Approximate mean curvature flows of varifolds and limit Brakke flows

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Résumé

Le flot de courbure moyenne d'une surface régulière est caractérisé par le déplacement de chaque point de la surface selon une vitesse vectorielle égale au vecteur de courbure moyenne, c'est-à-dire au gradient L^2 de la fonctionnelle d'aire de la surface. On rappelle en effet que, pour une d-sous-variété $\mathcal{M} \subset \mathbb{R}^n$ et un champ de vecteurs $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, la courbure moyenne H de $\mathcal{M}$ est caractérisée par l'équation

$$\frac{d}{dt}\mathcal{H}^d\left(\Phi_X^t(\mathcal{M})\right)_{|t=0} = -\int_{\mathcal{M}} X \cdot H \, d\mathcal{H}^d, \tag{1}$$

où $(\Phi_X^t)_{t\geq 0}$ est le flot engendré par le champ de vecteurs X et $\mathcal{H}^d$ désigne la mesure de Hausdorff d-dimensionnelle. Par conséquent, le flot de courbure moyenne de $\mathcal{M}$ consiste en l'évolution de la surface en chacun de ses points dans la direction qui permet de réduire son aire le plus rapidement possible.

Le flot de courbure moyenne est un modèle emblématique pour de nombreux phénomènes physiques, il est très utilisé en ingénierie et en sciences du numérique, et il est depuis des années le sujet de travaux mathématiques à l'interface de l'analyse, de la géométrie, de la théorie de la mesure, de l'optimisation et du calcul scientifique.

Il est connu que, même partant d'une surface régulière, le flot de courbure moyenne peut produire des singularités en temps fini. C'est ce qui a motivé l'introduction et l'étude de flots faibles de courbure moyenne permettant de donner un sens à l'évolution au delà des singularités. L'un des plus emblématiques de ces flots faibles est le flot introduit par Kenneth Brakke pour les varifolds [12]. On rappelle qu'un varifold est une mesure de Radon dans $\mathbb{R}^n \times G_{d,n}$, où $G_{d,n}$ désigne la grassmanienne des d-plans de $\mathbb{R}^n$. C'est une notion faible de surface qui est utilisée en théorie de la mesure géométrique car elle présente, sous des hypothèses supplémentaires très naturelles, des propriétés très intéressantes. En particulier, l'équation (1) peut être étendue aux varifolds afin de définir une notion généralisée de courbure moyenne. S'appuyant sur cette notion généralisée, un flot de Brakke est alors caractérisé par une inéquation variationnelle qui permet d'étendre, bien au delà des surfaces régulières, la notion de flot de courbure moyenne.

Un flot de Brakke partant d'un varifold donné peut être construit comme la limite d'un schéma itératif sophistiqué qui combine deux étapes : une étape de désingularisation et une étape de déformation dépendant d'une courbure moyenne approchée [12]. Cette construction, et l'alternative en codimension 1 proposée par Kim & Tonegawa [36], n'est toutefois pas adaptée à des varifolds qui ne sont pas entiers, c'est-à-dire rectifiables mais à multiplicité non entière, ou non rectifiables. Or, de nombreuses applications en informatique graphique et en traitement d'images font intervenir des structures discrètes (par exemple des nuages de points) de dimension et codimension très variées et qui ne peuvent pas être associées à des varifolds entiers. L'objectif de cette thèse est précisément de dépasser cette limitation.

Au chapitre 1, nous rappelons les notions et résultats de théorie de la mesure et de théorie des varifolds utilisés dans cette thèse. Nous revenons au chapitre 2 sur les notions fortes et faibles de flot de courbure moyenne, en détaillant plus particulièrement la construction d'un flot de Brakke due à Kim & Tonegawa [36]. Nous synthétisons au chapitre 3 les différents résultats obtenus dans la thèse. Nous y discutons aussi des conséquences de ces résultats dans le cas particulier des

nuages de points (flots de nuages de points, résultat de consistance, schéma numérique) et nous proposons plusieurs perspectives de recherche.

Il est connu qu'un flot de courbure moyenne régulière est un flot au sens de Brakke. La contribution présentée au chapitre 4 porte sur la préservation d'une propriété à la Brakke quand on discrétise un flot régulier. Plus précisément, nous considérons un flot de courbure moyenne $\mathcal{M}(t)_{t\in[0,T]}$ issu d'une sous-variété $\mathcal{M}$ de classe C^3 et, pour chaque temps t, une discrétisation volumique spatiale $V_h(t)$ de $\mathcal{M}(t)$. Nous montrons que la discrétisation $V_h(t)$ vérifie une inégalité de Brakke intégrale approchée avec un terme d'erreur dépendant de la géométrie de la sous-variété initiale, d'une échelle de régularisation ε utilisée pour la définition d'une courbure moyenne approchée de $V_h(t)$, et des noyaux utilisés dans la définition de cette courbure moyenne approchée.

Les chapitres 5 et 6 contiennent les résultats principaux de la thèse. Nous réussissons d'abord à définir au chapitre 5, par un procédé constructif et itératif inspiré des travaux de Brakke [12] et Kim & Tonegawa [36], un flot approché de courbure moyenne pour une classe très générale de varifolds pouvant représenter des données très variées, que ce soit des surfaces régulières ou singulières, des surfaces discrètes, des nuages de points, etc. Plus précisément, pour une échelle donnée d'approximation $\varepsilon \in]0,1[$ et une subdivision $\mathcal{T}$ de l'intervalle de temps choisi (par exemple $[0,1]$), étant donné un varifold initial général V, nous utilisons une variante de la courbure moyenne approchée définie dans [36] pour appliquer des poussées en-avant successives à V relativement à ε et à $\mathcal{T}$. Une interpolation linéaire en temps conduit à un flot approché de courbure moyenne discret en temps noté $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$. Nous prouvons la stabilité de ce flot par rapport à la subdivision et à la donnée initiale. Puis, grâce à ce résultat de stabilité et pour ε fixé, nous montrons la convergence, lorsque le pas de subdivision tend vers 0, de $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ vers un unique flot noté $(V_\varepsilon(t))_{t\in[0,1]}$. Nous prouvons que $(V_\varepsilon(t))_{t\in[0,1]}$ satisfait une égalité de Brakke par rapport à son vecteur de courbure moyenne approchée.

Nous considérons ensuite la notion de *flot de Brakke spatio-temporel*, une mesure sur $\mathbb{R}^n \times G_{d,n} \times [0,1]$ qui étend naturellement la notion de flot de Brakke. En particulier, si $W(t)_{t\geq 0}$ est un flot de Brakke alors $W(t) \otimes dt$ est un flot de Brakke spatio-temporel. Nous montrons dans le théorème 5.3.7 que, si $(V_\varepsilon(t))_{t\in[0,1]}$ désigne l'ensemble des flots limites obtenus par la construction décrite ci-dessus pour une suite d'échelles d'approximation ε tendant vers 0, alors la suite $\lambda_\varepsilon := V_\varepsilon(t) \otimes dt$ converge (après extraction) vers une limite λ qui admet un vecteur de courbure moyenne borné dans $L^2(\|\lambda\|)$. En outre, si sa restriction à $\mathbb{R}^n \times G_{d,n}$ est rectifiable, alors λ est un flot de Brakke spatio-temporel.

Dans le chapitre 6, nous nous intéressons aux propriétés en codimension 1 du flot limite λ et, plus généralement, de flots de Brakke spatio-temporels. Nous prouvons en particulier un résultat de non trivialité de λ – la possible trivialité est un des inconvénients de la notion de flot de Brakke – lorsque le varifold de départ est associé à la frontière d'une partition d'ensembles ouverts de périmètres finis de $\mathbb{R}^n$. Nous prouvons que le support de la composante spatiale d'un flot de Brakke spatio-temporel est inclus dans l'évolution de la donnée initiale par ensembles de niveau (*motion of level sets*) au sens de [22]. Cela découle du fait que le support évite les flots de courbure moyenne réguliers et, en particulier, nous en déduisons par un principe de comparaison fort que le support de la composante spatiale d'un flot de Brakke spatio-temporel partant d'une hypersurface régulière coïncide avec l'évolution par courbure moyenne (au sens classique) de l'hypersurface.

Contents

Chapter 1

Introduction

1.1 Notations

Throughout the thesis, we let $d, n \in \mathbb{N}$ be such that $1 \le d \le n$, $2 \le n$ and we adopt the following notations:

- $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$.
- $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure in $\mathbb{R}^n$.
- $B_r(x)$ and $B(r, x)$ denote the open ball of radius $r > 0$ and center $x \in \mathbb{R}^n$. Conventionally B_r denotes the open ball of radius $r > 0$ and center 0. For closed balls we keep the same notations and replace B by $\overline{B}$.
- For $k \in \mathbb{N}$, ω_k denotes the volume of the $k-$dimensional unit ball.
- For any two sets A and B of $\mathbb{R}^n$ we define $A + B := \{a + b,\ (a, b) \in A \times B\}$.
- For a set A, $A^\delta := \bigcup\limits_{x \in A} B_\delta(x) = \{y \in \mathbb{R}^n, d(y, A) < \delta\}$.
- $\mathcal{M}_{p,q}$ is the space of real matrices with p rows, q columns.
- For $A \in \mathcal{M}_{p,q}$, $B \in \mathcal{M}_{q,r}$ we denote either by $A \circ B$ or by AB the product of A and B.
- For two matrices M and N we define the matrix scalar product by

$$M : N = \operatorname{tr}\big(MN^t\big) = \operatorname{tr}\big(NM^t\big).$$

- The default matrix norm $\|\cdot\|$ considered in $\mathcal{M}_{p,q}$ is the operator 2-norm associated with the Euclidean norms $|\cdot|$ in $\mathbb{R}^p$ and $\mathbb{R}^q$. We also consider the norm $|\cdot|_\infty$ defined as $|M|_\infty = \max_{\substack{i=1\dots p\\ j=1\dots q}} |M_{ij}|$, for $M \in \mathcal{M}_{p,q}$ and we recall the classical relation:

$$\forall M \in \mathcal{M}_{p,q}, \qquad |M|_\infty \le \|M\| \le \sqrt{pq}\, |M|_\infty\, . \tag{1.1}$$

- For $k \in \mathbb{N}$ and $v, w \in \mathbb{R}^k$, $\langle v, w\rangle := \sum\limits_{i=1}^{k} v_i w_i$ is the scalar product of v and w.

- The supremum norm of a measurable function u is $\|u\|_\infty := \text{ess-sup}_x |u(x)|$.
- C^0, C^0_c denote the space of continuous functions, and compactly supported continuous functions, respectively.
- $C^{0,1}$, $C^{0,1}_c$ denote the space of Lipschitz functions, and compactly supported Lipschitz functions, respectively.
- C^1 denotes the space of continuously differentiable functions and, for $\varphi \in C^1$, $\|\varphi\|_{C^1} := \|\varphi\|_\infty + \|D\varphi\|_\infty$.
- C^2 denotes the space of continuously differentiable functions and, for $\varphi \in C^2$, $\|\varphi\|_{C^2} := \|\varphi\|_\infty + \|D\varphi\|_\infty + \|D^2\varphi\|_\infty$.
- $G_{d,n}$ is the Grassmannian manifold of d-dimensional vector subspaces of $\mathbb{R}^n$. We identify every element $S \in G_{d,n}$ with its orthogonal projection on the d-subspace $S \in \mathcal{M}_n(\mathbb{R})$. The distance considered between $S, T \in G_{d,n}$ is $\|S - T\|$, where $\|\cdot\|$ is the default matrix norm introduced above.
- For $\Omega \subset \mathbb{R}^n$, $\underset{\Omega}{\mathrm{Lip}}(f)$ denotes the Lipschitz coefficient of f on Ω.
- The functions and vectors involved may depend on the space and the time, we conventionally use ∇ for space derivation and ∂_t for time derivation.
- We mean by a closed submanifold, a compact boundaryless submanifold.
- Throughout the introduction and chapter 4, $\rho \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ is a non-increasing function supported on $[0,1]$ and $\xi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is a function supported on $[0,1]$, positive on $]0,1[$. We define ρ_ε and ξ_ε on $\mathbb{R}^n$ by

$$\rho_\varepsilon(x) := \rho\left(\frac{|x|}{\varepsilon}\right), \quad \xi_\varepsilon(x) := \xi\left(\frac{|x|}{\varepsilon}\right) \ \forall x \in \mathbb{R}^n.$$

Definition 1.1.1 (Subdivision). *For $a < b \in \mathbb{R}$ and $m \geq 1$, $\mathcal{T} = \{t_i\}_{i=0}^m$ is called a subdivision of $[a,b]$ if: $a = t_0 < t_1 \cdots < t_m = b$. We denote:*

$$\delta(\mathcal{T}) := \max_i t_i - t_{i-1} \quad i \in \{1, \ldots, m\}.$$

1.2 Generalities on measure theory

The central tool used in our work is the space of varifolds. Before going through its definition and properties, we first introduce the notion of rectifiable sets after reviewing the necessary notions on convergence and distances between Radon measures.

1.2.1 Radon measures

A measure μ on a locally compact space X is called a Radon measure when it is Borel, regular and finite on compact sets. The Riesz representation theorem implies that one can view any Radon measure as a continuous linear form on $C_c^0(\mathbb{R}^n, \mathbb{R})$ and vice-versa.

The bounded Lipschitz distance will be useful to compare measures and prove convergence.

Definition 1.2.1. *(Bounded Lipschitz distance) Let (X, d) be a locally compact separable metric space. The bounded Lipschitz distance between two finite Radon measures ν, μ on X is defined as*

$$\Delta(\nu, \mu) := \sup \left\{ \left| \int_X \varphi(x) d\nu(x) - \int_X \varphi(x) d\mu(x) \right| \varphi \in C^{0,1}(X, \mathbb{R}^+) \text{ and } \max\{\|\varphi\|_\infty, \mathrm{Lip}(\varphi)\} \leq 1 \right\} \tag{1.2}$$

We similarly define a localized version of the bounded Lipschitz distance, for any open set $U \subset X$:

$$\Delta_U(\nu, \mu) := \sup \left\{ \left| \int_U \varphi(x) d\nu(x) - \int_U \varphi(x) d\mu(x) \right| \begin{array}{l} \varphi \in C^{0,1}(X, \mathbb{R}^+),\ \mathrm{spt}\, \varphi \subset U \\ \max\{\|\varphi\|_\infty, \mathrm{Lip}(\varphi)\} \leq 1 \end{array} \right\}$$

We use throughout our work the weak-$*$ convergence of measures, whose definition is now recalled.

Definition 1.2.2 (Weak-$*$ convergence)**.** *Let (X, d) be a locally compact and separable metric space. Let $(\mu_i)_{i \in \mathbb{N}}$, μ be Radon measures on X. We say that: $(\mu_i)_i$ converges weakly-$*$ to μ if*

$$\int \varphi \, d\mu_i \to \int \varphi \, d\mu$$

for every $\varphi \in C_c^0(X, \mathbb{R})$.

The following result is a general fact in measure theory (see for instance [42, Theorem 5.9]).

Proposition 1.2.3. *Let (X, d) be a locally compact separable metric space. Let $(\mu_i)_{i \in \mathbb{N}}$ and μ be finite Radon measures. Assume that the measures $(\mu_i)_{i \in \mathbb{N}}$ and μ are supported in a compact set of X. Then:*

$$\mu_i \text{ converges weakly-}* \text{ to } \mu \quad \Longleftrightarrow \quad \Delta(\mu_i, \mu) \xrightarrow[i \to \infty]{} 0.$$

1.2.2 Rectifiable sets

Rectifiability is a fundamental notion of weak regularity for sets in the measure theoretic setting. A rectifiable set admits an approximate tangent space at almost every point, which opens the way to consistent definitions of mean curvature and second fundamental form, see for instance [12, 32, 15, 16, 17].

Definition 1.2.4. *(Rectifiable sets [39, Chapter 10]) $\mathcal{M}$ is (countably) d-rectifiable if and only if there exist a Borel set $\mathcal{M}_0 \subset \mathbb{R}^n$, countably many Lipschitz maps $(f_h)_{h\in\mathbb{N}} : \mathbb{R}^d \to \mathbb{R}^n$ and Borel sets $(F_h)_{h\in\mathbb{N}} \subset \mathbb{R}^d$ such that*

$$\mathcal{M} = \mathcal{M}_0 \cup \Big(\bigcup_{h\in\mathbb{N}} f_h(F_h) \Big), \qquad \text{with} \quad \mathcal{H}^d(\mathcal{M}_0) = 0.$$

One can view rectifiable sets differently thanks to Whitney's extension theorem:

Proposition 1.2.5. *[43, Lemma 1.1] $\mathcal{M}$ is (countably) d-rectifiable if and only if*

$$\mathcal{M} = \mathcal{M}_0 \cup \left(\bigcup_{j\in\mathbb{N}} N_j \right)$$

where $\mathcal{H}^d(\mathcal{M}_0) = 0$ and $\forall j \in \mathbb{N}, N_j$ is contained in some C^1 d-submanifold of $\mathbb{R}^n$.

As mentioned before, rectifiable sets admit approximate tangent spaces almost everywhere. To make this statement more precise, let us first recall the definition of the push-forward operation on Radon measures. Let (X, μ) be a measurable space and $f : X \to Y$ be a measurable map; the push-forward measure of μ by the map f is defined by

$$(f_{\#}\mu)(B) = \mu(f^{-1}(B)), \quad \text{for every Borel set } B \subset Y. \tag{1.3}$$

We are now able to define the notion of approximate tangent space.

Definition 1.2.6. *(Approximate tangent space) Let μ be a Radon measure on $\mathbb{R}^n$, P a d-plane in $\mathbb{R}^n$. P is said to be the approximate tangent space of μ with multiplicity θ at a point $x \in \mathbb{R}^n$ if*

$$r^{-d} \int_{\mathbb{R}^n} \varphi \left(\frac{y - x}{r} \right) d\mu(y) \xrightarrow[r\to 0^+]{} \theta \int_P \varphi(y)\, d\mathcal{H}^d(y), \quad \forall\, \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}^+).$$

In other words

$$r^{-d}(\tau_{x,r})_{\#}\mu \xrightarrow[r\to 0^+]{*} \theta \mathcal{H}^d_{|P}$$

where $\tau_{x,r}(\cdot) = \dfrac{\cdot - x}{r}$.

Roughly speaking, the blow up of the measure near x gets closer and closer to the Lebesgue measure on P (up to a constant) as $r \to 0^+$. Conversely, a measure that admits an approximate tangent space almost everywhere is supported on a rectifiable set, and this is the reason why rectifiable sets and measures hold a significant interest in measure theory. Here is the rigorous statement:

Theorem 1.2.7. *[43, Theorem 1.6] Let $\mathcal{M}$ be a $\mathcal{H}^d$ -measurable set with $\mathcal{H}^d(\mathcal{M} \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^n$. Then $\mathcal{M}$ is (countably) d-rectifiable if and only if the approximate tangent space exists for $\mathcal{H}^d$-a.e. $x \in \mathcal{M}$.*

1.3 Varifolds

1.3.1 Definitions

Definition 1.3.1. *(Varifolds) A d-varifold in $\mathbb{R}^n$ is a non-negative Radon measure on $\mathbb{R}^n \times G_{d,n}$. $V_d(\mathbb{R}^n)$ denotes the space of d-varifolds in $\mathbb{R}^n$.*

The mass measure $\|V\|$ associated with a varifold V is defined as follows:

$$\|V\|(\varphi) := \int_{\mathbb{R}^n \times G_{d,n}} \varphi(x)\, dV(x, S) \quad \forall \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}).$$

In other words, for every Borel set $A \subset \mathbb{R}^n$, we have $\|V\|(A) = V(A \times G_{d,n})$.

In the following, we list classical examples of varifolds.

1. Smooth varifolds: to a d-dimensional submanifold $\mathcal{M}$ in $\mathbb{R}^n$, we associate the varifold $V = \mathcal{H}^d_{|\mathcal{M}} \otimes \delta_{T.\mathcal{M}}$ defined by

   $$V:\ \varphi \in C_c^0(\mathbb{R}^n \times G_{d,n}, \mathbb{R}) \mapsto \int_{\mathcal{M}} \varphi(y, T_y\mathcal{M})\, d\mathcal{H}^d(y).$$

   The associated mass varifold is $\|V\| = \mathcal{H}^d_{|\mathcal{M}}$ such that

   $$\|V\|:\ \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}) \mapsto \int_{\mathcal{M}} \varphi(y)\, d\mathcal{H}^d(y).$$

2. Rectifiable varifolds: to a d-dimensional rectifiable set $\mathcal{M}$ in $\mathbb{R}^n$ (Definition 1.2.4), and a non-negative $\mathcal{H}^d$-integrable function θ on $\mathcal{M}$, we associate the varifold $V = \theta\mathcal{H}^d_{|\mathcal{M}} \otimes \delta_{T.\mathcal{M}}$ such that

   $$V:\ \varphi \in C_c^0(\mathbb{R}^n \times G_{d,n}, \mathbb{R}) \mapsto \int_{\mathcal{M}} \varphi(y, T_y\mathcal{M})\theta(y)\, d\mathcal{H}^d(y),$$

   where $T_y\mathcal{M}$ is the approximate tangent space of $\mathcal{M}$ at y. The associated mass varifold is defined as $\|V\| = \theta\mathcal{H}^d_{|\mathcal{M}}$ such that

   $$V:\ \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}) \mapsto \int_{\mathcal{M}} \varphi(y)\theta(y)\, d\mathcal{H}^d(y).$$

   Integral varifolds: a rectifiable varifold is called *integral* if $\theta \in \mathbb{N}$ a.e.

3. Point cloud varifolds: to a distribution of points $\{x_j\}_{j=1}^N$ in $\mathbb{R}^n$, d-planes $\{P_j\}_{j=1}^N$ in $G_{d,n}$ and masses $\{m_j\}_{j=1}^N$ in $\mathbb{R}^+$, we associate the varifold: $V = \sum_{j=1}^N m_j\delta_{x_j} \otimes \delta_{P_j}$ such that

   $$V:\ \varphi \in C_c^0(\mathbb{R}^n \times G_{d,n}, \mathbb{R}) \mapsto \sum_{j=1}^N m_j\varphi(x_j, P_j).$$

   Then, the associated mass varifold is defined as: $\|V\| = \sum_{j=1}^N m_j\delta_{x_j}$ such that

   $$V:\ \varphi \in C_c^0(\mathbb{R}^n, \mathbb{R}) \mapsto \sum_{j=1}^N m_j\varphi(x_j).$$

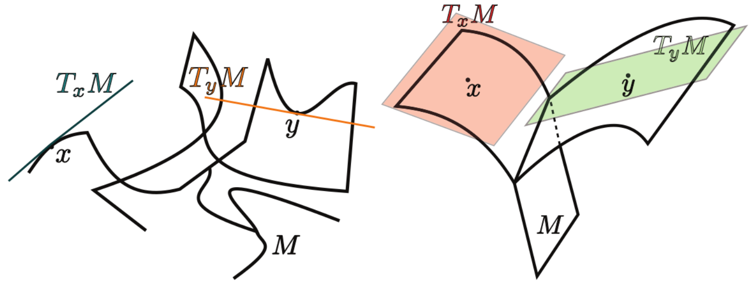

Figure 1.1: A rectifiable 1-varifold (left) and a rectifiable 2-varifold (right) (illustration from [13]).

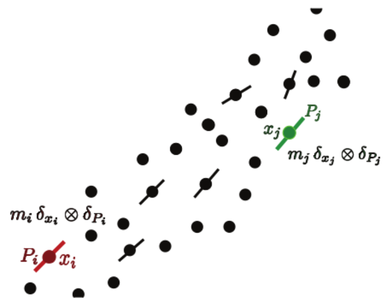

Figure 1.2: A point cloud varifold (illustration from [13]).

From now on, and for a smooth submanifold $\mathcal{M}$, we denote $M := \mathcal{H}^d_{|\mathcal{M}} \otimes \delta_{T_y\mathcal{M}}$ and $\|M\| := \mathcal{H}^d_{|\mathcal{M}}$.

We recall the notion of Ahlfors regularity in the setting of varifolds.

Definition 1.3.2. *(Ahlfors regularity) Let $V \in V_d(\mathbb{R}^n)$. We say that V is Ahlfors regular if its mass measure $||V||$ is d-Ahlfors regular, i.e. there exist $C_0 > 1$, $r_0 > 0$ such that for all $x \in \operatorname{spt}\|V\|$ and $0 < r \leq r_0$,*

$$C_0^{-1} r^d \leq \|V\|(B(x,r)) \leq r^d C_0. \tag{1.4}$$

Note that r_0 can be chosen as large as needed: if condition (1.4) holds for some $r_0 > 0$ then it holds for any $r_1 \geq r_0$ possibly adapting the regularity constant C_0.

1.3.2 Approximation by discrete varifolds

Volumetric varifolds were introduced to discretize, in the spirit of varifolds, submanifolds of any dimension and co-dimension, possibly with singularities. In this section, we present some of the results of [13] on the discretization and the approximation by volumetric varifolds.

Consider an open set $\Omega \subset \mathbb{R}^n$ and $h > 0$. A mesh $\mathcal{K}$ of Ω of size h is a locally finite partition of Ω into cells, with each cell $K \in \mathcal{K}$ of diameter less than h.
Given a d-submanifold $\mathcal{M}$ in Ω, we define a volumetric discretization V_h of $\mathcal{M}$ as follows:

$$V_h = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \tag{1.5}$$

where: $|K| = \mathcal{L}^n(K)$, $m_K = \mathcal{H}^d(\mathcal{M} \cap K)$, and $P_K \in \underset{S \in G_{d,n}}{\arg\min} \int_{\mathcal{M} \cap K} |T_y\mathcal{M} - S| \, d\mathcal{H}^d(y)$ for any cell K in $\mathcal{K}$.

The following proposition results from [13, Theorem 2.1] when the approximated rectifiable varifold is smooth.

Proposition 1.3.3. *Let Ω be a open set of $\mathbb{R}^n$, $\mathcal{K}_h$ a mesh of Ω. Let $\mathcal{M} \subset \Omega$ be a closed d-submanifold and M be the varifold associated with $\mathcal{M}$. Let V_h be a volumetric discretization of $\mathcal{M}$ defined as in (1.5). We have, for any Lipschitz function φ on Ω:*

$$\Big| \|M\|(\varphi) - \|V_h\|(\varphi) \Big| \leq h \underset{\Omega}{\mathrm{Lip}}(\varphi) \|M\|(\mathrm{spt}(\varphi)). \tag{1.6}$$

In addition, for $C > 0$ such that,

$$\big| T_x\mathcal{M} - T_y\mathcal{M} \big| \leq C|x - y| \quad \forall x, y \in \mathcal{M},$$

and if we denote $\Pi : \Omega \times G_{d,n} \to \mathbb{R}^n, (y, S) \mapsto y$, for every Lipschitz function φ on $\Omega \times G_{d,n}$, we have

$$\Big| M(\varphi) - V_h(\varphi) \Big| \leq h \underset{\Omega}{\mathrm{Lip}}(\varphi)(1 + 2C) \|M\|(\Pi(\mathrm{spt}(\varphi))). \tag{1.7}$$

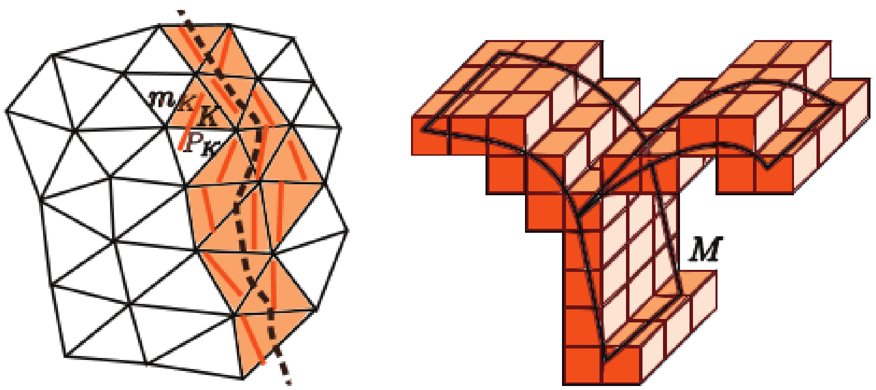

Figure 1.3: Approximation by volumetric varifolds [13].

We note that in the original statement of [13, Theorem 2.1], $y \mapsto T_y\mathcal{M}$ is a Hölder map with Hölder coefficient $\beta \in (0, 1)$. One can easily notice that the proof still holds for $\beta = 1$. This explains why the result presented in this section is different from the original one from [13].

1.4 Mean curvature of varifolds: definition and approximations

We introduce in this section the notion of generalized mean curvature for varifolds. Let us first recall the notions of push-forward (see [46, Section 1.4]) and first variation for varifolds.

Definition 1.4.1. *(Push-forward of a varifold) Let V be a varifold in $\mathbb{R}^n$ and f a C^1 diffeomorphism of $\mathbb{R}^n$. The push-forward of V by f is the varifold $f_\# V$ defined for every $\varphi \in C_c(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$ by*

$$f_\# V(\varphi) := \int_{\mathbb{R}^n \times G_{d,n}} \varphi(f(x), Df(x)(S)) J_S f(x) \, dV(x, S), \tag{1.8}$$

where $Df(x)(S)$ is the image of S in $G_{d,n}$ by the linear isomorphism $Df(x)$ and the tangential Jacobian $J_S f(x)$ is the determinant of the isomorphism $Df(x)$ from S to $Df(x)(S)$ defined as follows: if we write $\tilde{S} = (s_1|\dots|s_d)^t \in \mathcal{M}_{d,n}$ where $\{s_i\}_{i=1}^d$ is an orthonormal basis of S, and if we set $Y = Df(x) \circ \tilde{S}^t$, we have

$$J_S f(x) := \det \left(Y^t Y\right)^{\frac{1}{2}} \tag{1.9}$$

where $Df(x) \circ \tilde{S}^t$ is the product of the matrices $Df(x) \in \mathcal{M}_n$ (identified to a matrix using the canonical basis of $\mathcal{M}_n$) and $\tilde{S}^t \in \mathcal{M}_{n,d}$. Moreover, if we denote by P the projection on the space $Df(x)(S)$, we have (see [45, p. 184])

$$P = Y(Y^t Y)^{-1} Y^t. \tag{1.10}$$

Remark 1.4.2. (Push-forward of varifolds vs push-forward of Radon measures).
Let $\mathcal{M} \subset \mathbb{R}^n$ be a d-submanifold and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. By the area formula, we have

$$\int_{f(\mathcal{M})} \varphi(y)\, d\mathcal{H}^d(y) = \int_{\mathcal{M}} \varphi(f(x)) J_{T_x\mathcal{M}} f(x)\, d\mathcal{H}^d(x) \quad \forall \varphi \in C_0^1(\mathbb{R}^n, \mathbb{R}^+), \tag{1.11}$$

where $J_{T_x\mathcal{M}} f(x)$ is the tangential Jacobian of f with respect to $T_x\mathcal{M}$. The previous formula expresses the change of a measure of dimension d by a C^1 map and inspired Definition 1.4.1.
Now, let A be a measurable set of $\mathbb{R}^n$ and $\mu = \mathcal{L}^n_{|A}$. By the change of variable formula, which is a particular case of the area formula, the push-forward measure $f_{\#}\mu$ defined in (1.3) satisfies:

$$f_{\#}\mu(\varphi) = \int_{f(A)} \varphi(y)\, d\mathcal{L}^n(y) = \int_A \varphi(f(x)) Jf(x)\, d\mathcal{L}^n(x), \quad \forall \varphi \in C_0^1(\mathbb{R}^n, \mathbb{R}^+),$$

where Jf is the Jacobian of the map f. Note that $G_{n,n} = \mathbb{R}^n$, $Jf = J_{\mathbb{R}^n} f$ and the n-varifold $\mu \otimes \delta_{\mathbb{R}^n}$ can be identified naturally with the measure μ. We deduce that the push-forward of varifolds extends the notion of push-forward for Radon measures in order to take account more accurately of the dimensionality of the varifold measure.

In the following lemma we exhibit a compatibility property of the push-forward operation.

Lemma 1.4.3. *Let $V \in V_d(\mathbb{R}^n)$ and let f, g be two C^1 diffeomorphisms of $\mathbb{R}^n$, we have*

$$f_{\#}(g_{\#}V) = (f \circ g)_{\#}(V).$$

Proof. Take a test function $\varphi \in \mathrm{C}_c^0(\mathbb{R}^n \times G_{d,n}, \mathbb{R}^+)$, we have

$$\begin{aligned}
\int \varphi\, d(f_{\#}(g_{\#}V)) &= \int \varphi\left(f(y), Df(y)(T)\right)\ J_T f(y)\, d(g_{\#}V)(y,T) \\
&= \int \varphi\left((f(g(x)), Df(g(x))(Dg(x)(S))\right)\ J_{Dg(x)(S)} f(g(x))\ J_S g(x)\, dV(x,S) \\
&= \int \varphi\left(f \circ g(x), D(f \circ g)(x)(S)\right)\ J_S(f \circ g)(x)\, dV(x,S)
\end{aligned}$$

where we used that for $(x,S) \in \mathbb{R}^n \times G_{d,n}$,

$$Df(g(x))(Dg(x)(S)) = D(f \circ g)(x)(S) \quad \text{and} \quad J_{Dg(x)(S)} f(g(x)) J_S g(x) = J_S(f \circ g)(x)$$

thanks to the multiplicative property of the determinant. □

We present the formula that expresses the infinitesimal change of the mass measure of a varifold when pushing by the flow generated by a C^1 vector field. For $V \in V_d(\mathbb{R}^n)$ and $X \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, we set for t small:

$$f_t = \mathrm{id} + tX.$$

We have the following formula

$$\begin{aligned}
\partial_t \|(f_t)_\# V\|(\mathbb{R}^n)_{|t=0} &= \lim_{t\to 0^+} \frac{1}{t}\Big(\int_{\mathbb{R}^n\times G_{d,n}} J_S f_t(x)\, dV(x,S) - \int_{\mathbb{R}^n\times G_{d,n}} 1\, dV(x,S)\Big) \\
&= \int_{\mathbb{R}^n\times G_{d,n}} \lim_{t\to 0^+} \frac{1}{t}(J_S f_t(x) - 1)\, dV(x,S) \\
&= \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S X(x)\, dV(x,S).
\end{aligned}$$

where $\mathrm{div}_S X := \mathrm{tr}(S\circ DX)$ is the tangential divergence. This motivates the definition of the first variation of a varifold that measures the mass variations:

Definition 1.4.4 (First variation). *Let $V \in V_d(\mathbb{R}^n)$ of finite mass. The first variation of V is the map $\delta V :\ \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n) \to \mathbb{R}$ defined by*

$$\delta V :\ X \in C^1(\mathbb{R}^n,\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S(X)(x)\, dV(x,S). \tag{1.12}$$

Given $\varphi \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R})$ and $X \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, we denote:

$$\begin{aligned}
\delta(V,\varphi)(X) &:= \int_{\mathbb{R}^n\times G_{d,n}} \varphi(x)\ \mathrm{div}_S(X)(x)\, dV(x,S) + \int_{\mathbb{R}^n\times G_{d,n}} \nabla\varphi(x)\cdot X(x)\, dV(x,S) \\
&= \delta V(\varphi X) + \int_{\mathbb{R}^n\times G_{d,n}} X(x)\cdot S^\perp(\nabla\varphi(x))\, dV(x,S).
\end{aligned} \tag{1.13}$$

The map

$$\delta(V,\cdot)(\cdot) : (\mathrm{C}^1(\mathbb{R}^n,\mathbb{R}), \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n)) \to \mathbb{R},$$

is called the *weighted first variation* of the varifold V, it expresses the change of the mass of the varifold weighted by the function φ.

Remark 1.4.5. Note that $\delta(V,\cdot)(\cdot)$ is bilinear and

$$|\delta(V,\varphi)(X)| \le n\|X\|_{\mathrm{C}^1}\|V\|(\mathbb{R}^n)\ \|\varphi\|_{\mathrm{C}^1}\ , \quad \text{for all } \varphi \in C^1(\mathbb{R}^n,\mathbb{R}),\ X \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n).$$

Given a smooth d-manifold $\mathcal{M}$, we recall the notation $M := \mathcal{H}^d_{|\mathcal{M}} \otimes \delta_{T_x\mathcal{M}}$. For $X \in C^1(\mathbb{R}^n,\mathbb{R}^n)$,

$$\delta M(X) := \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S(X)(x)\, dM(x,S) = \int_{\mathcal{M}} \mathrm{div}_{T_x\mathcal{M}}(X)(x)\, d\mathcal{H}^d(x) = -\int_{\mathcal{M}} H(x,\mathcal{M})\cdot X(x)\, d\mathcal{H}^d(x), \tag{1.14}$$

where $H(\cdot,\mathcal{M})$ is the mean curvature vector of $\mathcal{M}$, defined by

$$H(\cdot,\mathcal{M}) = -\sum_{j=1}^{n-d}(\mathrm{div}_{\mathcal{M}}\,\nu_j)\nu_j, \tag{1.15}$$

where $\{\nu_j\}_j$ is any orthonormal basis of $T\mathcal{M}^\perp$.
When the first variation δV of a varifold V is bounded, it can be represented thanks to Riesz's representation theorem by a vector Radon measure $\|\delta V\|$, i.e.

$$\delta V(X) = \int_{\mathbb{R}^n} X \cdot \|\delta V\|.$$

Then, the Radon-Nikodym theorem implies the existence of a vector $H(\cdot, V) \in L^1(\mathbb{R}^n, \mathbb{R}^n, \|V\|)$ such that

$$\|\delta V\| = -H(\cdot, V)\|V\| + \|\delta V\|_s,$$

where $\|\delta V\|_s$ is a vector measure singular with respect to $\|V\|$. $H(\cdot, V)$ is called the (generalized) mean curvature of V in reference to (1.14).

Note that δV is not always bounded, for instance when V is a point cloud varifold, thus the generalized mean curvature cannot be defined in such a situation. It remains however possible to define an approximate mean curvature by convolution of the mass and the first variation.
Let $\varphi \in C^0(\mathbb{R}^n, \mathbb{R})$, we have

$$(\xi_\varepsilon * \|V\|)(\varphi) = V(\xi_\varepsilon * \varphi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \xi_\varepsilon(y-x)\,\varphi(y)\,dx d\|V\|(y) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \xi_\varepsilon(y-x)\,d\|V\|(y)\right)dx$$

hence we can represent the convolution of the mass by the function

$$(\xi_\varepsilon * \|V\|)(x) := \int_{\mathbb{R}^n} \xi_\varepsilon(y-x)\,d\|V\|(y). \tag{1.16}$$

Similarly, let $\varphi \in C^0(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$, we have

$$\begin{aligned}(\rho_\varepsilon * \delta V)(\varphi) = \delta V(\rho_\varepsilon * \varphi) &= \int_{\mathbb{R}^n \times G_{d,n}}\int_{\mathbb{R}^n} S(\nabla\rho_\varepsilon(y-x))\varphi(y)\;dxdV(y,S)\\ &= \int_{\mathbb{R}^n \times G_{d,n}}\left(\int_{\mathbb{R}^n} S(\nabla\rho_\varepsilon(y-x))\varphi(y)\,dV(y,S)\right)dx.\end{aligned}$$

We represent the convolution of the first variation by the vector

$$(\rho_\varepsilon * \delta V)(x) := \int_{\mathbb{R}^n} S(\nabla\rho_\varepsilon(y-x))\,dV(y,S)dx. \tag{1.17}$$

Taking the quotient of the convolved first variation and the convolved mass is a natural way of approximating the mean curvature [15]. For $V \in V_d(\mathbb{R}^n)$, the approximate mean curvature vector of V at a point $x \in \mathbb{R}^n$ is defined by

$$H_{\rho,\xi,\varepsilon}(x, V) = -\frac{C_\xi}{C_\rho}\frac{(\rho_\varepsilon * \delta V)(x)}{(\xi_\varepsilon * \|V\|)(x)} \tag{1.18}$$

whenever $\xi_\varepsilon * \|V\|(x) > 0$, and 0 otherwise, with C_ξ and C_ρ two normalization constants. [15, Theorem 4.3] states that if V is rectifiable and has bounded first variation, then

$$H_{\rho,\xi,\varepsilon}(x, V) \xrightarrow[\varepsilon \to 0]{} H(x, V)$$

for $\|V\|$-a.e $x \in \mathbb{R}^n$. This definition has other convergence and stability properties; also, it demonstrates very satisfactory numerical results for the computation of consistent mean curvature vectors of point clouds and the approximation of mean curvature flow, see [15, 17]. However, because of a slight lack of regularity, this definition is not well-suited to induce easily a consistent notion of mean curvature flow for which convergence results can be proved. In section 2.1 we recall the slightly different definition of approximate mean curvature used by Brakke [12] and Kim & Tonegawa [36] to construct weak mean curvature flows.

Chapter 2

Overview on mean curvature flow and Brakke flow

2.1 Mean curvature flow

2.1.1 Definition

The mean curvature flow (commonly denoted by MCF) is a natural geometric evolution where the evolving submanifold has a (vector) velocity equal to the mean curvature. From the variational characterization of the mean curvature vector (Equation (1.14)), we infer that this flow reduces the total area of a submanifold in the fastest way possible.

Definition 2.1.1. *(Mean curvature flow) Let $\mathcal{M}$ be a C^k manifold, $k \geq 2$. A family of C^k-embeddings of $\mathcal{M}$ into $\mathbb{R}^n$*

$$F : \mathcal{M} \times [0, T] \to \mathbb{R}^n$$

is said to be a mean curvature flow if

$$\frac{\partial F}{\partial t}(x, t) = H(F(x, t), F(\mathcal{M}, t))$$

where $H(F(x, t), F(\mathcal{M}, t))$ is the mean curvature vector of $\mathcal{M}_t = F(\mathcal{M}, t)$ at $x_t = F(x, t)$ as in (1.15).

The mean curvature flow equation can be written equivalently as :

$$\frac{\partial F}{\partial t}(x, t) = \Delta_{g_t} F(x, t) \tag{2.1}$$

where Δ_{g_t} is the Laplace-Beltrami operator. By the theory of pseudo-parabolic PDEs (see [21, Chapter 3]), Equation (2.1) has a smooth solution defined on a nontrivial time interval.

Examples of mean curvature evolutions:

1. Minimal surfaces, i.e. surfaces such that "$H = 0$": the mean curvature flow is constant equal to the initial submanifold.

2. Spheres: the motion by the mean curvature flow of a d-sphere in $\mathbb{R}^n$ of radius R is the family of spheres of radius $R(t)$, where

$$R(t) = \sqrt{R^2 - 2dt}, \quad t \in [0, R^2/2d].$$

3. Cylinders: let S_R^k be a k-sphere of radius R and define $C_R := S_R^k \times \mathbb{R}^m$, where $k + m < n$. The motion of the cylinder C_R is the family of cylinders $C_{R(t)} = S_{R(t)}^k \times \mathbb{R}^m$ where

$$R(t) = \sqrt{R^2 - 2kt}, \quad t \in [0, R^2/2k].$$

4. Translating solitons: they correspond to mean curvature flows of hypersurfaces satisfying

$$H(x) = \langle \vec{n}(x), v \rangle \, \vec{n}(x)$$

along the flow, where v is a constant vector in $\mathbb{R}^n$ and $\vec{n}$ is the normal (to the submanifold) at the point x, see [31, 26, 29] for more details and illustrations.

To see why spheres move in such a way, first we note that the mean curvature equation (2.1) is invariant under isometric transformations, hence, if starting from a sphere, the flow must be a flow of spheres at any time (as long as it exists). We now compute the radius of the evolving sphere, denoted by $R(t)$. We recall that, for a d-sphere in $\mathbb{R}^n$, of radius R, one has for any x

$$H(x) = -d\frac{x}{R^2}.$$

The mean curvature vector is pointing inward, hence the radius is decreasing (which is not surprising as the area decreases along the flow) and $\partial_t R(t) = -|H|$. Thus,

$$\partial_t R(t)^2 = 2\,\partial_t R(t) R(t) = -2|H|R(t) = -2d$$

which yields $R(t)^2 = R(0)^2 - 2dt$ and concludes the proof. We use the same reasoning (i.e. exploiting the symmetries of the equation) to deduce the law describing the evolution of cylinders.

In the following, we list out some of the important results regarding the mean curvature flow of submanifolds. We start with the case of curves in $\mathbb{R}^2$.

Theorem 2.1.2 (Grayson, [26, 27])**.** *The mean curvature flow shrinks embedded closed curves in $\mathbb{R}^2$ into single points in finite time. The extinction time is equal to $A/2\pi$ where A is the area of the region enclosed by the curve.*

For hypersurfaces, we have the following result:

Theorem 2.1.3 (Huisken, [30])**.** *A hypersurface is said to be mean convex if $H \cdot \vec{n} \geq 0$, $\vec{n}$ being the inward normal vector. The mean curvature flow shrinks compact mean convex hypersurfaces of $\mathbb{R}^n$ into round points in finite time, i.e when rescaling to have a constant volume inside the hypersurfaces, the flow converges to a $(n-1)$-sphere.*

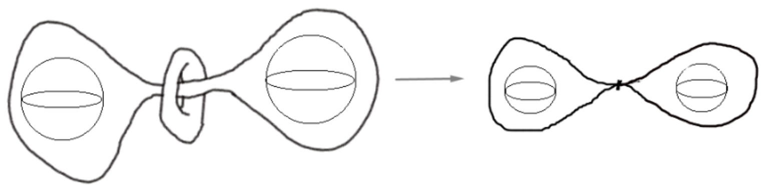

Figure 2.1: Appearance of singularities in a mean curvature flow.

2.1.2 Appearance of singularities

From the previous theorems, one could think that the mean curvature flow transforms every compact submanifold into a point in finite time. However, apart from closed curves in $\mathbb{R}^2$ and mean convex hypersurfaces, there is no guarantee that the flow shrinks to a point in finite (or even infinite) time. Often, the mean curvature flow develops complex singularities in finite time. Before exhibiting an example, we first need to recall two of the main results on mean curvature flow.

Theorem 2.1.4 (Comparison principle)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two compact hypersurfaces of $\mathbb{R}^n$. If $\mathcal{M}$ and $\mathcal{N}$ are disjoint then their mean curvature flows are disjoint.*

Proof. The proof is an application of the comparison principle valid for parabolic PDEs [40, Theorem 2.2.1]. □

Proposition 2.1.5 (Angenent torus, [7])**.** *Under the mean curvature flow, there exists a torus of $\mathbb{R}^2$ that shrinks homothetically to a point in finite time. Every torus with this property is called an Angenent torus.*

Following Angenent, we are now able to construct in $\mathbb{R}^3$ an example of a mean curvature flow that develops a singularity in finite time. We consider as in Figure 2.1:

1. a surface diffeomorphic to S^2 with a neck;
2. an Angenent torus circling the neck of the surface;
3. two spheres inside the surface, one on each side of the neck.

We choose the radius of the spheres big enough so that the flow of the torus disappears before the flow of the two spheres. The spheres shrinks to their centers, hence by the comparison principle (Theorem 2.1.4), the spheres prevent the surface from passing through the torus during the flow. Thus, the neck pinches at the extinction point of the torus as shown in Figure 2.1.

2.2 Weak mean curvature flows: Brakke's approach and Kim & Tonegawa's adaptation

To extend the definition of the mean curvature flow beyond singularities, several approaches have been proposed that yield weak notions of mean curvature flow:

- The Brakke flow [12, 36], defined in the setting of rectifiable varifolds and for arbitrary codimensions. It is obtained as the limit of approximating flows constructed by successive pushforwards applied on an initial varifold, with velocity equal to the approximate mean curvature (Definition 2.6).

- The level set formulation, based on an implicit representation of the evolving interface and the PDE satisfied by this representation [22, 23, 25, 24, 18, 6].
- Phase fields methods, based on diffuse representations of evolving interfaces and associated reaction-diffusion PDEs [3, 33, 28].
- The elliptic regularization of the mean curvature equation [35].
- A discretization of the mean curvature equation in the BV setting [38, 4, 20].
- De Giorgi's method of minimal barriers [19, 6, 11, 10].

Further references and details can be found in [9, 44].

Among the above mentioned approaches, only Brakke flows, level set flows, and flows based on De Giorgi's barriers can be considered in higher codimension. However, Brakke flows seem more easily extendable to handle the case of unstructured data such as point clouds.

Based on a precise study of the Brakke flow, and the adaptation due to Kim & Tonegawa, we have been able to propose the construction detailed in this manuscript of mean curvature flows or approximate mean curvature flows valid for general varifolds, including point clouds and surfaces with singularities, in arbitrary codimension. More precisely, Chapter 5 contains the following: for any varifold with compact support and arbitrary codimension

1. we provide a definition of a time-discrete approximate mean curvature flow with respect to a scale of approximation $\varepsilon > 0$ and a time subdivision $\mathcal{T}$;
2. we provide a definition of an approximate mean curvature flow with respect to ε as a limit of the flow constructed in 1. when the time step $\delta(\mathcal{T}) \to 0$.
3. we consider the measure defined in 2. coupled with the time measure dt and exhibit a limit as the scale of approximation ε tends to 0. Under a rectifiability assumption on the limit measure, we prove that it satisfies a spacetime Brakke inequality. This limit measure can be interpreted as a spacetime track of a generalized Brakke flow.

Kim & Tonegawa adapted Brakke's construction in the codimension 1 case when the initial datum is a boundary of an open partition of $\mathbb{R}^n$ (Definition 2.2.3). They managed to construct a Brakke flow that is not trivial, i.e. does not vanish instantly (see Section 2.2.6). We give below a brief overview of their construction, based on [46, 36], because it is important for our own construction. As we go along, we highlight the changes with respect to the original construction of Brakke.

2.2.1 Definition of Brakke's mean curvature flow

Let $(\mathcal{M}_t)_{t\in[0,T]}$ be a mean curvature flow (section 2.1) and $(M_t)_{t\in[0,T]}$ the associated family of varifolds, i.e. $M_t = \mathcal{H}^d_{|\mathcal{M}_t} \otimes \delta_{T.\mathcal{M}_t}$, $\forall t \in [0,1]$. The idea behind Brakke's solution to the mean curvature flow is to look for an integral version of the mean curvature equation (see [46, Section 2.1] for more details). In the measure theoretic setting, the quantities

$$\left\{ \int_{\mathcal{M}_t} \varphi(y,t)\, d\mathcal{H}^d(y),\ \varphi \in C_c(\mathbb{R}^n \times [0,T], \mathbb{R}^+) \right\}$$

fully characterize the family $(M_t)_{t\in[0,T]}$ of measures in $\mathbb{R}^n$. Taking the derivative when $\varphi \in C_c^1(\mathbb{R}^n \times [0,T], \mathbb{R}^+)$ gives:

$$\begin{aligned}
\partial_t \int_{\mathcal{M}_t} \varphi(y,t)\, d\mathcal{H}^d(y) &= \delta\left(M_t, \varphi\right)\left(H(\cdot, \mathcal{M}_t)\right) + \int_{\mathcal{M}_t} \partial_t \varphi(y,t)\, d\mathcal{H}^d(y) \\
&= \int_{\mathcal{M}_t} \Big(\varphi(y,t)\ \mathrm{div}_{T_y\mathcal{M}_t} H(y,\mathcal{M}_t) + \nabla\varphi(y,t)\cdot H(y,\mathcal{M}_t) + \partial_t\varphi(y,t)\Big)\, d\mathcal{H}^d(y) \\
&= \delta M_t(\varphi(\cdot,t) H(\cdot,\mathcal{M}_t)) + \int_{\mathcal{M}_t} \Big((T_y\mathcal{M}_t)^\perp\big(\nabla\varphi(y,t)\big)\cdot H(y,\mathcal{M}_t) + \partial_t\varphi(y,t)\Big)\, d\mathcal{H}^d(y),
\end{aligned}$$

where we used (1.13) with X a compactly supported smooth extension of $H(\cdot,\mathcal{M}_t)$ on $\mathbb{R}^n$. From (1.14) we obtain

$$\partial_t \int_{\mathcal{M}_t} \varphi(y,t)\, d\mathcal{H}^d(y) = \int_{\mathcal{M}_t} \Big(-\varphi(y,t)|H(y,\mathcal{M}_t)|^2 + (T_y\mathcal{M}_t)^\perp\big(\nabla\varphi(y,t)\big)\cdot H(y,\mathcal{M}_t) + \partial_t\varphi(y,t)\Big)\, d\mathcal{H}^d(y). \tag{2.2}$$

Integrating between t_1 and t_2 in the interval $[0,T]$ gives

$$\begin{aligned}
\|M_t\|\left(\varphi(\cdot,t)\right)\Big|_{t=t_1}^{t=t_2} &= \int_{t_1}^{t_2}\int_{\mathcal{M}_t} -\varphi(y,t)|H(y,\mathcal{M}_t)|^2 + (T_y\mathcal{M}_t)^\perp\big(\nabla\varphi(y,t)\big)\cdot H(y,\mathcal{M}_t)\, d\mathcal{H}^d(y)dt \\
&\quad + \int_{t_1}^{t_2}\int_{\mathcal{M}_t} \partial_t\varphi(y,t)\, d\mathcal{H}^d(y)dt.
\end{aligned} \tag{2.3}$$

One can prove (see [46, Chapter 2]) that any family of submanifolds $(\mathcal{M}_t)_{t\in[0,T]}$ whose spacetime track is C^2 and satisfies (2.3) is a classical mean curvature flow. This proves the consistency in the regular case. Equation (2.3) leads naturally to a weak notion of mean curvature flow for varifolds of bounded variation provided by the following definition from [46, Section 2].

Definition 2.2.1. *A family $(V_t)_{t\in[0,T]} \subset V_d(\mathbb{R}^n)$ is called a Brakke flow if,*

1. *For a.e. $t \in [0,T]$, V_t is integral.*

2. *For any compact K and $t < T$, $\sup_{s\in[0,t]} \|V_s\|(K) < \infty$.*

3. *For a.e. $t \in [0,T]$, V_t has locally bounded first variation and $\|\delta V_t\| \ll \|V_t\|$.*

4. *$H(\cdot, V_t) \in L^2_{loc}(\|V_t\| \times dt)$.*

5. *$(V_t)_{t\in[0,T]}$ satisfies the Brakke inequality, i.e. for $0 \le t_1 \le t_2 \le T$ and $\varphi \in C_c^1(\mathbb{R}^n \times [0,T], \mathbb{R}^+)$, we have*

$$\|V_t\|(\varphi(\cdot,t))\Big|_{t=t_1}^{t=t_2} \le \int_{t_1}^{t_2}\int_{\mathbb{R}^n} -\varphi(x,t)|H(x,V_t)|^2 + \nabla\varphi(x,t)\cdot H(x,V_t) + \partial_t\varphi(x,t)\, d\|V_t\|(x)dt. \tag{2.4}$$

We make the following comments about the definition:

- Brakke worked in [12] with rectifiable varifolds, and with the inequality:

$$D^+\|V_t\|(\varphi(\cdot,t)) \leq \int_{\mathbb{R}^n} -\varphi(y,t)|H(y,V_t)|^2 + (T_yV_t)^\perp\big(\nabla\varphi(y,t)\big)\cdot H(y,V_t)\,d\|V_t\|(y), \tag{2.5}$$

  where D^+ is the upper derivative.

- The absence of $(T_yM_t)^\perp$ in inequality (2.4) compared to (2.5) is due to the orthogonality of the mean curvature of integral varifolds [12, Theorem 5.8].

- Brakke as well as Kim & Tonegawa use an inequality to characterize the flow, instead of an equality as in (2.3). The reason behind is to allow sudden loss of mass and changes of topology (cf. Section 2.2.6 for illustrations). The main problem arising from this convention is the possibility that the construction gives a trivial Brakke flow, i.e. $V_t = 0$, $\forall t > 0$. With Brakke's construction, the triviality issue is not excluded even in the very regular case, in contrast with Kim & Tonegawa's approach.

- In order to introduce more flexibility, Brakke worked with an upper derivative instead of a strict derivative, whereas Kim & Tonegawa used an integrated Brakke inequality to avoid the differentiation of $\|V_t\|$. An interesting consequence of Kim & Tonegawa's choice is that the definition can be more easily extended to handle a velocity of the form $h+v$, where v is a time independent integrable vector field. We note that both definitions (Brakke's inequality with derivation or integration) are somehow equivalent, see [37] for details.

- Brakke also showed that in the case where the initial datum is integral, the constructed flow is integral as well, hence the inequality holds with no orthogonal projection in $(T_yV_t)^\perp$.

Brakke's flow is obtained as the continuous limit of an iterated discrete two-step scheme. The first step involves a de-singularization map (Section 2.2.3) and is essential to go beyond singularities. The second step uses a map with velocity equal to the approximate mean curvature (see Section 2.2.2). According to the choice of a de-singularization made during the construction, several possible flows can be obtained in the limit, see Figure 2.2.

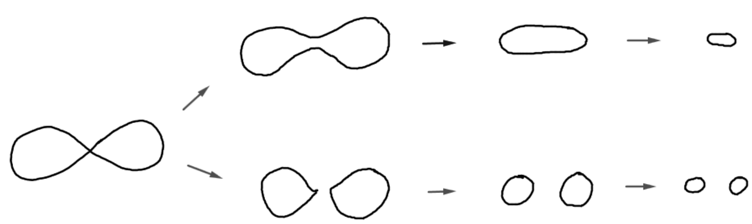

Figure 2.2: Non-uniqueness of the Brakke flow.

2.2.2 Approximate mean curvature

Define for each $\varepsilon \in (0,1)$:

$$\hat{\Phi}_\varepsilon(x) := \frac{1}{(2\pi\varepsilon^2)^{\frac{n}{2}}} \exp\Big(-\frac{|x|^2}{2\varepsilon^2}\Big).$$

and consider a truncation of $\hat{\Phi}_\varepsilon$ supported on the unit ball, denoted by Φ_ε. The approximate mean curvature vector at a point $x \in \mathbb{R}^n$ of a varifold V with finite mass is defined by Kim & Tonegawa as follows:

$$h_\varepsilon(x, V) = (\Phi_\varepsilon * \tilde{h}_\varepsilon(\cdot, V))(x)\,, \quad \text{where } \tilde{h}_\varepsilon(y, V) = -\frac{(\delta V * \Phi_\varepsilon)(y)}{(\|V\| * \Phi_\varepsilon)(y) + \varepsilon} \text{ for any } y \in \mathbb{R}^n. \tag{2.6}$$

The double convolution adds regularity, the ε in the denominator allows to define the approximate mean curvature on all $\mathbb{R}^n$, even when $(\|V\| * \Phi_\varepsilon)(y) = 0$.
We note that the original definition of Brakke has no ε in the denominator, it uses instead a non-truncated Gaussian-like kernel of the form $\exp\Big(-\dfrac{|x|^2}{\varepsilon^2 + \varepsilon^4|x|}\Big)$, up to a normalization constant $\beta(\varepsilon)$. The choice of a Gaussian kernel is made so that the derivatives of the kernel are bounded by the kernel up to ε to a negative power. Hence $||h_\varepsilon||_{C^2}$ is bounded by a negative power of ε.

The approximate mean curvature h_ε enjoys a C^1 boundedness property, stated as follows:

Proposition 2.2.2. *[46, Lemma 4.2] Let $M \geq 0$ and $V \in V_d(\mathbb{R}^n)$, assume that $\|V\|(\mathbb{R}^n) \leq M$. There exists a constant c depending only on n and M such that*

$$\sup_{x\in\mathbb{R}^n, \varepsilon\in(0,1)} \{\varepsilon^2|h_\varepsilon(x, V)|, \varepsilon^4|\nabla h_\varepsilon(x, V)|\} \leq c.$$

The previous property implies that, for a small Δt, the map $\mathrm{Id} + \Delta t h_\varepsilon$ is C^1.

