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Equivariant Bott–Chern currents and the Ray–Singer analytic torsion*

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0. Introduction

The purpose of this paper is to explain some relations between equivariant cohomology in complex geometry, the Ray–Singer analytic torsion [RS], Quillen metrics [Q], and Bott–Chern currents [BoC].

Let us recall that in [A], Atiyah and Witten gave a formal representation of the heat equation formula for the index of the Dirac operator acting on sections of spinors of a manifold M, as the integral on the loop space LM of a differential form which is equivariantly closed with respect to the natural action of S_1 on LM. By applying formally in infinite dimensions a localization formula of Duistermaat and Heckman [DH], Berline and Vergne [BeV], they obtained the right answer for the index of this Dirac operator. In [B2], we extended the observation of [A] to general twisted spin complexes. The heat equation formula for the index then appears as the pairing on the loop space LM of the Atiyah–Witten form with another equivariantly closed form, which is a natural lift to LM of the representative in Chern–Weil theory of the Chern character form on M of the considered twisting bundle. The results of [A] and [B2] only concern Dirac operators associated with the corresponding Levi–Civita connection on TM.

The local index theorem of Patodi [P1], Gilkey [Gi], Atiyah et al. [ABP] was known to hold for Dirac operators associated with the Levi–Civita connection of TM. In [B4], we gave a sufficient (and almost necessary) condition under which the local index theorem still holds for Dirac operators associated with connections on TM which have non zero torsion. If M is a complex Hermitian manifold, and if ω is the corresponding Kähler form, it was shown in [B4, Theorem 2.11] that the local Riemann–Roch–Hirzebruch Theorem holds if $\bar{\partial}\partial\omega = 0$, which relaxes the Kähler condition $d\omega = 0$ which was known since Patodi [P2].

When M is complex, the Kähler form ω of M lifts naturally into a Kähler form $\tilde{\omega}$ on the loop space LM, which is closed if and only if ω is closed. If $\bar{\partial} + \bar{\partial}^*$ is the Dirac operator acting on the bundle of spinors $\Lambda(T^{*(0,1)}M) \otimes (\det T^{(1,0)}M)^{-1/2}$,

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by McKean–Singer [MKS], for any $t>0$, the supertrace $\mathrm{Tr}_s[\exp(-t(\bar\partial+\bar\partial^*)^2)]$ coincides with the Euler characteristic of $(\det T^{(1,0)}M)^{-1/2}$. If we represent this expression as a formal integral on LM along the lines of [A], [B2], we obtain in (15) the formal formula

$$\mathrm{Tr}_s[\exp(-t(\partial+\partial^*)^2)]=C\int_{LM}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right). \tag{0.1}$$

In (0.1), X is the vector field on LM which generates the action of S_1 $\left(\text{i.e. } X(x)=\frac{dx}{ds}\right)$, $\partial_X, \bar\partial_X$ are the operators

$$\partial_X=\partial+i_{X^{(0,1)}};\quad \bar\partial_X=\bar\partial+i_{X^{(1,0)}} \tag{0.2}$$

and C is an infinite constant.

Equation (0.1) has far reaching consequences. In fact let us observe that formally

$$\frac{\partial}{\partial t}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)=-\frac{1}{t^2}\bar\partial_X\partial_X\left\{\sqrt{-1}\tilde\omega\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)\right\}. \tag{0.3}$$

Identities *formally* identical to (0.3) have been proved by Bott and Chern [BoC] in their study of secondary invariants in complex geometry. As shown in Bismut et al. [BGS2, Proposition 2.4], the Kähler form ω is intimately related with the number operator N of the complex $\Lambda T^{*(0,1)}M$ (which calculates the Z grading of this complex), and formally

$$C\int_{LM}\frac{\sqrt{-1}\tilde\omega}{t^2}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)$$

can be identified with

$$\frac{1}{t}\mathrm{Tr}_s\left[\left(N-\frac{\dim_{\mathbb{C}}M}{2}\right)\exp(-t(\bar\partial+\bar\partial^*)^2)\right]$$

which is precisely ... the integrand in the heat equation formula for the Ray–Singer analytic torsion of the $\bar\partial$ complex [RS].

In this paper, we develop this analogy further. Namely in the context of complex equivariant cohomology in finite dimensions, we integrate (0.3) by a zêta function technique. If M, LM are instead finite dimensional compact complex manifolds, and if M lies in LM as the zero set of a holomorphic Killing vector field X, we thus construct a current $\zeta'_{\tilde\omega}(0)$ on LM which is such that the following equation of currents of Bott–Chern type [BoC] holds on LM

$$\bar\partial_X\partial_X\zeta'_{\tilde\omega}(0)=1-\frac{\delta_{\{M\}}}{e}. \tag{0.4}$$

In (0.4), e is a Chern–Weil representative of the equivariant Euler class of the normal bundle to M in LM.

