APPROXIMATE MEAN CURVATURE FLOWS OF A GENERAL VARIFOLD, AND THEIR LIMIT SPACETIME BRAKKE FLOW

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ABSTRACT. We propose a construction of mean curvature flows by approximation for very general initial data, in the spirit of the works of Brakke and of Kim & Tonegawa based on the theory of varifolds. Given a general varifold, we construct by iterated push-forwards an approximate time-discrete mean curvature flow depending on both a given time step and an approximation parameter. We show that, as the time step tends to 0, this time-discrete flow converges to a unique limit flow, which we call the approximate mean curvature flow. An interesting feature of our approach is its generality, as it provides an approximate notion of mean curvature flow for very general structures of any dimension and codimension, ranging from continuous surfaces to discrete point clouds. We prove that our approximate mean curvature flow satisfies several properties: stability, uniqueness, Brakke-type equality, mass decay. By coupling this approximate flow with the canonical time measure, we prove convergence, as the approximation parameter tends to 0, to a spacetime limit measure whose generalized mean curvature is bounded. Under an additional rectifiability assumption, we further prove that this limit measure is a spacetime Brakke flow.

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1. Introduction

Let $\mathcal{M}$ be a d-dimensional manifold and $F_0 : \mathcal{M} \to \mathbb{R}^n$ a smooth embedding. The mean curvature flow starting from $\mathcal{M}_0 = F_0(\mathcal{M})$ is, see e.g. [31, 9], a smooth time-dependent family of embeddings $F : \mathcal{M} \times [0, T) \to \mathbb{R}^n$ satisfying

$$\begin{cases} \dfrac{\partial F}{\partial t}(x,t) &= H(x, \mathcal{M}_t) \\ F(x,0) &= F_0(x) \end{cases}$$

where $H(x, \mathcal{M}_t)$ is the mean curvature vector of $\mathcal{M}_t = F(\mathcal{M}, t)$ at $x_t = F(x,t)$ defined by

$$H(x, \mathcal{M}_t) = -\sum_{j=1}^{n-d} (\mathrm{div}_{\mathcal{M}_t} \nu_j)\nu_j, \tag{1}$$

with $\{\nu_j\}_j$ any orthonormal basis of the normal space $T_{x_t}\mathcal{M}_t^{\perp}$ at x_t and $\mathrm{div}_{\mathcal{M}_t}$ the tangential divergence on $\mathcal{M}_t$. The time evolution equation above is equivalent to $\frac{\partial F}{\partial t}(x,t) = \Delta_{\mathcal{M}_t} F(x,t)$, where $\Delta_{\mathcal{M}_t}$ is the Laplace-Beltrami operator on $\mathcal{M}_t$ associated to the metric induced by $F(\cdot, t)$. By the theory of pseudo-parabolic PDEs, this equation has a smooth solution defined on a nontrivial time interval [17].

Grayson proved that the mean curvature flow shrinks closed curves in $\mathbb{R}^2$ into single points in finite time [22]. The result extends to mean convex hypersurfaces of $\mathbb{R}^n$, by a result due to Huisken [24]. But there are well known examples of smooth hypersurfaces that may develop singularities, other than a limit point, in finite time, see for instance the construction by Grayson in [22] of a dumbbell flowing by mean curvature, whose neck pinches off before the two bells shrink.

To extend the definition of the mean curvature flow beyond singularities, several approaches have been proposed that yield weak notions of mean curvature flow:

- The Brakke flow [12, 28, 41, 27, 40], defined in the setting of rectifiable varifolds and, at least for the original construction of Brakke, in arbitrary codimension;
- The level set formulation, based on an implicit representation of the evolving interface and the PDE satisfied by this representation [18, 19, 21, 20, 14, 7];
- Phase fields methods, based on diffuse representations of evolving interfaces and associated reaction-diffusion PDEs [2, 26, 23];
- Diffusion generated motions, based on convolution-thresholding schemes [33, 29];
- De Giorgi's method of minimal barriers [15, 7, 11, 10];
- The minimizing movement approach, based on time-discrete variational approximations [30, 3].

Further references and details can be found in [9, 38, 16, 6, 4].

The starting point of our work was the need for a weak notion of mean curvature flow that is adapted to structured data (such as interfaces) and unstructured data (such as point clouds) in arbitrary dimension and codimension (including codimension 0 when dealing with volumetric data). Among the above mentioned approaches, only Brakke flows, level set flows, diffusion generated motions, and flows based on De Giorgi's barriers can be considered in higher codimension. None of them, though, has been explicitly adapted to handle in a consistent way both unstructured data such as point clouds and volumetric data. We propose in this paper a construction *à la* Brakke that achieves this goal.

The core notion in Brakke's [12] and Kim & Tonegawa's [28] constructions is the notion of varifold. Recall that a d–varifold V in $\mathbb{R}^n$ is a non-negative Radon measure on $\mathbb{R}^n \times G_{d,n}$, where $G_{d,n}$ is the Grassmannian manifold of d-dimensional vector subspaces of $\mathbb{R}^n$ [37]. $V_d(\mathbb{R}^n)$ denotes the space of d–varifolds in $\mathbb{R}^n$. A varifold V is associated with a mass measure $\|V\|$ characterized by its action on every Borel set $A \subset \mathbb{R}^n$: $\|V\|(A) = V(A \times G_{d,n})$. V is called rectifiable if it decomposes as $V = \theta \mathcal{H}^d \llcorner \mathcal{M} \otimes \delta_{T_x \mathcal{M}}$ where $\mathcal{M} \subset \mathbb{R}^n$ is d–rectifiable, $\theta \in L^1_{loc}(\mathcal{H}^d \llcorner \mathcal{M})$ is nonnegative, and $T_x\mathcal{M}$ is the approximate tangent space of $\mathcal{M}$ at x. If, in addition, $\theta \in \mathbb{N}$ a.e., then V is called integral.

The *first variation* of a varifold $V \in V_d(\mathbb{R}^n)$ of finite mass is the map

$$\delta V : \ X \in C^1(\mathbb{R}^n, \mathbb{R}^n) \longmapsto \int_{\mathbb{R}^n \times G_{d,n}} \operatorname{div}_S(X)(x)\, dV(x,S).$$

If δV is locally bounded, it can be represented by a vector Radon measure thanks to the Riesz representation theorem. Then the generalized mean curvature $H(\cdot, V)$ of V is (minus) the Radon-Nikodym derivative of δV with respect to $\|V\|$.

Let $(\mathcal{M}_t)_{t\in[0,T)}$ be a smooth mean curvature flow as at the beginning of this paper. Denote $(M_t)_{t\in[0,T)}$ the associated family of integral varifolds, i.e. $M_t = \mathcal{H}^d \llcorner \mathcal{M}_t \otimes \delta_{T_x \mathcal{M}_t}$, $\forall t \in [0,T)$. The mean curvature flow is fully characterized by the following Brakke integral equality [41]: for all test function $\varphi \in \mathrm{C}_c([0,T) \times \mathbb{R}^n, \mathbb{R}_+)$, and for any pair (t_1, t_2) such that $0 \le t_1 < t_2 < T$,

$$\|M_t\|(\varphi(t_2,\cdot) - \|M_t\|(\varphi(t_1,\cdot) = \int_{t_1}^{t_2} \int_{\mathcal{M}_t} \Big[-\varphi(t,x)|H(x,\mathcal{M}_t)|^2 + (T_x\mathcal{M}_t)^\perp\big(\nabla\varphi(t,x)\big) \cdot H(x,\mathcal{M}_t) + \partial_t\varphi(t,x)\Big]\, d\mathcal{H}^d(x)dt. \quad (2)$$

This leads to the notion of Brakke flow in the integrated sense of Kim & Tonegawa [28]:

Definition 1.1. *A one-parameter family $(V_t)_{t\in[0,T)}$ in $V_d(\mathbb{R}^n)$ is called a Brakke flow if*

(1) For a.e. $t \in [0,T)$, V_t is integral.
(2) For any compact $K \subset \mathbb{R}^n$ and $t < T$, $\sup_{s\in[0,t)} \|V_s\|(K) < \infty$.
(3) For a.e. $t \in [0,T)$, V_t has locally bounded first variation and $\delta V_t \ll \|V_t\|$.
(4) $H(\cdot, V_t) := -\delta V_t / \|V_t\| \in L^2_{loc}(dt \otimes \|V_t\|)$.
(5) $(V_t)_{t\in[0,T)}$ satisfies the Brakke inequality, i.e., for $0 \le t_1 \le t_2 < T$ and $\varphi \in \mathrm{C}^1_c([0,T) \times \mathbb{R}^n, \mathbb{R}_+)$,

$$\|V_{t_2}\|(\varphi(t_2,\cdot)) - \|V_{t_1}\|(\varphi(t_1,\cdot)) \le \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big[-\varphi(t,x)|H(x,V_t)|^2 + \nabla\varphi(t,x) \cdot H(x,V_t) + \partial_t\varphi(t,x)\Big]\, d\|V_t\|(x)dt. \quad (3)$$

The reason an inequality is required in (3), rather than an equality as in the smooth case (2), is to allow for sudden mass loss or topological changes, which is necessary to obtain a consistent definition of weak mean curvature flow.

Starting from an initial $(n-1)$-varifold associated with an open partition of $\mathbb{R}^n$, Kim & Tonegawa's construction of a Brakke flow is obtained as a continuous limit of an iterated two-step scheme. The first step involves a desingularization map which is essential to go beyond singularities (but different choices for the map may yield different flows, so the uniqueness cannot be guaranteed). The second step uses a push-forward map involving an approximate mean curvature depending on a scale ε. Brakke's own construction is more general, for it can move by mean curvature integral varifolds of arbitrary codimension. However, it does not exclude the triviality of the flow, i.e. $V_t = 0, \forall t > 0$, even starting from a smooth set. In contrast, Kim & Tonegawa's construction guarantees the nontriviality of the flow starting from a smooth $(n-1)$-set.

Neither Brakke's approach nor Kim & Tonegawa's approach can handle point clouds. For the latter, it is because isolated points do not define a proper open partition of $\mathbb{R}^n$. As for Brakke's construction, the flow starting from a point cloud is trivial. More precisely, in both approaches, for given parameters Δt (time step) and ε (regularization scale), alternating a time-discrete Euler scheme applied to an ε–regularization of the curvature with a desingularization step defines a time-discrete approximate flow. As a second step, a diagonal extraction procedure yields a flow in the joint limit $(\Delta t, \varepsilon) \to 0$, but this flow is unfortunately trivial when starting from a point cloud varifold. In this paper, we propose a construction that first lets $\Delta t \to 0$ alone, yielding a time-continuous ε-approximate flow, see Theorem 4.1. This limiting flow is well-defined starting from any varifold with compact support, and nontrivial in the setting of point clouds. We also study the limit $\varepsilon \to 0$, and naturally the limit flow is trivial when starting from a point cloud. However, our construction provides some control: if one considers as initial data a sequence of point cloud varifolds W_k converging to a submanifold $\mathcal{M}$, our construction allows us to quantify, at a given regularization scale ε, the discrepancy between the approximate flow starting from $\mathcal{M}$ and the approximate flows starting from the W_k's. In particular, there exists a sequence of scales $\varepsilon_k \to 0$ ensuring that this discrepancy vanishes as $k \to \infty$, see Remark 4.4.

A key advantage of the construction we propose in this paper is that approximate flows can be defined starting from *any* varifold with compact support and arbitrary codimension, which is not possible with Brakke's or Kim & Tonegawa's construction because of the aforementioned joint limit $(\Delta t, \varepsilon) \to 0$. However, the absence of a counterpart to Brakke's or Kim & Tonegawa's desingularization steps in our construction implies that stationary varifolds do not evolve under the approximate flow, no matter the choice of ε. Here is a typical example: two non-parallel lines will not evolve under the approximate flow at any scale ε. Indeed, a stationary varifold V_0 has zero first variation, $\delta V_0 = 0$, by definition. Therefore, the convolution of δV_0 with any mollifier is also zero, so that the approximate mean curvature h_ε introduced in Definition 3.2 is identically zero. As a consequence, the push-forward map in Definition 3.6 is the identity map and the resulting flow is constant equal to V_0.

Another consequence of the absence of a desingularization step is that we are unable to establish in general the rectifiability of the limit flows we introduce in Section 5. However, partial links between the smooth mean curvature motion of a codimension 1 interface and the flow obtained with our construction are shown in the companion paper [35].

Organization of the paper and main results.
Preliminary section 2 contains the notions necessary to our constructions. In Section 3, we construct a *time-discrete approximate mean curvature flow*, namely a family of varifolds $(V_{\varepsilon,\mathcal{T}}(t_i))_{i=0\dots m}$ starting from the initial varifold $V_0 = V(0)$ and depending on both a regularization parameter ε and a subdivision $\mathcal{T} = \{t_i\}_{i=0}^m$ of the time interval $[0,1]$. We first recall in Definition 3.2 the approximate notion of mean curvature $h_\varepsilon(\cdot, V)$ for a varifold V associated with the approximation

scale $\varepsilon > 0$, defined in [28] after [12]. Definition 3.6 then proposes to use an explicit Euler scheme strategy in order to define $V(t_i) = V_{\varepsilon,\mathcal{T}}(t_i)$: in each subinterval $[t_i, t_{i+1}]$, one consider the flow with respect to the constant vector field $h_\varepsilon(\cdot, V(t_i))$, which leads to map a point x in the support of $V(t_i)$ to $x + (t_{i+1} - t_i)h_\varepsilon(x, V(t_i))$. We consistently define the pushforward varifold $V(t_{i+1}) = f_{i\#}V(t_i)$ where $f_i = \mathrm{id} + (t_{i+1} - t_i)h_\varepsilon(\cdot, V(t_i))$. Assuming that the time subdivision $\mathcal{T}$ is sufficiently fine, Proposition 3.4 establishes a uniform upper bound on the masses of the varifolds $V(t_i)$ for all $i = 0, \dots, m$, while Lemma 3.5 shows that each map f_i is a diffeomorphism. We exhibit stability properties of this time-discrete flow with respect to time subdivisions and with respect to the initial varifold in Corollary 3.18.

We keep $\varepsilon > 0$ fixed and we let the time step $\delta(\mathcal{T}) := \max_i |t_{i+1} - t_i|$ go to 0 in Section 4 and we show in Theorem 4.1 that the time-discrete flow constructed in Section 3 has a consistent limit. More precisely, given a varifold V_0 with compact support and time subdivisions $\mathcal{T}_j$ of $[0, 1]$ whose time steps $\delta(\mathcal{T}_j)$ tend to 0, we prove that there exists a family of varifolds $(V_\varepsilon(t))_{t\in[0,1]}$ satisfying: for each $t \in [0, 1]$, $V_{\varepsilon,\mathcal{T}_j}(t)$ converges weakly star to $V_\varepsilon(t)$. Relying on the stability of the time-discrete flow with respect to the time subdivision, we furthermore prove that the limit flow $(V_\varepsilon(t))_{t\in[0,1]}$ is unique (independent of the choice of time subdivisions). We consider this limit as an *approximate mean curvature flow* for the approximation scale ε, and we investigate several properties. We prove that the stability property with respect to the initial varifold V_0 established in Section 3 passes to the limit and infer it for $(V_\varepsilon(t))_{t\in[0,1]}$ in Proposition 4.3. We establish a Brakke-type equality in Proposition 4.6 which yields as a corollary the decay of the mass along the approximate flow, as stated in Remark 4.7.

In Section 5, we investigate the limit $\varepsilon \to 0$ for the approximate mean curvature flow $(V_\varepsilon(t))_{t\in[0,1]}$ introduced in Section 4: to this end, we propose to couple it with the time measure and consider the limit of the measures $\mathrm{dt} \otimes V_\varepsilon(t)$ in $[0, 1] \times \mathbb{R}^n \times G_{d,n}$ when ε tends to 0. In Theorem 5.6, we prove that up to extracting a subsequence, $\mathrm{dt} \otimes V_\varepsilon(t)$ converges weakly star to a measure of the form $\mathrm{dt} \otimes V(t)$, where $(V(t))_{t\in[0,1]}$ is a family of d–varifolds in $\mathbb{R}^n$. Under a rectifiability assumption on the limit measure (more precisely on $V(t)$), we prove that it satisfies a spacetime Brakke inequality. This limit measure can be interpreted as a spacetime track of a generalized Brakke flow.

The appendix collects a few useful lemmas.

2. Preliminaries

2.1. **Notations.** Throughout the paper, we let $d, n \in \mathbb{N}$ be such that $1 \leq d \leq n$, $2 \leq n$, and we adopt the following notations:

- $\mathcal{L}^n$, $\mathcal{H}^d$ denote the n–dimensional Lebesgue measure and the d–dimensional Hausdorff measure in $\mathbb{R}^n$, respectively.
- $B_r(x)$ denotes the open ball of radius $r > 0$ and center $x \in \mathbb{R}^n$. We set $B_r = B_r(0)$. For closed balls, B is replaced by $\overline{B}$.
- For $k \in \mathbb{N}$, ω_k denotes the volume of the $k-$dimensional unit ball.
- For a set A and $\delta > 0$, $A^\delta := \bigcup_{x\in A} B_\delta(x) = \{y \in \mathbb{R}^n, d(y, A) < \delta\}$.
- $\mathcal{M}_{p,q}$ is the space of real matrices with p rows, q columns
- The default matrix norm $\|\cdot\|$ considered in $\mathcal{M}_{p,q}$ is the operator 2-norm associated with the Euclidean norms $|\cdot|$ in $\mathbb{R}^p$ and $\mathbb{R}^q$. We also consider the norm $|\cdot|_\infty$ defined as $|M|_\infty = \max_{\substack{i=1\dots p\\ j=1\dots q}} |M_{ij}|$, for $M \in \mathcal{M}_{p,q}$ and we recall the classical relation:

$$\forall M \in \mathcal{M}_{p,q}, \qquad |M|_\infty \leq \|M\| \leq \sqrt{pq}\,|M|_\infty\,. \tag{4}$$

- The space $\mathcal{M}_p = \mathcal{M}_{p,p}$ will be for some calculations equipped with the scalar product $M : N = \mathrm{tr}\big(MN^t\big) = \mathrm{tr}\big(NM^t\big)$, and the associated Frobenius norm.
- $G_{d,n}$ is the Grassmannian manifold of d-dimensional vector subspaces of $\mathbb{R}^n$. We identify $S \in G_{d,n}$ with its orthogonal projection on the d–subspace $S \in \mathcal{M}_n(\mathbb{R})$. The distance between $S, T \in G_{d,n}$ is $\|S - T\|$, where $\|\cdot\|$ is the matrix norm introduced above.
- The functions and vectors involved may depend on both space and time, we conventionally use ∇ for space derivation and ∂_t for time derivation.
- Given two topological spaces X and Y, $\mathrm{C}_c(X, Y)$ denotes the space of continuous and compactly supported functions $f : X \to Y$.
- Given an open subset U of $\mathbb{R}^n$, $\mathrm{C}^k(U, \mathbb{R}^m)$ denotes the space of functions $u : U \to \mathbb{R}^m$ that are of class C^k and, for $u \in \mathrm{C}^k(U, \mathbb{R}^m)$, $\|u\|_{\mathrm{C}^k} = \sum_{i=0}^k \|D^i u\|_\infty$.
- For $a, b \in \mathbb{R}$ with $a < b$ and $m \geq 1$, $\mathcal{T} = \{t_i\}_{i=0}^m$ is called a subdivision of $[a, b]$ if $a = t_0 < t_1 \cdots < t_m = b$. We denote $\delta(\mathcal{T}) := \max_{i \in \{1,\dots,m\}} t_i - t_{i-1}$.