2.2.3 De-singularizing maps and open partitions

To introduce the de-singularizing maps, we first introduce the notion of open partitions. The following definition corresponds [36, Definition 4.1] with an extra condition on the boundedness of the boundary to simplify the presentation. In other words, we impose that one and only one of the open sets of the collection is unbounded.

Definition 2.2.3. *Fix $N \geq 2$. A finite and ordered collection of sets $\mathcal{E} = \{E_i\}_{i=1}^N$ in $\mathbb{R}^n$ is called an open partition of N elements if:*

1. *$E_1, \ldots, E_N$ are open and mutually disjoint,*
2. *$\mathcal{H}^{n-1}\left(\mathbb{R}^n \setminus \bigcup_{i=1}^N E_i\right) < \infty$,*
3. *$\bigcup_{i=1}^N \partial E_i$ is countably $(n-1)$-rectifiable and bounded.*

The set of all open partitions of N elements is denoted by $\mathcal{OP}^N$.

The fact that $\mathcal{H}^{n-1}\left(\mathbb{R}^n \setminus \bigcup_{i=1}^N E_i\right) < \infty$ implies that $\mathbb{R}^n \setminus \bigcup_{i=1}^N E_i = \bigcup_{i=1}^N \partial E_i$.

We now introduce a set of maps that preserve the set of open partitions:

Definition 2.2.4. *(Admissible functions [46, Definition 4.4]) Given $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N$, a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathcal{E}$-admissible if it is Lipschitz and satisfies the following. Define $\tilde{E}_i :=$ int$(f(E_i))$ for each i. Then:*

(a) $\{\tilde{E}_i\}_{i=1}^N$ *are mutually disjoint,*

(b) $\mathbb{R}^n \setminus \bigcup_{i=1}^N \tilde{E}_i \subset f(\bigcup_{i=1}^N \partial E_i)$.

We denote $f_\star\mathcal{E} := \{\tilde{E}_i\}_{i=1}^N$ and it is easy to see that $f_\star\mathcal{E} \in \mathcal{OP}^N$. Also, we denote $f_\star(\partial\mathcal{E}) := \partial f_\star\mathcal{E}$.

Now we introduce the definition of the de-singularizing maps, also called "area-reducing admissible functions". Before that, we first introduce a set of test functions and vector fields that allow to localize the operation $f_\star$.

Definition 2.2.5. *(Restricted test functions and vector fields, [46, Definition 4.7])*
For $j \in \mathbb{N}$, define

$$\mathcal{A}_j := \{\varphi \in C^2(\mathbb{R}^n, \mathbb{R}^+) : \varphi(x) \leq 1,\ |\nabla\varphi(x)| \leq j\varphi(x),\ \|\nabla^2\varphi(x)\| \leq j\varphi(x) \text{ for all } x \in \mathbb{R}^n\},$$

$$\mathcal{B}_j := \{g \in C^2(\mathbb{R}^n; \mathbb{R}^n) : |g(x)| \leq j, \|\nabla g(x)\| \leq j, \|\nabla^2 g(x)\| \leq j \text{ for all } x \in \mathbb{R}^n \text{ and } \|g\|_{L^2(\mathbb{R}^n)} \leq j\}.$$

Definition 2.2.6. *(Area-reducing admissible functions [46, Definition 4.8]) For $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N$ and $j \in \mathbb{N}$, define $E(\mathcal{E}, j)$ to be the set of all $\mathcal{E}$-admissible functions $f : \mathbb{R}^n \to \mathbb{R}^n$ such that:*

1. $|f(x) - x| \leq j^{-2}, \ \forall x \in \mathbb{R}^n$;
2. $\mathcal{L}^n(\tilde{E}_i \Delta E_i) \leq j^{-1}$ *for all* $i \in \{1, \ldots, N\}$, *where* $\{\tilde{E}_i\}_{i=1}^N = f_\star\mathcal{E}$;
3. $\|\partial f_\star\mathcal{E}\|(\varphi) \leq \|\partial\mathcal{E}\|(\varphi)$ *for all* $\varphi \in \mathcal{A}_j$.

We now define a measure of how much the operation $f_\star$ can change the total mass of an open partition.

Definition 2.2.7. *For $\mathcal{E} \in \mathcal{OP}^N$ and $j \in \mathbb{N}$, we define*

$$\Delta_j\|\partial\mathcal{E}\| := \Delta_j\|\partial\mathcal{E}\|(\mathbb{R}^n) := \inf_{f \in E(\mathcal{E}, j)} \left(\|\partial f_\star\mathcal{E}\|(\mathbb{R}^n) - \|\partial\mathcal{E}\|(\mathbb{R}^n)\right).$$

As $f = \mathrm{id} \in E(\mathcal{E}, j)$, $\forall j \in \mathbb{N}$, we have $\Delta_j \leq 0$, $\forall j \in \mathbb{N}$.

We make the following comments about the definitions:

- Condition 2 in Definition 2.2.6 has no equivalent in Brakke's work, since Brakke uses rectifiable varifolds of arbitrary codimensions. In Brakke's work, the operation $f_\star$ is replaced by a standard varifold push-forward. Brakke distinguishes two models: the normal (i.e. standard) model and the reduced mass model. In the first model, the image varifold of a rectifiable varifold V is $f_\# V$. In the second model, the image varifold is the varifold associated to $\operatorname{spt} f_\# V$. The major difference between the two is that the second model does not change varifolds with zero mean curvature, see [12, Appendix C4] for details.

- (About condition 3 of Definition 2.2.6). As $\varphi \equiv 1 \in \mathcal{A}_j$ for all j, we have $\|\partial f_\star\mathcal{E}\|(\mathbb{R}^n) \leq \|\partial\mathcal{E}\|(\mathbb{R}^n)$, hence the decrease of the mass. Moreover, f cannot move smooth parts for j large enough and operates only around singularities (see [46, pp. 57-58] and [12, Theorem 4.15]). This explains why we qualify such functions as "de-singularizing". In figure 2.3 we present the effect of the de-singularizing maps on some famous configurations (junctions).

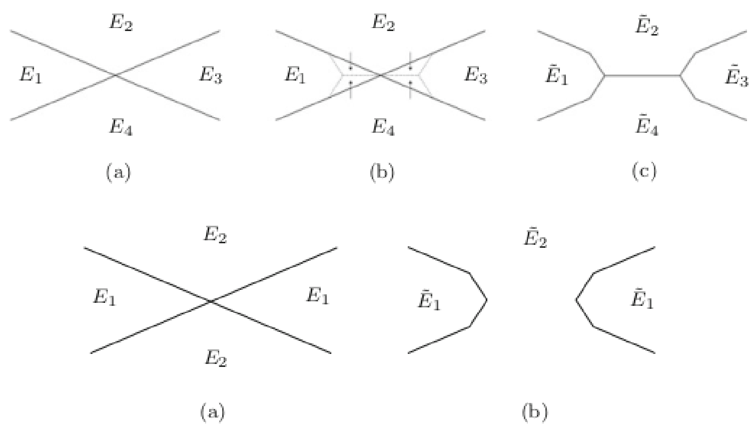

Figure 2.3: Two examples of de-singularization operations [46, p. 56].

2.2.4 Construction of the approximate flow

With the definitions that precede, we can construct an approximate flow starting from any open partition of finite mass. Given $j \in \mathbb{N}$, we choose $\varepsilon_j \approx j^{-6}$ and $\Delta t_j \approx \varepsilon_j^{3n+20}$ (the choices are justified in [36]); we choose for simplicity Δt_j of the form $2^{-p_j}, p_j \in \mathbb{N}, \forall j \in \mathbb{N}$. Let $\mathcal{E} \in \mathcal{OP}^N$ be such that $\mathcal{H}^{n-1}(\partial \mathcal{E}) := \mathcal{H}^{n-1}\left(\bigcup_{i=1}^{N} \partial E_i\right) < \infty$. We define an approximate flow $(\partial \mathcal{E}_{j,k})_k, k \in \{0, \ldots, j\Delta t_j^{-1}\}$ inductively as follows.

1. For $k = 0$, set $\mathcal{E}_{j,0} := \mathcal{E}$ and $\partial \mathcal{E}_{j,0} := \partial \mathcal{E}$.

2. Assuming inductively that $\mathcal{E}_{j,k-1}$ is defined until some $k \in \{0, \ldots, j\Delta t_j^{-1}\}$, choose

$$f_{j,k} \in E(\mathcal{E}_{j,k-1}, j)$$

such that

$$\|\partial (f_{j,k})_\star \mathcal{E}_{j,k-1}\|(\mathbb{R}^n) - \|\partial \mathcal{E}_{j,k-1}\|(\mathbb{R}^n) \leq (1 - j^{-5})\Delta_j \|\partial \mathcal{E}_{j,k-1}\|, \tag{2.7}$$

and define

$$\mathcal{E}^*_{j,k} := (f_{j,k})_\star \mathcal{E}_{j,k-1}.$$

3. Define $\hat{f}_{j,k}(x) := x + \Delta t_j h_{\varepsilon_j}(\partial \mathcal{E}^*_{j,k}, x)$, for j large enough the map $\hat{f}_{j,k}$ is a diffeomorphism, then define

$$\mathcal{E}_{j,k} := (\hat{f}_{j,k})_\# \mathcal{E}^*_{j,k}.$$

Studying the change of the mass of the flow $(\partial \mathcal{E}_{j,k})_k$ under the two operations above gives the following result, which combines [46, Lemmas 4.15 and 4.16].

Proposition 2.2.8. *Given $\mathcal{E} \in \mathcal{OP}^N$ such that $\mathcal{H}^{n-1}(\partial \mathcal{E}) \leq M < \infty$, $j \in \mathbb{N}$. For $\varepsilon_j \approx j^{-6}$ and $\Delta t_j \approx \varepsilon_j^{3n+20}$, the flow $(\partial \mathcal{E}_{j,k})_k$ satisfies:*

$$\frac{\|\partial \mathcal{E}_{j,k}\|(\mathbb{R}^n) - \|\partial \mathcal{E}_{j,k-1}\|(\mathbb{R}^n)}{\Delta t_j} + \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta(\partial \mathcal{E}_{j,k})|^2}{\Phi_{\varepsilon_j} * \|\partial \mathcal{E}_{j,k}\| + \varepsilon_j}\, dx - \frac{(1 - j^{-5})}{\Delta t_j} \Delta_j \|\mathcal{E}_{j,k-1}\| \leq c(n, M)\varepsilon_j \tag{2.8}$$

$$\frac{\|\partial\mathcal{E}_{j,k}\|(\varphi) - \|\partial\mathcal{E}_{j,k-1}\|(\varphi)}{\Delta t_j} \leq \delta\left(\partial\mathcal{E}_{j,k},\varphi\right)\left(h_{\varepsilon_j}(\cdot,\partial\mathcal{E}_{j,k})\right) + \varepsilon_j^{\frac{1}{8}}, \tag{2.9}$$

for $k \in \{0,\dots,j\Delta t_j^{-1}\}$ *and* $\varphi \in \mathcal{A}_j$.

Inequality (2.9) is the discrete analogous of

$$\|M_{t_1}\|(\varphi) - \|M_{t_2}\|(\varphi) \leq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi|H(\cdot,M_t)|^2 + T.M_t^{\perp}\nabla\varphi\cdot H(\cdot,M_t)\,d\mathcal{H}^d dt. \tag{2.10}$$

satisfied by any d-mean curvature flow $(M_t)_{t\geq 0}$ and for any $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^+)$.

We make the following comments about the construction:

- The factor $(1-j^{-5})$ is somewhat arbitrary; as the minimizer of Δ_j may not exist, one chooses a map that is almost a minimizer.
- The construction works fine for $\mathcal{E}$ with $\mathcal{H}^{n-1}(\partial\mathcal{E}\cap B_r) \leq e^{c\,r}$, $\forall r \geq 0$ for some $c \geq 0$, up to some slight changes. We adopted the finite mass case (as in [46]) for its simplicity.
- The construction of the approximate flow, ignoring the de-singularizing step, makes sense for any type of varifolds. One can construct an approximate flow starting from any arbitrary varifold and get the estimates (2.8) (without the term in Δ_j) and (2.9), this is precisely the starting point of the work presented in chapter 5.

2.2.5 Convergence to a Brakke flow

It is convenient to define the approximate flow for all $t \in [0,j], j \in \mathbb{N}$, instead of discrete times. Define

$$\mathcal{E}_j(t) := \mathcal{E}_{j,k} \quad \text{if} \quad t \in ((k-1)\Delta t_j, k\Delta t_j] \tag{2.11}$$

for $k \in \{0,\dots,j\Delta t_j^{-1}\}$, and consider the boundary $\partial\mathcal{E}_j(t)$ of $\mathcal{E}_j(t)$ for all $t \in [0,j]$. We now discuss the convergence of the flow represented by $\partial\mathcal{E}_j(t)$, and the proof that the limit flow is a Brakke flow (Definition 2.2.1). For simplicity, we restrict the study on a time interval $[0,T]$, with T dyadic, and we give a sketchy proof in several steps.

Step 1: Convergence of the mass measure $\partial\mathcal{E}_j(t)_{t\in[0,T]}$ and the proof of property (2) in Definition 2.2.1.
Note that the second and third terms on the left hand side (LHS) of (2.8) are positive, and removing them from the inequality gives:

$$\|\partial\mathcal{E}_j(t)\|(\mathbb{R}^n) \leq \|\partial\mathcal{E}\|(\mathbb{R}^n) + c(n,M)\Delta t_j\varepsilon_j. \tag{2.12}$$

Therefore, $\|\partial\mathcal{E}_j(t)\|(\mathbb{R}^n)$ is bounded for any $j \in \mathbb{N}$. By Banach-Alaoglu compactness theorem, one deduces the convergence of $\|\partial\mathcal{E}_j(t)\|(\mathbb{R}^n)$, up to an extraction depending on $t \in [0,T]$, to a Radon measure $\mu(t)$ on $\mathbb{R}^n$. The next goal is to find a subsequence (j') allowing the convergence independently of t. Let D be the set of dyadic numbers in $[0,T]$. By a diagonal extraction argument, one can choose a sequence (j') for which $\partial\mathcal{E}_j'(t)$ converges for every $t \in D$ to a limit measure, denoted by $\mu(t)$, $t \in D$. The following continuity result allows to extend the definition of $\mu(t)$ to all $t \in [0,T]$ and to prove that

$$\|\partial\mathcal{E}_{j'}(t)\| \rightharpoonup \mu(t) \quad \text{for all } t \in [0,T].$$

Lemma 2.2.9. *[46, Lemma 4.18] Let $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$. Define*

$$g(t) := \mu(t)(\varphi) - t\|\nabla^2\varphi\|_\infty \|\partial\mathcal{E}\|(\mathbb{R}^n).$$

Then g is a monotone decreasing function on D.

Taking the limit in (2.12) we prove that $\mu(t)(\mathbb{R}^n) \leq \|\partial\mathcal{E}\|(\mathbb{R}^n) < \infty$ for all $t \in [0,T]$. We note that the sequence $\partial\mathcal{E}_{j'}(t)$ does not converge necessarily. In fact, we are not sure of the existence of a sequence (j') not depending on t allowing the convergence of $\partial\mathcal{E}_{j'}(t)$ for a.e. $t \in [0,1]$. The reason behind is that we do not dispose of a continuity property for $t \mapsto \lim_j \partial\mathcal{E}_{j'}(t)$ similar to Lemma 2.2.9.
If $V(t)$ denotes the limit of $\partial\mathcal{E}_j(t)$ under any extraction, we already know that

$$\|V(t)\|(\mathbb{R}^n) = \mu(t)(\mathbb{R}^n) < \infty \quad \forall t \in [0,T]$$

which proves property (2) of Definition 2.2.1. The interested reader might refer to Proposition 5.3.10 for a similar and detailed proof of Step 1 (slightly adapted to our context).

Step 2: Proof of properties (3) and (4) of Definition 2.2.1.
We start with a property that links up $h_\varepsilon(\cdot, V)$ to δV.

Lemma 2.2.10. *[36, Proposition 5.5] There exists $\varepsilon_0 \in (0,1)$ depending on n and M with the following property. Consider $V \in V_{n-1}(\mathbb{R}^n)$ with $\|V\|(\mathbb{R}^n) \leq M$, $\varepsilon \in (0, \varepsilon_0)$, $g \in \mathcal{B}_j$ (Definition 2.2.5) and $j \in \mathbb{N}$ satisfying $\varepsilon \leq (2j)^{-6}$. Then we have*

$$\Big| \int_{\mathbb{R}^n} h_\varepsilon \cdot g \, d\|V\| + \delta V(g) \Big| \leq \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}} \left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon} \, dx \right)^{\frac{1}{2}}.$$

We note that by integrating 2.8 between 0 and T, and removing the positive term $-\Delta_j$, we have:

$$\int_0^T \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_j(t))|^2}{\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_j(t)\| + \varepsilon_j} \, dxdt \leq \|\partial\mathcal{E}\|(\mathbb{R}^n) + Tc(n,M)\varepsilon_j.$$

Fatou's lemma implies that

$$\liminf_j \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_j(t))|^2}{\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_j(t)\| + \varepsilon_j} \, dx < \infty \quad \text{for a.e } t \in [0,T]. \tag{2.13}$$

The following formal estimate is stated and proved rigorously in [36, Section 5]:

$$\int |h_\varepsilon(\cdot, V)|^2 d\|V\| \approx \int \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_j(t))|^2}{\left(\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_j(t)\| + \varepsilon_j\right)^2} \, d(\Phi_\varepsilon * \|V\|) \approx \int \frac{|\Phi_{\varepsilon_j} * \delta(\partial\mathcal{E}_j(t))|^2}{\Phi_{\varepsilon_j} * \|\partial\mathcal{E}_j(t)\| + \varepsilon_j} \, dx. \tag{2.14}$$

For a generic $t \in [0,T]$, let $(j_t)_{j \in \mathbb{N}}$ be a sequence such that $\partial\mathcal{E}_{j_t}(t)$ converges, denote its limit by $V(t)$, and assume in addition that

$$\limsup_j \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_{j_t}} * \delta(\partial\mathcal{E}_{j_t}(t))|^2}{\Phi_{\varepsilon_{j_t}} * \|\partial\mathcal{E}_{j_t}(t)\| + \varepsilon_{j_t}} \, dx < \infty. \tag{2.15}$$

From Lemma 2.2.10 with $V = \partial\mathcal{E}_{j_t}(t)$ and estimates (2.15) and (2.14), one can prove for a.e. $t \in [0,1]$ that $V(t)$ has bounded first variation,

$$\delta V(t)(X) = -\int_{\mathbb{R}^n} X \cdot H(\cdot, V(t))\, d\|V(t)\| \quad \forall X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n),$$

and that

$$\int_{\mathbb{R}^n} |H(\cdot, V(t))|^2\, d\|V(t)\| \leq \limsup_j \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_{j_t}} * \delta(\partial\mathcal{E}_{j_t}(t))|^2}{\Phi_{\varepsilon_{j_t}} * \|\partial\mathcal{E}_{j_t}(t)\| + \varepsilon_{j_t}}\, dx < \infty.$$

This concludes step 2.

Step 3: proof of the integrality of the limit (property (1) Definition 2.2.1). Let $(V(t))_{t\in[0,T]}$ be as defined in step 2. The proof of Kim & Tonegawa is a direct adaptation of the original proof of Brakke to the case of open partitions.
Integrating (2.8) in time, after removing the second term of the LHS, gives

$$\int_0^T \frac{(1-j^{-5})}{\Delta t_j}\Big|\Delta_j\|\mathcal{E}_j(t)\|\Big| \leq Tc(n,M)\varepsilon_j + \|\partial\mathcal{E}\|(\mathbb{R}^n).$$

Thus, for any sequence $(\alpha_j)_{j\in\mathbb{N}}$ such that $\lim_j \Delta t_j \alpha_j^{-1} = 0$, one has

$$\lim_j \int_0^T \Big|\Delta_j\|\mathcal{E}_j(t)\|\Big|\alpha_j^{-1} dt = 0.$$

thus possibly choosing a further subsequence

$$\lim_j \Delta_j\|\mathcal{E}_j(t)\|\alpha_j^{-1} = 0,\ \forall t \in [0,T]. \tag{2.16}$$

Fix a generic $t \in [0,T]$, we know from step 2 that $V(t)$ has bounded first variation. Then if we can prove that the density of $V(t)$ is bounded from below, Allard's rectifiability theorem [2, Theorem 5.5] implies that $V(t)$ is rectifiable. To do so, we assume by contradiction that the density of $V(t)$ is not bounded from below; hence, for any $c > 0$ there exists a set A_c of positive measure such that $\|V(t)\|(B(x,r)) \leq cr^{n-1}$, $\forall r \geq 0$, $\forall x \in A_c$. We recall that $V(t)$ is the limit of $\partial\mathcal{E}_j(t)$; then for any constant $c > 0$, for j large enough, there exists a set $A_{j,c}$ of positive measure (almost equal to the measure of A_c) such that

$$\|\partial\mathcal{E}_j(t)\|(B(x,r)) \leq cr^{n-1},\ \forall r \geq 0,\ \forall x \in A_{j,c}.$$

On the other hand, one can prove that there exists a constant $\tilde{c} > 0$ depending on n such that, if $\|\partial\mathcal{E}_j(t)\|(B(x,r)) \leq \tilde{c}r^{n-1}$, then there is a de-singularizing map which reduces the mass by $1/2$ in $B(x,r)$. This means that there is a de-singularizing map that reduces the measure of $\partial\mathcal{E}_j(t)$ by $\frac{1}{2}$ times the measure of $A_{j,c}$ and this contradicts (2.16) for $\alpha_j = 1$ (which says that $\partial\mathcal{E}_j(t)$ is almost a minimizer). The proof of the integrality is delicate and uses several arguments from geometric measure theory, we refer the reader to [46, Section 4.8] for a brief explanation of the proof and to [36, Section 8] for a complete proof.

Step 4: proof of the Brakke inequality (property (5) of Definition 2.2.1).

Fix a test function $\varphi \in C_c^1(\mathbb{R}^n \times [0,T], \mathbb{R}^+)$ and assume without loss of generality that $\|\varphi\|_\infty < 1$. Define $\varphi_i = \varphi + i^{-1}$, $i \in \mathbb{N}$; for i large enough we have $\|\varphi_i\|_\infty < 1$. Moreover, one can prove that there exists j_0 such that $\varphi_i(\cdot,t) \in \mathcal{A}_j$, $\forall j \geq j_0$ and $\forall t \in [0,T]$. To prove the Brakke inequality, we plug φ_i into the discrete Brakke inequality and take the limit first in j then in i.
Plugging $\varphi_i(\cdot,t)$ into (2.9), using (2.11) we obtain for any $t \in D$:

$$\|\partial\mathcal{E}_j(t)\|(\varphi_i(\cdot,t)) - \|\partial\mathcal{E}_j(t-\Delta t_j)\|(\varphi_i(\cdot,t)) \leq \Delta t_j \delta\left(\partial\mathcal{E}_j(t), \varphi_i(\cdot,t)\right)\left(h_{\varepsilon_j}(\cdot, \partial\mathcal{E}_j(t))\right) + \Delta t_j \varepsilon_j^{\frac{1}{8}}.$$

Therefore,

$$\begin{aligned} \|\partial\mathcal{E}_j(s)\|(\varphi_i(\cdot,s))\Big|_{s=t-\Delta t_j}^{s=t} &- \|\partial\mathcal{E}_j(t-\Delta t_j)\|\left(\varphi_i(\cdot,t) - \varphi_i(\cdot,t-\Delta t_j)\right) \\ &\leq \Delta t_j \delta\left(\partial\mathcal{E}_j(t), \varphi_i(\cdot,t)\right)\left(h_{\varepsilon_j}(\cdot, \partial\mathcal{E}_j(t))\right) + \Delta t_j \varepsilon_j^{\frac{1}{8}}. \end{aligned} \tag{2.17}$$

Formally we have

$$\|\partial\mathcal{E}_j(t-\Delta t_j)\|\left(\varphi_i(\cdot,t) - \varphi_i(\cdot,t-\Delta t_j)\right) \approx \Delta t_j \|\partial\mathcal{E}_j(t-\Delta t_j)\|\left(\partial_t\varphi_i(\cdot,t-\Delta t_j)\right). \tag{2.18}$$

Let $t_1, t_2 \in D$ be such that $t_1 < t_2$ and take j large enough such that $t_2 - t_1$ is a multiple of Δt_j. Summing (2.17) from $t_1 + \Delta t_j$ to t_2 and using (2.18) and (2.11) we obtain

$$\begin{aligned} \|\partial\mathcal{E}_j(s)\|(\varphi_i(\cdot,s))\Big|_{s=t_1}^{s=t_2} &- \int_{t_1-\Delta t_j}^{t_2-\Delta t_j} \|\partial\mathcal{E}_j(t)\|\left(\partial_t\varphi_i(\cdot,t)\right)\, dt \\ &\leq \int_{t_1}^{t_2} \delta\left(\partial\mathcal{E}_j(t), \varphi_i(\cdot,t)\right)\left(h_{\varepsilon_j}(\cdot, \partial\mathcal{E}_j(\cdot,t))\right) dt + (t_2-t_1)\varepsilon_j^{\frac{1}{8}}. \end{aligned}$$

By the uniform boundedness of the mass and the varifold convergence, we can prove that the limit of the LHS when $i, j \to \infty$ is

$$\|V(t_2)\|(\varphi(\cdot,t_2)) - \|V(t_1)\|(\varphi(\cdot,t_1)) - \int_{t_1}^{t_2} \|V(t)\|\left(\partial_t\varphi(\cdot,t)\right)\, dt.$$

The second term of the right hand side (RHS) tends to 0. The first term of the RHS converges to

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} -\varphi(\cdot,t)|H(\cdot,V(t))|^2 + \nabla\varphi(\cdot,t)\cdot H(\cdot,V(t))\, d\|V(t)\|\, dt.$$

The proof relies on step 2 and estimates from [36, Section 5], we refer the reader to [36, Section 9] for the details. We deduce by density the Brakke inequality for arbitrary t_1 and t_2, and this finishes the proof of step 4.
In our context, the proof of step 4 will be slightly adapted to prove a spacetime Brakke inequality for the limit flow constructed in this thesis (see the proof of Proposition 5.3.13 for details).

2.2.6 Triviality/non-triviality of the limit flow

A major issue about the Brakke construction is the triviality issue. In fact, even in the simple case of a sphere, we have no guarantees that the constructed Brakke flow is not trivial.

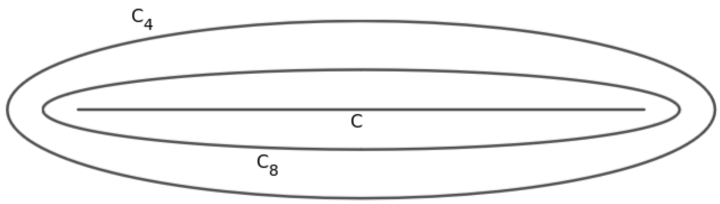

Figure 2.4: Triviality of the flow of the line segment.

One of the well-known cases where the instant vanishing of Brakke flows occurs is the case of embedded non-closed compact curves of $\mathbb{R}^2$. We consider the example of a line segment for simplicity: let C be the natural varifold associated to the line segment $[-1, 1]$. We will prove that any Brakke flow starting from C is trivial. To do so, consider the sequence of varifolds $C_j, j \in \mathbb{N}$ associated to the ellipsoids (see figure 2.4 for illustration)

$$(1 + j^{-1}) \left(\cos(t), j^{-1}\sin(t)\right), \, t \in [0, 1].$$

By Theorem 2.1.2 we know that the extinction time of C_j, under the mean curvature flow, tends to 0 as j tends to ∞. On the other hand, by the avoidance principle of Brakke flows to mean curvature flows ([35, Theorem 10.5]) we deduce that the Brakke flow of C vanishes immediately.

Kim & Tonegawa showed, mainly using Huisken's monotonicity formula ([46, Theorem 3.4]), that their constructed Brakke flow is nontrivial. The proof consists of two steps:

1. The open sets evolve continuously in time.

2. The constructed Brakke flow is equal to the boundary of the evolving open sets.

The nontriviality result is the core of the work of [36]. In the following theorem, the notion of "open partition" refers to [36, Definition 4.1], i.e. no boundedness condition on the open sets is imposed.

Theorem 2.2.11. *[36, Theorem 1.1] Let $\mathcal{E} = \bigcup_{i=1}^{N} E_i$ be an open partition such that the mass of $\partial\mathcal{E}$ is finite or grows at most exponentially near infinity. There exists a family of open partitions $\mathcal{E}(t) = \bigcup_{i=1}^{N} E_i(t)$ such that:*

1. *$\mathcal{E}(0) = \mathcal{E}$,*

2. *$\partial\mathcal{E}(t) = \bigcup_{i=1}^{N} \partial E_i(t)$ is a Brakke flow in the sense of Definition 2.2.1,*

3. *For any $i \in \{1, \ldots, N\}$, $t \mapsto E_i(t)$ evolves continuously with respect to the Lebesgue measure.*

In chapter 6, we prove in codimension one a non-triviality property for the flow that we introduce and study in this manuscript. The proof involves an approximate comparison principle with respect to evolving spheres. Roughly speaking, if $(V_\varepsilon(t))_{t\in[0,T]}$ is the constructed approximate mean curvature flow, and $(B(t))_{t\in[0,T]}$ is the flow of balls associated to mean curvature flow of spheres, we have

$$\|V_\varepsilon(t_2)\|(B(t_2)) \leq \|V_\varepsilon(t_1)\|(B(t_1)) + \varepsilon, \quad \text{where } 0 \leq t_1 \leq t_2 \leq T.$$

Combining the approximate avoidance principle with the relative isoperimetric inequality allows us to prove the nontriviality result.

Chapter 3

Contributions and perspectives

This PhD work adresses the possibility to extend to general data, in particular discrete data such as point clouds, the notion of mean curvature flow.

Our first contribution is a result of consistency for the flow obtained from a (space) volumetric discretization of smooth submanifolds flowing by mean curvature.

Our second and main contribution is the construction and the analysis of two new Brakke-type flows:

1. A time-discrete approximate mean curvature flow depending on a scale of approximation ε and a subdivision of the time interval;

2. A time-continuous approximate mean curvature flow depending only on ε, constructed as the limit of the time-discrete flow when the time step tends to 0.

These flows have the interesting and new property (in contrast with the original Brakke's or Kim & Tonewaga's flows) that their construction is valid for fairly general data with compact support: point clouds, singular surfaces, volumetric varifolds, etc.

We also study the limit as $\varepsilon \to 0$ of the time-continuous flow and some properties of the resulting flow, which under certain conditions is a spacetime Brakke flow.

In what follows, we give a slightly more detailed summary of our results.

3.1 Approximate Brakke equality for a volumetric discretization of mean curvature flow

We prove in Chapter 4 a consistency result for the mean curvature flow obtained from a space discretization of a smooth flow. More precisely, we discretize using volumetric varifolds defined in (1.5) the mean curvature flow of a C^3 submanifold of $\mathbb{R}^n$, and we prove the validity of an approximate Brakke equality for the discretized objects considered with their approximate mean curvature (see Definition 1.18) that makes more sense in a discrete setting.

Theorem (Thm 4.0.1, Chap 4)**.** *Let $\mathcal{M}$ be a C^3 closed d-submanifold in $\mathbb{R}^n$, let $\mathcal{M}(t)_{t\in[0,T]}$ be its mean curvature flow (Definition 2.1.1), and $M(t)_{t\in[0,T]}$ the varifolds associated with the family $\mathcal{M}(t)_{t\in[0,T]}$. Let*

$\mathcal{K}_h$ be a mesh of $\mathbb{R}^n$ of size $h \in (0,1)$ and $V_h(t)$ a discretization of $\mathcal{M}(t)$ for every $t \in [0,T]$ (Definition 1.5).
Let $\varepsilon, \gamma \in (0,1)$ be such that $h \leq \frac{\gamma}{2}\varepsilon$, and

- $\gamma \leq (8(1+C_0^{2/d}))^{-1}$, *where C_0 bounds the Ahlfors regularity constant of $\mathcal{M}(t), \forall t \in [0,T]$;*
- $\gamma \leq \max\limits_{x \in \mathcal{M}(t),\, t \in [0,T]} (\lambda(x,t))^{-1}$, *where $\lambda(x,t)$ is the maximal principal curvature of $\mathcal{M}(t)$ at x;*
- $\beta > \gamma 2^{3d} C_0^2(\mathrm{Lip}(\xi)+1)$, *where $\beta = \min\left\{\xi(s) \middle| s \in [\frac{C_0^{-2/d}}{4}, \frac{1}{2}]\right\} > 0$.*

Then, the following estimate holds:

$$\left| ||V_h(t_2)||(\varphi) - ||V_h(t_1)||(\varphi) + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi(x)|H_\varepsilon(x, V_h(t))|^2 - \nabla\varphi(x) \cdot H_\varepsilon(x, V_h(t)) d||V_h(t)||(x) dt \right|$$
$$\leq C\|\varphi\|_{C^2} \left(\max_{t \in \{t_1,t_2\}} \Delta(M(t), V_h(t)) + \varepsilon(||M(t_1)||(\mathbb{R}^n) - ||M(t_2)||(\mathbb{R}^n)) + (t_2 - t_1)\|M(0)\|(\mathbb{R}^n)(\varepsilon + \frac{h}{\varepsilon^3}) \right), \tag{3.1}$$

for every $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, $0 \leq t_1 \leq t_2 \leq T$, where C depends on n, d, γ, β, $||\rho||_{C^2}$, $||\xi||_{C^1}$, C_0 and other constants depending on the C^3-norm of $\mathcal{M}$.

3.2 Approximate mean curvature flows of a general varifold, and their limit spacetime Brakke flow

This section summarizes the main contributions of the thesis on approximate mean curvature flows that are presented in chapter 5:

1. Starting from any varifold of any codimension and finite mass, we give a definition of a time-discrete approximate mean curvature flow with respect to a scale of approximation and a time subdivision.

2. Starting from any varifold of compact support and of arbitrary codimension, we give a definition of an approximate mean curvature flow with respect to a scale of approximation obtained as a limit of the flow constructed in 1. The constructions 1 and 2 give sense to an approximate mean curvature flow for general data: point cloud, singular surfaces etc.

3. We introduce the definition of a spacetime Brakke flow. Roughly speaking, it consists of a generalization of the spacetime track of a Brakke flow. If we denote by $(V_\varepsilon(t))_{t\in[0,1]}$ the flow constructed in 2, then $\lim V_\varepsilon(t) \otimes dt$ is a spacetime Brakke flow given its $\mathbb{R}^n \times G_{d,n}$-component is rectifiable.

We now give a sketchy presentation of our achievements. We keep the same definition of the approximate mean curvature vector as in [36] (Definition 2.6).

Thanks to :

- the estimate $||h_\varepsilon(\cdot, V)||_{C^1} \leq c_1\|V\|(\mathbb{R}^n)\varepsilon^{-4}$ for any $V \in V_d(\mathbb{R}^n), \varepsilon \in (0,1)$ and for some constant c_1 depending only on n,

- the fact that the push-forward by the map $\mathrm{Id} + \Delta t h_\varepsilon$ increases the mass at most linearly in time (this is due to the fact that we approximate a MCF);

one can define a time-discrete approximate mean curvature flow as follows:

Definition (Time-discrete approximate mean curvature flow, Def 5.1.4, Chap 5)**.** *Let $M \geq 1$, $\varepsilon \in (0,1)$ and $a \in (0,1]$. Consider a subdivision $\mathcal{T} = \{t_i\}_{i=0}^m$ of $[0,a]$ (Definition 1.1.1) and assume that*

$$c_5\delta(\mathcal{T}) \leq (M+1)^{-3}\varepsilon^8 \quad \textit{with} \quad \delta(\mathcal{T}) = \max_{1\leq i\leq m} t_i - t_{i-1}\,. \tag{3.2}$$

for some constant c_5 depending only on n. Let $V_0 \in V_d(\mathbb{R}^n)$ satisfying $\|V_0\|(\mathbb{R}^n) \leq M$. Define $(V_{\varepsilon,\mathcal{T}}(t_i))_{i=0\ldots m}$ by $V_{\varepsilon,\mathcal{T}}(0) := V_0$ and, for $i = 1,\ldots,m$,

$$V_{\varepsilon,\mathcal{T}}(t_i) := f_{i\#}V_{\varepsilon,\mathcal{T}}(t_{i-1}) \quad \textit{with} \quad f_i = \mathrm{Id} + (t_i - t_{i-1})h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_{i-1}))\,.$$

We then define the family $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,a]}$ by linear interpolation between the points of the subdivision, and we call it a time-discrete approximate MCF*:*

$$V_{\varepsilon,\mathcal{T}}(t) := [\mathrm{Id} + (t-t_i)h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_i))]_\#\, V_{\varepsilon,\mathcal{T}}(t_i) \quad \textit{if} \quad t \in [t_i, t_{i+1}].$$

(3.2) is a technical condition allowing to define the push-forwards and guaranteeing at each time that the mass is bounded by $M+1$.
The following proposition encompasses the obtained results on the stability of the time-discrete approximate MCF with respect to the subdivision and the initial datum.

Proposition 3.2.1 (Stability, Prop 5.1.16, Chap 5)**.** *Let $\varepsilon \in (0,1)$, $M > 0$. Let V_0, W_0 be two varifolds in $V_d(\mathbb{R}^n)$ with $||V_0||(\mathbb{R}^n), ||W_0||(\mathbb{R}^n) \leq M$. Let $\mathcal{T}_1 = \{t_i\}_{i=1}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ be two subdivisions of $[0,1]$ satisfying (3.2). Let $(V_{\varepsilon,\mathcal{T}_1}(t))_{t\in[0,1]}$ (resp.$(W_{\varepsilon,\mathcal{T}_2}(t))_{t\in[0,1]}$) be the discrete approximate MCF with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) starting from V_0 (resp. W_0).*
If we set

$$\delta = \max\{\delta(\mathcal{T}_1, \mathcal{T}_2)\},$$

we have

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), W_{\varepsilon,\mathcal{T}_2}(t)) \leq \Delta(V_0, W_0)\exp\bigl(tC\varepsilon^{-n-7}\bigr) + Ct\delta\varepsilon^{-n-11}\exp\bigl(tC\varepsilon^{-n-7}\bigr), \tag{3.3}$$

for all $t \in [0,1]$ where C is a constant depending on M and n.

3.2.1 Approximate mean curvature flow for general varifolds

Thanks to the stability result 3.2.1, we prove that the time-discrete approximate mean curvature flow converges to a unique limit when the time step tends to 0 (with ε fixed). The next result summarizes Theorem 5.2.1 and Proposition 5.2.5.

Theorem (Convergence to approximate MCF, Thm 5.2.1 & Prop. 5.2.5). *Let $\varepsilon \in (0,1)$, $M > 0$. Let $V_0 \in V_d(\mathbb{R}^n)$ of bounded support with $||V_0||(\mathbb{R}^n) \leq M$. Let $(\mathcal{T}_j)_{j\in\mathbb{N}}$ be a sequence of subdivisions of the interval $[0,1]$ with step tending to 0 and consider $(V_{\varepsilon,\mathcal{T}_j}(t))_{t\in[0,1]}$, the discrete approximate MCF with respect to $\mathcal{T}_j$ starting from V_0.*
Then, as $j \to \infty$, $V_{\varepsilon,\mathcal{T}_j}(t)$ converges on $[0,1]$ to a unique limit that we denote by $V_\varepsilon(t)$ and call the approximate MCF of V_0.
In addition, $V_\varepsilon(t)$ satisfies a Brakke equality with respect to h_ε, i.e.

$$||V_\varepsilon(t_2)||(\varphi(\cdot,t_2)) - ||V_\varepsilon(t_1)||(\varphi(\cdot,t_1)) = \int_{t_1}^{t_2} \delta(V_\varepsilon(t),\varphi)(h_\varepsilon(\cdot,V_\varepsilon(t)))dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \frac{d}{dt}\varphi(\cdot,t)\, d||V_\varepsilon(t)||dt \tag{3.4}$$

for all $\varphi \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and $0 \leq t_1 \leq t_2 \leq 1$.

3.2.2 Spacetime Brakke flows

Before we resume the presentation of our results, we need to introduce some definitions.

Definition (Spacetime first variation and mean curvature, Def. 5.3.1**).**
Let $X \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ and λ be a finite Radon measure on $(\mathbb{R}^n \times G_{d,n} \times [0,1])$. The spacetime first variation of λ in the direction X is defined by:

$$\delta\lambda(X) = \int_{\mathbb{R}^n \times G_{d,n} \times [0,1]} \mathrm{div}_S\, X(y,t)\, d\lambda(y,S,t).$$

If the functional $\delta\lambda : C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n) \to \mathbb{R}$ is bounded with respect to $\|\cdot\|_\infty$ then by Riesz representation theorem and Radon-Nikodym decomposition, we can assert the existence of a vector field $h(\cdot,\cdot,\lambda)$ in $L^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n, d\|\lambda\|)$ such that

$$\int_{\mathbb{R}^n \times G_{d,n} \times [0,1]} \mathrm{div}_S\, X(y,t)\, d\lambda(y,S,t) = -\int_{\mathbb{R}^n \times [0,1]} X(y,t) \cdot h(y,t,\lambda)\, d\|\lambda\| + (\delta\lambda)_s(X) \tag{3.5}$$

$\forall X \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, where $\|\lambda\| = \Pi_\#\lambda$, Π being the canonical projection from $\mathbb{R}^n \times G_{d,n} \times [0,1]$ to $\mathbb{R}^n \times [0,1]$, and $(\delta\lambda)_s$ is a vector-valued Radon measure singular with respect to $\|\lambda\|$. The vector $h(\cdot,\cdot,\lambda)$ is called the spacetime mean curvature of λ.

We now introduce the notion of spacetime Brakke flow. One can think of this notion as a generalized spacetime track of Brakke flows.

Definition (Spacetime Brakke flow, Def 5.3.4, Chap 5**).** *Let λ be a finite Radon measure on $\mathbb{R}^n \times G_{d,n} \times [0,1]$. λ is called a spacetime Brakke flow if:*

(i) There exists $(\mu(t))_{t\in[0,1]}$, a family of Radon measures on $\mathbb{R}^n$ (we call it the mass measure of λ), and $(\nu_{(x,t)})_{(x,t)\in\mathbb{R}^n\times[0,1]}$ a family of probability measures such that $\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt$.

(ii) $\delta\lambda$ is bounded and $(\delta\lambda)_s = 0$.

(iii) (Integral Brakke inequality). For any $\varphi \in C^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, $0 \leq t_1 \leq t_2 \leq 1$ we have

$$\mu(t_2)(\varphi(\cdot,t_2)) - \mu(t_1)(\varphi(\cdot,t_1)) \leq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi(y,t)|h(y,t,\lambda)|^2\, d\mu(t)(y)dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n\times G_{d,n}} S^\perp(\nabla\varphi(y,t)) \cdot h(y,t,\lambda)\, d\lambda(y,S,t) + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \partial_t\varphi(\cdot,t)\, d\mu(t)dt \tag{3.6}$$

where $h(\cdot,\cdot,\lambda)$ is the spacetime mean curvature of λ. We say that λ starts from $V_0 = \mu(0) \otimes \nu_{(x,0)}$.

We prove an interesting property on the limit, stated as follows:

Theorem (Convergence, Thm 5.3.7, Chap 5)**.** *Let $\varepsilon \in (0,1)$ and $V_0 \in V_d(\mathbb{R}^n)$ with compact support and finite mass. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate mean curvature flow starting from V_0. We have:*

- *There exists a sequence $(\varepsilon_j)_j \xrightarrow[j\to\infty]{} 0$ such that*

$$V_{\varepsilon_j}(t) \otimes dt \xrightarrow[j\to\infty]{} \lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt, \quad \text{and } \|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{} \mu(t),$$

  where $\mu(t)$ is a Radon measure on $\mathbb{R}^n$ and $\nu_{(x,t)}$ a family of probability measures for $(x,t) \in \mathbb{R}^n \times [0,1]$.

- *$\delta\lambda$ is bounded, $(\delta\lambda)_s = 0$ and $\|h(\cdot,\cdot,\lambda)\|_{L^2(d\lambda)} \le V_0(\mathbb{R}^n)$.*

- *If we assume that $\mu(t) \otimes \nu_{(x,t)}$ is rectifiable for a.e. $t \in [0,1]$ then λ is a spacetime Brakke flow.*

The proof of the first part of the theorem relies on the boundedness in ε of the masses and the semi-continuity of $\mu(t)$. For the second and the third part, we adapt the proof of Kim & Tonegawa to our context. For technical reasons, the rectifiability assumption is essential to prove the Brakke inequality.

3.3 An avoidance principle for approximate MCFs and spacetime Brakke flows, and consequences

This section summarizes the contributions provided in Chapter 6 on avoidance principles satisfied by approximate mean curvature flows and spacetime Brakke flows. The main achievements we have obtained are the following:

1. The nontriviality of our constructed flow when the initial datum is the boundary of an open partition.

2. The avoidance of smooth mean curvature flows by the mass measures of spacetime Brakke flows. This is a generalisation of [35, Theorem 10.5] which was proved for rectifiable Brakke varifolds.

The first result is stated rigorously as follows:

Theorem 3.3.1 (Nontriviality, Thm 6.2.9, Chap 6)**.** *Let $\mathcal{E}$ be an open partition (Definition 2.2.3). Let $\varepsilon \in (0,1)$ and $(\partial\mathcal{E}_\varepsilon(t))_{t\in[0,1]}$ be the approximate mean curvature flow starting from $\partial\mathcal{E}$. Let λ be the limit of $\partial\mathcal{E}_\varepsilon(t) \otimes dt$ (by any possible extraction). Then λ is non-trivial measure. i.e. if $(\mu(t))_{t\in[0,1]}$ is the mass measure of λ, then $\mu(t)(\mathbb{R}^n) > 0$ for a nontrivial time interval.*

The proof of Theorem 3.3.1 relies on the approximate comparison principle with respect to spheres satisfied by the time-discrete approximate mean curvature flows.

The second result is stated rigorously as follows:

Theorem (Avoidance, Thm 6.4.3, Chap 6)**.** *Let $(\mathcal{M}_t)_{t\in[0,1]}$ be a mean curvature flow. Let λ be a spacetime Brakke flow and $(\mu(t))_{t\in[0,1]}$ be its mass measure. Assume that $\mathcal{M}_0$ and $\mu(0)$ have compact support and codimension* 1*. We have*

$$\operatorname{spt}\mu(0)\cap\mathcal{M}_0=\emptyset \implies \operatorname{spt}\mu(t)\cap\mathcal{M}_t=\emptyset\ \forall t\in[0,1].$$

The proof is a direct adaptation of [35, Theorem 10.5]. According to [35, Definition 10.1], a family $(F_t)_{t\geq 0}$ of closed sets is a set-theoretic subsolution to the mean curvature flow if

$$\mathcal{M}_0\cap F_0=\emptyset \implies \mathcal{M}_t\cap F_t=\emptyset\ \forall t\geq 0$$

for every compact hypersurface $\mathcal{M}_0$, where $(\mathcal{M}_t)_{t\geq 0}$ is its MCF. Hence, the mass measure of a spacetime Brakke flow is a *set-theoretic subsolution* of the mean curvature flow. This implies the following nice result:

Corollary 3.3.2 (Coincidence with smooth flows, Cor 6.4.5, Chap 6)**.** *Let $(\mu(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow (Definition 5.3.4) starting from $V_0\in V_{n-1}(\mathbb{R}^n)$. Assume that V_0 is the varifold associated to a compact hypersurface, and denote by $(\mathcal{M}_t)_{t\in[0,1]}$ its mean curvature flow.*

1. *If* $\operatorname{spt}\mu(0)\subsetneq\operatorname{spt}\|V_0\|$*, then* $\mu(t)=0$, $\forall t>0$.
2. *If we assume that* $\mu(s)(\mathbb{R}^n)>0$ *for some* $s\in(0,1]$ *then* $\operatorname{spt}\mu(t)=\mathcal{M}_t$, $\forall t\in[0,s)$.

3.4 Flows of point clouds

The spacetime Brakke flow of a point cloud varifold is trivial. To see that, we first note that a point cloud is contained in infinitely many smooth hypersurfaces. Corollary 3.3.2 implies that the evolution of the point cloud is contained in the MCF of each smooth hypersurface that contains the point cloud at time $t=0$. By the comparison principle, this implies the triviality of the flow.

Consequently, only approximate mean curvature flows can be considered for point clouds. But do they have a limit as the initial point cloud becomes more and more dense, converging in the limit to a smooth surface? An example of a point cloud with increasing densities is shown in Figure 3.1.

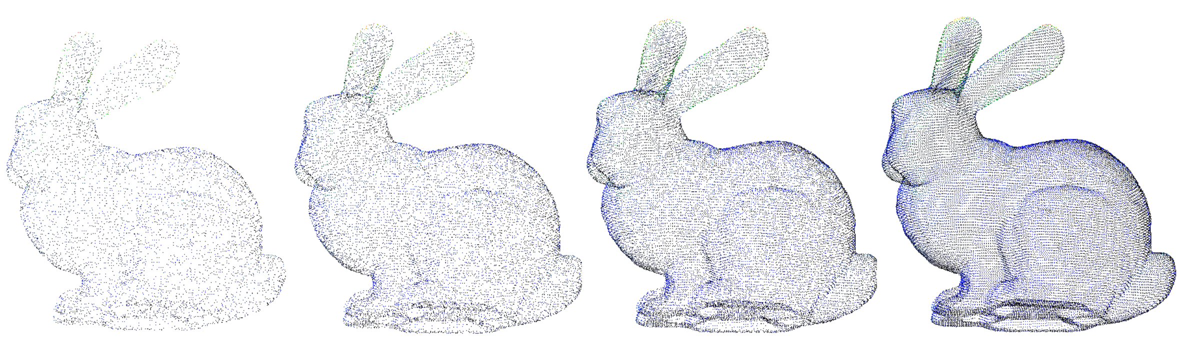

Figure 3.1: Point cloud approximations of a surface with increasing densities. If approximate mean curvature flows are computed starting from each point cloud in a collection with increasing resolutions, what can be said about the limit mean curvature flow in the limit of resolution?

We provide below an explicit construction of our time-discrete approximate mean curvature flow in the case of point cloud varifolds. We also synthesize what can be deduced for point clouds, in particular when they approximate a smooth surface, from the general consistency properties proved in Chapter 5. Then, we recall the definition of the approximate mean curvature flow of point cloud varifolds proposed in [17]. Finally, we make a comparison between the two flows.

3.4.1 An explicit scheme for the time-discrete approximate MCF of a point cloud

A time-discrete ε-approximate flow of a point cloud can be easily derived from Definition 3.2 as an explicit scheme. Let $V = \sum_{j=1}^{N} m_j \delta_{x_j} \otimes \delta_{P_j}$ a point cloud varifold. Let $\varepsilon \in (0,1)$ and $\mathcal{T} = \{t_i\}_{i=1}^m$ be such that $c_5 \delta(\mathcal{T})(\|V\|(\mathbb{R}^n)+1)^3 \leq \varepsilon^8$, and set

$$V_{\varepsilon,\mathcal{T}}(0) = \sum_{j=1}^{N} m_j(0) \delta_{x_j(0)} \otimes \delta_{P_j(0)} = V.$$

The time-discrete approximate flow starting from $V_{\varepsilon,\mathcal{T}}(0)$ is defined inductively as

$$V_{\varepsilon,\mathcal{T}}(t_i) = \sum_{j=1}^{N} m_j(t_i) \delta_{x_j(t_i)} \otimes \delta_{P_j(t_i)} \quad \forall i \in \{1, \ldots, m\}$$

with

- $m_j(t_{i+1}) = \Big(J_{P_j(t_i)}(\mathrm{id} + (t_{i+1}-t_i)h_\varepsilon)(x_j(t_i)) \Big) m_j(t_i)$,
- $P_j(t_{i+1}) = \Big(D(\mathrm{id} + (t_{i+1}-t_i)h_\varepsilon)(x_j(t_i)) \Big) \left(P_j(t_i)\right).$

Implementing and testing practically this flow on real point clouds is the purpose of future work.

3.4.2 Converging point clouds and limits of their approximate MCFs

Let $(W_k)_k$ be a sequence of point cloud varifolds converging to a varifold $V_0 \in V_{n-1}(\mathbb{R}^n)$ associated with a C^2 hypersurface $\mathcal{M}$. Let $(\mathcal{T}_k)_k$ be a sequence of subdivisions of $[0,1]$. We choose $(\varepsilon_k)_k, (\delta(\mathcal{T}_k))_k \in (0,1)$ such that

$$\Delta(V_0, W_k) \exp\big(C\varepsilon_k^{-n-7}\big) + C\delta(\mathcal{T}_k)\varepsilon^{-n-11} \exp\big(C\varepsilon_k^{-n-11}\big) \xrightarrow[k\to\infty]{} 0.$$

We consider for every $k \in \mathbb{N}$ the time-discrete approximate MCF $((W_k)_{\varepsilon_k,\mathcal{T}_k}(t))_{t\in[0,1]}$ starting from the point cloud W_k. Thanks to Proposition 3.2.1, Theorem 3.3.1 and Corollary 3.3.2 in Chapter 5, we can characterize the limit of these approximate mean curvature flows:

1. $(W_k)_{\varepsilon_k,\mathcal{T}_k}(t) \otimes dt$ converges to a non-trivial measure λ such that

$$\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt$$

   where $(\mu(t))_{t\in[0,1]}$ is a collection of Radon measures on $\mathbb{R}^n$, $\nu_{(x,t)}$ is a collection of probability measures on $G_{d,n}$, and $\mu(0) \otimes \nu_{(x,0)} = V_0$.

2. If we assume $\mu(t) \otimes \nu_{(x,t)}$ to be rectifiable for a.e. $t \in [0,1]$, then λ is a spacetime Brakke flow. Moreover, denoting as $(\mathcal{M}_t)_{t\geq 0}$ the mean curvature flow starting from $\mathcal{M}$, we prove that $\operatorname{spt} \mu(t) = \mathcal{M}_t$ on a non-trivial time interval.

3.4.3 Another construction: the Buet-Rumpf approximate mean curvature flow

The motion of point clouds by their approximate mean curvature is defined in [17] as follows: given a point cloud d-varifold $V = \sum_{i=1}^{N} m_i \delta_{(x_i,P_i)}$ in $\mathbb{R}^n$, a continuous motion of point cloud varifolds by approximate mean curvature flow starting from V is a family of varifolds $(V(t))_{t\geq 0}$ with $V(0) = V$ and:

$$V(t) = \sum_{i=1}^{N} m_i(t)\delta_{(x_i(t),P_i(t))} \quad \text{and} \quad X(t) = (x_1(t)\ldots x_N(t)) \in \mathbb{R}^{nN}$$

such that

$$\frac{d}{dt}x_i(t) = H_\varepsilon(x_i(t), V(t)). \tag{3.7}$$

If the masses and the tangents are Lipschitz functions of the positions, the existence of the motion for at least a small time interval is guaranteed by Cauchy-Lipschitz theorem. For numerical simulations the masses are defined by either formula $m_i = 1$ or $m_i = \dfrac{\varepsilon^d}{\sum_j \eta\left(\frac{|x_i - x_j|}{\varepsilon}\right)}$ for some positive function η supported on $[0, 1]$. As for the tangents, they are defined via regression.

Two time discretizations of Equation (3.7), an implicit and an explicit ones, are proposed in [17] . If $\tau > 0$ denotes the time step, the implicit scheme (implicit with respect to the positions) is defined as:

$$x_i^{k+1} = x_i^k + \tau H_\varepsilon\left(x_i^{k+1}, V\right) \tag{3.8}$$

where

$$V = \sum_{j=1}^{N} m_i^k \delta_{(x_i^{k+1},P_i^k)},$$

and $X^k := \left(x_1^k, \ldots, x_N^k\right) \in \mathbb{R}^{nN}$ being the positions at time $t_k := k\tau$.

3.4.4 Comparison of the two approaches

Numerical schemes are proposed in [17] for the above mean curvature flow, they are flexible, easy to implement and give satisfactory numerical results. A drawback of the approach proposed in [17] is that stability or convergence results (as the time step decreases and the density of the point cloud increases) seem to be hardly reachable, and possibly not true. In contrast, the method we propose is based on a significantly more difficult construction, but stability and convergence results can be proved due to the more rigid definitions of masses and tangent planes.

3.5 Further comments on our construction

Weak MCFs constructed in [22, 18, 35] (and some others) coincide with the classical mean curvature flow in codimension 1 as long as the latter exists. These approaches use the MCF equation to construct approximate solutions and rely on the theory of PDEs to prove convergence and consistency in the C^2 case. The approximate flow in our construction satisfies the approximate mean curvature flow equation only in the weak sense (approximate Brakke equality). Hence, we do not have strong estimates coming from PDE theory to prove the consistency. We recall that we choose to work with a weak PDE instead of the strong one in order to include various types of varifolds, such as point clouds and singular submanifolds .

3.6 Research perspectives

In a work in progress, we are addressing the following questions:

- (Coincidence with smooth flows) Let $\mathcal{M}$ be a compact C^2 $(n-1)$-submanifold of $\mathbb{R}^n$ and $(\mathcal{M}_t)_{t\geq 0}$ its mean curvature flow. Let $\mu(t)$ be the mass measure of a spacetime Brakke flow starting from $\mathcal{M}$. We prove in this manuscript that if $\mu(s)(\mathbb{R}^n) > 0$ for some s then $\operatorname{spt}\mu(t) = \mathcal{M}_t$ for any $t \in [0, s)$. Can we prove that $\mu(t) = \mathcal{H}^d_{|\mathcal{M}_t}$ for a.e. $t \in [0, s)$?

- (Rectifiability of $V(t)$) If λ is a limit of $V_\varepsilon(t) \otimes dt$, we proved that λ is a spacetime Brakke flow given that its $\mathbb{R}^n \times G_{d,n}$-component is rectifiable. Can we prove that the $\mathbb{R}^n \times G_{d,n}$-component is rectifiable given that the initial datum is rectifiable?

- (Nontriviality) We prove in this work that our constructed flow is nontrivial when the initial datum is a boundary of an open partition. Does the nontriviality of our constructed flow for C^2 surfaces still hold in higher codimension?

- (Disintegration) If $V(t)$ is a Brakke flow then $V(t) \otimes dt$ is a spacetime Brakke flow. Conversely, a spacetime Brakke flow can always be desintegrated as $V(t) \otimes dt$. But is it always true that $V(t)$ is a Brakke flow?

- (Dimension mismatch) Consider a d-varifold whose mass is exactly supported on a m-rectifiable set with $d \neq m$. Even in this context where there is a mismatch between the dimension of the support and the dimension of the tangent space, our construction still holds. But what kind of evolution is it? Can we say more?

- (Density change) We know by the avoidance principle that if we start from a C^2 submanifold with non-constant density, the support of its flow coincides with the support of the evolution with constant density. What can be said about the density evolution in time? Even in the case of spheres, the answer is not clear to us.

- (Numerical simulations) Based on the discussion of section 3.4.1, it would be interesting to simulate numerically the approximate mean curvature flow of various initial data.

Chapter 4

Approximate Brakke equality for a volumetric discretization of mean curvature flow

The goal of this chapter is to show a consistency result on the mean curvature flow evolution via space-discretization by volumetric varifolds as in (1.5).
Consider a C^3 closed submanifold $\mathcal{M}$ of $\mathbb{R}^n$ and $(\mathcal{M}(t))_{t\in[0,T]}$, $T > 0$ the mean curvature flow of $\mathcal{M}$ on $[0,T]$ (Definition 2.1.1). Let $\mathcal{K}_h$ be a mesh of $\mathbb{R}^n$ of size $h \in (0,1)$ and $V_h(t)$ a discretization of $\mathcal{M}(t)$ for every $t \in [0,T]$. The work consists of showing a Brakke approximate equality for the discretization with respect to its approximate mean curvature $H_\varepsilon(\cdot, V_h(t))$. The error term depends on ε, h, $\mathcal{M}$ and the kernels ρ and ξ involved in the definition of the approximate mean curvature (1.18).