Equation (0.1) suggests that in full generality, the Ray–Singer analytic torsion can be formally obtained by pairing on the loop space LM of the considered manifold M the current $\zeta'_{\tilde\omega}(0)$ with the lift of the Chern character form of the twisting bundle to LM, which was obtained in [B2].

The previous considerations strongly suggest that the result of Bismut–Gillet–Soulé [BGS3, Theorems 1.27 and 2.14] which calculates the curvature of the holomorphic Hermitian connection on the determinant of a direct image equipped with the Quillen metric [Q], if properly understood, expresses in fact identities verified by equivariant Bott–Chern currents on LM. The best evidence for this is the generalized anomaly formula for Quillen metrics, [BGS3, Theorems 1.22 and 1.23] which calculates how the metric on the determinant of a direct image depends on the Kähler metric of M. Its proof relies on complicate algebraic manipulations whose deep geometric significance is hidden in the operator theoretical formalism.

In this paper, we show how the anomaly formula of [BGS3] is the obvious extension to infinite dimensions of a non trivial result on equivariant Bott–Chern currents in finite dimensions. Its proof is non trivial, but still gives a transparent explanation for the proof of [BGS3, Theorems 1.22 and 1.23].

Our paper is organized as follows. In Sect. A, we recall the general Kodaira–Nakano formula proved in our previous work [B4]. In Sect. B, we give an argument to establish formula (0.1). In Sect. C, in a finite dimensional context, we construct equivariant Bott–Chern currents, which depend explicitly on the considered metric. In Sect. D, we study the dependence of these Bott–Chern currents on the metric. This finite dimensional result was used by us in joint work with Gillet and Soulé to elaborate the proof of [BGS3, Theorem 1.22] where the dependence of the Quillen metric on the metric of the considered manifold was established.

A. A Kodaira–Nakano formula

Let M be a compact connected complex manifold of complex dimension l. Let $T^{(1,0)}M$ denote the holomorphic tangent space of M, and let $T_R M$ be the corresponding real tangent space. We assume that M is equipped with a smooth Hermitian metric. Let ω be the associated Kähler form. If J is the complex structure on $T_R M$, if $X, Y \in T_R M$, then $\omega(X,Y) = \langle X, JY\rangle$.

Let ξ be a holomorphic Hermitian vector bundle on M, and let ∇^ξ be the holomorphic Hermitian connection on ξ. Let $\bar\partial$ be the Dolbeault operator acting on the set of smooth sections of $\Lambda(T^{*(0,1)}M)\otimes\xi$, and let $\bar\partial^*$ be its adjoint with respect to the natural Hermitian product associated with the given metrics on M and on ξ.

The exterior algebra $\Lambda(T_R^* M)$ and the Clifford algebra $c(T_R M)$ are canonically isomorphic as Z graded vector spaces. If $\alpha \in \Lambda(T_R^* M)$, let ${}^c\alpha$ denote its image in $c(T_R M)$. The map c can be extended into a map from $\Lambda(T_R^* M)\otimes \mathrm{End}\,\xi$ into $c(T_R M)\otimes \mathrm{End}\,\xi$.

Classically (see e.g. [B4, Sect. 2]), $\Lambda(T^{*(0,1)}M)\otimes\xi$ is a $c(T_R M)$ Clifford module. So $c(T_R M)$ acts naturally on $\Lambda(T^{*(0,1)}M)\otimes\xi$.

If E is a vector bundle on M with connection ∇^E, we denote by $(\nabla^E)^2$ the curvature of ∇^E.

Let K be the scalar curvature of M. We now recall the generalized Kodaira–Nakano identity proved in [B4, Theorem 2.3].

Let ∇^L be the Levi–Civita connection on $T_R M$. Then, as explained in [B4, Sect. 2], ∇^L lifts into a unitary connection on $\Lambda(T^{*(0,1)}M)\otimes\xi$, which we still note ∇^L. Let $e_1, \ldots, e_{2l}$ be an orthonormal base of $T_R M$.