2.2. **Varifolds.** We recall below basic definitions and results concerning varifolds, see [37].

Definition 2.1. *(Varifolds)*
A d–varifold in $\mathbb{R}^n$ is a nonnegative Radon measure on $\mathbb{R}^n \times G_{d,n}$. The space of d–varifolds in $\mathbb{R}^n$ is denoted as $V_d(\mathbb{R}^n)$. Every varifold V is associated with its mass measure $\|V\|$, a Radon measure on $\mathbb{R}^n$ defined for every Borel set $A \subset \mathbb{R}^n$ by $\|V\|(A) = V(A \times G_{d,n})$.

We collect below classical examples of varifolds, we refer to [12, 32] for more general, less trivial, examples.

(1) *Rectifiable varifolds*: $V \in V_d(\mathbb{R}^n)$ is rectifiable if it decomposes as $V = \theta \mathcal{H}^d \llcorner \mathcal{M} \otimes \delta_{T_x \mathcal{M}}$ where $\mathcal{M} \subset \mathbb{R}^n$ is a d–rectifiable set [37, 5], $T_x\mathcal{M}$ is the approximate tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$, and $\theta \in L^1_{loc}(\mathcal{H}^d \llcorner \mathcal{M})$ is a nonnegative function called the multiplicity of V. We denote $V = \mathbf{v}(\mathcal{M}, \theta)$. The associated mass measure is $\|V\| = \theta \mathcal{H}^d \llcorner \mathcal{M}$.
(2) *Integral varifolds*: a rectifiable d-varifold $V = \mathbf{v}(\mathcal{M}, \theta)$ is called *integral* if $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^d$–almost every $x \in \mathcal{M}$.
(3) *Point cloud varifolds*: to a finite collection of points $\{x_j\}_{j=1}^N$ in $\mathbb{R}^n$, d-planes $\{P_j\}_{j=1}^N$ in $G_{d,n}$ and masses $\{m_j\}_{j=1}^N$ in $\mathbb{R}_+$, we may associate the varifold $V = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$. The associated mass measure is $\|V\| = \sum_{j=1}^N m_j \delta_{x_j}$.

Remark 2.2. From now on, given a smooth submanifold $\mathcal{M}$, we denote the canonical associated varifold $\mathbf{v}(\mathcal{M}, 1)$ as $M := \mathcal{H}^d \llcorner \mathcal{M} \otimes \delta_{T_y \mathcal{M}}$ with mass measure $\|M\| := \mathcal{H}^d \llcorner \mathcal{M}$.

Each varifold in the three classes above decomposes as a generalized tensor product of measures. This property is actually true for any varifold, as stated in the result below that is a consequence of a general disintegration result, see for instance [5, Theorem 2.28].

Proposition 2.3 (Disintegration)**.** *Let V be a d–varifold in $\mathbb{R}^n$. There exists a family $(\nu_x)_x$ of probability measures in $G_{d,n}$, defined for $\|V\|$–a.e. $x \in \mathbb{R}^n$, such that $V = \|V\| \otimes \nu^x$ in the following sense*

$$\forall \varphi \in \mathrm{C}_c(\mathbb{R}^n \times G_{d,n}), \quad \int \varphi \, dV = \int_{x \in \mathbb{R}^n} \int_{S \in G_{d,n}} \varphi(x, S) \, d\nu^x(S) \, d\|V\|(x) \, .$$

We recall now the definition of the bounded Lipschitz distance for Radon measures, that will be used to compare varifolds regardless of their type (rectifiable, point clouds...).

Definition 2.4. *(Bounded Lipschitz distance) Let (X, d) be a locally compact separable metric space. The bounded Lipschitz distance between two finite Radon measures μ, ν on X is*

$$\Delta(\mu,\nu) := \sup\Big\{\Big|\int_X \varphi(x)d\mu(x) - \int_X \varphi(x)d\nu(x)\Big|,\ \varphi\in \mathrm{C}^0(X,\mathbb{R}_+),\ \max\{\|\varphi\|_\infty, \mathrm{Lip}(\varphi)\}\le 1\Big\} \quad (5)$$

The notion of convergence we will be dealing with throughout this paper is the weak-$*$ convergence of Radon measures:

Definition 2.5 (Weak-$*$ convergence)**.** *Let (X, d) be a locally compact and separable metric space and $(\mu_i)_{i\in\mathbb{N}}$, μ Radon measures on X. We say that $(\mu_i)_i$ converges weakly-$*$ to μ, and we write $\mu_i \xrightarrow[i\to\infty]{*} \mu$, if*

$$\forall \varphi\in \mathrm{C}_c(X,\mathbb{R}_+),\quad \int_X \varphi\, d\mu_i \to \int_X \varphi\, d\mu. \quad (6)$$

The bounded Lipschitz distance provides a local metrization of the weak-$*$ convergence, as stated in the following result, see [36, Thm 5.9].

Proposition 2.6. *Let (X, d) be a locally compact separable metric space and $(\mu_i)_{i\in\mathbb{N}}$, μ finite Radon measures with support included in a compact subset of X. Then*

$$(\mu_i) \text{ converges weakly-}* \text{ to } \mu \quad\Longleftrightarrow\quad \Delta(\mu_i,\mu)\xrightarrow[i\to\infty]{} 0.$$

The notion of push-forward of a varifold [41, Sec. 1.4] is crucial for our construction:

Definition 2.7. *(Push-forward of a varifold) Let V be a d-varifold in $\mathbb{R}^n$ and f a C^1 diffeomorphism of $\mathbb{R}^n$. The push-forward of V by f is the varifold $f_\# V$ defined for every $\varphi\in C_c(\mathbb{R}^n\times G_{d,n},\mathbb{R})$ by*

$$f_\# V(\varphi) := \int_{\mathbb{R}^n\times G_{d,n}} \varphi(f(x), Df(x)(S)) J_S f(x)\, dV(x,S), \quad (7)$$

where $Df(x)(S)$ is the image of S in $G_{d,n}$ by the linear isomorphism $Df(x)$, and the tangential Jacobian $J_S f(x)$ is the determinant of the isomorphism $Df(x)$ from S to $Df(x)(S)$, defined as follows: let $\tilde{S} = (s_1|\dots|s_d)^t \in \mathcal{M}_{d,n}$, with $\{s_i\}_{i=1}^d$ an orthonormal basis of S, and $Y = Df(x)\tilde{S}^t$, where $Df(x)$ is identified with its $n\times n$ matrix in the canonical basis of $\mathbb{R}^n$, then

$$J_S f(x) := \det\left(Y^t Y\right)^{\frac{1}{2}}. \quad (8)$$

Remark 2.8. Using the notations above, the orthogonal projection P onto the space $Df(x)(S)$ satisfies (see [39, p. 184])

$$P = Y(Y^tY)^{-1}Y^t. \quad (9)$$

Remark 2.9. Let $\mathcal{M}\subset\mathbb{R}^n$ be a d–submanifold and $f:\mathbb{R}^n\to\mathbb{R}^n$ a C^1 diffeomorphism of $\mathbb{R}^n$. By the area formula,

$$\int_{f(\mathcal{M})} \varphi(y)\, d\mathcal{H}^d(y) = \int_{\mathcal{M}} \varphi(f(x)) J_{T_x\mathcal{M}} f(x)\, d\mathcal{H}^d(x) \quad \forall\varphi\in C_0^1(\mathbb{R}^n,\mathbb{R}_+), \quad (10)$$

where $J_{T_x\mathcal{M}} f(x)$ is the tangential Jacobian of f with respect to $T_x\mathcal{M}$. Note that if $V_\mathcal{M} = v(\mathcal{M},1) = \mathcal{H}^d\llcorner\mathcal{M}\otimes\delta_{T_x\mathcal{M}}$ denotes the unit multiplicity rectifiable d–varifold associated with $\mathcal{M}$ then, according to Definition 2.7 and (10),

$$f_\# V_\mathcal{M} = V_{f(\mathcal{M})} = \mathcal{H}^d\llcorner f(\mathcal{M})\otimes\delta_{T_y f(\mathcal{M})}.$$

More generally, if a nonnegative multiplicity function $\theta\in L^1_{loc}(\mathcal{H}^d\llcorner\mathcal{M})$ is used, then

$$f_\# v(\mathcal{M},\theta) = \theta(f^{-1}(y))\mathcal{H}^d\llcorner f(\mathcal{M})\otimes\delta_{T_y f(\mathcal{M})}.$$

In particular,

$$\|f_{\#}v(\mathcal{M},\theta)\| = (\theta\circ f^{-1})\mathcal{H}^d \llcorner f(\mathcal{M}) = v(f(\mathcal{M}),\theta\circ f^{-1}).$$

The following lemma characterizes the composition of two push-forwards.

Lemma 2.10. *Let $V \in V_d(\mathbb{R}^n)$ and let f, g be two C^1 diffeomorphisms of $\mathbb{R}^n$. Then*

$$f_{\#}(g_{\#}V) = (f\circ g)_{\#}(V).$$

Proof. Let $\varphi \in \mathrm{C}_c(\mathbb{R}^n \times G_{d,n}, \mathbb{R}_+)$. Then

$$\begin{aligned}\int \varphi\, d(f_{\#}(g_{\#}V)) &= \int \varphi\left(f(y), Df(y)(T)\right)\ J_T f(y)\, d(g_{\#}V)(t,y)\\ &= \int \varphi\left((f(g(x)), Df(g(x))(Dg(x)(S))\right)\ J_{Dg(x)(S)}f(g(x))\, J_S g(x)\, dV(x,S)\\ &= \int \varphi\left(f\circ g(x), D(f\circ g)(x)(S)\right)\ J_S(f\circ g)(x)\, dV(x,S)\end{aligned}$$

where we used that for $(x,S) \in \mathbb{R}^n \times G_{d,n}$,

$$Df(g(x))(Dg(x)(S)) = D(f\circ g)(x)(S) \quad \text{and} \quad J_{Dg(x)(S)}f(g(x))J_S g(x) = J_S(f\circ g)(x)$$

thanks to the multiplicative property of the determinant. □

Given $X \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ and $t>0$, we define

$$f_t = \mathrm{id} + tX\ .$$

For t small enough, f_t is a C^1 diffeomorphism and we can examine the infinitesimal change of the mass measure of a varifold $V \in V_d(\mathbb{R}^n)$ pushed by f_t. We have the following formula [37]:

$$\partial_t \|(f_t)_{\#}V\|(\mathbb{R}^n)_{|t=0} = \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S\, X(x)\, dV(x,S)$$

where $\mathrm{div}_S\, X := \mathrm{tr}(S\, DX)$ is the tangential divergence. This computation motivates the definition of the first variation of a varifold.

Definition 2.11 (First variation)**.** *The first variation of a varifold $V \in V_d(\mathbb{R}^n)$ of finite mass is the map*

$$\delta V :\ X \in C^1(\mathbb{R}^n,\mathbb{R}^n) \longmapsto \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S\, X(x)\, dV(x,S). \tag{11}$$

Given $\varphi \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R})$ and $X \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, we denote:

$$\begin{aligned}\delta(V,\varphi)(X) &:= \int_{\mathbb{R}^n\times G_{d,n}} \varphi(x)\ \mathrm{div}_S\, X(x)\, dV(x,S) + \int_{\mathbb{R}^n\times G_{d,n}} \nabla\varphi(x)\cdot X(x)\, dV(x,S)\\ &= \delta V(\varphi X) + \int_{\mathbb{R}^n\times G_{d,n}} X(x)\cdot S^{\perp}(\nabla\varphi(x))\, dV(x,S).\end{aligned} \tag{12}$$

where $S^{\perp}$ denotes the orthogonal projection on the space orthogonal to S.

Definition 2.12 (Weighted first variation)**.** *Let $V \in V_d(\mathbb{R}^n)$ of finite mass. The map $\delta(V,\cdot)(\cdot)$ from $(\mathrm{C}^1(\mathbb{R}^n,\mathbb{R}), \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n))$ onto $\mathbb{R}$ defined above is called the weighted first variation of V.*

Remark 2.13. Note that $\delta(V,\cdot)(\cdot)$ is bilinear and

$$|\delta(V,\varphi)(X)| \le n\|X\|_{\mathrm{C}^1}\|V\|(\mathbb{R}^n)\ \|\varphi\|_{\mathrm{C}^1}\ , \quad \text{for all } \varphi \in C^1(\mathbb{R}^n,\mathbb{R}),\ X \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n).$$

Given a closed (compact and without boundary) smooth d-manifold $\mathcal{M}$, we recall the notation $M := \mathcal{H}^d \llcorner \mathcal{M} \otimes \delta_{T_x\mathcal{M}}$. For $X \in C^1(\mathbb{R}^n, \mathbb{R}^n)$,

$$\delta M(X) = \int_{\mathbb{R}^n \times G_{d,n}} \operatorname{div}_S X(x)\, dM(x,S) = \int_{\mathcal{M}} \operatorname{div}_{T_x\mathcal{M}} X(x)\, d\mathcal{H}^d(x) = -\int_{\mathcal{M}} H(x,\mathcal{M}) \cdot X(x)\, d\mathcal{H}^d(x), \tag{13}$$

where $H(\cdot, \mathcal{M})$ is the mean curvature vector of $\mathcal{M}$ defined in (1).

When the first variation δV of a varifold $V \in V_d(\mathbb{R}^n)$ is locally bounded, the Riesz representation theorem implies that δV can be represented by a vector Radon measure, which we continue to denote by δV for simplicity. Then, the Radon-Nikodym decomposition theorem implies the existence of a vector $H(\cdot, V) \in (L^1(\mathbb{R}^n, \|V\|))^n$ such that

$$\delta V = -H(\cdot, V)\|V\| + \delta^s V, \tag{14}$$

where $\delta^s V$ is a vector measure singular with respect to $\|V\|$. The analogy with (13) motivates the following definition.

Definition 2.14 (Generalized mean curvature)**.** *The vector $H(\cdot, V)$ defined in (14) for a varifold $V \in V_d(\mathbb{R}^n)$ with locally bounded first variation is called the generalized mean curvature of V*

Not every varifold V has locally bounded first variation, for instance if V is a point cloud varifold. For such a varifold, the generalized mean curvature is not defined. In this situation, the alternative is to define an approximate mean curvature, as in [13] or, with better regularity properties (which is what we shall need here), as in [12, 28]. The latter definition of approximate mean curvature is recalled at the beginning of the next section.

3. Definition and stability of time-discrete approximate mean curvature flows

In this section, we define, for a given time subdivision and approximation scale, a time-discrete approximate mean curvature flow starting from any varifold V with finite mass. The construction relies on iterated push-forwards of V by diffeomorphisms of the form $\mathrm{id} + \tau h_\varepsilon$, where $\varepsilon > 0$ is the approximation scale, τ is the local time step, and h_ε is the approximate mean curvature of V at scale ε introduced in [28] after [12], obtained using suitable mollifications by a regularization kernel. The definition and properties of the regularization kernel are recalled in Subsection 3.1, and the definition of h_ε in Subsection 3.2. Subsections 3.3, 3.4 and 3.5 are devoted to the definition of the time-discrete approximate mean curvature flow, and to the study of its stability properties with respect to initial data and time subdivision.

3.1. Preliminaries on the regularization kernel.

Let $\psi \in C^\infty(\mathbb{R}^n)$ be a radially symmetric function such that:

$$\begin{aligned} &\psi(x) = 1 \text{ for } |x| \le 1/2, \qquad \psi(x) = 0 \text{ for } |x| \ge 1, \\ &0 \le \psi(x) \le 1, \quad |\nabla \psi(x)| \le 3, \quad \|\nabla^2 \psi(x)\| \le 9, \qquad \text{for all } x \in \mathbb{R}^n. \end{aligned} \tag{15}$$

Define for each $\varepsilon \in (0,1)$:

$$\hat{\Phi}_\varepsilon(x) := \frac{1}{(2\pi\varepsilon^2)^{\frac{n}{2}}} \exp\Big(-\frac{|x|^2}{2\varepsilon^2}\Big) \qquad \text{and} \qquad \Phi_\varepsilon(x) := c(\varepsilon)\psi(x)\hat{\Phi}_\varepsilon(x), \tag{16}$$

where $c(\varepsilon) := \dfrac{1}{\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx}$ so that, because $\displaystyle\int_{\mathbb{R}^n} \hat{\Phi}_\varepsilon(x)dx = 1$, we have that

$$\int_{\mathbb{R}^n} \Phi_\varepsilon(x)\, dx = 1. \tag{17}$$

Then, using $\psi \leq 1$,

$$c(\varepsilon) = \frac{1}{\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx} \geq \frac{1}{\int_{\mathbb{R}^n} \hat{\Phi}_\varepsilon(x)dx} = 1.$$

Also, as $\psi = 1$ on $[0, \frac{1}{2}]$,

$$\int_{\mathbb{R}^n} \psi(x)\hat{\Phi}_\varepsilon(x)dx \geq \int_{B_{\frac{1}{2}}(0)} \hat{\Phi}_\varepsilon(x)dx.$$

By the change of variables $y = \varepsilon^{-1}x$ we obtain

$$\int_{B_{\frac{1}{2}}(0)} \hat{\Phi}_\varepsilon(x)dx = \int_{B_{\frac{1}{2\varepsilon}}(0)} \hat{\Phi}_1(y)dy \geq \int_{B_{\frac{1}{2}}(0)} \hat{\Phi}_1(y)dy =: c^{-1}. \tag{18}$$

c is a constant depending only on n, and by definition, $1 \leq c(\varepsilon) \leq c$.

