'} \) is a holomorphic submersion, it follows from [H, Theorem 8.2.13] that \( \pi_*(\mu T(h^2)) \) is a smooth current on \( B \), which can therefore be defined pointwise.

**Theorem 2.9.** We have the equality of smooth differential forms on \( B \)

\[
\pi_*(\mu T(h^2)) = \eta(h^2)(\mu)'(0).
\]

*Proof.* We proceed as in the proof of Theorem 2.7. Namely we replace the current \( T(h^2) \) by its approximations \( T^a(h^2) = \varphi(\zeta^a(0)) \) and \( \eta(h^2)(\mu)'(0) \) by approximating forms \( \eta^a(h^2)(\mu)'(0) \) which we define in the obvious way. We claim that

\[
\pi_*(\mu T^a(h^2)) = \eta^a(h^2)(\mu)'(0).
\]

In fact, \( T^a(h^2) \) is a sum of currents which are smooth respectively on \( M \) and \( M' \) and \( (2.29) \) is tautological.

Now by Theorem 2.5, as \( a \to +\infty \)

\[
\mu T^a(h^2) \to \mu T(h^2) \quad \text{in} \quad \mathcal{D}'(M).
\]

Then

\[
\pi_*(\mu T^a(h^2)) \to \pi_*(\mu T(h^2)) \quad \text{in} \quad \mathcal{D}'(B).
\]

Therefore equality \( (2.28) \) holds in \( \mathcal{D}'(B) \). Since both sides of \( (2.28) \) are smooth forms, equality \( (2.28) \) holds pointwise.

**Remark 2.10.** Of course a similar result holds for the current \( \gamma(h^2) \).

### III. The Bott-Chern current as a finite part.

In this section, we prove that, in general, the current \( T(h^2) \) is smooth on \( M \setminus M' \) and not locally integrable on \( M \). More precisely, we determine the singularity of \( T(h^2) \) near \( M' \). We show in particular that if \( Y \) is a normal coordinate to \( M' \), then near \( M' \), \( T(h^2) \approx |Y|^{-2 \dim N} \). The integral of \( T(h^2) \) on the complementary of an \( \eta \) neighborhood of \( M' \) in \( M \) is then equivalent to \( c \log \eta \) as \( \eta \to 0 \). We calculate \( T(h^2) \) as a finite part by subtracting off the logarithmic divergence.

Our results mostly rely on the techniques and the estimates of [B2].

This section is organized as follows. In a), we calculate the precise singularity of the current \( T(h^2) \) near \( M' \). In b), we calculate \( T(h^2) \) as a finite part.

Our assumptions and notations are the same as in Sections 1 and 2.

#### a) The singularity of the Bott-Chern current.

Let \( g^N \) be a smooth Hermitian metric on the vector bundle \( N \). Let \( g^{TM} \) be a smooth Hermitian metric on \( TM \). Identifying \( N \) with the orthogonal to \( TM' \) in \( TM \), we assume that \( g^N \) is exactly the metric induced by \( g^{TM} \).
If \( y \in N \), \( y \) represents \( Y = y + \bar{y} \in N_R \). Let \( |y|, |Y| \) be the norms of \( y, Y \) in \( N, N_R \). Then \( |Y|^2 = 2|y|^2 \).

Let \( S_N \) be the unit sphere in \( N_R \), \( S_N = \{ Y \in N_R; |Y| = 1 \} \). As the boundary of the unit ball in \( N_R \), \( S_N \) is naturally oriented. Let \( n \) be the unit vector in \( N_R \) normal to \( S_N \) and pointing outwards.

Let \( \alpha \) be a smooth differential form on \( N \) which is such that there exist \( c, C > 0 \) for which \( |\alpha| \leq c \exp(-C|y|^2) \).

Let \( \psi_t \) be the group of diffeomorphism of \( N: y \in N \to e^t y \). The group \( \psi_t \) is generated by the vector field \( Y = y + \bar{y} \).

Note that on the sphere bundle \( S_N \), the form \( \int_{-\infty}^{+\infty} i_Y \psi_t^* \alpha ds \) is well-defined. Indeed as \( s \to +\infty \), we use the fact that \( \alpha \) decays as indicated as \( |y| \to +\infty \). As \( s \to -\infty \), \( i_Y \psi_t^* \alpha = \psi_t^* i_Y \alpha \) decays exponentially fast.