We deal with an approximate mean curvature vector of the discretization $V_h(t)_{t\in[0,T]}$ since it makes more sense in the discrete setting. We consider the submanifold $\mathcal{M}$ of regularity C^3 and not only C^2 in order to have Lemma 4.0.4. The approximate mean curvature $H_\varepsilon(\cdot,\cdot)$ we use in this chapter is defined in (1.18), where we set $C_\rho = C_\xi = 1$ for simplicity.

The mass decay property of the mean curvature flow is an important tool, heavily used in this chapter. For a family of varifolds $(M(t))_{t\in[0,T]}$ associated to a mean curvature flow we have

$$\|M(t)\|(\mathbb{R}^n) \leq \|M(0)\|(\mathbb{R}^n), \forall t \in [0,T], \tag{4.1}$$

it stems from (2.3) for $\varphi \equiv 1$.

The following result forms the core of this chapter. We stress that the result concerns the error on the Brakke equality that results from space discretization, that it does not concern neither the error of time discretization nor the stability of the discretization scheme.

Theorem 4.0.1. *Let $\mathcal{M}$ be a C^3 closed d-submanifold in $\mathbb{R}^n$, $\mathcal{M}(t)_{t\in[0,T]}$ be its mean curvature flow (Definition 2.1.1). Let $M(t)_{t\in[0,T]}$ be the varifolds associated with the family $\mathcal{M}(t)_{t\in[0,T]}$. Let $\mathcal{K}_h$ be a mesh of $\mathbb{R}^n$ of size h and $V_h(t)$ a discretization of $\mathcal{M}(t)$ for every $t \in [0,T]$ (Definition (1.5)).*
Then, there exist two constants $\gamma \in (0,1)$ and C (whose dependence is discussed in Remark 4.0.2), such

that, for all $h, \varepsilon \in (0,1)$ satisfying $h \leq \frac{\gamma}{2}\varepsilon$,

$$\left| ||V_h(t_2)||(\varphi) - ||V_h(t_1)||(\varphi) + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi(x)|H_\varepsilon(x, V_h(t))|^2 - \nabla\varphi(x) \cdot H_\varepsilon(x, V_h(t))\, d||V_h(t)||(x) dt \right|$$
$$\leq C\|\varphi\|_{\mathcal{C}^2} \left(\max_{t \in \{t_1, t_2\}} \Delta(M(t), V_h(t)) + \varepsilon(||M(t_1)||(\mathbb{R}^n) - ||M(t_2)||(\mathbb{R}^n)) + (t_2 - t_1)\|M(0)\|(\mathbb{R}^n)(\varepsilon + \frac{h}{\varepsilon^3}) \right), \tag{4.2}$$

for every $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, $0 \leq t_1 \leq t_2 \leq T$.

Remark 4.0.2. The constant γ in Theorem 4.0.1 satisfies:

- $\gamma \leq (8(1 + C_0^{2/d}))^{-1}$, where C_0 bounds the Ahlfors regularity constant of $M(t)$ for all $t \in [0, T]$;
- $\gamma \leq \max_{x \in \mathcal{M}(t),\, t \in [0,T]} (\lambda(x,t))^{-1}$, where $\lambda(x,t)$ is the maximal principal curvature of $\mathcal{M}(t)$ at x;
- $\beta > \gamma 2^{3d} C_0^2(\mathrm{Lip}(\xi) + 1)$, where $\beta = \min\left\{\xi(s) \big| s \in [\frac{C_0^{-2/d}}{4}, \frac{1}{2}]\right\} > 0$.

The constant C in Theorem 4.0.1 depends on n, d, γ, β, $||\rho||_{\mathcal{C}^2}$, $||\xi||_{\mathcal{C}^1}$, C_0, C_1 (defined in Lemma 4.0.4) and C_2, where C_2 bounds the Lipschitz constants of the maps $y \mapsto T_y\mathcal{M}(t)$, $\forall t \in [0, T]$.

Remark 4.0.3. If we ignore the smallness of the right-hand side of (4.2) with respect to $|t_1 - t_2|$ we obtain a weaker (but simpler) estimate:

$$\left| ||V_h(t_2)||(\varphi) - ||V_h(t_1)||(\varphi) + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi(x)|H_\varepsilon(x, V_h(t))|^2 - \nabla\varphi(x) \cdot H_\varepsilon(x, V_h(t))\, d||V_h(t)||(x) dt \right|$$
$$\leq ||\varphi||_{\mathcal{C}^2} C' \left(\varepsilon + \frac{h}{\varepsilon^3} \right), \ \forall \varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+), \ \text{where } C' = C\left(3\|M(0)\|(\mathbb{R}^n) + T\right).$$

where we used: $\Delta(\|M(t)\|, \|V_h(t)\|) \leq h||M(t)||(\mathbb{R}^n)$ (1.6) and $||M(t)||(\mathbb{R}^n) \leq ||M(0)||(\mathbb{R}^n), \forall t \in [0, T]$ (4.1).

The following is an approximation result concerning the approximate mean curvature, it stems from [17, Proposition 3.3] (see also [15, Paragraph 5]).

Lemma 4.0.4. *Let $\varepsilon \in (0,1)$, let $\mathcal{M}$ C^3 closed d-submanifold of $\mathbb{R}^n$ and M its associated varifold. There exists a constant C_1 depending on the C^3- norm of $\mathcal{M}$ (seen locally as a graph over its tangent space) and uniform on $\mathcal{M}$ such that: for any $x \in \mathcal{M}$,*

$$\left|H(x, M) - H_\varepsilon(x, M)\right| \leq C_1\varepsilon.$$

Remark 4.0.5. The mean curvature flow is continuous with respect to the C^3-distance on the space of submanifolds, the fact that $[0, T]$ is compact allows to choose C_1 uniformly in t, i.e.

$$\left|H(x, M(t)) - H_\varepsilon(x, M(t))\right| \leq C_1\varepsilon, \qquad \forall t \in [0, T], \forall x \in \mathcal{M}(t), \tag{4.3}$$

for $(M(t))_{t \in [0,T]}$ being the varifolds associated to the mean curvature flow $(\mathcal{M}(t))_{t \in [0,T]}$.

Remark 4.0.6. We note that in the original statement of Lemma 4.0.4, the kernels ρ and ξ are supposed to form a *natural kernel pair*, i.e. they must satisfy the relation: $-n\xi(r) = r\rho'(r)$ for all $r \in [0,1]$. Looking carefully at the proof, we realize that this assumption on the pair of kernels is not necessary for this particular statement.

Sketch of the proof of Theorem 4.0.1: The starting point is the following equality satisfied by mean curvature flows. Let $(M(t))_{t\in[0,T]}$ be the varifolds associated with a mean curvature flow. We deduce from (2.3) (considering the orthogonality of the mean curvature) that

$$\begin{aligned} \|M(t_2)\|(\varphi) - \|M(t_1)\|(\varphi) + \int_{t_1}^{t_2}\int_{M(t)} \varphi(y)|H(y,M(t))|^2\, d\mathcal{H}^d(y)dt \\ - \int_{t_1}^{t_2}\int_{M(t)} \nabla\varphi(y)\cdot H(y,M(t))\, d\mathcal{H}^d(y)dt = 0, \end{aligned} \tag{4.4}$$

for any $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^+)$ and $0 \le t_1 \le t_2 \le T$. Then we measure the error made when, in (4.4),

- $H(\cdot, M(t))$ is replaced with $H_\varepsilon(\cdot, M(t))$ (Lemma 4.0.7),
- $M(t)$ is replaced with $V_h(t)$ (Lemma 4.0.9 and final part of the proof),

and the result ensues.

Let us now see the proof in detail. We first use Lemma 4.0.4 to prove the following result.

Lemma 4.0.7. *Let $\varepsilon \in (0,1)$. Let $\mathcal{M}$ be a C^3 closed d-submanifold and $\mathcal{M}(t)_{t\in[0,T]}$ be the mean curvature flow of $\mathcal{M}$. Let $M(t)_{t\in[0,T]}$ be the varifolds associated to $\mathcal{M}(t)_{t\in[0,T]}$. We have*

$$\begin{aligned} &\left| ||M(t_2)||(\varphi) - ||M(t_1)||(\varphi) + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi(x)|H_\varepsilon(x,M(t))|^2 - \nabla\varphi(x)\cdot H_\varepsilon(x,M(t))\, d||M(t)||(x)dt \right| \\ &\le \|\varphi\|_{\mathcal{C}^1}\tilde{C}_1\varepsilon\left(||M(t_1)||(\mathbb{R}^n) - ||M(t_2)||(\mathbb{R}^n) + (t_2-t_1)\|M(0)\|(\mathbb{R}^n)\right), \end{aligned} \tag{4.5}$$

for any $0 \le t_1 \le t_2 \le T$ and $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^+)$, where $\tilde{C}_1 = C_1(2+C_1)$.

Proof. Let $t_1, t_2 \in [0,T]$ be such that $0 \le t_1 \le t_2 \le T$, let $\varphi \in C_c^1(\mathbb{R}^n,\mathbb{R}^+)$. We have by (4.3)

$$\begin{aligned} &\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \nabla\varphi(x)\cdot H(x,M(t))d||M(t)||(x)dt - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \nabla\varphi(x)\cdot H_\varepsilon(x,M(t))d||M(t)||(x)\, dt\right| \\ &\le ||\nabla\varphi||_\infty \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left|H(x,M(t)) - H_\varepsilon(x,M(t))\right| d||M(t)||(x)\, dt \\ &\le ||\nabla\varphi||_\infty C_1\varepsilon \int_{t_1}^{t_2}\int_{\mathbb{R}^n} d||M(t)||(x)\, dt \le ||\nabla\varphi||_\infty C_1 ||M(0)||(\mathbb{R}^n)(t_2-t_1)\varepsilon, \end{aligned} \tag{4.6}$$

where we used $||M(t)||(\mathbb{R}^n) \le ||M(0)||(\mathbb{R}^n), \forall t \in [0,T]$ (4.1). Also, by (4.3)

$$\left| |H(x,M(t))|^2 - |H_\varepsilon(x,M(t))|^2 \right| \le C_1\varepsilon \left| |H(x,M(t))| + |H_\varepsilon(x,M(t))| \right| \le C_1\varepsilon\left(2|H(x,M(t))| + C_1\varepsilon\right),$$

so that, using (4.1)

$$
\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(x)|H(x,M(t))|^2\,d||M(t)||(x)dt-\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(x)|H_\varepsilon(x,M(t))|^2\,d||M(t)||(x)dt\Big|\\
&\le\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\big|\varphi(x)|H(x,M(t))|^2-\varphi(x)|H_\varepsilon(x,M(t))|^2\big|d||M(t)||(x)dt\\
&\le||\varphi||_\infty 2C_1\varepsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\big|H(x,M(t))\big|\,d||M(t)||(x)dt+||\varphi||_\infty(C_1\varepsilon)^2\int_{t_1}^{t_2}\int_{\mathbb{R}^n}d||M(t)||(x)dt\\
&\le||\varphi||_\infty 2C_1\varepsilon\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\big|H(x,M(t))\big|\,d||M(t)||(x)dt+||\varphi||_\infty(C_1\varepsilon)^2||M(0)||(\mathbb{R}^n)(t_2-t_1).
\end{aligned}
$$

For the first term, using (4.4) with $\varphi\equiv 1$ and the inequality $ab\le a^2+b^2$

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\big|H(x,M(t))\big|\,d||M(t)||(x)dt&\le\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\Big(\big|H(x,M(t))\big|^2+1\Big)\,d||M(t)||(x)dt\\
&\le(||M(t_1)||(\mathbb{R}^n)-||M(t_2)||(\mathbb{R}^n))+(t_2-t_1)||M(0)||(\mathbb{R}^n).
\end{aligned}
$$

It yields, as $\varepsilon\le 1$

$$
\begin{aligned}
&\Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(x)|H(x,M(t))|^2d||M(t)||(x)dt-\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(x)|H_\varepsilon(x,M(t))|^2\,d||M(t)||(x)dt\Big|\\
&\le||\varphi||_\infty 2C_1\varepsilon\Big((||M(t_1)||(\mathbb{R}^n)-||M(t_2)||(\mathbb{R}^n))+(t_2-t_1)||M(0)||(\mathbb{R}^n)+C_1||M(0)||(\mathbb{R}^n)(t_2-t_1)\Big).
\end{aligned}
\tag{4.7}
$$

Finally, we deduce from (4.4), (4.6) and (4.7) that:

$$
\begin{aligned}
&\Big|||M(t_2)||(\varphi)-||M(t_1)||(\varphi)+\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(x)|H_\varepsilon(x,M(t))|^2-\nabla\varphi(x)\cdot H_\varepsilon(x,M(t))\,d||M(t)||(x)dt\Big|\\
&\le||\varphi||_\infty 2C_1\varepsilon(||M(t_1)||(\mathbb{R}^n)-||M(t_2)||(\mathbb{R}^n))+||\varphi||_{\mathcal{C}^1}\varepsilon||M(0)||(\mathbb{R}^n)2C_1(2+C_1)(t_2-t_1)\\
&\le||\varphi||_{\mathcal{C}^1}\tilde{C}_1\varepsilon\left(||M(t_1)||(\mathbb{R}^n)-||M(t_2)||(\mathbb{R}^n)+(t_2-t_1)||M(0)||(\mathbb{R}^n)\right),
\end{aligned}
$$

where $\tilde{C}_1=2C_1(2+C_1)$. This concludes the proof of Lemma 4.0.7. □

In the following lemma, we bound the $C^{0,1}$-norm of $H_\varepsilon(\cdot,V)$ for $V\in V_d(\mathbb{R}^n)$ Ahlfors regular; the proof is inspired from [17, Proposition 4.6].

Lemma 4.0.8. *Let V be a d-Ahlfors regular varifold in $\mathbb{R}^n$ with Ahlfors constant C_0. Let $\varepsilon,\gamma\in(0,1)$ with $\gamma\le(8(1+C_0^{2/d}))^{-1}$, we have:*

$$
\max_{\operatorname{spt}\|V\|^{\gamma\varepsilon}}|H_\varepsilon(\cdot,V)|\le c_5\varepsilon^{-1},\qquad \operatorname*{Lip}_{\operatorname{spt}\|V\|^{\gamma\varepsilon}}(H_\varepsilon(\cdot,V))\le c_5\varepsilon^{-2},
$$

where c_5 is a constant that depends only on ρ,ξ,C_0 and d.

Proof. We start with $\max_{\operatorname{spt}\|V\|^{\gamma\varepsilon}}|H_\varepsilon(\cdot,V)|$, we first prove that there exists a constant c_7 such that:

$$
\varepsilon^n\|V\|*\xi_\varepsilon(z)\ge c_7\varepsilon^d\quad\forall z\in\operatorname{spt}\|V\|^{\gamma\varepsilon}.
$$

Denote

$$\beta = \min\left\{\xi(s) \big| s \in \left[\frac{C_0^{-2/d}}{4}, \frac{1}{2}\right]\right\} > 0.$$

Let $z \in \operatorname{spt} \|V\|^{\gamma\varepsilon}$, let $x \in \operatorname{spt} \|V\|$ be such that $|x-z| \le \gamma\varepsilon$, we write

$$\begin{aligned}
\varepsilon^n \|V\| * \xi_\varepsilon(z) &= \int_{\mathbb{R}^n} \xi\left(\frac{|y-z|}{\varepsilon}\right) d\|V\|(y) \\
&\ge \beta\left(\|V\|(B(z,\frac{\varepsilon}{2})) - \|V\|(B(z, C_0^{-2/d}\frac{\varepsilon}{4}))\right) \\
&\ge \beta\left(\|V\|\left(B(x, \frac{\varepsilon}{2} - |x-z|)\right) - \|V\|\left(B(x, C_0^{-2/d}\frac{\varepsilon}{4} + |x-z|)\right)\right) \\
&\text{using the Ahlfors property} \\
&\ge \beta\left(C_0^{-1}\left(\frac{\varepsilon}{2} - |x-z|\right)^d - C_0\left(C_0^{-2/d}\frac{\varepsilon}{4} + |x-z|\right)^d\right) \\
&\ge \beta C_0^{-1} 2^{-d}\left((\varepsilon - 2|x-z|)^d - (\frac{\varepsilon}{2} + 2C_0^{2/d}|x-z|)^d\right) \\
&\ge \beta C_0^{-1} 2^{-d}\left((\varepsilon - 2\gamma\varepsilon)^d - (\frac{\varepsilon}{2} + 2C_0^{2/d}\gamma\varepsilon)^d\right) \\
&\ge \beta C_0^{-1} 2^{-d}\varepsilon^d\left((1-2\gamma)^d - (\frac{1}{2} + 2C_0^{2/d}\gamma)^d\right) \\
&\text{using "}a^d - b^d \ge (a-b)a^{d-1} \text{ when } a \ge b \ge 0\text{" we obtain:} \\
&\ge \beta C_0^{-1} 2^{-d}\varepsilon^d\left((\frac{1}{2} - 2\gamma(1 + C_0^{2/d}))(1-2\gamma)^{d-1}\right) \\
&\ge \beta C_0^{-1} 2^{-d}\varepsilon^d\left(\frac{1}{2}\right)^{d+1} \ge \beta C_0^{-1} 2^{-2d-1}\varepsilon^d,
\end{aligned}$$

thus,

$$\varepsilon^n \|V\| * \xi_\varepsilon(z) \ge c_7 \varepsilon^d, \tag{4.8}$$

with $c_7 = \beta C_0^{-1} 2^{-2d-1}$.
Now, let $z \in \operatorname{spt} \|V\|^{\gamma\varepsilon}$, let $x \in \operatorname{spt} \|V\|$ be such that $|x-z| \le \gamma\varepsilon$, we have by the Ahlfors regularity of V

$$\begin{aligned}
\left|\varepsilon^n(\delta V * \rho_\varepsilon)(z)\right| = \Big|\int_{\mathbb{R}^n} S(\nabla\rho_\varepsilon(z-y)) dV(y,S)\Big| &\le \varepsilon^{-1}||\rho'||_\infty \|V\|\left(B(z,\varepsilon)\right) \\
&\le \varepsilon^{-1}||\rho'||_\infty \|V\|\left(B(x,2\varepsilon)\right) \le \varepsilon^{-1}||\rho'||_\infty C_0(2\varepsilon)^d = 2^d||\rho'||_\infty C_0 \varepsilon^{d-1}.
\end{aligned}$$

Thus, for any $z \in \operatorname{spt} \|V\|^{\gamma\varepsilon}$,

$$\left|\varepsilon^n(\delta V * \rho_\varepsilon)(z)\right| \le 2^d ||\rho'||_\infty C_0 \varepsilon^{d-1}.$$

Consequently,

$$\max_{z \in \operatorname{spt} \|V\|^{\gamma\varepsilon}} |H_\varepsilon(z,V)| = \max_{z \in \operatorname{spt} \|V\|^{\gamma\varepsilon}} \frac{|(\delta V * \rho_\varepsilon)(z)|}{|(\|V\| * \xi_\varepsilon)(z)|} \le \beta^{-1} C_0^2 2^{3d+1} ||\rho'||_\infty \varepsilon^{-1} \le c_5 \varepsilon^{-1},$$

with $c_5 \ge \beta^{-1} C_0^2 2^{3d+1} ||\rho||_{\mathcal{C}^2}$, we conclude the proof of the first part of our lemma.

We now deal with $\operatorname{Lip}_{\operatorname{spt}\|V\|^{\gamma\varepsilon}} H_\varepsilon(\cdot, V)$. For any $x, z \in \operatorname{spt}\|V\|^{\gamma\varepsilon}$, we have by the Ahlfors regularity of V

$$\begin{aligned}
\varepsilon^n \big| \|V\| * \xi_\varepsilon(z) - \|V\| * \xi_\varepsilon(x) \big| &= \Big| \int_{\mathbb{R}^n} \xi\left(\frac{z-y}{\varepsilon}\right) d\|V\|(y) - \int_{\mathbb{R}^n} \xi\left(\frac{x-y}{\varepsilon}\right) d\|V\|(y) \Big| \\
&\leq |x-z|\varepsilon^{-1} ||\xi'||_\infty \|V\| \left(B(x,\varepsilon) \cup B(z,\varepsilon)\right) \\
&\leq |x-z|\varepsilon^{-1} ||\xi'||_\infty \|V\| \left(B(x',2\varepsilon) \cup B(z',2\varepsilon)\right) \text{ for some } x', z' \in \operatorname{spt}\|V\| \\
&\leq |x-z|\varepsilon^{-1} ||\xi'||_\infty \left(2C_0 (2\varepsilon)^d\right) \leq 2^{d+1} C_0 ||\xi'||_\infty |x-z| \varepsilon^{d-1}.
\end{aligned} \tag{4.9}$$

Similar computations give

$$\varepsilon^n \big| \delta V * \rho_\varepsilon(z) - \delta V * \rho_\varepsilon(x) \big| \leq 2^{d+1} C_0 ||\rho''||_\infty |x-z| \varepsilon^{d-2}, \tag{4.10}$$

therefore,

$$\begin{aligned}
\Big| H_\varepsilon(z,V) - H_\varepsilon(x,V) \Big| &= \Big| \frac{\delta V * \rho_\varepsilon(z)}{\|V\| * \xi_\varepsilon(z)} - \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)} \Big| \\
&\leq \frac{\big| \delta V * \rho_\varepsilon(z) - \delta V * \rho_\varepsilon(x) \big|}{\|V\| * \xi_\varepsilon(z)} + |\delta V * \rho_\varepsilon(x)| \Big| \frac{1}{\|V\| * \xi_\varepsilon(z)} - \frac{1}{\|V\| * \xi_\varepsilon(x)} \Big|.
\end{aligned}$$

On the one hand, (4.10) and (4.8) imply

$$\begin{aligned}
\frac{\big| \delta V * \rho_\varepsilon(z) - \delta V * \rho_\varepsilon(x) \big|}{\|V\| * \xi_\varepsilon(z)} &\leq \frac{2^{d+1} C_0 ||\rho''||_\infty |x-z| \varepsilon^{d-2}}{\varepsilon^n \|V\| * \xi_\varepsilon(z)} \\
&\leq \frac{2^{d+1} C_0 ||\rho''||_\infty |x-z| \varepsilon^{d-2}}{\beta C_0^{-1} 2^{-2d-1} \varepsilon^d} \\
&\leq \beta^{-1} C_0^2 2^{3d+1} ||\rho'||_\infty |x-z| \varepsilon^{-2}.
\end{aligned}$$

On the other hand, (4.9) and (4.8) imply

$$\begin{aligned}
|\delta V * \rho_\varepsilon(x)| \Big| \frac{1}{\|V\| * \xi_\varepsilon(z)} - \frac{1}{\|V\| * \xi_\varepsilon(x)} \Big| &\leq \varepsilon^{-n} 2^d ||\rho'||_\infty C_0 \varepsilon^{d-1} \Big| \frac{\|V\| * \xi_\varepsilon(z) - \|V\| * \xi_\varepsilon(x)}{\|V\| * \xi_\varepsilon(z) \|V\| * \xi_\varepsilon(x)} \Big| \\
&\leq 2^d ||\rho'||_\infty C_0 \varepsilon^{d-1} \frac{2^{d+1} C_0 ||\xi'||_\infty |x-z| \varepsilon^{d-1}}{\varepsilon^{2n} \|V\| * \xi_\varepsilon(z) \|V\| * \xi_\varepsilon(x)} \\
&\leq 2^d ||\rho'||_\infty C_0 \varepsilon^{d-1} \frac{2^{d+1} C_0 ||\xi'||_\infty |x-z| \varepsilon^{d-1}}{\beta C_0^{-1} 2^{-2d-1} \varepsilon^d \beta C_0^{-1} 2^{-2d-1} \varepsilon^d} \\
&\leq 2^{6d+3} C_0^4 \beta^{-2} ||\rho'||_\infty ||\xi'||_\infty |x-z| \varepsilon^{-2}.
\end{aligned}$$

In conclusion,

$$\operatorname{Lip}_{\operatorname{spt}\|V\|^{\gamma\varepsilon}} H_\varepsilon(\cdot, V) \leq c_5 \varepsilon^{-2},$$

for $c_5 = \beta^{-1} C_0^2 2^{3d+1} ||\rho||_{\mathcal{C}^2} \left(1 + \beta^{-1} C_0^2 2^{3d+2} ||\xi'||_\infty\right)$, and this concludes the proof of Lemma 4.0.8. $\square$

We now substitute $M(t)$ with $V_h(t)$ in (4.4) and estimate the resulting error. We consider a fixed $t \in [0, T]$ and temporarily drop time dependence.

Lemma 4.0.9. *Let $\mathcal{M}$ be a closed C^2 d-submanifold of $\mathbb{R}^n$ and $M \in V_d(\mathbb{R}^n)$ its associated varifold. Let $h \in (0, 1)$, $\mathcal{K}_h$ be a mesh of $\mathbb{R}^n$ of size h and V_h be a volumetric discretization of $\mathcal{M}$ of parameter h (defined in 1.5).*
Let $\varepsilon, \gamma \in (0, 1)$ be such that $2h \leq \gamma\varepsilon$, and

- $\gamma \leq \max_{x \in \mathcal{M}} \lambda(x)^{-1}$, *where $\lambda(x)$ is the maximal principal curvature of $\mathcal{M}$ at x;*
- $\gamma \leq (8(1 + C_0^{d/2}))^{-1}$, *where C_0 is the Ahlfors regularity constant of M;*
- $\beta > \gamma 2^{3d} C_0^2(\mathrm{Lip}(\xi) + 1)$, *where $\beta = \min\left\{\xi(s) \middle| s \in [\frac{C_0^{-2/d}}{4}, \frac{1}{2}]\right\} > 0$.*

Given $V \in V_d(\mathbb{R}^n)$, $\varepsilon \in (0, 1)$, $x \in \mathbb{R}^n$ and $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, we use the notation:

$$\varphi_\varepsilon(x, V) := -\varphi(x)|H_\varepsilon(x, V)|^2 + \nabla\varphi(x) \cdot H_\varepsilon(x, V). \tag{4.11}$$

Then,

$$\Big| \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)\, d||M|| - \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)\, d||V_h|| \Big| \leq c_4 ||\varphi||_{\mathcal{C}^2} \|M\|(\mathbb{R}^n) \frac{h}{\varepsilon^3}, \tag{4.12}$$

and

$$\Big| \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M) d||V_h|| - \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, V_h) d||V_h|| \Big| \leq c_8 ||\varphi||_{\mathcal{C}^1} \|M\|(\mathbb{R}^n) \frac{h}{\varepsilon^3}, \tag{4.13}$$

where c_4 only depends on ρ, ξ, C_0, d and c_5 from Lemma 4.0.8; c_8 is defined in (4.22).

Proof. We start with the proof of 4.12. By Lemma 1.3.3, we have:

$$\Big| \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M) d||M|| - \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M) d||V_h|| \Big| \leq \underset{\mathbb{R}^n}{\mathrm{Lip}}(\varphi_\varepsilon(\cdot, M)) \Delta(||M||, ||V_h||) \leq h \underset{\mathbb{R}^n}{\mathrm{Lip}}(\varphi_\varepsilon(\cdot, M)) ||M||(\mathbb{R}^n).$$

Note that on the boundary of $\mathcal{M}^\varepsilon$, the quantity $||M|| * \xi_\varepsilon$ might tend to 0 faster than $\delta M * \rho_\varepsilon$, this would make the norm of $H_\varepsilon(\cdot, M)$ explode, so does $\mathrm{Lip}\, \varphi_\varepsilon(\cdot, M)$. As both $\mathcal{M}$ and $\mathrm{spt}\, \|V_h\|$ are included in $\mathcal{M}^h$, we will introduce a cut-off function to measure the difference only on a small neighborhood containing $\mathcal{M}^h$. As $2h \leq \gamma\varepsilon$, we have $\mathcal{M}^h \subset \mathcal{M}^{\gamma\varepsilon}$ and we define ψ as follows:

$$\psi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{M}^h, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \mathcal{M}^{\gamma\varepsilon}. \end{cases}$$

Note that $\mathrm{Lip}(\psi) \leq \frac{1}{\gamma\varepsilon - h} \leq \frac{2}{\gamma}\varepsilon^{-1}$ since $2h \leq \gamma\varepsilon$. Plugging the cut-off function into the expression, using (1.6) we obtain

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)\, d||M|| - \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)\, d||V_h|| \Big| &= \Big| \int_{\mathbb{R}^n} \psi\varphi_\varepsilon(\cdot, M)\, d||M|| - \int_{\mathbb{R}^n} \psi\varphi_\varepsilon(\cdot, M)\, d||V_h|| \Big| \\ &\leq h\, \mathrm{Lip}(\psi\varphi_\varepsilon(\cdot, M)) ||M||(\mathbb{R}^n). \end{aligned} \tag{4.14}$$

As $\mathrm{Lip}(\psi) \leq \frac{2}{\gamma}\varepsilon^{-1}$ (temporarily the dependence of φ_ε and H_ε on M in the notations), we have

$$\mathrm{Lip}(\psi\varphi_\varepsilon) \leq (\max|\psi|) \operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} \varphi_\varepsilon + (\mathrm{Lip}\,\psi) \max_{\mathcal{M}^{\gamma\varepsilon}} |\varphi_\varepsilon| \leq \operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} \varphi_\varepsilon + \frac{2}{\gamma}\varepsilon^{-1} \max_{\mathcal{M}^{\gamma\varepsilon}} |\varphi_\varepsilon| \tag{4.15}$$

and recalling (4.11) and Lemma 4.0.8,

$$\max_{\mathcal{M}^{\gamma\varepsilon}} \varphi_\varepsilon \leq \max \varphi(\max_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon|^2) + \max \nabla\varphi \max_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon| \leq c_5\varepsilon^{-1}\|\varphi\|_{\mathcal{C}^1}\left(1 + c_5\varepsilon^{-1}\right) \tag{4.16}$$

and

$$\operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} \varphi_\varepsilon \leq \mathrm{Lip}\,\varphi(\max_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon|)^2 + \max\varphi(\operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon|^2) + \mathrm{Lip}(\nabla\varphi) \max_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon| + \max(|\nabla\varphi|)(\operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} H_\varepsilon),$$

using $\operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}}(|H_\varepsilon|^2) \leq 2\max_{\mathcal{M}^{\gamma\varepsilon}} |H_\varepsilon| \operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}}(H_\varepsilon)$ we obtain

$$\operatorname*{Lip}_{\mathcal{M}^{\gamma\varepsilon}} \varphi_\varepsilon \leq ||\nabla\varphi||_\infty c_5^2\varepsilon^{-2} + ||\varphi||_\infty 2c_5^2\varepsilon^{-1}\varepsilon^{-2} + ||\nabla^2\varphi||_\infty c_5\varepsilon^{-1} + ||\nabla\varphi||_\infty c_5\varepsilon^{-2}. \tag{4.17}$$

From (4.15), (4.16) and (4.17), using $\gamma \leq 1$, we obtain

$$\mathrm{Lip}(\psi\varphi_\varepsilon(\cdot, M)) \leq ||\varphi||_{\mathcal{C}^2}\varepsilon^{-3}2(1+\gamma^{-1})\left(c_5^2 + c_5\right).$$

Finally, (4.14) yields,

$$\left|\int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)d||M|| - \int_{\mathbb{R}^n} \varphi_\varepsilon(\cdot, M)d||V_h||\right| \leq c_4||\varphi||_{\mathcal{C}^2}||M||(\mathbb{R}^n)\frac{h}{\varepsilon^3}$$

where $c_4 = 2(1+\gamma^{-1})\left(c_5^2 + c_5\right)$, this concludes the proof of (4.12).
We now prove (4.13). We start by estimating: $\left|H_\varepsilon(z, M) - H_\varepsilon(z, V_h)\right|$ on $\mathcal{M}^h$. To do so, let $z \in \mathcal{M}^h$ and $x \in \mathcal{M}$ be such that $|x - z| \leq h$, we have by [15, Lemma 4.4] with $B = B(x, \varepsilon + |x - z|)$ and using that ξ_ε is $\varepsilon^{-n-1}\,\mathrm{Lip}(\xi)$-Lipschitz,

$$\varepsilon^n||V_h|| * \xi_\varepsilon(z) + \varepsilon^{-1}\,\mathrm{Lip}(\xi)\left(\Delta_B(\|M\|, ||V_h||) + |x - z|||M||(B)\right) \geq \varepsilon^n\|M\| * \xi_\varepsilon(x).$$

From Lemma 1.3.3, we have for $\varphi \in C^{0,1}(B, \mathbb{R})$, $\left|\|M\|(\varphi) - ||V_h||(\varphi)\right| \leq h \operatorname*{Lip}_B(\varphi)\|M\|(B)$, hence

$$\Delta_B(\|M\|, ||V_h||) \leq h\|M\|(B),$$

so that, using (4.8) (with $V = M$) we obtain since $2h \leq \gamma\varepsilon$

$$\begin{aligned}
\varepsilon^n||V_h|| * \xi_\varepsilon(z) &\geq -\varepsilon^{-1}\,\mathrm{Lip}(\xi)\left(2h\|M\|(B)\right) + c_7\varepsilon^d \\
&\geq -\gamma\,\mathrm{Lip}(\xi)\|M\|(B) + \beta C_0^{-1}2^{-2d-1}\varepsilon^d \\
&\geq -\gamma\,\mathrm{Lip}(\xi)C_0(2\varepsilon)^d + \beta C_0^{-1}2^{-2d-1}\varepsilon^d \\
&\geq \varepsilon^d(-\gamma\,\mathrm{Lip}(\xi)C_0 2^d + \beta C_0^{-1}2^{-2d-1}).
\end{aligned}$$

Thus, as γ satisfies $\beta > \gamma C_0^2 2^{3d+1}\,\mathrm{Lip}(\xi)$, we obtain:

$$\varepsilon^n||V_h|| * \xi_\varepsilon(z) \geq c_9\varepsilon^d \tag{4.18}$$

where $c_9 = \beta - \gamma C_0^2 2^{3d+1}\,\mathrm{Lip}(\xi)$. Now we write

$$\begin{aligned}\big|H_\varepsilon(z,M) - H_\varepsilon(z,V_h)\big| &= \Big|\frac{\delta M * \rho_\varepsilon(z)}{\|M\| * \xi_\varepsilon(z)} - \frac{\delta V_h * \rho_\varepsilon(z)}{||V_h|| * \xi_\varepsilon(z)}\Big| \\ &\le |\delta M * \rho_\varepsilon(z)|\Big|\frac{1}{\|M\| * \xi_\varepsilon(z)} - \frac{1}{||V_h|| * \xi_\varepsilon(z)}\Big| + \frac{\big|\delta M * \rho_\varepsilon(z) - \delta V_h * \rho_\varepsilon(z)\big|}{||V_h|| * \xi_\varepsilon(z)}.\end{aligned}$$

On the one hand, we have from (1.6), (4.8) (with $V = M$), (4.18) and the Ahlfors property of $\mathcal{M}$

$$\begin{aligned}|\delta M * \rho_\varepsilon(z)|\Big|\frac{1}{\|M\| * \xi_\varepsilon(z)} - \frac{1}{||V_h|| * \xi_\varepsilon(z)}\Big| &= |\delta M * \rho_\varepsilon(z)|\Big|\frac{\|M\| * \xi_\varepsilon(z) - ||V_h|| * \xi_\varepsilon(z)}{\|M\| * \xi_\varepsilon(z)||V_h|| * \xi_\varepsilon(z)}\Big| \\ &\le \varepsilon^{-1}||\rho'||_\infty \|M\|(B)\frac{h||\xi'||_\infty \|M\|(B)}{\varepsilon^{2n}\|M\| * \xi_\varepsilon(z)||V_h|| * \xi_\varepsilon(z)} \\ &\le \frac{\varepsilon^{-1}||\rho'||_\infty||\xi'||_\infty C_0 2^d\varepsilon^d h C_0 2^d \varepsilon^d}{c_7\varepsilon^d c_9 \varepsilon^d} \\ &\le \frac{h}{\varepsilon}\frac{||\rho'||_\infty||\xi'||_\infty 2^{2d} C_0^2}{c_7 c_9}.\end{aligned}$$

On the other hand, we have from (1.7), if we denote by C_2 the Lipschitz constant of the map $x \in \mathcal{M} \mapsto T_x\mathcal{M}$, we have

$$\begin{aligned}\varepsilon^n\Big|\delta M * \rho_\varepsilon(z) - \delta V_h * \rho_\varepsilon(z)\Big| &= \Big|\int_{\mathbb{R}^n} S\left(\nabla\rho_\varepsilon(y-z)\right)\,dM(y,S) - \int_{\mathbb{R}^n} S\left(\nabla\rho_\varepsilon(y-z)\right)\,dV_h(y,S)\Big| \\ &\le \mathrm{Lip}(\Theta) h(1+2C_2)\|M\|(B)\end{aligned} \tag{4.19}$$

where $\Theta : \mathbb{R}^n \times G_{d,n} \to \mathbb{R}^n$, $(y,S) \mapsto S\left(\nabla\rho_\varepsilon(y-z)\right)$, we know that $\mathrm{Lip}(\Theta) \le \varepsilon^{-2}\|\rho\|_{\mathcal{C}^2}$, so that

$$\varepsilon^n\Big|\delta M * \rho_\varepsilon(z) - \delta V_h * \rho_\varepsilon(z)\Big| \le \varepsilon^{-2}\|\rho\|_{\mathcal{C}^2} h(1+2C_2)\|M\|(B).$$

Next, using (4.19), (4.18) and the Ahlfors property of $\mathcal{M}$

$$\begin{aligned}\frac{\big|\delta M * \rho_\varepsilon(z) - \delta V_h * \rho_\varepsilon(z)\big|}{||V_h|| * \xi_\varepsilon(z)} &\le \frac{\varepsilon^{-2}||\rho||_{\mathcal{C}^2} h(1+2C_2)\|M\|(B)}{\varepsilon^n||V_h|| * \xi_\varepsilon(z)} \\ &\le \frac{\varepsilon^{-2}||\rho||_{\mathcal{C}^2} h(1+2C_2)C_0 2^d\varepsilon^d}{c_9\varepsilon^d} \\ &\le \frac{h}{\varepsilon^2}\frac{2^d||\rho||_{\mathcal{C}^2}(1+2C_2)C_0}{c_9}\end{aligned}$$

Thus, for

$$c_{10} = \frac{||\rho'||_\infty||\xi'||_\infty 2^{2d}C_0^2}{c_7 c_9} + \frac{2^d||\rho||_{\mathcal{C}^2}(1+2C_2)C_0}{c_9}, \tag{4.20}$$

we have

$$\big|H_\varepsilon(z,M) - H_\varepsilon(z,V_h)\big| \le c_{10}\frac{h}{\varepsilon^2} \quad \forall z \in \mathcal{M}^h. \tag{4.21}$$

We now carry on with the proof of (4.13). We have from (4.21) and Lemma 4.0.8 with $V = M$

$$\begin{aligned} \Big||H_\varepsilon(z,M)|^2 - |H_\varepsilon(z,V_h)|^2\Big| &\leq c_{10}\frac{h}{\varepsilon^2}\Big||H_\varepsilon(z,M)| + |H_\varepsilon(z,V_h)|\Big| \\ &\leq c_{10}\frac{h}{\varepsilon^2}\Big|2|H_\varepsilon(z,M)| + c_{10}\frac{h}{\varepsilon^2}\Big| \\ &\leq c_{10}\frac{h}{\varepsilon^2}\left(2c_5\frac{1}{\varepsilon} + c_{10}\frac{h}{\varepsilon^2}\right). \end{aligned}$$

Recalling (4.11), we have

$$\begin{aligned} \big|\varphi_\varepsilon(z,M) - \varphi_\varepsilon(z,M)\big| &\leq \|\varphi\|_\infty\Big||H_\varepsilon(z,M)|^2 - |H_\varepsilon(z,V_h)|^2\Big| + \|\nabla\varphi\|_\infty\Big||H_\varepsilon(z,M)| - |H_\varepsilon(z,V_h)|\Big| \\ &\leq \|\varphi\|_\infty c_{10}\frac{h}{\varepsilon^2}\left(2c_5\frac{1}{\varepsilon} + c_{10}\frac{h}{\varepsilon^2}\right) + \|\nabla\varphi\|_\infty c_{10}\frac{h}{\varepsilon^2} \\ &\leq \|\varphi\|_{\mathcal{C}^1} c_{10}(2c_5 + c_{10} + 1)\frac{h}{\varepsilon^3} \quad \text{since } h \leq \varepsilon \leq 1. \end{aligned}$$

Finally, using $||V_h||(\mathbb{R}^n) = ||M||(\mathbb{R}^n)$, we obtain

$$\Big|\int_{\mathbb{R}^n}\varphi_\varepsilon(\cdot,M)d||V_h|| - \int_{\mathbb{R}^n}\varphi_\varepsilon(\cdot,V_h)d||V_h||\Big| \leq c_8||\varphi||_{\mathcal{C}^1}||M||(\mathbb{R}^n)\frac{h}{\varepsilon^3},$$

where

$$c_8 = c_{10}\left(2c_5 + c_{10} + 1\right), \tag{4.22}$$

this concludes the proof of (4.13). □

Proof of Theorem 4.0.1. Let $\mathcal{M}$ be a C^3 closed d-submanifold in $\mathbb{R}^n$ and $\mathcal{M}(t)_{t\in[0,T]}$ its mean curvature flow (Definition 2.1.1). Let $M(t)_{t\in[0,T]}$ be the varifolds associated with the family $\mathcal{M}(t)_{t\in[0,T]}$. Let $\mathcal{K}_h$ be a mesh of $\mathbb{R}^n$ of size $h \in (0,1)$ and $V_h(t)$ a discretization of $\mathcal{M}(t)$ for every $t \in [0,T]$ (Definition (1.5)).

The mean curvature flow is continuous in time with respect to the C^3-distance on the space of d-submanifold of $\mathbb{R}^n$ (see [21, Chapter 3]), thus, one can bound uniformly on time, thanks to the compactness of $[0,T]$, the constants C_0, C_2 and the maximal principal curvature (as they depend on the C^2-norm of the submanifold). The constant C_1 also evolves continuously with respect to to the C^3-distance on the space of d-submanifolds of $\mathbb{R}^n$, as a consequence, it also can be bounded uniformly on time (see Remark 4.0.5).

Let $\varepsilon, \gamma \in (0,1)$, assume $2h \leq \gamma\varepsilon$ and that γ fulfils the requirements in Remark 4.0.2. Using $||M(t)||(\mathbb{R}^n) \leq ||M(0)||(\mathbb{R}^n)$ (4.1) and Lemma 4.0.9 with $M = M(t)$, we have:

$$\begin{aligned} &\Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^n} -\varphi(z)|H_\varepsilon(z,M(t))|^2 + \nabla\varphi(z)\cdot H_\varepsilon(z,M(t))d||M(t)||dt \\ &- \int_{t_1}^{t_2}\int_{\mathbb{R}^n} -\varphi(z)|H_\varepsilon(z,V_h(t))|^2 + \nabla\varphi(z)\cdot H_\varepsilon(z,V_h(t))d||V_h(t)||dt\Big| \\ &\leq (c_4 + c_8)(t_2 - t_1)||\varphi||_{\mathcal{C}^2}||M(0)||(\mathbb{R}^n)\frac{h}{\varepsilon^3}, \end{aligned}$$

for all $0 \leq t_1 \leq t_2 \leq T$ and $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$. Combining the previous inequality with (4.5), (4.4) and $\big|\|V_h(t)(\varphi) - \|M\|(\varphi)\big| \leq \mathrm{Lip}(\varphi)\Delta(\|V_h(t)\|, \|M\|)$, we obtain:

$$\begin{aligned}
&\left| ||V_h(t_2)||(\varphi) - ||V_h(t_1)||(\varphi) + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi(x)|H_\varepsilon(x, V_h(t))|^2 - \nabla\varphi(x) \cdot H_\varepsilon(x, V_h(t)) d||V_h(t)||(x)dt \right| \\
&\leq 2\,\mathrm{Lip}(\varphi) \max_{t\in\{t_1,t_2\}} \Delta(M(t), V_h(t)) + ||\varphi||_{\mathcal{C}^1} \tilde{C}_1 \varepsilon(||M(t_1)||(\mathbb{R}^n) - ||M(t_2)||(\mathbb{R}^n)) \\
&+ ||\varphi||_{\mathcal{C}^1} \tilde{C}_1 \varepsilon\left((t_2 - t_1)\|M(0)\|(\mathbb{R}^n)\right) + (c_4 + c_8)(t_2 - t_1)||\varphi||_{\mathcal{C}^2}\|M(0)\|(\mathbb{R}^n)(\varepsilon + \frac{h}{\varepsilon^3}) \\
&\leq C\|\varphi\|_{\mathcal{C}^2} \left(\max_{t\in\{t_1,t_2\}} \Delta(M(t), V_h(t)) + \varepsilon(||M(t_1)||(\mathbb{R}^n) - ||M(t_2)||(\mathbb{R}^n)) + (t_2 - t_1)\|M(0)\|(\mathbb{R}^n)(\varepsilon + \frac{h}{\varepsilon^3}) \right),
\end{aligned} \tag{4.23}$$

for all $0 \leq t_1 \leq t_2 \leq T$ and $\varphi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, where $C = 2 + \tilde{C}_1 + c_4 + c_8$, and we conclude the proof of Theorem 4.0.1. □

List of constants used in the chapter

- C_0 The Ahlfors regularity constant of $\mathcal{M}$.
- C_1 Measures how well H_ε approximates H (Lemma 4.0.4).
- C_2 The Lipschitz constant of the map $y \mapsto T_y\mathcal{M}$.
- $\beta = \min\left\{\xi(s) \big| s \in [\frac{C_0^{-2/d}}{4}, \frac{1}{2}]\right\} > 0$.
- γ any constant satisfying:

  $$\gamma < \min\left\{(8(1 + C_0^{2/d}))^{-1}, \max_{x\in\mathcal{M}(t)} (\lambda(x,t))^{-1}, \beta\left(2^{3d} C_0^2(\mathrm{Lip}(\xi) + 1)\right)^{-1}\right\},$$

  $\lambda(x,t)$ is the maximal principal curvature of $\mathcal{M}(t)$ at x.
- $\tilde{C}_1 = 2C_1(2 + C_1)$ (4.5).
- $c_4 = 2(1 + \gamma^{-1})\left(c_5^2 + c_5\right)$.
- $c_5 = \beta^{-1} C_0^2 2^{3d+1} ||\rho||_{\mathcal{C}^2} \left(1 + \beta^{-1} C_0^2 2^{3d+2} ||\xi'||_\infty\right)$.
- $c_7 = \beta C_0^{-1} 2^{-2d-1}$ (4.8).
- $c_8 = c_{10}\left(2c_5 + c_{10} + 1\right)$ (4.22).
- $c_9 = \beta - \gamma C_0^2 2^{3d+1}\,\mathrm{Lip}(\xi)$ (4.18).
- $c_{10} = \dfrac{||\rho'||_\infty ||\xi'||_\infty 2^{2d} C_0^2}{c_7 c_9} + \dfrac{2^d ||\rho||_{\mathcal{C}^2}(1 + 2C_2)C_0}{c_9}$ (4.20).

- $C = 2 + \tilde{C}_1 + c_4 + c_8$ appears in the Brakke approximate inequality (Theorem 4.0.1).
- $C' = ||M(0)||(\mathbb{R}^n)(2 + C_1) + CT$ appears in the weak version of the Brakke approximate inequality (Remark 4.0.3).

Chapter 5

Approximate mean curvature flows of a general varifold, and their limit spacetime Brakke flow

This chapter is constituted of an article in preparation [14], in collaboration with B. Buet, G-P. Leonardi and S. Masnou. The aim of this work is to define a weak notion of mean curvature flow for general data. Starting from any varifold, we provide a notion of time-discrete approximate mean curvature flow depending on the time step and an approximation scale ε. By letting the time step tend to 0, we obtain an approximate mean curvature flow or, in other words and roughly speaking, a mean curvature flow with speed equal to the approximate mean curvature depending on ε. Furthermore we study the limit as $\varepsilon \to 0$ and we prove a few properties in relation with Brakke flows and the theory of mean curvature flow.

Organization of the chapter

We recall the definition at a point $x \in \mathbb{R}^n$ and at a scale $\varepsilon > 0$ of the approximate mean curvature vector of a varifold V defined on $\mathbb{R}^n$, see [36, Sec. 5] and Sec. 5.1:

$$h_\varepsilon(x, V) := -\left(\Phi_\varepsilon(\cdot) * \frac{(\Phi_\varepsilon * \delta V)(\cdot)}{\Phi_\varepsilon * \|V\|(\cdot) + \varepsilon} \right)(x)$$

where Φ_ε is a truncated Gaussian defined on $\mathbb{R}^n$ and δV is the first variation of the mass of V (see 1.12). Then we introduce a *time-discrete approximate mean curvature flow* with respect to the approximate mean curvature. We start from an initial varifold V_0, we let $\varepsilon > 0$ and $\mathcal{T} = \{t_i\}_{i=0}^m$ be a subdivision of the interval $[0,1]$, and we define by iterative push-forwards the time-discrete approximate mean curvature flow with parameters $V_0, \varepsilon, \mathcal{T}$:

$$\begin{cases} V(0) := V_0 \\ V(t_i) := (\mathrm{id} + (t_i - t_{i-1}) h_\varepsilon(\cdot, V(t_{i-1})))_\# V(t_{i-1}). \end{cases}$$

Since h_ε has a bounded C^2-norm and is continuous with respect to the bounded Lipschitz distance on varifolds, we prove that the time-discrete approximate mean curvature flow is stable with respect to the subdivision and the initial datum.

In Section 5.2, for a fixed scale ε and for any initial datum, we prove thanks to the stability property that the discrete approximate MCF converges as the subdivision's step tends to 0 to a limit flow that is *unique*, i.e. independent of the sequence of subdivisions. We call this limit flow the approximate mean curvature flow of V_0, and we denote it by $(V_\varepsilon(t))_{t\in[0,1]}$.
We prove that $(V_\varepsilon(t))_{t\in[0,1]}$ satisfies a Brakke equality for the approximate mean curvature vector. In view of the definition of generalized normal velocity given in [46, Definition 2.2], we can interpret this equality as: "the approximate MCF has a generalized normal velocity equal to $h_\varepsilon^\perp$". We deduce from the previous equality that the total mass is decreasing as the time increases, such a property is inherited from the mean curvature flow evolution.

In Section 5.3, we study the property of the limit flow when the smoothing scale ε goes to 0. We prove that $V_\varepsilon(t) \otimes dt$ converges to a limit (up to an extraction on ε) and that the limit satisfies a Brakke inequality with respect to its spacetime mean curvature (to be defined) if its $\mathbb{R}^n \times G_{d,n}$-component is assumed to be rectifiable.

In Section 5.4, we show a consistency result on the approximate mean curvature flow.

The appendix in Section 5.5 contains a few technical lemmas used throughout the chapter. 333

5.1 Definition and stability of a time-discrete approximate mean curvature flow

In this section, we will define a time-discrete approximate MCF starting from any varifold V with finite mass. The construction relies on iterated push-forwards of V by diffeomorphisms of the form $\mathrm{id} + \tau h_\varepsilon$, where τ is a given time step and h_ε is close to the mean curvature of V understood in a regularization of the distributional mean curvature of V. In Section 5.1.1, we start by recalling the definition (5.5) of the approximate mean curvature h_ε introduced in [36] after [12]. We then recall in Proposition 5.1.2 the C^2 estimates on the approximate mean curvature established in [12, 36]. Section 5.1.2 then investigates the effect of push-forwarding the varifold once: in Proposition 5.1.3, we evidence the relation (5.34) between the time step and ε that allows to iterate such push-forwards and leads to Definition 5.1.4 of time-discrete approximate mean curvature flow.

The authors in [36] worked on varifolds of co-dimension 1, here we work on varifolds of arbitrary co-dimension in $\mathbb{R}^n$.

5.1.1 Basic properties of the approximate mean curvature

Let $\psi \in C^\infty(\mathbb{R}^n)$ be a radially symmetric function such that:

$$\begin{aligned} &\psi(x) = 1 \text{ for } |x| \leq 1/2,\ \ \psi(x) = 0 \text{ for } |x| \geq 1, \\ &0 \leq \psi(x) \leq 1,\ \ |\nabla \psi(x)| \leq 3,\ \ \|\nabla^2 \psi(x)\| \leq 9 \text{ for all } x \in \mathbb{R}^n. \end{aligned} \tag{5.1}$$

Define for each $\varepsilon \in (0,1)$:

$$\hat{\Phi}_\varepsilon(x) := \frac{1}{(2\pi\varepsilon^2)^{\frac{n}{2}}} \exp\Big(-\frac{|x|^2}{2\varepsilon^2}\Big),\ \ \Phi_\varepsilon(x) := c(\varepsilon)\psi(x)\hat{\Phi}_\varepsilon(x), \tag{5.2}$$

we have $\int_{\mathbb{R}^n} \hat{\Phi}_\varepsilon(x)dx = 1$ and $c(\varepsilon) = \dfrac{1}{\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx}$, $c(\varepsilon)$ is chosen so that:

$$\int_{\mathbb{R}^n} \Phi_\varepsilon(x)\, dx = 1. \tag{5.3}$$

We have, using $\psi \leq 1$,

$$c(\varepsilon) = \frac{1}{\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx} \geq \frac{1}{\int_{\mathbb{R}^n} \hat{\Phi}_\varepsilon(x)dx} = 1.$$

Also, as $\psi = 1$ on $[0, \frac{1}{2}]$:

$$\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx \geq \int_{B(0,\frac{1}{2})} \hat{\Phi}_\varepsilon(x)dx$$

by the change of variables $y = \varepsilon^{-1}x$ we obtain

$$\int_{B(0,\frac{1}{2})} \hat{\Phi}_\varepsilon(x)dx = \int_{B(0,\frac{1}{2\varepsilon})} \hat{\Phi}_1(y)dy \geq \int_{B(0,\frac{1}{2})} \hat{\Phi}_1(y)dy =: c^{-1} \tag{5.4}$$

we have then, $1 \leq c(\varepsilon) \leq c$ where c is a constant depending only on n.
This kernel has a remarkable property which is that the derivatives are bounded by a power of ε times the kernel + an exponentially small term (see [36, Lemma 4.13 and Lemma 4.14]). This property is the key ingredient that make the computations in section 5 [36] work. We now define the approximate mean curvature vector for any varifold V as follows, for any $x \in \mathbb{R}^n$:

$$h_\varepsilon(x, V) = (\Phi_\varepsilon * \tilde{h}_\varepsilon(\cdot, V))(x)\,, \quad \text{where } \tilde{h}_\varepsilon(y, V) = -\frac{(\delta V * \Phi_\varepsilon)(y)}{(\|V\| * \Phi_\varepsilon)(y) + \varepsilon} \text{ for any } y \in \mathbb{R}^n. \tag{5.5}$$

The second convolution guarantees the decay of the mass (up to a small error) as we will see in (5.26) and later in remark 5.2.7, it also reduces computing $\|h_\varepsilon\|_{C^2}$ to computing $\|\tilde{h}_\varepsilon\|_\infty$ and $\|\Phi_\varepsilon\|_{C^2}$ avoiding to differentiate the fraction $\tilde{h}_\varepsilon$ (see Proposition 5.1.2 for details).
In the following lemma we list some of the properties of the kernel Φ_ε, the first two estimates simplify the estimates in [36, Lemma 4.13].

Lemma 5.1.1 (Kernel properties). *Let $\varepsilon \in (0, 1)$ and Φ_ε defined as in (5.2). There exists a constant c_0 depending only on n such that*

$$|\nabla\Phi_\varepsilon| \leq \varepsilon^{-2}\Phi_\varepsilon + c_0\chi_{B(0,1)}, \tag{5.6}$$

$$|\nabla^2\Phi_\varepsilon| \leq 2\varepsilon^{-4}\Phi_\varepsilon + 2c_0\chi_{B(0,1)}. \tag{5.7}$$

As a consequence,

$$\|\nabla\Phi_\varepsilon\|_{L^1} \leq (1 + \omega_n c_0)\varepsilon^{-2} \quad \text{and} \quad \|\nabla^2\Phi_\varepsilon\|_{L^1} \leq 2(1 + \omega_n c_0)\varepsilon^{-4}\,, \tag{5.8}$$

and

$$\mathrm{Lip}(\Phi_\varepsilon) \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\,, \quad \mathrm{Lip}(\nabla\Phi_\varepsilon) \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}. \tag{5.9}$$

Proof. Define

$$c_0 := \sup_{\varepsilon\in(0,1)} c(\varepsilon)\frac{9\varepsilon^{-2-n}}{(2\pi)^{n/2}}\exp\left(-\frac{1}{8\varepsilon^2}\right) < \infty. \tag{5.10}$$

By (5.2) for all $x \in \mathbb{R}^n$

$$\nabla\Phi_\varepsilon(x) = -\varepsilon^{-2}\Phi_\varepsilon(x)x + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x) \tag{5.11}$$

Φ_ε is supported on $B(0,1)$ therefore for all $x \in \mathbb{R}^n$

$$\left|\varepsilon^{-2}\Phi_\varepsilon(x)x\right| \le \varepsilon^{-2}\Phi_\varepsilon(x)$$

also by construction $\nabla\psi = 0$ on $[0,\frac{1}{2}] \cup [1,\infty)$ and $|\nabla\psi| \le 3$ on $[\frac{1}{2},1]$, this yields for all $x \in \mathbb{R}^n$

$$\begin{aligned}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| &\le 3c(\varepsilon)\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\sup_{\frac{1}{2}\le|x|\le 1}\exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B(0,1)}(x)\\ &\le 3c(\varepsilon)\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B(0,1)}(x) \le c_0\chi_{B(0,1)}(x)\end{aligned} \tag{5.12}$$

this proves (5.6). For (5.7), differentiating (5.11) gives for all $x \in \mathbb{R}^n$

$$\begin{aligned}\nabla^2\Phi_\varepsilon(x) &= -\varepsilon^{-2}x\otimes\left(-\varepsilon^{-2}\Phi_\varepsilon(x)x + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)\right) - \varepsilon^{-2}\Phi_\varepsilon(x)I_n\\ &\quad + c(\varepsilon)\left(\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x) - \hat{\Phi}_\varepsilon(x)\varepsilon^{-2}\nabla\psi(x)\ \otimes x\right)\\ &= \varepsilon^{-4}\Phi_\varepsilon(x)x\otimes x - 2\varepsilon^{-2}c(\varepsilon)\hat{\Phi}_\varepsilon(x)x\otimes\nabla\psi(x) - \varepsilon^{-2}\Phi_\varepsilon(x)I_n\\ &\quad + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x).\end{aligned}$$

We know that $\|v\otimes w\| \le \|v\|w\|$ for every two vectors v and w, the fact that Φ_ε and ψ are supported on $[0,1]$ implies, for all $x \in \mathbb{R}^n$

$$\begin{aligned}\|\nabla^2\Phi_\varepsilon(x)\| &\le \varepsilon^{-4}\Phi_\varepsilon(x) + 2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| + \Phi_\varepsilon(x) + \|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\|\\ &\le 2\varepsilon^{-4}\Phi_\varepsilon(x) + 2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| + \|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\|.\end{aligned}$$

Similarly to (5.12), we have for all $x \in \mathbb{R}^n$

$$\begin{aligned}2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| &\le c(\varepsilon)6\frac{\varepsilon^{-2-n}}{(2\pi)^{\frac{n}{2}}}\sup_{\frac{1}{2}\le|x|\le 1}\exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B(0,1)}(x)\\ &\le c(\varepsilon)6\frac{\varepsilon^{-2-n}}{(2\pi)^{\frac{n}{2}}}\exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B(0,1)}(x) \le c_0\chi_{B(0,1)}(x)\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\| &\le c(\varepsilon)9\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\sup_{\frac{1}{2}\le|x|\le 1}\exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B(0,1)}(x)\\ &\le c(\varepsilon)9\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B(0,1)}(x) \le c_0\chi_{B(0,1)}(x).\end{aligned} \tag{5.14}$$

Finally, (5.13) and (5.14) imply

$$\|\nabla^2\Phi_\varepsilon\| \leq 2\varepsilon^{-4}\Phi_\varepsilon + 2c_0\chi_{B(0,1)}$$

and this concludes the proof of (5.7). For the L^1-estimates, we write using (5.6)

$$\begin{aligned}\|\nabla\Phi\|_{L^1} = \int_{\mathbb{R}^n} |\nabla\Phi_\varepsilon(x)|dx &\leq \int_{\mathbb{R}^n} \left(\varepsilon^{-2}\Phi_\varepsilon(x) + c_0\chi_{B(0,1)}(x)\right) dx \\ &\leq \varepsilon^{-2}\int_{\mathbb{R}^n} \Phi_\varepsilon(x)\, dx + c_0 \int_{B(0,1)} 1 \; dx \\ &\leq \varepsilon^{-2} + \omega_n c_0 \leq (1+\omega_n c_0)\varepsilon^{-2}.\end{aligned} \tag{5.15}$$

Similarly, we prove that $\|\nabla^2\Phi\|_{L^1} \leq 2(1+c_0\omega_n)\varepsilon^{-4}$, and this finishes the proof of (5.8).
For the Lipschitz constant, we use the definition of Φ_ε (see (5.2) to get:

$$\mathrm{Lip}(\Phi_\varepsilon) \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\,, \quad \mathrm{Lip}(\nabla\Phi_\varepsilon) \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}$$

and this finishes the proof of (5.9) and Lemma 5.1.1. □

The following property is a mere adaptation of [36, Lemma 5.1], we bound the C^2-norm of the approximate mean curvature for $\varepsilon \in (0,1)$, here we impose no smallness requirement on ε contrarily to the original statement.