The kernel Φ_ε has a remarkable property: its derivatives are bounded by a power of ε times the kernel plus an exponentially small term (see [28, Lemmas 4.13 and 4.14]). This property is the key ingredient for the computations of [28, Section 5].

In the following lemma we list some of the properties of the kernel Φ_ε, similar to the estimates in [28, Lemma 4.13].

Lemma 3.1 (Kernel properties). *Let $\varepsilon \in (0,1)$ and Φ_ε be defined as in (16). There exists a constant c_0 depending only on n such that*

$$|\nabla\Phi_\varepsilon| \leq \varepsilon^{-2}\Phi_\varepsilon + c_0\chi_{B_1(0)}, \tag{19}$$

$$|\nabla^2\Phi_\varepsilon| \leq 2\varepsilon^{-4}\Phi_\varepsilon + 2c_0\chi_{B_1(0)}. \tag{20}$$

As a consequence,

$$\|\nabla\Phi_\varepsilon\|_{L^1} \leq (1+\omega_n c_0)\varepsilon^{-2} \quad \textit{and} \quad \|\nabla^2\Phi_\varepsilon\|_{L^1} \leq 2(1+\omega_n c_0)\varepsilon^{-4}\,, \tag{21}$$

and

$$\mathrm{Lip}(\Phi_\varepsilon) \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\,, \quad \mathrm{Lip}(\nabla\Phi_\varepsilon) \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}. \tag{22}$$

Proof. Define

$$c_0 := \sup_{\varepsilon\in(0,1)} c(\varepsilon)\frac{9\varepsilon^{-2-n}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{8\varepsilon^2}\right) < \infty. \tag{23}$$

By (16) for all $x \in \mathbb{R}^n$

$$\nabla\Phi_\varepsilon(x) = -\varepsilon^{-2}\Phi_\varepsilon(x)x + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x) \tag{24}$$

Φ_ε is supported on $B_1(0)$ therefore for all $x \in \mathbb{R}^n$

$$\left|\varepsilon^{-2}\Phi_\varepsilon(x)x\right| \leq \varepsilon^{-2}\Phi_\varepsilon(x)$$

also by construction $\nabla\psi = 0$ on $[0, \frac{1}{2}] \cup [1, \infty)$ and $|\nabla\psi| \leq 3$ on $[\frac{1}{2}, 1]$, this yields for all $x \in \mathbb{R}^n$

$$\begin{aligned} |c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| &\leq 3c(\varepsilon)\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}} \sup_{\frac{1}{2}\leq|x|\leq 1} \exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B_1(0)}(x) \\ &\leq 3c(\varepsilon)\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B_1(0)}(x) \leq c_0\chi_{B_1(0)}(x) \end{aligned} \tag{25}$$

this proves (19). For (20), differentiating (24) gives for all $x \in \mathbb{R}^n$

$$\begin{aligned}\nabla^2\Phi_\varepsilon(x) &= -\varepsilon^{-2}x \otimes \left(-\varepsilon^{-2}\Phi_\varepsilon(x)x + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)\right) - \varepsilon^{-2}\Phi_\varepsilon(x)I_n \\ &\quad + c(\varepsilon)\left(\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x) - \hat{\Phi}_\varepsilon(x)\varepsilon^{-2}\nabla\psi(x) \otimes x\right) \\ &= \varepsilon^{-4}\Phi_\varepsilon(x)x \otimes x - 2\varepsilon^{-2}c(\varepsilon)\hat{\Phi}_\varepsilon(x)x \otimes \nabla\psi(x) - \varepsilon^{-2}\Phi_\varepsilon(x)I_n \\ &\quad + c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x).\end{aligned}$$

We know that $\|v\otimes w\| \le \|v\|w\|$ for every two vectors v and w, the fact that Φ_ε and ψ are supported on $[0,1]$ implies, for all $x \in \mathbb{R}^n$

$$\begin{aligned}\|\nabla^2\Phi_\varepsilon(x)\| &\le \varepsilon^{-4}\Phi_\varepsilon(x) + 2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| + \Phi_\varepsilon(x) + \|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\| \\ &\le 2\varepsilon^{-4}\Phi_\varepsilon(x) + 2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| + \|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\|.\end{aligned}$$

Similarly to (25), we have for all $x \in \mathbb{R}^n$

$$\begin{aligned}2\varepsilon^{-2}|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla\psi(x)| &\le c(\varepsilon)6\frac{\varepsilon^{-2-n}}{(2\pi)^{\frac{n}{2}}}\sup_{\frac{1}{2}\le|x|\le 1}\exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B_1(0)}(x) \\ &\le c(\varepsilon)6\frac{\varepsilon^{-2-n}}{(2\pi)^{\frac{n}{2}}}\exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B_1(0)}(x) \le c_0\chi_{B_1(0)}(x)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\|c(\varepsilon)\hat{\Phi}_\varepsilon(x)\nabla^2\psi(x)\| &\le c(\varepsilon)9\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\sup_{\frac{1}{2}\le|x|\le 1}\exp\left(-\frac{|x|^2}{\varepsilon^2}\right)\chi_{B_1(0)}(x) \\ &\le c(\varepsilon)9\frac{\varepsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\exp\left(-\frac{1}{8\varepsilon^2}\right)\chi_{B_1(0)}(x) \le c_0\chi_{B_1(0)}(x).\end{aligned} \tag{27}$$

Finally, (26) and (27) imply

$$\|\nabla^2\Phi_\varepsilon\| \le 2\varepsilon^{-4}\Phi_\varepsilon + 2c_0\chi_{B_1(0)}$$

and this concludes the proof of (20). For the L^1-estimates, we write using (19)

$$\begin{aligned}\|\nabla\Phi\|_{L^1} &= \int_{\mathbb{R}^n}|\nabla\Phi_\varepsilon(x)|dx \le \int_{\mathbb{R}^n}\left(\varepsilon^{-2}\Phi_\varepsilon(x) + c_0\chi_{B_1(0)}(x)\right)dx \\ &\le \varepsilon^{-2}\int_{\mathbb{R}^n}\Phi_\varepsilon(x)\,dx + c_0\int_{B_1(0)}1\,dx \\ &\le \varepsilon^{-2} + \omega_n c_0 \le (1+\omega_n c_0)\varepsilon^{-2}.\end{aligned} \tag{28}$$

Similarly, we prove $\|\nabla^2\Phi\|_{L^1} \le 2(1+c_0\omega_n)\varepsilon^{-4}$, and this completes the proof of (21).
For the Lipschitz constant, we use the definition of Φ_ε (see (16) to get:

$$\mathrm{Lip}(\Phi_\varepsilon) \le (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\,, \quad \mathrm{Lip}(\nabla\Phi_\varepsilon) \le 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}$$

and this finishes the proof of (22) and Lemma 3.1. □

3.2. **Approximate mean curvature: definition and properties.** We can now define the approximate mean curvature vector at scale $\varepsilon > 0$ for any varifold V.

Definition 3.2 (Approximate mean curvature)**.** *The approximate mean curvature of a varifold $V \in V_d(\mathbb{R}^n)$ at $x \in \mathbb{R}^n$ for the approximation scale $\varepsilon > 0$ is*

$$h_\varepsilon(x,V) = (\Phi_\varepsilon * \tilde{h}_\varepsilon(\cdot,V))(x)\,, \quad \textit{where } \tilde{h}_\varepsilon(y,V) = -\frac{(\delta V * \Phi_\varepsilon)(y)}{(\|V\| * \Phi_\varepsilon)(y) + \varepsilon} \textit{ for any } y \in \mathbb{R}^n. \tag{29}$$

The double convolution guarantees the decay of the mass (up to a small error), as we will see in (42), and later in Remark 4.7. It also reduces the calculation of $\|h_\varepsilon\|_{C^2}$ to that of $\|\tilde{h}_\varepsilon\|_\infty$ and $\|\Phi_\varepsilon\|_{C^2}$, thereby avoiding the differentiation of the fraction $\tilde{h}_\varepsilon$ (see Proposition 3.3 for details).

The following property is a mere adaptation of [28, Lemma 5.1]: we bound the C^2–norm of the approximate mean curvature for $\varepsilon \in (0,1)$, but we impose here no smallness requirement on ε contrary to the original statement.

Proposition 3.3 (C^2 boundedness of h_ε)**.** *There exists a constant $c_1 \geq 2$ depending only on n with the following property: for any $\varepsilon \in (0,1)$ and $M \in [1,+\infty)$, if $V \in V_d(\mathbb{R}^n)$ is a d–varifold with total mass $\|V\|(\mathbb{R}^n) \leq M$, then*

$$\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty \leq c_1 M \varepsilon^{-2}, \quad \|h_\varepsilon(\cdot,V)\|_\infty \leq c_1 M \varepsilon^{-2}, \tag{30}$$

$$\|Dh_\varepsilon(\cdot,V)\|_\infty \leq c_1 M \varepsilon^{-4}, \tag{31}$$

$$\|D^2 h_\varepsilon(\cdot,V)\|_\infty \leq c_1 M \varepsilon^{-6}. \tag{32}$$

Proof. Let $\varepsilon \in (0,1)$, $M \geq 1$ and let V be a d–varifold satisfying $\|V\|(\mathbb{R}^n) \leq M$. We start with the proof of (30), setting $c_1 = 2(1+\omega_n c_0)(1+c_0)$. For any $\varphi \in \mathrm{C}(\mathbb{R}^n,\mathbb{R})$, we recall that Φ_ε is radial and then

$$(\Phi_\varepsilon * \|V\|)(\varphi) = \|V\|(\Phi_\varepsilon * \varphi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_\varepsilon(y-x)\,\varphi(y)\,dx\,d\|V\|(y)$$

hence we can associate the convoluted measure $\Phi_\varepsilon * \|V\|$ with the function

$$x \in \mathbb{R}^n \mapsto (\Phi_\varepsilon * \|V\|)(x) := \int_{\mathbb{R}^n} \Phi_\varepsilon(y-x)\,d\|V\|(y). \tag{33}$$

Similarly,

$$\begin{aligned}(\Phi_\varepsilon * \delta V)(\varphi) = \delta V(\Phi_\varepsilon * \varphi) &= \int_{\mathbb{R}^n\times G_{d,n}}\int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon(y-x))\varphi(y)\;dxdV(y,S)\\ &= \int_{\mathbb{R}^n\times G_{d,n}}\left(\int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon(y-x))\varphi(y)\,dV(y,S)\right)dx.\end{aligned}$$

hence the convoluted first variation $\Phi_\varepsilon * \delta V$ can be associated with the function

$$x \in \mathbb{R}^n \mapsto (\Phi_\varepsilon * \delta V)(x) := \int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon(y-x))\,dV(y,S)dx. \tag{34}$$

Using that, for all $S \in G_{d,n}$, $\|S\| \leq 1$, we have

$$|(\Phi_\varepsilon * \delta V)(x)| = \left|\int_{\mathbb{R}^n\times G_{d,n}} S(\nabla\Phi_\varepsilon(x-y))\,dV(y,S)\right| \leq \int_{\mathbb{R}^n} |\nabla\Phi_\varepsilon(x-y)|\,d\|V\|(y)\,.$$

Therefore, applying (19) and then (33) we obtain

$$|(\Phi_\varepsilon * \delta V)(x)| \leq \int_{\mathbb{R}^n}\big(\varepsilon^{-2}\Phi_\varepsilon(x-y) + c_0\chi_{B_1(0)}(x-y)\big)d\|V\|(y) \leq \varepsilon^{-2}(\Phi_\varepsilon * \|V\|)(x) + c_0 M\,. \tag{35}$$

It remains to write the definition (29) of $\tilde{h}_\varepsilon(\cdot,V)$ and apply (35) to infer

$$\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty = \sup_{x\in\mathbb{R}^n}\frac{|(\Phi_\varepsilon * \delta V)(x)|}{\Phi_\varepsilon * \|V\|(x) + \varepsilon} \leq \varepsilon^{-2} + \varepsilon^{-1}c_0 M \leq (1+c_0)M\varepsilon^{-2}\,. \tag{36}$$

We can now use $h_\varepsilon := \Phi_\varepsilon * \tilde{h}_\varepsilon$ (see (29)), $\|\Phi_\varepsilon\|_{L^1} = 1$, and (36) to obtain

$$\|h_\varepsilon(\cdot,V)\|_\infty \leq \|\Phi_\varepsilon\|_{L^1}\,\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty \leq (1+c_0)M\varepsilon^{-2}\,, \tag{37}$$

and noting that $(1+c_0) \le c_1$ concludes the proof of (30).
We similarly have both $Dh_\varepsilon(\cdot,V) = \nabla\Phi_\varepsilon * \tilde{h}_\varepsilon$ and $D^2h_\varepsilon(\cdot,V) = \nabla^2\Phi_\varepsilon * \tilde{h}_\varepsilon$ so that applying (21) together with (36) concludes the proof of Proposition 3.3 as follows:

$$\|Dh_\varepsilon(\cdot,V)\|_\infty \le \|\nabla\Phi_\varepsilon\|_{L^1}\,\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty \le (1+c_0\omega_n)(1+c_0)M\,\varepsilon^{-4} \le c_1 M\,\varepsilon^{-4}\,,$$
$$\|D^2h_\varepsilon(\cdot,V)\|_\infty \le \|\nabla^2\Phi_\varepsilon\|_{L^1}\,\|\tilde{h}_\varepsilon(\cdot,V)\|_\infty \le 2(1+c_0\omega_n)(1+c_0)M\,\varepsilon^{-6} \le c_1 M\,\varepsilon^{-6}\,.$$

□

3.3. **Definition of a time-discrete approximate mean curvature flow.** The goal of this subsection is to define a time-discrete approximate mean curvature flow (see Definition 3.6) starting from an initial varifold $V_0 \in V_d(\mathbb{R}^n)$, for a given time subdivision $\mathcal{T}$ and approximation parameter ε. Such a definition relies on iterating push-forwards, starting from the initial varifold V_0, with velocity equal to the approximate mean curvature vector. To this end, we first investigate the effect of a single push-forward: in Proposition 3.4, we derive an expansion with respect to Δt of the push-forward of the mass of a varifold under the map $f = \mathrm{id} + \Delta t\, h_\varepsilon$. Computations rely on the Taylor expansion of the tangential Jacobian (see Lemma A.2), from which follows estimate (40). It is then possible to prove that the mass of the push-forward varifold decays up to a small error Δt (see (42)), hence allowing to iterate push-forwards for suitable time steps (see condition (49)) and resulting in Definition 3.6.

Given $\varepsilon \in (0,1)$ and $V \in V_d(\mathbb{R}^n)$, we introduce the notation

$$f_{\varepsilon,V} = \mathrm{id} + \Delta t\, h_\varepsilon(\cdot,V)\,,$$

Depending on the context, we may drop the ε or V index dependency for simplicity.

Proposition 3.4. *Let ε, $\Delta t \in (0,1)$ and $M \ge 1$. Let $V \in V_d(\mathbb{R}^n)$ satisfy $\|V\|(\mathbb{R}^n) \le M$ and $S \in G_{d,n}$. There exists a constant $c_3 > 0$ that depends only on n and such that, if*

$$c_3 M \Delta t \le \varepsilon^4 \tag{38}$$

then $f = \mathrm{id} + \Delta t\, h_\varepsilon(\cdot,V)$ is a diffeomorphism, and

$$\forall (x,S) \in \mathbb{R}^n \times G_{d,n}, \quad J_S f(x) \in \left[\frac{1}{2},\frac{3}{2}\right] \cap \left[1 - c_3\Delta t\|Dh_\varepsilon\|_\infty, 1 + c_3\Delta t\|Dh_\varepsilon\|_\infty\right], \tag{39}$$

$$\forall \varphi \in \mathrm{C}^2(\mathbb{R}^n,\mathbb{R}_+), \quad \left|\|f_\# V\|(\varphi) - \|V\|(\varphi) - \Delta t\,\delta(V,\varphi)(h_\varepsilon(\cdot,V))\right| \le c_3\|\varphi\|_{\mathrm{C}^2} M^3 (\Delta t)^2 \varepsilon^{-8}\,, \tag{40}$$

and

$$\delta V(h_\varepsilon(\cdot,V)) = -\int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V)(y)|^2}{(\Phi_\varepsilon * \|V\|)(y) + \varepsilon} dy \le 0\,. \tag{41}$$

If we furthermore assume $c_3 M^3 \Delta t \le \varepsilon^8$ (that implies (38)) then,

$$\|f_\# V\|(\mathbb{R}^n) \le \|V\|(\mathbb{R}^n) + \Delta t\,. \tag{42}$$

Proof. Throughout the proof, we may increase $c_3 > 0$ to meet the requirements, provided that we respect the exclusive dependency in n. Let ε, $\Delta t \in (0,1)$, $M \ge 1$ and $V \in V_d(\mathbb{R}^n)$ satisfy $\|V\|(\mathbb{R}^n) \le M$. As V is fixed, we write h_ε for $h_\varepsilon(\cdot,V)$ hereafter as well as $f = \mathrm{id} + \Delta t\, h_\varepsilon$.
Step 1: We first prove that f is a diffeomorphism under the condition (38). To do so, we only need to check the hypothesis of Lemma 3.5 (see below) with $h = h_\varepsilon$. From (30) and (38) we can infer that

$$\Delta t\,\|h_\varepsilon\|_\infty \le c_1 M \Delta t \varepsilon^{-2} \le \frac{c_1}{c_3} < 1.$$

From (31) and (38)

$$\Delta t\,\|Dh_\varepsilon\|_\infty \le c_1 M \Delta t \varepsilon^{-4} \le \frac{c_1}{c_3} < 1, \tag{43}$$

and we can then apply (161) (with $k = n$ and $Q = \Delta t\, Dh_\varepsilon$) together with (4), (38) and (31) so that for all $x \in \mathbb{R}^n$,

$$|Jf(x) - 1| = |\det(I_n + \Delta t\, Dh_\varepsilon(x)) - \det(I_n)| \leq c_2 \Delta t |Dh_\varepsilon(x)|_\infty \leq \frac{c_1 c_2}{c_3} < 1$$

and f is a diffeomorphism of $\mathbb{R}^n$ thanks to Lemma 3.5.