Theorem 1. *The following identity holds*

$$2(\bar{\partial}+\bar{\partial}^*)^2 = -\sum_1^{2l}\left(\nabla^L_{e_i} - \frac{\sqrt{-1}}{4}{}^c(i_{e_i}(\partial-\bar{\partial})\omega)\right)^2 + \frac{K}{4} + {}^c((\nabla^\xi)^2 + \tfrac{1}{2}\mathrm{Tr}[(\nabla^{TM})^2]I_\xi)$$
$$-\frac{\sqrt{-1}}{2}{}^c(\bar{\partial}\partial\omega) - \tfrac{1}{8}\|(\partial-\bar{\partial})\omega\|^2. \tag{1}$$

B. Path integrals and the Riemann–Roch–Hirzebruch Theorem

We first briefly recall the formalism of Atiyah [A]. Let M be a Riemannian manifold. LM denotes the set of smooth loops $s\in S_1 = R/Z \to x_s \in M$. If $x\in LM$, the tangent space T_xLM is identified with the set of smooth periodic vector fields $Y: s\in S_1 \to Y_s \in T_{x_s}M$. T_xLM is equipped with the L_2 scalar product

$$Y, Z\in T_xLM \to \langle Y, Z\rangle = \int_0^1 \langle Y_s, Z_s\rangle ds. \tag{2}$$

LM is then a Riemannian manifold. For $s\in S_1$, set

$$k_s x = x_{s+\cdot}.$$

k_s is a group of isometries of LM, which is generated by the vector field X given by $X = \frac{dx}{ds}$. X is a Killing vector field on LM.

If d is the exterior differentiation operator on LM, if i_X is the interior multiplication operator associated with X, then the Lie operator L_X is given by

$$L_X = (d+i_X)^2. \tag{3}$$

Assume now that M is a compact connected complex manifold equipped with a Hermitian metric, whose Kähler form is denoted ω. If J is the complex structure on TM, J induces an almost complex structure on $T\,LM$

$$Y\in TLM \to JY\in TLM. \tag{4}$$

One easily verifies that the almost complex structure (4) is integrable, and so LM is also a complex manifold. The Kähler form $\tilde{\omega}$ of LM is given by $Y, Z\in TLM \to \tilde{\omega}(Y,Z) = \int_0^1 \omega(Y_s, Z_s)ds$. Note that if the manifold (M,ω) is Kähler, then $(LM,\tilde{\omega})$ is also Kähler.

The group k_s is now a group of holomorphic isometries. The vector field X is then Killing *and* holomorphic.

d splits into $d = \partial + \bar{\partial}$. Let $X^{(1,0)}, X^{(0,1)}$ be the components of X in $T^{(1,0)}LM$ and $T^{(0,1)}LM$. Equivalently, if $x\in LM$, for every $s\in S_1$

$$X_s = X_s^{(1,0)} + X_s^{(0,1)}$$
$$X_s^{(1,0)} \in T_{x_s}^{(1,0)}M; \quad X_s^{(0,1)} \in T_{x_s}^{(0,1)}M.$$

Set

$$\partial_X = \partial + i_{X^{(0,1)}}; \quad \bar{\partial}_X = \bar{\partial} + i_{X^{(1,0)}} \tag{5}$$

Since X is a holomorphic vector field, then

$$\partial_X^2 = \bar{\partial}_X^2 = 0. \tag{6}$$

From (3), (6), we find that

$$L_X = \partial_X \bar{\partial}_X + \bar{\partial}_X \partial_X. \tag{7}$$

Let α be a smooth differential form on LM which is X invariant, so that $L_X \alpha = 0$. From (7), we deduce that

$$\bar{\partial}_X \partial_X \alpha = -\partial_X \bar{\partial}_X \alpha. \tag{8}$$

On X invariant forms, the operators $\partial_X, \bar{\partial}_X$ behave like the usual operators $\partial, \bar{\partial}$. In particular

$$\bar{\partial}_X \partial_X \tilde{\omega} = -\partial_X \bar{\partial}_X \tilde{\omega}. \tag{9}$$

To simplify our notations, we will assume that M is spin, or equivalently that the line bundle $\det(T^{(1,0)}M)$ has a square root λ. We equip λ with the metric induced by the metric of M. Let ∇^λ be the corresponding holomorphic Hermitian connection.

Let β be the smooth differential form on LM constructed in [B2, Definition 3.6] naturally associated with the vector bundle $(\xi \otimes \lambda, \nabla^\xi \otimes 1 + 1 \otimes \nabla^\lambda)$. By [B2, Theorem 3.9], we know that $(\partial_X + \bar{\partial}_X)\beta = 0$.