Proposition 5.1.2 (C^2 boundedness of h_ε)**.** *There exists a constant $c_1 \geq 2$ depending only on n with the following property. For any $\varepsilon \in (0,1)$ and $M \in [1,+\infty)$, if $V \in V_d(\mathbb{R}^n)$ is a d–varifold with total mass $\|V\|(\mathbb{R}^n) \leq M$, then*

$$\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty \leq c_1 M\varepsilon^{-2}, \quad \|h_\varepsilon(\cdot,V)\|_\infty \leq c_1 M\varepsilon^{-2}, \tag{5.16}$$

$$\|Dh_\varepsilon(\cdot,V)\|_\infty \leq c_1 M\varepsilon^{-4}, \tag{5.17}$$

$$\|D^2h_\varepsilon(\cdot,V)\|_\infty \leq c_1 M\varepsilon^{-6}. \tag{5.18}$$

Proof. Let $\varepsilon \in (0,1)$, $M \geq 1$ and let V be a d–varifold satisfying $\|V\|(\mathbb{R}^n) \leq M$.
We set $c_1 = 2(1+\omega_n c_0)(1+c_0)$ and we start with the proof of (5.16). By (1.17) and using that for all $S \in G_{d,n}$, $\|S\| \leq 1$, we have

$$|(\Phi_\varepsilon * \delta V)(x)| = \left|\int_{\mathbb{R}^n\times G_{d,n}} S(\nabla\Phi_\varepsilon(x-y))\, dV(y,S)\right| \leq \int_{\mathbb{R}^n} |\nabla\Phi_\varepsilon(x-y)|\, d\|V\|(y)\,.$$

Therefore, applying (5.6) and then (1.16) we obtain

$$|(\Phi_\varepsilon * \delta V)(x)| \leq \int_{\mathbb{R}^n} \left(\varepsilon^{-2}\Phi_\varepsilon(x-y) + c_0\chi_{B(0,1)}(x-y)\right) d\|V\|(y) \leq \varepsilon^{-2}(\Phi_\varepsilon * \|V\|)(x) + c_0 M\,. \tag{5.19}$$

It remains to write the definition (5.5) of $\tilde{h}_\varepsilon(\cdot,V)$ and apply (5.19) to infer

$$\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty = \sup_{x\in\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V)(x)|}{\Phi_\varepsilon * \|V\|(x) + \varepsilon} \leq \varepsilon^{-2} + \varepsilon^{-1}c_0 M \leq (1+c_0)M\varepsilon^{-2}\,. \tag{5.20}$$

We can now write definition (5.5) $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$ and use $\|\Phi_\varepsilon\|_{L^1} = 1$ and (5.20) to obtain

$$\|h_\varepsilon(\cdot, V)\|_\infty \leq \|\Phi_\varepsilon\|_{L^1} \, \|\tilde{h}_\varepsilon(\cdot, V)\|_\infty \leq (1 + c_0) M \varepsilon^{-2} \,, \tag{5.21}$$

and noting that $(1 + c_0) \leq c_1$ concludes the proof of (5.16).
We similarly have both $Dh_\varepsilon(\cdot, V) = \nabla\Phi_\varepsilon * \tilde{h}_\varepsilon$ and $D^2 h_\varepsilon(\cdot, V) = \nabla^2 \Phi_\varepsilon * \tilde{h}_\varepsilon$ so that applying (5.8) together with (5.20) concludes the proof of Proposition 5.1.2 as follows

$$\|Dh_\varepsilon(\cdot, V)\|_\infty \leq \|\nabla\Phi_\varepsilon\|_{L^1} \, \|\tilde{h}_\varepsilon(\cdot, V)\|_\infty \leq (1 + c_0\omega_n)(1 + c_0) M \, \varepsilon^{-4} \leq c_1 M \, \varepsilon^{-4} \,,$$
$$\|D^2 h_\varepsilon(\cdot, V)\|_\infty \leq \|\nabla^2\Phi_\varepsilon\|_{L^1} \, \|\tilde{h}_\varepsilon(\cdot, V)\|_\infty \leq 2(1 + c_0\omega_n)(1 + c_0) M \, \varepsilon^{-6} \leq c_1 M \, \varepsilon^{-6} \,.$$

□

5.1.2 Definition of the time-discrete approximate mean curvature flow

The goal of Section 5.1.2 is to define a time-discrete approximate MCF (see Definition 5.1.4) starting from an initial varifold $V_0 \in V_d(\mathbb{R}^n)$, for a given time discretization $\mathcal{T}$ and a regularization parameter ε. Such a definition relies on iterating push-forwards, starting with the initial varifold V_0, with velocity equal to the approximate mean curvature vector h_ε. To this end, we first investigate the effect of a single push-forward: in Proposition 5.1.3, we derive an expansion of the push-forward of the mass of a varifold under the map $f = \mathrm{id} + \Delta t \, h_\varepsilon$ and with respect to Δt. Computations rely on the Taylor expansion of the tangential Jacobian (see Lemma 5.5.3) and provide estimate (5.24). It is then possible to prove that the mass of the push-forward varifold decays up to a small error Δt (see (5.26)), hence allowing to iterate push-forwards for suitable time steps (see condition (5.34)) and resulting in Definition 5.1.4.

Given $\varepsilon \in (0, 1)$ and $V \in V_d(\mathbb{R}^n)$, we introduce the notation

$$f_{\varepsilon,V} = \mathrm{id} + \Delta t \, h_\varepsilon(\cdot, V) \,,$$

and depending on the context, we will possibly drop the ε or V index dependency.

Proposition 5.1.3. *Let $\varepsilon \in (0, 1)$, $M \geq 1$. Let $V \in V_d(\mathbb{R}^n)$ with $\|V\|(\mathbb{R}^n) \leq M$ and $S \in G_{d,n}$. For $\Delta t \geq 0$, if*

$$c_1 c_4 M \Delta t \leq \varepsilon^4 \tag{5.22}$$

then $f = \mathrm{id} + \Delta t \, h_\varepsilon(\cdot, V)$ is a diffeomorphism, and

$$J_S f \in \left[\frac{1}{2}, \frac{2}{3}\right] \cap \left[1 - c_4 \Delta t \|Dh_\varepsilon\|_\infty, 1 + c_4 \Delta t \|Dh_\varepsilon\|_\infty\right]. \tag{5.23}$$

Furthermore, let $c_5 = 4c_1^2 c_4$, then for any $\varphi \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}_+)$,

$$\left| \|f_\# V\|(\varphi) - \|V\|(\varphi) - \Delta t \, \delta(V, \varphi)(h_\varepsilon(\cdot, V)) \right| \leq c_5 M^3 \|\varphi\|_{\mathrm{C}^2} (\Delta t)^2 \varepsilon^{-8}, \tag{5.24}$$

and

$$\delta V(h_\varepsilon(\cdot, V)) = - \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V)(y)|^2}{(\Phi_\varepsilon * \|V\|)(y) + \varepsilon} dy \leq 0 \,. \tag{5.25}$$

Assume that Δt satisfies $c_5 \Delta t M^3 \leq \varepsilon^8$ then, f is a diffeomorphism and

$$\|f_\# V\|(\mathbb{R}^n) \leq \|V\|(\mathbb{R}^n) + \Delta t. \tag{5.26}$$

Proof. Let $\varepsilon \in (0,1)$, $\Delta t \geq 0$, $M \geq 1$ and $V \in V_d(\mathbb{R}^n)$ satisfying $\|V\|(\mathbb{R}^n) \leq M$. As V is fixed, we write h_ε for $h_\varepsilon(\cdot, V)$ hereafter as well as $f = \mathrm{id} + \Delta t\, h_\varepsilon$.
Step 1**:** We first prove that f is a diffeomorphism under the condition (5.22). To do so, we only need to check the hypothesis of Lemma 5.5.6 with $h = h_\varepsilon$. From (5.16) and (5.22) we can infer that

$$\Delta t \|h_\varepsilon\|_\infty \leq c_1 M \varepsilon^{-2} \leq \frac{1}{2c_4} < 1.$$

From (5.17) and (5.22)

$$\Delta t \, \|Dh_\varepsilon\|_\infty \leq c_1 M \varepsilon^{-4} \leq \frac{1}{2c_4} < 1, \tag{5.27}$$

and we can then apply (5.143) (with $k = n$ and $Q = \Delta t\, Dh_\varepsilon$) we infer by (1.1), (5.22) and (5.17)

$$|Jf(x) - 1| = |\det(I_n + \Delta t\, Dh_\varepsilon(x)) - \det(I_n)| \leq c_2 \Delta t |Dh_\varepsilon(x)|_\infty \leq \frac{c_2}{2c_4} < 1$$

for any $x \in \mathbb{R}^n$ (using $c_2 \leq c_4$). By Lemma 5.5.6 f is a diffeomorphism of $\mathbb{R}^n$.
Step 2**:** Let $(x, S) \in \mathbb{R}^n \times G_{d,n}$, we now prove (5.23) and

$$|J_S f(x) - 1 - \Delta t \ \mathrm{div}_S(h_\varepsilon(x))| \leq c_4 (\Delta t \|Dh_\varepsilon\|_\infty)^2. \tag{5.28}$$

Let us write $\tilde{S} = (\tau_1 | \ldots | \tau_d)^t \in \mathcal{M}_{d,n}$ where $\{\tau_i\}_{i=1}^d$ is an orthonormal basis of S. We recall that we denote by S the orthogonal projector onto the subspace S and then by construction,

$$\tilde{S} \circ \tilde{S}^t = I_d \in \mathcal{M}_d \quad \text{and} \quad \tilde{S}^t \circ \tilde{S} = S \in \mathcal{M}_n \, .$$

We recall that by definition of tangential Jacobian (1.9),

$$J_S f(x) = \det \Big(((I_n + \Delta t\, Dh_\varepsilon(x)) \circ \tilde{S}^t)^t \circ ((I_n + \Delta t\, Dh_\varepsilon(x)) \circ \tilde{S}^t) \Big)^{\frac{1}{2}}$$

and we can apply (5.145) with $R = \Delta t\, Dh_\varepsilon(x)$ and $L = \tilde{S}$, indeed, $c_3 |R|_\infty \leq \frac{c_3}{2c_4} \leq 1$ thanks to (5.27). We obtain, again using (5.27),

$$|J_S f(x) - 1| \leq c_4 \Delta t |Dh_\varepsilon(x)|_\infty \leq c_4 \Delta t \|Dh_\varepsilon\|_\infty \leq \frac{1}{2}\, , \tag{5.29}$$

hence proving (5.23).
Similarly to the proof of (5.23), we are allowed to use (5.146) with $R = \Delta t\, Dh_\varepsilon(x)$ and $L = \tilde{S}$. Noting that

$$\mathrm{tr}\Big(Dh_\varepsilon(x) \circ \tilde{S}^t \circ \tilde{S} \Big) = \mathrm{tr}(Dh_\varepsilon(x) \circ S) = \mathrm{div}_S(h_\varepsilon(x))$$

we can infer that

$$|J_S f(x) - 1 - \Delta t \ \mathrm{div}_S(h_\varepsilon(x))| \leq c_4 (\Delta t\, |Dh_\varepsilon(x)|_\infty)^2 \leq c_4 (\Delta t \|Dh_\varepsilon\|_\infty)^2 \, .$$

Step 3**:** We now prove (5.24). Let $\varphi \in C^2(\mathbb{R}^n, \mathbb{R}^+)$ and assume $\|\varphi\|_{\mathrm{C}^2} < \infty$ (otherwise there is nothing to prove). Coming back to the definitions of push-forward varifold (Definition 1.4.1) and weighted first variation (1.13), we have

$$\begin{aligned} &\|f_\# V\|(\varphi) - \|V\|(\varphi) - \Delta t\, \delta(V, \varphi)(h_\varepsilon) \\ &\quad = \int_{\mathbb{R}^n \times G_{d,n}} \varphi(f(x)) J_S f(x) - \varphi(x) - \Delta t\, \varphi(x)\, \mathrm{div}_S(h_\varepsilon(x)) - \Delta t\, \nabla \varphi(x) \cdot h_\varepsilon(x)\, dV(x, S). \end{aligned} \tag{5.30}$$

Let $(x,S) \in \mathbb{R}^n \times G_{d,n}$. We first recall that $f(x) - x = \Delta t\, h_\varepsilon(x)$ so that

$$|f(x) - x| \le \Delta t\, \|h_\varepsilon\|_\infty \le c_1 \Delta t M \varepsilon^{-2}$$

thanks to (5.16). We can then apply Taylor's inequality to φ between x and $f(x)$ to obtain

$$|\varphi(f(x)) - \varphi(x)| \le |f(x) - x| \|\nabla\varphi\|_\infty \le c_1 M \|\varphi\|_{\mathrm{C}^2} \Delta t\, \varepsilon^{-2} \tag{5.31}$$

and

$$\begin{aligned} |\varphi(f(x)) - \varphi(x) - \Delta t\, h_\varepsilon(x) \cdot \nabla\varphi(x)| &= |\varphi(f(x)) - \varphi(x) - (f(x) - x) \cdot \nabla\varphi(x)| \\ &\le \frac{1}{2}|f(x) - x|^2 \|\nabla^2\varphi\|_\infty \le \frac{c_1^2}{2} M^2 \|\varphi\|_{\mathrm{C}^2} \Delta t^2\, \varepsilon^{-4} . \end{aligned} \tag{5.32}$$

Now rewriting the integrand in the right-hand side of (5.30) and using (5.31), (5.29),(5.28), (5.32) and Proposition 5.1.2 we have

$$\begin{aligned} |\varphi(f(x)) J_S f(x) \; &-\varphi(x) - \Delta t\, \varphi(x) \operatorname{div}_S(h_\varepsilon(x)) - \Delta t\, \nabla\varphi(x) \cdot h_\varepsilon(x)| \\ &\le |\varphi(f(x)) - \varphi(x)|\, |J_S f(x) - 1| + \varphi(x)\, |J_S f(x) - 1 - \Delta t\, \operatorname{div}_S(h_\varepsilon(x))| \\ &\quad + |\varphi(f(x)) - \varphi(x) - \Delta t\, h_\varepsilon(x) \cdot \nabla\varphi(x)| \\ &\le c_1 M \|\varphi\|_{\mathrm{C}^2} \Delta t\, \varepsilon^{-2}\, c_4 c_1 M \Delta t\, \varepsilon^{-4} + \|\varphi\|_\infty c_4 (c_1 M \Delta t\, \varepsilon^{-4})^2 + \frac{c_1^2}{2} M^2 \|\varphi\|_{\mathrm{C}^2} \Delta t^2\, \varepsilon^{-4} \\ &\le 3 c_1^2 c_4 \|\varphi\|_{\mathrm{C}^2} M^2 \Delta t^2\, \varepsilon^{-8} \le c_5 \|\varphi\|_{\mathrm{C}^2} M^2 \Delta t^2\, \varepsilon^{-8} \end{aligned}$$

and integrating the previous inequality together with (5.30) leads to (5.24).

Step 4**:** By definition (5.5), $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$ and thus, for all $S \in G_{d,n}$, $\operatorname{div}_S(h_\varepsilon) = S(\nabla\Phi_\varepsilon) * \tilde{h}_\varepsilon$. Then, by definition of δV and (1.17),

$$\begin{aligned} \delta V(h_\varepsilon) &= \int_{\mathbb{R}^n \times G_{d,n}} \operatorname{div}_S(h_\varepsilon(x))\, dV(x,S) = \int_{\mathbb{R}^n \times G_{d,n}} \int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon(y - x)) \cdot \tilde{h}_\varepsilon(y)\, dy\, dV(x,S) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \times G_{d,n}} S(\nabla\Phi_\varepsilon(y - x))\, dV(x,S) \cdot \tilde{h}_\varepsilon(y)\, dy \\ &= \int_{\mathbb{R}^n} (\Phi_\varepsilon * \delta V)(y) \cdot \tilde{h}_\varepsilon(y)\, dy = -\int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V)(y)|^2}{(\Phi_\varepsilon * \|V\|)(y) + \varepsilon}\, dy \le 0. \end{aligned}$$

We are left with the proof of (5.26). We recall that $c_5 = 4 c_1^2 c_4$ and we assume that Δt satisfies $c_5 \Delta t \le M^{-3} \varepsilon^8$ then Δt in particular satisfies (5.22), assumption under which the map f is a diffeomorphism of $\mathbb{R}^n$ and (5.28) holds. Consequently, applying Definition 1.4.1 of push-forward varifold and using (5.28), (5.25), and (5.17), we obtain

$$\begin{aligned} \|f_\# V\|(\mathbb{R}^n) &= \int_{\mathbb{R}^n \times G_{d,n}} J_S f(x)\, dV(x,S) \\ &= \int_{\mathbb{R}^n \times G_{d,n}} 1 + \Delta t\, \operatorname{div}_S(h_\varepsilon(x)) + (J_S f(x) - 1 - \Delta t\, \operatorname{div}_S(h_\varepsilon(x)))\, dV(x,S) \\ &\le \|V\|(\mathbb{R}^n) + \Delta t\, \delta V(h_\varepsilon) + c_4 M (\Delta t \|D h_\varepsilon\|_\infty)^2 \\ &\le \|V\|(\mathbb{R}^n) + M^3 c_1^2 c_4 \Delta t^2 \varepsilon^{-8} \\ &\le \|V\|(\mathbb{R}^n) + \Delta t \end{aligned}$$

hence concluding the proof of (5.26). □

Given $M \geq 1$ and a d-varifold V_0 satisfying $\|V_0\|(\mathbb{R}^n) \leq M$, Proposition 5.1.3 gives the condition $c_5 \Delta t < M^{-3} \varepsilon^8$ allowing to define $V_1 = f_{0\#} V_0$ with $f_0 = f_{\varepsilon, V_0} = \mathrm{id} + \Delta t\, h_\varepsilon(\cdot, V_0)$. We would like to iterate on several time steps and thus push the varifold V_1 by the map $f_1 = f_{\varepsilon, V_1} = \mathrm{id} + \Delta t h_\varepsilon(\cdot, V_1)$. However, note that $\|f_{0\#} V_0\|(\mathbb{R}^n) \leq M + \Delta t$ and not necessarily $\|f_{0\#} V_0\|(\mathbb{R}^n) \leq M$: the choice of Δt is no longer suitable. To rule out this issue, we can initially choose Δt satisfying

$$c_5 \Delta t < (M+1)^{-3} \varepsilon^8. \tag{5.33}$$

(5.26) thus ensuring $\|V_1\|(\mathbb{R}^n) = \|f_{0\#} V_0\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) + \Delta t \leq M + 1$, and we can iterate the process as long as the mass remains less than $M+1$, thus at least $\lfloor 1/\Delta t \rfloor$ times when considering uniform time discretizations of $[0,1]$. Considering a possibly non uniform time discretization $(\Delta t_i)_{i=1\ldots m} \in (0,1)$ of $[0,a]$ for $a \leq 1$: $\sum_{i=1}^{m} \Delta t_i = a \leq 1$, one can iterate the process m times with Δt_i being the time step at step i, this justifies the following definition.

Definition 5.1.4 (Time-discrete approximate MCF)**.** *Let $M \geq 1$, $\varepsilon \in (0,1)$ and $a \in (0,1]$. Consider a subdivision $\mathcal{T} = \{t_i\}_{i=0}^m$ of $[0,a]$ (see Definition 1.1.1) and assume*

$$c_5 \delta(\mathcal{T}) \leq (M+1)^{-3} \varepsilon^8 \tag{5.34}$$

where $\Delta t_i = t_i - t_{i-1}$ for $i = 1, \ldots, m$ and $\delta(\mathcal{T}) = \max_{1 \leq i \leq m} \Delta t_i$.
Let $V_0 \in V_d(\mathbb{R}^n)$ satisfy $\|V_0\|(\mathbb{R}^n) \leq M$. Define $(V_{\varepsilon,\mathcal{T}}(t_i))_{i=0\ldots m}$ by $V_{\varepsilon,\mathcal{T}}(0) := V_0$ ($t_0 = 0$) and, for $i = 1, \ldots, m$,

$$V_{\varepsilon,\mathcal{T}}(t_i) := f_{i\#} V_{\varepsilon,\mathcal{T}}(t_{i-1}) \quad \textit{with} \quad f_i = \mathrm{id} + \Delta t_i\, h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_{i-1})) \,.$$

We then define the family $(V_{\varepsilon,\mathcal{T}}(t))_{t \in [0,a]}$ by linear interpolation between the points of the subdivision, and we call it a time-discrete approximate MCF*:*

$$V_{\varepsilon,\mathcal{T}}(t) := \left[\mathrm{id} + (t - t_i) h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_i))\right]_\# V_{\varepsilon,\mathcal{T}}(t_i) \quad \textit{if} \quad t \in [t_i, t_{i+1}].$$

Remark 5.1.5. We note that under the assumptions of Definition 5.1.4 (and using the same notations), we have

$$\|V_{\varepsilon,\mathcal{T}}(t)\|(\mathbb{R}^n) \leq M + 1, \quad \forall t \in [0,a] \,, \tag{5.35}$$

and we will use (5.35) extensively throughout the chapter.
Moreover, if we assume that there exists $R_0 > 0$ such that $\operatorname{spt} V_0 \subset B(0, R_0) \times G_{d,n}$, then

$$\forall t \in [0,a], \quad \operatorname{spt} V_{\varepsilon,\mathcal{T}} \subset B(0, R_0 + c_1(M+1)\varepsilon^{-2}) \times G_{d,n} \,.$$

Indeed, thanks to Proposition 5.1.2 and (5.35), for $t \in [0,a] \subset [0,1]$,

$$\|h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t))\|_\infty \leq c_1(M+1)\varepsilon^{-2}$$

and therefore, $\operatorname{spt} V_{\varepsilon,\mathcal{T}}(t) \subset B\left(0, R_0 + c_1\, t\, (M+1)\varepsilon^{-2}\right) \times G_{d,n}$. Such compactness property will be used when letting Δt go to 0 to define a limit flow in Section 5.2.

Remark 5.1.6 (Piecewise constant flow)**.** Note that in Definition 5.1.4, we first define $V_{\varepsilon,\mathcal{T}}(t_i)$ at the points t_i of the subdivision $\mathcal{T}$ and we then define $V_{\varepsilon,\mathcal{T}}(t)$ for $t \in [t_i, t_{i+1}]$ by a linear interpolation between t_i and t_{i+1}. It is possible to consider an alternative definition of the flow between t_i and t_{i+1}, simply taking the following piecewise constant extension: for $i \in \{0, 1, \ldots, m-1\}$,

$$V_{\varepsilon,\mathcal{T}}^{pc}(t) := V_{\varepsilon,\mathcal{T}}(t_i) \quad \text{if} \quad t \in (t_i, t_{i+1}) \,.$$

As we will see in Proposition 5.2.2, both $V_{\varepsilon,\mathcal{T}}$ and $V^{pc}_{\varepsilon,\mathcal{T}}$ lead to the same limit flow V_ε when the size of the subdivision tends to zero. We consequently restrict our study to only one of the two flows and we choose to investigate $V_{\varepsilon,\mathcal{T}}$ introduced in Definition 5.1.4.

Hereafter, $M \geq 1$ and $\varepsilon \in (0,1)$ are fixed, all subdivisions we consider satisfy (5.34) and we define time-discrete approximate MCF starting from a varifold of mass less than M.

5.1.3 Stability of the time-discrete approximate MCF with respect to the initial datum

When investigating a discrete scheme of a flow, the stability arises as a crucial issue. More precisely, we consider in Proposition 5.1.9 two time-discrete approximate MCF $(V(t))_t$ and $(W(t))_t$ respectively from V_0 and W_0 and we prove that the stability holds in terms of bounded Lipschitz distance: $\Delta(V(t),W(t)) \leq \exp(\Lambda t)\Delta(V_0,W_0)$, where $\Lambda \sim \varepsilon^{-n-7}$. Up to a constant, Λ is an upper bound of the Lipschitz constant of $V \mapsto h_\varepsilon(\cdot,V)$ with respect to the C^1–norm, as established in Lemma 5.1.7. In Remark 5.1.10, we draw a parallel with the classical time discretization of ODEs showing that Λ is the expected constant in our setting.

Lemma 5.1.7. *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V and W be two varifolds of $V_d(\mathbb{R}^n)$ satisfying $\|V\|(\mathbb{R}^n) \leq M$, $\|W\|(\mathbb{R}^n) \leq M$. There exists $c_6 \geq 4c_1$ only depending on n such that*

$$\left\|h_\varepsilon(\cdot,V) - h_\varepsilon(\cdot,W)\right\|_\infty \leq c_6 M\varepsilon^{-n-5}\Delta(V,W) \text{ and } \left\|Dh_\varepsilon(\cdot,V) - Dh_\varepsilon(\cdot,W)\right\|_\infty \leq c_6 M\varepsilon^{-n-7}\Delta(V,W).$$

Proof. We set

$$c_6 = \max\{4c_1, (1+c_0\omega_n)(c(2\pi)^{-n/2}+c_0)(2+c_1)\} \tag{5.36}$$

thus $c_6 \geq 4c_1$. Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V and W be two varifolds of $V_d(\mathbb{R}^n)$ satisfying $\|V\|(\mathbb{R}^n) \leq M$, $\|W\|(\mathbb{R}^n) \leq M$. We first show that

$$\begin{aligned} &\|\Phi_\varepsilon * \|V\| - \Phi_\varepsilon * \|W\|\|_\infty \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\Delta(V,W), \text{ and} \\ &\|\Phi_\varepsilon * \delta V - \Phi_\varepsilon * \delta W\|_\infty \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}\Delta(V,W)\,. \end{aligned} \tag{5.37}$$

We have for any $x \in \mathbb{R}^n$ by (5.9) and the definition (5.2) of Φ_ε

$$\begin{aligned} \Big|\Phi_\varepsilon * \|V\|(x) - \Phi_\varepsilon * \|W\|(x)\Big| &= \Big|\int_{\mathbb{R}^n}\Phi_\varepsilon(x-y)d\|V\|(y) - \int_{\mathbb{R}^n}\Phi_\varepsilon(x-y)d\|W\|(y)\Big| \\ &= \Big|\|V\|\left(\Phi_\varepsilon(\cdot - x)\right) - \|W\|\left(\Phi_\varepsilon(\cdot - x)\right)\Big| \\ &\leq \max\{\|\Phi_\varepsilon\|_\infty, \mathrm{Lip}(\Phi_\varepsilon)\}\Delta(\|V\|,\|W\|) \\ &\leq \max\{c(2\pi)^{-\frac{n}{2}}\varepsilon^{-n}, (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\}\Delta(\|V\|,\|W\|) \\ &\leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\Delta(V,W)\,, \end{aligned}$$

since $\Delta(\|V\|,\|W\|) \leq \Delta(V,W)$, this gives the first estimate of (5.37). For the second estimate we first recall that for $x \in \mathbb{R}^n$, $\Phi_\varepsilon * \delta V(x) = \displaystyle\int_{\mathbb{R}^n \times G_{d,n}} S\nabla\Phi_\varepsilon(x-y)\, dV(y,S)$ and we thus compute the Lipschitz constant of the map $\Theta : (y,S) \mapsto S(\nabla\Phi_\varepsilon(y))$ (the map $y \mapsto x-y$ being an isometry), we have for $(y,S),(z,T)\in\mathbb{R}^n\times G_{d,n}$, using $\|S\|=1$

$$\begin{aligned}|\Theta(y,S)-\Theta(y,T)| &= |S(\nabla\Phi_\varepsilon(y))-T(\nabla\Phi_\varepsilon(z))|\\ &\le |S(\nabla\Phi_\varepsilon(y))-S(\nabla\Phi_\varepsilon(z))|+|S(\nabla\Phi_\varepsilon(z))-T(\nabla\Phi_\varepsilon(z))|\\ &\le \|S\|\,|\nabla\Phi_\varepsilon(y)-\nabla\Phi_\varepsilon(z)|+\|S-T\|\,|\nabla\Phi_\varepsilon(z)|\\ &\le \mathrm{Lip}(\nabla\Phi_\varepsilon)|y-z|+\|\nabla\Phi_\varepsilon\|_\infty\|S-T\|\\ &\le 2(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-4}\quad \text{thanks to Lemma 5.1.1.}\end{aligned}$$

Therefore $\mathrm{Lip}(\Theta)\le 2(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-4}$, also from (5.6) we have $\|\Theta\|_\infty\le(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-2}$. We can now carry on with the proof the the second part of (5.37), for $x\in\mathbb{R}^n$,

$$\begin{aligned}\Big|\Phi_\varepsilon*\delta V(x)-\Phi_\varepsilon*\delta W(x)\Big| &= \Big|\int_{\mathbb{R}^n}S(\nabla\Phi_\varepsilon)(x-y)dV(y,S)-\int_{\mathbb{R}^n}S(\nabla\Phi_\varepsilon)(x-y)dW(y,S)\Big|\\ &\le \max\{\|\Theta\|_\infty,\mathrm{Lip}(\Theta)\}\Delta(V,W)\\ &\le 2(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-4}\Delta(V,W)\,,\end{aligned}$$

which gives the desired result. We carry on with the proof of Lemma 5.1.7. Let $x\in\mathbb{R}^n$, from (5.37), (5.16), we have

$$\begin{aligned}\Big|\tilde h_\varepsilon(x,V)-\tilde h_\varepsilon(x,W)\Big| &= \Big|\frac{\Phi_\varepsilon*\delta V(x)}{\Phi_\varepsilon*\|V\|(x)+\varepsilon}-\frac{\Phi_\varepsilon*\delta W(x)}{\Phi_\varepsilon*\|W\|(x)+\varepsilon}\Big|\\ &\le \frac{|\Phi_\varepsilon*\delta V(x)-\Phi_\varepsilon*\delta W(x)|}{\Phi_\varepsilon*\|V\|(x)+\varepsilon}+\Big|\frac{\Phi_\varepsilon*\delta W(x)}{\Phi_\varepsilon*\|W\|(x)+\varepsilon}\Big|\frac{\big|\Phi_\varepsilon*\|V\|(x)-\Phi_\varepsilon*\|W\|(x)\big|}{\Phi_\varepsilon*\|V\|(x)+\varepsilon}\\ &\le \frac{1}{\varepsilon}\|\Phi_\varepsilon*\delta V-\Phi_\varepsilon*\delta W\|_\infty+\|\tilde h_\varepsilon(\cdot,W)\|_\infty\frac{1}{\varepsilon}\,\|\Phi_\varepsilon*\|V\|-\Phi_\varepsilon*\|W\|\|_\infty\\ &\le \frac{1}{\varepsilon}\left(2(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-4}+c_1M\varepsilon^{-2}(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-2}\right)\Delta(V,W)\\ &\le (c(2\pi)^{-\frac{n}{2}}+c_0)\,(2+c_1M)\,\varepsilon^{-n-5}\Delta(V,W) \qquad (5.38)\\ &\le c_6M\varepsilon^{-n-5}\Delta(V,W)\,.\end{aligned}$$

We recall that $h_\varepsilon=\Phi_\varepsilon*\tilde h_\varepsilon$ and we obtain thanks to (5.3) and (5.38):

$$\|h_\varepsilon(\cdot,V)-h_\varepsilon(\cdot,W)\|_\infty\le\|\Phi_\varepsilon\|_{L^1}\|\tilde h_\varepsilon(\cdot,V)-\tilde h_\varepsilon(\cdot,W)\|_\infty\le c_6M\varepsilon^{-n-5}\Delta(V,W)\,.$$

Similarly $Dh_\varepsilon=\nabla\Phi_\varepsilon*\tilde h_\varepsilon$ and using (5.8) and (5.38), we obtain

$$\begin{aligned}\|Dh_\varepsilon(\cdot,V)-Dh_\varepsilon(\cdot,W)\|_\infty &\le \|\nabla\Phi_\varepsilon\|_{L^1}\|\tilde h_\varepsilon(\cdot,V)-\tilde h_\varepsilon(\cdot,W)\|_\infty\\ &\le \varepsilon^{-2}(1+c_0\omega_n)\,(c(2\pi)^{-n/2}+c_0)(2+c_1M)\,\Delta(V,W)\varepsilon^{-n-5}\\ &\le c_6M\varepsilon^{-n-7}\Delta(V,W),\end{aligned}$$

hence concluding the proof. □

In Proposition 5.1.8, we investigate the evolution of the bounded Lipschitz distance between two varifolds V and W through one step of the time-discrete approximate flow introduced in Definition 5.1.4. The proof relies on the Lemma 5.5.5, Lemma 5.1.7 and on careful estimates of the Lipschitz constant of the map $(x,S,V)\in\mathbb{R}^n\times G_{d,n}\times V_d(\mathbb{R}^n)\mapsto J_Sf_{\varepsilon,V}(x)$.

Proposition 5.1.8. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let $V \in V_d(\mathbb{R}^n)$ with $\|V\|(\mathbb{R}^n) \leq M$. For $\Delta t \geq 0$, such that*

$$c_5 \Delta t M^3 \leq \varepsilon^8. \tag{5.39}$$

We recall the notation

$$f_{\varepsilon,V} := \mathrm{Id} + \Delta t\, h_\varepsilon(\cdot, V).$$

Let $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\|Dg - I_n\|_\infty \leq 2c_1 \Delta t M \varepsilon^{-4}$. Let $c_7 = 6(128 n c_2 c_3 c_6 + c_1 c_4)$. Then, for any $(x,S),(y,T) \in \mathbb{R}^n \times G_{d,n}$

$$\left| J_S f_{\varepsilon,V}(x) - J_T g(y) \right| \leq c_7 \left(\Delta t M \varepsilon^{-4} \|S - T\| + \Delta t M \varepsilon^{-6} |x - y| + \|D f_{\varepsilon,V} - Dg\|_\infty \right). \tag{5.40}$$

Let $W \in V_d(\mathbb{R})$, $\|W\|(\mathbb{R}^n) \leq M$, we have

$$\left| J_S f_{\varepsilon,V}(x) - J_T f_{\varepsilon,W}(y) \right| \leq c_7 \Delta t M \left(\varepsilon^{-4} \|S - T\| + \varepsilon^{-6} |x - y| + \varepsilon^{-n-7} \Delta(V,W) \right). \tag{5.41}$$

Moreover, we have

$$\Delta((f_{\varepsilon,V})_\# V, g_\# V) \leq \|V\|(\mathbb{R}^n) \left(28 \|f_{\varepsilon,V} - g\|_{C^1} + \|J.f_{\varepsilon,V} - J.g\|_\infty \right), \tag{5.42}$$

and,

$$\Delta\left((f_{\varepsilon,V})_\# V, (f_{\varepsilon,W})_\# W \right) \leq (1 + c_7 \Delta t M^2 \varepsilon^{-n-7}) \Delta(V,W). \tag{5.43}$$

Proof. As previously, ε is fixed throughout the proof and consequently, we can write f_V (resp. f_W) instead of $f_{\varepsilon,V}$ (resp. $f_{\varepsilon,W}$). We recall that for $S,T \in G_{d,n}$ we choose $\tilde{S}, \tilde{T}$ as in Lemma 5.5.5 such that that $\|\tilde{S} - \tilde{T}\| \leq 2\|S - T\|$. In the proof, we use extensively the formulas

$$\tilde{S} \circ \tilde{S}^t = \tilde{T} \circ \tilde{T}^t = I_d, \ \ \|\tilde{S}\| = \|\tilde{T}\| = 1, \ \text{ and } \|A\| = \|A^t\| \text{ for any matrix } A.$$

We define $G := Dg - I_n$ and recall that by hypothesis one has

$$\|G\|_\infty \leq 2 c_1 M \Delta t \varepsilon^{-4}. \tag{5.44}$$

Step 1: We prove

$$\begin{aligned} &\|\tilde{S} \circ D f_{\varepsilon,V}(x)^t \circ D f_{\varepsilon,V}(x) \circ \tilde{S}^t - \tilde{T} \circ Dg(y)^t \circ Dg(y) \circ \tilde{T}^t\| \\ &\leq 4 \left(8 c_1 M \Delta t \varepsilon^{-4} \|S - T\| + c_1 M \Delta t \varepsilon^{-6} |x - y| + \|D f_{\varepsilon,V} - Dg\|_\infty \right). \end{aligned} \tag{5.45}$$

We set

$$P := \tilde{S} \circ D f_V(x)^t \circ D f_V(x) \circ \tilde{S}^t \text{ and } N := \tilde{T} \circ Dg(y)^t \circ Dg(y) \circ \tilde{T}^t\,.$$

Setting $F := \Delta t D h_\varepsilon(\cdot, V)$ for simplicity, we have:

$$\begin{aligned} P &= I_d + \tilde{S} \circ \left(F(x)^t + F(x) \right) \circ \tilde{S}^t + \tilde{S} \circ F(x)^t \circ F(x) \circ \tilde{S}^t \\ &= I_d + A_f + B_f, \end{aligned}$$

and similarly

$$\begin{aligned} N &= I_d + \tilde{T} \circ \left(G^t(y) + G(y) \right) \circ \tilde{T}^t + \tilde{T} \circ G^t(y) \circ G(y) \circ \tilde{T}^t \\ &= I_d + A_g + B_g. \end{aligned}$$

Then

$$\|P - N\| \le \|A_f - A_g\| + \|B_f - B_g\|.$$

We have from (5.142)

$$\|A_f - A_g\| \le 2\left(\|F(x)\| + \|G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 2\|F(x) - G(y)\|. \tag{5.46}$$

Similarly, from (5.142) we can infer that

$$\|B_f - B_g\| \le 2\left(\|F(x)\|^2 + \|G(y)\|^2\right)\|\tilde{S} - \tilde{T}\| + \left(\|F(x)\| + \|G(y)\|\right)\|F(x) - G(y)\|. \tag{5.47}$$

We note that using (5.18) (we recall that $F(x) = \Delta t Dh_\varepsilon(x, V)$)

$$\|F(x) - G(y)\| \le \|F(x) - F(y)\| + \|F(y) - G(y)\| \le c_1 M \Delta t \varepsilon^{-6}|x - y| + \|Df_V - Dg\|_\infty. \tag{5.48}$$

Finally, from (5.46), (5.47) , (5.48) and Lemma 5.5.5 we obtain (using $\|F(x)\| \le c_1 M \Delta t \varepsilon^{-4} \le 1$ and $\|G(y)\| \le 2c_1 M \Delta t \varepsilon^{-4} \le 1$)

$$\begin{aligned}
\|P - N\| &\le 4\left(\|F(x)\| + \|G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 4\|F(x) - G(y)\| \\
&\le 4\left(c_1 M \Delta t \varepsilon^{-4} + 2c_1 M \Delta t \varepsilon^{-4}\right)\|\tilde{S} - \tilde{T}\| + 4c_1 M \Delta t \varepsilon^{-6}|x - y| + 4\|Df - Dg\|_\infty \\
&\le 32 c_1 M \Delta t \varepsilon^{-4}\|S - T\| + 4c_1 M \Delta t \varepsilon^{-6}|x - y| + 4\|Df_V - Dg\|_\infty
\end{aligned}$$

and this finishes the proof of (5.45).
Step 2**:** We prove

$$\left|J_S f_{\varepsilon,V}(x) - J_T g(y)\right| \le 512 c_1 c_2 \Delta t M \varepsilon^{-4}\|S - T\| + 64 c_1 c_2 \Delta t M \varepsilon^{-6}|x - y| + 64 c_2 \|Df_{\varepsilon,V} - Dg\|_\infty, \tag{5.49}$$

then (5.40) follows directly by definition of c_7 (recalling that $4c_1 \le c_6$).
Let $V \in V_d(\mathbb{R}^n)$ be such that $\|V\|(\mathbb{R}^n) \le M$, let (x, S), $(y, T) \in G_{d,n}$ and set

$$P = \tilde{S} \circ Df_V(x)^t \circ Df_V(x) \circ \tilde{S}^t \text{ and } N = \tilde{T} \circ Dg(y)^t \circ Dg(y) \circ \tilde{T}^t\,.$$

Let us show that

$$\|P - I_d\| \le \frac{1}{4}, \quad \|N - I_d\| \le \frac{1}{4} \quad \text{and } P \text{ is invertible with } \|P^{-1}\| \le 2\,.$$

To this end, we apply Lemma 5.5.3 with $L = \tilde{S}$ and $R = \Delta t\, Dh_\varepsilon(x, V)$. Using (5.17) and (5.39), we first note that

$$\forall z \in \mathbb{R}^n, \quad \Delta t\ |Dh_\varepsilon(z, V)|_\infty \le \Delta t\, c_1 M \varepsilon^{-4} \le \frac{1}{4c_1 c_4} \le \frac{1}{4c_4} < 1 \quad \text{since } c_5 = 4c_1^2 c_4 \text{ and } c_1 \ge 1\,. \tag{5.50}$$

In particular $|R|_\infty \le \dfrac{1}{4c_4} \le 1$ allows to apply (5.144) so that

$$\begin{aligned}
\|P - I_d\| \le d\,|P - I_d|_\infty &\le d\left|\tilde{S} \circ \left(F(x)^t + F(x)\right) \circ \tilde{S}^t + \tilde{S} \circ F(x)^t \circ F(x) \circ \tilde{S}^t\right|_\infty \\
&\le nc_3\,|F(x)|_\infty \le \frac{nc_3}{4c_4} \le \frac{1}{16} < \frac{1}{4} \quad \text{since } c_4 \ge 4nc_3\,.
\end{aligned} \tag{5.51}$$

As $c_3|R|_\infty \leq 1$, we can also apply (5.145) to conclude that $\left|\det(P)^{\frac{1}{2}} - 1\right| \leq c_4|R|_\infty \leq \frac{1}{2} < 1$ and thus P is invertible. Furthermore, using (5.51), we have

$$\|P^{-1}\| \leq \|P^{-1} - I_d\| + \|I_d\| \leq \|P^{-1}\|\|P - I_d\| + 1 \leq \frac{1}{2}\|P^{-1}\| + 1 \quad \Rightarrow \quad \|P^{-1}\| \leq 2\,. \tag{5.52}$$

We recall that $Dg(y) = I_n + G(y)$, from (5.44) and (5.34) we can assert that

$$|G(y)|_\infty \leq 2c_1 M \Delta t \varepsilon^{-4} \leq \frac{1}{2c_4} \leq 1$$

(where we used $c_5 = 4c_1^2 c_4$, $c_4 \geq 2$ and $c_1 \geq 1$). We can thus apply (5.144) to obtain

$$\begin{aligned} \|N - I_d\| \leq d\,|N - I_d|_\infty &= d\left|\tilde{T} \circ (G(y)^t + G(y)) \circ \tilde{T}^t + \tilde{T} \circ G(y)^t \circ G(y) \circ \tilde{T}^t\right|_\infty \\ &\leq nc_3\,|G(y)|_\infty \leq \frac{nc_3}{c_4} \leq \frac{1}{4} \ \text{ since } c_4 \geq 4nc_3. \end{aligned} \tag{5.53}$$

We can now show that P and N satisfy the condition given in (5.152). Indeed, from (5.51), (5.52) and (5.53), we have

$$\|P^{-1}\|\|P - N\| \leq 2\|(P - I_d) - (N - I_d)\| \leq 2\left(\frac{1}{4} + \frac{1}{4}\right) \leq 1$$

and thus applying (5.152) leads to

$$|\det(P) - \det(N)| \leq c_2\,|\det(P)|\ \|P^{-1}\|\ \|P - N\| \leq 8c_2\,\|P - N\| \tag{5.54}$$

since $\det(P) = J_S f_V(x)^2 \leq 4$ by (5.23). We know that for $a \geq \frac{1}{2}, b \geq 0$

$$|a - b| = \frac{|a^2 - b^2|}{a + b} \leq 2|a^2 - b^2|. \tag{5.55}$$

We have by (5.23) $\det(P)^{\frac{1}{2}} = J_S f_V(x) \geq \frac{1}{2}$ and by the positivity of the tangential Jacobian $\det(N) \geq 0$, applying (5.55) with $a = \det(P)^{\frac{1}{2}}$ and $b = \det(N)^{\frac{1}{2}}$ we obtain using (5.54),

$$\left|\det(P)^{\frac{1}{2}} - \det(N)^{\frac{1}{2}}\right| \leq 2\,|\det(P) - \det(N)| \leq 16c_2\|P - N\|. \tag{5.56}$$

Finally we obtain (5.49) from (5.56), (5.45) and the definition of P and N.
Step 3: *proof of* (5.41). We fix $W \in V_d(\mathbb{R}^n)$ satisfying $\|W\|(\mathbb{R}^n) \leq M$ and $(x, S), (y, T) \in \mathbb{R}^n \times G_{d,n}$. From (5.17) we can state that $\|Df_W\|_\infty \leq 2c_1 M \Delta t \varepsilon^{-4}$, this allows to apply (5.49) with f_V and $g = f_W$ so that

$$\left|J_S f_V(x) - J_T f_W(y)\right| \leq 512c_1c_2 M\Delta t\varepsilon^{-4}\|S - T\| + 64c_1c_2 M\Delta t\varepsilon^{-6}|x - y| + 64c_2\|Df_V - Df_W\|_\infty. \tag{5.57}$$

From Lemma 5.1.7 we deduce that

$$\|Df_V - Df_W\|_\infty \leq \Delta t\|Dh_\varepsilon(\cdot, V) - Dh_\varepsilon(\cdot, W)\|_\infty \leq c_6 M \Delta t \varepsilon^{-n-7}\Delta(V, W) \tag{5.58}$$

recalling that $c_7 = 6(128nc_2c_3c_6 + c_1c_4)$ we obtain (noting that $4c_1 \le c_6$)

$$\begin{aligned}\left|J_S f_V(x) - J_T f_W(y)\right| &\le \underbrace{512c_1c_2}_{\le 128c_2c_6} M\Delta t\varepsilon^{-4}\|S-T\| + 64c_1c_2M\Delta t\varepsilon^{-6}|x-y| + 64c_2c_6M\Delta t\varepsilon^{-n-7}\Delta(V,W) \\ &\le c_7\Delta tM\left(\varepsilon^{-6}|x-y| + \varepsilon^{-4}\|S-T\| + \varepsilon^{-n-7}\Delta(V,W)\right)\end{aligned} \tag{5.59}$$

and we finish the proof of (5.41).

Step 4**:** We now study the Lipschitz constant of the map $(x,S,f) \mapsto Df(x)(S)$, which will be crucial in proving the remaining estimates of the proposition. We namely prove

$$\|Df_{\varepsilon,V}(x)(S) - Dg(y)(T)\| \le (1+326nc_1c_3M\Delta t\varepsilon^{-4})\|S-T\| + 41c_1M\Delta t\varepsilon^{-6}|x-y| + 41\|Df_{\varepsilon,V} - Dg\|_\infty. \tag{5.60}$$

By Definition (1.10) we have

$$Df(x)(S) = Y(Y^tY)^{-1}Y^t \quad \text{and} \quad Dg(y)(T) = Z(Z^tZ)^{-1}Z^t$$

where $Y = Df(x)\circ \tilde{S}^t$ and $Z = Dg(y)\circ\tilde{T}^t$. We recall the following notations:

$$P = \tilde{S}\circ Df_V(x)^t \circ Df_V(x)\circ\tilde{S}^t (= Y^tY) \text{ and } N = \tilde{T}\circ Dg(y)^t\circ Dg(y)\circ\tilde{T}^t (= Z^tZ)\,,$$

and $F(x) := \Delta t Dh_\varepsilon(x,V)$, $G(y) := Dg(y) - I_n$. By the formulas $\tilde{S}^t\circ\tilde{S} = S$ and $\tilde{T}^t\circ\tilde{T} = T$, if we set $\tilde{P} = P^{-1} - I_d$ and $\tilde{N} = N^{-1} - I_d$ we obtain

$$\begin{aligned}Y(Y^tY)^{-1}Y^t &= \left(\tilde{S}^t + F(x)\circ\tilde{S}^t\right)\circ P^{-1}\circ\left(\tilde{S} + \tilde{S}\circ F(x)^t\right) \\ &= S + \tilde{S}^t\circ\tilde{P}\circ\tilde{S} + \underbrace{F(x)\circ\tilde{S}^t\circ P^{-1}\circ\tilde{S} + \tilde{S}^t\circ P^{-1}\circ\tilde{S}\circ F(x)^t}_{=R_f} + F(x)\circ\tilde{S}^t\circ P^{-1}\circ\tilde{S}\circ F(x)^t \\ &= S + C_f + R_f + E_f\end{aligned}$$

and similarly

$$\begin{aligned}Z(Z^tZ)^{-1}Z^t &= \left(\tilde{T}^t + G(y)\circ\tilde{T}^t\right)\circ N^{-1}\circ\left(\tilde{T} + \tilde{T}\circ G(y)^t\right) \\ &= T + \tilde{T}^t\circ\tilde{N}\circ\tilde{T} + \underbrace{G(y)\circ\tilde{T}^t\circ N^{-1}\circ\tilde{T} + \tilde{T}^t\circ N^{-1}\circ\tilde{T}\circ G(y)^t}_{=R_g} + G(y)\circ\tilde{T}^t\circ N^{-1}\circ\tilde{T}\circ G(y)^t \\ &= T + C_g + R_g + E_g.\end{aligned}$$

We have then,

$$\|Y(Y^tY)^{-1}Y^t - Z(Z^tZ)^{-1}Z^t\| \le \|S-T\| + \|C_f - C_g\| + \|D_f - D_g\| + \|E_f - E_g\|. \tag{5.61}$$

We first prove that

$$\|C_f - C_g\| \le 144nc_1c_3M\Delta t\varepsilon^{-4}\|S-T\| + 16c_1M\Delta t\varepsilon^{-6}|x-y| + 16\|Df_V - Dg\|_\infty. \tag{5.62}$$

From (5.51) and using (5.17) we have

$$\|P - I_d\| \le dc_3\|F(x)\|_\infty \le dc_3\|F(x)\| \le dc_1c_3M\Delta t\varepsilon^{-4}.$$

In particular, (5.52) implies that $\|P^{-1}\| \leq 2$. Therefore,

$$\|\tilde{P}\| = \|P^{-1} - I_d\| \leq \|P^{-1}\|\|P - I_d\| \leq 2dc_1c_3M\Delta t\varepsilon^{-4}.$$

Similarly (recalling that $\|G(y)\| \leq 3c_1M\Delta t\varepsilon^{-4} \leq 1$) (5.53) implies

$$\|N - I_d\| \leq nc_3|G(y)|_\infty \leq nc_3\|G(y)\| \leq 2nc_1c_3M\Delta t\varepsilon^{-4}.$$

In particular, same computations as in (5.52) implies that $\|N^{-1}\| \leq 2$. Therefore,

$$\|\tilde{N}\| = \|N^{-1} - I_d\| \leq \|N^{-1}\|\|N - I_d\| \leq 4nc_1c_3M\Delta t\varepsilon^{-4}.$$

Finally, by (5.45) we have

$$\begin{aligned}\|\tilde{P} - \tilde{N}\| &= \|(P^{-1} - I_d) - (N^{-1} - I_d)\| \leq \|P^{-1}(P - N)N^{-1}\| \leq \|P^{-1}\|\|N^{-1}\|\|P - N\| \\ &\leq 16\left(8c_1M\Delta t\varepsilon^{-4}\|S - T\| + c_1M\Delta t\varepsilon^{-6}|x - y| + \|Df_V - Dg\|_\infty\right).\end{aligned} \tag{5.63}$$

We carry on with the proof of (5.62), we write using the previous estimates, Lemma 5.5.5 and (5.142)

$$\begin{aligned}\|C_f - C_g\| &\leq \|\tilde{S}^t \circ \tilde{P} \circ \tilde{S} - \tilde{T}^t \circ \tilde{N} \circ \tilde{T}\| \\ &\leq 2(\|\tilde{P}\| + \|\tilde{N}\|)\|S - T\| + \|\tilde{P} - \tilde{N}\| \leq 16nc_1c_3M\Delta t\varepsilon^{-4}\|S - T\| + \|\tilde{P} - \tilde{N}\| \\ &\leq 144nc_1c_3M\Delta t\varepsilon^{-4}\|S - T\| + 16c_1M\Delta t\varepsilon^{-6}|x - y| + 16\|Df_V - Dg\|_\infty\end{aligned}$$

and we are done with (5.62). We now prove

$$\|R_f - R_g\| \leq 144c_1M\Delta t\varepsilon^{-4}\|S - T\| + 20c_1M\Delta t\varepsilon^{-6}|x - y| + 20\|Df_V - Dg\|_\infty. \tag{5.64}$$

Indeed, using that $P^t = P$, $N^t = N$, and recalling that $\|P^{-1}\| \leq 2$ and $\|N^{-1}\| \leq 2$, we deduce that

$$\begin{aligned}\|R_f - R_g\| &= \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} + \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} - \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\ &\leq 2\|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| \\ &\leq 2\|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} - F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{T}\| + 2\|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{T} - F(x) \circ \tilde{S}^t \circ N^{-1} \circ \tilde{T}\| \\ &+ 2\|F(x) \circ \tilde{S}^t \circ N^{-1} \circ \tilde{T} - F(x) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| + 2\|F(x) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} - G(y) \circ \tilde{T} \circ N^{-1} \circ \tilde{T}\| \\ &\leq 2\|F(x)\|\|P^{-1}\|\|\tilde{S} - \tilde{T}\| + 2\|F(x)\|\|P^{-1} - N^{-1}\| \\ &+ 2\|F(x)\|\|N^{-1}\|\|\tilde{S} - \tilde{T}\| + 2\|N^{-1}\|\|F(x) - G(y)\| \\ &\leq 16c_1M\Delta t\varepsilon^{-4}\|S - T\| + \underbrace{2c_1M\Delta t\varepsilon^{-4}}_{\leq 1}\|\tilde{P} - \tilde{N}\| + 4\,\mathrm{Lip}(F)|x - y| + 4\|Df_V - Dg\|_\infty\end{aligned}$$

then the estimate (5.64) follows from (5.63) and $\mathrm{Lip}(F) \leq c_1M\Delta t\varepsilon^{-6}$ (estimate (5.18)).
We now show that

$$\|E_f - E_g\| \leq 38c_1M\Delta t\varepsilon^{-4}\|S - T\| + 5c_1M\Delta t\varepsilon^{-6}|x - y| + 5\|Df_V - Dg\|_\infty. \tag{5.65}$$

Indeed, recalling that $\|P^{-1}\| \le 2$, $\|N^{-1}\| \le 2$, $\|F\|_\infty \le c_1 M \Delta t \varepsilon^{-4}$ and $\|G\|_\infty \le 2c_1 M \Delta t \varepsilon^{-4}$ and using Lemma 5.5.5 one has

$$
\begin{aligned}
&\|E_f - E_g\| = \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&\le \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t - F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ G(y)^t\| \\
&+ \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ G(y)^t - F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&+ \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{T} \circ G(y)^t - F(x) \circ \tilde{S}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&+ \|F(x) \circ \tilde{S}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t - F(x) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&+ \|F(x) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&\le \left(\|F(x)\|\|P^{-1}\| + \|G(y)\|\|N^{-1}\|\right) \|F(x) - G(y)\| + \|F(x)\|\|G(y)\|(\|P^{-1}\| + \|N^{-1}\|)\|\tilde{S} - \tilde{T}\| \\
&+ \|F(x)\|\|G(y)\|(\|P^{-1} - N^{-1}\|) \\
&\le 6c_1 M \Delta t \varepsilon^{-4} \|F(x) - G(y)\| + 16 c_1^2 M^2 \Delta t^2 \varepsilon^{-8} \|S - T\| + 2c_1^2 M^2 \Delta t^2 \varepsilon^{-8} \|P^{-1} - N^{-1}\| \\
&\le \|F(x) - G(y)\| + \Delta t \|S - T\| + \|\tilde{P} - \tilde{N}\|,
\end{aligned}
$$

where we used $c_5 \Delta t M^3 \varepsilon^{-8} \le 1$, $c_5 = 4c_1^2 c_4$, $c_1 \ge 2$ and $c_4 \ge 4$. Now, we use $\|F(x) - G(y)\| \le c_1 M \Delta t \varepsilon^{-6} |x - y| + \|Df_V - Dg\|$ and (5.63) to deduce (5.65). Finally, plugging (5.62), (5.64) and (5.65) into (5.61) finishes the proof of (5.60).