**Proposition 3.1.** The following identity of differential forms on \( M' \) holds

\[
\int_N \alpha = \int_{S_N} \int_{-\infty}^{+\infty} i_{\psi_t^*} \alpha ds.
\]

**Proof.** Let \( F \) be the map: \( (s, y) \in R \times S_N \to \psi_s(y) \in N_R \). Clearly

\[
\int_N \alpha = \int_{R \times S_N} F^* \alpha.
\]

Now if \( j \) is the embedding \( S_N \to N_R \),

\[
(F^* \alpha)(s, y) = j^* \psi_s^* \alpha + dsj^* (i_Y \psi_s^* \alpha).
\]

Using (3.2) and (3.3), we get (3.1). \( \square \)

Take \( x_0 \in M' \). Let \( U \) be an open neighborhood of \( x_0 \) in \( M \), and let \( z = (z^1, \ldots, z^l) \) be a holomorphic system of coordinates on \( U \) such that \( V = M' \cap U \) is represented by \( (z^{k+1} = 0, \ldots, z^l = 0) \). Set

\[
x = (z^1, \ldots, z^k)
\]

\[
y = (z^{k+1}, \ldots, z^l).
\]

Then \( x \) is a coordinate system on \( V \). Set \( e = l - k \). For \( \varepsilon > 0 \), let \( B_\varepsilon \) be the open ball of center 0 and radius \( \varepsilon \) in \( \mathbb{C}^e \). We may and we will assume that for \( \varepsilon > 0 \) small enough, \( V \times B_\varepsilon \subset U \).

In the sequel, we always assume that \( x \in V, y \in B_\varepsilon \). If \( (x, y) \in V \times B_\varepsilon \), we consider \( y \) as an element of \( N_x \).

On \( V \times B_\varepsilon \), \( \Lambda^p(T^*_R M) \) splits into

\[
\Lambda^p(T^*_R M) = \bigoplus_{i+j=p} \Lambda^i(T^*_R M') \otimes \Lambda^j(N_x^*).
\]
If \( \alpha \in \Lambda_{(x,y)}(T^*_R M) \), let \( \alpha^{\text{max}} \) be the component of \( \alpha \) which has maximal degree 2 \( \dim N = 2e \) in the Grassmann variables of \( N^*_R \). Note that at \( (x, 0) \), i.e., on \( M' \), \( \alpha^{\text{max}} \) does not depend on the coordinate system \( (x, y) \).

For \( u > 0 \), let \( B_u \) be the superconnection on the vector bundle \( F \) on \( N \)

\[
B_u = \nabla + \sqrt{u} \partial_y V.
\]

In the sequel, the differential form on \( N \), \( \text{Tr}_s[N_H \exp(-B_u^2)] \) will be considered as a form on \( V \times \mathbb{C}^e \).

Note that on \( N \)

\[
\text{Tr}_s[N_H \exp(-(\nabla^2))^2]^{\text{max}} = 0,
\]

and so as \( u \downarrow 0 \)

\[
\text{Tr}_s[N_H \exp(-B_u^2)]^{\text{max}} = 0(u).
\]

**Definition 3.2.** \( \beta_F \) denotes the smooth form on \( N \setminus \{0\} \)

\[
\beta_F = \int_0^{+\infty} \varphi(\text{Tr}_s[N_H \exp(-B_u^2)])^{\text{max}} \frac{du}{u}.
\]

Because of (3.5) and of the fact that on \( N \setminus \{0\} \), the forms \( \text{Tr}_s[N_H \exp(-B_u^2)] \) decay exponentially fast as \( u \to +\infty \), \( \beta_F \) is indeed well-defined. Of course \( \beta_F \) depends on the coordinate system \( (x, y) \). Clearly

\[
\text{Tr}_s[N_H \exp(-B_u^2)] = \psi_{(\log u)u}^\# \text{Tr}_s[N_H \exp(-B^2)]
\]

and so by Proposition 3.1, we get the equality of differential forms on \( M' \)

\[
\int_{S_u} i_n \beta_F = 2 \int_N \varphi(\text{Tr}_s[N_H \exp(-B^2)]).
\]

Let \( \lambda \) be the volume form on \( N \) with respect to the metric \( g^N \). In the coordinate system \( (x, y) \), \( \lambda \) extends into a smooth form on \( V \times \mathbb{C}^e \), which is purely vertical (i.e., only involves the Grassmann variables \( dy^\alpha, d\bar{y}^\alpha, 1 \leq \alpha \leq e \)).

**Theorem 3.3.** For any \( a > 0, y \in N \setminus \{0\} \),

\[
\beta_F(ay) = \frac{1}{a^{\dim N}} \beta_F(y).
\]

Let \( \theta(h^4), \omega(h^4) \) be the smooth forms on \( M \setminus M' \) which are the restrictions of the currents \( \gamma(h^4), T(h^4) \) to \( M \setminus M' \). Then \( \theta(h^4) \) is locally integrable on \( M \), and coincides as a current with \( \gamma(h^4) \).
If $V$ and $\varepsilon$ are small enough, there exists $C > 0$ such that if $(x,y) \in V \times B_{\varepsilon}$,

$$|y|^{2 \dim N} |\omega(h^t) (x,y) - \beta_F(x,y)| \leq C|y|.$$  

(3.10)

The current $\omega(h^t) - \beta_F$ is integrable on $V \times B_{\varepsilon}$. If the metrics $(h^{5_0}, \ldots, h^{5_m})$ verify assumption (A) with respect to the metrics $g^N$ and $g^n$, then

$$\beta_F = - (\dim N - 1)! \frac{1}{|y|^{2 \dim N}} (Td^{-1} y)(g^N) \chi(g^n) \frac{\lambda}{\pi^{\dim N}}.$$  

(3.11)

Proof. In the proof, the constants $C$ may vary from line to line.