Also one verifies that β is a sum of forms of type (p,p) so that

$$\partial_X \beta = 0, \quad \bar{\partial}_X \beta = 0. \tag{10}$$

Let X' be the one form on LM dual to X, so that if $Y \in T_X LM$

$$X'(Y) = \int_0^1 \langle Y_s, dx_s \rangle. \tag{11}$$

Let C be the infinite constant

$$C = \frac{\left(\prod_1^{+\infty} m^2\right)^l}{(2\pi)^l} i^l.$$

Take $t > 0$. We now will give a formal path integral representation for $A_t = \operatorname{Tr}_s[\exp(-t(\bar{\partial} + \bar{\partial}^*)^2)]$. Using the generalized Lichnerowicz formula of Theorem 1, and by proceeding as in [A], [B2, Sect. 2], [B3, Sect. 2e)] we find that formally

$$A_t = C \int_{LM} \exp\left\{ -\frac{(d + i_X)X'}{2t} + \frac{\sqrt{-1}}{2t} i_X(\partial - \bar{\partial})\tilde{\omega} + \sqrt{-1}\frac{\bar{\partial}\partial\tilde{\omega}}{t} \right\} \beta. \tag{12}$$

In (12), we have eliminated the contribution of $(K/4) - \frac{1}{8}\|(\partial - \bar{\partial})\omega\|^2$ which is irrelevant here. Observe that

$$X' = (i_{X^{(1,0)}} - i_{X^{(0,1)}})\sqrt{-1}\,\tilde{\omega}. \tag{13}$$

Therefore using (9), we get

$$-\frac{(d+i_X)X'}{2}+\sqrt{-1}\frac{i_X(\partial-\bar\partial)\tilde\omega}{2}+\sqrt{-1}\bar\partial\partial\tilde\omega$$
$$=-\frac{(d+i_X)}{2}(\bar\partial_X-\partial_X)\sqrt{-1}\tilde\omega=\bar\partial_X\partial_X\sqrt{-1}\tilde\omega. \tag{14}$$

We then get

$$A_t=C\int_{LM}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)\beta. \tag{15}$$

C. The finite dimensional case: equivariant Bott–Chern currents

In the sequel, we will assume that LM is instead a compact connected complex manifold equipped with a Hermitian metric whose Kähler form is noted $\tilde\omega$.

X is a holomorphic Killing vector field on LM. Note that the Kähler form $\tilde\omega$ is then X invariant. Set

$$M=\{x\in LM; X(x)=0\}$$

M is a union of compact connected complex totally geodesic submanifolds of LM. If ω is the restriction of $\tilde\omega$ to M, then ω is the Kähler form of M. We otherwise define the operators $\partial_X, \bar\partial_X$ as in (5).

Let N be the normal bundle to M in LM. N is a complex holomorphic vector bundle on M. Also N inherits a Hermitian metric g^N from the metric of LM. Let ∇^N be the holomorphic Hermitian connection on (N, g^N) and let R^N be is curvature. Let J_X be the infinitesimal action of X in N. In local coordinates, if $y\in N$, $J_X y=\frac{\partial X^{(1,0)}}{\partial x}y\cdot J_X$ is a skew-adjoint endomorphism of N.

We now give an application of a result of [B3, Theorem 1.3], where formulas of Berline and Vergne [BeV] and Duistermaat and Heckmann [DH] were proved.

Theorem 2. *Assume that the form $\tilde\omega$ is closed, so that $(LM,\tilde\omega)$ is Kähler. Then for any smooth differential form μ on LM*

$$\lim_{t\downarrow 0}\int_{LM}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)\mu=\int_M\frac{\mu}{\det\left[-\frac{J_X+R^N}{2i\pi}\right]}. \tag{16}$$

Proof. By (14), we know that since $\tilde\omega$ is closed, then

$$\int_{LM}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{t}\right)\mu=\int_{LM}\exp\left\{-\frac{(d+i_X)X'}{2t}\right\}\mu. \tag{17}$$

Let $\nabla^{N_{\mathbb R}}$ be the connection on $N_{\mathbb R}$ enduced by the Levi–Civita connection of LM, and let $L^{N_{\mathbb R}}$ be its curvature. Let $J_X^{\mathbb R}$ be the action of J_X on $N_{\mathbb R}$. By [B3, proof of

Theorem 1.3], we know that

$$\lim_{t\downarrow 0}\int_{LM}\exp\left\{-\frac{(d+i_X)X'}{2t}\right\}\mu=\int_M\frac{\mu}{Pf\left[\frac{J_X^R+L^{N_R}}{2\pi}\right]}. \tag{18}$$

To obtain (16), we will prove that ∇^{N_R} is the extension to N_R of the holomorphic Hermitian connection ∇^N. Since LM is Kähler, the Levi–Civita connection of LM is the extension to T_RLM of the holomorphic Hermitian connection ∇^{LM} on TLM. We now claim that on M, we have the identification of holomorphic Hermitian vector bundles

$$TLM=TM\oplus N \tag{19}$$

(where the sum in the right hand side is orthogonal). In fact if J_X is considered as acting on TLM

$$TM=\mathrm{Ker}\,[J_X];\quad N=\mathrm{Im}\,[J_X]. \tag{20}$$

Since X is a holomorphic vector field, J_X is a holomorphic section of $\mathrm{End}(TLM)$, and so $\mathrm{Ker}\,J_X$ and $\mathrm{Im}\,J_X$ are holomorphic subbundles of TLM. We have thus proved (19). It immediately follow that ∇^{N_R} is the extension to N_R of the connection ∇^N. Equation (16) follows from (18). □