Step 5**:** *proof of* (5.42). Let $\varphi \in \mathrm{C}^{0,1}(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$ satisfying $\|\varphi\|_\infty \le 1$ and $\mathrm{Lip}(\varphi) \le 1$. For $(x, S) \in \mathbb{R}^n \times G_{d,n}$, from (5.60) and (5.23) we have

$$
\begin{aligned}
&\left|\varphi(f_V(x), Df_V(x)(S)) J_S f_V(x) - \varphi(g(x), Dg(x)(S)) J_S g(x)\right| \\
&\le \left|\varphi(f_V(x), Df_V(x)(S)) - \varphi(g(x), Dg(x)(S))\right| |J_S f_V(x)| + \varphi(g(x), Dg(x)(S)) |J_S f_V(x) - J_S g(x)| \\
&\le \mathrm{Lip}(\varphi) \left(|f_V(x) - g(x)| + \|Df_V(x)(S) - Dg(x)(S)\|\right) \|J.f_V\|_\infty + \|\varphi\|_\infty |J_S f_V(x) - J_S g(x)| \\
&\le \|f_V - g\|_\infty + 41\frac{2}{3} \|Df_V - Dg\|_\infty + \|J.f_V - J.g\|_\infty \le 28 \|f_V - g\|_{\mathrm{C}^1} + \|J.f_V - J.g\|_\infty
\end{aligned}
$$

where $\|\cdot\|_\infty$ is taken over all $(x, S) \in \mathbb{R}^n \times G_{d,n}$. From the previous inequality and the definition of the push-forward, one has

$$
\left|f_{V\#}V(\varphi) - g_\# V(\varphi)\right| \le \|V\|(\mathbb{R}^n) \left(28 \|f_V - g\|_{\mathrm{C}^1} + \|J.f_V - J.g\|_\infty\right).
$$

We can eventually take the supremum with respect to φ and conclude

$$
\Delta(f_{V\#}V, g_\# V) = \sup_{\substack{\|\varphi\|_\infty \le 1, \\ \mathrm{Lip}(\varphi) \le 1}} \left|f_{V\#}V(\varphi) - g_\# V(\varphi)\right| \le \|V\|(\mathbb{R}^n) \left(28 \|f_V - g\|_{C^1} + \|J.f_V - J.g\|_\infty\right).
$$

Step 6**:** *proof of* (5.43) . We first apply (5.42) with $V = W$, $f = f_V$ and $g = f_W$ so that (recalling that $c_7 = 6(128 n c_2 c_3 c_6 + c_1 c_4)$)

$$
\begin{aligned}
\Delta\left((f_V)_\# W, (f_W)_\# W\right) &\le M \left(28 \|f_V - f_W\|_{C^1} + \|J.f_V - J.f_W\|_\infty\right) \\
&\le 28 M \Delta t \|h_\varepsilon(\cdot, V) - h_\varepsilon(\cdot, W)\|_{\mathrm{C}^1} + 64 c_2 c_6 M^2 \Delta t \varepsilon^{-n-7} \Delta(V, W) \text{ by (5.59)} \\
&\le (56 c_6 + 64 c_2 c_6) M^2 \Delta t\, \varepsilon^{-n-7} \Delta(V, W) \text{ by Lemma 5.1.7} \\
&\le \frac{c_7}{2} M^2 \Delta t\, \varepsilon^{-n-7} \Delta(V, W). \qquad (5.66)
\end{aligned}
$$

We now prove that

$$\Delta\left((f_V)_\# V, (f_V)_\# W\right) \le \left(1 + \frac{c_7}{2} M^2 \Delta t \varepsilon^{-6}\right) \Delta(V, W)\,. \tag{5.67}$$

Let $\varphi \in \mathrm{C}_c^{0,1}(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$ satisfying $\|\varphi\|_\infty \le 1$ and $\mathrm{Lip}(\varphi) \le 1$ and coming back to the definition of Δ (see Definition 1.2), we consider $\psi : (x, S) \mapsto \varphi(f_V(x), Df_V(x)(S))\, J_S f_V(x)$. As f_V is a C^1-diffeomorphism, we have $\psi \in \mathrm{C}_c^0(\mathbb{R}^n \times G_{d,n})$ and by definition of varifold push-forward,

$$\left| \int \varphi \, d(f_V)_\# V - \int \varphi \, d(f_V)_\# W \right| = \left| \int \psi \, dV - \int \psi \, dW \right| \le \max(\|\psi\|_\infty, \mathrm{Lip}(\psi)) \Delta(V, W)\,. \tag{5.68}$$

One has to pay attention to the fact that the $\|\cdot\|_\infty$ and $\mathrm{Lip}(\cdot)$ refer to both variables $(x, S) \in \mathbb{R}^n \times G_{d,n}$. Introducing the notations $\psi_1 : (x, S) \mapsto Df_V(x)(S) = (I_n + \Delta t\, Dh_\varepsilon(x, V))\,(S)$ and $\psi_2 : (x, S) \mapsto J_S f_V(x)$, we have

$$\begin{aligned} \mathrm{Lip}(\psi_1) \le (1 + 326 n c_1 c_3 M \Delta t \varepsilon^{-6})\,, \quad \mathrm{Lip}(\psi_2) \le 128 c_2 c_6 M \Delta t\, \varepsilon^{-6}\,, \\ \|\psi_2\|_\infty \le 1 + c_1 c_4 M \Delta t\, \varepsilon^{-4} \quad \text{and} \quad \mathrm{Lip}(f_V) \le 1 + c_1 M \Delta t\, \varepsilon^{-4}\,. \end{aligned} \tag{5.69}$$

Indeed, let $(x, S), (y, T) \in \mathbb{R}^n \times G_{d,n}$, then (5.60) with f_V and $g = f_W$ imply

$$\begin{aligned} |\psi_1(x, S) - \psi_1(y, T)| &\le (1 + 326 n c_1 c_3 M \Delta t \varepsilon^{-4}) \|S - T\| + 41 c_1 M \Delta t \varepsilon^{-6} |x - y| \\ &\le (1 + 326 n c_1 c_3 M \Delta t \varepsilon^{-6})(\|S - T\| + |x - y|). \end{aligned}$$

Furthermore, by (5.59)

$$|\psi_2(x, S) - \psi_2(y, T)| \le 128 c_2 c_6 M \Delta t\, \varepsilon^{-6} \left(|x - y| + \|S - T\|\right)\,,$$

and by (5.23) and (5.17), $|J_S f_V(x)| \le 1 + c_4 \Delta t\, \|Dh_\varepsilon(\cdot, V)\|_\infty \le 1 + c_1 c_4 M \Delta t\, \varepsilon^{-4}$. We also have $\mathrm{Lip}(f_V) \le 1 + \Delta t\ \mathrm{Lip}(h_\varepsilon) \le 1 + c_1 M \Delta t\, \varepsilon^{-4}$ thanks to (5.17). With (5.69) in hand, we can estimate $\mathrm{Lip}(\psi)$ as follows:

$$\begin{aligned} &|\psi(x, S) - \psi(y, T)| \\ &\quad \le |\varphi(f_V(x), \psi_1(x, S)) - \varphi(f_V(y), \psi_1(y, T))|\, \|\psi_2\|_\infty + \|\varphi\|_\infty\, \mathrm{Lip}(\psi_2)\,(|x - y| + \|S - T\|) \\ &\quad \le [\mathrm{Lip}(\varphi)\,(\max\{\mathrm{Lip}(f_V), \mathrm{Lip}(\psi_1)\})\, \|\psi_2\|_\infty + \mathrm{Lip}(\psi_2)]\,(|x - y| + \|S - T\|) \end{aligned}$$

and therefore, recalling that $c_7 = 6(128 n c_2 c_3 c_6 + c_1 c_4)$, we have using $c_6 \ge 4 c_1$:

$$\begin{aligned} \mathrm{Lip}(\psi) &\le \left(1 + 326 n c_1 c_3 M \Delta t \varepsilon^{-6}\right) \left(1 + c_1 c_4 M \Delta t\, \varepsilon^{-4}\right) + 128 c_2 c_6 M \Delta t\, \varepsilon^{-6} \\ &\le 1 + (326 n c_1 c_3 + c_1 c_4 + 128 c_2 c_6) M \Delta t \varepsilon^{-6} + 326 n c_1^2 c_3 c_4 M^2 \Delta t^2\, \varepsilon^{-10} \\ &\le 1 + (82 n c_3 c_6 + c_1 c_4 + 128 c_2 c_6 + 21 n c_3 c_6) M \Delta t \varepsilon^{-6} \\ &\le \left(1 + \frac{c_7}{2} M^2 \Delta t \varepsilon^{-6}\right)\,, \end{aligned} \tag{5.70}$$

since by assumption $1 \le M$ and (noting that $c_6 \ge 4$ and by definition $c_5 = 4 c_1^2 c_4$)

$$326 n c_1^2 c_3 c_4 M^2 \Delta t^2\, \varepsilon^{-10} \le 82 n c_3 c_5 M^4 \Delta t^2 \varepsilon^{-10} \le 82 n c_3 M \Delta t \varepsilon^{-2} \le 21 n c_3 c_6 M \Delta t \varepsilon^{-6}$$

$$\|\psi\|_\infty \le \|\varphi\|_\infty \|\psi_2\|_\infty \le \|\psi_2\|_\infty \le \left(1 + \frac{c_7}{2} M \Delta t \varepsilon^{-6}\right) \tag{5.71}$$

we insert (5.70) and (5.71) in (5.68) and take the supremum over all φ to infer (5.67). Combining (5.67) and (5.66) we conclude the proof of (5.43), and subsequently the proof of Proposition 5.1.8.
□

Iterating Proposition 5.1.8 leads to the following stability result on the time-discrete approximate MCF.

Proposition 5.1.9 (Stability with respect to the initial datum)**.** *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V_0 and W_0 be two varifolds in $V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M, \|W_0\|(\mathbb{R}^n) \leq M$.*
Let $\mathcal{T} = \{t_i\}_{i=0}^m$ be a subdivision of $[0,1]$ (in the sense of Definition 1.1.1) satisfying (5.34). Denote by $V_{\varepsilon,\mathcal{T}}(t)$ (resp. $W_{\varepsilon,\mathcal{T}}(t)$) the time-discrete approximate MCF with respect to $\mathcal{T}$ starting from V_0 (resp. W_0) as introduced in Definition 5.1.4. Then, for any $t \in [0,1]$, one has

$$\Delta\left(V_{\varepsilon,\mathcal{T}}(t), W_{\varepsilon,\mathcal{T}}(t)\right) \leq \exp\bigl(c_{7,M} t\, \varepsilon^{-n-7}\bigr) \Delta(V_0, W_0), \tag{5.72}$$

where $c_{7,M} = c_7(M+1)^2$ and c_7 was introduced in Proposition 5.1.8.

Proof. As $\varepsilon \in (0,1)$ and the subdivision $\mathcal{T}$ are fixed, we write $V(t)$ (resp. $W(t)$) for $V_{\varepsilon,\mathcal{T}}(t)$ (resp. $W_{\varepsilon,\mathcal{T}}(t)$) hereafter. From (5.43) applied with $V = V(t_{i-1}), W = W(t_{i-1})$ and $\Delta t = d_i = t_i - t_{i-1}$, and noting that $\|V(t_{i-1})\|(\mathbb{R}^n) \leq M+1$ and $\|W(t_{i-1})\|(\mathbb{R}^n) \leq M+1$ (see Remark 5.1.5), we infer that for any $i \in \{1\ldots,m\}$ we have:

$$\Delta\left(V(t_i), W(t_i)\right) \leq (1 + c_{7,M} d_i\, \varepsilon^{-n-7}) \Delta(V(t_{i-1}), W(t_{i-1}))\,.$$

By iteration of the previous inequality for $k \in \{1\ldots,i\}$ and applying the inequality $1 + a \leq \exp(a)$ in $\mathbb{R}$, we obtain

$$\Delta\left(V(t_i), W(t_i)\right) \leq \prod_{k=1}^{i} (1 + c_{7,M} d_k\, \varepsilon^{-n-7}) \Delta(V(0), W(0)) \underbrace{\leq}_{\sum_{k=1}^i d_k = t_i} \exp\bigl(c_{7,M} t_i\, \varepsilon^{-n-7}\bigr) \Delta(V_0, W_0)\,.$$

Let now $t \in (t_i, t_{i+1}]$ and apply once again Proposition 5.1.8 (with $\Delta t = t - t_i$) so that

$$\begin{aligned}\Delta\left(V(t), W(t)\right) &\leq (1 + c_{7,M}(t - t_i)\, \varepsilon^{-n-7}) \Delta\left(V(t_i), W(t_i)\right) \\ &\leq (1 + c_{7,M}(t - t_i) \varepsilon^{-n-7}) \exp\bigl(c_{7,M} t_i\, \varepsilon^{-n-7}\bigr) \Delta(V_0, W_0) \\ &\leq \exp\bigl(c_{7,M} t\, \varepsilon^{-n-7}\bigr) \Delta(V_0, W_0)\,,\end{aligned}$$

thus ending the proof of the stability of the time-discrete approximate MCF with respect to the initial datum. □

Remark 5.1.10 (Analogy with ODE discretization)**.** The construction of the time-discrete approximate MCF defined in our work can be compared to the discretization of the classical Cauchy problem in $[0,T]$:

$$\begin{cases} y'(t) = f(y,t), \\ y(0) = y_0. \end{cases}$$

It is known that the stability constant (with respect to the supremum norm on $[0,T]$) for the explicit Euler discretization of the ODE is $\exp(LT)$ with $L = \max_{t \leq T} \mathrm{Lip}(f(\cdot,t))$ (see for instance [8, Section 2.4]). Comparing with the stability estimate (5.72) we obtain in Proposition 5.1.9, we observe that $c_{7,M}\varepsilon^{-n-7}$ is indeed a bound on the Lipschitz constant of $V \mapsto H_\varepsilon(\cdot, V)$ when $V_d(\mathbb{R}^n)$ is endowed with the Bounded Lipschitz distance Δ and $\mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is endowed with $\|\cdot\|_{\mathrm{C}^1}$, see Lemma 5.1.7.

5.1.4 Stability with respect to the subdivision

In this section, we investigate the robustness of the time-discrete approximate MCF (introduced in Definition 5.1.4) with respect to the choice of the subdivision $\mathcal{T}$. It is a natural property to expect for a numerical scheme and it is furthermore crucial in order to take the limit "$\delta(\mathcal{T}) \to 0$" and obtain a well-defined "time-continuous" approximate MCF as subsequently done in Theorem 5.2.1. We establish in Proposition 5.1.11 that time-discrete approximate MCF are stable with respect to subdivisions. The proof of Proposition 5.1.11 is split into several steps: the section starts with two lemmas (Lemma 5.1.12 and 5.1.13) aiming to compare two flows corresponding to a fine subdivision and the trivial subdivision of small time interval $[0, \delta]$. Then, in Lemma 5.1.15 we extend the comparison to the case of two nested subdivisions of the interval $[0, 1]$ from which Proposition 5.1.11 can be inferred straightforwardly.

Proposition 5.1.11 (Stability with respect to the subdivision)**.** *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$, let $\mathcal{T}_1 = \{t_i\}_{i=1}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ be two subdivisions (Definition 1.1.1) of $[0,1]$ satisfying (5.34). Let $V_{\varepsilon,\mathcal{T}_1}(t)$ (resp.$V_{\varepsilon,\mathcal{T}_2}(t)$) be the time-discrete approximate MCF with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) starting from V_0. We set:*

$$\delta = \max\left\{\delta(\mathcal{T}_1), \delta(\mathcal{T}_2)\right\}.$$

Then, for all $t \in [0,1]$, one has:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \leq c_{10,M} t \delta \varepsilon^{-n-11} \exp\bigl(c_{7,M} t \varepsilon^{-n-7}\bigr).$$

where $c_{10,M} = c_{10}(M+1)^5$ and c_{10} is a constant depending only on n and $c_{7,M}$ was introduced in Proposition 5.1.9.

Before proving our Proposition 5.1.11, we shall introduce some preliminary lemmas.

In the following lemma, we measure how far the push-forward operation is from satisfying the semigroup property. In practice, we measure how far apart are two time-discrete approximate MCFs constructed with respect to two subdivisions including the trivial subdivision.

Lemma 5.1.12. *Let $\varepsilon \in (0,1)$, $M \geq 1$ and $\delta \geq 0$ such that $\delta c_5 (M+1)^3 \varepsilon^8 \leq 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$. Consider $\mathcal{T} = \{t_i\}_{i=1}^m$ a given subdivision (1.1.1) of $[0,\delta]$ and $\mathcal{T}'$ the trivial subdivision of $[0,\delta]$. For $i \in \{1, \ldots m\}$, we introduce*

$$d_i = t_i - t_{i-1} \quad \text{and} \quad \tilde{f}_i = \left(\mathrm{id} + d_i\, h_\varepsilon(\cdot, V_0)\right) .$$

We then consider two different flows:

- $\left(V_{\varepsilon,\mathcal{T}'}(t_i)\right)_{i=0\ldots m}$ *where $V_{\varepsilon,\mathcal{T}'}$ is the time-discrete approximate MCF of V_0 with respect to $\mathcal{T}'$ according to Definition 5.1.4,*
- $\left(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i)\right)_{i=0\ldots m}$ *is defined as follows:*

$$\begin{cases} \widetilde{V}_{\varepsilon,\mathcal{T}}(0) := V_0 \\ \widetilde{V}_{\varepsilon,\mathcal{T}}(t_i) := (\tilde{f}_i)_\# \widetilde{V}_{\varepsilon,\mathcal{T}}(t_{i-1}) \quad \forall i \in \{1, \ldots m\}. \end{cases} \tag{5.73}$$

Then,

$$\Delta\left(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i), V_{\varepsilon,\mathcal{T}'}(t_i)\right) \leq c_8 M^3 t_i^2 \varepsilon^{-10} \quad \forall\, i \in \{0, \ldots m\},$$

where $c_8 = 112 c_1^2 + 3 c_1^2 c_7$.

It is important to note that using the previous notation with $\Delta t = d_i$, we have $\tilde{f}_i = f_{\varepsilon,V_0}$ while the definition of time-discrete approximate MCF would involve $f_{\varepsilon,V_{\varepsilon,\mathcal{T}}(t_{i-1})}$ instead. The velocity h_ε is taken with respect to the initial varifold all along the subdivision when defining $\tilde{V}_{\varepsilon,\mathcal{T}}$.

Proof. We introduce the following notations: $g_0 = \tilde{g}_0 = f_0 = \tilde{f}_0 = \mathrm{id}$ and

$$\forall i \in \{1, \dots m\} \quad g_i = \mathrm{id} + t_i h_\varepsilon(\cdot, V_0) \quad \text{and} \quad \tilde{g}_i = \tilde{f}_i \circ \dots \circ \tilde{f}_1 \, .$$

Step 1**:** we first prove that for all $i \in \{1, \dots, m\}$,

$$\|\tilde{g}_i - g_i\|_{\mathrm{C}^1} \leq 4 c_1^2 M^2 t_i^2 \varepsilon^{-10} \, . \tag{5.74}$$

Indeed, let $i \in \{1, \dots m\}$ and $x \in \mathbb{R}^n$, we have by definition

$$\tilde{g}_i(x) = \tilde{f}_i \circ \ldots \circ \tilde{f}_1(x) = \tilde{f}_i\left(\tilde{g}_{i-1}(x)\right) = \tilde{g}_{i-1}(x) + d_i\, h_\varepsilon(\tilde{g}_{i-1}(x), V_0) = x + \sum_{k=1}^{i} d_k h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \,,$$

$$\text{hence} \quad |\tilde{g}_i(x) - x| \leq \sum_{k=1}^{i} d_k \left|h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right)\right| \leq t_i \, \|h_\varepsilon(\cdot, V_0)\|_\infty \,, \tag{5.75}$$

$$g_i(x) = x + t_i\, h_\varepsilon(x, V_0) = x + \sum_{k=1}^{i} d_k\, h_\varepsilon\left(x, V_0\right) \,,$$

and applying the mean value theorem we infer

$$\begin{aligned} |\tilde{g}_i(x) - g_i(x)| &\leq \sum_{k=1}^{i} d_k \left|h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) - h_\varepsilon\left(x, V_0\right)\right| \leq \sum_{k=1}^{i} d_k \|D h_\varepsilon(\cdot, V_0)\|_\infty \, |\tilde{g}_{k-1}(x) - x| \\ &\leq t_i\, t_{i-1} \|D h_\varepsilon(\cdot, V_0)\|_\infty \|h_\varepsilon(\cdot, V_0)\|_\infty \\ &\leq t_i^2 c_1^2 M^2 \varepsilon^{-6} \quad \text{thanks to Proposition 5.1.2.} \end{aligned} \tag{5.76}$$

We proceed similarly to bound the derivatives but we have to handle the term $D\tilde{g}_i(x)$ arising from the chain rule applied to $x \mapsto h_\varepsilon(\tilde{g}_i(x), V_0)$: recalling that $\tilde{g}_i = (\mathrm{id} + d_i\, h_\varepsilon(\cdot, V_0)) \circ \tilde{g}_{i-1}$, we infer for $x \in \mathbb{R}^n$ and $i \in \{1, \dots, m\}$,

$$\begin{aligned} \|D\tilde{g}_i(x)\| &\leq \|\mathrm{id} + d_i\, D h_\varepsilon(\tilde{g}_i(x), V_0)\| \|D\tilde{g}_{i-1}(x)\| \leq (1 + d_i\, c_1 M \varepsilon^{-4}) \|D\tilde{g}_{i-1}(x)\| \text{ using (5.17)} \\ &\leq \prod_{k=1}^{i} \left(1 + d_k\, c_1 M \varepsilon^{-4}\right) \|D\tilde{g}_0(x)\| \leq \prod_{k=1}^{i} \exp\left(d_k\, c_1 M \varepsilon^{-4}\right) \\ &\leq \exp\left(t_i c_1 M \varepsilon^{-4}\right) \leq 2 \end{aligned} \tag{5.77}$$

where we used $\tilde{g}_0 = \mathrm{id}$, $1 + s \leq \exp(s)$ for $s \in \mathbb{R}$ and

$$\exp\left(t_i c_1 M \varepsilon^{-4}\right) \leq \exp\left(\delta \frac{c_5}{4} (M+1)^3 \varepsilon^{-8}\right) \leq \exp(1/4) \leq 2.$$

We can now expand $D\tilde{g}_i$ and Dg_i as we did previously for $\tilde{g}_i$ and g_i:

$$D\tilde{g}_i(x) = \mathrm{id} + \sum_{k=1}^{i} d_k\, D h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \circ D\tilde{g}_{k-1}(x) \,,$$

(5.17) and (5.77) implies

$$\|D\tilde{g}_i(x) - \mathrm{id}\| \leq \sum_{k=1}^{i} d_k \, \|Dh_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right)\| \, \|D\tilde{g}_{k-1}(x)\| \leq 2t_i c_1 M \varepsilon^{-4}, \tag{5.78}$$

$$Dg_i(x) = \mathrm{id} + t_i \, Dh_\varepsilon(x, V_0) = \mathrm{id} + \sum_{k=1}^{i} d_k \, Dh_\varepsilon\left(x, V_0\right).$$

We can then apply the mean value theorem to $Dh_\varepsilon(\cdot, V_0)$, together with (5.75), (5.78) and Proposition 5.1.2 to infer

$$\begin{aligned}
\|D\tilde{g}_i(x) - Dg_i(x)\| &\leq \sum_{k=1}^{i} d_k \, \|Dh_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \circ D\tilde{g}_{k-1}(x) - Dh_\varepsilon\left(x, V_0\right)\| \\
&\leq \sum_{k=1}^{i} d_k \|Dh_\varepsilon(\cdot, V_0)\|_\infty \, \|D\tilde{g}_{k-1}(x) - \mathrm{id}\| + \|D^2 h_\varepsilon(\cdot, V_0)\|_\infty \, |\tilde{g}_{k-1}(x) - x| \\
&\leq 2t_i \, t_{i-1}(c_1 M \varepsilon^{-4})^2 + c_1 M \varepsilon^{-4} t_i t_{i-1} c_1 M \varepsilon^{-6} \\
&\leq 3c_1^2 M^2 \, t_i^2 \, \varepsilon^{-10}
\end{aligned} \tag{5.79}$$

where we used $t_{i-1} \leq t_i$, this ends step1.
Step2: we prove that $||J.g_i - J.\tilde{g}_i||_\infty \leq 3c_1^2 c_7 M^2 t_i^2 \varepsilon^{-10}$.
Estimate (5.78) allows to use (5.40) with $V = V_0$, $f = g_i$ and $g = \tilde{g}_i$, together with (5.79) we obtain

$$||J.g_i - J.\tilde{g}_i|| \leq c_7 \left(3c_1^2 M^2 t_i^2 \varepsilon^{-10}\right) = 3c_1^2 c_7 M^2 t_i^2 \varepsilon^{-10} \tag{5.80}$$

and we finish the proof of step 2.

By (5.73) and Lemma 1.4.3 we have $\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i) = (\tilde{f}_i)_\# \left((\tilde{f}_{i-1})_\# \cdots {}_\# \left((\tilde{f}_1)_\# V_0\right)\right) = (\tilde{g}_i)_\# V_0$, (5.79) allows to use (5.42) with $V = V_0$, $f = g_i$ and $g = \tilde{g}_i$ so that

$$\Delta\left(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i), V_{\varepsilon,\mathcal{T}'}(t_i)\right) = \Delta\left((\tilde{g}_i)_\# V_0, (g_i)_\# V_0\right) \leq \|V_0\|(\mathbb{R}^n)\left(28\|g_i - \tilde{g}_i\|_{\mathrm{C}^1} + \|J.g_i - J.\tilde{g}_i\|_\infty\right).$$

By plugging (5.74) and (5.80) into the previous formula, by using $\|V_0\|(\mathbb{R}^n) \leq M$, and by recalling that $c_8 = 112c_1^2 + 3c_1^2 c_7$, we obtain the desired result. ☐

In the following lemma we compare the time-discrete approximate MCFs of two given varifolds on a small interval of time, one defined with respect to the trivial subdivision and the other with respect to a finer subdivision of the time interval. The proof is based on Lemma 5.1.12.

Lemma 5.1.13. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let V_0 and W_0 in $V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$,$\|W_0\|(\mathbb{R}^n) \leq M$. Let $\delta \geq 0$ be such that $c_5\delta(M+1)^3 < \varepsilon^8$ and consider $\mathcal{T} = \{t_i\}_{i=1}^m$ a subdivision (1.1.1) of $[0,\delta]$.*
Denote by $V_{\varepsilon,\mathcal{T}'}(t)$ the time-discrete approximate MCF of V_0 with respect to the trivial subdivision of $[0,\delta]$ and by $W_{\varepsilon,\mathcal{T}}(t)$ the time-discrete approximate MCF of W_0 with respect to $\mathcal{T}$.
Then, for any $i \in \{0, 1 \ldots m\}$:

$$\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) \leq \Delta(V_0, W_0)\exp\left(c_{7,M} t_i \varepsilon^{-n-7}\right) + 2c_{9,M} t_i^2 \varepsilon^{-n-11} \exp\left(c_{7,M} t_i \varepsilon^{-n-7}\right) \tag{5.81}$$

where $c_{9,M} = c_9(M+1)^5$ and $c_9 = 2c_7(57c_1c_4 + c_8)$.

Remark 5.1.14. A particular case of (5.81) is when $i = m$:

$$\Delta(V(\delta), W(\delta)) \le \Delta(V_0, W_0)\exp\bigl(c_{7,M}\delta\varepsilon^{-n-7}\bigr) + 2c_{9,M}\delta^2\varepsilon^{-n-11}\exp\bigl(c_{7,M}\delta\varepsilon^{-n-7}\bigr). \tag{5.82}$$

The last inequality will be useful in the sequel.

Proof. The assumption $c_5\delta(M+1)^3 < \varepsilon^8$ implies that the subdivisions involved ($\mathcal{T}'$ and $\mathcal{T}$) satisfy (5.34) and allows to define time-discrete approximate MCF for both subdivisions. For every $i \in \{1 \dots m\}$, set:

$$d_i = t_i - t_{i-1}$$

and define $\tilde{V}_{\varepsilon,\mathcal{T}}(t)$ the auxiliary flow as in (5.73). We have by Lemma 5.1.12

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) &\le \Delta(V_{\varepsilon,\mathcal{T}'}(t_i), \tilde{V}_{\varepsilon,\mathcal{T}}(t_i)) + \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i))\\ &\le c_8M^3t_i^2\varepsilon^{-10} + \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)).\end{aligned} \tag{5.83}$$

Now we only need to bound $\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i))$. Similarly to Lemma 5.1.12 we introduce the notation $\tilde{f}_i = \mathrm{id} + d_ih_\varepsilon(\cdot, V_0)$. We recall that for any $l \in \{0, 1 \dots m\}$

$$\tilde{V}_{\varepsilon,\mathcal{T}}(t_l) = (\mathrm{id} + d_lh_\varepsilon(\cdot, V_0))_{\#}\,\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}) = \tilde{f}_{l\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})$$

hence,

$$\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_l), W_{\varepsilon,\mathcal{T}}(t_l)) = \Delta\left(\tilde{f}_{l\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), (\mathrm{id} + d_lh_\varepsilon(\cdot, W_{\varepsilon,\mathcal{T}}(t_{l-1})))_{\#}\,W_{\varepsilon,\mathcal{T}}(t_{l-1})\right)$$

by (5.43) for $V = \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})$, $W = W_{\varepsilon,\mathcal{T}}(t_{l-1})$ and $\Delta t = d_l$

$$\begin{aligned}&\Delta\left(\left(\mathrm{id} + d_lh_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))\right)_{\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), (\mathrm{id} + d_lh_\varepsilon(\cdot, W_{\varepsilon,\mathcal{T}}(t_{l-1})))_{\#}\,W_{\varepsilon,\mathcal{T}}(t_{l-1})\right)\\ &\le \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), W_{\varepsilon,\mathcal{T}}(t_{l-1}))\left(1 + c_{7,M}d_l\varepsilon^{-n-7}\right).\end{aligned}$$

By (5.42) applied with $V = \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})$, $f = \tilde{f}_l$ and $g = \mathrm{id} + d_lh_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))$, Lemma 5.1.7 and (5.41) we assert that :

$$\begin{aligned}&\Delta\left(\left(\mathrm{id} + d_lh_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))\right)_{\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), \tilde{f}_{l\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})\right)\\ &\le \|\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})\|(\mathbb{R}^n)\left(28d_l\|h_\varepsilon(\cdot, V_0) - h_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))\|_{C^1} + \|J.\tilde{f}_l - J.(\mathrm{id} + d_lh_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})))\|_\infty\right)\\ &\le 56c_6d_l(M+1)^2\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), V_0)\varepsilon^{-n-7} + c_7d_l(M+1)^2\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), V_0)\varepsilon^{-n-7}\\ &\le 2c_7d_l(M+1)^2\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), V_0)\varepsilon^{-n-7},\end{aligned} \tag{5.84}$$

where we used (5.35) and $c_7 \ge 56c_6$. Finally, we have by Lemma 5.1.12

$$\begin{aligned}\Delta(V_0, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})) &\le \Delta(V_0, V_{\varepsilon,\mathcal{T}'}(t_{l-1})) + \Delta(V_{\varepsilon,\mathcal{T}'}(t_{l-1}), \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))\\ &\le \Delta(V_0, V_{\varepsilon,\mathcal{T}'}(t_{l-1})) + c_8M^3t_{l-1}^2\varepsilon^{-10}\\ &\le \Delta(V_0, V_{\varepsilon,\mathcal{T}'}(t_{l-1})) + c_8M^3t_{l-1}\varepsilon^{-4}.\end{aligned} \tag{5.85}$$

Using (5.42) with $V = V_0$, $f = \mathrm{id}$ and $g = \mathrm{id} + t_{l-1}h_\varepsilon(\cdot, V_0)$ combined with Proposition 5.1.2 and (5.23)

$$\begin{aligned}\Delta(V_0, V_{\varepsilon,\mathcal{T}'}(t_{l-1})) &\le \|V_0\|(\mathbb{R}^n)\left(28\|t_{l-1}h_\varepsilon(\cdot, V_0)\|_{C^1} + \|J.(\mathrm{id} + t_{l-1}h_\varepsilon(\cdot, V_0)) - J.(\mathrm{id})\|_\infty\right)\\ &\le M\left(56c_1Mt_{l-1}\varepsilon^{-4} + \|J.(\mathrm{id} + t_{l-1}h_\varepsilon(\cdot, V_0)) - 1\|_\infty\right)\end{aligned}$$

from (5.23) together with (5.1.2) we infer that

$$\|J.(\mathrm{id} + t_{l-1}h_\varepsilon(\cdot, V_0)) - 1\|_\infty \le c_4 t_{l-1}\|Dh_\varepsilon(\cdot, V_0)\|_\infty \le c_1c_4Mt_{l-1}\varepsilon^{-4}$$

thus

$$\Delta(V_0, V_{\varepsilon,\mathcal{T}'}(t_{l-1})) \le M\left(56c_1Mt_{l-1}\varepsilon^{-4} + c_1c_4Mt_{l-1}\varepsilon^{-4}\right) \le 57c_1c_4M^2t_{l-1}\varepsilon^{-4}. \tag{5.86}$$

Summing up (5.84), (5.85) and (5.86) we obtain (recalling that $c_9 = 2c_7(57c_1c_4 + c_8)$)

$$\begin{aligned}&\Delta\left(\left(\mathrm{id} + d_lh_\varepsilon(\cdot, \tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}))\right)_{\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), \tilde{f}_{l\#}\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1})\right)\\ &\le 2c_7d_l(M+1)^2(57c_1c_4M^2t_{l-1}\varepsilon^{-4} + c_8M^3t_{l-1}\varepsilon^{-4})\varepsilon^{-n-7}\\ &\le c_9d_l(M+1)^5t_{l-1}\varepsilon^{-n-11}.\end{aligned}$$

Thus,

$$\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_l), W_{\varepsilon,\mathcal{T}}(t_l)) \le \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_{l-1}), W_{\varepsilon,\mathcal{T}}(t_{l-1}))\left(1 + c_{7,M}d_l\varepsilon^{-n-7}\right) + c_{9,M}d_lt_l\varepsilon^{-n-11}. \tag{5.87}$$

Iterating 5.87 for $l \in \{1, \dots i\}$,

$$\begin{aligned}\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) &\le \Delta(V(0), W(0))\prod_{l=1}^{i}\left(1 + c_{7,M}d_l\varepsilon^{-n-7}\right)\\ &\quad + \sum_{l=0}^{i-1}(c_{9,M}d_{i-l}t_{i-l}\varepsilon^{-n-11})\prod_{j=0}^{l-1}\left(1 + c_{7,M}d_{i-j}\varepsilon^{-n-7}\right)\\ &= \Delta(V(0), W(0))\exp\left(c_{7,M}t_i\varepsilon^{-n-7}\right) + A.\end{aligned}$$

To bound A, we note first that $t_{i-l} \le t_i$ for any $l \in \{0, \dots, i\}$ and that

$$\prod_{j=0}^{l-1}\left(1 + c_{7,M}d_{i-j}\varepsilon^{-n-7}\right) \le \prod_{j=0}^{i-1}\left(1 + c_{7,M}d_{i-j}\varepsilon^{-n-7}\right) \underbrace{\le}_{\sum_{j=1}^{i} d_j = t_i} \exp\left(c_{7,M}t_i\varepsilon^{-n-7}\right).$$

Therefore,

$$\begin{aligned}A &= \sum_{l=0}^{i-1}(c_{9,M}d_{i-l}t_{i-l}\varepsilon^{-n-11})\left(\prod_{j=0}^{l-1}\left(1 + c_{7,M}d_{i-j}\varepsilon^{-n-7}\right)\right)\\ &\le \sum_{l=0}^{i-1}(c_{9,M}d_{i-l}t_i\varepsilon^{-n-11})\exp\left(c_{7,M}t_i\varepsilon^{-n-7}\right) \le c_{9,M}t_i^2\varepsilon^{-n-11}\exp\left(c_{7,M}t_i\varepsilon^{-n-7}\right).\end{aligned}$$

This yields,

$$\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) \le \Delta(V_0, W_0)\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big) + c_{9,M}t_i^2\varepsilon^{-n-11}\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big).$$

Finally, from (5.83) we affirm that $\forall i \in \{0,1\dots m\}$,

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) &\le \Delta(V_{\varepsilon,\mathcal{T}'}(t_i), \tilde{V}_{\varepsilon,\mathcal{T}}(t_i)) + \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i))\\ &\le c_8M^3t_i^2\varepsilon^{-10} + \Delta(V_0,W_0)\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big) + c_{9,M}t_i^2\varepsilon^{-n-11}\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big).\end{aligned} \tag{5.88}$$

Therefore, noting that $c_9 \ge c_8$ we can affirm that $c_{9,M} = c_9(M+1)^5 \ge c_8M^3$ and that $\forall i \in \{0,1\dots m\}$,

$$\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) \le \Delta(V_0, W_0)\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big) + 2c_{9,M}t_i^2\varepsilon^{-n-11}\exp\big(c_{7,M}t_i\varepsilon^{-n-7}\big),$$

this finishes the proof. □

In the following lemma, we use 5.1.13 to show Proposition 5.1.11 (stability with respect to subdivision) in the special case where the two subdivisions are nested (one included in the other).

Lemma 5.1.15. *Let $\varepsilon \in (0,1)$, $M \ge 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \le M$. Consider $\mathcal{T}_1 = \{t_i\}_{i=1}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ two subdivisions (1.1.1) of $[0,1]$ satisfying (5.34), assume that $\mathcal{T}_1 \subset \mathcal{T}_2$, set $\delta = \delta(\mathcal{T}_1)$. Then, if we denote by $V_{\varepsilon,\mathcal{T}_1}(t)$ the time-discrete approximate MCF of V_0 with respect to $\mathcal{T}_1$ and by $V_{\varepsilon,\mathcal{T}_2}(t)$ the time-discrete approximate MCF of V_0 with respect to $\mathcal{T}_2$. We have*

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \le 2c_{9,M}t\delta\varepsilon^{-n-11}\exp\big(c_{7,M}t\varepsilon^{-n-7}\big)$$

for all $t \in [0,1]$.

Proof. We set

$$d_i := t_i - t_{i-1},\ \forall i \in \{1,\dots,m\}.$$

Step1: we bound $\Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i))$ for $i \in \{0,1\dots,m\}$.
Fix $i \in \{0,1\dots,m\}$, for any $l \in \{1,\dots,i\}$, we use (5.82) on the interval $[t_{l-1},t_l]$ with $V_0 = V_{\varepsilon,\mathcal{T}_1}(t_{l-1})$, $W_0 = V_{\varepsilon,\mathcal{T}_2}(t_{l-1})$, $\mathcal{T}'$ being the trivial subdivision of $[t_{l-1},t_l]$ and $\mathcal{T} = \mathcal{T}_2 \cap [t_{l-1},t_l]$ (where we replace δ by $t_l - t_{l-1}$) to obtain

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t_l), V_{\varepsilon,\mathcal{T}_2}(t_l)) \le \Delta(V_{\varepsilon,\mathcal{T}_1}(t_{l-1}), V_{\varepsilon,\mathcal{T}_2}(t_{l-1}))\exp\big(c_{7,M}d_l\varepsilon^{-n-7}\big) + 2c_{9,M}d_l^2\varepsilon^{-n-11}\exp\big(c_{7,M}d_l\varepsilon^{-n-7}\big). \tag{5.89}$$

Iterating (5.89) for $l \in \{0,1\dots i\}$ we get:

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i)) &\le \Delta(V_{\varepsilon,\mathcal{T}_1}(0), V_{\varepsilon,\mathcal{T}_2}(0))\prod_{l=1}^{i}\exp\big(c_{7,M}d_l\varepsilon^{-n-7}\big)\\ &\quad + \sum_{l=0}^{i-1}(2c_{9,M}d_{i-l}^2\varepsilon^{-n-11})\left(\prod_{j=0}^{l-1}\exp\big(c_{7,M}d_{i-j}\varepsilon^{-n-7}\big)\right)\\ &= \sum_{l=0}^{i-1}(2c_{9,M}d_{i-l}^2\varepsilon^{-n-11})\left(\prod_{j=0}^{l-1}\exp\big(c_{7,M}d_{i-j}\varepsilon^{-n-7}\big)\right) =: A,\end{aligned}$$

where we used $\Delta(V_{\varepsilon,\mathcal{T}_1}(0), V_{\varepsilon,\mathcal{T}_2}(0)) = \Delta(V_0, V_0) = 0$. To bound A we write

$$\prod_{j=0}^{l-1} \exp\big(c_{7,M} d_{i-j} \varepsilon^{-n-7}\big) \le \prod_{j=0}^{i-1} \exp\big(c_{7,M} d_{i-j} \varepsilon^{-n-7}\big) \underbrace{\le}_{\sum_{j=1}^{i} d_j = t_i} \exp\big(c_{7,M} t_i \varepsilon^{-n-7}\big).$$

Using again $\sum_{j=1}^{i} d_j = t_i$, with $d_j \le \delta$, we infer that

$$A = \sum_{l=0}^{i-1} (2c_{9,M} d_{i-l}^2 \varepsilon^{-n-11}) \left(\prod_{j=0}^{l-1} \exp\big(c_{7,M} d_{i-j} \varepsilon^{-n-7}\big) \right) \le 2c_{9,M} t_i \delta \varepsilon^{-n-11} \exp\big(c_{7,M} t_i \varepsilon^{-n-7}\big)$$

thus, for $i \in \{0, 1 \ldots m\}$, we have:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i)) \le 2c_{9,M} t_i \delta \varepsilon^{-n-11} \exp\big(c_{7,M} t_i \varepsilon^{-n-7}\big). \tag{5.90}$$

Step2: we bound $\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k))$ for $k \in \{0, 1 \ldots, m'\}$.
Fix $k \in \{0, 1 \ldots, m'\}$, for $s_k \in \mathcal{T}_2$, let $i \in \{0, 1 \ldots m-1\}$ be such that $s_k \in [t_i, t_{i+1}]$ applying Lemma 5.1.13 on the interval $[t_i, t_{i+1}]$ with $V_0 = V_{\varepsilon,\mathcal{T}_1}(t_i)$, $W_0 = V_{\varepsilon,\mathcal{T}_2}(t_i)$, $\mathcal{T}'$ being the trivial subdivision of $[t_i, t_{i+1}]$, $\mathcal{T} = \mathcal{T}_2 \cap [t_i, t_{i+1}]$, and using (5.90)

$$\begin{aligned}
&\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \le \Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i)) \exp\big(c_{7,M}(s_k - t_i)\varepsilon^{-n-7}\big) \\
&+ 2c_{9,M}(s_k - t_i)^2 \varepsilon^{-n-11} \exp\big(c_{7,M}(s_k - t_i)\varepsilon^{-n-7}\big) \\
&\le 2c_{9,M} t_i \delta \varepsilon^{-n-11} \exp\big(c_{7,M} t_i \varepsilon^{-n-7}\big) \exp\big(c_{7,M}(s_k - t_i)\varepsilon^{-n-7}\big) \\
&+ 2c_{9,M}(s_k - t_i)^2 \varepsilon^{-n-11} \exp\big(c_{7,M}(s_k - t_i)\varepsilon^{-n-7}\big).
\end{aligned}$$

Noting that $\exp\big(c_{7,M} t_i \varepsilon^{-n-7}\big) \exp\big(c_{7,M}(s_k - t_i)\varepsilon^{-n-7}\big) = \exp\big(c_{7,M} s_k \varepsilon^{-n-7}\big)$ and

$$t_i \delta + (s_k - t_i)^2 \le \delta(t_i + s_k - t_i) \le \delta s_k$$

therefore, for all $s_k \in \mathcal{T}_2$ we have:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \le 2c_{9,M} s_k \delta \varepsilon^{-n-11} \exp\big(c_{7,M} s_k \varepsilon^{-n-7}\big). \tag{5.91}$$

Step3: we bound $\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t))$ for $t \in [0, 1]$.
Let $t \in [s_k, s_{k+1}]$ for some $k \in \{0, 1, \ldots, m'-1\}$, applying Proposition 5.1.8 with $V = V_{\varepsilon,\mathcal{T}_1}(s_k)$, $W = V_{\varepsilon,\mathcal{T}_2}(s_k)$ and $\Delta t = t - s_k$ we obtain:

$$\begin{aligned}
\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) &\le \Delta\Big((\mathrm{id} + (t - s_k) h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}_1}(s_k)))_\# V_{\varepsilon,\mathcal{T}_1}(s_k), (\mathrm{id} + (t - s_k) h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}_2}(s_k)))_\# V_{\varepsilon,\mathcal{T}_2}(s_k)\Big) \\
&\le (1 + (t - s_k) c_{7,M} \varepsilon^{-n-7}) \Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \\
&\le \exp\left((t - s_k) c_{7,M} \varepsilon^{-n-7}\right) \Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)).
\end{aligned}$$

From (5.91) we conclude that for all $t \in [0, 1]$.

$$\begin{aligned}
\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) &\le \exp\left((t - s_k) c_{7,M} \varepsilon^{-n-7}\right) 2c_{9,M} s_k \delta \varepsilon^{-n-11} \exp\big(c_{7,M} s_k \varepsilon^{-n-7}\big) \\
&\le 2c_{9,M} t \delta \varepsilon^{-n-11} \exp\big(c_{7,M} t \varepsilon^{-n-7}\big)
\end{aligned}$$

and this ends the proof of Lemma 5.1.15. □

The proof of Proposition 5.1.11 comes as a direct consequence of Lemma 5.1.15 by introducing a union subdivision and using the triangle inequality.

proof of Proposition 5.1.11 . We start by setting $c_{10,M} = 4c_{9,M}$ and $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$. Let $V_{\varepsilon,\mathcal{T}_3}(t)$ be the time-discrete approximate MCF with respect to $\mathcal{T}_3$ starting from V_0. By Proposition 5.1.15 we infer that $\forall t \in [0,1]$:

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) &\leq \Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_3}(t)) + \Delta(V_{\varepsilon,\mathcal{T}_3}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \\ &\leq 2c_{9,M} t\delta\varepsilon^{-n-11} \exp\big(c_{7,M} t\varepsilon^{-n-7}\big) + 2c_{9,M} t\delta\varepsilon^{-n-11} \exp\big(c_{7,M} t\varepsilon^{-n-7}\big) \\ &\leq \underbrace{4c_{9,M}}_{=c_{10,M}} t\delta\varepsilon^{-n-11} \exp\big(c_{7,M} t\varepsilon^{-n-7}\big)\end{aligned} \tag{5.92}$$

which concludes the proof of Proposition 5.1.11. □

We conclude the section with the following corollary. It encompasses the previous results on the stability of the time-discrete approximate MCF.

Corollary 5.1.16. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let V_0, W_0 be two varifolds in $V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$, $\|W_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T}_1 = \{t_i\}_{i=1}^{m}$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ be two subdivisions (1.1.1) of $[0,1]$ satisfying (5.34). Let $V_{\varepsilon,\mathcal{T}_1}(t)$ (resp.$W_{\varepsilon,\mathcal{T}_2}(t)$) be the time-discrete approximate MCF with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) starting from V_0 (resp. W_0).*
If we set: $\delta = \max\{\delta(\mathcal{T}_1), \delta(\mathcal{T}_2)\}$, we have:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), W_{\varepsilon,\mathcal{T}_2}(t)) \leq \Delta(V_0, W_0) \exp\big(tc_{7,M}\varepsilon^{-n-7}\big) + c_{10,M} t\delta\varepsilon^{-n-11} \exp\big(tc_{7,M}\varepsilon^{-n-7}\big),$$

for all $t \in [0,1]$.

Proof. We start by setting $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$. Let $V_{\varepsilon,\mathcal{T}_3}(t)$ (resp. $W_{\varepsilon,\mathcal{T}_3}(t)$) be the time-discrete approximate MCF with respect to $\mathcal{T}_3$ and starting from V_0 (resp. W_0). By Lemma 5.1.15 we infer that: for all $t \in [0,1]$,

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_3}(t)) \leq 2c_{9,M} t\delta\varepsilon^{-n-11} \exp\big(c_{7,M} t\varepsilon^{-n-7}\big),$$

and,

$$\Delta(W_{\varepsilon,\mathcal{T}_2}(t), W_{\varepsilon,\mathcal{T}_3}(t)) \leq 2c_{9,M} t\delta\varepsilon^{-n-11} \exp\big(c_{7,M} t\varepsilon^{-n-7}\big).$$

By Proposition 5.1.9 we obtain:

$$\Delta(V_{\varepsilon,\mathcal{T}_3}(t), W_{\varepsilon,\mathcal{T}_3}(t)) \leq \exp\big(tc_{7,M}\varepsilon^{-n-7}\big)\Delta(V_0, W_0).$$

Summing up, one deduces:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), W_{\varepsilon,\mathcal{T}_2}(t)) \leq \Delta(V_0, W_0) \exp\big(tc_{7,M}\varepsilon^{-n-7}\big) + \underbrace{4c_{9,M}}_{c_{10,M}} t\delta\varepsilon^{-n-11} \exp\big(tc_{7,M}\varepsilon^{-n-7}\big)$$

for all $t \in [0,1]$, this concludes the proof. □

5.2 Existence, uniqueness and properties of the limit approximate flow when the time step tends to 0

For any given varifold of finite mass, $\varepsilon \in (0,1)$ and $\mathcal{T}$ a subdivision of $[0,1]$ we constructed a time-discrete approximate MCF that we denoted by $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ (see Definition 5.1.4). The goal of this section is to prove that, no matter how the successive finer subdivisions are chosen, the flow converges to a unique limit (Theorem 5.2.1), we call it the approximate MCF and we denote it by $(V_\varepsilon(t))_{t\in[0,1]}$. We then exhibit some properties of this limit, namely, the stability with respect to the initial datum (Proposition 5.2.3) and the decay of the mass (Remark 5.2.7). We prove in Proposition 5.2.5 that $V_\varepsilon(t)$ satisfies a Brakke-type inequality (referring to inequality (2.3)) with respect to its approximate mean curvature.

5.2.1 Existence and uniqueness of a limit approximate flow

In the following theorem, we show that the time-discrete approximate MFCs starting from a given varifold V_0 of compact support converges to the same limit no matter how the time step tends to 0. The proof is based on the uniform boundedness of the masses (see Remark 5.35) and the stability result with respect to the subdivision (Proposition 5.1.11).

Theorem 5.2.1 (Convergence)**.** *Let $\varepsilon \in (0,1)$, $M \geq 1$ and let $V_0 \in V_d(\mathbb{R}^n)$ be a varifold of compact support and satisfying $\|V_0\|(\mathbb{R}^n) \leq M$. For each $j \in \mathbb{N}$,*

- *let $\mathcal{T}_j^D = \{k\, 2^{-j}\}_{k=0,1,\dots,2^j}$ be the dyadic subdivision (1.1.1) of the interval $[0,1]$ of size $\delta(\mathcal{T}_j^D) = 2^{-j} \xrightarrow[j\to\infty]{} 0$,*
- *let $V_{\varepsilon,\mathcal{T}_j^D}(t)_{t\in[0,1]}$ be the time-discrete approximate MCF with respect to $\mathcal{T}_j^D$ starting from V_0. Note that according to condition (5.34) in Definition 5.1.4, such a flow is well-defined for j large enough so that $c_5 2^{-j} \leq (M+1)^{-3}\varepsilon^8$.*

Then,

(i) there exists a family $(V_\varepsilon(t))_{t\in[0,1]}$ in $V_d(\mathbb{R}^n)$ such that for any $t\in[0,1]$:

1. *$\|V_\varepsilon(t)\|(\mathbb{R}^n) \leq M+1$,*
2. *$V_{\varepsilon,\mathcal{T}_j^D}(t) \xrightharpoonup{*} V_\varepsilon(t)$,*
3. *$\Delta\left(V_{\varepsilon,\mathcal{T}_j^D}(t), V_\varepsilon(t)\right) \to 0 \quad$ as $j \to +\infty$.*

(ii) If $(\mathcal{T}_j)_{j\in\mathbb{N}}$ is any other sequence of subdivisions of size $\delta(\mathcal{T}_j)$ tending to 0, then $V_{\varepsilon,\mathcal{T}_j}(t)_{t\in[0,1]}$ (the time-discrete approximate MCF with respect to $\mathcal{T}_j$ starting from V_0) converges to the same family $(V_\varepsilon(t))_{t\in[0,1]}$ as for the dyadic subdivisions: for any $t\in[0,1]$,

$$V_{\varepsilon,\mathcal{T}_j}(t) \xrightharpoonup{*} V_\varepsilon(t) \quad \text{and} \quad \Delta\left(V_{\varepsilon,\mathcal{T}_j}(t), V_\varepsilon(t)\right) \to 0 \quad \text{as } j \to +\infty\,.$$

In other words, there exists a unique limit flow $(V_\varepsilon(t))_{t\in[0,1]}$ starting from V_0, that we call the approximate MCF of V_0.

Proof. Let $\varepsilon \in (0,1)$, $M \geq 1$ and let $V_0 \in V_d(\mathbb{R}^n)$ be a varifold with compact support and satisfying $\|V_0\|(\mathbb{R}^n) \leq M$.
We start with the proof of (i). For $j \in \mathbb{N}$, let $\mathcal{T}_j^D$ and $V_{\varepsilon,\mathcal{T}_j^D}$ be as in the statement of the theorem. Let $t \in [0,1]$. By construction, we know that for any $t \in [0,1]$:

$$\|V_{\varepsilon,\mathcal{T}_j^D}(t)\|(\mathbb{R}^n) \leq M + 1.$$

Thus, by Banach-Alaoglu's theorem, there exists a subsequence $a_t(j)$ (depending on t) for which the sequence $V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t)$ converges weakly-* to a certain limit denoted by $V_\varepsilon(t)$. Note that up to this point, such a limit could depend on the extraction a_t and on the specific choice of the dyadic subdivisions $(\mathcal{T}_j^D)_j$. We first show that the whole sequence $V_{\varepsilon,\mathcal{T}_j^D}(t)$ (and not only the extracted one) converges to $V_\varepsilon(t)$ as $j \to \infty$.
As V_0 has compact support, there exists $R_0 > 0$ such that $\operatorname{spt} V_0 \subset B(0,R_0) \times G_{d,n}$. Then, thanks to Remark 5.1.5, all the varifolds we are considering hereafter are supported in the common bounded set $B\left(0, R_0 + c_1(M+1)\varepsilon^{-2}\right) \times G_{d,n}$. Applying Proposition 1.2.3, we can deduce that,

$$\Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t)) \xrightarrow[j\to\infty]{} 0 \,.$$

Note that $a_t(j) \geq j$ and therefore the dyadic subdivision $\mathcal{T}^D_{a_t(j)}$ is finer than $\mathcal{T}_j^D$. For j large enough, so that $c_5 2^{-j} \leq (M+1)^{-3}\varepsilon^8$, we can apply Lemma 5.1.15 with $\mathcal{T}_j^D \subset \mathcal{T}^D_{a_t(j)}$ and obtain

$$\Delta(V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t), V_{\varepsilon,\mathcal{T}_j^D}(t)) \leq 2c_9 t \delta_j \varepsilon^{-n-11} \exp\left(c_7 t \varepsilon^{-n-7}\right) \quad \text{with } \delta_j = 2^{-j} \,.$$

This implies

$$\Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}_j^D}(t)) \leq \Delta(V_\varepsilon(t), V_{\mathcal{T}^D_{a_t(j)}}(t)) + \Delta(V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t), V_{\varepsilon,\mathcal{T}_j^D}(t)) \xrightarrow[j\to\infty]{} 0.$$

Thus, the full sequence $V_{\varepsilon,\mathcal{T}_j^D}(t)$ converges to $V_\varepsilon(t)$ for each $t \in [0,1]$ in the bounded Lipschitz topology and thus in the weakly-* topology (again thanks to Proposition 1.2.3).
We now prove (ii). Let $\left(\mathcal{T}_j\right)_{j\in\mathbb{N}}$ be a sequence of subdivisions of size $\delta\left(\mathcal{T}_j\right)$ tending to 0. For j large enough so that $c_5\delta(\mathcal{T}_j) \leq (M+1)^{-3}\varepsilon^8$, let $V_{\varepsilon,\mathcal{T}_j}(t)_{t\in[0,1]}$ be the time-discrete approximate MCF with respect to $\mathcal{T}_j$ starting from V_0. Let $t \in [0,1]$ and set $\tilde{\delta}_j = \max\{\delta(\mathcal{T}_j), \delta(\mathcal{T}_j^D)\}$; we apply Proposition 5.1.11 and obtain

$$\Delta(V_{\varepsilon,\mathcal{T}_j^D}(t), V_{\varepsilon,\mathcal{T}_j}(t)) \leq c_{10,M} t \tilde{\delta}_j \varepsilon^{-n-11} \exp\left(c_{7,M} t \varepsilon^{-n-7}\right) \xrightarrow[j\to\infty]{} 0.$$

This implies that for any $t \in [0,1]$, $V_{\varepsilon,\mathcal{T}_j}(t)$ converges to $V_\varepsilon(t)$ both in the bounded Lipschitz topology and thus in the weak-* topology (again thanks to Proposition 1.2.3).
We conclude that independently of how the time step goes to 0, the limit flow exists and is *unique*, we call it the approximate MCF and we will denote it by $V_\varepsilon(t)$. □

Given $\varepsilon \in (0,1)$ and a subdivision $\mathcal{T}$ of $[0,1]$, we proposed in Remark 5.1.6 an alternative definition $V^{pc}_{\varepsilon,\mathcal{T}}$ of time-discrete approximate MCF: we recall that the difference with $V_{\varepsilon,\mathcal{T}}$ lies in the way the flow is extended from the points $t_0, t_1, \ldots, t_m \in \mathcal{T}$ of the subdivision to any $t \in [0,1]$. While $V_{\varepsilon,\mathcal{T}}$ is defined through a kind of linear interpolation between the flow at time t_i and t_{i+1}, $V^{pc}_{\varepsilon,\mathcal{T}}$ is set to be constant in between such subdivision times. In the following proposition, we derive an error term estimate between both extensions and infer that they lead to the same definition of limit flow $(V_\varepsilon(t))_{t\in[0,1]}$.

Proposition 5.2.2. *Let $\varepsilon \in (0,1)$. Let $\mathcal{T} = \{t_i\}_{i=0}^m$ be a subdivision (1.1.1) of $[0,1]$ satisfying (5.34). Let $V_0 \in V_d(\mathbb{R}^n)$ of compact support and $V_{\varepsilon,\mathcal{T}}(t)$ its time-discrete approximate MCF with respect to $\mathcal{T}$. Let $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ be the associated piecewise constant flow with respect to $\mathcal{T}$ (Remark 5.1.6). Then if we set: $\delta = \delta(\mathcal{T})$, we have:*

$$\Delta(V_{\varepsilon,\mathcal{T}}(t), V^{pc}_{\varepsilon,\mathcal{T}}(t)) \leq c_{11}\delta\varepsilon^{-4}, \quad \forall t \in [0,1],$$

where $c_{11} = (56c_1 + c_1c_4)(M+1)^2$. As a consequence, when the step of the subdivision goes to 0, $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ converges to $V_\varepsilon(t)$ (defined in Theorem 5.2.1): for any $t \in [0,1]$,

$$V^{pc}_{\varepsilon,\mathcal{T}_j}(t) \text{ converges weakly-* to } V_\varepsilon(t) \quad \text{and} \quad \Delta\left(V^{pc}_{\varepsilon,\mathcal{T}_j}(t), V_\varepsilon(t)\right) \to 0 \quad \text{as } j \to +\infty\,. \tag{5.93}$$

Proof. As ε and $\mathcal{T}$ are fixed, we denote $V_{\varepsilon,\mathcal{T}}(t)$ and $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ by $V(t)$ and $V(t)^{pc}$ throughout the proof. Let $t \in [t_i, t_{i+1})$ for some $i \in \{0,\dots,m-1\}$, denote $f = \mathrm{id} + (t-t_i)h_\varepsilon(\cdot, V(t_i))$ and $g = \mathrm{id}$. We have:

$$\Delta(V(t), V^{pc}(t)) = \Delta(V(t), V(t_i)) = \Delta\Big(f_\# V(t_i), g_\# V(t_i))\Big).$$

Using (5.17) we can check that (noting that $\|V(t_i)\| \leq M+1$)

$$\|Df - Dg\|_\infty = (t-t_i)\|Dh_\varepsilon(\cdot, V(t_i))\|_\infty \leq 2c_1(t-t_i)(M+1)\varepsilon^{-4}$$

we then can apply (5.42) with $V = V(t_i)$, to obtain

$$\Delta\Big(f_\# V(t_i), V(t_i)\Big) \leq \|V(t_i)\|(\mathbb{R}^n)\left(28(t-t_i)\|h_\varepsilon(\cdot, V(t_i))\|_{C^1} + \|J.f - 1\|_\infty\right). \tag{5.94}$$

Therefore, by $\|V(t_i)\|(\mathbb{R}^n) \leq M+1$, Proposition 5.1.2 and (5.23) we infer that

$$\begin{aligned}\Delta(V_{\varepsilon,T}(t), V^{pc}_{\varepsilon,T}(t)) &\leq 56(M+1)^2c_1(t-t_i)\varepsilon^{-4} + (M+1)^2c_1c_4(t-t_i)\varepsilon^{-4} \\ &\leq (56c_1 + c_1c_4)(M+1)^2(t-t_i)\varepsilon^{-4} \leq c_{11}\delta\varepsilon^{-4}\end{aligned} \tag{5.95}$$

where we set $c_{11} = (56c_1 + c_1c_4)(M+1)^2$, this concludes the proof. □

Thereafter $(V_\varepsilon(t))_{t\in[0,1]}$ denotes the approximate MCF starting from V_0 as defined in Theorem 5.2.1. We now investigate the properties of this flow, starting with the stability with respect to the initial varifold.