Using (3.6), (3.7), it is clear that if $\tau_u$ is the map $(x,y) \rightarrow (x, ay)$, then $\tau_u^* \beta_F = \beta_F$. Also $\tau_u^* \beta_F(x,y) = a^{2 \dim N} \beta_F(x, ay)$. (3.9) follows.

Let $\alpha_u$ (resp. $\delta_u$) be the form on $M$ (resp. $N$),

$$\alpha_u = \text{Tr}_u[N_H \exp(-A_u^2)],$$  

(3.12)

(resp.

$$\delta_u = \text{Tr}_u[N_H \exp(-B_u^2)].$$  

(3.13)

Set $\sigma_u = \tau_{u_2}$. By the proofs of [B2, Theorem 3.2] and more specifically by using equations [B2, (3.89)-(3.104) and (3.107)-(3.109)], we know that if $V$ and $\varepsilon$ are small enough, there exist $C, C' > 0$ such that for $u > 1, x \in V, |y| \leq \varepsilon u$, then

$$|\sigma_u^* \alpha_u(x, y) - \delta_1(x,y)| \leq \frac{C}{\sqrt{u}} \exp(-C'|y|^2).$$  

(3.14)

We temporarily decompose $\alpha_u$ according to the partial degree in the Grassmann variables of $N_H^*$, so that

$$\alpha_u = \sum_{0 \leq p \leq 2 \dim N} \alpha_u^p.$$  

(3.15)

In (3.15), $\alpha_u^p$ ($0 \leq p \leq 2 \dim N$) is of partial degree $p$ in the Grassmann variables $dy^s, d\bar{y}^s$. Then

$$\sigma_u^* (\sigma_u^p)(x,y) = \sum_{0 \leq p \leq 2 \dim N} u^{-p/2} \alpha_u^p \left( x, \frac{y}{\sqrt{u}} \right).$$  

(3.16)

For $0 < \eta \leq 1, |y| \leq \varepsilon$, we use (3.14) with $u = 1/\eta^2$ and $y$ replaced by $y/\eta$. We get

$$\left| \sum_{0 \leq p \leq 2 \dim N} \eta^p \alpha_u^p(x, y) - \delta_1 \left( x, \frac{y}{\eta} \right) \right| \leq C\eta \exp \left( -C' \frac{|y|^2}{\eta^2} \right).$$  

(3.17)
In the right-hand side of (2.3), the first integral \( \int_0^1 \frac{du}{u} \) defines a smooth form on \( M \), and so does not contribute to the singular part of the current \( \zeta'_\epsilon(0) \) near \( M' \). Also

\[
(3.18) \quad \int_{1}^{+\infty} \alpha_u(x, y) \frac{du}{u} = \int_{|y|^2/\epsilon^2}^{+\infty} \alpha_{x\epsilon u/|y|^2}(x, y) \frac{du}{u}.
\]

Now if \( |y| \leq \epsilon \sqrt{u} \), then \( \eta = \frac{|y|}{\epsilon \sqrt{u}} \leq 1 \) and so using (3.17), we find that if \( |y| \leq \epsilon(1 \wedge \sqrt{u}) \), then

\[
(3.19) \quad \sum_{\alpha}^{2 \dim N} \left( \frac{\epsilon}{\sqrt{u}} \right)^p \alpha_{\epsilon u/|y|^2}(x, y) - \delta_\epsilon \left( x, \epsilon \sqrt{u} \frac{y}{|y|} \right) \leq C \frac{|y|}{\epsilon \sqrt{u}} \exp(-C' \epsilon^2 u).
\]

From (3.18), (3.19), we deduce that if \( \delta^p \) denotes the component of \( \delta_1 \) with partial vertical degree \( p \), then if \( |y| \leq \epsilon \), \( 0 \leq p \leq 2 \dim N \)

\[
(3.20) \quad \left| \epsilon \right|^p \int_1^{+\infty} \alpha_u(x, y) \frac{du}{u} - \int_{|y|^2/\epsilon^2}^{+\infty} \left( \epsilon \sqrt{u} \right)^p \delta^p \left( x, \epsilon \sqrt{u} \frac{y}{|y|} \right) \frac{du}{u} \leq C |y| \int_{|y|^2/\epsilon^2}^{+\infty} \exp(-C' \epsilon^2 u) \epsilon^{-p} \frac{du}{u}.
\]

Observe that

\[
(3.21) \quad \delta_u = \tau_u^{*} \delta_1.
\]

From (3.21), we get for \( y \neq 0 \)

\[
(3.22) \quad \left( \epsilon \sqrt{u} \right)^p \delta^p \left( x, \epsilon \sqrt{u} \frac{y}{|y|} \right) = \delta^p_{\epsilon u} \left( x, \frac{y}{|y|} \right).
\]

Clearly

\[
(3.23) \quad \left| \int_{|y|^2/\epsilon^2}^{+\infty} \delta^0_{\epsilon u} \left( x, \frac{y}{|y|} \right) \frac{du}{u} \right| \leq C (\log |y| + C').
\]

Also for \( p = 0 \), the expression in the right-hand side of (3.20) is bounded as \( |y| \to 0 \).