Remark 3. As the reader may suspect in view of [B4, Theorem 2.11] and of Theorem 2, a similar result holds under the weaker condition

$$\bar\partial\partial\tilde\omega=0. \tag{21}$$

The identities to be proven in this case are closely related to [B4, Theorem 1.6].

Remark 4. Observe that even if $(LM,\tilde\omega)$ is Kähler, it is not equivariantly Kähler, i.e. in general

$$\bar\partial_X\partial_X\tilde\omega\neq 0. \tag{22}$$

This is why ideas of Hermitian geometry are relevant in this context, even if $(LM,\tilde\omega)$ is Kähler.

We now prove an identity which is a finite dimensional analogue of a result given in [BGS2, Theorem 2.9] on the analytic torsion forms of direct images.

Proposition 5. *For any $t>0$*

$$\frac{\partial}{\partial t}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{2t}\right)=-\bar\partial_X\partial_X\left(\frac{\sqrt{-1}\tilde\omega}{2t^2}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{2t}\right)\right). \tag{23}$$

Proof. Clearly

$$\begin{aligned}\frac{\partial}{\partial t}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{2t}\right)&=-\frac{\bar\partial_X\partial_X(\sqrt{-1}\tilde\omega)}{2t^2}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{2t}\right)\\&=-\bar\partial_X\partial_X\left(\frac{\sqrt{-1}\tilde\omega}{2t^2}\exp\left(\frac{\bar\partial_X\partial_X\sqrt{-1}\tilde\omega}{2t}\right)\right).\end{aligned} \tag{24}$$

The proposition follows. □

Let now μ be a smooth form on LM. From the methods of the proof of [B3, Theorem 1.3], we find that as $t \downarrow 0$, we have an asymptotic expansion

$$\int_M \frac{\sqrt{-1}\tilde{\omega}}{2t} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right)\mu = \sum_{-l-1}^{k} \frac{M_k}{t^k} + o(t^k). \tag{25}$$

Therefore the function of $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$

$$F_1(s) = \frac{-1}{\Gamma(s)} \int_0^1 t^{s-1} \left\{ \int_{LM} \frac{\sqrt{-1}\tilde{\omega}}{2t} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right)\mu \right\} dt \tag{26}$$

extends into a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$. Also for $s \in \mathbb{C}$, $\mathrm{Re}(s) < 1$, the function

$$F_2(s) = \frac{-1}{\Gamma(s)} \int_1^{+\infty} t^{s-1} \left\{ \int_{LM} \frac{\sqrt{-1}\tilde{\omega}}{2t} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right)\mu \right\} dt \tag{27}$$

is holomorphic. Therefore the function $F_1(s) + F_2(s)$ is holomorphic at $s = 0$.

Let $\zeta'_{\tilde{\omega}}(0)$ be the current on LM, which is defined by the relation

$$\int_{LM} \mu \zeta'_{\tilde{\omega}}(0) = (F_1 + F_2)'(0). \tag{28}$$

Let $\delta_{\{M\}}$ be the current of integration on the oriented manifold M, so that if μ is a smooth form on LM $\int_{LM} \mu \delta_{\{M\}} = \int_M \mu$.

Theorem 6. *If the form $\tilde{\omega}$ is closed,* i.e. *if $(LM, \tilde{\omega})$ is Kähler, then*

$$\bar{\partial}_X \partial_X \zeta'_{\tilde{\omega}}(0) = 1 - \frac{\delta_{\{M\}}}{\det\left(-\frac{J_X + R^N}{2i\pi}\right)}. \tag{29}$$

Proof. If μ is a smooth differential form on LM which is a sum of forms of type (p, p), then using Proposition 5, for $s \in \mathbb{C}$ and $\mathrm{Re}(s)$ large enough

$$-\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left\{ \int_{LM} \frac{\sqrt{-1}\tilde{\omega}}{2t} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right) \bar{\partial}_X \partial_X \mu \right\} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^1 t^s \frac{\partial}{\partial t} \left\{ \int_{LM} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right)\mu \right\} dt$$

$$= \frac{1}{\Gamma(s)} \int_{LM} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2}\right)\mu - \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} \left\{ \int_{LM} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right)\mu \right\} dt. \tag{30}$$