Step 2: Let $(x, S) \in \mathbb{R}^n \times G_{d,n}$, we now prove (39) and

$$|J_S f(x) - 1 - \Delta t\ \mathrm{div}_S(h_\varepsilon(x))| \leq c_2 (\Delta t\, \|Dh_\varepsilon\|_\infty)^2. \tag{44}$$

Let us write $\tilde{S} = (\tau_1 | \dots | \tau_d)^t \in \mathcal{M}_{d,n}$ where $\{\tau_i\}_{i=1}^d$ is an orthonormal basis of S. We recall that we also denote by S the orthogonal projector onto the subspace S. For clarity, we denote by $\circ$ the matrix product in the calculations below. By construction,

$$\tilde{S} \circ \tilde{S}^t = I_d \in \mathcal{M}_d \quad \text{and} \quad \tilde{S}^t \circ \tilde{S} = S \in \mathcal{M}_n.$$

We recall that by definition of tangential Jacobian (8),

$$J_S f(x) = \det\left(((I_n + \Delta t\, Dh_\varepsilon(x)) \circ \tilde{S}^t)^t \circ ((I_n + \Delta t\, Dh_\varepsilon(x)) \circ \tilde{S}^t)\right)^{\frac{1}{2}}$$

and we can apply (163) with $R = \Delta t\, Dh_\varepsilon(x)$ and $L = \tilde{S}$, indeed, $c_2 |R|_\infty \leq \frac{c_1 c_2}{c_3} \leq 1$ thanks to (43). We obtain, again using (43),

$$|J_S f(x) - 1| \leq c_2 \Delta t |Dh_\varepsilon(x)|_\infty \leq c_3 \Delta t \|Dh_\varepsilon\|_\infty \leq \frac{1}{2}\,, \tag{45}$$

hence proving (39).

Similarly to the proof of (39), we are allowed to use (164) with $R = \Delta t\, Dh_\varepsilon(x)$ and $L = \tilde{S}$. Noting that

$$\mathrm{tr}\Big(Dh_\varepsilon(x) \circ \tilde{S}^t \circ \tilde{S}\Big) = \mathrm{tr}(Dh_\varepsilon(x) \circ S) = \mathrm{div}_S(h_\varepsilon(x))$$

we can infer that

$$|J_S f(x) - 1 - \Delta t\ \mathrm{div}_S(h_\varepsilon(x))| \leq c_2 (\Delta t\, |Dh_\varepsilon(x)|_\infty)^2 \leq c_2 (\Delta t\, \|Dh_\varepsilon\|_\infty)^2\,.$$

Step 3: We now prove (40). Let $\varphi \in C^2(\mathbb{R}^n, \mathbb{R}_+)$ and assume $\|\varphi\|_{\mathrm{C}^2} < \infty$ (otherwise there is nothing to prove). Coming back to the definitions of push-forward varifold (Definition 2.7) and weighted first variation (12), we have

$$\begin{aligned} &\|f_\# V\|(\varphi) - \|V\|(\varphi) - \Delta t\, \delta(V, \varphi)(h_\varepsilon) \\ &\quad = \int_{\mathbb{R}^n \times G_{d,n}} \Big[\varphi(f(x)) J_S f(x) - \varphi(x) - \Delta t\, \varphi(x)\, \mathrm{div}_S(h_\varepsilon(x)) - \Delta t\, \nabla\varphi(x) \cdot h_\varepsilon(x)\Big]\, dV(x, S). \end{aligned} \tag{46}$$

Let $(x, S) \in \mathbb{R}^n \times G_{d,n}$. We first recall that $f(x) - x = \Delta t\, h_\varepsilon(x)$ so that

$$|f(x) - x| \leq \Delta t\, \|h_\varepsilon\|_\infty \leq c_1 \Delta t M \varepsilon^{-2}$$

thanks to (30). We can then apply Taylor's inequality to φ between x and $f(x)$ to obtain

$$|\varphi(f(x)) - \varphi(x)| \leq |f(x) - x| \|\nabla\varphi\|_\infty \leq c_1 \|\varphi\|_{\mathrm{C}^2} M \Delta t\, \varepsilon^{-2} \tag{47}$$

and

$$\begin{aligned} |\varphi(f(x)) - \varphi(x) - \Delta t\, h_\varepsilon(x) \cdot \nabla\varphi(x)| &= |\varphi(f(x)) - \varphi(x) - (f(x) - x) \cdot \nabla\varphi(x)| \\ &\leq \frac{1}{2} |f(x) - x|^2 \|\nabla^2\varphi\|_\infty \leq c_1^2 \|\varphi\|_{\mathrm{C}^2} M^2 \Delta t^2\, \varepsilon^{-4}\,. \end{aligned} \tag{48}$$

Now rewriting the integrand in the right-hand side of (46) and using (43) to (48)

$$\begin{aligned}
|\varphi(f(x))J_S f(x) \; &-\varphi(x) - \Delta t\, \varphi(x)\, \mathrm{div}_S(h_\varepsilon(x)) - \Delta t\, \nabla\varphi(x)\cdot h_\varepsilon(x)| \\
&\leq |\varphi(f(x)) - \varphi(x)|\, |J_S f(x) - 1| + \varphi(x)\, |J_S f(x) - 1 - \Delta t\ \mathrm{div}_S(h_\varepsilon(x))| \\
&\quad + |\varphi(f(x)) - \varphi(x) - \Delta t\, h_\varepsilon(x)\cdot\nabla\varphi(x)| \\
&\leq c_1\|\varphi\|_{\mathrm{C}^2} M\Delta t\, \varepsilon^{-2}\, c_3 M\Delta t\, \varepsilon^{-4} + \|\varphi\|_\infty c_2 (c_1 M\Delta t\, \varepsilon^{-4})^2 + c_1^2 M^2 \|\varphi\|_{\mathrm{C}^2}\Delta t^2\, \varepsilon^{-4} \\
&\leq c_3\|\varphi\|_{\mathrm{C}^2} M^2\Delta t^2\, \varepsilon^{-8}
\end{aligned}$$

and integrating the previous inequality together with (46) leads to (40).
Step 4: By definition (29), $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$ and thus, for all $S \in G_{d,n}$, $\mathrm{div}_S(h_\varepsilon) = S(\nabla\Phi_\varepsilon) * \tilde{h}_\varepsilon$. Then, by definition of δV and (34), we obtain (41):

$$\begin{aligned}
\delta V(h_\varepsilon) &= \int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S(h_\varepsilon(x))\, dV(x,S) = \int_{\mathbb{R}^n\times G_{d,n}} \int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon(y-x))\cdot\tilde{h}_\varepsilon(y)\, dy\, dV(x,S) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\times G_{d,n}} S(\nabla\Phi_\varepsilon(y-x))\, dV(x,S)\cdot\tilde{h}_\varepsilon(y)\, dy \\
&= \int_{\mathbb{R}^n} (\Phi_\varepsilon * \delta V)(y)\cdot\tilde{h}_\varepsilon(y)\, dy = -\int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V)(y)|^2}{(\Phi_\varepsilon * \|V\|)(y) + \varepsilon}\, dy \leq 0.
\end{aligned}$$

We are left with the proof of (42) and we assume $c_3\Delta t M^3 \leq \varepsilon^8$ then Δt in particular satisfies (38), assumption under which the map f is a diffeomorphism of $\mathbb{R}^n$ and (44) holds. Consequently, applying Definition 2.7 of push-forward varifold and using (43), (44) and (41), we obtain

$$\begin{aligned}
\|f_\# V\|(\mathbb{R}^n) &= \int_{\mathbb{R}^n\times G_{d,n}} J_S f(x)\, dV(x,S) \\
&= \int_{\mathbb{R}^n\times G_{d,n}} 1 + \Delta t\ \mathrm{div}_S(h_\varepsilon(x)) + (J_S f(x) - 1 - \Delta t\ \mathrm{div}_S(h_\varepsilon(x)))\ dV(x,S) \\
&\leq \|V\|(\mathbb{R}^n) + \Delta t\, \delta V(h_\varepsilon) + c_2 M(\Delta t\|Dh_\varepsilon\|_\infty)^2 \\
&\leq \|V\|(\mathbb{R}^n) + c_1^2 c_2 M^3\Delta t^2\varepsilon^{-8} \\
&\leq \|V\|(\mathbb{R}^n) + \Delta t
\end{aligned}$$

hence concluding the proof of (42). □

Lemma 3.5. *Let $f \in \mathrm{C}(\mathbb{R}^n,\mathbb{R}^n)$ such that $f := \mathrm{id} + \Delta t\, h$, $\Delta t > 0$, $h \in \mathrm{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, assume that*

$$\max\{\Delta t\|h\|_\infty, \Delta t\|Dh\|_\infty, \|Jf - 1\|_\infty\} < 1.$$

Then, f is a diffeomorphism of $\mathbb{R}^n$.

Proof. For any $x \in \mathbb{R}^n$, we have $Jf(x) \neq 0$, therefore f is a local diffeomorphism by the inverse mapping theorem. We now prove that f is injective, indeed, for any $x, y \in \mathbb{R}$ one has

$$|f(x) - f(y)| = |x - y - \Delta t\,(h(x) - h(y))\,| \geq \big||x-y| - \Delta t|h(x) - h(y)|\big| \geq |x-y|\big|1 - \Delta t\ \mathrm{Lip}(h)\big| > 0,$$

this proves the injectivity of f.
Up to now, we have checked that f is injective and a local diffeomorphism at every point therefore f is a global diffeomorphism from $\mathbb{R}^n$ onto $f(\mathbb{R}^n)$; as $f(\mathbb{R}^n)$ is open, it remains to show that $f(\mathbb{R}^n)$ is closed. We have by assumption that $\|f - \mathrm{id}\|_\infty = \Delta t\|h\|_\infty < 1$, this implies that f is proper, by [34] it is closed, therefore $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$, recalling that it was open we have $f(\mathbb{R}^n) = \mathbb{R}^n$ and f is a diffeomorphism of $\mathbb{R}^n$. □

Given $M \geq 1$ and a d-varifold V_0 satisfying $\|V_0\|(\mathbb{R}^n) \leq M$, Proposition 3.4 gives the condition $c_3\Delta t < M^{-3}\varepsilon^8$ allowing to define $V_1 = f_{0\#}V_0$ with $f_0 = f_{\varepsilon,V_0} = \mathrm{id} + \Delta t\, h_\varepsilon(\cdot, V_0)$. We would like to iterate on several time steps and thus push the varifold V_1 by the map $f_1 = f_{\varepsilon,V_1} = \mathrm{id} + \Delta t\, h_\varepsilon(\cdot, V_1)$. However, note that $\|f_{0\#}V_0\|(\mathbb{R}^n) \leq M + \Delta t$ and not necessarily $\|f_{0\#}V_0\|(\mathbb{R}^n) \leq M$: the choice of Δt is no longer suitable. To rule out this issue, we can initially choose Δt satisfying

$$c_3\Delta t < (M+1)^{-3}\varepsilon^8,$$

and (42) thus ensuring $\|V_1\|(\mathbb{R}^n) = \|f_{0\#}V_0\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) + \Delta t \leq M + 1$, and we can iterate the process as long as the mass remains less than $M+1$, thus at least $\lfloor 1/\Delta t \rfloor$ times when considering uniform time discretizations of $[0,1]$. Considering a possibly non uniform time discretization $(\Delta t_i)_{i=1\dots m} \in (0,1)$ of $[0,a]$ for $a \leq 1$: $\sum_{i=1}^{m} \Delta t_i = a \leq 1$, one can iterate the process m times with Δt_i being the time step at step i, this justifies the following definition.

Definition 3.6 (Time-discrete approximate mean curvature flow)**.** *Let $M \geq 1$, $\varepsilon \in (0,1)$ and $a \in (0,1]$. Consider a subdivision $\mathcal{T} = \{t_i\}_{i=0}^m$ of $[0,a]$ and assume*

$$c_3\delta(\mathcal{T}) \leq (M+1)^{-3}\varepsilon^8 \tag{49}$$

where $\Delta t_i = t_i - t_{i-1}$ for $i = 1,\dots,m$ and $\delta(\mathcal{T}) = \max_{1\leq i\leq m} \Delta t_i$.
Let $V_0 \in V_d(\mathbb{R}^n)$ satisfy $\|V_0\|(\mathbb{R}^n) \leq M$. Define $(V_{\varepsilon,\mathcal{T}}(t_i))_{i=0\dots m}$ by $V_{\varepsilon,\mathcal{T}}(0) := V_0$ ($t_0 = 0$) and, for $i = 1,\dots,m$,

$$V_{\varepsilon,\mathcal{T}}(t_i) := f_{i\#}V_{\varepsilon,\mathcal{T}}(t_{i-1}) \quad \textit{with} \quad f_i = \mathrm{id} + \Delta t_i\, h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_{i-1}))\,.$$

We then define the family $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,a]}$ by linear interpolation between the points of the subdivision, and we call it a time-discrete approximate mean curvature flow*:*

$$V_{\varepsilon,\mathcal{T}}(t) := [\mathrm{id} + (t-t_i)h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t_i))]_{\#}\, V_{\varepsilon,\mathcal{T}}(t_i) \quad \textit{if} \quad t \in [t_i, t_{i+1}].$$

Remark 3.7. We note that under the assumptions of Definition 3.6 (and using the same notations), we have

$$\|V_{\varepsilon,\mathcal{T}}(t)\|(\mathbb{R}^n) \leq M+1, \quad \forall t \in [0,a]\,, \tag{50}$$

and we will use (50) extensively throughout the chapter.
Moreover, if we assume that there exists $R_0 > 0$ such that $\mathrm{spt}\, V_0 \subset B_{R_0}(0) \times G_{d,n}$, then

$$\forall t \in [0,a], \quad \mathrm{spt}\, V_{\varepsilon,\mathcal{T}} \subset B_{R_0 + c_1(M+1)\varepsilon^{-2}}(0) \times G_{d,n}\,.$$

Indeed, thanks to Proposition 3.3 and (50), for $t \in [0,a] \subset [0,1]$,

$$\|h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}}(t))\|_\infty \leq c_1(M+1)\varepsilon^{-2}$$

and therefore, $\mathrm{spt}\, V_{\varepsilon,\mathcal{T}}(t) \subset B\left(0, R_0 + c_1\, t\, (M+1)\varepsilon^{-2}\right) \times G_{d,n}$. Such compactness property will be used when letting Δt go to 0 (for a fixed ε) to define a limit flow in Section 4.

Remark 3.8 (Piecewise constant flow)**.** Note that in Definition 3.6, we first define $V_{\varepsilon,\mathcal{T}}(t_i)$ at the points t_i of the subdivision $\mathcal{T}$ and we then define $V_{\varepsilon,\mathcal{T}}(t)$ for $t \in [t_i, t_{i+1}]$ by a linear interpolation between t_i and t_{i+1}. It is possible to consider an alternative definition of the flow between t_i and t_{i+1}, simply taking the following piecewise constant extension: for $i \in \{0,1,\dots,m-1\}$,

$$V^{pc}_{\varepsilon,\mathcal{T}}(t) := V_{\varepsilon,\mathcal{T}}(t_i) \quad \text{if} \quad t \in (t_i, t_{i+1})\,.$$

As we will see in Proposition 4.2, both $V_{\varepsilon,\mathcal{T}}$ and $V^{pc}_{\varepsilon,\mathcal{T}}$ lead to the same limit flow V_ε when the size of the subdivision tends to zero. We consequently restrict our study to only one of the two flows and we choose to investigate $V_{\varepsilon,\mathcal{T}}$ introduced in Definition 3.6.

Hereafter, $M \geq 1$ and $\varepsilon \in (0,1)$ are fixed, all subdivisions we consider satisfy (49) and we define time-discrete approximate mean curvature flow starting from a varifold of mass less than M.

3.4. **Stability with respect to initial data.** When investigating a discrete scheme for a flow, the stability arises as a crucial issue. More precisely, we consider in Proposition 3.11 two time-discrete approximate mean curvature flows $(V(t))_t$, $(W(t))_t$ starting from V_0, W_0, respectively, and we prove that the stability holds in terms of bounded Lipschitz distance: $\Delta(V(t), W(t)) \leq \exp(\Lambda t)\Delta(V_0, W_0)$, where $\Lambda \sim \varepsilon^{-n-7}$. Up to a constant, Λ is an upper bound of the Lipschitz constant of $V \mapsto h_\varepsilon(\cdot, V)$ with respect to the C^1–norm, as established in Lemma 3.9. In Remark 3.12, we draw a parallel with the classical time discretization of ODEs showing that Λ is the expected constant in our setting.