Since \( 2 \dim N \geq 2 \), we deduce from (3.20)–(3.23)

\[
(3.24) \quad \left| y \right|^{2 \dim N} \left| \int_1^{+\infty} \alpha^0_u(x, y) \frac{du}{u} \right| \leq C |y|.
\]
For $p > 1$, the integrals

\begin{equation}
(3.25) \int_{|y|^2/\epsilon^2}^{+\infty} (\epsilon \sqrt{u})^p \delta_f^p \left( x, \frac{\epsilon \sqrt{u} \frac{y}{|y|}}{u} \right) \frac{du}{u}
\end{equation}

remain bounded as $|y| \to 0$. Also for $p > 1$, the expressions in the right hand side of (3.20) are bounded. We thus find that for $1 \leq p \leq 2 \dim N - 1$,

\begin{equation}
(3.26) \ |y|^{2 \dim N} \int_1^{+\infty} \alpha^p_u(x, y) \frac{du}{u} \leq C|y|.
\end{equation}

Clearly, since $2 \dim N > 2$,

\begin{equation}
(3.27) \ \left| \int_0^{|y|^2/\epsilon^2} (\epsilon \sqrt{u})^{2 \dim N} \delta_f^2 \left( x, \frac{\epsilon \sqrt{u} \frac{y}{|y|}}{u} \right) \frac{du}{u} \right| \leq C|y|^2.
\end{equation}

Using (3.20), (3.22), (3.27), we find that

\begin{equation}
(3.28) \ |y|^{2 \dim N} \int_1^{+\infty} \alpha^2 u^{2 \dim N} \left( x, \frac{\epsilon \sqrt{u} \frac{y}{|y|}}{u} \right) \frac{du}{u} - \int_0^{+\infty} \delta^2 \left( x, \frac{\epsilon \sqrt{u} \frac{y}{|y|}}{u} \right) \frac{du}{u} \leq C|y|.
\end{equation}

Equivalently, we get

\begin{equation}
(3.29) \ \left| \int_1^{+\infty} \varphi \left( \alpha^2 u^{2 \dim N} \left( x, \frac{\epsilon \sqrt{u} \frac{y}{|y|}}{u} \right) \right) \frac{du}{u} - \beta \left( x, \frac{y}{|y|} \right) \right| \leq C|y|.
\end{equation}

Using (3.9), (3.24), (3.26), (3.29), we obtain (3.10). Since the function $1/|y|^{2 \dim N - 1}$ is integrable near 0, we find that the current $\omega(h^2) - \beta_\nu$ is integrable.

In [B2, eq. (4.9), (4.11)], it was noted that

\begin{equation}
(3.30) \ \sqrt{u} \ Tr_\nu [\partial_\gamma V \exp(-B^2_\nu)] = -i_\gamma \ Tr_\nu [\exp(-B^2_\nu)],
\end{equation}

so that

\begin{equation}
(3.31) \ \sqrt{u} \ Tr_\nu [\partial_\gamma V \exp(-B^2_\nu)]^\max = 0.
\end{equation}

Using (3.30), (3.31) and the proof of [B2, Theorem 4.1] (which is in fact identical to the proof of [B2, Theorem 3.2]), we find by the same arguments as before that

\begin{equation}
(3.32) \ |y|^{2 \dim N} |\theta(h^2)(x, y)| \leq C|y|,
\end{equation}

so that $\theta(h^2)$ is integrable near $M'$. Also by replacing in the analogue of (3.18) the
integrals \( \int_{t}^{+\infty} \{ \} \frac{du}{u} \) by integrals \( \int_{t}^{a} \{ \} \frac{du}{u} \), we find that if \( \theta^a(h^2) \) denotes the density of the smooth approximating current \( \gamma^a(h^2) \), then we have the uniform estimate for \( a \geq 1, x \in V, |y| \leq \varepsilon \),

\[
|y|^{2 \dim N} |\theta^a(h^2)(x, y)| \leq C|y|.
\]

Using (3.33) and the Dominated Convergence Theorem, we find that as \( a \to +\infty \), \( \theta^a(h^2) \) converges in the sense of distributions to \( \theta(h^2) \). Since by Theorem 1.5, as \( a \to +\infty \), \( \gamma^a(h^2) \to \gamma(h^2) \) and since \( \gamma^a(h^2) = \theta^a(h^2) \), we deduce that \( \gamma(h^2) = \theta(h^2) \).