Proposition 5.2.3. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let V_0, W_0 in $V_d(\mathbb{R}^n)$ satisfying $\|V_0\|(\mathbb{R}^n) \leq M$, $\|W_0\|(\mathbb{R}^n) \leq M$ and both compactly supported. Then, for all $t \in [0,1]$,*

$$\Delta(V_\varepsilon(t), W_\varepsilon(t)) \leq \Delta(V_0, W_0)\exp\left(c_{7,M}\, t\, \varepsilon^{-n-7}\right),$$

where V_ε (resp. W_ε) is the approximate MCF starting from V_0 (resp. W_0).

Proof. We fix a sequence of subdivisions $(\mathcal{T}_j)_{j\in\mathbb{N}}$ with time step $\delta_j \to 0$ as $j \to \infty$ (we can take the dyadic subdivisions for instance). Let $\left(V_{\varepsilon,\mathcal{T}_j}(t)\right)_{t\in[0,1]}$ (resp. $\left(W_{\varepsilon,\mathcal{T}_j}(t)\right)_{t\in[0,1]}$) be the time-discrete approximate MCF with respect to $\mathcal{T}_j$ starting from V_0 (resp. W_0). Let j be large enough so that (5.34) holds: $c_5\delta_j \leq (M+1)^{-3}\varepsilon^8$, then we can simply apply Proposition 5.1.9 and obtain for all $t \in [0,1]$,

$$\Delta(V_{\varepsilon,\mathcal{T}_j}(t), W_{\varepsilon,\mathcal{T}_j}(t)) \leq \Delta(V_0, W_0)\exp\left(c_{7,M}\, t\, \varepsilon^{-n-7}\right),$$

and therefore, by the triangle inequality and Theorem 5.2.1, letting j tend to ∞, we can conclude that

$$\begin{aligned}\Delta(V_\varepsilon(t), W_\varepsilon(t)) &\le \Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}_j}(t)) + \Delta(V_{\varepsilon,\mathcal{T}_j}(t), W_{\varepsilon,\mathcal{T}_j}(t)) + \Delta(W_{\varepsilon,\mathcal{T}_j}(t), W_\varepsilon(t)) \\ &\le \Delta(V_0, W_0)\exp\big(c_{7,M}\, t\, \varepsilon^{-n-7}\big)\,.\end{aligned}$$

□

5.2.2 Equality à la Brakke

In Proposition 5.2.5, we show a Brakke-type equality for the flow $(V_\varepsilon(t))_{t\in[0,1]}$ with respect to its approximate mean curvature. The proof consists of taking the limit in inequality (5.97) which results from the expansion of the push-forward varifold formula (5.24) and Theorem 5.2.1. We conclude the section with the decay property of mass $t \mapsto \|V_\varepsilon(t)\|(\mathbb{R}^n)$, which follows directly from (5.96) and (5.25).

We first introduce the following lemma on the regularity of the weighted first variation with respect to the varifold.

Lemma 5.2.4. *Let $\varphi \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}^+)$, $X \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $V, W \in V_d(\mathbb{R}^n)$ of finite mass. Then*

$$|\delta(V,\varphi)(X) - \delta(W,\varphi)(X)| \le 2n\|\varphi\|_{\mathrm{C}^2}\|X\|_{\mathrm{C}^2}\Delta(V,W)\,.$$

Proof. Let $\varphi \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^+)$, $X \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and set $\Theta(x, S) := \varphi(x)\operatorname{div}_S X(x) + \nabla\varphi(x)\cdot X(x)$. From Definition (1.13) one has

$$|\delta(V,\varphi)(X) - \delta(W,\varphi)(X)| \le \operatorname{Lip}(\Theta)\Delta(V,W).$$

It is only left to prove that $\operatorname{Lip}(\Theta) \le n\|\varphi\|_{\mathrm{C}^1}\|X\|_{\mathrm{C}^2}$.
Let $x, y \in \mathbb{R}^n$ and $(S,T) \in G_{d,n}$, we have

$$\begin{aligned}|\varphi(x)\operatorname{div}_S X(x) - \varphi(y)\operatorname{div}_T X(y)| &\le |\varphi(x)-\varphi(y)||\operatorname{div}_S X(x)| + |\varphi(y)||\operatorname{div}_S X(x) - \operatorname{div}_S X(y)| \\ &\quad + |\varphi(y)||\operatorname{div}_S X(y) - \operatorname{div}_T X(y)| := A + B + C.\end{aligned}$$

We recall that $\operatorname{div}_S X = \operatorname{tr}(S\circ DX)$. For the first term we have Lemma 5.5.1

$$\begin{aligned}&A \le n\|\nabla\varphi\|_\infty\|DX(x)\||x-y| \le n\|\nabla\varphi\|_\infty\|DX\|_\infty|x-y|,\\ &B \le n\|\varphi\|_\infty\|S\|\|DX(x)-DX(y)\| \le n\|\varphi\|_\infty\|D^2X\|_\infty|x-y|, \text{ and}\\ &C \le n\|\varphi\|_\infty\|DX(y)\|\|S-T\| \le n\|\varphi\|_\infty\|DX\|_\infty\|S-T\|.\end{aligned}$$

We carry on with the Lipschitz constant of the second term of Θ, we have

$$\operatorname{Lip}(\nabla\varphi\cdot X) \le \|\nabla^2\varphi\|_\infty\|X\|_\infty + \|\nabla\varphi|_\infty\|DX\|_\infty.$$

Finally, from the previous estimates we have

$$\operatorname{Lip}(\Theta) \le (n+1)\|\varphi\|_{\mathrm{C}^2}\|X\|_{\mathrm{C}^2} \le 2n\|\varphi\|_{\mathrm{C}^2}\|X\|_{\mathrm{C}^2}.$$

□

Proposition 5.2.5. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let V_0 in $V_d(\mathbb{R}^n)$ of compact support with $\|V_0\|(\mathbb{R}^n) \leq M$. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate MCF starting from V_0. For any $\varphi \in \mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and $0 \leq a \leq b \leq 1$ we have*

$$\|V_\varepsilon(b)\|(\varphi(\cdot,b)) - \|V_\varepsilon(a)\|(\varphi(\cdot,a)) = \int_a^b \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt + \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\,dt. \tag{5.96}$$

Proof. Let $\varepsilon \in (0,1)$, $M \geq 1$ and $V_0 \in V_d(\mathbb{R}^n)$ of compact support satisfying $\|V_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T} = \{t_i\}_{i=1}^m$ be a uniform subdivision of $[0,1]$ of size Δt satisfying (5.34). Let $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ be the piecewise constant approximate MCF with respect to to $\mathcal{T}$ starting from V_0. We first prove that (5.96) holds for $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ up to an error term of order Δt. More precisely, we prove in Steps 1 and 2 that $\exists C > 0$ (only depending on n and M) such that for any $\varphi \in \mathrm{C}^2(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and $0 \leq a \leq b \leq 1$ we have

$$\begin{aligned}\Bigg| \|V^{pc}_{\varepsilon,\mathcal{T}}(b)\|(\varphi(\cdot,b)) - \|V^{pc}_{\varepsilon,\mathcal{T}}(a)\|(\varphi(\cdot,a)) - \int_a^b \delta(V^{pc}_{\varepsilon,\mathcal{T}}(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V^{pc}_{\varepsilon,\mathcal{T}}(t)))\,dt \\ - \int_a^b \int_{\mathbb{R}^n} \partial_t\varphi(\cdot,t)\,d\|V^{pc}_{\varepsilon,\mathcal{T}}(t)\|dt\Bigg| \leq C\|\varphi\|_{\mathrm{C}^2}\Delta t\varepsilon^{-8}\,.\end{aligned} \tag{5.97}$$

In Step 3, recalling that $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ converges to $(V_\varepsilon(t))_{t\in[0,1]}$ when considering subdivisions $\mathcal{T}$ whose size tends to 0, we take the limit in (5.97) and establish (5.96) for φ of regularity C^2. We conclude the proof of Proposition 5.2.5 applying density of $\mathrm{C}^2(\mathbb{R}^n\times[0,1],\mathbb{R}_+)$ in $\mathrm{C}^1(\mathbb{R}^n\times[0,1],\mathbb{R}_+)$ in Step 4. Remark 5.1.5 states that $\bigcup_{t\in[0,1]} \operatorname{spt} V^{pc}_{\varepsilon,\mathcal{T}}(t)$ is contained in a compact set, denote it by K_ε, for ε fixed independently of the subdivision $\mathcal{T}$, hence $\bigcup_{t\in[0,1]} \operatorname{spt} V_\varepsilon(t)$ is contained in K_ε as well. In the proof, the C^k-norms ($k \in \{1,2\}$) of the test functions ψ and φ are implicitly taken with respect to the set K_ε, hence are finite. In both Steps 1 and 2, $\mathcal{T}$ and ε are fixed and we denote for simplicity $V(t) := V^{pc}_{\varepsilon,\mathcal{T}}(t)$.

Step 1: We prove the inequality (5.97) for $a, b \in \mathcal{T}$: let $\ell \in \{0,1,\dots,m-1\}$ and $\varphi \in \mathrm{C}^2(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$.

We can apply (5.24) in Proposition 5.1.3 to a spatial test function $\psi \in \mathrm{C}^2(\mathbb{R}^n,\mathbb{R}^+)$. We recall that the piecewise constant MCF coincides with the time discrete approximate MCF at the points of the subdivision and furthermore, $f_\# V(t_\ell) = V(t_{\ell+1})$ for f as in (5.24). Therefore,

$$\Big|\|V(t_{\ell+1})\|(\psi) - \|V(t_\ell)\|(\psi) - \Delta t\;\delta(V(t_\ell),\psi)(h_\varepsilon(\cdot,V(t_\ell)))\Big| \leq c_5(M+1)^3\|\psi\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8}.$$

We now recall that $V(t)$ is piecewise constant and thus, for all $t \in (t_\ell, t_{\ell+1})$, $V(t) = V(t_\ell)$ and

$$\int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\psi)(h_\varepsilon(\cdot,V(t)))\,dt = \Delta t\,\delta(V(t_\ell),\psi)(h_\varepsilon(\cdot,V(t_\ell)))$$

so that taking $\psi = \varphi(\cdot,t_{\ell+1})$, we obtain

$$\begin{aligned}\Bigg|\|V(t_{\ell+1})\|(\varphi(\cdot,t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot,t_{\ell+1})) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\varphi(\cdot,t_{\ell+1}))(h_\varepsilon(\cdot,V(t)))\,dt\Bigg| \\ \leq c_5(M+1)^3\|\varphi(\cdot,t_{\ell+1})\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8}\,.\end{aligned} \tag{5.98}$$

Applying the mean value theorem to φ and $\nabla\varphi$:

$$\int_{t_\ell}^{t_{\ell+1}} \|\varphi(\cdot,t)-\varphi(\cdot,t_{\ell+1})\|_{C^1}\,dt \le \int_{t_\ell}^{t_{\ell+1}} \|\varphi\|_{C^2}|t-t_{\ell+1}|\,dt \le \|\varphi\|_{C^2}(\Delta t)^2\,,$$

and then using Remark 1.4.5, Proposition 5.1.2 and (5.98), we have

$$\begin{aligned}
&\left|\|V(t_{\ell+1})\|(\varphi(\cdot,t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot,t_{\ell+1})) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt\right| \\
&\le c_5(M+1)^3\|\varphi(\cdot,t_{\ell+1})\|_{C^2}(\Delta t)^2\varepsilon^{-8} + \int_{t_\ell}^{t_{\ell+1}} |\delta(V(t),\varphi(\cdot,t)-\varphi(\cdot,t_\ell))(h_\varepsilon(\cdot,V(t)))|\,dt \\
&\le c_5(M+1)^3\|\varphi\|_{C^2}(\Delta t)^2\varepsilon^{-8} + \int_{t_\ell}^{t_{\ell+1}} n\|h_\varepsilon(\cdot,V(t))\|_{C^1}\|V(t)\|(\mathbb{R}^n)\|\varphi(\cdot,t)-\varphi(\cdot,t_\ell)\|_{C^1}\,dt \\
&\le c_5(M+1)^3\|\varphi\|_{C^2}(\Delta t)^2\varepsilon^{-8} + nc_1(M+1)^2\|\varphi\|_{C^2}(\Delta t)^2\varepsilon^{-4} \\
&\le (c_5+nc_1)(M+1)^3\|\varphi\|_{C^2}(\Delta t)^2\varepsilon^{-8}.
\end{aligned} \tag{5.99}$$

Writing for $x\in\mathbb{R}^n$, $\varphi(x,t_{\ell+1})-\varphi(x,t_\ell) = \int_{t_\ell}^{t_{\ell+1}} \partial_t\varphi(x,t)\,dt$ and integrating with respect to $\|V(t_\ell)\| = \|V(t)\|$ for all $t\in(t_\ell,t_{\ell+1})$,

$$\begin{aligned}
\|V(t_\ell)\|(\varphi(\cdot,t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot,t_\ell)) &= \int_{x\in\mathbb{R}^n}\int_{t_\ell}^{t_{\ell+1}} \partial_t\varphi(x,t)\,dt\,d\|V(t_\ell)\|(x) \\
&= \int_{t_\ell}^{t_{\ell+1}}\int_{x\in\mathbb{R}^n} \partial_t\varphi(x,t)\,d\|V(t_\ell)\|(x)\,dt = \int_{t_\ell}^{t_{\ell+1}} \|V(t)\|\left(\partial_t\varphi(\cdot,t)\right)\,dt\,.
\end{aligned} \tag{5.100}$$

We can now combine (5.99) and (5.100) to obtain that for $\ell\in\{0,1,\dots,m-1\}$,

$$\begin{aligned}
&\Bigg|\|V(t_{\ell+1})\|(\varphi(\cdot,t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot,t_\ell)) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt \\
&\qquad - \int_{t_\ell}^{t_{\ell+1}} \|V(t)\|(\partial_t\varphi(\cdot,t))\,dt\Bigg| \\
&= \left|\|V(t_{\ell+1})\|(\varphi(\cdot,t_\ell)) - \|V(t_\ell)\|(\varphi(\cdot,t_\ell)) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt\right| \\
&\le (c_5+nc_1)(M+1)^3\|\varphi\|_{C^2}(\Delta t)^2\,\varepsilon^{-8}\,,
\end{aligned}$$

which is (5.97) for $a=t_\ell$ and $b=t_{\ell+1}$. Let now $a=t_p\le t_q=b$, note that if $p=q$, (5.97) is trivial and otherwise, summing up the previous inequality for $\ell\in\{p,\dots,q-1\}$ and using $(q-p)\Delta t = t_q-t_p\le 1$ leads to the inequality (5.97), which concludes the proof of Step 1 (case where $a,b\in\mathcal{T}$).

Step 2**:** We recover the approximate Brakke-type equality (5.97) for any arbitrary a,b. Let $0\le a<b\le 1$ and $\varphi\in C^2(\mathbb{R}^n\times[0,1],\mathbb{R}^+)$. Let the points t_p, $t_q\in\mathcal{T}$ be such that $t_p\le a<t_{p+1}$ and $t_q\le b<t_{q+1}$. We then have $|a-t_p|<\Delta t$ and $|t_q-b|<\Delta t$, and recalling that $V(t)$ is piecewise constant on intervals of the form $[t_\ell, t_{\ell+1})$, we also have $V(a) = V(t_p)$ and $V(b) = V(t_q)$ so that

$$\begin{aligned}\Big| \|V(b)\|(\varphi(\cdot,b)) - \|V(a)\|(\varphi(\cdot,a)) - \|V(t_q)\|(\varphi(\cdot,t_q)) + \|V(t_p)\|(\varphi(\cdot,t_p))\Big| \\ = \Big| \|V(t_q)\|(\varphi(\cdot,b) - \varphi(\cdot,t_q)) - \|V(t_p)\|(\varphi(\cdot,a) - \varphi(\cdot,t_p))\Big| \\ \le 2(M+1)\|\varphi\|_{\mathrm{C}^1} \Delta t \end{aligned} \tag{5.101}$$

thanks to the mean value theorem applied to φ.
Furthermore, for all $t \in [0,1]$, using Remark 1.4.5 and Proposition 5.1.2, we have

$$|\delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))| \le n\|h_\varepsilon(\cdot,V(t))\|_{\mathrm{C}^1}(M+1)\|\varphi(\cdot,t)\|_{\mathrm{C}^1} \le nc_1(M+1)^2\|\varphi\|_{\mathrm{C}^1}\,\varepsilon^{-4}$$

and therefore

$$\begin{aligned}\left| \int_a^b \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt - \int_{t_p}^{t_q} \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt \right| \\ \le (\underbrace{|t_p - a| + |t_q - b|}_{\le 2\Delta t}) \sup_{t\in[0,1]} |\delta(V(t),\varphi)(h_\varepsilon(\cdot,V(t)))| \le 2nc_1(M+1)^2\|\varphi\|_{\mathrm{C}^1}\Delta t\,\varepsilon^{-4}\,. \end{aligned} \tag{5.102}$$

We are left with estimating

$$\left| \int_a^b \|V(t)\|(\partial_t\varphi(\cdot,t))\,dt - \int_{t_p}^{t_q} \|V(t)\|(\partial_t\varphi(\cdot,t))\,dt \right| \le 2(M+1)\|\varphi\|_{\mathrm{C}^1}\Delta t\,. \tag{5.103}$$

We can complete the proof of Step 2 and establish (5.97) by combining (5.101) and (5.102) and (5.103) together with Step 1.
Step 3**:** We show (5.96) restricted to C^2 test functions.
We first recall that the approximate MCF $(V_\varepsilon(t))_{t\in[0,1]}$ starting from V_0 can be obtained as the limit flow ($j \to \infty$) of any time-discrete approximate MCF $(V_{\varepsilon,\mathcal{T}_j}(t))_t$ for subdivisions $\mathcal{T}_j$ of size $\delta(\mathcal{T}_j)$ tending to 0, as stated in Theorem 5.2.1. We can thus choose a sequence of uniform subdivisions $\mathcal{T}_j$ of size $\Delta t_j := \delta(\mathcal{T}_j) \xrightarrow[j\to\infty]{} 0$, and we fix the subdivisions $\mathcal{T}_j = \{t_{\ell,j}\}_{\ell=0,\dots,m_j}$ hereafter, we will write t_ℓ instead of $t_{\ell,j}$ in the proof in order to lighten notations. We additionally recall that the piecewise constant flow $(V^{pc}_{\varepsilon,\mathcal{T}_j}(t))_t$ introduced in Remark 5.1.6 converges as well to $(V_\varepsilon(t))_t$ thanks to Proposition 5.2.2 and as it is more convenient in this proof, we work with $(V^{pc}_{\varepsilon,\mathcal{T}_j}(t))_t$. For the sake of lightening the notation we will denote $V_j(t) := V^{pc}_{\varepsilon,\mathcal{T}_j}(t)$ hereafter.
We carry on with the proof of Step 3, let $\varphi \in \mathrm{C}^2(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ and $0 \le a < b \le 1$, from (5.97) we have for any $j \in \mathbb{N}$,

$$\begin{aligned}\Big| \|V_j(b)\|(\varphi(\cdot,b)) - \|V_j(a)\|(\varphi(\cdot,a)) - \int_a^b \delta(V_j(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_j(t)))\,dt \\ - \int_a^b \|V_j(t)\|(\partial_t\varphi(\cdot,t))\,dt \Big| \le C\|\varphi\|_{\mathrm{C}^2}\Delta t\,\varepsilon^{-8}\,. \end{aligned} \tag{5.104}$$

When j goes to ∞ we know thanks to (5.93) in Proposition 5.2.2 that for all $t \in [0,1]$,

$$V_j(t) \text{ converges weakly-* to } V_\varepsilon(t) \quad \text{and} \quad \Delta(V_j(t),V_\varepsilon(t)) \xrightarrow[j\to\infty]{} 0\,.$$

As a first consequence, we obtain that

$$\|V_j(b)\|(\varphi(\cdot,b)) \xrightarrow[j\to\infty]{} \|V_\varepsilon(b)\|(\varphi(\cdot,b)) \quad \text{and} \quad \|V_j(a)\|(\varphi(\cdot,a)) \xrightarrow[j\to\infty]{} \|V_\varepsilon(a)\|(\varphi(\cdot,a)) \,. \tag{5.105}$$

We recall that $\|V_\varepsilon(t)\|(\mathbb{R}^n) \leq M+1$ and thanks to Proposition 5.1.2 and Lemma 5.1.7, we have for all $t \in [0,1]$,

$$\|h_\varepsilon(\cdot, V_\varepsilon(t))\|_{\mathrm{C}^2} \leq 3c_1(M+1)\varepsilon^{-6} \text{ and } \|h_\varepsilon(\cdot, V_j(t)) - h_\varepsilon(\cdot, V_\varepsilon(t))\|_{\mathrm{C}^1} \leq 2c_6(M+1)\varepsilon^{-n-7}\Delta(V_j(t), V_\varepsilon(t))$$

and therefore, applying Remark 1.4.5, Lemma 5.2.4 and Lemma 5.1.7 (with $V = V_j(t)$ and $W = V_\varepsilon(t)$) we infer

$$\begin{aligned}
&\left|\delta(V_j(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_j(t))) - \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\right| \\
\leq & n(M+1)\|\varphi\|_{\mathrm{C}^1}\, \|h_\varepsilon(\cdot,V_j(t)) - h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^1} + 2n(M+1)\|\varphi\|_{\mathrm{C}^1}\, \|h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^2}\, \Delta(V_j(t),V_\varepsilon(t)) \\
\leq & \,(6c_1 + 2c_6)\, n(M+1)^2 \|\varphi\|_{\mathrm{C}^1}\, \varepsilon^{-n-7}\, \Delta(V_j(t),V_\varepsilon(t)) \,.
\end{aligned}$$

Integrating the previous inequality, we obtain by dominated convergence, noting that for all t, $\Delta(V_j(t), V_\varepsilon(t)) \leq 2(M+1)$:

$$\left| \int_a^b \delta(V_j(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_j(t)))\, dt - \int_a^b \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\, dt \right| \quad \xrightarrow[j\to\infty]{} \quad 0 \quad . \tag{5.106}$$

It remains to let j tend to ∞ in the following term

$$\lim_{j\to\infty} \int_a^b \|V_j(t)\|(\partial_t \varphi(\cdot,t))\, dt = \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\, dt \tag{5.107}$$

where the convergence holds by dominated convergence: for each $t \in [0,1]$ the weakly-* convergence of $V_j(t)$ to $V_\varepsilon(t)$ implies that $\lim_{j\to\infty} \|V_j(t)\|(\partial_t\varphi(\cdot,t)) = \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))$ and the integrands are uniformly bounded by the constant $\|\varphi\|_{\mathrm{C}^1}(M+1)$.
We can eventually let j tend to $+\infty$ in (5.104) and conclude the proof of Step 3 (i.e. (5.96) for all C^2 test function φ) combining the convergence of the 3 terms given by (5.105), (5.106) and (5.107) in the l.h.s. while the r.h.s. tends to 0.
Step 4**:** It remains to check that we can pass from C^2 to C^1 test functions φ in (5.96) to conclude the proof (5.96) for C^1 test functions.
Let $\varphi \in \mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ and apply density of $\mathrm{C}^2(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ in $\mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ to have a sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ such that for all k, $\varphi_k \in \mathrm{C}^2(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ and

$$\|\varphi - \varphi_k\|_{\mathrm{C}^1} \xrightarrow[k\to\infty]{} 0 \,.$$

Thanks to Step 3, we know that (5.96) holds for C^2 functions and thus for all $k \in \mathbb{N}$,

$$\|V_\varepsilon(b)\|(\varphi_k(\cdot,b)) - \|V_\varepsilon(a)\|(\varphi_k(\cdot,a)) = \int_a^b \delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\, dt + \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t))\, dt \tag{5.108}$$

and it remains to check that we can let k tend to $+\infty$, which basically follows from the fact that each term involved in (5.96) is linear continuous with respect to $\varphi \in \mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ endowed with $\|\cdot\|_{\mathrm{C}^1}$. Indeed, we first note that for any $t \in [0,1]$,

$$\Big| \|V_\varepsilon(t)\|(\varphi_k(\cdot,t)) - \|V_\varepsilon(t)\|(\varphi(\cdot,t)) \Big| \leq (M+1)\|\varphi_k(\cdot,t) - \varphi(\cdot,t)\|_\infty \leq (M+1)\|\varphi - \varphi_k\|_{\mathrm{C}^1}$$

and consequently,

$$\lim_{k\to\infty} \|V_\varepsilon(b)\|(\varphi_k(\cdot,b)) = \|V_\varepsilon(b)\|(\varphi(\cdot,b)) \quad \text{and} \quad \lim_{k\to\infty} \|V_\varepsilon(a)\|(\varphi_k(\cdot,a)) = \|V_\varepsilon(a)\|(\varphi(\cdot,a))\,. \tag{5.109}$$

Similarly, for all $t \in [0,1]$,

$$\Big| \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t)) - \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t)) \Big| \leq (M+1)\|\partial_t\varphi_k(\cdot,t) - \partial_t\varphi(\cdot,t)\|_\infty \leq (M+1)\|\varphi - \varphi_k\|_{\mathrm{C}^1}$$

and consequently,

$$\lim_{k\to\infty} \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t))\,dt = \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\,dt\,. \tag{5.110}$$

We can apply Remark 1.4.5 and Proposition 5.1.2 to the remaining term:

$$\begin{aligned} &\Big| \delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t))) - \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t))) \Big| \\ &\qquad \leq n(M+1)\,\|h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^1}\,\|\varphi_k(\cdot,t) - \varphi(\cdot,t)\|_{\mathrm{C}^1} \leq 2c_1 n(M+1)^2\varepsilon^{-4}\|\varphi - \varphi_k\|_{\mathrm{C}^1} \end{aligned}$$

and consequently,

$$\lim_{k\to\infty} \int_a^b \delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt = \int_a^b \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt\,. \tag{5.111}$$

We eventually conclude the proof of Step 4 and hence of the Proposition 5.2.5 thanks to (5.108), (5.109), (5.110) and (5.111).

□

The following result stems from the proof of Proposition 5.2.5. We will use it later to prove Lemma 6.2.2 on the approximate avoidance principle.

Corollary 5.2.6. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let V_0 in $V_d(\mathbb{R}^n)$ of compact support with $\|V_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T}$ be a subdivision of $[0,1]$ satisfying (5.34). Let $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ be the piecewise constant approximate MCF starting from V_0. For any $\varphi \in \mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and $0 \leq a \leq b \leq 1$ we have*

$$\begin{aligned} \Big| \|V^{pc}_{\varepsilon,\mathcal{T}}(b)\|(\varphi(\cdot,b)) - \|V^{pc}_{\varepsilon,\mathcal{T}}(a)\|(\varphi(\cdot,a)) - \int_a^b \delta(V^{pc}_{\varepsilon,\mathcal{T}}(t),\varphi(\cdot,t))(h_\varepsilon(t))\,dt \\ - \int_a^b \int_{\mathbb{R}^n} \partial_t\varphi(\cdot,t)\,d\|V^{pc}_{\varepsilon,\mathcal{T}}(t)\|dt \Big| \leq c_{12}\|\varphi\|_{\mathrm{C}^2}\Delta t\varepsilon^{-8}. \end{aligned} \tag{5.112}$$

We conclude this section by noting a straightforward though important consequence of the Brakke-type equality we established in Proposition 5.2.5: the mass $t \mapsto \|V_\varepsilon(t)\|(\mathbb{R}^n)$ is non-increasing along the flow.

Remark 5.2.7. Let $\varepsilon \in (0,1)$, let $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$ and assume that V_0 is compactly supported. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate MCF starting from V_0. Then, for all $0 \leq a < b \leq 1$,

$$\|V_\varepsilon(b)\|(\mathbb{R}^n) - \|V_\varepsilon(a)\|(\mathbb{R}^n) = -\int_a^b \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y) + \varepsilon}\, dy\, dt \leq 0\,.$$

In particular,

$$\|V_\varepsilon(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n),\ \forall t \in [0,1], \tag{5.113}$$

and

$$\int_0^1 \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y) + \varepsilon}\, dy\, dt \leq \|V_0\|(\mathbb{R}^n).$$

Indeed, let $0 \leq a < b \leq 1$, applying Proposition 5.2.5 with the constant test function $\varphi : (y,t) \mapsto 1 \in \mathrm{C}^1(\mathbb{R}^n \times [0,1], \mathbb{R}_+)$ we have $\partial_t \varphi = 0$, $\delta(V_\varepsilon(t), \varphi(\cdot,t)) = \delta V_\varepsilon(t)$ (see (1.13)) and then using (5.25) in Proposition 5.1.3, we infer

$$\|V_\varepsilon(b)\|(\mathbb{R}^n) - \|V_\varepsilon(a)\|(\mathbb{R}^n) = \int_a^b \delta V_\varepsilon(t)\left(h_\varepsilon(\cdot, V_\varepsilon(t))\right)\, dt = -\int_a^b \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y) + \varepsilon}\, dy\, dt.$$

5.3 Convergence of the approximate mean curvature flows (as $\varepsilon \to 0$), spacetime Brakke flows

For a varifold $V_0 \in V_d(\mathbb{R}^n)$ of compact support, we constructed an approximate MCF $(V_\varepsilon(t))_{t\in[0,1]}$ (Theorem 5.2.1) as the limit of a time-discrete approximate MCF (Definition 5.1.4) when the time step of the subdivision goes to 0 for a fixed ε. In this section we investigate the behaviour of $(V_\varepsilon(t))_{t\in[0,1]}$ when $\varepsilon \to 0$ and the properties of the limit.
Following the works of Brakke and Kim & Tonegawa, one can prove the convergence, up to extraction independently of t, of $(\|V_\varepsilon(t)\|)_{t\in[0,1]}$ to a limit measure $\mu(t)$ for all $t \in [0,1]$. The convergence is first established for a common sequence $(\varepsilon_j)_j$ for all dyadic numbers of $[0,1]$ thanks to the uniform boundedness of the mass and a diagonal extraction argument. Then it extends to all $t \in [0,1]$ using the continuity property of $t \mapsto \|V_\varepsilon(t)\|$. The previous procedure does not work for the measures $(V_\varepsilon(t))_{t\in[0,1]}$ because of the lack of the continuity property. This issue is very common when studying the compactness of Brakke-type flows (cf. for instance [46, Theorem 3.7]). To circumvent this issue, we consider the tensor product measure $V_\varepsilon(t) \otimes dt$.

In section 5.3.1, we introduce the notions of spacetime mean curvature and spacetime Brakke flow and list some of their properties. In section 5.3.2, we prove that the measures $V_\varepsilon(t) \otimes dt$ converges due to uniform boundedness of the mass in ε, up to an extraction, to a limit measure denoted by λ. We prove that λ has a bounded spacetime mean curvature in L^2, and that λ is a spacetime Brakke flow given that its $\mathbb{R}^n \times G_{d,n}$-component is rectifiable.

We introduce as in [36] classes of test functions and vector fields that are suitable for studying the behaviour of the approximate MCFs. For $j \in \mathbb{N}$ we define

$$\mathcal{A}_j := \{\varphi \in C^2(\mathbb{R}^n; \mathbb{R}^+) : \varphi(x) \leq 1, |\nabla\varphi(x)| \leq j\varphi(x), \|\nabla^2\varphi(x)\| \leq j\varphi(x) \text{ for all } x \in \mathbb{R}^n\},$$

$$\mathcal{B}_j := \{g \in C^2(\mathbb{R}^n; \mathbb{R}^n) : |g(x)| \leq j, \|\nabla g(x)\| \leq j, \|\nabla^2 g(x)\| \leq j \text{ for all } x \in \mathbb{R}^n \text{ and } \|g\|_{L^2(\mathbb{R}^n)} \leq j\}.$$

5.3.1 Spacetime mean curvature and spacetime Brakke flows

Definition 5.3.1 (Spacetime mean curvature)**.** *Let β be a Radon measure on $\mathbb{R}^n \times G_{d,n} \times [0,1]$.*

- *We define* the first variation of β*: for any vector field $X \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$,*

$$\delta\beta(X) := \int_0^1 \int_{\mathbb{R}^n \times G_{d,n}} \mathrm{div}_S X(y,t) d\beta(y,S,t).$$

- *We introduce* the spacetime mass $\|\beta\|$ *of β: $\|\beta\| := \Pi_{\#}\beta$ is a Radon measure in $\mathbb{R}^n \times [0,1]$, where $\Pi : \mathbb{R}^n \times G_{d,n} \times [0,1] \to \mathbb{R}^n \times [0,1]$ is the canonical projection.*

- *If in addition, $\delta\beta$ is bounded in the $\mathrm{C}_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$–topology, we say that β has bounded first variation and then by Riesz representation theorem $\delta\beta$ is identified with a vector-valued Radon measure (also denoted by $\delta\beta$)*

$$\delta\beta(X) = \int_{\mathbb{R}^n \times [0,1]} X \cdot \, d\delta\beta \quad \forall X \in \mathrm{C}(\mathbb{R}^n \times [0,1], \mathbb{R}^n) \, .$$

  In this case, thanks to Radon Nikodym decomposition, we can assert that there exists a Radon measure $(\delta\beta)_s$ singular with respect to $\|\beta\|$ and $h(\cdot,\cdot,\beta) \in L^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n, \|\beta\|)$ that we call the spacetime mean curvature*, satisfying $\delta\beta = -h\|\beta\| + (\delta\beta)_s$ that is:*

$$\delta\beta(X) := -\int_0^1 \int_{\mathbb{R}^n} h(y,t,\beta) \cdot X(y,t) d\|\beta\|(y,t) + (\delta\beta)_s(X), \tag{5.114}$$

  for any $X \in \mathrm{C}_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$.

The following remark, and more precisely (5.115), justifies the consistency of Definition 5.3.1 with respect to the usual first variation and mean curvature of varifolds, considering the case where $\beta = W(t) \otimes dt$ for a family of varifolds $(W(t))_t$ with bounded first variation.

Remark 5.3.2. Let $(W(t))_{t\in[0,1]}$ be a family of d–varifolds with bounded variation for a.e $t \in [0,1]$, and write for a.e $t \in [0,1]$,

$$\delta W(t) = -h(\cdot, W)\|W(t)\| + (\delta W(t))_s \text{ with } h(\cdot, W(t)) \in L^1(\mathbb{R}^n, \|W(t)\|) \text{ and } (\delta W(t))_s \perp \|W(t)\| \, .$$

Let $\beta := W(t) \otimes dt$, and we write $\|\beta\| = \|W(t)\| \otimes dt$, $\delta\beta = \delta W(t) \otimes dt$. Assume that β has a bounded first variation in the sense of Definition 5.3.1, then

$$h(y,t,\beta) = h(y, W(t)) \quad \text{for } \|\beta\|\text{–a.e } (y,t) \in \mathbb{R}^n \times [0,1]. \tag{5.115}$$

Indeed, we have for any $X \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$

$$\begin{aligned}
\delta\beta(X) &= \int_{t=0}^1 \left(\int_{\mathbb{R}^n \times G_{d,n}} \mathrm{div}_S X(y,t) \, dW(t)(y,S) \right) dt = \int_{t=0}^1 \delta W(t)(X) \, dt \\
&= -\int_0^1 \int_{\mathbb{R}^n} h(y, W(t)) \cdot X(y,t) \, d\|W(t)\|(y) dt + \int_0^1 (\delta W(t))_s(X) \, dt. \\
&= -\int_{\mathbb{R}^n \times [0,1]} h(y, W(t)) \cdot X(y,t) d\|\beta\|(y,t) + \int_0^1 (\delta W(t))_s(X) \, dt \, .
\end{aligned}$$

We finally obtain $\delta\beta = -h(\cdot, W(t))\|\beta\| + (\delta W(t))_s \otimes dt$. As $(\delta W(t))_s$ is singular with respect to $\|W(t)\|$ for a.e $t \in [0,1]$ implies that $(\delta W(t))_s \otimes dt$ is singular with respect to $\|W\| \otimes dt = \|\beta\|$ (Lemma 5.3.3), (5.115) follows from the uniqueness of the Radon measure decomposition (up to a set of zero measure).

Lemma 5.3.3. *Let $(\alpha_t)_{t\in[0,1]}$ and $(\beta_t)_{t\in[0,1]}$ be two families of measures on $\mathbb{R}^n$ such that $\alpha_t \perp \beta_t$ for a.e $t \in [0,1]$. Then, $\alpha_t \otimes dt \perp \beta_t \otimes dt$.*

Proof. For a.e $t \in [0,1]$, let A_t be such that $\operatorname{spt}\alpha_t \subset A_t$ and $\beta_t\,(\operatorname{spt}\beta_t \setminus A_t^c) = 0$. We know that $\operatorname{spt}\alpha_t \otimes dt \subset \bigcup_{t\in[0,1]}(A_t \times \{t\})$ and

$$(\beta_t \otimes dt)\left(\operatorname{spt}(\beta_t \otimes dt) \setminus (\bigcup_{t\in[0,1]} A_t^c \times \{t\})\right) = 0$$

thus,

$$(\beta_t \otimes dt)\left(\operatorname{spt}(\beta_t \otimes dt) \setminus (\bigcup_{t\in[0,1]} A_t \times \{t\})^c\right) = 0,$$

and this proves the lemma.
Notice that $\left(\bigcup_{t\in[0,1]}(A_t \times \{t\})\right)^c = \bigcup_{t\in[0,1]}(\operatorname{spt} A_t \times \{t\})^c$, this finishes the proof. □

We now give the definition of spacetime Brakke flows.

Definition 5.3.4. *(Spacetime Brakke flows) Let λ be a finite Radon measure on $\mathbb{R}^n \times G_{d,n} \times [0,1]$. λ is called a spacetime Brakke flow if:*

(i) *There exist $(\mu(t))_{t\in[0,1]}$, a family of Radon measures on $\mathbb{R}^n$, $\forall t \in [0,1]$, we call it the mass measure of λ, and $\nu_{(x,t)}$ a family of probability measures for $(x,t) \in \mathbb{R}^n \times [0,1]$ such that $\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt$.*

(ii) *$\delta\lambda$ is bounded and $(\delta\lambda)_s = 0$.*

(iii) *(Integral Brakke inequality). For any $\varphi \in C_c^1(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, $0 \le t_1 \le t_2 \le 1$ we have*

$$\mu(t_2)(\varphi(\cdot,t_2)) - \mu(t_1)(\varphi(\cdot,t_1)) \le -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi(y,t)|h(y,t,\lambda)|^2\, d\mu(t)(y)dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n\times G_{d,n}} S^\perp(\nabla\varphi(y,t))\cdot h(y,t,\lambda)\, d\lambda(y,S,t) + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \partial_t\varphi(\cdot,t)\, d\mu(t)dt \tag{5.116}$$

where $h(\cdot,\cdot,\lambda)$ is the spacetime mean curvature of λ. We say that λ starts from $V_0 = \mu(0) \otimes \nu_{(x,0)}$.

Remark 5.3.5. We give a remark on the definition of the spacetime Brakke flow and two important direct consequences:

(i) One could assume only that $\lambda = V(t) \otimes dt$, with $(V(t))_{t\in[0,1]}$ being a family of measures on $\mathbb{R}^n \times G_{d,n}$. Actually, using Young's disintegration theorem [5, Theorem 2.28] we infer that there exists a family of probability measures on $G_{d,n}$, $\left(\nu_{(x,t)}\right)_{(x,t)}$, such that:

$$V(t) = \|V(t)\| \otimes \nu_{(x,t)},$$

(ii) (Mass decay)

$$\mu(t_2)(\mathbb{R}^n) \leq \mu(t_1)(\mathbb{R}^n) \leq \mu(0)(\mathbb{R}^n) \ \text{ for all } \ 0 \leq t_1 \leq t_2 \leq 1. \tag{5.117}$$

(iii) (L^2 bound) $h \in L^2(\|\lambda\|)$ and,

$$\int_0^1 \int_{\mathbb{R}^n} |h(y,t,\lambda)|^2 \, d\mu(t)(y)dt \leq \mu(0)(\mathbb{R}^n) = \|V_0\|(\mathbb{R}^n).$$

Proof of (ii) and (iii). Define for every $r \in \mathbb{R}^+$, a $C^1(\mathbb{R}^n, \mathbb{R}^+)$ function as :

$$\varphi_r(x) = \begin{cases} 1 & x \in B_r, \\ 0 & x \in B_{3r}^c, \end{cases}$$

and $\|\nabla \varphi_r\|_\infty \leq r^{-1}$. Plugging φ_r in (5.116) we obtain (denoting $h := h(\cdot,\cdot,\lambda)$ for simplicity)

$$\begin{aligned} \mu(t_2)(\varphi_r) - \mu(t_1)(\varphi_r) &\leq -\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi_r |h|^2 \, d\mu(t)dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla \varphi_r) \cdot h \, d\lambda \\ &\leq -\int_{t_1}^{t_2} \int_{B_r} |h|^2 \, d\mu(t)dt + r^{-1} \|h\|_{L^1(d\|\lambda\|)}. \end{aligned}$$

We now let $r \to +\infty$, we obtain

$$\mu(t_2)(\mathbb{R}^n) + \lim_{r \to +\infty} \int_{t_1}^{t_2} \int_{B_r} |h|^2 \, d\mu(t)dt \leq \mu(t_1)(\mathbb{R}^n).$$

This proves the decay property and the desired L^2-bound. □

Remark 5.3.6 (Brakke flows and spacetime Brakke flows)**.** Let $(V(t))_{t\in[0,1]}$ be a Brakke flow (or particularly a MCF), $\lambda = (V(t))_{t\in[0,1]} \otimes dt$ is a spacetime Brakke flow where

$$\mu(t) = \|V(t)\| \ \text{and} \ \ h(y,t,\lambda) = h(y, V(t)) \quad \forall (y,t) \in \operatorname{spt} \|V(t)\| \times [0,1].$$

5.3.2 Convergence of the approximate mean curvature flows, properties of the limit

We now study the convergence of $(V_\varepsilon(t))_{t\in[0,1]} \otimes dt$ and show the properties of the limit.

Theorem 5.3.7. *(Convergence) Let $\varepsilon \in (0,1)$ and $V_0 \in V_d(\mathbb{R}^n)$ of compact support and finite mass. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate mean curvature flow starting from V_0. We have:*

1. *There exists a sequence $(\varepsilon_j)_j \xrightarrow[j\to\infty]{} 0$ such that*

$$V_{\varepsilon_j}(t) \otimes dt \xrightarrow[j\to\infty]{} \lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt, \quad \text{and } \|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{} \mu(t),$$

   where $\mu(t)$ is a Radon measure on $\mathbb{R}^n$ and $\nu_{(x,t)}$ a family of probability measures for $(x,t) \in \mathbb{R}^n \times [0,1]$.

2. *$\delta\lambda$ is bounded, $(\delta\lambda)_s = 0$ and $\|h(\cdot,\cdot,\lambda)\|_{L^2(d\lambda)} \leq V_0(\mathbb{R}^n)$.*

3. *If we assume that $\mu(t) \otimes \nu_{(x,t)}$ is rectifiable for a.e $t \in [0,1]$ then λ is a spacetime Brakke flow.*

For simplicity, we split Theorem 5.3.7 into several propositions and prove each one separately.

Remark 5.3.8. The proof of Theorem 5.3.7 is crucially based on Propositions 5.4 and 5.5 in [36]. Note however that they are stated in [36] for codimension 1 varifolds, while we work with varifolds of arbitrary codimension in $\mathbb{R}^n$. Actually, the proofs of Propositions 5.4 and 5.5 in [36] do not require at all the varifolds to be 1-codimensional because they are based on results due to Brakke, which are valid for general varifolds.

The next proposition combines the generalized estimates (in terms of codimension) of [36, Proposition 5.4] and [36, Proposition 5.5].

Proposition 5.3.9. *There exists some $\varepsilon_* = \varepsilon_*(n, M) > 0$ (depending only on n and M) such that for any varifold $W \in V_d(\mathbb{R}^n)$ such that $\|W\|(\mathbb{R}^n) \leq M$; if $0 < \varepsilon \leq \varepsilon_*$ and $\varepsilon^{-\frac{1}{6}} \geq 2m$, we have*

- *for any $\psi \in \mathcal{A}_m$*

$$\left|\delta W\left(\psi h_\varepsilon(\cdot, W)\right) + \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\, dx\right| \leq \varepsilon^{\frac{1}{4}} \left(1 + \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\, dx\right) \tag{5.118}$$

  and

$$\int_{\mathbb{R}^n} \psi |h_\varepsilon(\cdot, W)|^2 \, d\|W\| \leq \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}(1 + \varepsilon^{\frac{1}{4}})\, dx + \varepsilon^{\frac{1}{4}}, \tag{5.119}$$

- *for any $X \in \mathcal{B}_m$*

$$\left|\int_{\mathbb{R}^n} h_\varepsilon \cdot X \, d\|W\| + \delta W(X)\right| \leq \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}} \left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\, dx\right)^{\frac{1}{2}}. \tag{5.120}$$

Proposition 5.3.10 (Existence of a limit for the mass measure)**.** *Let $V_0 \in V_d(\mathbb{R}^n)$ of bounded support, for any $\varepsilon \in (0,1)$, let $V_\varepsilon(t)$ be the approximate mean curvature flow starting from V_0.*
There exists a sequence $(\varepsilon_j)_{j=1}^\infty$ (not depending on t) converging to 0 as $j \to \infty$, and a family of Radon measures $(\mu(t))_{t\in[0,1]}$ on $\mathbb{R}^n$ such that:

$$\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t) \tag{5.121}$$

for all $t \in [0,1]$.

Proof. Proposition 5.3.10 states that, up to an extraction independent of t, the mass measure $\|V_\varepsilon(t)\|$ converges as ε goes to 0 to a limit time-dependent measure $\mu(t)$. The proof is a direct adaptation of the proof of [36, Proposition 6.4 (1)], which is itself based on the results of Section 5 in [36] and two estimates [36, (6.3)], [36, (6.5)]. In short, the proof of [36] is based on the following arguments: a limit measure $\mu(t)$ is defined for a countable, dense collection D of times using an extraction argument, the extension of $\mu(t)$ to almost all times, more precisely the complement of a countable set of times, follows from a continuity property of $\mu(t)$ on D, the convergence of the subsequence $\|V_\varepsilon(t)\|$ to $\mu(t)$ for all these times t follows from an approximate continuity property satisfied by $\|V_\varepsilon(t)\|$, and a last extraction is used to recover $\mu(t)$ for all t. The adaptation of the proof to our framework uses the following arguments:

- $V_0 \in V_d(\mathbb{R}^n)$ is not necessarily an open partition in our case, but the proof in [36] remains valid in this case;
- the proof of [36] is based on the results of [36, Section 5] which are valid for varifolds of any codimension (and not only for codimension 1 varifolds which are the subject of [36, Proposition 6.4(1)]);
- the weight function Ω is identically equal to 1 in our case because we work with varifolds of finite mass;
- estimate [36, Inequality (6.3)] is replaced by a decrease property of the mass (cf. Remark 5.2.7);
- estimate [36, Inequality (6.5)] is replaced by (5.96) (with test functions depending only on the space variable).

Step 1: Let $\varepsilon \in (0,1)$, $V \in V_d(\mathbb{R}^n)$. We prove the following inequality

$$\delta(V,\varphi)(h_\varepsilon(\cdot,V)) \leq 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\|,$$

for any $\varphi \in \mathcal{A}_j$ and $2j \leq \varepsilon^{-\frac{1}{6}}$.

Indeed, let $\varphi \in \mathcal{A}_j$ with $2j \leq \varepsilon^{-\frac{1}{6}}$ and set for simplicity $b := \displaystyle\int_{\mathbb{R}^n} \frac{\varphi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * ||V|| + \varepsilon} dx$, we have

$$\begin{aligned}
\delta\left(V,\varphi\right)\left(h_\varepsilon(\cdot,V)\right) &= \delta V(\varphi h_\varepsilon) + \int_{\mathbb{R}^n} S^\perp(\nabla\varphi)\cdot h_\varepsilon(\cdot,V)\, dV(x,S) \\
&\leq -b + \varepsilon^{\frac{1}{4}} b + \varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \varphi|h_\varepsilon(\cdot,V)|^2\, d||V|| + \frac{1}{2}\int_{\mathbb{R}^n}\frac{|\nabla\varphi|^2}{\varphi}\, d||V|| \text{ by (5.118)} \\
&\leq -b + \varepsilon^{\frac{1}{4}} b + \varepsilon^{\frac{1}{4}} + \frac{1}{2} b(1+\varepsilon^{\frac{1}{4}}) + \frac{1}{2}\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n}\frac{|\nabla\varphi|^2}{\varphi}\, d||V|| \text{ by (5.119)} \\
&\leq \frac{1}{2}(-1+3\varepsilon^{\frac{1}{4}})b + 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n}\frac{|\nabla\varphi|^2}{\varphi}\, d||V|| \\
&\leq 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n}\frac{|\nabla\varphi|^2}{\varphi}\, d||V||.
\end{aligned}$$

Step 2: We define the limit measure $\mu(t)$ for a.e $t \in [0,1]$.
Let $D \cap [0,1]$ be the set of dyadic numbers in $[0,1]$. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence converging to 0 such that, $\|V_{\varepsilon_j}(t)\|$ converges for any $t \in D$, denote the limit $\mu(t)$. The previous claim stems from the Banach-Alaoglu theorem and the uniform boundedness of the mass (5.113).
Let $Z := (\varphi_q)_{q\in\mathbb{N}}$ be a countable subset of $C_c^2(\mathbb{R}^n,\mathbb{R}^+)$ which is dense in $C_c(\mathbb{R}^n,\mathbb{R}^+)$. Take $\varphi_q \in Z$ and assume without loss of generality $\varphi_q < 1$, then, for any $i \in \mathbb{N}$ large enough we have $\varphi_q + i^{-1} \in \mathcal{A}_m$ for any $m \geq m_0$, m_0 depends on i and φ_q. We apply (5.96) with $\varphi(\cdot,t) = \varphi_q$, together with step 1 to obtain

$$\|V_{\varepsilon_j}(b)\|(\varphi_q + i^{-1}) - \|V_{\varepsilon_j}(a)\|(\varphi_q + i^{-1}) \leq (b-a)2\varepsilon_j^{\frac{1}{4}} + \frac{1}{2}\int_a^b\int_{\mathbb{R}^n}\frac{|\nabla(\varphi_q + i^{-1})|^2}{\varphi_q + i^{-1}}\, d\|V_{\varepsilon_j}(t)\| dt,$$

for any $a, b \in [0,1], a \leq b$ and $2m_0 \leq \varepsilon_j^{-\frac{1}{6}}$. We obtain from [46, Lemma 3.1]

$$\frac{|\nabla(\varphi_q + i^{-1})|^2}{\varphi_q + i^{-1}} \leq \frac{|\nabla \varphi_q|^2}{\varphi_q} \leq 2\|\nabla^2 \varphi_q\|_\infty.$$

Therefore, for any $a, b \in [0,1], a \leq b$

$$\|V_{\varepsilon_j}(b)\|(\varphi_q + i^{-1}) - \|V_{\varepsilon_j}(a)\|(\varphi_q + i^{-1}) \leq (b-a)2\varepsilon_j^{\frac{1}{4}} + (b-a)\|\nabla^2 \varphi_q\|_\infty \|V_0\|(\mathbb{R}^n). \tag{5.122}$$

We let $j \to \infty$, we deduce for $a, b \in D, a \leq b$ that

$$\mu(b)(\varphi_q + i^{-1}) - \mu(a)(\varphi_q + i^{-1}) \leq (b-a)\|\nabla^2 \varphi_q\|_\infty \|V_0\|(\mathbb{R}^n).$$

We let $i \to \infty$, using the uniform boundedness of the mass (5.113), we deduce for $a, b \in D, a \leq b$ that

$$\mu(b)(\varphi_q) - \mu(a)(\varphi_q) \leq (b-a)\|\nabla^2 \varphi_q\|_\infty \|V_0\|(\mathbb{R}^n).$$

The previous inequality tells us that the map $g_q : t \mapsto \mu(t)(\varphi_q) - t(b-a)\|\nabla^2 \varphi_q\|_\infty \|V_0\|(\mathbb{R}^n)$ is nonincreasing for $t \in D$. Define

$$\mathcal{C} := \{t \in [0,1], \text{for some } q \in \mathbb{N} \lim_{s \to t^-} g_q(s) > \lim_{s \to t^+} g_q(s)\}.$$

By the monotonicity property of g_q, $\mathcal{C}$ is a countable set in $[0,1]$, and $\mu(t)(\varphi_q)$ may be defined continuously on the complement of $\mathcal{C}$ uniquely from the values on D; then, one can define the measure $\mu(t)$ for a.e $t \in [0,1]$ by density of Z in $C_c(\mathbb{R}^n, \mathbb{R}^+)$.
Step 3: We prove that for any $t \in [0,1] \setminus \mathcal{C}$

$$\|V_{\varepsilon_j}(t)\| \stackrel{*}{\rightharpoonup} \mu(t).$$

Let $t \in [0,1] \setminus \mathcal{C}$ and $s \in D, t < s$. From (5.122) we have

$$\|V_{\varepsilon_j}(s)\|(\varphi_q + i^{-1}) \leq \|V_{\varepsilon_j}(t)\|(\varphi_q + i^{-1}) + O(s-t).$$

We let $j \to$ so that

$$\mu(s)(\varphi_q + i^{-1}) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q + i^{-1}) + O(s-t).$$

Then we take the limit in i to obtain

$$\mu(s)(\varphi_q) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q) + O(s-t).$$

We now let $s \to t^-$ and use the continuity of g_q at t so that

$$\mu(t)(\varphi_q) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q).$$

The same reasoning for $s < t$ gives $\mu(t)(\varphi_q) \geq \limsup_j \|V_{\varepsilon_j}(t)\|(\varphi_q)$; hence

$$\lim_j \|V_{\varepsilon_j}(t)\|(\varphi_q) = \mu(t)(\varphi_q), \quad \forall \varphi_q \in Z \text{ and } \forall t \in [0,1] \setminus \mathcal{C};$$

we conclude the proof of step 3 by density of Z in $C_c(\mathbb{R}^n, \mathbb{R}^+)$.

The set $\mathcal{C}$ is countable, hence, by further extraction of the sequence $(\varepsilon_j)_j$, we can define $\mu(t)$ on $[0,1]$ entirely and ensure the convergence for <u>all</u> $t \in [0,1]$. □

Proposition 5.3.11 (The limit flow)**.** *Let $V_0 \in V_d(\mathbb{R}^n)$ be of compact support. For any $\varepsilon \in (0,1)$,*

- *let $V_\varepsilon(t)$ be the approximate mean curvature flow starting from V_0,*
- *set $\lambda_\varepsilon = V_\varepsilon(t) \otimes dt$ that is λ_ε is the Radon measure on $\mathbb{R}^n \times G_{d,n} \times [0,1]$ satisfying for all $\varphi \in \mathrm{C}_c(\mathbb{R}^n \times G_{d,n} \times [0,1], \mathbb{R})$,*

$$\int \varphi \, d\lambda_\varepsilon = \int_0^1 \left(\int_{(x,S)\in\mathbb{R}^n\times G_{d,n}} \varphi(x,S,t) \, dV_\varepsilon(t) \right) dt.$$

Then, there exists a sequence $(\varepsilon_j)_j$ for which

$$\lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{*} \lambda \tag{5.123}$$

where λ is a Radon measure of the form

$$\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt.$$

Here $\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t), \forall t \in [0,1]$ and $(\nu_{(x,t)})_{(x,t)}$ is a family of probability measures on $G_{d,n}$ defined for $(x,t) \in \mathbb{R}^n \times [0,1]$ (up to a set of $(\mu(t)\otimes dt)$–zero measure).