Lemma 3.9. *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V and W be two varifolds of $V_d(\mathbb{R}^n)$ satisfying $\|V\|(\mathbb{R}^n) \leq M$, $\|W\|(\mathbb{R}^n) \leq M$. There exists $c_4 > 0$ only depending on n such that*

$$\big\|h_\varepsilon(\cdot, V) - h_\varepsilon(\cdot, W)\big\|_\infty \leq c_4 M \varepsilon^{-n-5}\Delta(V,W) \text{ and } \big\|Dh_\varepsilon(\cdot, V) - Dh_\varepsilon(\cdot, W)\big\|_\infty \leq c_4 M \varepsilon^{-n-7}\Delta(V,W).$$

Proof. As previously, we may increase $c_3 > 0$ throughout the proof to meet the requirements, provided that we respect the exclusive dependency in n. Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V and W be two varifolds of $V_d(\mathbb{R}^n)$ satisfying $\|V\|(\mathbb{R}^n) \leq M$, $\|W\|(\mathbb{R}^n) \leq M$. We first show that

$$\begin{aligned} &\|\Phi_\varepsilon * \|V\| - \Phi_\varepsilon * \|W\|\|_\infty \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\Delta(V,W), \text{ and} \\ &\|\Phi_\varepsilon * \delta V - \Phi_\varepsilon * \delta W\|_\infty \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}\Delta(V,W)\,. \end{aligned} \tag{51}$$

We have for any $x \in \mathbb{R}^n$ by (22) and the definition (16) of Φ_ε

$$\begin{aligned} \Big|\Phi_\varepsilon * \|V\|(x) - \Phi_\varepsilon * \|W\|(x)\Big| &= \Big|\int_{\mathbb{R}^n} \Phi_\varepsilon(x-y)d\|V\|(y) - \int_{\mathbb{R}^n} \Phi_\varepsilon(x-y)d\|W\|(y)\Big| \\ &= \Big|\|V\|\left(\Phi_\varepsilon(\cdot - x)\right) - \|W\|\left(\Phi_\varepsilon(\cdot - x)\right)\Big| \\ &\leq \max\{\|\Phi_\varepsilon\|_\infty, \mathrm{Lip}(\Phi_\varepsilon)\}\Delta(\|V\|, \|W\|) \\ &\leq \max\{c(2\pi)^{-\frac{n}{2}}\varepsilon^{-n}, (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\}\Delta(\|V\|, \|W\|) \\ &\leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}\Delta(V,W)\,, \end{aligned}$$

since $\Delta(\|V\|, \|W\|) \leq \Delta(V,W)$, this gives the first estimate of (51). For the second estimate we first recall that for $x \in \mathbb{R}^n$, $\Phi_\varepsilon * \delta V(x) = \displaystyle\int_{\mathbb{R}^n \times G_{d,n}} S\nabla\Phi_\varepsilon(x-y)\, dV(y,S)$ and we thus compute the Lipschitz constant of the map $\Theta : (y,S) \mapsto S(\nabla\Phi_\varepsilon(y))$ (the map $y \mapsto x - y$ being an isometry), we have for $(y,S), (z,T) \in \mathbb{R}^n \times G_{d,n}$, using $\|S\| = 1$

$$\begin{aligned} |\Theta(y,S) - \Theta(z,T)| &= |S(\nabla\Phi_\varepsilon(y)) - T(\nabla\Phi_\varepsilon(z))| \\ &\leq \|S\|\,|\nabla\Phi_\varepsilon(y) - \nabla\Phi_\varepsilon(z)| + \|S - T\|\,|\nabla\Phi_\varepsilon(z)| \\ &\leq \mathrm{Lip}(\nabla\Phi_\varepsilon)|y - z| + \|\nabla\Phi_\varepsilon\|_\infty\|S - T\| \\ &\leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4} \quad \text{thanks to Lemma 3.1.} \end{aligned}$$

Therefore $\mathrm{Lip}(\Theta) \leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}$, also from (19) we have $\|\Theta\|_\infty \leq (c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-2}$. We can now carry on with the proof the the second inequality of (51): for $x \in \mathbb{R}^n$,

$$\begin{aligned} \Big|\Phi_\varepsilon * \delta V(x) - \Phi_\varepsilon * \delta W(x)\Big| &= \Big|\int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon)(x-y)dV(y,S) - \int_{\mathbb{R}^n} S(\nabla\Phi_\varepsilon)(x-y)dW(y,S)\Big| \\ &\leq \max\{\|\Theta\|_\infty, \mathrm{Lip}(\Theta)\}\Delta(V,W) \\ &\leq 2(c(2\pi)^{-\frac{n}{2}} + c_0)\varepsilon^{-n-4}\Delta(V,W)\,, \end{aligned}$$

which gives the desired result. From (51) and (30), we have for $x \in \mathbb{R}^n$,

$$\begin{aligned}
\left|\tilde{h}_\varepsilon(x,V)-\tilde{h}_\varepsilon(x,W)\right| &= \left|\frac{\Phi_\varepsilon * \delta V(x)}{\Phi_\varepsilon * \|V\|(x)+\varepsilon} - \frac{\Phi_\varepsilon * \delta W(x)}{\Phi_\varepsilon * \|W\|(x)+\varepsilon}\right| \\
&\leq \frac{|\Phi_\varepsilon * \delta V(x) - \Phi_\varepsilon * \delta W(x)|}{\Phi_\varepsilon * \|V\|(x)+\varepsilon} + \left|\frac{\Phi_\varepsilon * \delta W(x)}{\Phi_\varepsilon * \|W\|(x)+\varepsilon}\right| \frac{\left|\Phi_\varepsilon * \|V\|(x) - \Phi_\varepsilon * \|W\|(x)\right|}{\Phi_\varepsilon * \|V\|(x)+\varepsilon} \\
&\leq \frac{1}{\varepsilon}\|\Phi_\varepsilon * \delta V - \Phi_\varepsilon * \delta W\|_\infty + \|\tilde{h}_\varepsilon(\cdot,W)\|_\infty \frac{1}{\varepsilon}\|\Phi_\varepsilon * \|V\| - \Phi_\varepsilon * \|W\|\|_\infty \\
&\leq \frac{1}{\varepsilon}\left(2(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-4} + c_1 M\varepsilon^{-2}(c(2\pi)^{-\frac{n}{2}}+c_0)\varepsilon^{-n-2}\right)\Delta(V,W) \\
&\leq \left(c(2\pi)^{-\frac{n}{2}}+c_0\right)(2+c_1 M)\,\varepsilon^{-n-5}\Delta(V,W) \qquad (52) \\
&\leq c_4 M \varepsilon^{-n-5}\Delta(V,W)\,.
\end{aligned}$$

We recall that $h_\varepsilon = \Phi_\varepsilon * \tilde{h}_\varepsilon$ and we obtain thanks to (17) and (52):

$$\|h_\varepsilon(\cdot,V) - h_\varepsilon(\cdot,W)\|_\infty \leq \|\Phi_\varepsilon\|_{L^1}\|\tilde{h}_\varepsilon(\cdot,V) - \tilde{h}_\varepsilon(\cdot,W)\|_\infty \leq c_4 M\varepsilon^{-n-5}\Delta(V,W)\,.$$

Similarly $Dh_\varepsilon = \nabla\Phi_\varepsilon * \tilde{h}_\varepsilon$ and using (21) and (52), we obtain

$$\begin{aligned}
\|Dh_\varepsilon(\cdot,V) - Dh_\varepsilon(\cdot,W)\|_\infty &\leq \|\nabla\Phi_\varepsilon\|_{L^1}\|\tilde{h}_\varepsilon(\cdot,V) - \tilde{h}_\varepsilon(\cdot,W)\|_\infty \\
&\leq \varepsilon^{-2}(1+c_0\omega_n)\,(c(2\pi)^{-n/2}+c_0)(2+c_1M)\,\Delta(V,W)\varepsilon^{-n-5} \\
&\leq c_4 M\varepsilon^{-n-7}\Delta(V,W),
\end{aligned}$$

hence concluding the proof. □

In Proposition 3.10, we investigate the evolution of the bounded Lipschitz distance between two varifolds V and W through one step of the time-discrete approximate flow introduced in Definition 3.6. The proof relies on Lemma A.4, Lemma 3.9 and on careful estimates of the Lipschitz constant of the map $(x,S,V) \in \mathbb{R}^n \times G_{d,n} \times V_d(\mathbb{R}^n) \mapsto J_S f_{\varepsilon,V}(x)$.

Proposition 3.10. *Let $\varepsilon, \Delta t \in (0,1)$ and $M \geq 1$. Let $V, W \in V_d(\mathbb{R}^n)$ satisfy $\|V\|(\mathbb{R}^n) \leq M$, $\|W\|(\mathbb{R}^n) \leq M$. Let $g \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ be such that $\|Dg - I_n\|_\infty \leq c_1 M\Delta t\varepsilon^{-4}$ and recall the notation $f_{\varepsilon,V} = I_n + \Delta t\, h_\varepsilon(\cdot,V)$. There exists $c_5 > 0$ depending only on n such that if Δt satisfies*

$$c_5 M^3 \Delta t \leq \varepsilon^8\,, \qquad (53)$$

then, for any $(x,S),(y,T) \in \mathbb{R}^n \times G_{d,n}$

$$\left|J_S f_{\varepsilon,V}(x) - J_T g(y)\right| \leq c_5 M\Delta t\left(\varepsilon^{-4}\|S-T\| + \varepsilon^{-6}|x-y|\right) + c_5\|Df_{\varepsilon,V} - Dg\|_\infty, \qquad (54)$$

$$\left|J_S f_{\varepsilon,V}(x) - J_T f_{\varepsilon,W}(y)\right| \leq c_5 M\Delta t\left(\varepsilon^{-4}\|S-T\| + \varepsilon^{-6}|x-y| + \varepsilon^{-n-7}\Delta(V,W)\right). \qquad (55)$$

and

$$\Delta((f_{\varepsilon,V})_\# V, g_\# V) \leq M\left(c_5\|f_{\varepsilon,V} - g\|_{C^1} + \|J.f_{\varepsilon,V} - J.g\|_\infty\right), \qquad (56)$$

$$\Delta\left((f_{\varepsilon,V})_\# V, (f_{\varepsilon,W})_\# W\right) \leq (1 + c_5 M^2\Delta t\varepsilon^{-n-7})\Delta(V,W). \qquad (57)$$

Proof. Note that throughout the proof, we adapt i.e. increase $c_5 > 0$ with the constraint that c_5 only depends on n (like other c_i) whenever needed. As previously, ε is fixed throughout the proof and we can write f_V (resp. f_W, h) instead of $f_{\varepsilon,V}$ (resp. $f_{\varepsilon,W}$, h_ε). We fix $(x,S),(y,T) \in \mathbb{R}^n \times G_{d,n}$ and thanks to Lemma A.4, we choose $\tilde{S}, \tilde{T} \in \mathcal{M}_{d,n}$ such that, as orthogonal projectors, $S = \tilde{S}^t \circ \tilde{S}$, $T = \tilde{T}^t \circ \tilde{T}$, where $\circ$ denotes the matrix product for clarity. Moreover,

$$\tilde{S} \circ \tilde{S}^t = \tilde{T} \circ \tilde{T}^t = I_d \quad \text{and} \quad \|\tilde{S} - \tilde{T}\| \leq 2\|S - T\|\,.$$

We introduce some additional notations: we write $G = Dg - I_n$, $F = Df_V - I_n = \Delta t Dh(,V)$, and we recall that, by hypothesis on g, (53) and Proposition 3.3, one has (noting that $\Delta t M\varepsilon^{-4} \leq \Delta t M^3\varepsilon^{-8}$ and taking c_5 large enough),

$$\|G(y)\| \leq c_1 M\Delta t\varepsilon^{-4} \leq \frac{c_1}{c_5} \leq 1 \quad \text{and} \quad \|F(x)\| \leq c_1 M\Delta t\varepsilon^{-4} \leq 1\,. \tag{58}$$

We also use the notations

$$P = \tilde{S} \circ Df_V(x)^t \circ Df_V(x) \circ \tilde{S}^t \quad \text{and} \quad N = \tilde{T} \circ Dg(y)^t \circ Dg(y) \circ \tilde{T}^t\,.$$

and we recall that $J_S f_V(x) = \det(P)^{\frac{1}{2}}$ and similarly $J_T g(y) = \det(N)^{\frac{1}{2}}$.

Step 1: We first prove that

$$\|P - N\| \leq 3\left(4c_1 M\Delta t\varepsilon^{-4}\|S - T\| + c_1 M\Delta t\varepsilon^{-6}|x - y| + \|Df_V - Dg\|_\infty\right). \tag{59}$$

Let us decompose P and N as follows, using $Df_V(x) = I_n + F(x)$,

$$P = \tilde{S} \circ (I_n + F(x)) \circ (I_n + F(x))^t \circ \tilde{S}^t = I_d + \tilde{S} \circ \left(F(x)^t + F(x)\right) \circ \tilde{S}^t + \tilde{S} \circ F(x)^t \circ F(x) \circ \tilde{S}^t \tag{60}$$

and similarly

$$N = I_d + \tilde{T} \circ \left(G^t(y) + G(y)\right) \circ \tilde{T}^t + \tilde{T} \circ G^t(y) \circ G(y) \circ \tilde{T}^t\,. \tag{61}$$

On one hand, a simple computation (see Lemma A.1) together with (58) give

$$\begin{aligned}
&\|\tilde{S} \circ \left(F(x)^t + F(x)\right) \circ \tilde{S}^t - \tilde{T} \circ \left(G(x)^t + G(x)\right) \circ \tilde{T}^t\| \\
&\quad \leq 2\left(\|F(x)^t + F(x)\| + \|G(y)^t + G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 2\|F(x)^t + F(x) - G(y)^t - G(y)\| \\
&\quad \leq 2\left(\|F(x)\| + \|G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 2\|F(x) - G(y)\|
\end{aligned} \tag{62}$$

Similarly using Lemma A.1 and (58), we obtain

$$\begin{aligned}
&\|\tilde{S} \circ F(x)^t \circ F(x) \circ \tilde{S}^t - \tilde{T} \circ G(x)^t \circ G(x) \circ \tilde{T}^t\| \\
&\qquad \leq \left(\|F(x)^t \circ F(x)\| + \|G(y)^t \circ G(y)\|\right)\|\tilde{S} - \tilde{T}\| + \|F(x)^t \circ F(x) - G(x)^t \circ G(x)\| \\
&\qquad \leq \left(\|F(x)\| + \|G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 2\|F(x) - G(y)\|
\end{aligned} \tag{63}$$

Consequently, from (60) to (63), we infer

$$\|P - N\| \leq 3\left(\|F(x)\| + \|G(y)\|\right)\|\tilde{S} - \tilde{T}\| + 3\|F(x) - G(y)\| \tag{64}$$

so that we can conclude the proof of (59) (Step 1) recalling that $F(x) = \Delta t Dh_\varepsilon(x, V)$ and applying (31) and (32):

$$\begin{aligned}
&\|F(x) - G(y)\| \leq \|F(x) - F(y)\| + \|F(y) - G(y)\| \leq c_1 M\Delta t\varepsilon^{-6}|x - y| + \|Df_V - Dg\|_\infty, \\
&\|F(x)\| + \|G(y)\| \leq 2c_1 M\Delta t\varepsilon^{-4}\|\tilde{S} - \tilde{T}\| \leq 4c_1 M\Delta t\varepsilon^{-4}\|S - T\|\,.
\end{aligned} \tag{65}$$

Step 2: We now prove (54) and (55).
Let us show that

$$\|P - I_d\| \leq \frac{1}{4}, \quad \|N - I_d\| \leq \frac{1}{4} \quad \text{and } P,\ N \text{ are invertible with } \|P^{-1}\| \leq 2 \text{ and } \|N^{-1}\| \leq 2\,. \tag{66}$$

To this end, we apply Lemma A.2 with $L = \tilde{S}$ and $R = F(x) = \Delta t\, Dh(x, V)$: using (31) and (53),

$$|F(x)|_\infty = \Delta t\ |Dh_\varepsilon(x, V)|_\infty \leq c_1\Delta t M\varepsilon^{-4} \leq \frac{c_1}{c_5} \leq 1 \quad \text{and} \quad c_2|F(x)|_\infty \leq \frac{c_1 c_2}{c_5} \leq 1\,,$$

which allows to apply (162) so that

$$\begin{aligned}\|P-I_d\| \le d\,|P-I_d|_\infty &\le d\Big|\tilde S\circ\big(F(x)^t+F(x)\big)\circ\tilde S^t+\tilde S\circ F(x)^t\circ F(x)\circ\tilde S^t\Big|_\infty\\ &\le nc_2\,|F(x)|_\infty \le nc_2\frac{c_1}{c_5}<\frac14,\end{aligned} \tag{67}$$

where $\circ$ denotes the matrix product for clarity. As $c_2|F(x)|_\infty \le 1$, we can also apply (163) to conclude that $\left|\det(P)^{\frac12}-1\right| \le c_2|F(x)|_\infty \le \frac{c_1c_2}{c_5} < 1$ and thus P is invertible. Furthermore, using (67), we have

$$\|P^{-1}\| \le \|P^{-1}-I_d\|+\|I_d\| \le \|P^{-1}\|\|I_d-P\|+1 \le \frac12\|P^{-1}\|+1 \quad\Rightarrow\quad \|P^{-1}\|\le 2\,. \tag{68}$$

We similarly proceed with G, we recall (58) $|G(y)|_\infty \le \|G(y)\| \le 2c_1M\Delta t\varepsilon^{-4} \le \frac{2c_1}{c_5} \le 1$ and we can apply (162) to obtain

$$\begin{aligned}\|N-I_d\| \le d\,|N-I_d|_\infty &= d\Big|\tilde T\circ\big(G(y)^t+G(y)\big)\circ\tilde T^t+\tilde T\circ G(y)^t\circ G(y)\circ\tilde T^t\Big|_\infty\\ &\le nc_2\,|G(y)|_\infty \le nc_2\frac{2c_1}{c_5}\le\frac14\,.\end{aligned} \tag{69}$$

Note that similarly to (68), we also have that N is invertible and $\|N^{-1}\|\le 2$.
We recall that $\frac14 \le |\det(P)| = J_Sf_V(x)^2 \le 4$ thanks to (39) (which applies since (38) holds for c_5 large enough). We can now show that P and N satisfy the condition given in (170). Indeed, from (67), (68) and (69), we have

$$\|P^{-1}\|\|P-N\| \le 2\|(P-I_d)-(N-I_d)\| \le 2\left(\frac14+\frac14\right)\le 1$$

and thus applying (170) leads to

$$|\det(P)-\det(N)| \le c_2\,|\det(P)|\;\|P^{-1}\|\,\|P-N\| \le 8c_2\,\|P-N\|\,. \tag{70}$$

We note that for $a\ge\frac12$, $b\ge 0$

$$|a-b| = \frac{|a^2-b^2|}{a+b} \le 2|a^2-b^2|\,,$$

and we apply it with $a=\det(P)^{\frac12}=J_Sf_V(x)\ge\frac12$ and $b=\det(N)^{\frac12}=J_Tg(y)$ so that, using (70), we obtain

$$|J_Sf_V(x)-J_Tg(y)| = \left|\det(P)^{\frac12}-\det(N)^{\frac12}\right| \le 2\,|\det(P)-\det(N)| \le 16c_2\|P-N\|. \tag{71}$$

We conclude the proof of (54) gathering (71) and (59). Then, applying (54) with $g=f_W$ allows to prove (55). Indeed, $f_W\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ and from (31) we have $\|Df_W-I_n\|_\infty = \Delta t\|Dh(\cdot,W)\|_\infty \le c_1M\Delta t\varepsilon^{-4}$ so that

$$\big|J_Sf_V(x)-J_Tf_W(y)\big| \le c_5M\Delta t\left(\varepsilon^{-4}\|S-T\|+\varepsilon^{-6}|x-y|\right)+c_5\|Df_V-Df_W\|_\infty$$

and by Lemma 3.9:

$$\|Df_V-Df_W\|_\infty \le \Delta t\|Dh(\cdot,V)-Dh(\cdot,W)\|_\infty \le c_4M\Delta t\varepsilon^{-n-7}\Delta(V,W)$$

hence concluding the proof of (55) and Step 2.