From (37) and from Theorem 2, we find that

$$\left[-\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left\{ \int_{LM} \frac{\sqrt{-1}\tilde{\omega}}{2t} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2t}\right) \bar{\partial}_X \partial_X \mu \right\} dt \right]'(0)$$

$$= \int_{LM} \exp\left(\frac{\bar{\partial}_X \partial_X \sqrt{-1}\tilde{\omega}}{2}\right)\mu - \int_M \frac{\mu}{\det\left(-\frac{J_X + R^N}{2i\pi}\right)}. \tag{31}$$

Similarly from Proposition 5, we find easily that

$$\left[-\frac{1}{\Gamma(s)}\int_1^{+\infty} t^{s-1}\left\{\int_{LM}\frac{\sqrt{-1}\tilde{\omega}}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\bar{\partial}_X\partial_X\mu\right\}\right]'(0)$$

$$=\int_{LM}\left\{1-\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2}\right)\right\}\mu \tag{32}$$

(29) follows from (31)–(32). □

D. The dependence of $\zeta'_{\tilde{\omega}}(0)$ on the Kähler metric

As we saw in (20), the normal bundle N to M in LM is given by

$$N = \operatorname{Im} J_X. \tag{33}$$

Since J_X is a skew-adjoint complex tensor acting on (N, g^N), J_X has non zero purely imaginary eigenvalues. One verifies easily that the eigenvalues of J_X (counted with multiplicity) are constant on each connected component of M. Therefore on each connected component of M, N splits holomorphically as a direct sum

$$N = \bigoplus_j N_j \tag{34}$$

and J_X acts on each N_j as a purely imaginary scalar matrix $J_{X,j}$. Moreover the splitting (34) is orthogonal with respect the metric g^N. The holomorphic Hermitian connection ∇^N on (N, g^N) also splits into $\nabla^N = \oplus \nabla^{N_j}$. If R^{N_j} is the curvature of ∇^{N_j}, we have the obvious identity

$$\frac{1}{\det\left(-\dfrac{J_X + R^N}{2i\pi}\right)} = \prod_j \frac{1}{\det\left(-\dfrac{J_{X,j} + R^{N_j}}{2i\pi}\right)}. \tag{35}$$

Since the $J_{X,j}$ are constant scalar matrices, the right hand-side of (35) is written as a standard characteristic class in Chern–Weil theory. We note $\eta(\tilde{\omega})$ the smooth form (35) on M (which depends explicitly on $\tilde{\omega}$ via the metric g^N).

Let $\tilde{P}^{LM}$ (resp. P^M) be the set of sums of X invariant (p,p) currents on LM (resp. of sums of (p,p) smooth forms on M). Let $\tilde{P}^{LM,0}$ (resp. $P^{M,0}$) be the set of currents α in $\tilde{P}^{LM}$ (resp. of smooth forms α' in P^M), such that there exist X invariant currents $\beta, \tilde{\beta}$ (resp. smooth forms $\beta', \tilde{\beta}'$) for which $\alpha = \partial_X\beta + \bar{\partial}_X\tilde{\beta}$ (resp. $\alpha' = \partial\beta' + \bar{\partial}\tilde{\beta}'$).

Let $\tilde{\omega}, \tilde{\omega}'$ be two Kähler forms on LM, which are both X invariant. By results of Bott and Chern [BoC], Donaldson [D], Bismut et al. [BGS1, Theorem 1.29], we know that there exists a uniquely defined class $\gamma(\tilde{\omega}, \tilde{\omega}') \in P^M/P^{M,0}$ such that

$$\bar{\partial}\partial\gamma(\tilde{\omega}, \tilde{\omega}') = \eta(\tilde{\omega}') - \eta(\tilde{\omega}). \tag{36}$$

This follows from (35), in which $\eta(\tilde{\omega})$ is expressed as a standard characteristic class in P^M.

We now prove a result whose proof is very closely related to the proof of the generalized anomaly formula for the Quillen metric given in Bismut et al. [BGS3, Theorem 1.22].