Proof. We know by Proposition 5.3.10 that there exists a sequence $(\varepsilon_j)_j$ for which $\|V_{\varepsilon_j}(t)\|$ converges to a limit measure $\mu(t)$ for all $t \in [0,1]$. Using for all $j \in \mathbb{N}$,

$$\lambda_{\varepsilon_j}(\mathbb{R}^n \times G_{d,n} \times [0,1]) = \int_0^1 V_{\varepsilon_j}(t)(\mathbb{R}^n \times G_{d,n}) \, dt \le \|V_0\|(\mathbb{R}^n),$$

we can assert by Banach-Alaoglu's compactness theorem that, up to a further extraction, $\lambda_{\varepsilon_j} \rightharpoonup \lambda$, where λ is a Radon measure on $\mathbb{R}^n \times G_{d,n} \times [0,1]$.
Denoting by Π the canonical projection $(x,S,t) \in \mathbb{R}^n \times G_{d,n} \times [0,1] \mapsto (x,t) \in \mathbb{R}^n \times [0,1]$ we now show that $\Pi_\# \lambda = \mu(t) \otimes dt$. Indeed, on the one hand, as a consequence of (5.123), we have

$$\Pi_\# \lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{*} \Pi_\# \lambda \, . \tag{5.124}$$

On the other hand, by definition of push-forward measure, we have for $\varphi \in \mathrm{C}_c(\mathbb{R}^n \times [0,1], \mathbb{R})$,

$$\begin{aligned} \Pi_\# \lambda_{\varepsilon_j}(\varphi) &= \int_{\mathbb{R}^n \times G_{d,n} \times [0,1]} \varphi \circ \Pi \, d\lambda_{\varepsilon_j} = \int_{(x,S,t)\in\mathbb{R}^n\times G_{d,n}\times[0,1]} \varphi(\Pi(x,S,t)) \, dV_{\varepsilon_j}(t) \, dt \\ &= \int_{t=0}^1 \int_{x\in\mathbb{R}^n} \varphi(x,t) \, d\|V_{\varepsilon_j}(t)\| \, dt \\ &\xrightarrow[j\to\infty]{} \int_{t=0}^1 \int_{x\in\mathbb{R}^n} \varphi(x,t) \, d\mu(t) \, dt. \end{aligned} \tag{5.125}$$

From (5.124) and (5.125) we obtain

$$\Pi_\# \lambda = \mu(t) \otimes dt.$$

It follows by Young's disintegration theorem [5, Theorem 2.28] that there exists a family of probability measures $\left(\nu_{(x,t)}\right)_{(x,t)}$ on $G_{d,n}$, such that:

$$\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt,$$

which completes the proof. □

Proposition 5.3.12. *Let λ be the limit measure defined in Proposition 5.3.11. We have:*

$$\forall X \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n), \quad \delta(V_{\varepsilon_j}(t) \otimes dt)(X) \xrightarrow[j \to \infty]{} \delta\lambda(X)\,. \tag{5.126}$$

Moreover, λ has bounded first variation (i.e. $\delta\lambda$ is a Radon measure) and

$$\delta\lambda = h(\cdot,t,\lambda)\|\lambda\| = h(\cdot,t,\lambda)\mu(t)\otimes dt \quad \text{and} \quad \int_0^1 \int_{\mathbb{R}^n} |h(y,t,\lambda)|^2 \, d\mu(t)\, dt \leq \|V_0\|(\mathbb{R}^n) < \infty, \tag{5.127}$$

in particular $(\delta\lambda)_s = 0$; for all bounded $\psi \in \mathrm{C}(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, for all $0 \leq t_1 < t_2 \leq 1$,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \psi(y,t)|h(y,t,\lambda)|^2 \, d\mu(t)\, dt \leq \liminf_{j \to \infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \psi(y,t) \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j} \, dy\, dt. \tag{5.128}$$

Proof. Let us first check that (5.126) is a consequence of $V_{\varepsilon_j}(t) \otimes dt \xrightarrow[j\to\infty]{*} \lambda$. Indeed, let $X \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, then $g : (y,S,t) \mapsto \operatorname{div}_S X(y) \in \mathrm{C}_c(\mathbb{R}^n \times G_{d,n} \times [0,1])$ and thus

$$\delta(V_{\varepsilon_j}(t) \otimes dt)(X) = \int g \, d(V_{\varepsilon_j}(t) \otimes dt) \xrightarrow[j \to \infty]{} \int g \, d\lambda = \delta\lambda(X)\,.$$

Let us now consider the sequences $(\mu_j)_{j \in \mathbb{N}}$ of Radon measures in $\mathbb{R}^n \times [0,1]$ and $(f_j)_{j\in\mathbb{N}} \in \mathrm{C}^\infty(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ defined as

$$\mu_j = (\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j) \otimes dt \quad \text{and} \quad f_j(\cdot,t) = \frac{\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)} \quad \text{for all } j \in \mathbb{N}.$$

Let $\varphi \in \mathrm{C}_c(\mathbb{R}^n \times [0,1])$. First note that by definition,

$$\int_{\mathbb{R}^n \times [0,1]} \varphi f_j \mu_j = \int_0^1 \int_{\mathbb{R}^n} \varphi(y,t) \Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\, dy\, dt \quad \Longrightarrow \quad f_j\mu_j = (\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)) \otimes dt.$$

We obtain by standard arguments (bicontinuity of distributional bracket to be more specific) that μ_j converge to $\mu(t) \otimes dt$ as Radon measures and $f_j\mu_j$ converges to $\delta\lambda$ as distributions of order 1. Indeed, as Φ_ε is a mollifier, we recall that for all $t \in [0,1]$, $\|\varphi(\cdot,t) * \Phi_{\varepsilon_j} - \varphi(\cdot,t)\|_{\mathrm{C}^1} \xrightarrow[j\to\infty]{} 0$ and therefore, by dominated convergence and $\|V_{\varepsilon_j}\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n)$,

$$\int_0^1 \left| \int_{\mathbb{R}^n} \varphi \, \left(d\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}\|(t) - d\|V_{\varepsilon_j}\|(t) \right) \right| \, dt \leq \|V_0\|(\mathbb{R}^n) \int_0^1 \|\varphi(\cdot,t) * \Phi_{\varepsilon_j} - \varphi(\cdot,t)\|_\infty \, dt \xrightarrow[j\to\infty]{} 0$$

so that recalling that $\|V_{\varepsilon_j}(t)\| \otimes dt$ converges to $\mu(t) \otimes dt = \|\lambda\|$,

$$\begin{aligned} \left| \int \varphi \, d\mu_j - \int \varphi \, d\mu(t)\, dt \right| \leq & \int_0^1 \left| \int_{\mathbb{R}^n} \varphi \, \left(d\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}\|(t) - d\|V_{\varepsilon_j}\|(t) \right) \right| \, dt \\ & + \left| \int \varphi \, \left(d\|V_{\varepsilon_j}\|(t)\, dt - \, d\mu(t)\, dt \right) \right| + \varepsilon_j \left| \int \varphi \, dy\, dt \right| \xrightarrow[j\to\infty]{} 0\,. \end{aligned}$$

We hence checked that $\mu_j \xrightarrow[j\to\infty]{*} \mu(t) \otimes dt$. In a very similar way, we can check that for all $X \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, $\int X \cdot f_j \, d\mu_j \xrightarrow[j\to\infty]{} \delta\lambda(X)$ since

$$\begin{aligned} \left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)(X) - \delta V_{\varepsilon_j}(t)(X)\right| = \left|\delta V_{\varepsilon_j}(t)(\Phi_{\varepsilon_j} * X) - \delta V_{\varepsilon_j}(t)(X)\right| &\leq \|V_{\varepsilon_j}(t)\|(\mathbb{R}^n)\|\Phi_{\varepsilon_j} * X - X\|_{\mathrm{C}^1} \\ &\leq \|V_0\|(\mathbb{R}^n)\|\Phi_{\varepsilon_j} * X - X\|_{\mathrm{C}^1} \xrightarrow[j\to\infty]{} 0 \end{aligned}$$

where we used Remark 5.2.7, recalling (5.126) we obtain the desired distributional convergence

$$\begin{aligned} \left|\int X \cdot f_j \, d\mu_j - \delta\lambda(X)\right| &\leq \int_0^1 \left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)(X) - \delta V_{\varepsilon_j}(t)(X)\right| \, dt + \left|\delta(V_{\varepsilon_j}(t) \otimes dt)(X) - \delta\lambda(X)\right| \\ &\xrightarrow[j\to\infty]{} 0 \, . \end{aligned}$$

Let $\psi \in \mathrm{C}(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ be bounded and consider $F : ((y,t),q) \mapsto \psi(y,t)|q|^2$, then F is non-negative continuous, and with respect to q, it is convex and has superlinear growth, hence satisfying the assumptions of [32] 4.1.2. We additionally have by Remark 5.2.7 that for all j,

$$\begin{aligned} \int_{\mathbb{R}^n \times [0,1]} F((y,t), f_j(y,t)) \, d\mu_j(y,t) &= \int_{\mathbb{R}^n \times [0,1]} \psi(y,t) \frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)} \, dy \, dt \\ &\leq \|\psi\|_\infty \|V_{\varepsilon_j}(0)\|(\mathbb{R}^n) = \|\psi\|_\infty \|V_0\|(\mathbb{R}^n) < \infty \end{aligned}$$

and we can apply [32] 4.4.2(i) and (ii) (see also 2.36 in [5]): there exists $f \in L^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n, \mu(t) \otimes dt)$ such that, up to extraction, the sequence of vector measures $f_j \mu_j$ converge to $f(\mu(t) \otimes dt)$ and

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}^n} \psi(y,t)|f(y,t)|^2 \, d\mu(t) \, dt &= \int F((y,t), f(y,t)) \, d(\mu(t) \otimes dt) \\ &\leq \liminf_{j\to\infty} \int F((y,t), f_j(y,t)) \, d\mu_j(y,t) \\ &\leq \liminf_{j\to\infty} \int_{\mathbb{R}^n \times [0,1]} \psi(y,t) \frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)} \, dy \, dt \leq \|\psi\|_\infty \|V_0\|(\mathbb{R}^n) \, . \end{aligned} \tag{5.129}$$

Thanks to the uniqueness of the distributional limit: $f(\mu(t) \otimes dt) = \delta\lambda$ so that $(\delta\lambda)_s = 0$ and $f = h(\cdot, t, \lambda)$, and we obtain (5.127) plugging $\psi = 1$ in (5.129).
We are left with proving (5.128) for $0 \leq t_1 \leq t_2 \leq 1$, and we can take an affine cut–off approximating $\mathbf{1}_{[t_1,t_2]}$ from below: for k large enough with respect to $t_2 - t_1$, let χ_k be a continuous piecewise-affine function satisfying $\mathbf{1}_{[t_1+\frac{1}{k}, t_2-\frac{1}{k}]} \leq \chi_k \leq \mathbf{1}_{[t_1,t_2]}$, then applying (5.129) to $\chi_k\psi$ gives

$$\int_0^1 \int_{\mathbb{R}^n} \chi_k \psi |h|^2 \, d\mu(t) \, dt \leq \liminf_{j\to\infty} \int_{\mathbb{R}^n \times [0,1]} \chi_k \psi \frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)} \, dy \, dt$$

and we can take the limit $k \to \infty$ in the l.h.s. by dominated convergence while we use $\chi_k \leq \mathbf{1}_{[t_1,t_2]}$ in the r.h.s. to conclude the proof of (5.128), and hence the current proof. □

Proposition 5.3.13 (Spacetime Brakke inequality for the limit flow.)**.** *Let λ be the limit measure defined in Proposition 5.3.11, we write $\lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt$ and we assume that $\mu(t) \otimes \nu_{(\cdot,t)}$ is rectifiable a.e $t \in [0,1]$, hence, if $\mathcal{M}_t$ denotes the support of $\mu(t)$ one has:*

$$\mu(t) \otimes \nu_{(\cdot,t)} = \mu(t) \otimes \delta_{T.\mathcal{M}_t}.$$

Then, for any $\varphi \in \mathrm{C}^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and $0 \leq t_1 \leq t_2 \leq 1$,

$$\mu(t_2)(\varphi(\cdot,t_2)) - \mu(t_1)(\varphi(\cdot,t_1)) \leq -\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi(y,t) |h(y,t,\lambda)|^2 \, d\mu(t)(y) dt$$
$$+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} T_y \mathcal{M}_t^\perp (\nabla \varphi(y,t)) \cdot h(y,t,\lambda) \, d\mu(y) dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \partial_t \varphi(\cdot,t) \, d\mu(t) dt.$$

Proof. Denote $\lambda_{\varepsilon_j} = V_{\varepsilon_j}(t) \otimes dt$ and choose (as in Proposition 5.3.11) a sequence $(\varepsilon_j)_j$ satisfying:

$$\lim_{j\to\infty} \lambda_{\varepsilon_j} = \lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt \quad \text{and} \quad \lim_{j \to \infty} \|V_{\varepsilon_j}(t)\| = \mu(t).$$

Consider $\varphi \in C^1_c(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq 1$. The inequality we are seeking to prove is linear in φ, without loss of generality we assume $\varphi < 1$, and for all sufficiently large $i \in \mathbb{N}$ we define $\varphi_i := \varphi + i^{-1} < 1$. We can plug φ_i in (5.96), also recalling (1.13), we obtain:

$$\|V_{\varepsilon_j}(t_2)\|(\varphi_i(\cdot,t_2)) - \|V_{\varepsilon_j}(t_1)\|(\varphi_i(\cdot,t_1)) - \int_{t_1}^{t_2} \|V_{\varepsilon_j}(t)\|(\partial_t \varphi_i(\cdot,t)) \, dt$$
$$= \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla_x \varphi_i) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \, dV_{\varepsilon_j}(t) \, dt + \int_{t_1}^{t_2} \delta(V_{\varepsilon_j}(t)) \Big[\varphi_i(\cdot,t) h_\varepsilon(\cdot, V_\varepsilon(t)) \Big] \, dt \quad (5.130)$$

and the proof now consists in taking the limit, first in j and then in i.

Step 1**:** We take the limit in the l.h.s. of (5.130), that is, we prove

$$\|V_{\varepsilon_j}(t_2)\|(\varphi_i(\cdot,t_2)) - \|V_{\varepsilon_j}(t_1)\|(\varphi_i(\cdot,t_1)) - \int_{t_1}^{t_2} \|V_{\varepsilon_j}(t)\|(\partial_t \varphi_i(\cdot,t)) \, dt$$
$$\xrightarrow[i,j\to\infty]{} \|V(t_2)\|(\varphi(\cdot,t_2)) - \|V(t_1)\|(\varphi(\cdot,t_1)) - \int_{t_1}^{t_2} \|V(t)\|(\partial_t \varphi(\cdot,t)) \, dt. \quad (5.131)$$

First note that $\partial_t \varphi_i = \partial_t \varphi$ and recall that for all $t \in [0,1]$,

$$\|V_{\varepsilon_j}\|(t) \xrightarrow[j\to\infty]{*} \mu(t) \quad \Longrightarrow \quad \begin{cases} \|V_{\varepsilon_j}(t)\|(\partial_t \varphi_i(\cdot,t)) = \|V_{\varepsilon_j}(t)\|(\partial_t \varphi(\cdot,t)) \xrightarrow[j\to\infty]{} \mu(t)(\partial_t \varphi(\cdot,t)) \\ \|V_{\varepsilon_j}(t)\|(\varphi(\cdot,t)) \xrightarrow[j\to\infty]{} \mu(t)(\varphi(\cdot,t)) \end{cases}$$

and since $\|V_{\varepsilon_j}(t)\|(\partial_t \varphi(\cdot,t)) \leq \|\partial_t \varphi\|_\infty \|V_0\|(\mathbb{R}^n)$ by the decay of the mass (Remark 5.2.7), we infer by dominated convergence that for any i,

$$\int_{t_1}^{t_2} \|V_{\varepsilon_j}(t)\|(\partial_t \varphi_i(\cdot,t)) \, dt \xrightarrow[j\to\infty]{} \int_{t_1}^{t_2} \|V(t)\|(\partial_t \varphi(\cdot,t)) \, dt \, .$$

Using again the decay of the mass and $\varphi_i = \varphi + i^{-1}$, we obtain

$$\left| \|V_{\varepsilon_j}(t)\|(\varphi_i(\cdot,t)) - \mu(t)(\varphi(\cdot,t)) \right| \leq i^{-1}\|V_0\|(\mathbb{R}^n) + \left| \|V_{\varepsilon_j}(t)\|(\varphi(\cdot,t)) - \mu(t)(\varphi(\cdot,t)) \right| \xrightarrow[i,j\to\infty]{} 0$$

and with $t = t_1, t_2$ we can conclude the proof of (5.131) (Step 1). We now deal with the two terms involving the mean curvature.

Step 2**:** We now prove that

$$\limsup_{i\to\infty} \limsup_{j\to\infty} \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)))dt \leq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi(y,t)|h(y,t,\lambda)|^2 d\mu(t)(y)dt. \quad (5.132)$$

First, we note that $\varphi_i = \varphi + i^{-1}$ and then, there exists $m_{i,\varphi} \in \mathbb{N}$ (large enough, depending on i and φ: e.g. $m_{i,\varphi} \geq i\|\nabla_x \varphi\|_\infty$) such that for all $m \geq m_{i,\varphi}$, $\varphi_i \in \mathcal{A}_m$. We apply (5.118) with $\varepsilon = \varepsilon_j$ and $\psi = \varphi_i$ whence, for fixed φ and i, one has to take j large enough to ensure $\varepsilon_j \leq \varepsilon_*$ and $\varepsilon_j^{-\frac{1}{6}} \geq 2m_{i,\varphi}$: this is the reason why we have to take $\lim_{j\to\infty}$ before $\lim_{i\to\infty}$ hereafter. Concerning the varifold, we apply (5.118) with $W = V_{\varepsilon_j}(t)$ (for $t \in [0,1]$) and $M = \|V_0\|(\mathbb{R}^n)$ since we know that $\|V_{\varepsilon_j}(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) \leq M$. We obtain, for all $t \in [0,1]$ and for all j large enough,

$$\left| \delta V_{\varepsilon_j}(t)\left(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\right) + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx \right| \leq \varepsilon_j^{\frac{1}{4}} \left(1 + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx \right)$$

which we integrate between t_1 and t_2 so that using $0 \leq \varphi_i \leq 1$ and Remark 5.2.7,

$$\begin{aligned}\int_{t_1}^{t_2} & \left| \delta V_{\varepsilon_j}(t)\left(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\right) + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx \right| dt \\ & \leq \varepsilon_j^{\frac{1}{4}} \left(1 + \int_{\mathbb{R}^n \times [0,1]} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\, dt \right) \leq \varepsilon_j^{\frac{1}{4}} \left(1 + \|V_0\|(\mathbb{R}^n)\right) \xrightarrow[j\to\infty]{} 0\,.\end{aligned}$$

We infer:

$$\begin{aligned}\limsup_{j\to\infty} \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)))dt &= -\liminf_{j\to\infty} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j} dx dt \\ &\leq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi_i |h(\cdot,\lambda)|^2 d\mu(t) dt \quad \text{by (5.128) in Proposition 5.3.12}\end{aligned}$$

and the proof of (5.132) (Step 2) follows from $-\varphi_i \leq -\varphi$.

Step 3**:** As $\nabla_x \varphi_i = \nabla_x \varphi$, we are left with the proof of

$$\limsup_{j\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla_y \varphi) \cdot h_{\varepsilon_j}(y, V_{\varepsilon_j}(t))\, dV_{\varepsilon_j}(t)\, dt \leq \int_{t_1}^{t_2}\int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla_y \varphi) \cdot h(y,t,\lambda)\, d\lambda\,. \quad (5.133)$$

We fix an arbitrary $g \in C_c^2(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, there exists $m_g \in \mathbb{N}$ such that $g(\cdot, t)) \in \mathcal{B}_m \, \forall m \geq m_g$ and $\forall t \in [0,1]$, this is due to the compactness of $[0,1]$. We apply (5.120) with $\varepsilon = \varepsilon_j$ and $X = g$ and we take j large enough to ensure that $\varepsilon_j \leq \varepsilon_*$ and $\varepsilon_j^{\frac{1}{6}} \geq 2m_g$. Concerning the varifold, we apply

(5.120) with $W = V_{\varepsilon_j}(t)$ (for $t \in [0,1]$) and $M = \|V_0\|(\mathbb{R}^n)$ since we know that $\|V_{\varepsilon_j}(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) \leq M$ by Remark 5.2.7. We obtain, for j large enough

$$\left| \int_{\mathbb{R}^n} h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \cdot g(\cdot,t) \, d\|V_{\varepsilon_j}(t)\| + \delta V_{\varepsilon_j}(t)(g(\cdot,t)) \right| \leq \varepsilon_j^{\frac{1}{4}} + \varepsilon_j^{\frac{1}{4}} \left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta V_{\varepsilon_j}(t)|^2}{\Phi_\varepsilon * \|V_{\varepsilon_j}(t)\| + \varepsilon} \, dx \right)^{\frac{1}{2}}$$

which we integrate between t_1 and t_2 so that

$$\begin{aligned}
\int_{t_1}^{t_2} &\left| \int_{\mathbb{R}^n} h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \cdot g(\cdot,t) \, d\|V_{\varepsilon_j}(t)\| + \delta V_{\varepsilon_j}(t)(g(\cdot,t)) \right| dt \\
&\leq \int_{t_1}^{t_2} \varepsilon_j^{\frac{1}{4}} \, dt + \varepsilon_j^{\frac{1}{4}} \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j} dx \right)^{\frac{1}{2}} dt \\
&\leq \varepsilon_j^{\frac{1}{4}} + \varepsilon_j^{\frac{1}{4}} \left(\int_0^1 \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j} \, dx \, dt \right)^{\frac{1}{2}} \\
&\leq \varepsilon_j^{\frac{1}{4}} \left(1 + \|V_0\|(\mathbb{R}^n)^{\frac{1}{2}} \right) \xrightarrow[j \to \infty]{} 0 \, ,
\end{aligned} \tag{5.134}$$

where we used Jensen inequality and Remark 5.2.7.

We now observe that the map $g : (y,t) \mapsto (T_y \mathcal{M}_t)^\perp(\nabla\varphi)$ is $\mu(t) \otimes dt$–measurable and belongs to $L^2(\mu(t) \otimes dt)$ ($\mu(t)$ is finite and $\varphi \in \mathrm{C}_c^1$, hence g is bounded by $\|\nabla\varphi\|$), we can assert that, for any $\eta \in (0,1)$, there exists a map $g_\eta \in \mathrm{C}_c^2(\mathbb{R}^n \times [t_1,t_2], \mathbb{R}^n)$ such that:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left| (T_y \mathcal{M}_t)^\perp(\nabla\varphi(y)) - g_\eta(y,t) \right|^2 d\mu(t)(y) dt < \eta^2. \tag{5.135}$$

Now we compute

$$\begin{aligned}
\int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla\varphi) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \, d\lambda_{\varepsilon_j} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} \left(S^\perp(\nabla\varphi) - g_\eta \right) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \, d\lambda_{\varepsilon_j} \\
&+ \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} g_\eta \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \, d\|V_{\varepsilon_j}(t)\| dt + \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta) \, dt \right) \\
&- \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta) \, dt + \delta\lambda(g_\eta) \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} h(\cdot,\cdot,\lambda) \cdot \left(g_\eta - (T_y \mathcal{M}_t)^\perp(\nabla\varphi) \right) d\mu(t) dt \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (T_y \mathcal{M}_t)^\perp(\nabla\varphi(y,t)) \cdot h(y,t,\lambda) \, d\mu(t) dt.
\end{aligned} \tag{5.136}$$

By the varifold convergence, (5.135) and (5.119) we have

$$\begin{aligned}
&\overline{\lim_j} \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} \left(S^\perp(\nabla\varphi) - g_\eta\right) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\lambda_{\varepsilon_j} \\
&\leq \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} |S^\perp(\nabla\varphi) - g_\eta|^2\, d\lambda\right)^{\frac{1}{2}} \left(\overline{\lim_j} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))|^2\, d\|V_{\varepsilon_j}(t)\|\right)^{\frac{1}{2}} \\
&= \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |(T_y\mathcal{M}_t)^\perp(\nabla\varphi) - g_\eta|^2\, d\mu(t)dt\right)^{\frac{1}{2}} \left(\overline{\lim_j} \int_0^1 \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dxdt\right)^{\frac{1}{2}} \\
&\leq \eta\, (\|V_0\|(\mathbb{R}^n))^{\frac{1}{2}} \xrightarrow[\eta \to 0]{} 0.
\end{aligned} \tag{5.137}$$

By (5.134) we have:

$$\overline{\lim_j} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} g_\eta \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\|V_{\varepsilon_j}\|(t) + \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt\right) = 0. \tag{5.138}$$

By the varifold convergence we have:

$$\overline{\lim_j} \left| \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt - \delta\lambda(g_\eta)\right| = 0, \tag{5.139}$$

and finally, the Cauchy-Schwarz inequality and (5.135) imply

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathbb{R}^n} h(\cdot, \cdot, \lambda) \cdot \left(g_\eta - (T_y\mathcal{M}_t)^\perp(\nabla\varphi)\right) d\mu(t)dt \\
&\leq \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |h(\cdot, \cdot, \lambda)|^2\, d\mu(t)dt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |g_\eta - (T_y\mathcal{M}_t)^\perp(\nabla\varphi)|^2\, d\mu(t)dt\right)^{\frac{1}{2}} \\
&\leq (\|V_0\|(\mathbb{R}^n))^{\frac{1}{2}} \eta \xrightarrow[\eta \to 0]{} 0.
\end{aligned} \tag{5.140}$$

From (5.136)-(5.140) we deduce (5.133) (Step 3), this concludes the proof of Proposition 5.3.13. □

Remark 5.3.14. From the proof, we notice that it only suffices to assume that $\nu_{\cdot,t}$ is a Dirac's measure, as we do not use the fact that $T.\mu(t)$ is the tangent space nor the properties of $\mu(t)$ as a rectifiable measure.

Remark 5.3.15 (Non uniqueness of the limit spacetime Brakke flows)**.** We recall that the limit measure λ in Theorem 5.3.7 depends on the choice of the subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$, hence is not unique (in general). This, somehow, is related to the non-uniqueness of Brakke flows, as Brakke flows themselves are spacetime Brakke flows when tensored with the measure "dt".

5.4 Consistency of approximate mean curvature flows

The following is a consistency result on the approximate MCFs.

Proposition 5.4.1. *Let $V_0, (W_k)_{k\in\mathbb{N}} \in V_d(\mathbb{R}^n)$ of compact supports with $\|W_k\|(\mathbb{R}^n) \le M$, and $(\varepsilon_k)_{k\in\mathbb{N}} \in (0,1)$ such that*

$$W_k \xrightarrow[k\to\infty]{*} V_0 \quad \textit{and} \quad \varepsilon_k \xrightarrow[k\to\infty]{} 0.$$

Define for every $k \in \mathbb{N}$,

- *$(V_{\varepsilon_k}(t))_{t\in[0,1]}$: the approximate MCF starting from V_0.*
- *$((W_k)_{\varepsilon_k}(t))_{t\in[0,1]}$: the approximate MCF starting from W_k.*

If we assume that $V_{\varepsilon_k}(t)$ converges weakly- for a.e $t \in [0,1]$ to a certain limit $V(t)$ of bounded support for a.e $t \in [0,1]$, and that*

$$\Delta(W_k, V_0)\exp\bigl(c_7\varepsilon_k^{-n-7}\bigr) \to 0 \quad \textit{as} \quad k \to \infty. \tag{5.141}$$

Then, $(W_k)_{\varepsilon_k}(t) \xrightarrow[k\to\infty]{*} V(t)$ for a.e $t \in [0,1]$. In particular, if $V(t)$ is a Brakke flow, $(W_k)_{\varepsilon_k}(t)$ converges weakly- to a Brakke flow starting from V_0 (which in this case is $V(t)$).*

Proof. By the weak-*convergence and the fact that $\operatorname{spt} V_0$ is compact we can infer that $\|V_0\|(\mathbb{R}^n) \le M$. By Proposition 5.2.3 we have:

$$\Delta(V_{\varepsilon_k}(t), (W_k)_{\varepsilon_k}(t)) \le \Delta(V_0, W_k)\exp\bigl(tc_{7,M}\varepsilon_k^{-n-7}\bigr),$$

for all $t \in [0,1]$. We deduce that

$$\begin{aligned}\Delta(V(t), (W_k)_{\varepsilon_k}(t)) &\le \Delta(V(t), V_{\varepsilon_k}(t)) + \Delta(V_{\varepsilon_k}(t), (W_k)_{\varepsilon_k}(t))\\ &\le \Delta(V(t), V_{\varepsilon_k}(t)) + \Delta(V_0, W_k)\exp\bigl(tc_{7,M}\varepsilon_k^{-n-7}\bigr).\end{aligned}$$

We can prove by Proposition 1.2.3, coupled with a truncation argument as $\operatorname{spt} V(t)$ is bounded, that

$$\Delta(V(t), V_{\varepsilon_k}(t)) \to 0,$$

for a.e $t \in [0,1]$. Thus, $(W_k)_{\varepsilon_k}(t) \rightharpoonup V(t)$ for a.e $t \in [0,1]$. ☐

Remark 5.4.2. In fact, we can prove that for any fixed t, under the same notations and hypothesis of Proposition 5.4.1

$$\text{if } V_{\varepsilon_k}(t) \xrightarrow[k\to\infty]{*} V(t) \text{ then } (W_k)_{\varepsilon_k}(t) \xrightarrow[k\to\infty]{*} V(t).$$

In the proposition we require a.e (with respect to time) convergence to stay consistent with general convergence results.

5.5 Appendix

The following lemma is used to prove Lemma 5.2.4.

Lemma 5.5.1. *Let $S \in G_{d,n}$, $A \in \mathcal{M}_n$, one has*

$$|\operatorname{tr}(S \circ A)| \le n\|A\|.$$

Proof. Indeed, it is clear that for any matrix $B \in \mathcal{M}_n$ that $|\operatorname{tr}(B)| \leq n|B|_\infty \leq n\|B\|$. Therefore,

$$|\operatorname{tr}(S \circ A)| \leq n\|S \circ A\| \leq n\|S\|\|A\| \leq n\|A\|,$$

where we used $\|S\| = 1$, this finishes the proof. □

The following lemma stems directly from the triangle inequality, it helps simplifying the proofs of Proposition 5.1.8.

Lemma 5.5.2. *Let $A, B \in \mathcal{M}_n$ and $S, T \in \mathcal{M}_{d,n}$ be such that $\|S\| = \|T\| = 1$, one has*

$$\|SAS^t - TBT^t\| \leq (\|A\| + \|B\|)\,\|S - T\| + \|A - B\|. \tag{5.142}$$

Proof. Using the triangle inequality, and the fact that $\|M^t\| = \|M\|$ for any matrix M we infer that

$$\begin{aligned}\|SAS^t - TBT^t\| &\leq \|SAS^t - SAT^t\| + \|SAT^t - SBT^t\| + \|SBT^t - TBT^t\| \\ &\leq \|S\|\|A\|\|S^t - T^t\| + \|S\|\|A - B\|\|T^t\| + \|S - T\|\|B\|\|T^t\| \\ &\leq (\|A\| + \|B\|)\,\|S - T\| + \|A - B\|.\end{aligned}$$

This concludes the proof of (5.142). □

The following lemma contains several properties on determinant's expansions, used mainly to prove Propositions 5.1.3 and 5.1.8.

Lemma 5.5.3. *Let $1 \leq k \leq n$ and $Q \in \mathcal{M}_k$ be such that $|Q|_\infty \leq 1$. There exists $c_2 \geq 1$ only depending on n such that,*

$$|\det(I_k + Q) - \det(I_k)| \leq c_2|Q|_\infty \quad \textit{and} \quad |\det(I_k + Q) - \det(I_k) - \operatorname{tr}(Q)| \leq c_2(|Q|_\infty)^2\,. \tag{5.143}$$

Let $1 \leq d \leq n$, $L \in \mathcal{M}_{d,n}$, $R \in \mathcal{M}_n$ be such that $L \circ L^t = I_d$ and $|R|_\infty \leq 1$. Then

$$((I_n + R) \circ L^t)^t \circ ((I_n + R) \circ L^t) = I_d + Q \quad \textit{with} \quad Q = L \circ \left(R^t + R\right) \circ L^t + L \circ R^t \circ R \circ L^t\,.$$

and there exists $c_3 \geq c_2$ only depending on n such that

$$|Q|_\infty \leq c_3|R|_\infty\,. \tag{5.144}$$

Moreover, if we assume that $c_3|R|_\infty \leq 1$ then there exists $c_4 \geq 4nc_3$ only depending on n such that

$$\left|\det\left(((I_n + R) \circ L^t)^t \circ ((I_n + R) \circ L^t)\right)^{\frac{1}{2}} - 1\right| \leq c_4|R|_\infty, \tag{5.145}$$

and

$$\left|\det\left(((I_n + R) \circ L^t)^t \circ ((I_n + R) \circ L^t)\right)\right)^{\frac{1}{2}} - 1 - \operatorname{tr}\left(R \circ L^t \circ L\right)\right| \leq c_4|R|_\infty^2. \tag{5.146}$$

Proof. We consider the normed space $(\mathcal{M}_{p,q}, |\cdot|_\infty)$ and we recall that for $M \in \mathcal{M}_{p,q}$ and $N \in \mathcal{M}_{q,r}$,

$$|MN|_\infty \leq q\,|M|_\infty|N|_\infty\,. \tag{5.147}$$

Let $Q \in \mathcal{M}_k$ be such that $|Q|_\infty \leq 1$ and let $B = \{M \in \mathcal{M}_k \ : \ |I_k - M|_\infty \leq 1\}$ be the closed unit ball centered at I_k. By compactness of B, we can introduce

$$c_{2,k} = \max\left\{1, \max_{M \in B} \|D\det(M)\|, \frac{1}{2}\max_{M \in B}\|D^2\det(M)\|\right\}\,,$$

where $\|\cdot\|$ are the linear and bilinear operator norms associated with $(\mathcal{M}_{p,q}, |\cdot|_\infty)$. Note that $c_{2,k}$ depends on k (since B depends on k) though this can be avoided by defining $c_2 = \max_{1\le k\le n} c_{2,k}$. We recall that the differential of the determinant map $\det$ at I_k is the trace map: $D\det(I_k) = \mathrm{tr}$ and therefore, applying Taylor-Lagrange inequality to $\det$ on the line segment $[I_k, I_k + Q] \subset B$ yields (5.143). Let $L \in \mathcal{M}_{d,n}$, $R \in \mathcal{M}_n$ be such that $L \circ L^t = I_d$ and $|R|_\infty \le 1$ and let us use the notation $Q = L \circ \big(R^t + R\big) \circ L^t + L \circ R^t \circ R \circ L^t \in \mathcal{M}_d$ hereafter. First note that $|L|_\infty \le 1$, indeed, the assumption $L \circ L^t = I_d$ can be reformulated as follows: the columns of L^t (i.e. the rows of L) constitute an orthonormal family $(v_1, \ldots, v_d)$ of $\mathbb{R}^n$ so that $|L|_\infty = \max_{ij} |L_{ij}| = \max_{ij} |v_i \cdot e_j| \le 1$. Let $c_3 = (2n^2 + n^3)(1 + c_2)$, using (5.147) and $|M^t|_\infty = |M|_\infty$, we have

$$\begin{aligned}|Q|_\infty &= \big|L \circ \big(R^t + R\big) \circ L^t + L \circ R^t \circ R \circ L^t\big|_\infty \le n^2|R + R^t|_\infty|L|^2_\infty + n^3|L|^2_\infty|R|^2_\infty \\ &\le (2n^2 + n^3|R|_\infty)|R|_\infty \le (2n^2 + n^3)|R|_\infty \le c_3|R|_\infty ,\end{aligned}$$

that is (5.144). We now set $c_4 = (1 + c_2)c_3^2 + n^4 \ge (1 + c_2)^2(2n + n^3)c_3 \ge 4nc_3$, and we moreover assume $c_3|R|_\infty \le 1$, therefore

$$((I_n + R) \circ L^t)^t \circ ((I_n + R) \circ L^t) = I_d + Q \quad \text{with} \quad |Q|_\infty \le 1$$

so that the first part of (5.143) gives

$$|\det(I_d + Q) - 1| \le c_2|Q|_\infty \le c_2(2n^2 + n^3)|R|_\infty \le c_3|R|_\infty \le 1 \quad \text{and in particular} \quad \det(I_d + Q) \ge 0\,. \tag{5.148}$$

We infer (5.145) applying $|a - 1| \le |a^2 - 1|$ (note that $a = \det(I_d + Q) \ge 0$). We are left with the proof of (5.146). We now apply the second inequality in (5.143) to obtain

$$|\det(I_d + Q) - 1 - \mathrm{tr}\, Q| \le c_2|Q|^2_\infty \le c_2c_3^2|R|^2_\infty\,. \tag{5.149}$$

Furthermore, using $\mathrm{tr}(A) = \mathrm{tr}\big(A^t\big)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ when both products make sense, we have

$$\mathrm{tr}\big(L \circ R^t \circ L^t\big) = \mathrm{tr}\big(L \circ R \circ L^t\big) = \mathrm{tr}\big(R \circ L^t \circ L\big)$$

and thus, by definition of Q and (5.147),

$$\begin{aligned}\big|\mathrm{tr}\, Q - 2\,\mathrm{tr}\big(R \circ L^t \circ L\big)\big| &= \big|\mathrm{tr}\big(L \circ R^t \circ R \circ L^t\big)\big| \le d|L \circ R^t \circ R \circ L^t|_\infty \le dn^3|L|^2_\infty|R|^2_\infty \\ &\le n^4|R|^2_\infty\,.\end{aligned} \tag{5.150}$$

From (5.149) and (5.150) we obtain

$$\big|\det(I_d + Q) - 1 - 2\,\mathrm{tr}\big(R \circ L^t \circ L\big)\big| \le (c_2c_3^2 + n^4)|R|^2_\infty \tag{5.151}$$

We now apply the following inequality, valid for any $z \ge -1$,

$$\left|\sqrt{1 + z} - 1 - \frac{1}{2}z\right| \le \frac{1}{2}z^2$$

with $z = \det(I_d + Q) - 1$, from (5.148) we know that $-1 \le z \le c_3|R|_\infty$, we hence obtain

$$\begin{aligned}\left|\sqrt{\det(I_d + Q)} - 1 - \mathrm{tr}\big(R \circ L^t \circ L\big)\right| &\le \left|\sqrt{1 + z} - 1 - \frac{1}{2}z\right| + \left|\frac{1}{2}z - \mathrm{tr}\big(R \circ L^t \circ L\big)\right| \\ &\le \frac{1}{2}z^2 + \frac{1}{2}(c_2c_3^2 + n^4)|R|^2_\infty \text{ thanks to (5.151)}, \\ &\le \frac{1}{2}(c_3^2 + c_2c_3^2 + n^4)|R|^2_\infty \le c_4|R|^2_\infty\,,\end{aligned}$$

which concludes the proof of (5.146). □

The following lemma stems directly from Lemma 5.5.3.

Lemma 5.5.4. *Let $P, N \in \mathcal{M}_d$ and assume that P is invertible, then*

$$\|P^{-1}\|\,\|P-N\| \le 1 \quad \Rightarrow \quad |\det(P)-\det(N)| \le c_2|\det(P)|\|P^{-1}\|\,\|P-N\|\,. \tag{5.152}$$

Proof. Indeed, first note that

$$|\det(P)-\det(N)| = |\det(P)|\,\left|1-\det\left(P^{-1}N\right)\right| \quad \text{and} \quad P^{-1}N = I_d + P^{-1}(N-P)\,.$$

Furthermore $\left|P^{-1}(N-P)\right|_\infty \le \|P^{-1}(N-P)\| \le \|P^{-1}\|\,\|P-N\| \le 1$ so that applying (5.143) with $k=d$ and $Q = P^{-1}(N-P)$ we can assert that

$$\left|1-\det\left(P^{-1}N\right)\right| \le c_2\left|P^{-1}(N-P)\right|_\infty \le c_2\,\|P^{-1}\|\,\|N-P\|$$

which concludes the proof of (5.152). □

The following is a crucial step to prove Proposition 5.1.8.

Lemma 5.5.5. *Let $S, T \in G_{d,n}$, there exist $\tilde{S} = (s_1|\dots|s_d)^t\,, \tilde{T} = (t_1|\dots|t_d)^t \in \mathcal{M}_{d,n}$ where $\{s_i\}_{i=1}^d$ and $\{t_i\}_{i=1}^d$ are two orthonormal basis of S and T such that*

$$\|\tilde{S}-\tilde{T}\| \le 2\|S-T\|.$$

Proof. Let θ be the largest principal angle between the subspaces S and T, which can be characterized by:

$$\sin(\theta) = \max_s \min_t \sqrt{1-\langle s,t\rangle^2},\ \ s\in S, t\in T \quad \text{and}\ |s|=|t|=1.$$

We infer from [1, Proposition III.29] that $\|S-T\| = \sin(\theta)$, furthermore, there exists a rotation r of $\mathbb{R}^n$ such that $r(S)=T$, with

$$\|r-I_n\| \le 2\sin(\theta/2).$$

Let $\tilde{S} = (s_1|\dots|s_d)^t \in \mathcal{M}_{d,n}$, with $\{s_i\}_{i=1}^d$ being an orthonormal basis of S, the matrix $\tilde{T} = r\circ\tilde{S} \in \mathcal{M}_{d,n}$ can be written as $(t_1|\dots|t_d)^t$ where $\{t_i\}_{i=1}^d$ is an orthonormal basis of and T. We have then, using that $\|\tilde{S}\|=1$

$$\|\tilde{S}-\tilde{T}\| = \|\tilde{S}-r\circ\tilde{S}\| \le \|I_n-r\|\|\tilde{S}\| \le 2\sin(\theta/2);$$

the result follows from noting that $2\sin(\theta/2) \le 2\sin(\theta)$ as $\theta\in[0,\pi/2]$. □

The following is a key lemma to prove the diffeomorphic character of the pushforward maps involved in the construction of the approximate MCF (Proposition 5.1.3). We prove that the map $\mathrm{id}+\Delta t h$, $h\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ is a diffeomorphism under a condition on Δt and h. We use this lemma to obtain a sufficient condition on Δt to have that the map $\mathrm{id}+\Delta t h_\varepsilon$ is a diffeomorphism, where h_ε is the approximate mean curvature.

Lemma 5.5.6. *Let $f\in C(\mathbb{R}^n,\mathbb{R}^n)$ such that $f := \mathrm{id}+\Delta t h,\ \Delta t>0,\ h\in C^1(\mathbb{R}^n,\mathbb{R}^n)$, assume that*

$$\max\{\Delta t\|h\|_\infty, \Delta t\|Dh\|_\infty, \|Jf-1\|_\infty\} < 1.$$

Then, f is a diffeomorphism of $\mathbb{R}^n$.

Proof. For any $x \in \mathbb{R}^n$, we have $Jf(x) \neq 0$, therefore f is a local diffeomorphism by the inverse mapping theorem. We now prove that f is injective, indeed, for any $x, y \in \mathbb{R}$ one has

$$|f(x) - f(y)| = |x - y - \Delta t\,(h(x) - h(y))\,| \geq \big||x - y| - \Delta t|h(x) - h(y)|\big| \geq |x - y|\big|1 - \mathrm{Lip}(h)\big| > 0,$$

this proves the injectivity of f.
Up to now, we have checked that f is injective and a local diffeomorphism at every point therefore f is a global diffeomorphism from $\mathbb{R}^n$ onto $f(\mathbb{R}^n)$; as $f(\mathbb{R}^n)$ is open, it remains to show that $f(\mathbb{R}^n)$ is closed. We have by assumption that $||f - \mathrm{id}||_\infty = \Delta t\|h\|_\infty < 1$, this implies that f is proper, by [41] it is closed, therefore $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$, recalling that it was open we have $f(\mathbb{R}^n) = \mathbb{R}^n$ and f is a diffeomorphism of $\mathbb{R}^n$. □

5.6 List of constants used in the chapter and their properties

- $\forall \varepsilon \in (0,1)$, $c(\varepsilon) = \dfrac{1}{\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx}$ (5.2), $c(\varepsilon) \geq 1$.
- $c = \left(\displaystyle\int_{B(0,\frac{1}{2})} \hat{\Phi}_1(y)dy\right)^{-1}$ (5.4), $\forall \varepsilon \in (0,1)\ \ 1 \leq c(\varepsilon) \leq c$.
- $c_0 := \sup\limits_{\varepsilon\in(0,1)} c(\varepsilon)\frac{9\varepsilon^{-2-n}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{8\varepsilon^2}\right) < \infty$ (5.10).
- $c_1 = 2(1 + \omega_n c_0)(1 + c_0) \geq 2,$ Proposition 5.1.2.
- $c_2 = \max\limits_{1\leq k\leq d} \left\{1, \max\limits_{M\in B_1(\mathcal{M}_k)} \|D\det(M)\|, \frac{1}{2} \max\limits_{M\in B_1(\mathcal{M}_k)} \|D^2\det(M)\|\right\} \geq 1$ Proposition 5.1.3.
- $c_3 = (2n^2 + n^3)(1 + c_2)$ Lemma 5.5.3.
- $c_4 = (1 + c_2)c_3^2 + n^4$ Proposition 5.1.3 $c_4 \geq 4nc_3$, $c_4 \geq c_2$, $c_4 \geq 2$.
- $c_5 = 4c_1^2 c_4$ (5.33).
- $c_6 = \max\{4c_1, (1 + c_0\omega_n)(c(2\pi)^{-n/2} + c_0)(2 + c_1)\} \geq 4c_1$ (5.36).
- $c_7 = 6(128nc_2c_3c_6 + c_1c_4)$ Proposition 5.1.8, $c_7 \geq 56c_6$ $c_{7,M} = c_7(M + 1)^2$.
- $c_8 = 112c_1^2 + 3c_1^2 c_7$ Lemma 5.1.12.
- $c_9 = 2c_7(57c_1c_4 + c_8)$ Lemma 5.1.13 $c_9 \geq c_8$. $c_{9,M} = c_9(M + 1)^5$, $c_{9,M} \geq c_8M^3$.
- $c_{10} = 4c_9$, $c_{10,M} = c_{10}(M + 1)^5$ Proposition 5.1.11.
- $c_{11} = (56c_1 + c_1c_4)(M + 1)^2$ Proposition 5.2.2.
- $c_{12} = (c_6 + 3nc_1 + 4)(M + 1)^3$ Corollary 5.2.6.

Chapter 6

Avoidance principle for approximate MCFs and spacetime Brakke flows

The classical avoidance principle for the mean curvature flow says that if $\mathcal{M}$ and $\mathcal{N}$ are two disjoint smooth compact hypersurfaces, then their respective mean curvature flows are disjoint (see Theorem 2.1.4). This avoidance property is a consequence of the maximum principle. This principle fails in higher codimensions, take for instance the MCF of two enlaced disjoint circles in $\mathbb{R}^3$ (see Figure 6.1). Ilmanen ([34, Lemma 4E]) generalized the avoidance principle to arbitrary "set-theoretic subsolutions of mean curvature flow". He showed in [35, Theorem 10.5] that the support of a codimension 1 integral Brakke flow is a set-theoretic subsolution of mean curvature flow, so the avoidance principle also applies to such a Brakke flow.

In this chapter, inspired by the works of Ilmanen and Brakke, we prove certain avoidance and approximate avoidance principles for spacetime Brakke flows and their approximations defined in chapter 5.

We start by proving the nontriviality of the limit of the spacetime approximate MCF (Definition 6.2.1, Theorem 6.2.9) when starting from the boundary of an open partition of $\mathbb{R}^n$ (Definition 2.2.3). The proof is based on a ε- approximate comparison principle with respect to spheres evolving by the MCF satisfied by the piecewise discrete approximate MCF. This solves the triviality issue (discussed in section 2.2.6) that might occur in the Brakke construction.

We show in Proposition 6.3.1 different avoidance principles for the mass measure of d-spacetime Brakke flows with respect to spheres evolving by the law $R(t)^2 = R(0)^2 - 2dt$. Theorem 6.4.3 states a general avoidance principle satisfied by spacetime Brakke flows of codimension 1 with respect to smooth mean curvature flows of codimension 1. The theorem has several consequences, notably, the inclusion of the mass measure of a spacetime Brakke flow in the level set flow of the starting varifold ([22]).

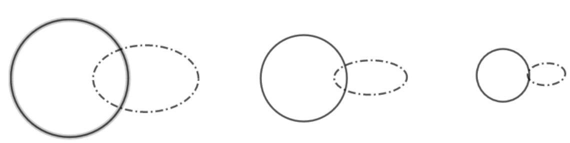

Figure 6.1: The MCFs of two enlaced disjoint circles in $\mathbb{R}^3$ collide.

6.1 Preliminaries

We state the following technical lemmas that will help later in proving different results of the chapter.

Lemma 6.1.1. *Let $h \in C(\mathbb{R}^n, \mathbb{R}^n)$, $\varphi \in C^1(\mathbb{R}^n, (0,\infty))$ and $S \in G_{d,n}$. We have:*

$$-|h|^2\varphi + S^\perp \nabla\varphi \cdot h \leq \frac{1}{4}\frac{|S\nabla\varphi|^2}{\varphi} + \nabla\varphi \cdot h.$$

Proof. We can simply write, as φ does not vanish,

$$\begin{aligned}-|h|^2\varphi + S^\perp\nabla\varphi\cdot h = -|h|^2\varphi - S\nabla\varphi\cdot h + \nabla\varphi\cdot h &= -\left|\varphi^{\frac{1}{2}}h + \frac{1}{2}\frac{S\nabla\varphi}{\varphi^{\frac{1}{2}}}\right|^2 + \frac{1}{4}\frac{|S\nabla\varphi|^2}{\varphi} + \nabla\varphi\cdot h \\ &\leq \frac{1}{4}\frac{|S\nabla\varphi|^2}{\varphi} + \nabla\varphi\cdot h,\end{aligned}$$

this completes the proof. □

We introduce the notion of barrier functions and we highlight one of their main properties. For convenience reasons, we use the definition of Brakke ([12, Section 3.6]) for barrier functions even though it might refer to a more general class of functions (see [9]).

Definition 6.1.2. *A function $\psi \in C^2(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ is called a barrier function if there exists $a \in \mathbb{R}^n$ and $\gamma \in C^2(\mathbb{R}, \mathbb{R}^+)$ such that*

$$\psi(t,x) = \gamma(|x-a|^2 + 2dt) \quad \textit{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

and

$$\left(\frac{d}{dr}\gamma(r)\right)^2 \leq 4\gamma(r)\frac{d^2}{dr^2}\gamma(r) \quad \textit{for all } r \in \mathbb{R}.$$

Lemma 6.1.3. *(Sphere barriers) Let ψ be a barrier function, for every $S \in G_{d,n}$. We have*

$$\frac{1}{4}\frac{|S\nabla\psi|^2}{\psi} - S : \nabla^2\psi + \partial_t\psi \leq 0$$

on $\{(x,t) \in \mathbb{R}^n \times \mathbb{R}, \psi(x,t) \neq 0\}$.

Proof. We set $a = 0$ for simplicity. Let $\gamma \in C^2(\mathbb{R}, \mathbb{R}^+)$ be such that

$$\psi(t,x) = \gamma(|x|^2 + 2dt) \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

We compute the derivatives of ψ:

$$\begin{aligned}\psi(x,t) &= \gamma(|x|^2 + 2dt), \\ \partial_t\psi(x,t) &= 2d\gamma'(|x|^2 + 2dt), \\ \nabla\psi(x,t) &= 2\gamma'(|x|^2 + 2dt)x, \text{ and} \\ \nabla^2\psi(x,t) &= 4\gamma''(|x|^2 + 2dt)x \otimes x + 2\gamma'(|x|^2 + 2dt)I_n.\end{aligned}$$

We have then, on $\{(x,t) \in \mathbb{R}^n \times \mathbb{R}, \psi(x,t) \neq 0\}$, using $S : I_n = \mathrm{tr}(S) = d$ and $\gamma'(r)^2 \leq 4\gamma(r)\gamma''(r)$ for all $r \in \mathbb{R}$,

$$\begin{aligned}
&\frac{1}{4}\frac{|S\nabla\psi(x,t)|^2}{\psi(x,t)} - S : \nabla^2\psi(x,t) + \partial_t\psi(x,t) \\
&= \frac{|S(x)|^2\gamma'(|x|^2+2dt)^2}{\gamma(|x|^2+2dt)} - 4|S(x)|^2\gamma''(|x|^2+2dt) - 2d\gamma'(|x|^2+2dt) + 2d\gamma'(|x|^2+2dt) \\
&= |S(x)|^2\left(\frac{\gamma'(|x|^2+2dt)^2}{\gamma(|x|^2+2dt)} - 4\gamma''(|x|^2+2dt)\right) \leq 0,
\end{aligned}$$

and this completes the proof. □

6.2 Nontriviality of the limit spacetime approximate MCF of open partitions

We start by defining the notion of a limit of the spacetime approximate MCF.

Definition 6.2.1. *(Limit of the spacetime approximate MCF) Let $V_0 \in V_d(\mathbb{R}^n)$ of compact support, let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate MCF starting from V_0. We say that a Radon measure λ in $\mathbb{R}^n \times G_{d,n} \times [0,1]$ is a limit of the spacetime approximate MCF if there exists a sequence $(\varepsilon_j)_j \to 0$, a family of Radon measures $(\mu(t))_{t\in[0,1]}$ on $\mathbb{R}^n$ and a family of probability measures $(\nu_{(x,t)})_{(x,t)}$ on $G_{d,n}$ such that*

- $\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to+\infty]{*} \mu(t)$*, for every* $t \in [0,1]$ *and*
- $V_{\varepsilon_j}(t) \otimes dt \xrightarrow[j\to+\infty]{*} \lambda = \mu(t) \otimes \nu_{(x,t)} \otimes dt.$

We recall that we have shown the existence of such sequence and limits in Proposition 5.3.10. In analogy to Definition 5.3.4, the family $(\mu(t))_{t\in[0,1]}$ is called the mass measure of λ.

We aim to prove that limits of the spacetime approximate MCF are nontrivial when the starting varifold is the boundary of an open partition (Definition 2.2.3). The sense we give to the nontriviality is that $\mu(t)(\mathbb{R}^n) > 0$ for a certain time interval $[0, t_0], t_0 > 0$.

The following lemma is the key to prove the nontriviality property. We prove a ε-avoidance principle for a limit of the spacetime approximate MCF with respect to spheres evolving by the law

$$R(t)^2 = R(0)^2 - 2dt,\ \ t \in [0, R(0)^2/2d]; \tag{6.1}$$

where $R(t)$ denotes the radius of the sphere at time t. This is an adaptation of Brakke's sphere barrier to external varifolds principle [12, Theorem 3.7]. The proof consists of injecting a suitable test function encoding the evolution of the spheres in the ε-Brakke inequality satisfied by approximate MCFs (Corollary 5.2.6).

Lemma 6.2.2. *Let $M \geq 1$, $\varepsilon \in (0,1)$ and $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T} = \{t_i\}_{i=1}^m$ be a subdivision of $[0,1]$ satisfying (5.34) and $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ be the piecewise constant approximate MCF with*

respect to $\mathcal{T}$ starting from V_0.
Define $\psi(x,t) = \gamma(|x-a|^2 + 2dt)$ with

$$\gamma(r) = \begin{cases} (R^2 - r)^4 & \text{for } r \le R^2, \\ 0 & \text{for } r > R^2. \end{cases}$$

Assume that $c_{12}\delta(\mathcal{T})\varepsilon^{-8} \le \varepsilon$. Then, there exists $\varepsilon_0 \in (0,1)$ depending only on n and M such that, for any $\varepsilon \in (0, \varepsilon_0)$, we have:

$$\|V^{pc}_{\varepsilon,\mathcal{T}}(b)\|(\psi(\cdot,b)) - \|V^{pc}_{\varepsilon,\mathcal{T}}(a)\|(\psi(\cdot,a)) \le c_{13}\varepsilon^{\frac{1}{6}} \quad \text{for any } a,b \in [0,1],\ a \le b;$$

where c_{13} depends only on n, M and R.

Proof. We assume that $\mathcal{T}$ satisfies (5.34) to give sense to the approximate MCF. We note that the choice of the power in the definition of γ is not relevant, as long as it is strictly larger than 3 (so that ψ is C^3). In the proof, we denote for simplicity $V(t) = V^{pc}_{\varepsilon,\mathcal{T}}(t)$.
Step 1**:** We first prove that

$$\int_0^1 \int_{\mathbb{R}^n} \frac{|(\delta V(t) * \Phi_\varepsilon)(y)|^2}{(\|V(t)\| * \Phi_\varepsilon)(y) + \varepsilon}\, dydt \le M + 1. \tag{6.2}$$

From Remark 5.1.5, one can state that $\forall i \in \{0, \dots, m\}$

$$\|V(t_i)\|(\mathbb{R}^n) \le M + 1.$$

From (5.24) with $\varphi \equiv 1$, $V = V(t_i)$ and $\Delta t = t_{i+1} - t_i$, we have $\forall i \in \{0, \dots, m-1\}$

$$\|V(t_{i+1})\|(\mathbb{R}^n) - \|V(t_i)\|(\mathbb{R}^n) - \Delta t \delta V(t_i)\left(h_\varepsilon(\cdot, V(t_i))\right) \le c_5(M+1)^3\left(t_{i+1} - t_i\right)^2 \varepsilon^{-8}.$$

Then, by (5.34) and (5.25), and the fact that $V(t)$ is constant on $[t_i, t_{i+1})$, we infer that $\forall i \in \{0, \dots, m-1\}$

$$\|V(t_{i+1})\|(\mathbb{R}^n) - \|V(t_i)\|(\mathbb{R}^n) + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^n} \frac{|(\delta V(t) * \Phi_\varepsilon)(y)|^2}{(\|V(t)\| * \Phi_\varepsilon)(y) + \varepsilon}\, dydt \le t_{i+1} - t_i.$$

Summing up the inequalities for $i \in \{0, \dots, m-1\}$ we obtain

$$\|V(1)\|(\mathbb{R}^n) - \|V(0)\|(\mathbb{R}^n) + \int_0^1 \int_{\mathbb{R}^n} \frac{|(\delta V(t) * \Phi_\varepsilon)(y)|^2}{(\|V(t)\| * \Phi_\varepsilon)(y) + \varepsilon}\, dydt \le 1.$$

Then, step 1 follows from $\|V(0)\|(\mathbb{R}^n) = \|V_0\|(\mathbb{R}^n) \le M$.
Step 2**:** Let $\psi_\varepsilon = c^{-1}\left(\psi + 4\varepsilon^{\frac{1}{6}}\|\psi\|_{C^3}\right)$ with $c = c(n,R) = 2\max\{\|\psi\|_{C^3}, \|\nabla\psi\|_{L^2}, 1\} < \infty$. We prove that $\psi_\varepsilon \in \mathcal{A}_j$ and $\nabla\psi_\varepsilon \in \mathcal{B}_j$ (Definition 2.2.5) with $j = \lfloor \frac{1}{2}\varepsilon^{-\frac{1}{6}} \rfloor$ and $\varepsilon \in (0, 4^{-6})$.
We know that $\psi \ge 0$, and $4\varepsilon^{\frac{1}{6}} j \ge 1$, this implies:

$\psi(\cdot,t) \in C^3(\mathbb{R}^n, \mathbb{R}^+)\ \forall t \in [0,1]$;
$\|\psi_\varepsilon(\cdot,t)\|_\infty \le 1\ \forall t \in [0,1]$;
$\|\nabla^m\psi_\varepsilon(\cdot,t)\|_\infty = c^{-1}\|\nabla^m\psi(\cdot,t)\|_\infty \le j\psi_\varepsilon(x,t) \le j,\ \forall (x,t) \in \mathbb{R}^n \times [0,1]$ and $m \in \{1,2,3\}$;
$\|\nabla\psi_\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^n)} = c^{-1}\|\nabla\psi(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le 1 \le j\ \forall t \in [0,1]$.