Step 3: We now study the Lipschitz constant of the map $(x,S,f)\mapsto Df(x)(S)\in G_{d,n}$, which will be crucial in proving the remaining estimates of the proposition. We namely prove

$$\|Df_V(x)(S)-Dg(y)(T)\| \le (1+c_5M\Delta t\varepsilon^{-4})\|S-T\|+c_5M\Delta t\varepsilon^{-6}|x-y|+c_5\|Df_V-Dg\|_\infty\,. \tag{72}$$

We recall (see (9)) that we can write

$$Df_V(x)(S) = Y(Y^tY)^{-1}Y^t = Y \circ P^{-1} \circ Y^t \quad \text{and} \quad Dg(y)(T) = Z(Z^tZ)^{-1}Z^t = Z \circ N^{-1} \circ Z^t$$
$$\text{with} \quad Y = Df(x) \circ \tilde{S}^t = \tilde{S}^t + F(x) \circ \tilde{S}^t \quad \text{and} \quad Z = Dg(y) \circ \tilde{T}^t = \tilde{T}^t + G(y) \circ \tilde{T}^t \,,$$

so that we can further decompose $Df_V(x)(S)$ and $Dg(y)(T)$ as follows. By the formulas $\tilde{S}^t \circ \tilde{S} = S$ and $\tilde{T}^t \circ \tilde{T} = T$, if we set $\tilde{P} = P^{-1} - I_d$ and $\tilde{N} = N^{-1} - I_d$ we obtain

$$\begin{aligned} Df_V(x)(S) &= \left(\tilde{S}^t + F(x) \circ \tilde{S}^t\right) \circ P^{-1} \circ \left(\tilde{S} + \tilde{S} \circ F(x)^t\right) \\ &= \underbrace{\tilde{S}^t \circ P^{-1} \circ \tilde{S}}_{C_f} + \underbrace{F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} + \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t}_{R_f} + \underbrace{F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t}_{E_f} \\ &= C_f + R_f + E_f \end{aligned}$$

and similarly

$$\begin{aligned} Dg(y)(T) &= \left(\tilde{T}^t + G(y) \circ \tilde{T}^t\right) \circ N^{-1} \circ \left(\tilde{T} + \tilde{T} \circ G(y)^t\right) \\ &= \underbrace{\tilde{T}^t \circ N^{-1} \circ \tilde{T}}_{C_g} + \underbrace{G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} + \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t}_{R_g} + \underbrace{G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t}_{E_g} \\ &= C_g + R_g + E_g \,, \end{aligned}$$

so that

$$\|Df_V(x)(S) - Dg(y)(T)\| \leq \|C_f - C_g\| + \|D_f - D_g\| + \|E_f - E_g\| \,, \tag{73}$$

We first prove that

$$\|C_f - C_g\| \leq (1 + c_5 M \Delta t \varepsilon^{-4})\|S - T\| + c_5 M \Delta t \varepsilon^{-6}|x - y| + c_5 \|Df_V - Dg\|_\infty \,. \tag{74}$$

Since $\tilde{S}^t \circ \tilde{S} = S$ and $\tilde{T}^t \circ \tilde{T} = T$, we have $C_f - C_g = S - T + \tilde{S}^t \circ (P^{-1} - I_d) \circ \tilde{S} - \tilde{T}^t \circ (N^{-1} - I_d) \circ \tilde{T}$ so that applying Lemma A.1 and A.4, we obtain

$$\begin{aligned} \|C_f - C_g\| &\leq \|S - T\| + \left(\|P^{-1} - I_d\| + \|N^{-1} - I_d\|\right) \|\tilde{S} - \tilde{T}\| + \|(P^{-1} - I_d) - (N^{-1} - I_d)\| \\ &\leq \left(1 + 2\left(\|P^{-1} - I_d\| + \|N^{-1} - I_d\|\right)\right) \|S - T\| + \|P^{-1} - N^{-1}\| \end{aligned} \tag{75}$$

From (67), (31) and $\|P^{-1}\| \leq 2$ on one hand and from (69), the assumption on g and $\|N^{-1}\| \leq 2$ on the other hand, we infer

$$\begin{aligned} \|P^{-1} - I_d\| &\leq 2nc_2|F(x)|_\infty \leq 2nc_1c_2M\Delta t\varepsilon^{-4} \\ \|N^{-1} - I_d\| &\leq 2nc_2|G(y)|_\infty \leq 2nc_1c_2M\Delta t\varepsilon^{-4} \,. \end{aligned} \tag{76}$$

Furthermore, as $P^{-1} - N^{-1} = P^{-1}(N - P)N^{-1}$, we have

$$\|P^{-1} - N^{-1}\| \leq \|P^{-1}\|\|N^{-1}\|\|P - N\| \leq 4\|P - N\| \,. \tag{77}$$

We conclude the proof of (74) putting together (75), (76), (77) and (59).
We now prove

$$\|R_f - R_g\| \leq c_5 M \Delta t \left(\varepsilon^{-4}\|S - T\| + \varepsilon^{-6}|x - y|\right) + c_5\|Df_V - Dg\|_\infty. \tag{78}$$

We note that by definition $P^t = P$ and then $(P^{-1})^t = P^{-1}$, and similarly, $(N^{-1})^t = N^{-1}$ so that (using $\|P^{-1}\| \le 2$ and $\|N^{-1}\| \le 2$),

$$\begin{aligned}
&\|R_f - R_g\| \\
&\quad = \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} + \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} - \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&\quad \le 2\|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T}\|
\end{aligned}$$

and decomposing the term above into 4 terms and recalling $\|\tilde{S} - \tilde{T}\| \le 2\|S - T\|$ and (77), we obtain

$$\begin{aligned}
&\|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| \\
&\quad \le \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ (\tilde{S} - \tilde{T})\| + \|F(x) \circ \tilde{S}^t \circ (P^{-1} - N^{-1}) \circ \tilde{T}\| \\
&\qquad + \|F(x) \circ (\tilde{S}^t - \tilde{T}^t) \circ N^{-1} \circ \tilde{T}\| + \|(F(x) - G(y)) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| \\
&\quad \le 2\|F(x)\|\|\tilde{S} - \tilde{T}\| + \|F(x)\|\|P^{-1} - N^{-1}\| + 2\|F(x)\|\|\tilde{S} - \tilde{T}\| + 2\|F(x) - G(y)\| \\
&\quad \le \|F(x)\| \left(8\|S - T\| + 4\|P - N\|\right) + 2\|F(x) - G(y)\| .
\end{aligned}$$

The estimate (78) then follows from (58), (59) and (65).
We are left with checking that

$$\|E_f - E_g\| \le c_5 M \Delta t \left(\varepsilon^{-4}\|S - T\| + \varepsilon^{-6}|x - y|\right) + c_5\|Df_V - Dg\|_\infty . \tag{79}$$

Indeed, recalling $\|P^{-1}\| \le 2$, $\|N^{-1}\| \le 2$ (see (66)), $\|F(x)\| \le c_1 M \Delta t \varepsilon^{-4} \le 1$ and $\|G(y)\| \le 2c_1 M \Delta t \varepsilon^{-4} \le 1$ (see (58)), (65) and using Lemma A.4 one has

$$\begin{aligned}
&\|E_f - E_g\| = \|F(x) \circ \tilde{S}^t \circ P^{-1} \circ \tilde{S} \circ F(x)^t - G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ G(y)^t\| \\
&\quad \le \|F(x) \circ \left(\tilde{S}^t \circ P^{-1} \circ \tilde{S} - \tilde{T}^t \circ N^{-1} \circ \tilde{T}\right) \circ F(x)^t\| + \|(F(x) - G(y)) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ F(x)^t\| \\
&\qquad + \|G(y) \circ \tilde{T}^t \circ N^{-1} \circ \tilde{T} \circ (F(x)^t - G(y)^t\| \\
&\quad \le \|F(x)\|^2\|\tilde{S}^t \circ P^{-1} \circ \tilde{S} - \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| + 2(\|F(x)\| + \|G(y)\|)\|F(x) - G(y)\| \\
&\quad \le c_1 M \Delta t \varepsilon^{-4}\|\tilde{S}^t \circ P^{-1} \circ \tilde{S} - \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| + 4\left(c_1 M \Delta t \varepsilon^{-6}|x - y| + \|Df_V - Dg\|_\infty\right) .
\end{aligned} \tag{80}$$

We can then apply Lemma A.1, (77) and $\|\tilde{S} - \tilde{T}\| \le 2\|S - T\|$ to obtain

$$\begin{aligned}
\|\tilde{S}^t \circ P^{-1} \circ \tilde{S} - \tilde{T}^t \circ N^{-1} \circ \tilde{T}\| &\le \left(\|P^{-1}\| + \|N^{-1}\|\right)\|\tilde{S} - \tilde{T}\| + \|P^{-1} - N^{-1}\| \\
&\le 8\|S - T\| + 4\|P - N\| .
\end{aligned} \tag{81}$$

Therefore, (79) follows from (80), (81) and (59).

Step 4: We now prove (56).
Let $\varphi \in \mathrm{C}(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$ satisfying $\|\varphi\|_\infty \le 1$ and $\mathrm{Lip}(\varphi) \le 1$. For $(x, S) \in \mathbb{R}^n \times G_{d,n}$, we recall (39) $|J_S f_V(x)| \le 2$ and thanks to (72) we have

$$\begin{aligned}
&\left|\varphi(f_V(x), Df_V(x)(S))J_S f_V(x) - \varphi(g(x), Dg(x)(S))J_S g(x)\right| \\
&\quad \le \mathrm{Lip}(\varphi)\left(|f_V(x) - g(x)| + \|Df_V(x)(S) - Dg(x)(S)\|\right)|J_S f_V(x)| + \|\varphi\|_\infty |J_S f_V(x) - J_S g(x)| \\
&\quad \le \|f_V - g\|_\infty + 2c_5\|Df_V - Dg\|_\infty + \|J.f_V - J.g\|_\infty \\
&\quad \le c_5\|f_V - g\|_{\mathrm{C}^1} + \|J.f_V - J.g\|_\infty
\end{aligned} \tag{82}$$

where $\|\cdot\|_\infty$ is taken over all $(x, S) \in \mathbb{R}^n \times G_{d,n}$ (and the last inequality holds up to doubling c_5). Integrating (82) over $\mathbb{R}^n \times G_{d,n}$ one has by definition of push-forward varifold (see Definition 2.7)

$$\left|f_{V\#}V(\varphi) - g_\# V(\varphi)\right| \le M\left(c_5\|f_V - g\|_{\mathrm{C}^1} + \|J.f_V - J.g\|_\infty\right) .$$

Taking the supremum with respect to such Lipschitz functions φ leads to (56) by definition of Δ.

Step 5: We are left with the proof of (57). We first apply (56) with $V = W$, $f = f_V$ and $g = f_W$ so that

$$\begin{aligned}\Delta\left((f_V)_\# W, (f_W)_\# W\right) &\leq M\left(c_5\|f_V - f_W\|_{C^1} + \|J.f_V - J.f_W\|_\infty\right)\\ &\leq c_5 M\Delta t\|h_\varepsilon(\cdot, V) - h_\varepsilon(\cdot, W)\|_{\mathrm{C}^1} + c_5 M^2\Delta t\varepsilon^{-n-7}\Delta(V,W) \text{ (by (55) with } x = y,\, S = T)\\ &\leq (2c_4 + c_5)M^2\Delta t\,\varepsilon^{-n-7}\Delta(V,W) \text{ by Lemma 3.9} \end{aligned} \tag{83}$$

We now prove that

$$\Delta\left((f_V)_\# V, (f_V)_\# W\right) \leq \left(1 + c_5 M\Delta t\varepsilon^{-6}\right)\Delta(V,W)\,. \tag{84}$$

Let $\varphi \in \mathrm{C}^0_c(\mathbb{R}^n \times G_{d,n}, \mathbb{R})$ be a Lipschitz function satisfying $\|\varphi\|_\infty \leq 1$ and $\mathrm{Lip}(\varphi) \leq 1$, then, coming back to the definition of Δ (Definition 5), we consider $\psi : (x,S) \mapsto \varphi(f_V(x), Df_V(x)(S))\, J_S f_V(x)$. As f_V is a C^1- diffeomorphism, we have $\psi \in \mathrm{C}_c(\mathbb{R}^n \times G_{d,n})$ and by definition of varifold push-forward,

$$\left|\int \varphi\, d(f_V)_\# V - \int \varphi\, d(f_V)_\# W\right| = \left|\int \psi\, dV - \int \psi\, dW\right| \leq \max(\|\psi\|_\infty, \mathrm{Lip}(\psi))\Delta(V,W)\,. \tag{85}$$

One has to pay attention to the fact that the $\|\cdot\|_\infty$ and $\mathrm{Lip}(\cdot)$ refer to both variables $(x,S) \in \mathbb{R}^n \times G_{d,n}$. Introducing the notations $\psi_1 : (x,S) \mapsto Df_V(x)(S) = \left(I_n + \Delta t\, Dh_\varepsilon(x,V)\right)(S)$ and $\psi_2 : (x,S) \mapsto J_S f_V(x)$, we have

$$\begin{aligned}\mathrm{Lip}(\psi_1) \leq \left(1 + c_5 M\Delta t\varepsilon^{-6}\right), \quad \mathrm{Lip}(\psi_2) \leq c_5 M\Delta t\,\varepsilon^{-6}\,,\\ \|\psi_2\|_\infty \leq 1 + c_5 M\Delta t\,\varepsilon^{-4} \quad \text{and} \quad \mathrm{Lip}(f_V) \leq 1 + c_5 M\Delta t\,\varepsilon^{-4}\,. \end{aligned} \tag{86}$$

Indeed, let $(x,S), (t,y) \in \mathbb{R}^n \times G_{d,n}$, and apply (72) and (55) with f_V and $g = f_V$ so that

$$\begin{aligned}|\psi_1(x,S) - \psi_1(t,y)| &\leq (1 + c_5 M\Delta t\varepsilon^{-4})\|S - T\| + c_5\Delta t\varepsilon^{-6}|x - y|\\ &\leq (1 + c_5 M\Delta t\varepsilon^{-6})(\|S - T\| + |x - y|)\\ |\psi_2(x,S) - \psi_2(t,y)| &\leq c_5 M\Delta t\,\left(\varepsilon^{-4}\|S - T\| + \varepsilon^{-6}|x - y|\right)\,.\end{aligned}$$

Furthermore, by (39) and (31), $\|\psi_2\|_\infty = \|J.f_V\|_\infty \leq 1 + c_2\Delta t\,\|Dh_\varepsilon(\cdot,V)\|_\infty \leq 1 + c_1 c_2 M\Delta t\,\varepsilon^{-4}$ and $\mathrm{Lip}(f_V) \leq 1 + \Delta t\ \mathrm{Lip}(h_\varepsilon) \leq 1 + c_1 M\Delta t\,\varepsilon^{-4}$. With (86) in hand, we already note that

$$\|\psi\|_\infty \leq \|\varphi\|_\infty\|\psi_2\|_\infty \leq \|\psi_2\|_\infty \leq \left(1 + c_5 M\Delta t\varepsilon^{-6}\right) \tag{87}$$

and then, we can estimate $\mathrm{Lip}(\psi)$ as follows:

$$\begin{aligned}&|\psi(x,S) - \psi(t,y)|\\ &\quad\leq |\varphi(f_V(x), \psi_1(x,S)) - \varphi(f_V(y), \psi_1(t,y))|\,\|\psi_2\|_\infty + \|\varphi\|_\infty\,\mathrm{Lip}(\psi_2)\,(|x - y| + \|S - T\|)\\ &\quad\leq [\mathrm{Lip}(\varphi)\,(\max\{\mathrm{Lip}(f_V), \mathrm{Lip}(\psi_1)\})\,\|\psi_2\|_\infty + \mathrm{Lip}(\psi_2)]\,(|x - y| + \|S - T\|)\\ &\quad\leq \left(1 + c_5 M\Delta t\varepsilon^{-6}\right)\left(1 + c_5 M\Delta t\,\varepsilon^{-4}\right) + c_5 M\Delta t\,\varepsilon^{-6} \leq \left(1 + 4c_5 M\Delta t\varepsilon^{-6}\right)\,,\end{aligned} \tag{88}$$

recalling that $c_5 M\Delta t\,\varepsilon^{-4} \leq 1$ by (53). We conclude the proof of (84) thanks to (85), (87) and (88) (up to increasing c_5). Combining (83) and (84) we conclude the proof of (57), and subsequently the proof of Proposition 3.10. □

Iterating Proposition 3.10 leads to the following stability result on the time-discrete approximate mean curvature flow.