Theorem 7. *If the forms $\tilde{\omega}$ and $\tilde{\omega}'$ are closed, then*

$$\zeta'_{\tilde{\omega}'}(0)-\zeta'_{\tilde{\omega}}(0)=-\gamma(\tilde{\omega},\tilde{\omega}')\delta_{\{M\}} \quad \text{in} \quad \tilde{P}^{LM}/\tilde{P}^{LM,0}. \tag{37}$$

Proof. Let $l\in[0,1]\to\tilde{\omega}_l$ be a smooth family of Kähler X invariant closed forms on LM such that $\tilde{\omega}_0=\tilde{\omega}$, $\tilde{\omega}_1=\tilde{\omega}'$. A possible choice for $\tilde{\omega}_l$ is $\tilde{\omega}_l=(1-l)\tilde{\omega}+l\tilde{\omega}'$. Let μ be a smooth form is P^{LM}. For $s\in\mathbb{C}$ close enough to 0, we know by (26)–(28) how to define unambiguously the expression $\int_M \mu\zeta_{\tilde{\omega}_l}(s)$, which we write in the form

$$\int_{LM}\mu\zeta_{\tilde{\omega}_l}(s)=-\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left\{\int_{LM}\frac{\sqrt{-1}\tilde{\omega}_l}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}_l}{2t}\right)\mu\right\}dt. \tag{38}$$

The same procedure as in (26)–(28) and in (38) will be used to define the integrals which follow.

In the sequel, we note $\equiv$ instead of the equality sign $=$ every time that currents in $\tilde{P}^{LM,0}$ are neglected, and we do not note the subscript l.

Set $\dot{\tilde{\omega}}=\frac{\partial\tilde{\omega}_l}{\partial l}$. Using Proposition 5, we find that for $s\in\mathbb{C}$ close enough to 0

$$\begin{aligned}
&\frac{\partial}{\partial l}\int_{LM}\mu\zeta_{\tilde{\omega}}(s)\\
&=-\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left\{\int_{LM}\frac{\sqrt{-1}}{2t}\left(\dot{\tilde{\omega}}+\tilde{\omega}\frac{\bar{\partial}_X\partial_X\sqrt{-1}\dot{\tilde{\omega}}}{2t}\right)\exp\left(\sqrt{-1}\frac{\bar{\partial}_X\partial_X\tilde{\omega}}{2t}\right)\mu\right\}dt\\
&\equiv-\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left\{\int_{LM}\left(\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)-\frac{\sqrt{-1}}{2}\dot{\tilde{\omega}}\frac{\partial}{\partial t}\right.\right.\\
&\left.\left.\cdot\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\right)\mu\right\}dt=\frac{-s}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left\{\int_{LM}\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu\right\}dt.
\end{aligned} \tag{39}$$

From the proof of [B3, Theorem 1.3] and from Theorem 2, we know that as $t\downarrow 0$, there is an asymptotic expansion

$$\int_{LM}\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu=\sum_{-1}^{k}M_jt^j+o(t^k). \tag{40}$$

From (39), we find that

$$\frac{\partial}{\partial l}\int_{LM}\mu\zeta'_{\tilde{\omega}_l}(0)\equiv -M_0. \tag{41}$$

Let g_l^N be the metric on N induced by $\tilde{\omega}_l$, and let R_l^N be the curvature of the holomorphic Hermitian connection on (N,g_l^N). Then if we write $R^N=R_l^N$, by Theorem 2 we find that

$$M_{-1}=\int_M\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2}\frac{\mu}{\det\left(-\frac{J_X+R^N}{2i\pi}\right)}. \tag{42}$$

To calculate M_0, we now will use the fact that $\tilde{\omega}$ and $\tilde{\omega}'$ are closed. By Proposition 5, we get

$$\frac{\partial}{\partial t}t\int_{LM}\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2t}\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu$$
$$=-\int_{LM}\frac{\sqrt{-1}\dot{\tilde{\omega}}}{2t}\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu$$
$$\equiv\int_{LM}\left(\frac{\sqrt{-1}}{2t}\bar{\partial}_X\dot{\tilde{\omega}}\right)\left(\frac{\sqrt{-1}}{2t}\partial_X\tilde{\omega}\right)\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu. \tag{43}$$

Using (40), (43), and the fact that the asymptotic expansion (40) can be differentiated, and also the relations $\bar{\partial}\dot{\tilde{\omega}}=0$, $\partial\tilde{\omega}=0$, we get

$$M_0\equiv\lim_{t\downarrow 0}\int_{LM}\left(\frac{\sqrt{-1}\,i_{X^{(1,0)}}\dot{\tilde{\omega}}}{2t}\right)\left(\frac{\sqrt{-1}\,i_{X^{(0,1)}}\tilde{\omega}}{2t}\right)\exp\left(\frac{\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}}{2t}\right)\mu. \tag{44}$$