Also by [MQ, Theorem 4.5, [B2, eq. (3.138)-(3.141), (4.22)], we find that under assumption (A)

\[
(3.34) \quad \text{Tr}_s[N_H \exp(-B^2)]^{\max}
\]

\[= -(iu)^{\dim N}(T^{-1})^\prime(-(V^N)^2) \text{Tr}[\exp(-(V^N)^2)] \exp\left(-\frac{u|Y|^2}{2}\right)\lambda.\]

From (3.6), (3.34), we get (3.11).

\( \boxdot \)

b) The current \( T(h^2) \) as a principal part. The form \( \omega(h^2) \) is in general not integrable on \( M \). Still by using (3.10), it has a well-defined principal part, which defines a current. We now compare the current \( T(h^2) \) with such a principal part.

For \( \eta > 0 \), let \( M^\eta \) denote the set of points of \( M \) whose Riemannian distance to \( M' \) is larger than \( \eta \).

**Theorem 3.4.** Let \( \mu \) be a smooth even form on \( M \). Then as \( \eta > 0 \) converges to 0

\[
\int_M \mu \omega(h^2) + 2 \log \eta \int_{M'} i^* \mu \int_N \varphi(\text{Tr}_s[N_H \exp(-B^2)])
\]

has a limit which we note \( \int_M \mu \omega^f(h^2) \). Moreover the following identities hold

\[
(3.36) \quad \int_M \mu T(h^2) = \int_M \mu \omega^f(h^2) - \int_{M'} i^* \mu \int_N (2 \log |Y| - \Gamma'(1))\varphi(\text{Tr}_s[N_H \exp(-B^2)]).
\]

If the metrics \( h^2_0, \ldots, h^2_m \) verify assumption (A) with respect to the metrics \( g^N \) and \( g^n \), then the following identities hold

\[
\varphi\left(\int_N \text{Tr}_s[N_H \exp(-B^2)]\right) = -(T^{-1})^\prime(g^N) \text{ch}(g^n)
\]
\begin{align*}
\int_N (2 \log |Y| - \Gamma'(1)) \varphi(\text{Tr}_e[N_H \exp(-B^2)]) \\
= -(T_{d^{-1}})'(g^N) \text{ch}(g^n) \left( \sum_{k=1}^{\dim N - 1} \frac{1}{k} + \log 2 \right).
\end{align*}

**Proof.** Take $x_0 \in M'$. Let $V$ be a small open ball in $M'$ of center $x_0$. We will use the same notations as in the proof of Theorem 3.3. In particular, $e$ denotes the dimension of $N$ over $V$. We also choose geodesic coordinates in the directions of $T_xM$ which are normal to $T_xM'$ with respect to the given Euclidean scalar product of $T_xM$. For $\varepsilon > 0$, set $B^R_\varepsilon = \{ Y \in R^{2\varepsilon}; |Y| \leq \varepsilon \}$. $U = V \times B^R_\varepsilon$ is then a small real neighborhood of $x$ in $M$. Again, we identify $R^{2\varepsilon}$ with the real normal bundle $N_R$ to $M'$ in $M$. Set

$$\omega = \varphi^{-1}(\omega(h^2))$$

$$\beta = \varphi^{-1}(\beta_F).$$

We may and we will assume that the support of $\mu$ is included in $U$. The form $i^*\mu$ on $V$ lifts naturally into a form on $V \times R^{2\varepsilon}$. Then

\begin{align*}
\int_{M^*} \mu \omega = \int_{M^*} (\mu - 1_{|Y| \leq \varepsilon} i^*\mu) \omega + \int_{M^*} 1_{|Y| \leq \varepsilon} (i^*\mu) \omega.
\end{align*}

We know that since $\mu$ is smooth

$$|\mu^0(x, y) - \mu^0(x, 0)| \leq C |y|.$$

In the sequel, we lift any differential form on $V$ into a differential form on $V \times R^{2\varepsilon}$. In particular, $i^*\mu$ is considered as a form on $V \times R^{2\varepsilon}$. Then $i^*\mu$ has partial vertical degree 0 and coincides with $\mu^0(x, 0)$.

Using Theorem 3.3, we find that

\begin{align*}
\lim_{\eta \to 0} \int_{M^*} (\mu - 1_{|Y| \leq \varepsilon} i^*\mu) \omega &= \int_M (\mu - 1_{|Y| \leq \varepsilon} i^*\mu) \omega.
\end{align*}

Moreover

\begin{align*}
\int_{M^*} 1_{|Y| \leq \varepsilon} (i^*\mu) \omega &= \int_{M^*} 1_{|Y| \leq \varepsilon} i^*\mu(\omega - \beta) + \int_{M^*} 1_{|Y| \leq \varepsilon} (i^*\mu) \beta.
\end{align*}