Proposition 3.11 (Stability with respect to initial data)**.** *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V_0 and W_0 be two varifolds in $V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$, $\|W_0\|(\mathbb{R}^n) \leq M$.*
Let $\mathcal{T} = \{t_i\}_{i=0}^m$ be a subdivision of $[0,1]$ satisfying (49). Denote by $V_{\varepsilon,\mathcal{T}}(t)$ (resp. $W_{\varepsilon,\mathcal{T}}(t)$) the time-discrete approximate mean curvature flow with respect to $\mathcal{T}$ starting from V_0 (resp. W_0) as introduced in Definition 3.6. Then, for any $t \in [0,1]$, one has

$$\Delta\left(V_{\varepsilon,\mathcal{T}}(t), W_{\varepsilon,\mathcal{T}}(t)\right) \leq \exp\left(c_{5,M} t\, \varepsilon^{-n-7}\right) \Delta(V_0, W_0), \tag{89}$$

where $c_{5,M} = c_5(M+1)^2$ and c_5 was introduced in Proposition 3.10.

Proof. As $\varepsilon \in (0,1)$ and the subdivision $\mathcal{T}$ are fixed, we write $V(t)$ (resp. $W(t)$) for $V_{\varepsilon,\mathcal{T}}(t)$ (resp. $W_{\varepsilon,\mathcal{T}}(t)$) hereafter. From (57) applied with $V = V(t_{i-1}), W = W(t_{i-1})$ and $\Delta t = d_i = t_i - t_{i-1}$, and noting that $\|V(t_{i-1})\|(\mathbb{R}^n) \leq M+1$ and $\|W(t_{i-1})\|(\mathbb{R}^n) \leq M+1$ (see Remark 3.7), we infer that for any $i \in \{1\ldots,m\}$ we have:

$$\Delta\left(V(t_i), W(t_i)\right) \leq (1 + c_{5,M} d_i\, \varepsilon^{-n-7}) \Delta(V(t_{i-1}), W(t_{i-1}))\,.$$

By iteration of the previous inequality for $k \in \{1\ldots,i\}$ and applying the inequality $1+a \leq \exp(a)$ in $\mathbb{R}$, we obtain

$$\Delta\left(V(t_i), W(t_i)\right) \leq \prod_{k=1}^{i} (1 + c_{5,M} d_k\, \varepsilon^{-n-7}) \Delta(V(0), W(0)) \underbrace{\leq}_{\sum_{k=1}^i d_k = t_i} \exp\left(c_{5,M} t_i\, \varepsilon^{-n-7}\right) \Delta(V_0, W_0)\,.$$

Let now $t \in (t_i, t_{i+1}]$ and apply once again Proposition 3.10 (with $\Delta t = t - t_i$) so that

$$\begin{aligned}
\Delta\left(V(t), W(t)\right) &\leq (1 + c_{5,M}(t-t_i)\, \varepsilon^{-n-7}) \Delta\left(V(t_i), W(t_i)\right) \\
&\leq (1 + c_{5,M}(t-t_i)\varepsilon^{-n-7}) \exp\left(c_{5,M} t_i\, \varepsilon^{-n-7}\right) \Delta(V_0, W_0) \\
&\leq \exp\left(c_{5,M} t\, \varepsilon^{-n-7}\right) \Delta(V_0, W_0)\,,
\end{aligned}$$

thus ending the proof of the stability of the time-discrete approximate mean curvature flow with respect to initial data. □

Remark 3.12 (Analogy with ODE discretization)**.** The construction of the time-discrete approximate mean curvature flow defined in this paper can be compared to the discretization of the classical Cauchy problem in $[0,T]$:

$$\begin{cases} y'(t) = f(t,y), \\ y(0) = y_0. \end{cases}$$

It is known that a stability constant (with respect to the supremum norm on $[0,T]$) for the explicit Euler discretization of the ODE is $\exp(LT)$ with $L = \max_{t \leq T} \mathrm{Lip}(f(\cdot,t))$ (see for instance [8, Section 2.4]). Comparing with the stability estimate (89) obtained in Proposition 3.11, we observe that $c_{5,M}\varepsilon^{-n-7}$ is indeed a bound on the Lipschitz constant of $V \mapsto H_\varepsilon(\cdot, V)$ when $V_d(\mathbb{R}^n)$ is endowed with the bounded Lipschitz distance Δ, and $\mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is endowed with $\|\cdot\|_{\mathrm{C}^1}$, see Lemma 3.9.

3.5. **Stability with respect to time subdivision.** In this section, we investigate the robustness of the time-discrete approximate mean curvature flow (introduced in Definition 3.6) with respect to the choice of the subdivision $\mathcal{T}$. It is a natural property to expect for a numerical scheme and it is furthermore crucial in order to take the limit "$\delta(\mathcal{T}) \to 0$" and obtain a well-defined "time-continuous" approximate mean curvature flow as subsequently done in Theorem 4.1. We establish in Proposition 3.13 that time-discrete approximate mean curvature flows are stable with respect to subdivisions. The proof of Proposition 3.13 is split into several steps: the section starts with two lemmas (Lemma 3.14 and 3.15) aiming to compare two flows corresponding to a fine

subdivision and the trivial subdivision of small time interval $[0,\delta]$. Then, in Lemma 3.17 we extend the comparison to the case of two nested subdivisions of the interval $[0,1]$ from which Proposition 3.13 can be inferred straightforwardly.

Proposition 3.13 (Stability with respect to time subdivision). *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ with $\|V_0\|(\mathbb{R}^n) \leq M$, let $\mathcal{T}_1 = \{t_i\}_{i=0}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=0}^{m'}$ be two subdivisions of $[0,1]$ satisfying (49). Let $V_{\varepsilon,\mathcal{T}_1}(t)$ (resp.$V_{\varepsilon,\mathcal{T}_2}(t)$) be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) starting from V_0. We set:*

$$\delta = \max\big\{\delta(\mathcal{T}_1), \delta(\mathcal{T}_2)\big\}.$$

Then, for all $t \in [0,1]$, one has:

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \leq c_{7,M}\, t\, \delta\, \varepsilon^{-n-11} \exp\big(c_{5,M}\, t\, \varepsilon^{-n-7}\big).$$

where $c_{7,M} = c_7(M+1)^5$, $c_{5,M} = c_5(M+1)^2$ and c_7, c_5 are constants depending only on n introduced in Proposition 3.11 and Lemma 3.15.

Before proving Proposition 3.13, we shall introduce some preliminary lemmas.

We first estimate in the following lemma how far the push-forward operation is from satisfying the semigroup property. In practice, we measure how far apart are two time-discrete approximate mean curvature flows constructed with respect to two subdivisions including the trivial subdivision.

Lemma 3.14. *Let $\varepsilon, \delta \in (0,1)$ and $M \geq 1$ such that $c_3(M+1)^3\delta\varepsilon^8 \leq 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ satisfy $\|V_0\|(\mathbb{R}^n) \leq M$. Consider $\mathcal{T} = \{t_i\}_{i=0}^m$ a given subdivision of $[0,\delta]$ and $\mathcal{T}'$ the trivial subdivision of $[0,\delta]$. For $i \in \{1,\dots m\}$, we introduce*

$$d_i = t_i - t_{i-1} \quad \text{and} \quad \tilde{f}_i = (\mathrm{id} + d_i\, h_\varepsilon(\cdot, V_0)) \ .$$

We then consider two different flows:

- $\big(V_{\varepsilon,\mathcal{T}'}(t_i)\big)_{i=0\dots m}$ *where $V_{\varepsilon,\mathcal{T}'}$ is the time-discrete approximate mean curvature flow of V_0 with respect to $\mathcal{T}'$ according to Definition 3.6,*
- $\Big(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i)\Big)_{i=0\dots m}$ *is defined as follows:*

$$\begin{cases} \widetilde{V}_{\varepsilon,\mathcal{T}}(0) := V_0 \\ \widetilde{V}_{\varepsilon,\mathcal{T}}(t_i) := (\tilde{f}_i)_{\#}\widetilde{V}_{\varepsilon,\mathcal{T}}(t_{i-1}) \quad \forall i \in \{1,\dots m\}. \end{cases} \tag{90}$$

Then, there exists $c_6 > 0$ that depends only on n and such that

$$\Delta\left(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i), V_{\varepsilon,\mathcal{T}'}(t_i)\right) \leq c_6 M^3 t_i^2 \varepsilon^{-10} \quad \forall\ i \in \{0,\dots m\}\ .$$

It is important to note that, using the previous notation with $\Delta t = d_i$, we have $\tilde{f}_i = f_{\varepsilon,V_0}$, while the definition of time-discrete approximate mean curvature flow would involve $f_{\varepsilon,V_{\varepsilon,\mathcal{T}}(t_{i-1})}$ instead. The velocity h_ε is taken with respect to the initial varifold all along the subdivision when defining $\tilde{V}_{\varepsilon,\mathcal{T}}$.

Proof. As previously, the constant $c_6 > 0$ is adapted throughout the proof, provided that it depends only on n. We introduce the following notations: $g_0 = \tilde{g}_0 = f_0 = \tilde{f}_0 = \mathrm{id}$ and

$$\forall i \in \{1,\dots m\} \quad g_i = \mathrm{id} + t_i h_\varepsilon(\cdot, V_0) \quad \text{and} \quad \tilde{g}_i = \tilde{f}_i \circ \cdots \circ \tilde{f}_1\ .$$

We first prove that for all $i \in \{1,\dots,m\}$,

$$\|\tilde{g}_i - g_i\|_{\mathrm{C}^1} \leq c_6\, M^2 t_i^2 \varepsilon^{-10}\ . \tag{91}$$

Indeed, let $i \in \{1, \dots m\}$ and $x \in \mathbb{R}^n$, we have by definition

$$\tilde{g}_i(x) = \tilde{f}_i \circ \ldots \circ \tilde{f}_1(x) = \tilde{f}_i\left(\tilde{g}_{i-1}(x)\right) = \tilde{g}_{i-1}(x) + d_i\, h_\varepsilon(\tilde{g}_{i-1}(x), V_0) = x + \sum_{k=1}^{i} d_k h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \,,$$

$$\text{hence} \quad |\tilde{g}_i(x) - x| \le \sum_{k=1}^{i} d_k \left|h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right)\right| \le t_i \left\|h_\varepsilon(\cdot, V_0)\right\|_\infty \,, \tag{92}$$

$$g_i(x) = x + t_i\, h_\varepsilon(x, V_0) = x + \sum_{k=1}^{i} d_k\, h_\varepsilon\left(x, V_0\right) \,,$$

and applying the mean value theorem we infer

$$\begin{aligned}
|\tilde{g}_i(x) - g_i(x)| &\le \sum_{k=1}^{i} d_k \left|h_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) - h_\varepsilon\left(x, V_0\right)\right| \le \sum_{k=1}^{i} d_k \|Dh_\varepsilon(\cdot, V_0)\|_\infty \left|\tilde{g}_{k-1}(x) - x\right| \\
&\le t_i\, t_{i-1} \|Dh_\varepsilon(\cdot, V_0)\|_\infty \|h_\varepsilon(\cdot, V_0)\|_\infty \\
&\le c_1^2\, t_i^2 M^2 \varepsilon^{-6} \quad \text{thanks to Proposition 3.3.}
\end{aligned} \tag{93}$$

We proceed similarly to bound the derivatives but we have to handle the term $D\tilde{g}_i(x)$ arising from the chain rule applied to $x \mapsto h_\varepsilon(\tilde{g}_i(x), V_0)$: recalling that $\tilde{g}_i = (\mathrm{id} + d_i\, h_\varepsilon(\cdot, V_0)) \circ \tilde{g}_{i-1}$, we infer for $x \in \mathbb{R}^n$ and $i \in \{1, \dots, m\}$,

$$\begin{aligned}
\|D\tilde{g}_i(x)\| &\le \|\mathrm{id} + d_i\, Dh_\varepsilon(\tilde{g}_i(x), V_0)\| \|D\tilde{g}_{i-1}(x)\| \le (1 + d_i\, c_1 M \varepsilon^{-4}) \|D\tilde{g}_{i-1}(x)\| \text{ using (31)} \\
&\le \prod_{k=1}^{i} \left(1 + d_k\, c_1 M \varepsilon^{-4}\right) \|D\tilde{g}_0(x)\| \le \prod_{k=1}^{i} \exp\left(d_k\, c_1 M \varepsilon^{-4}\right) \\
&\le \exp\left(t_i c_1 M \varepsilon^{-4}\right) \le 2
\end{aligned} \tag{94}$$

where we used $\tilde{g}_0 = \mathrm{id}$, $1 + s \le \exp(s)$ for $s \in \mathbb{R}$ and

$$\exp\left(t_i c_1 M \varepsilon^{-4}\right) \le \exp\left(\delta c_3 (M+1)^3 \varepsilon^{-8}\right) \le e \le 3.$$

We can now expand $D\tilde{g}_i$ and Dg_i as we did previously for $\tilde{g}_i$ and g_i:

$$D\tilde{g}_i(x) = x + \sum_{k=1}^{i} d_k\, Dh_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \circ D\tilde{g}_{k-1}(x) \,,$$

so that (31) and (94) imply

$$\|D\tilde{g}_i(x) - \mathrm{id}\| \le \sum_{k=1}^{i} d_k \left\|Dh_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right)\right\| \left\|D\tilde{g}_{k-1}(x)\right\| \le 3 c_1 t_i M \varepsilon^{-4} \,, \tag{95}$$

and

$$Dg_i(x) = \mathrm{id} + t_i\, Dh_\varepsilon(x, V_0) = \mathrm{id} + \sum_{k=1}^{i} d_k\, Dh_\varepsilon\left(x, V_0\right).$$

We can then apply the mean value theorem to $Dh_\varepsilon(\cdot, V_0)$, together with (92), (95) and Proposition 3.3 to infer

$$\|D\tilde{g}_i(x) - Dg_i(x)\| \le \sum_{k=1}^{i} d_k \left\|Dh_\varepsilon\left(\tilde{g}_{k-1}(x), V_0\right) \circ D\tilde{g}_{k-1}(x) - Dh_\varepsilon\left(x, V_0\right)\right\|$$

$$\leq \sum_{k=1}^{i} d_k \|Dh_\varepsilon(\cdot, V_0)\|_\infty \|D\tilde{g}_{k-1}(x) - \mathrm{id}\| + \|D^2 h_\varepsilon(\cdot, V_0)\|_\infty |\tilde{g}_{k-1}(x) - x|$$
$$\leq 3t_i\, t_{i-1}(c_1 M\varepsilon^{-4})^2 + c_1 M\varepsilon^{-4} t_i t_{i-1} c_1 M\varepsilon^{-6}$$
$$\leq 4c_1^2 M^2\, t_i^2\, \varepsilon^{-10}\ . \tag{96}$$

We conclude the proof of (91) thanks to (93) and (96).
By definition of $\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i)$ (see (90)) and Lemma 2.10 we have

$$\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i) = (\tilde{f}_i)_\# \left((\tilde{f}_{i-1})_\# \cdots_\# \left((\tilde{f}_1)_\# V_0 \right) \right) = (\tilde{g}_i)_\# V_0\ ,$$

and thanks to (95) we can apply Proposition 3.10 with $V = V_0$, $\|V_0\|(\mathbb{R}^n) \leq M$ $f = g_i$ and $g = \tilde{g}_i$ so that by (56)

$$\Delta\left(\widetilde{V}_{\varepsilon,\mathcal{T}}(t_i), V_{\varepsilon,\mathcal{T}'}(t_i)\right) = \Delta\left((\tilde{g}_i)_\# V_0, (g_i)_\# V_0\right) \leq M\left(c_5 \|g_i - \tilde{g}_i\|_{\mathrm{C}^1} + \|J.g_i - J.\tilde{g}_i\|_\infty\right) \tag{97}$$

and by (54)

$$\|J.g_i - J.\tilde{g}_i\| \leq c_6\, M^2 t_i^2 \varepsilon^{-10}\ . \tag{98}$$

We conclude the proof of Lemma 3.14 thanks to (97), (91) and (98). □

In the following lemma we compare the time-discrete approximate mean curvature flows of two given varifolds on a small time interval, one of them being defined with respect to the trivial subdivision and the other one being defined with respect to a finer subdivision of the time interval. The proof relies on Lemma 3.14.

Lemma 3.15. *Let $\varepsilon, \delta \in (0,1)$ and $M \geq 1$. Let V_0 and W_0 in $V_d(\mathbb{R}^n)$ be two varifolds satisfying $\|V_0\|(\mathbb{R}^n) \leq M$ and $\|W_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T} = \{t_i\}_{i=1}^m$ be a subdivision of $[0,\delta]$ and denote by*

- *$V_{\varepsilon,\mathcal{T}'}(t)$ the time-discrete approximate mean curvature flow of V_0 w.r.t. the trivial subdivision $\mathcal{T}' = \{0,\delta\}$ of $[0,\delta]$,*
- *$W_{\varepsilon,\mathcal{T}}(t)$ the time-discrete approximate mean curvature flow of W_0 w.r.t. $\mathcal{T}$.*

If

$$c_3\delta(M+1)^3 < \varepsilon^8$$

then, for any $i \in \{0, 1 \dots m\}$:

$$\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) \leq \left(\Delta(V_0, W_0) + c_{7,M}\, t_i^2\, \varepsilon^{-n-11}\right) \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right) \tag{99}$$

where $c_7 > 0$ is a constant that depends only on n and $c_{7,M} = c_7(M+1)^5$.

Remark 3.16. A particular case of (99) is when $i = m$:

$$\Delta(V(\delta), W(\delta)) \leq \left(\Delta(V_0, W_0) + c_{7,M}\, \delta^2\, \varepsilon^{-n-11}\right) \exp\left(c_{5,M}\, \delta\, \varepsilon^{-n-7}\right). \tag{100}$$

The last inequality will be useful in the sequel.