Now since $\tilde{\omega}$ is closed, by (19), we know that

$$\bar{\partial}_X\partial_X\sqrt{-1}\tilde{\omega}=-\frac{(d+i_X)X'}{2}.$$

Note that the forms $i_{X^{(1,0)}}\dot{\tilde{\omega}}, i_{X^{(0,1)}}\tilde{\omega}$ vanish on M. Also TM and N are orthogonal in TLM for all the ω_l simultaneously, so that if $Z\in T_RM, Z'\in N_R$

$$\tilde{\omega}(Z,Z')=\dot{\tilde{\omega}}(Z,Z')=0. \tag{45}$$

In particular for $y\in N$, the forms $i_{J_Xy}\dot{\tilde{\omega}}$ and $i_{J_X\bar{y}}\tilde{\omega}$ both vanish on T_RM. Using now (45) and the previous considerations, we find that the methods of [B3, proof n°2 of Theorem 1.3] can be used to calculate M_0. For $y\in N$, set $Y=y+\bar{y}\in N_R$. From (44), we get

$$M_0\equiv\int_M\mu\int_{N_R}\sqrt{-1}i_{J_Xy}\dot{\tilde{\omega}}\wedge\sqrt{-1}i_{J_X\bar{y}}\tilde{\omega}\exp\left\{-\frac{\langle R^N(.,.)Y,J_XY\rangle}{2}-\frac{|J_XY|^2}{2}+J_X\right\}. \tag{46}$$

If η denotes a sum of forms in $\Lambda(N_R^*)$, let $\eta^{\max}$ be the component of top degree $\dim N_R^*$. One verifies easily that

$$\{\sqrt{-1}i_{J_Xy}\dot{\tilde{\omega}}\wedge\sqrt{-1}i_{J_X\bar{y}}\omega\exp(J_X)\}^{\max}=\left\langle(g^N)^{-1}\left(\frac{\partial g^N}{\partial l}\right)J_Xy,J_X^{-1}(J_X\bar{y})\right\rangle\{\exp(J_X)\}^{\max}. \tag{47}$$

Now since the vector field X is Killing for all the metrics $\tilde{\omega}_l, J_X$ commutes with $(g^N)^{-1}\left(\frac{\partial g^N}{\partial l}\right)$. From (46), we find that

$$M_0\equiv\int_M\mu\frac{\partial}{\partial b}\left[\int_{N_R}\exp\left\{-\frac{1}{2}\left\langle\left(R^N(.,.)+b(g^N)^{-1}\left(\frac{\partial g^N}{\partial l}\right)\right)Y,J_XY\right\rangle-\frac{|J_XY|^2}{2}+J_X\right\}\right]_{b=0}. \tag{48}$$

The right-hand side of (48) contains a Gaussian integral which can be calculated as in [B3, (1.22)]. We thus obtain

$$M_0 \equiv \int_M \mu \frac{\partial}{\partial b} \left[\frac{1}{\det\left(- \dfrac{J_X + R^N + b(g^N)^{-1} \dfrac{\partial g^N}{\partial l}}{2i\pi} \right)} \right]_{b=0} . \tag{49}$$

From (41), (49), we find that

$$\int_{LM} \mu(\zeta'_{\tilde{\omega}'}(0) - \zeta'_{\tilde{\omega}}(0)) \equiv - \int_M \mu \int_0^1 \frac{\partial}{\partial b} \left[\frac{1}{\det\left(- \dfrac{J_X + R^N + b(g)^{N-1} \dfrac{\partial g^N}{\partial l}}{2i\pi} \right)} \right]_{b=0} dl. \tag{50}$$

By [BGS1, Theorem 1.27, Remarks 1.28 and 1.31] using (50), we find that

$$\int_{LM} \mu(\zeta'_{\tilde{\omega}'}(0) - \zeta'_{\tilde{\omega}}(0)) \equiv - \int_M \mu \gamma(\tilde{\omega}, \tilde{\omega}'). \tag{51}$$

Our Theorem is proved. □

Remark 8. The proof of Theorem 3.6 contains in a simple finite dimensional context the main tools used in [BGS3, Sect. 1h)] in an infinite dimensional situation. The analogue of (42) for M_{-1} was obtained in [BGS3, Theorem 1.22]. The analogue of the key formula (43) was obtained in [BGS3, Theorem 1.20] as a formula of linear algebra on supertraces of finite or infinite dimensional operators. Formula (4) is intimately related to the generalized Lichnerowicz formula of [BGS3, Theorem 1.21]. Equation (48) appears in an infinite dimensional form in [BGS3, (1.142)].

In fact if we use again the notations of Sect. A and B, if τ denotes the Ray–Singer analytic torsion [RS] of the Dolbeault complex $(\bar{\partial}, \xi)$, then we have the formal representation of $\log(\tau)$ as a path integral on the loop space LM

$$\log(\tau) = C \int_{LM} \zeta'_{\tilde{\omega}}(0)\beta. \tag{52}$$

Most of the results of [BGS2] and [BGS3] on Quillen metrics and on the Bott–Chern forms of direct images can be formally derived from (29) and (52).

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