Using Theorem 3.3 again, we get

\begin{align*}
\lim_{\eta \to 0} \int_{M^*} 1_{|Y| \leq \varepsilon} i^*\mu(\omega - \beta) &= \int_M 1_{|Y| \leq \varepsilon} i^*\mu(\omega - \beta).
\end{align*}
Clearly, if we use (3.9), we find that if \( \eta \leq \varepsilon \)

\[
\int_{M_n} 1_{|y| \leq \varepsilon} (i^* \mu) \beta(y) = \int_{M^n} 1_{|y| \leq \varepsilon} (i^* \mu) |Y|^{-2 \dim N} \beta \left( \frac{y}{|y|} \right)
\]

\[
= (\log \varepsilon - \log \eta) \int_{M'} i^* \mu \int_{S^n} i_n \beta.
\]

Using (3.8) and (3.39)-(3.42), we see that as \( u \to 0 \), \( \int_{M'} \mu \omega + \log \eta \int_{M'} i^* \mu \int_{S^n} i_n \beta \) has a limit \( \int_M \mu \omega_f \) given by

\[
\int_M \mu \omega_f = \int_M (\mu - 1_{|y| \leq \varepsilon} i^* \mu) \omega + \int_M 1_{|y| \leq \varepsilon} i^* \mu (\omega - \beta)
\]

\[
+ 2 \log \varepsilon \int_{M'} i^* \mu \left( \int_N \text{Tr}_s [N_H \exp(-B^2)] \right).
\]

Also if we still use the notations in the proof of Theorem 3.3, then

\[
\int_M \mu \int_1^T \alpha_u \frac{du}{u} = \int_M \int_{|y|^2 |c^2} \mu \alpha_u \frac{du}{u}.
\]

From (3.19), (3.44), we deduce easily that for \( 0 \leq p \leq 2N - 1 \), then as \( T \to +\infty \)

\[
\int_M \mu \int_1^T \alpha_u^p \frac{du}{u} \to \int_M \mu \int_1^{+\infty} \alpha_u^p \frac{du}{u}.
\]

where the right-hand side defines a locally integrable current. Set

\[
\bar{\alpha} = \alpha^{2 \dim N}, \quad \bar{\delta} = \delta^{2 \dim N}.
\]

Then

\[
\int_M \int_1^T \mu^0 \bar{\alpha}_u \frac{du}{u} = \int_M \int_1^T (\mu^0 \mu \mid_{|y| \leq \varepsilon i^* \mu} ) \bar{\alpha}_u \frac{du}{u}
\]

\[
+ \int_M \int_1^T 1_{|y| \leq \varepsilon \mu} (x, 0) \bar{\delta}_u \frac{du}{u}
\]

\[
+ \int_M \int_1^T 1_{|y| \leq \varepsilon \mu} (x, 0) \delta_u \frac{du}{u}.
\]
By using (3.19) and (3.44), we find that as \( T \to +\infty \)

\[
(3.48) \quad \int_M \int_1^T (\mu^0 - 1_{|y| \leq \varepsilon} \mu) \overline{\delta_y} \frac{du}{u} \to \int_M \int_1^{+\infty} (\mu^0 - 1_{|y| \leq \varepsilon} \mu) \overline{\delta_y} \frac{du}{u}.
\]

where the right-hand side is of course integrable in all variables. Similarly using (3.19), (3.22), (3.44), we see that as \( T \to +\infty \)

\[
(3.49) \quad \int_M \int_1^T 1_{|y| \leq \varepsilon} \mu^0(x, 0) (\overline{\delta_y} - \overline{\delta_u}) \frac{du}{u} \to \int_M \int_1^{+\infty} 1_{|y| \leq \varepsilon} \mu^0(x, 0) (\overline{\delta_y} - \overline{\delta_u}) \frac{du}{u}.
\]

Also using (3.21), we know that

\[
(3.50) \quad \overline{\delta_y}(x, y) = \frac{1}{|Y|^2 \dim N} \delta_{|y|^2} \left( \frac{x}{|Y|}, \frac{y}{|Y|} \right).
\]

Therefore

\[
(3.51) \quad \int_M \int_1^T 1_{|y| \leq \varepsilon} \mu^0(x, 0) \overline{\delta_y} \frac{du}{u} = \int_M \int_{|y|^2}^{T|y|^2} 1_{|y| \leq \varepsilon} \mu^0(x, 0) \frac{1}{|y|^2 \dim N} \overline{\delta_y} \left( \frac{x}{|y|}, \frac{y}{|y|} \right) \frac{du}{u}.
\]

Now for \( 0 \leq u \leq Te^2 \)

\[
(3.52) \quad \int_M 1_{u/T \leq |y| \leq (\sqrt{u} \wedge \varepsilon)} \mu^0(x, 0) \frac{1}{|y|^2 \dim N} \overline{\delta_y} \left( \frac{x}{|y|}, \frac{y}{|y|} \right) = \left( \log(\sqrt{u} \wedge \varepsilon) - \log \left( \frac{u}{T} \right) \right) \int_M i^* \mu \int_{S^N} i_n \overline{\delta_y}(x, y).
\]