Thus $\psi_\varepsilon(\cdot,t) \in \mathcal{A}_j$ and $\nabla\psi_\varepsilon(\cdot,t) \in \mathcal{B}_j$ $\forall t \in [0,1]$.
Step 3**:** We inject ψ_ε into the inequality (5.112) and deduce the desired result.
Indeed, we have for $a, b \in [0,1],\ a \leq b$

$$\begin{aligned}\|V(b)\|(\psi_\varepsilon(\cdot,b)) - \|V(a)\|(\psi_\varepsilon(\cdot,a)) - \int_a^b \delta(V(t),\psi_\varepsilon(\cdot,t))(h_\varepsilon(t))\,dt \\ - \int_a^b \int_{\mathbb{R}^n} \partial_t\psi_\varepsilon(\cdot,t)\,d\|V(t)\|dt \leq c_{12}\|\psi_\varepsilon\|_{C^2}\delta(\mathcal{T})\varepsilon^{-8} \leq \varepsilon.\end{aligned} \tag{6.3}$$

where we used $\|\psi_\varepsilon\|_{\mathcal{C}^2} \leq 1$. We recall that by (1.13) we have

$$\delta(V(t),\psi_\varepsilon(\cdot,t))(h_\varepsilon(t)) = \delta V(t)(\psi_\varepsilon h_\varepsilon) + \int_{\mathbb{R}^n\times G_{d,n}} S^\perp\nabla\psi_\varepsilon \cdot h_\varepsilon(t)\,dV(t). \tag{6.4}$$

Applying (5.118) and (5.119) with $V = V(t)$, $\varphi = \psi_\varepsilon(\cdot,t)$ knowing that $j \leq \frac{1}{2}\varepsilon^{-\frac{1}{6}}$ we deduce that there exists $\varepsilon_0 \in (0, 4^{-6})$ depending only on M and n such that, for any $\varepsilon \in (0,\varepsilon_0)$ one has, for all $t \in [0,1]$

$$\delta V(t)(\psi_\varepsilon(\cdot,t)h_\varepsilon(t)) \leq -\int_{\mathbb{R}^n} \frac{\psi_\varepsilon(\cdot,t)|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_\varepsilon * \|V(t)\| + \varepsilon}dx + \varepsilon^{\frac{1}{4}}\left(\int_{\mathbb{R}^n} \frac{\psi_\varepsilon(\cdot,t)|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_\varepsilon * \|V(t)\| + \varepsilon}dx + 1\right) \tag{6.5}$$

and that

$$-\int_{\mathbb{R}^n} \frac{\psi_\varepsilon(\cdot,t)|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_{\varepsilon_j} * \|V(t)\| + \varepsilon}dx \leq -\int_{\mathbb{R}^n} \psi_\varepsilon(\cdot,t)|h_\varepsilon(t)|^2 d\|V(t)\| + \varepsilon^{\frac{1}{4}}\left(\int_{\mathbb{R}^n} \frac{\psi_\varepsilon(\cdot,t)|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_\varepsilon * \|V(t)\| + \varepsilon}dx + 1\right) \tag{6.6}$$

thus combining (6.5), (6.6) and using $\|\psi_\varepsilon\|_\infty \leq 1$ and (6.2) we deduce that

$$\int_a^b \delta V(t)(\psi_\varepsilon(\cdot,t)h_\varepsilon(t))\,dt \leq -\int_a^b \int_{\mathbb{R}^n} \psi_\varepsilon(\cdot,t)|h_\varepsilon(t)|^2\,d\|V(t)\|dt + \varepsilon^{\frac{1}{4}}\,(2M+4)\,. \tag{6.7}$$

From (6.3) , (6.4) and (6.7), using $\varepsilon \leq \varepsilon^{\frac{1}{4}}$ we infer that

$$\begin{aligned}\|V(b)\|(\psi_\varepsilon(\cdot,b)) - \|V(a)\|(\psi_\varepsilon(\cdot,a)) \leq \int_a^b \int_{\mathbb{R}^n\times G_{d,n}} -\psi_\varepsilon(\cdot,t)|h_\varepsilon(t)|^2 + S^\perp\nabla\psi_\varepsilon \cdot h_\varepsilon(t)\,dV(t)dt \\ + \int_a^b \int_{\mathbb{R}^n} \partial_t\psi_\varepsilon(\cdot,t)\,d\|V(t)\|dt + \varepsilon^{\frac{1}{4}}\,(2M+5)\,.\end{aligned}$$

From Lemma 6.1.1 we infer that

$$\begin{aligned}\|V(b)\|(\psi_\varepsilon(\cdot,b)) - \|V(a)\|(\psi_\varepsilon(\cdot,a)) \leq \int_a^b \int_{\mathbb{R}^n} \frac{1}{4}\frac{|S\nabla\psi_\varepsilon(\cdot,t)|^2}{\psi_\varepsilon(\cdot,t)} + \nabla\psi_\varepsilon(\cdot,t)\cdot h_\varepsilon(t) + \partial_t\psi_\varepsilon(\cdot,t)\,dV(t)dt \\ + \varepsilon^{\frac{1}{4}}\,(2M+5)\,,\end{aligned} \tag{6.8}$$

where dividing by ψ_ε is possible as $\psi_\varepsilon > 0$. From (5.120) for $V = V(t)$ and $g = \nabla\psi_\varepsilon(\cdot, t)$ we have, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}
&\Big| \int_a^b \int_{\mathbb{R}^n} h_\varepsilon(t) \cdot \nabla\psi_\varepsilon(\cdot, t)\, d\|V\|(t) + \delta V(t)\left(\nabla\psi_\varepsilon(\cdot, t)\right)\, dt \Big| \\
&\le \varepsilon^{\frac{1}{4}}(b-a) + \varepsilon^{\frac{1}{4}} \int_a^b \left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_\varepsilon * \|V\|(t) + \varepsilon} dx \right)^{\frac{1}{2}} dt \\
&\le \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}} \left(\int_0^1 \int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta V(t)|^2}{\Phi_\varepsilon * \|V\|(t) + \varepsilon} dx dt \right)^{\frac{1}{2}} \\
&\le \varepsilon^{\frac{1}{4}}(1 + (2M)^{\frac{1}{2}}) \le \varepsilon^{\frac{1}{4}}(1+2M), \quad \text{as } 1 \le M \le 2M,
\end{aligned}$$

where we used Jensen inequality. This gives,

$$\begin{aligned}
\int_a^b \int_{\mathbb{R}^n} h_\varepsilon(t) \cdot \nabla\psi_\varepsilon(\cdot, t)\, d\|V\|(t) dt &\le - \int_a^b \delta V(t)\left(\nabla\psi_\varepsilon(\cdot, t)\right)\, dt + \varepsilon^{\frac{1}{4}}(1+2M) \\
&= \int_a^b \int_{\mathbb{R}^n} -S : \nabla^2\psi_\varepsilon(\cdot, t) dV(t) dt + \varepsilon^{\frac{1}{4}}(1+2M).
\end{aligned}$$

From (6.8) we infer that

$$\begin{aligned}
\|V(b)\|(\psi_\varepsilon(\cdot, b)) - \|V(a)\|(\psi_\varepsilon(\cdot, a)) \le \int_a^b \int_{\mathbb{R}^n} \frac{1}{4} \frac{|S\nabla\psi_\varepsilon(\cdot, t)|^2}{\psi_\varepsilon(\cdot, t)} - S : \nabla^2\psi_\varepsilon(\cdot, t) + \partial_t\psi_\varepsilon(\cdot, t)\, dV(t) dt \\
+ \varepsilon^{\frac{1}{4}}\left(4M + 6\right).
\end{aligned}$$

We note that $\psi_\varepsilon \ge c^{-1}\psi$, $\nabla^m\varphi_\varepsilon = c^{-1}\nabla^m\varphi$, $\forall m \in \{1, 2\}$ and that ψ is a barrier function, hence Lemma 6.1.3 implies

$$\frac{1}{4} \frac{|S\nabla\psi_\varepsilon(\cdot, t)|^2}{\psi_\varepsilon(\cdot, t)} - S : \nabla^2\psi_\varepsilon(\cdot, t) + \partial_t\psi_\varepsilon(\cdot, t) \le c^{-1} \left(\frac{1}{4} \frac{|S\nabla\psi(\cdot, t)|^2}{\psi(\cdot, t)} - S : \nabla^2\psi(\cdot, t) + \partial_t\psi(\cdot, t) \right) \le 0$$

this implies

$$\|V(b)\|(\psi_\varepsilon(\cdot, b)) - \|V(a)\|(\psi_\varepsilon(\cdot, a)) \le \varepsilon^{\frac{1}{4}}\left(4M + 6\right).$$

We conclude from the uniform boundedness of the mass (Remark 5.35) and from $\|\psi\|_{C^2} \le c$ that

$$\|V(b)\|(\psi(\cdot, b)) - \|V(a)\|(\psi(\cdot, a)) \le 4c\varepsilon^{\frac{1}{6}}(M+1) + c\varepsilon^{\frac{1}{4}}\left(4M+6\right) \le c\varepsilon^{\frac{1}{6}}(8M+10) \le c_{13}\varepsilon^{\frac{1}{6}}$$

where we set $c_{13} = c(8M + 10)$. This finishes the proof. □

We show that the mass measure of a limit of the spacetime approximate MCF satisfies the external varifold comparison principle.

Corollary 6.2.3. *(External varifold comparison principle) Let $V \in V_d(\mathbb{R}^n)$ of compact support and $(\mu(t))_{t\in[0,1]}$ be the mass measure of a limit of the spacetime approximate MCF starting from V (Definition 6.2.1). We have the following:*

1. *$(\mu(t))_{t\in[0,1]}$ satisfies the the external varifold comparison principle, i.e. if*

$$\mu(0)(B(a, R)) = 0 \implies \mu(t)(B(a, \sqrt{R^2 - dt})) = 0, \quad t \in [0, 1] \cap [0, R^2/2d].$$

2. $\operatorname{spt}\mu(t)$ *is bounded by the convex hull of* $\operatorname{spt} V$ *for all* $t \in [0,1]$. *In particular* $\bigcup_{t\in[0,1]} \operatorname{spt}\mu(t)$ *is bounded.*

Proof. We start by proving the external varifold varifold principle. Let $\varepsilon \in (0,1)$, $(V_\varepsilon(t))_{t\in[0,1]}$ the approximate MCF starting from $\mathcal{E}$. Define, $\psi(x,t) = \gamma(|x-a|^2 + 2dt)$ such that

$$\gamma(r) = \begin{cases} (R^2 - r)^4 & \text{for } r \le R^2, \\ 0 & \text{for } r > R^2. \end{cases}$$

Proposition 5.2.2 ensures that the limit of the piecewise constant approximate MCF converges to $(V_\varepsilon(t))_{t\in[0,1]}$, hence, Lemma 6.2.2 infers that

$$\|V_\varepsilon(t)\|(\psi(\cdot,t)) \le \|V_\varepsilon(0)\|(\psi(\cdot,0)) + c_{13}\varepsilon^{\frac{1}{6}}.$$

Taking the limit in ε, we infer from Proposition 5.3.10

$$\mu(t)(\psi(\cdot,t)) \le \mu(0)(\psi(\cdot,0)).$$

By construction, $\psi(\cdot,t) > 0$ on $B(a, \sqrt{R^2 - 2dt})$ hence

$$\mu(0)(B(a,R)) = 0 \implies \mu(t)(B(a,\sqrt{R^2 - dt})) = 0, \quad t \in [0,1] \cap [0, R^2/2d].$$

The convex barrier principle is a direct adaptation of the proof of [12, Theorem 3.8] using the sphere barrier to external varifolds. □

Corollary 6.2.4. *(Decay of the mass) Let* $V_0 \in V_d(\mathbb{R}^n)$ *of compact support and* $(\mu(t))_{t\in[0,1]}$ *the mass measure of a limit spacetime approximate MCF starting from* V_0. *From Remark 5.2.7 and the boundedness of* $\operatorname{spt}\mu(t)$ $\forall t \in [0,1]$ *(Corollary 6.2.3), we deduce that the function* $t \mapsto \mu(t)(\mathbb{R}^n)$ *is nonincreasing.*

In order to prove the nontriviality of the limit of the spacetime MCFs, we first extract a sufficient relation between $\delta(\mathcal{T})$ and ε allowing the convergence of the mass measure of the piecewise constant approximate MCF $\left(V^{pc}_{\varepsilon,\mathcal{T}}(t)\right)_{t\in[0,1]}$ to the mass measure of the limit of the spacetime approximate MCF $(\mu(t))_{t\in[0,1]}$ when ε and $\delta(\mathcal{T})$ converge to 0 simultaneously.

Lemma 6.2.5. *Let* $V_0 \in V_d(\mathbb{R}^n)$ *be of compact support. Let* $(\varepsilon_j)_{j\in\mathbb{N}} \in (0,1)$ *be a sequence converging to* 0 *and* $\forall j \in \mathbb{N}$, *let* $(V_{\varepsilon_j}(t))_{t\in[0,1]}$ *be the approximate MCF starting from* V_0 *converging to* $(\mu(t))_{t\in[0,1]}$ *the mass measure of a limit of the spacetime approximate MCF starting from* V_0 *(Definition 6.2.1). Let* $(\mathcal{T}_j)_{j\in\mathbb{N}}$ *be a sequence of subdivisions satisfying* (5.34) *and* $\left(V^{pc}_{\varepsilon_j,\mathcal{T}_j}(t)\right)_{t\in[0,1]}$ *be the corresponding piecewise constant approximate MCF. Assume that*

$$\delta(\mathcal{T}_j)\varepsilon_j^{-n-11}\exp\left(c_{10,M}\varepsilon_j^{-n-7}\right) \xrightarrow[j\to\infty]{} 0.$$

Then

$$\|V^{pc}_{\varepsilon_j,\mathcal{T}_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t),\ \forall t \in [0,1].$$

Proof. Let $(V_{\varepsilon,\mathcal{T}_1}(t))_{t\in[0,1]}$ and $(V_{\varepsilon,\mathcal{T}_2}(t))_{t\in[0,1]}$ be two time-discrete approximate MCFs associated with two subdivisions $\mathcal{T}_1, \mathcal{T}_2$ satisfying (5.34) and starting from $V_0 \in V_d(\mathbb{R}^n)$. We know from Proposition 5.1.11 that for all $t \in [0,1]$ and for $\delta = \max\{\delta(\mathcal{T}_1), \delta(\mathcal{T}_2)\}$

$$\Delta(\|V_{\varepsilon,\mathcal{T}_1}(t)\|, \|V_{\varepsilon,\mathcal{T}_2}(t)\|) \leq c_{10,M}\delta\varepsilon^{-n-11}\exp\big(c_{10,M}\varepsilon^{-n-7}\big).$$

We let $\delta(\mathcal{T}_2) \to 0$ and infer from Theorem 5.2.1 that for all $t \in [0,1]$,

$$\Delta(\|V_{\varepsilon,\mathcal{T}_1}(t)\|, \|V_{\varepsilon}(t)\|) \leq c_{10,M}\delta(\mathcal{T}_1)\varepsilon^{-n-11}\exp\big(c_{10,M}\varepsilon^{-n-7}\big).$$

Let $(\varepsilon_j)_{j\in\mathbb{N}} \to 0$ such that $\|V_{\varepsilon_j}\|(t) \xrightarrow{*} \mu(t)$, $\forall t \in [0,1]$, where $(\mu(t))_{t\in[0,1]}$ is a mass measure of a limit of the spacetime approximate MCF starting from V_0. The boundedness of $\operatorname{spt}\mu(t)$, $t \in [0,1]$ (Corollary 6.2.3) implies that $\Delta(\|V_{\varepsilon_j}(t)\|, \mu(t)) \xrightarrow[j\to\infty]{} 0\,\forall t \in [0,1]$.

Let $(\mathcal{T}_j)_{j\in\mathbb{N}}$ be a sequence of subdivisions satisfying (5.34) and the condition

$$\delta(\mathcal{T}_j)\varepsilon_j^{-n-11}\exp\Big(c_{10,M}\varepsilon_j^{-n-7}\Big) \xrightarrow[j\to\infty]{} 0, \tag{6.9}$$

we can assert that $\forall t \in [0,1]$

$$\Delta(\mu(t), \|V_{\varepsilon_j,\mathcal{T}_j}(t)\|) \leq \Delta(\mu(t), \|V_{\varepsilon_j}(t)\|) + \Delta(\|V_{\varepsilon_j}(t)\|, \|V_{\varepsilon_j,\mathcal{T}_j}(t)\|) \xrightarrow[j\to\infty]{} 0.$$

Again, by the boundedness of $\operatorname{spt}\mu(t)$, we deduce that $\|V_{\varepsilon_j,\mathcal{T}_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t)$, $\forall t \in [0,1]$. Finally, from Proposition 5.2.2 we deduce that $\|V^{pc}_{\varepsilon_j,\mathcal{T}_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t)$, $\forall t \in [0,1]$.

□

The following lemma provides a quantitative continuity property on evolution by C^1 diffeomorphisms of open sets with respect to balls.

Lemma 6.2.6. *Let E be an open set in $\mathbb{R}^n$. Let f be a C^1 diffeomorphism of $\mathbb{R}^n$, assume that $\delta := \max\{\|f - \mathrm{id}\|_\infty, \|Jf - 1\|_\infty\} < 1$. Let $B = B(a,R)$ for some $a \in \mathbb{R}^n$ and $R \in \mathbb{R}^+$. One has*

$$|\mathcal{L}^n(B \cap f(E)) - \mathcal{L}^n(B \cap E)| \leq \delta c_{14},$$

where c_{14} is a constant depending only on n and R.

Proof. Indeed, we have by the area formula:

$$\mathcal{L}^n(B \cap f(E)) = \int_{f(E)} \chi_B \, d\mathcal{L}^n = \int_E \chi_{f^{-1}(B)} \, Jf \, d\mathcal{L}^n.$$

This gives

$$\begin{aligned}|\mathcal{L}^n(B \cap f(E)) - \mathcal{L}^n(B \cap E)| &= \Big|\int_E \chi_{f^{-1}(B)} \, Jf \, d\mathcal{L}^n - \int_E \chi_B \, d\mathcal{L}^n\Big| \\ &\leq \int_E \chi_B \|Jf - 1\|_\infty d\mathcal{L}^n + \|Jf\|_\infty \int_E |\chi_{f^{-1}(B)} - \chi_B| \, d\mathcal{L}^n \\ &\leq \delta\mathcal{L}^n(B) + 2\mathcal{L}^n(B(a, R+\delta) \setminus B(a, R-\delta)),\end{aligned}$$

where we used

$$\|Jf\| \leq \|Jf - 1\| + 1 \leq \delta + 1 \leq 2.$$

If $\delta > R$ then $\mathcal{L}^n(B(a, R+\delta) \setminus B(a, R-\delta)) \leq \mathcal{L}^n(B(a, 2\delta)) \leq \delta 2^n \omega_n$. Otherwise, by the mean value theorem applied on the function $x \mapsto x^n$ we have

$$(R+\delta)^n - (R-\delta)^n \leq 2n\delta(R+\delta)^{n-1},$$

hence,

$$\begin{aligned}
&\mathcal{L}^n(B(a, R+\delta) \setminus B(a, R-\delta)) \\
&\leq \omega_n \left((R+\delta)^n - (R-\delta)^n\right) \\
&\leq 2n\omega_n\delta(R+\delta)^{n-1} \leq 2n\omega_n\delta(R+1)^{n-1},
\end{aligned}$$

this implies that

$$|\mathcal{L}^n(B \cap f(E)) - \mathcal{L}^n(B \cap E)| \leq c_{14}\delta,$$

with $c_{14} = \omega_n R^n + 2\max\{2^n\omega_n, 2n\omega_n(R+1)^{n-1}\}$, this concludes the proof of the claim. □

We make the following remark about open partitions and the approximate flow of their boundaries.

Remark 6.2.7. Given an open partition $\mathcal{E}$, one can define a piecewise approximate MCF for the integral varifold associated to $\partial\mathcal{E} = \bigcup_{i=1}^{N} \partial E_i$, we denote it by $(\partial\mathcal{E})_{\varepsilon,\mathcal{T}}(t) = \bigcup_{i=1}^{N} (\partial E_i)_{\varepsilon,\mathcal{T}}(t)$. The open partition character is preserved through the flow as the push-forward maps involved in the construction of the flow are C^1 diffeomorphisms.

We investigate the change of volume, restricted to fixed balls of $\mathbb{R}^n$, of the evolution by approximate mean curvature of an open partition. For simplicity, we state the result for uniform subdivisions.

Corollary 6.2.8 (Change of volume). *Let $\varepsilon \in (0,1)$, $1 \leq M$ and $\mathcal{T}$ be a uniform subdivision of $[0,1]$, of time step $\Delta t > 0$, satisfying (5.34). Let $\mathcal{E} = \bigcup_{i=1}^{N} E_i$ be an open partition of $\mathbb{R}^n$ such that $\mathcal{L}^{n-1}(\partial\mathcal{E}) \leq M$. Let $(\partial\mathcal{E})_{\varepsilon,\mathcal{T}}(t) = \bigcup_{i=1}^{N} (\partial E_i)_{\varepsilon,\mathcal{T}}(t)$ be the piecewise-constant approximate MCF with respect to $\mathcal{T}$ starting from $\mathcal{E}$, and $(\mathcal{E})_{\varepsilon,\mathcal{T}}(t) = \bigcup_{i=1}^{N} (E_i)_{\varepsilon,\mathcal{T}}(t)$ the corresponding open partition.*
Then, for any $i \in \{1, \ldots, N\}$ and $(a, R) \in (\mathbb{R}^n, \mathbb{R}^+)$ one has

$$|\mathcal{L}^n(B(a,R) \cap (E_i)_{\varepsilon,\mathcal{T}}(t+\Delta t)) - \mathcal{L}^n\left(B(a,R) \cap (E_i)_{\varepsilon,\mathcal{T}}(t)\right)| \leq c_{14}\varepsilon,$$

for any $t \in [0, 1-\Delta t]$, where c_{14} is a constant depending only on n and R.

Proof. Applying Lemma 6.2.6 in our context with $f = \mathrm{id} + \Delta t\, h_\varepsilon\left(\cdot, \bigcup_{i=1}^{N} (\partial E_i)_{\varepsilon,\mathcal{T}}(t)\right)$. From Proposition 5.1.2, (5.23), (5.34), (5.35) one has

$$\max\{\|f - \mathrm{Id}\|_\infty, \|Jf - 1\|_\infty\} \leq c_4(M+1)\varepsilon^{-4} \leq \varepsilon < 1,$$

this gives

$$|\mathcal{L}^n(B(a,R)\cap(E_i)_{\varepsilon,\mathcal{T}}(t+\Delta t))-\mathcal{L}^n\left(B(a,R)\cap(E_i)_{\varepsilon,\mathcal{T}}(t)\right)|\le c_{14}\varepsilon,$$

for any $t\in[0,1-\Delta t]$ and this finishes the proof of Corollary 6.2.8. □

Theorem 6.2.9 (Nontriviality of the limit of the spacetime approximate MCF)**.**
Let $\mathcal{E}=\bigcup_{i=1}^N E_i$ be an open partition of $\mathbb{R}^n$ and let $\mu(t)$ be the mass measure of a limit of the spacetime approximate MCF flow starting from $\partial\mathcal{E}$ (Definition 6.2.1). Then, there exists $t_0>0$ such that $\mu(t)(\mathbb{R}^n)>0,\forall t\in[0,t_0]$.

Proof. Let $p\in\{1,\dots,N\}$ with E_p bounded. Denote $\mathcal{O}:=E_p$ and by $\partial\mathcal{O}$ the varifold associated to its boundary. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence converging to 0 such that if we denote the approximate MCF starting from $\partial\mathcal{E}$ by $\big((\partial\mathcal{E})_{\varepsilon_j}(t)\big)_{t\in[0,1]}$, we have:

$$\|(\partial\mathcal{E})_{\varepsilon_j}(t)\|\xrightarrow[j\to\infty]{*}\mu(t)\ \forall t\in[0,1].$$

Let $(\mathcal{T}_j)_{j\in\mathbb{N}}$ be a family of subdivisions of $[0,1]$, of uniform time step

$$\Delta t_j:=\left\lceil\varepsilon_j^{-n-12}\exp\Big(c_{10,M}\varepsilon_j^{-n-7}\Big)\right\rceil^{-1}.$$

For j large enough, $\mathcal{T}_j$ satisfies (5.34). In the proof, we denote for simplicity $(\partial\mathcal{E})_j(t):=(\partial\mathcal{E})^{pc}_{\varepsilon_j,\mathcal{T}_j}(t)$ to be the piecewise approximate MCF starting from $\partial\mathcal{E}$; as

$$\Delta t_j\le\varepsilon_j^{n+12}\exp\Big(-c_{10,M}\varepsilon_j^{-n-7}\Big)=\varepsilon_j^{n+11}\exp\Big(-c_{10,M}\varepsilon_j^{-n-7}\Big)\,o(1),$$

we deduce from Lemma 6.2.5 that

$$\|(\partial\mathcal{E})_j(t)\|\xrightarrow[j\to\infty]{*}\mu(t)\quad\forall t\in[0,1].$$

The goal now is to prove that there exists $t_0\in(0,1]$, a constant $\tilde{\omega}>0$ such that $\mu(t_0)(\mathbb{R}^n)\ge\tilde{\omega}$, by the decay property of the mass (Corollary 6.2.4) we obtain $\mu(t)(\mathbb{R}^n)\ge\mu(t_0)(\mathbb{R}^n)\ge\tilde{\omega}$ for every $t\in[0,t_0]$ and this finishes the proof of the theorem.
We carry on with the proof, let $(a,R)\subset\mathbb{R}^n\times\mathbb{R}^+$ be such that $B(a,R)\subset\mathcal{O}$.
Define $\psi(x,t)=\gamma(|x-a|^2+2dt)$ such that

$$\gamma(r)=\begin{cases}(R^2-r)^4 & \text{for } r\le R^2,\\ 0 & \text{for } r>R^2.\end{cases}$$

Let $(\partial\mathcal{O}_j(t))_{t\in[0,1]}$ (resp. $(\mathcal{O}_j(t))_{t\in[0,1]}$) denotes the evolution in time of $\partial\mathcal{O}$ (resp. $\mathcal{O}$) under the piece-wise approximate MCF of $\partial\mathcal{E}$. For j large enough, we have $c_{12}\Delta t_j\,\varepsilon_j^{-8}\le\varepsilon_j$ and $\varepsilon_j\in(0,\varepsilon_0)$, hence from Proposition 6.2.2

$$\|(\partial\mathcal{E})_j(t)\|(\psi(\cdot,t))-\|(\partial\mathcal{E})_j(0)\|(\psi(\cdot,0))\le c_{13}\varepsilon_j^{\frac{1}{6}}\quad\forall t\in[0,1].$$

By construction of ψ, we can assert that $\|(\partial\mathcal{E})_j(0)\|(\psi(\cdot,0)) = 0$. We set $t_0 = \frac{R^2}{8d}$, for any $t \in [0,t_0]$ and $x \in B := B\left(a, \frac{R}{2}\right)$ we have

$$\psi(x,t) = \varphi(|x-a|^2 + 2dt) \geq \left(R^2 - \left(\frac{R^2}{4} + 2dt_0\right)\right)^4 = \frac{R^8}{16}.$$

This yields:

$$\|(\partial\mathcal{E})_j(t)\|(\psi(\cdot,t)) \geq \frac{R^8}{16}\|(\partial\mathcal{E})_j(t)\|(B) \geq \frac{R^8}{16}\mathcal{L}^{n-1}\left(\partial\mathcal{O}_j(t) \cap B\right)$$

and that $\mathcal{L}^{n-1}\left(\partial\mathcal{O}_j(t) \cap B\right) \xrightarrow[j\to\infty]{} 0$ uniformly on $[0,t_0]$.

We have the following two cases:
Case 1 : There exists a subsequence $(\alpha(j))_{j\in\mathbb{N}} \xrightarrow[j\to+\infty]{} +\infty$ such that $\forall j \in \mathbb{N}$, $\mathcal{L}^n\left(\mathcal{O}_{\alpha(j)}(t_0) \cap B\right) \geq \frac{1}{4}\mathcal{L}^n(B)$; then, we can infer from the isoperimetric inequality that

$$\mathcal{L}^{n-1}\left(\partial\mathcal{O}_{\alpha(j)}(t_0)\right) \geq c_n\left(\mathcal{L}^n\left(\mathcal{O}_{\alpha(j)}(t_0)\right)\right)^{\frac{n-1}{n}} \geq c_n\left(\mathcal{L}^n\left(\mathcal{O}_{\alpha(j)}(t_0) \cap B\right)\right)^{\frac{n-1}{n}} \geq c_n\left(\frac{1}{4}\mathcal{L}^n(B)\right)^{\frac{n-1}{n}} := \tilde{\omega}$$

for some constant $c_n > 0$ depending only on n, taking j to $+\infty$ we deduce that

$$\mu(t_0)(\mathbb{R}^n) = \lim_j \|(\partial\mathcal{E})_{\alpha(j)}(t_0)\|(\mathbb{R}^n) \geq \liminf_j \mathcal{L}^{n-1}\left(\partial\mathcal{O}_{\alpha(j)}(t_0)\right) \geq \tilde{\omega}$$

and this finishes the proof.
Case 2 : There is no such sequence, this means that there exists $j_0 \in \mathbb{N}$ such that

$$\forall j \geq j_0,\ \mathcal{L}^n\left(\mathcal{O}_j(t_0) \cap B\right) < \frac{1}{4}\mathcal{L}^n(B).$$

We have $\mathcal{L}^n\left(\mathcal{O}_j(0) \cap B\right) = \mathcal{L}^n(B)$ and $\mathcal{L}^n\left(\mathcal{O}_j(t_0) \cap B\right) < \frac{1}{4}\mathcal{L}^n(B)$, by Corollary 6.2.8 we infer that for any $t \in [0, 1-\Delta t_j]$

$$\left|\mathcal{L}^n\left(\mathcal{O}_j(t+\Delta t_j) \cap B\right) - \mathcal{L}^n\left(\mathcal{O}_j(t) \cap B\right)\right| \leq c_{14}\varepsilon_j.$$

Then, taking j_0 larger so that $c_{14}\varepsilon_j \leq \frac{1}{4}\mathcal{L}^n(B)$ we can infer that there exists s (depending on j) $\in [0,t_0]$ such that

$$\frac{1}{2}\mathcal{L}^n(B) \geq \mathcal{L}^n\left(\mathcal{O}_j(s) \cap B\right) \geq \frac{1}{4}\mathcal{L}^n(B)$$

for all $j \geq j_0$. By the relative isoperimetric inequality ([5, Remark 3.50]), there exists a constant $\tilde{c}_n > 0$ (depending only on n) such that for any $j \geq j_0$

$$\mathcal{L}^{n-1}\left(\partial\mathcal{O}_j(s) \cap B\right) \geq \tilde{c}_n \min\{\mathcal{L}^n(\mathcal{O}_j(s) \cap B)^{\frac{n-1}{n}}, \mathcal{L}^n\left(B \setminus \mathcal{O}_j(s)\right)^{\frac{n-1}{n}}\} \geq \tilde{c}_n\left(\frac{1}{4}\mathcal{L}^n(B)\right)^{\frac{n-1}{n}}$$

this yields a contradiction as $\mathcal{L}^{n-1}\left(\partial\mathcal{O}_j(t) \cap B\right) \xrightarrow[j\to+\infty]{} 0$ uniformly on $[0,t_0]$.
Conclusion: there exists $\tilde{\omega} > 0$ such that $\mu(t)(\mathbb{R}^n) \geq \tilde{\omega} > 0\,\forall t \in [0,t_0]$, this finishes the proof of Theorem 6.2.9.

□

Remark 6.2.10 (Lower bound on the extinction time)**.** The proof of Theorem (6.2.9), up to a slight technical change, shows that the flow is nontrivial for $t \in [0, t_0)$ where

$$t_0 = \max\{R > 0,\ B(a, R) \subset E_j \in \mathcal{E}, \text{ for some } a \in \mathbb{R}^n \text{ and } E_j \text{ bounded}\}.$$

Concretely, instead of $t_0 = \frac{R^2}{8d}$ one could set $t_0 = \dfrac{R^2}{2d} - \eta$ with η very small and instead of $B := B(a, \frac{R}{2})$ one could set $B := B(a, \eta)$. We then conclude by letting $\eta \to 0$.

6.3 Avoidance of evolving spheres

In Proposition 6.3.1 we prove the avoidance principles with respect to spheres evolving by the law in (6.1), either when the varifold is supported outside the sphere, or when it is supported inside the sphere; as a consequence, we deduce the convex set barrier principle. The proof consists of plugging a suitable barrier function in the spacetime Brakke inequality and using Lemmas 6.1.1 and 6.1.3.

Proposition 6.3.1. *(Barriers) Let $(\mu(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow (Definition 5.3.4) starting from a varifold $V_0 \in V_d(\mathbb{R}^n)$ of compact support. We have:*

1. *Sphere barrier to external varifolds:*

$$\text{if } \mu(0)(B(a, R)) = 0 \quad \text{then } \mu(t)(B(a, (R^2 - 2dt)^{\frac{1}{2}}) = 0\ \forall t \in [0, 1] \cap [0, R^2/2d]. \tag{6.10}$$

2. *Convex set barriers:*

$$\bigcup_{t\in[0,1]} \operatorname{spt} \mu(t) \text{ is contained in the convex hull of } \operatorname{spt} \|V_0\|. \tag{6.11}$$

3. *Sphere barrier to internal varifolds:*

$$\text{if } \operatorname{spt} \mu(0) \subset \overline{B}(a, R) \quad \text{then } \operatorname{spt} \mu(t) \subset \overline{B}(a, (R^2 - 2dt)^{\frac{1}{2}})\ \forall t \in [0, 1] \cap [0, R^2/2d]. \tag{6.12}$$

Proof. We start with the proof of the sphere barrier to external varifolds (6.10). Define $\psi(x, t) = \gamma(|x - a|^2 + 2dt)$ such that

$$\gamma(r) = \begin{cases} (R^2 - r)^\beta & \text{for } r \le R^2, \\ 0 & \text{for } r > R^2 \end{cases}$$

with $\beta > 2$ (so that both ψ and γ are C^2), then, easy computations show that ψ is a barrier function. Plugging ψ into the integral Brakke inequality (5.116) we obtain for any $t_1 \in [0, 1] \cap [0, R^2/2d]$ (removing the dependence on variables and noting $h = h(\cdot, \cdot, \lambda)$ for simplicity)

$$\mu(t_1)(\psi(\cdot, t_1)) - \mu(0)(\psi(\cdot, 0)) \le \int_0^{t_1} \int_{\mathbb{R}^n \times G_{d,n}} -\psi|h|^2 + S^\perp \nabla\psi \cdot h + \partial_t \psi \, d\lambda.$$

We note that by assumption, $\mu(0)(\psi(\cdot,0)) = 0$ as $\psi(\cdot,0)$ vanishes outside $B(a,R)$. By Lemma 6.1.1, (5.114) and $(\delta\lambda)_s = 0$ we have

$$\begin{aligned}\mu(t_1)(\psi(\cdot,t_1)) &\leq \int_0^{t_1}\int_{\{(x,S,t),\psi\neq 0\}} \frac{1}{4}\frac{|S\nabla\psi|^2}{\psi} + \nabla\psi\cdot h + \partial_t\psi\, d\lambda \\ &= \int_0^{t_1}\int_{\{(x,S,t),\psi\neq 0\}} \frac{1}{4}\frac{|S\nabla\psi|^2}{\psi} - S:\nabla^2\psi + \partial_t\psi\, d\lambda.\end{aligned}$$

Hence, by Lemma 6.1.3 we deduce that $\mu(t_1)(\psi(\cdot,t_1)) = 0$ (as $\psi \geq 0$). By construction, $\psi(\cdot,t_1) > 0$ on $B(a,(R^2-2dt_1)^{\frac{1}{2}})$, this implies that $\mu(t_1)(B(a,(R^2-2dt_1)^{\frac{1}{2}})) = 0$ and finishes the proof of (6.10).

Proof of the convex set barriers : using the sphere barrier to external varifolds (6.10), the proof is a direct adaptation of the proof of [12, Theorem 3.8].

The proof of (6.12) is similar to the proof of (6.10). We define a test function $\psi(x,t) = \gamma(|x-a|^2 + 2dt)$ such that

$$\gamma(r) = \begin{cases} 0 & \text{for } r \leq R^2, \\ (r-R^2)^\beta & \text{for } r > R^2 \end{cases}$$

with $\beta > 2$ (so that both ψ and γ are C^2). Construct a C^2 function $\tilde{\psi}$ equal to ψ on $B(a,2R)$ and to 0 outside $B(a,3R)$. By the convex barrier principle, $\bigcup_{t\in[0,1]} \operatorname{spt}\mu(t) \subset \overline{B}(a,R)$ thus on the supports on $\operatorname{spt}\mu(t) \cap \{\tilde{\psi} > 0\}$ we have the formula

$$\frac{1}{4}\frac{|S\nabla\tilde{\psi}|^2}{\tilde{\psi}} - S:\nabla^2\tilde{\psi} + \partial_t\tilde{\psi} \leq 0. \tag{6.13}$$

Finally, by the spacetime Brakke inequality (5.116), Lemma 6.1.1 and (6.13), we infer that for any $t \in [0,1] \cap [0, R^2/2d]$

$$\mu(t)(\tilde{\psi}(\cdot,t)) \leq \mu(0)(\tilde{\psi}(\cdot,0)) = 0 \quad \text{and} \quad \mu(t)(\tilde{\psi}(\cdot,t)) = 0.$$

Hence, as $\tilde{\psi}(\cdot,t) > 0$ on $B(a,2R)\setminus\overline{B}(a,(R^2-2dt)^{\frac{1}{2}})$ and $\operatorname{spt}\mu(t) \subset \overline{B}(a,R)$ we infer that $\operatorname{spt}\mu(t) \subset \overline{B}(a,(R^2-2dt)^{\frac{1}{2}})$ and we finish the proof. □

Corollary 6.3.2 (Non-existence of compact stationary varifolds)**.** *We call a varifold $V \in V_d(\mathbb{R}^n)$ stationary if:*

$$\forall\, X \in C_c^1(\mathbb{R}^n,\mathbb{R}^n), \quad \delta V(X) = 0.$$

Consider the measure $\lambda = V(t)\otimes dt$ on $\mathbb{R}^n \times G_{d,n} \times [0,1]$ with $V(t) = V$, $\forall t \in [0,1]$, clearly $h(\cdot,\cdot,\lambda) = 0$ and λ is a spacetime Brakke flow. The sphere barrier to internal varifolds property ((6.12)) implies that $(\|V(t)\|)_{t\in[0,1]} = (\|V\|)_{t\in[0,1]}$ avoids the motion of spheres, hence $V = \emptyset$. Conclusion: there exists no compact stationary varifold of any codimension. In particular, it proves that there exists no compact minimal surface (a stationary Brakke flow in general), of any codimension, in $\mathbb{R}^n$ (which was already known).

6.4 Avoidance principle for codimension 1 spacetime Brakke flows, inclusion in level set flows

In this section we prove that the mass measure of a codimension 1 spacetime Brakke flow avoids smooth codimension 1 mean curvature flows. The proof is a slight adaptation of the proof of [35, 10.5].

We start by showing the lower semi-continuity property of the map $t \mapsto \mu(t)$.

Proposition 6.4.1 (Lower semi-continuity). *Let $V_0 \in V_d(\mathbb{R}^n)$ of compact support, $(\mu(t))_{t\in[0,1]}$ the mass measure of a spacetime Brakke flow starting from V_0 (Definition 5.3.4). Then*

- (i) *For any $\psi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, the map $t \mapsto \mu(t)(\psi) - Ct$ is nonincreasing for any $C \geq \|\nabla^2\psi\|_\infty ||V_0||(\mathbb{R}^n)$.*
- (ii) *For any $\varphi \in C_c^2(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$ and any $s \in (0,1]$,*

$$\lim_{t\to s^-} \mu(t)(\varphi(\cdot,t)) \geq \mu(s)(\varphi(\cdot,s)). \tag{6.14}$$

Proof. Let $\psi \in C_c^2(\mathbb{R}^n, \mathbb{R}^+)$, from the integral Brakke inequality (5.116), and using $ab \leq \frac{1}{2}(a^2+b^2)$ we obtain for every $s, r \in [0,1]$ such that $0 \leq r \leq s \leq 1$ (for simplicity we denote $h = h(\cdot,\cdot,\lambda)$)

$$\begin{aligned}
\mu(s)(\psi) - \mu(r)(\psi) &\leq -\int_r^s \int_{\mathbb{R}^n} \psi |h|^2 \, d\mu(t) dt + \int_r^s \int_{\mathbb{R}^n} S^\perp(\nabla\psi) \cdot h \, d\lambda \\
&\leq \int_r^s \int_{\mathbb{R}^n} -\psi|h|^2 + \frac{1}{2}\psi|h|^2 + \frac{1}{2}\frac{|\nabla\psi|^2}{\psi} d\mu(t) dt \\
&\leq (s-r)\|\nabla^2\psi\|_\infty ||V_0||(\mathbb{R}^n) \quad \text{by [46, Lemma 3.1]}
\end{aligned}$$

where we used $\mu(t)(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n)$ (Remark 5.3.5 (ii)), this proves (i).
Let $s \in [0,1]$, we first prove (ii) for time-independent test functions. Let $\varphi \in C_c^2(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, for $\psi = \varphi(\cdot, s)$ and $C := \|\nabla^2\psi\|_\infty ||V_0||(\mathbb{R}^n)$ we know from (i) that the map $t \mapsto \|\mu(t)\|(\psi) - Ct$ is nonincreasing. Thus, $\lim\limits_{t\to s^-} (\mu(t)(\psi) - Ct) \geq \mu(s)(\psi) - Cs$ which yields $\lim\limits_{t\to s^-} \mu(t)(\varphi(\cdot,s)) \geq \mu(s)(\varphi(\cdot,s))$. Then, (ii) follows from

$$\Big| \lim_{t\to s^-} \mu(t)(\varphi(\cdot,s) - \varphi(\cdot,t)) \Big| \leq \lim_{t\to s} (s-t)\|\varphi\|_{C^1} \|V_0\|(\mathbb{R}^n) = 0.$$

□

In the following lemma we prove a continuity property of Brakke flows and the mass measures of spacetime Brakke flows (seen as a subset of $\mathbb{R}^n$).

Lemma 6.4.2 (A continuity property). *Let $(\mu(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow (Definition 5.3.4) or a mass measure of a Brakke flow and starting from some $V_0 \in V_d(\mathbb{R}^n)$. Then, for any $r > 0$ and a set A of $\mathbb{R}^n$, we have $\forall\, t \in [0,1] \cap \left[0, r^2/2d\right]$*

$$\operatorname{spt} \mu(0) \cap (A + B_r) = \emptyset \implies \operatorname{spt} \mu(t) \cap \left(A + B_{\sqrt{r^2 - 2dt}}\right) = \emptyset.$$

Proof. Let $\mu(t)$ be defined as in the lemma, let A be a set of $\mathbb{R}^n$ such that $\operatorname{spt}\mu(0) \cap (A + B_r) = \emptyset$. We write $A + B_r = \bigcup_{x \in A} B_r(x)$, hence $\operatorname{spt}\mu(0) \cap B_r(x) = \emptyset\, \forall x \in A$. By the avoidance principle for external varifolds (6.10) we can infer that

$$\operatorname{spt}\mu(t) \cap B_{\sqrt{r^2-2dt}}(x) = \emptyset \ \forall t \in [0,1] \cap [0, r^2/2d] \ \forall x \in A,$$

the result follows from noting that $A + B_{\sqrt{r^2-2dt}} = \bigcup_{x\in A} B_{\sqrt{r^2-2dt}}(x)$. The result is valid for Brakke flows and mean curvature flows as it is based only on the avoidance principle for external varifolds (which is true for Brakke flows and MCFs by [12, Theorem 3.7]). □

We now show that the mass measures of a codimension 1 spacetime Brakke flow avoids the MCF of C^2 hypersurfaces.

Theorem 6.4.3 (Avoidance of smooth MCFs). *Let $V_0 \in V_{n-1}(\mathbb{R}^n)$ of compact support, let $(\mu(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow starting from V_0 (see Definition 5.3.4). Let $(\mathcal{M}_t)_{t\in[0,1]}$ be the MCF of a compact C^2 hypersurface $\mathcal{M}$ ($\mathcal{M}_0 = \mathcal{M}$). We have:*

$$\operatorname{spt}\mu(0) \cap \operatorname{spt}\mathcal{M} = \emptyset \implies \operatorname{spt}\mu(t) \cap \operatorname{spt} M_t = \emptyset \quad \forall\, t \in [0,1].$$

Proof. The idea is to construct a test function $\psi(\cdot,t)$ out of the distance function to $\mathcal{M}_t$ vanishing outside a neighborhood of $\mathcal{M}_t$ and prove that $\mu(t)(\psi(\cdot,t)) = 0\, \forall\, t \in [0,1]$. We assume that $\operatorname{spt}\mu(0) \cap \mathcal{M} = \emptyset$.

Step 1: (Construction, properties of the test function). Let E_t be the compact region bounded by $\mathcal{M}_t$, define

$$r(x,t) = \begin{cases} -\operatorname{dist}(x, \mathcal{M}_t), & x \in E_t, \\ \operatorname{dist}(x, \mathcal{M}_t), & x \in E_t^c. \end{cases}$$

Fix $\gamma > 0$ enough small so that

$$\operatorname{dist}(\operatorname{spt}\mu(0), \mathcal{M}) > \gamma,$$

and that $r(x,t)$ is smooth on a open spacetime-neighborhood of U, where

$$U \equiv \{(x,t) : -\gamma < r(x,t) < \gamma, 0 \le t \le 1\}.$$

This is possible due to the compactness of $[0,1]$. Let $\beta > 0$ be such that

$$\beta > \max\big\{2\,,\ \frac{3}{4}(1 + \gamma \max_{\overline{U}} \|\nabla^2 r\|)\big\}. \tag{6.15}$$

Define the test function

$$\psi = \varphi \circ r = \begin{cases} (\gamma - |r|)^\beta & |r| \le \gamma; \\ 0 & |r| \ge \gamma. \end{cases} \tag{6.16}$$

Then $\psi \in C_c^0(\mathbb{R}^n \times [0,1], \mathbb{R}^+)$, ψ vanishes except on U, and ψ is C^2 except along $\mathcal{M}_t$.

Observe that $\mu(0)(\psi(\cdot,0)) = 0$, we aim to show that this remains true for $t \in [0,1]$ and this completes the proof. We derive the expressions of the first two derivatives of ψ and φ and a property between those of φ that will be used later in the proof. We have (outside $\mathcal{M}_t$) the following identities:

$$\nabla\psi = \varphi'(r)\nabla r, \quad \nabla^2\psi = \varphi''(r)\nabla r \otimes \nabla r + \varphi'(r)\nabla^2 r, \quad \partial_t\varphi = \varphi'(r)\partial_t r, \tag{6.17}$$

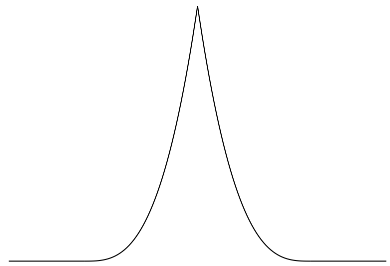

Figure 6.2: Graph of φ used in the definition of the test function ψ in (6.16)

and

$$\begin{aligned} &\varphi(t) = (\gamma - |t|)^{\beta}, \\ &\varphi'(t) = -\text{sign}(t)\beta(\gamma - |t|)^{\beta-1}, \\ &\varphi''(t) = \beta(\beta-1)(\gamma - |t|)^{\beta-2}, \\ &\frac{(\varphi')^2}{\varphi}(t) = \beta^2(\gamma - |t|)^{\beta-2}. \end{aligned} \tag{6.18}$$

$\mathbf{Step2}$: We show that if $\mu(s)(\psi(\cdot,s)) = 0,\ s \in [0,1)$ then there exists $\tau > 0$ such that $\operatorname{spt}\mu(t) \cap \mathcal{M}_t = \emptyset,\ \forall t \in [s, s+\tau]$.
Indeed, $\mu(s)(\psi(\cdot,s)) = 0$ for some $s \in [0,1)$ implies that $\operatorname{dist}(\operatorname{spt}\mu(s), \mathcal{M}_s) \geq \gamma$. Assume without loss of generality that $\operatorname{spt}\mu(s)$ lies outside E_s (the compact region bounded by $\mathcal{M}_s$). Applying Lemma 6.4.2 once with $A = E_s + B_{\gamma/2}, r = \gamma/2$ and $(\mu(t))_{t\in[0,1]}$ and another with $A = \left(E_s + B_{\gamma/2}\right)^c, r = \gamma/2$ and $(\mathcal{M}_s)_{t\in[0,1]}$ yields that

$$\exists \tau > 0,\ \operatorname{spt}\mu(t) \cap (E_s + B_{\gamma/2}) = \emptyset \text{ and } \mathcal{M}_t \cap (E_s + B_{\gamma/2})^c = \emptyset \quad \forall t \in [s, s+\tau],$$

thus $\operatorname{spt}\mu(t) \cap \mathcal{M}_t = \emptyset,\ \forall t \in [s, s+\tau]$, this concludes step 2.
$\mathbf{Step3}$: We first prove that if $\mu(s)(\psi(\cdot,s)) = 0,\ s \in [0,1)$ then $\exists \tau_0 > 0$ such that $\mu(t)(\psi(\cdot,t)) = 0\, \forall t \in [s, s+\tau_0]$. Let $s \in [0,1)$ be such that $\mu(s)(\psi(\cdot,s)) = 0$, step 2 implies:

$$\exists \tau_0 > 0,\ \operatorname{spt}\mu(t) \cap \mathcal{M}_t = \emptyset,\ \forall t \in [s, s+\tau_0]. \tag{6.19}$$

(6.19) implies that $\psi(\cdot,t)$ is C^2 on $\operatorname{spt}\mu(t)$, $\forall\, t \in [s, s+\tau_0]$, we have from (5.116)

$$\begin{aligned} \mu(t)(\psi(\cdot,t)) - \mu(s)(\psi(\cdot,s)) \leq &-\int_s^t \int_{\mathbb{R}^n} \psi(y,t)|h(y,t,\lambda)|^2\, d\mu(t)(y)dt \\ &+ \int_s^t \int_{\mathbb{R}^n\times G_{d,n}} S^{\perp}(\nabla\psi(y,t)) \cdot h(y,t,\lambda)\, d\lambda(y,S,t) + \int_s^t \int_{\mathbb{R}^n} \partial_t\psi(y,t)\, \mu(t)(y)dt. \end{aligned}$$

By definition of s and Lemma 6.1.1 (we identify by abuse of notation $f(y,S,t)$ with $f(y,t)$ for $f = \psi(y,t)|h(y,t,\lambda)|^2$ and $f = \partial_t\psi(y,t)$ so that the integrals make sense), we have

$$\mu(t)(\psi(\cdot,t)) \leq \int_s^t \int_{\mathbb{R}^n\times G_{d,n}} \frac{1}{4}\frac{|S(\nabla\psi(y,t))|^2}{\psi(y,t)} + \nabla\psi(y,t)\cdot h(y,t,\lambda) + \partial_t\psi(y,t)\, d\lambda(y,S,t).$$

We know that $\psi(\cdot,t)$ is C^2 on $\operatorname{spt}\mu(t)$ for every $t\in[s,s+\tau]$, we have by (5.114) and $(\delta\lambda)_s=0$ ([Definition 5.3.4,(ii)])

$$\int_s^t\int_{\mathbb{R}^n}\nabla\psi(y,t)\cdot h(y,t,\lambda)\,d\mu(t)dt=\int_s^t\int_{\mathbb{R}^n\times G_{d,n}}-S:\nabla^2\psi(y,t)\,d\lambda(y,S,t).$$

It yields,

$$\mu(t)(\psi(\cdot,t))\le\int_s^{s+\tau}\int_{\mathbb{R}^n\times G_{d,n}}\frac{1}{4}\frac{|S(\nabla\psi(y,t))|^2}{\psi(y,t)}-S:\nabla^2\psi(y,t)+\partial_t\psi(y,t)\,d\lambda(y,S,t). \tag{6.20}$$

We now prove that the integrand of (6.20), that we denote by I, is non-positive for all $(x,S,t)\in\mathbb{R}^n\times G_{d,n}\times[s,s+\tau]$ (in fact $I\le 0$ whenever ψ is C^2).
Plugging $\psi=\varphi(r)$ into I, and using (6.17) we get

$$I=\left(\frac{1}{4}\frac{\varphi'^2}{\varphi}-\varphi''\right)|S(\nabla r)|^2+\varphi'\left(-S:\nabla^2 r+\partial_t r\right).$$

Since $(\mathcal{M}_t)_{t\in[0,1]}$ is a smooth mean curvature flow, a standard calculation (see for instance [23, Identity (6.4)]) tells us that

$$r\left(\partial_t r-\Delta r\right)\ge 0,$$

on U, where $\Delta=\Delta^{\mathbb{R}^n}$. Since $\varphi'(r)r\le 0$, we have

$$\varphi'(r)\left(\partial_t r-\Delta r\right)\le 0.$$

Now $1=|\nabla r|^2=|S(\nabla r)|^2+|\nabla r(\vec{n})|^2$ and $\Delta r=S:\nabla^2 r+\nabla^2 r(\vec{n})\cdot\vec{n}$, where $\vec{n}$ is the unit normal to the hyperplane S, and therefore

$$I\le\left(\frac{1}{4}\frac{\varphi'^2}{\varphi}-\varphi''\right)(1-|\nabla r(\vec{n})|^2)+\varphi'\nabla^2 r(\vec{n})\cdot\vec{n}. \tag{6.21}$$

For $x\in U$, define the hyperplane $T(x)=\nabla r^\perp(x)$, note that $\vec{n}=T(\vec{n})+(\vec{n}\cdot\nabla r)\,\nabla r$, thus

$$1=|\nabla r|^2=|T(\vec{n})|^2+|\nabla r(\vec{n})|^2, \tag{6.22}$$

the identity $|\nabla r|=1$ yields $\nabla^2 r(\nabla r)=\nabla|\nabla r|^2=0$ and

$$\begin{aligned}\nabla^2 r(\vec{n})\cdot\vec{n}&=\nabla^2 r\left(T(\vec{n}),T(\vec{n})\right)+2\nabla^2 r\left(\nabla r(\vec{n}),T(\vec{n})\right)+\nabla^2 r\left(\nabla r(\vec{n}),\nabla r(\vec{n})\right)\\&=\nabla^2 r\left(T(\vec{n}),T(\vec{n})\right)\le||\nabla^2 r|||T(\vec{n})|^2.\end{aligned} \tag{6.23}$$

Injecting (6.22) and (6.23) into (6.21) we obtain

$$I\le\left(\frac{1}{4}\frac{\varphi'^2}{\varphi}-\varphi''+|\varphi'|||\nabla^2 r||\right)|T(\vec{n})|^2\le\left(\frac{1}{4}\frac{\varphi'^2}{\varphi}-\varphi''+|\varphi'|||\nabla^2 r||\right). \tag{6.24}$$

Substituting the computations of (6.18) into (6.24) we obtain

$$\begin{aligned}I&\le\beta(\gamma-|r|)^{\beta-2}\left(1-\frac{3}{4}\beta+(\gamma-|r|)\|\nabla^2 r\|\right)\\&\le\beta(\gamma-|r|)^{\beta-2}\left(1-\frac{3}{4}\beta+\gamma\|\nabla^2 r\|\right).\end{aligned}$$

We obtain by the choice of β that $I \leq 0$, therefore from (6.20) we have $\mu(t)(\psi(\cdot,t)) = 0, \forall t \in [s, s+\tau_0]$.

Conclusion: Define $T = \sup\{s \in [0,1],\ \mu(t)(\psi(\cdot,t)) = 0\, \forall t \in [0,s]\}$. On the one hand, Proposition 6.4.1 implies that $\mu(T)(\psi(\cdot,T)) = 0$. On the other hand, and if $T < 1$, step 3 implies that there exists $t > T$ such that $\mu(t)(\psi(\cdot,t)) = 0$, hence $T = 1$ and the proof is complete. □

Remark 6.4.4. We notice from the proof of Theorem 6.4.3 that the distance between the mass measure of a spacetime Brakke flow and a smooth flow is nondecreasing. We will see below in Corollary 6.4.5 that, more generally, the distance between the masses of two spacetime Brakke flows is nondecreasing.

According to [35, Definition 10.1], a family $(F_t)_{t\geq 0}$ of closed sets is a set-theoretic subsolution to the mean curvature flow if

$$\mathcal{M}_0 \cap F_0 = \emptyset \implies \mathcal{M}_t \cap F_t = \emptyset\ \forall t \geq 0$$

for every compact hypersurface $\mathcal{M}_0$, where $(\mathcal{M}_t)_{t\geq 0}$ is its MCF. Theorem 6.4.3 states that the mass measure of a spacetime Brakke flow is a *set-theoretic subsolution* of the mean curvature flow. This has some major consequences that we list in the following corollary:

Corollary 6.4.5. *Let $(\mu(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow (Definition 5.3.4) starting from $V_0 \in V_{n-1}(\mathbb{R}^n)$ of compact support. Let $(\mathcal{M}_t)_{t\in[0,1]}$ be a MCF.*

1. *(Inclusion) Assume that $\operatorname{spt}\mu(0) \subset \mathcal{M}_0$, then*

$$\operatorname{spt}\mu(t) \subset \mathcal{M}_t\ \forall\, t \in [0,1].$$

2. *(Coincidence) Assume that $\operatorname{spt}\mu(s) \subsetneq \mathcal{M}_s$ for some $s \in [0,1]$, then $\operatorname{spt}\mu(t) = \emptyset, \forall t > s$. In particular, if $\mu(s)(\mathbb{R}^n) > 0$ for some $s \in [0,1]$ then $\operatorname{spt}\mu(t) = \mathcal{M}_t, \forall t \in [0,s)$.*

3. *(Avoidance principle between spacetime Brakke flows) Let $(\mu_1(t))_{t\in[0,1]}$ be the mass measure of a spacetime Brakke flow, then*

$$\operatorname{spt}\mu(0) \cap \operatorname{spt}\mu_1(0) = \emptyset \quad \implies \quad \operatorname{dist}(\operatorname{spt}\mu(t), \operatorname{spt}\mu_1(t)) \text{ is nondecreasing}.$$

Proof.

1. More generally, set-theoretic subsolutions to MCF are included in the evolution by level set flow of the initial data, this is a consequence of [35, Inclusion Theorem 10.7]. We then obtain our result noting that the level set flow and the mean curvature flow coincide on C^2 hypersurfaces by [22, Theorem 6.1].

2. Assume that $\operatorname{spt}\mu(s) \neq \mathcal{M}_s$ for some $s \in [0,1]$, as $\operatorname{spt}\mu(s)$ is closed in $\mathcal{M}_s$, there exists an open set of $\mathcal{M}_s$ that we denote by o satisfying $\operatorname{spt}\mu(s) \subset \mathcal{M}_s \setminus o$. Perturbing $\mathcal{M}_s$ smoothly on o, one can construct a C^2 hypersurface $\mathcal{M}'_s$ containing $\operatorname{spt}\mu(s)$ and contained in the domain bounded by $\mathcal{M}_s$. By [23, Theorem 4.1] we know that the level set flows (which are equal to the MCFs in this case) of $\mathcal{M}_s$ and $\mathcal{M}'_s$ split instantaneously. The result follows directly as $(\operatorname{spt}\mu(t))_{t\geq s}$ is included in both level set flows.

3. It is a consequence of [35, 10.1].

□

6.5 List of constants used in the chapter

- c_{13} (defined in Lemma 6.2.2).
- $c_{14} = \omega_n R^n + 2\max\{2^n \omega_n, 2n\omega_n (R+1)^{n-1}\}$ (defined in Lemma 6.2.6).

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Approximate mean curvature flows of varifolds and limit Brakke flows

Abstract: This thesis deals with the construction by approximation of mean curvature flows for very general initial data, in the spirit of Brakke and Kim & Tonegawa's works based on the theory of varifolds. We construct, for general varifolds and by iterated push-forwards, an approximate time-discrete mean curvature flow depending on both a given time step and an approximation parameter. We show the convergence, when the time step tends to 0, of this time-discrete flow to a unique limit flow, called the approximate mean curvature flow. An interesting feature of our approach is its generality, since it provides an approximate notion of mean curvature flow for very general structures of any dimension and codimension, whether continuous surfaces in the classical sense or point clouds. By coupling this approximate flow with the canonical time measure, we prove the convergence to a spacetime limit measure whose generalized mean curvature is bounded. Under an additional rectifiability assumption, we prove that this limit measure is a spacetime Brakke flow. Finally, we study in codimension 1 its properties of non-triviality and coincidence with smooth mean curvature flows.

Keywords: Geometric measure theory; varifolds ; approximate mean curvature flow ; Brakke flow.

Flots approchés de courbure moyenne pour les varifolds et flots de Brakke limites

Résumé : Cette thèse porte sur la construction par approximation de flots de courbure moyenne pour des données initiales très générales, dans l'esprit des travaux de Brakke et Kim & Tonegawa qui utilisent la théorie des varifolds. Nous construisons, pour des varifolds généraux et par itérations de poussées en avant, un flot approché de courbure moyenne discret en temps dépendant à la fois d'un pas de temps et d'un paramètre d'approximation donnés. Nous montrons la convergence, lorsque le pas de temps tend vers 0, de ce flot discret vers un flot limite unique, appelé flot approché de courbure moyenne. Un intérêt de notre approche est sa généralité puisqu'elle fournit une notion approchée de flot de courbure moyenne pour des structures très générales de dimension et codimension quelconques, que ce soit des surfaces continues au sens classique ou des nuages de points. En couplant le flot approché obtenu avec la mesure temporelle canonique, nous prouvons la convergence, lorsque le paramètre d'approximation tend vers 0, vers une mesure spatio-temporelle limite dont la courbure moyenne généralisée est bornée. Sous une hypothèse supplémentaire de rectifiabilité, nous prouvons que cette mesure limite est un flot de Brakke spatio-temporel. Enfin, nous étudions en codimension 1 ses propriétés de non trivialité et de coïncidence avec les flots réguliers de courbure moyenne.

Mots clés: Théorie de la mesure géométrique ; varifolds ; flot approché de courbure moyenne; flot de Brakke.

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