So, from (3.51) and (3.52), we get

\[
(3.53) \quad \int_M \int_1^T 1_{|y| \leq \varepsilon} \mu^0(x, 0) \overline{\delta_y} \frac{du}{u} = \frac{1}{2} \log T \int_Y i^* \mu \int_{S^N} \int_0^{Te^2} i_n \overline{\delta_y} \frac{du}{u}
\]

\[
+ \int_Y i^* \mu \int_{S^N} \int_{e^2}^{Te^2} \frac{1}{2} \log \left( \frac{e^2}{u} \right) i_n \overline{\delta_y} \frac{du}{u}.
\]

There is \( C > 0 \) such that for \( u \geq 1 \)

\[
(3.54) \quad \left| \int_{S^N} i_n \overline{\delta_y} \right| \leq \exp(-Cu).
\]
Also using (3.50) again, we get

\[
(3.55) \quad \int_{\mathbb{S}_n} \int_{\varepsilon^2}^{+\infty} \frac{1}{2} \log \left( \frac{e^2}{u} \right) i_n \delta_u \frac{du}{u} = -\int_{\mathbb{S}_n} \int_{\varepsilon^2}^{+\infty} 1_{\varepsilon < |Y|} \frac{1}{|Y|^2 \dim N} \delta_u \left( x, \frac{y}{|Y|} \right) \frac{du}{u}
\]

\[
= -\int_1^{+\infty} \frac{du}{u} \int_{\mathbb{S}_n} 1_{|Y| > \varepsilon} \delta_u.
\]

From (3.7), (3.8), (3.53)–(3.55), we find that

\[
(3.56) \quad \lim_{T \to +\infty} \left\{ \int_M \int_1^T 1_{|Y| < \varepsilon} \mu^0 (x, 0) \delta_u \frac{du}{u} - \log T \int_Y i^* \mu \int_N \delta_1 \right\}
\]

\[
= -\int_Y i^* \mu \int_N 1_{|Y| > \varepsilon} \int_1^{+\infty} \delta_u \frac{du}{u}.
\]

Also by (2.3), we know that

\[
(3.57) \quad \int_M \mu^0 \zeta (0) = \lim_{T \to +\infty} \left\{ \int_0^1 \int_M (\alpha_u - \alpha_0) \frac{du}{u} + \int_1^T \mu \alpha_u \frac{du}{u}
\]

\[
- \Gamma (1) \left( \int_M \mu \alpha_0 - \int_Y i^* \mu \int_N \delta_1 \right) - \log T \int_Y i^* \mu \int_N \delta_1 \right\}.
\]

Using (3.45), (3.47)–(3.57), we find that

\[
(3.58) \quad \int_M \mu^0 \zeta (0) = \int_0^1 \mu (\alpha_u - \alpha_0) \frac{du}{u} + \sum_{\varepsilon} \int_M \mu \int_1^{+\infty} \alpha_u \frac{du}{u}
\]

\[
+ \int_M (\mu^0 - 1_{|Y| < \varepsilon} i^* \mu) \int_1^{+\infty} \alpha_u \frac{du}{u}
\]

\[
+ \int_M 1_{|Y| < \varepsilon} i^* \mu \int_1^{+\infty} (\alpha_u - \delta_u) \frac{du}{u} - \int_M 1_{|Y| > \varepsilon} i^* \mu \int_1^{+\infty} \delta_u \frac{du}{u}
\]

\[
- \Gamma (1) \left[ \int_M \mu \alpha_0 - \int_Y i^* \mu \int_N \delta_1 \right].
\]

So from (3.43), (3.58), we find that

\[
(3.59) \quad \int_M \mu^0 \zeta (0) = \int_M \mu \omega^0 + \int_{M'} i^* \mu \int_N \left\{ 1_{|Y| \leq \varepsilon} \left( \beta - \int_1^{+\infty} \delta_u \frac{du}{u} \right)
\]

\[
- 1_{|Y| > \varepsilon} \int_1^{+\infty} \delta_u \frac{du}{u} + \left( \Gamma (1) - 2 \log \varepsilon \right) \delta_1 \right\}.
\]
Now

\begin{equation}
(3.60) \quad \int_N \left\{ 1_{|Y| \leq \varepsilon} \left( \beta - \int_1^{+\infty} \delta_u \frac{du}{u} \right) - 1_{|Y| > \varepsilon} \int_1^{+\infty} \delta_u \frac{du}{u} \right\}
\end{equation}

\begin{equation}
= \int_N \left\{ 1_{|Y| \leq \varepsilon} \int_0^{+\infty} \delta_u \frac{du}{u} - 1_{|Y| > \varepsilon} \int_1^{+\infty} \delta_u \frac{du}{u} \right\}.
\end{equation}

Also by (3.7), \( \delta_u = \psi^{\star}_{(\log u) / 2} \delta_1 \) and so

\begin{equation}
(3.61) \quad \int_N 1_{|Y| \leq \varepsilon} \int_0^{+\infty} \delta_u \frac{du}{u} = \int_0^{+\infty} \int_N 1_{|Y| \leq \varepsilon, \sqrt{u} \delta_1} \delta_u \frac{du}{u} \int_N 1_{|Y| \leq \sqrt{u} \delta_1} = 2 \int_N 1_{|Y| \leq \varepsilon} \left( \log \varepsilon - \log |Y| \right) \delta_1.
\end{equation}

\begin{equation}
= 2 \int_N 1_{|Y| \geq \varepsilon} \left( \log |Y| - \log \varepsilon \right) \delta_1.
\end{equation}

From (3.61), we get

\begin{equation}
(3.62) \quad \int_N 1_{|Y| \leq \varepsilon} \int_0^{+\infty} \delta_u \frac{du}{u} - \int_N 1_{|Y| > \varepsilon} \int_1^{+\infty} \delta_u \frac{du}{u} = 2 \int_N \left( \log \varepsilon - \log |Y| \right) \delta_1.
\end{equation}

From (3.59), (3.62), we deduce that

\begin{equation}
(3.63) \quad \int_M \mu \gamma(0) \gamma(1) = \int_M \mu \gamma(1) = \int_M i \gamma(1) = \int_N \left( \gamma(1) - 2 \log |Y| \right) \delta_1.
\end{equation}

From (3.63), we obtain (3.36).

Assume now that assumption (A) is verified. Then, the first line in (3.37) was proved in [B2, Theorem 4.3].

With the same arguments as in [MQ, Theorem 4.5] and [B2, eq. (3.140), (4.22)], we get

\begin{equation}
(3.64) \quad \int_N \exp \left( \frac{-t|Y|^2}{2} \right) \text{Tr}_N[NH \exp(-B^2)]
= -\frac{1}{(1 + n)^{\text{dim}N}(2i\pi)^{\text{dim}N}(T^{-1}l)(-\nabla^N)^2} \text{Tr}[\exp(-\nabla^N)^2].
\end{equation}
Now for $0 < \text{Re}(s) < \dim N$

\[(3.65) \quad \frac{1}{\Gamma(s)} \int_{0}^{+\infty} t^{s-1} \exp \left( -\frac{t \mid Y \mid^2}{2} \right) \frac{dt}{(1 + t)^{\dim N}} = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} t^{s-1} dt \left\{ \frac{1}{\Gamma(\dim N)} \int_{0}^{+\infty} u^{\dim N - 1} e^{-u(1 + t)} \frac{du}{u} \right\} = \frac{\Gamma(\dim N - s)}{\Gamma(\dim N)}.
\]

By (3.64), (3.65), we find that if $0 < \text{Re}(s) < \dim N$

\[(3.66) \quad \int_{N} \left( \frac{2}{\mid Y \mid^2} \right)^s \text{Tr}_s[N_H \exp(-B^2)]
= -2(2\pi)^{\dim N}(Td^{-1})'(-(V^N)^2) \text{Tr}[\exp(-(V^N)^2)] \frac{\Gamma(\dim N - s)}{\Gamma(\dim N)}.
\]

Both sides of (3.66) extend into meromorphic functions of $s$ which are holomorphic at $s = 0$. By differentiating (3.66) at $s = 0$, we get

\[(3.67) \quad \int_{N} (2 \log \mid Y \mid - \log 2) \text{Tr}_s[N_H \exp(-B^2)]
= -2(2\pi)^{\dim N}(Td^{-1})'(-(V^N)^2) \text{Tr}[\exp(-(V^N)^2)] \frac{\Gamma'(\dim N)}{\Gamma(\dim N)}.
\]

Since $\Gamma(s + 1) = s \Gamma(s)$, we get

\[(3.68) \quad \frac{\Gamma'(\dim N)}{\Gamma(\dim N)} = \frac{\Gamma'(1)}{\Gamma(1)} + \sum_{k=1}^{\dim N - 1} \frac{1}{k}.
\]

The second line in (3.37) follows from (3.67), (3.68).

**Remark 3.5.** Clearly the current $T(h^4)$ does not depend on the metric of $TM$. Therefore the right hand side of (3.36) does not depend on the metric of $TM$. Also $\int_{M} i^*\mu \int_{N} (2 \log \mid Y \mid - \Gamma'(1)) \text{Tr}_x[N_H \exp(-B^2)]$ only depends on the metric of $N$. Therefore $\int_{M} \mu \omega \xi'$ only depends on the metric of $N$. Equivalently two metrics on $TM$ which induce the same metric on $N$ define the same current $\omega'(h^4)$. This fact has a simple geometric interpretation.

**REFERENCES**


Bismut: Université Paris-Sud, Mathématique, Bâtiment 425, F-91405 Orsay Cedex, FRANCE.

Gillet: Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois, 60638.

Soulé: CNRS and IHES, 35 route de Chartres, F-91440 Bures/Yvette, FRANCE.

Bismut: Université Paris-Sud, Mathématique, Bâtiment 425, F-91405 Orsay Cedex, FRANCE.

Gillet: Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois, 60638.

Soulé: CNRS and IHES, 35 route de Chartres, F-91440 Bures/Yvette, FRANCE.