Proof. As previously, the constant $c_7 > 0$ is adapted throughout the proof, provided that it depends only on n. The assumption $c_3\delta(M+1)^3 < \varepsilon^8$ implies that the involved subdivisions ($\mathcal{T}'$ and $\mathcal{T}$) satisfy (49), which allows to define time-discrete approximate mean curvature flows for both subdivisions. For every $i \in \{1 \dots m\}$, set:

$$d_i = t_i - t_{i-1}$$

and define $\tilde{V}_{\varepsilon,\mathcal{T}}(t)$ the auxiliary flow as in (90). We have by Lemma 3.14

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) &\leq \Delta(V_{\varepsilon,\mathcal{T}'}(t_i), \tilde{V}_{\varepsilon,\mathcal{T}}(t_i)) + \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) \\ &\leq c_6 M^3 t_i^2 \varepsilon^{-10} + \Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)).\end{aligned} \tag{101}$$

We are left with estimating $\Delta(\tilde{V}_{\varepsilon,\mathcal{T}}(t_i), W_{\varepsilon,\mathcal{T}}(t_i))$. As in Lemma 3.14, we introduce the notations $\tilde{f}_i = \mathrm{id} + d_i\, h_\varepsilon(\cdot, V_0)$ and $g_i = \mathrm{id} + t_i\, h_\varepsilon(\cdot, V_0)$, and we also use the following shortened notations up to the end of the current proof: for $i \in \{0, 1, \dots, m\}$,

$$W_i = W_{\varepsilon,\mathcal{T}}(t_i) \quad \text{and} \quad \tilde{V}_i = \tilde{V}_{\varepsilon,\mathcal{T}}(t_i) \quad \text{and} \quad V_i' = V_{\varepsilon,\mathcal{T}'}(t_i)$$
$$\text{and then} \quad f_{\varepsilon,W_i} = \mathrm{id} + d_{i+1} h_\varepsilon(\cdot, W_i) \quad \text{and} \quad f_{\varepsilon,\tilde{V}_i} = \mathrm{id} + d_{i+1} h_\varepsilon(\cdot, \tilde{V}_i)$$

so that for $l \in \{1, \dots, m\}$,

$$\tilde{V}_l = (\tilde{f}_l)_\# \tilde{V}_{l-1} \quad \text{and} \quad W_l = (f_{\varepsilon,W_{l-1}})_\# W_{l-1}\,.$$

Coming back to (101) we have by triangular inequality

$$\begin{aligned}\Delta\left(\tilde{V}_{\varepsilon,\mathcal{T}}(t_l), W_{\varepsilon,\mathcal{T}}(t_l)\right) = \Delta\left(\tilde{V}_l, W_l\right) &= \Delta\left((\tilde{f}_l)_\# \tilde{V}_{l-1}, (f_{\varepsilon,W_{l-1}})_\# W_{l-1}\right) \\ &\leq \Delta\left((\tilde{f}_l)_\# \tilde{V}_{l-1}, (f_{\varepsilon,\tilde{V}_{l-1}})_\# \tilde{V}_{l-1}\right) + \Delta\left((f_{\varepsilon,\tilde{V}_{l-1}})_\# \tilde{V}_{l-1}, (f_{\varepsilon,W_{l-1}})_\# W_{l-1}\right)\,.\end{aligned} \tag{102}$$

We can directly handle the second term in the previous inequality thanks to (57) that we apply with $V = \tilde{V}_{l-1}$, $W = W_{l-1}$ and $\Delta t = d_l$ so that

$$\Delta\left((f_{\varepsilon,\tilde{V}_{l-1}})_\# \tilde{V}_{l-1}, (f_{\varepsilon,W_{l-1}})_\# W_{l-1}\right) \leq \left(1 + c_{5,M}\, d_l\, \varepsilon^{-n-7}\right) \Delta\left(\tilde{V}_{l-1}, W_{l-1}\right)\,. \tag{103}$$

We now apply (56) and (55) with $V = \tilde{V}_{l-1}$, $g = \tilde{f}_l$ and $\Delta t = d_l$ so that together with Lemma 3.9 and (50), we can assert that:

$$\begin{aligned}\Delta\left((\tilde{f}_l)_\# \tilde{V}_{l-1}, (f_{\varepsilon,\tilde{V}_{l-1}})_\# \tilde{V}_{l-1}\right) &\leq (M+1)\left(c_5\, d_l\, \|h_\varepsilon(\cdot, V_0) - h_\varepsilon(\cdot, \tilde{V}_{l-1})\|_{C^1} + \|J.\tilde{f}_l - J.f_{\varepsilon,\tilde{V}_{l-1}}\|_\infty\right) \\ &\leq (2c_4+1)c_5\,(M+1)^2\, d_l\, \varepsilon^{-n-7} \Delta(\tilde{V}_{l-1}, V_0)\,.\end{aligned} \tag{104}$$

We recall that $\mathcal{T}'$ being the trivial subdivision, we have $V_{l-1}' = (f_{\varepsilon,V_0})_\# V_0$ with $(f_{\varepsilon,V_0})_\# V_0 = \mathrm{id} + t_{l-1} h_\varepsilon(\cdot, V_0)$ and we can thus apply (56) once more, with $V = V_0$, $g = \mathrm{id}$ and $\Delta t = t_{l-1}$:

$$\begin{aligned}\Delta(V_{l-1}', V_0) &\leq M\,(c_5\, t_{l-1} \|h_\varepsilon(\cdot, V_0)\|_{C^1} + \|J.f_{\varepsilon,V_0} - 1\|_\infty) \\ &\leq c_7\, M^2\, t_{l-1}\, \varepsilon^{-4} \quad \text{thanks to (39) and Proposition 3.3.}\end{aligned}$$

Consequently, by triangular inequality and Lemma 3.14, we obtain (with $c_3 M^3 t_{l-1} \leq \varepsilon^8$)

$$\Delta(\tilde{V}_{l-1}, V_0) \leq \Delta(\tilde{V}_{l-1}, V_{l-1}') + \Delta(V_{l-1}', V_0) \leq c_6\, M^3 t_{l-1}^2 \varepsilon^{-10} + \Delta(V_{l-1}', V_0) \leq c_7\, M^3 t_{l-1} \varepsilon^{-4} \tag{105}$$

Collecting (102), (103) and (104), (105) we obtain the inductive relation:

$$\Delta(\tilde{V}_l, W_l) \leq \underbrace{\left(1 + c_{5,M}\, d_l\, \varepsilon^{-n-7}\right)}_{=a_l} \Delta(\tilde{V}_{l-1}, W_{l-1}) + \underbrace{c_{7,M}\, d_l\, t_{l-1}\, \varepsilon^{-n-11}}_{=b_l}\,. \tag{106}$$

Iterating (106) for $l \in \{1, \dots i\}$, we infer

$$\Delta(\tilde{V}_i, W_i) \leq \Delta(V_0, W_0) \prod_{l=1}^{i} a_l + \sum_{l=0}^{i-1} b_{i-l} \prod_{j=0}^{l-1} a_{i-j}$$

We first note that for $l \leq i$

$$\prod_{j=0}^{l-1} a_{i-j} \leq \prod_{j=0}^{i-1} a_{i-j} \prod_{j=1}^{i} a_j = \prod_{j=1}^{i} \left(1 + c_{5,M}\, d_j \varepsilon^{-n-7}\right) \leq \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right).$$

where we used again that for $x \in \mathbb{R}$, $1 + x \leq \exp(x)$ and $\sum_{j=1}^{i} d_j = t_i$. As we obtain an upper bound that does not depend on l, then

$$\sum_{l=0}^{i-1} b_{i-l} = c_{7,M}\, \varepsilon^{-n-11} \sum_{l=0}^{i-1} d_{i-l} t_{i-l-1} \leq c_{7,M}\, \varepsilon^{-n-11} t_i \sum_{l=0}^{i-1} d_{i-l} = c_{7,M}\, t_i^2\, \varepsilon^{-n-11}\,,$$

yields

$$\Delta(\tilde{V}_l, W_l) \leq c_{7,M}\, t_i^2\, \varepsilon^{-n-11} \left(\Delta(V_0, W_0) + \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right)\right)\,.$$

Coming back to (101) we conclude that $\forall i \in \{0, 1 \dots m\}$,

$$\begin{aligned}\Delta(V_{\varepsilon,\mathcal{T}'}(t_i), W_{\varepsilon,\mathcal{T}}(t_i)) &\leq c_6 M^3 t_i^2 \varepsilon^{-10} + c_{7,M}\, t_i^2\, \varepsilon^{-n-11} \left(\Delta(V_0, W_0) + \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right)\right) \\ &\leq c_{7,M}\, t_i^2\, \varepsilon^{-n-11} \left(\Delta(V_0, W_0) + \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right)\right)\,.\end{aligned}$$

□

In the following lemma, we use 3.15 to show Proposition 3.13 (stability with respect to subdivision) in the special case where the two subdivisions are nested (one included in the other).

Lemma 3.17. *Let $\varepsilon \in (0,1)$, $M \geq 1$. Let $V_0 \in V_d(\mathbb{R}^n)$ satisfy $\|V_0\|(\mathbb{R}^n) \leq M$. Consider $\mathcal{T}_1 = \{t_i\}_{i=1}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ two subdivisions of $[0,1]$ satisfying (49), assume that $\mathcal{T}_1 \subset \mathcal{T}_2$ and set $\delta = \delta(\mathcal{T}_1)$. Then, for all $t \in [0,1]$,*

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \leq c_{7,M}\, t\, \delta\, \varepsilon^{-n-11} \exp\left(c_{5,M}\, t\, \varepsilon^{-n-7}\right)$$

where $V_{\varepsilon,\mathcal{T}_1}(t)$ denotes the time-discrete approximate mean curvature flow of V_0 with respect to $\mathcal{T}_1$ and $V_{\varepsilon,\mathcal{T}_2}(t)$ the time-discrete approximate mean curvature flow of V_0 with respect to $\mathcal{T}_2$.

Proof. We set for $i \in \{1, \dots, m\}$, $d_i = t_i - t_{i-1}$

Step 1: We first compare the flows at time t_i of the subdivision $\mathcal{T}_1$, more precisely, we show that for $i \in \{0, 1 \dots, m\}$,

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i)) \leq c_{7,M}\, t_i\, \delta \varepsilon^{-n-11} \exp\left(c_{5,M}\, t_i\, \varepsilon^{-n-7}\right). \tag{107}$$

Fix $i \in \{0, 1 \dots, m\}$, for any $l \in \{1, \dots, i\}$, we use (100) on the interval $[t_{l-1}, t_l]$ of length $d_l = t_l - t_{l-1}$ with $V_0 = V_{\varepsilon,\mathcal{T}_1}(t_{l-1})$, $W_0 = V_{\varepsilon,\mathcal{T}_2}(t_{l-1})$, $\mathcal{T}'$ being the trivial subdivision of $[t_{l-1}, t_l]$ and $\mathcal{T} = \mathcal{T}_2 \cap [t_{l-1}, t_l]$ to obtain

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t_l), V_{\varepsilon,\mathcal{T}_2}(t_l)) \leq \left[\Delta(V_{\varepsilon,\mathcal{T}_1}(t_{l-1}), V_{\varepsilon,\mathcal{T}_2}(t_{l-1})) + c_{7,M}\, d_l^2\, \varepsilon^{-n-11}\right] \exp\left(c_{5,M} d_l \varepsilon^{-n-7}\right).$$

Noting that $u_l = \exp\left(-c_{5,M}\, t_l\, \varepsilon^{-n-7}\right) \Delta(V_{\varepsilon,\mathcal{T}_1}(t_l), V_{\varepsilon,\mathcal{T}_2}(t_l))$ hence satisfies the inductive relation

$$u_l \leq u_{l-1} + c_{7,M}\, d_l^2\, \varepsilon^{-n-11}$$

and we infer that $u_i \leq u_0 + c_{7,M} \sum_{l=0}^{i} d_l^2\, \varepsilon^{-n-11} \leq u_0 + c_{7,M}\, \delta t_i\, \varepsilon^{-n-11} = c_{7,M}\, \delta t_i\, \varepsilon^{-n-11}$ since $u_0 = \Delta(V_{\varepsilon,\mathcal{T}_1}(0), V_{\varepsilon,\mathcal{T}_2}(0)) = 0$ yielding (107).

Step 2: We now compare the flows at time s_k of the finer subdivision $\mathcal{T}_2$, more precisely, for $k \in \{0, 1 \dots, m'\}$,

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \leq c_{7,M}\, s_k \delta\, \varepsilon^{-n-11} \exp\left(c_{5,M}\, s_k\, \varepsilon^{-n-7}\right). \tag{108}$$

Fix $k \in \{0, 1 \dots, m'\}$ and for $s_k \in \mathcal{T}_2$, let $i \in \{0, 1 \dots m-1\}$ be such that $s_k \in [t_i, t_{i+1}]$. Then, applying Lemma 3.15 on the interval $[t_i, t_{i+1}]$ with $V_0 = V_{\varepsilon,\mathcal{T}_1}(t_i)$, $W_0 = V_{\varepsilon,\mathcal{T}_2}(t_i)$, $\mathcal{T}'$ being the

trivial subdivision of $[t_i, t_{i+1}]$ and $\mathcal{T} = \mathcal{T}_2 \cap [t_i, t_{i+1}]$, we obtain

$$\begin{aligned}
&\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \\
&\quad \leq \left[\Delta(V_{\varepsilon,\mathcal{T}_1}(t_i), V_{\varepsilon,\mathcal{T}_2}(t_i)) + c_{7,M}\,(s_k - t_i)^2\,\varepsilon^{-n-11}\right] \exp\left(c_{5,M}\,(s_k - t_i)\varepsilon^{-n-7}\right) \\
&\quad \leq c_{7,M}\,t_i\,\delta\varepsilon^{-n-11}\exp\left(c_{5,M}\,s_k\,\varepsilon^{-n-7}\right) + c_{7,M}\,(s_k - t_i)^2\,\varepsilon^{-n-11}\exp\left(c_{5,M}\,(s_k - t_i)\,\varepsilon^{-n-7}\right),
\end{aligned}$$

thanks to (107), that leads to (108) noting that $\exp\left(c_{5,M}(s_k - t_i)\varepsilon^{-n-7}\right) \leq \exp\left(c_{5,M}s_k\varepsilon^{-n-7}\right)$ and $t_i\delta + (s_k - t_i)^2 \leq \delta(t_i + s_k - t_i) \leq \delta s_k$.

Step 3: Let $t \in [s_k, s_{k+1}]$ for some $k \in \{0, 1, \ldots, m' - 1\}$, applying Proposition 3.10 with $V = V_{\varepsilon,\mathcal{T}_1}(s_k)$, $W = V_{\varepsilon,\mathcal{T}_2}(s_k)$ and $\Delta t = t - s_k$ we obtain:

$$\begin{aligned}
&\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \\
&\qquad \leq \Delta\Big((\mathrm{id} + (t - s_k)h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}_1}(s_k)))_{\#}\,V_{\varepsilon,\mathcal{T}_1}(s_k), (\mathrm{id} + (t - s_k)h_\varepsilon(\cdot, V_{\varepsilon,\mathcal{T}_2}(s_k)))_{\#}\,V_{\varepsilon,\mathcal{T}_2}(s_k)\Big) \\
&\qquad \leq (1 + (t - s_k)c_{5,M}\varepsilon^{-n-7})\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)) \\
&\qquad \leq \exp\left((t - s_k)c_{5,M}\varepsilon^{-n-7}\right)\Delta(V_{\varepsilon,\mathcal{T}_1}(s_k), V_{\varepsilon,\mathcal{T}_2}(s_k)).
\end{aligned}$$

From (108) we conclude that for all $t \in [0, 1]$.

$$\begin{aligned}
\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) &\leq \exp\left(c_{5,M}\,(t - s_k)\varepsilon^{-n-7}\right)\,c_{7,M}\,s_k\delta\varepsilon^{-n-11}\exp\left(c_{5,M}s_k\varepsilon^{-n-7}\right) \\
&\leq c_{7,M}\,t\,\delta\,\varepsilon^{-n-11}\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right)
\end{aligned}$$

and this ends the proof of Lemma 3.17. □

The proof of Proposition 3.13 is now a straightforward consequence of Lemma 3.17 introducing the union of both subdivisions that is finer than each of them.

Proof of Proposition 3.13. Let $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$ be the union of both subdivisions, so that $\mathcal{T}_1,\ \mathcal{T}_2 \subset \mathcal{T}_3$ and let $V_{\varepsilon,\mathcal{T}_3}(t)$ be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_3$ starting from V_0. By Lemma 3.17 we infer that $\forall t \in [0, 1]$:

$$\begin{aligned}
\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_2}(t)) &\leq \Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_3}(t)) + \Delta(V_{\varepsilon,\mathcal{T}_3}(t), V_{\varepsilon,\mathcal{T}_2}(t)) \\
&\leq 2c_{7,M}\,t\,\delta\,\varepsilon^{-n-11}\exp\left(c_{5,M}\,t,\varepsilon^{-n-7}\right)
\end{aligned}$$

which concludes the proof of Proposition 3.13 (up to doubling $c_7 > 0$). □

We conclude the section with the following corollary. It encompasses the previous results on the stability of the time-discrete approximate mean curvature flow.

Corollary 3.18. *Let $\varepsilon \in (0, 1)$, $M \geq 1$. Let V_0, W_0 be two varifolds in $V_d(\mathbb{R}^n)$ satisfying $\|V_0\|(\mathbb{R}^n) \leq M$ and $\|W_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T}_1 = \{t_i\}_{i=1}^m$ and $\mathcal{T}_2 = \{s_j\}_{j=1}^{m'}$ be two subdivisions of $[0, 1]$ satisfying (49). Let $V_{\varepsilon,\mathcal{T}_1}(t)$ (resp.$W_{\varepsilon,\mathcal{T}_2}(t)$) be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) starting from V_0 (resp. W_0).*
If we set: $\delta = \max\{\delta(\mathcal{T}_1), \delta(\mathcal{T}_2)\}$, we have for all $t \in [0, 1]$,

$$\Delta(V_{\varepsilon,\mathcal{T}_1}(t), W_{\varepsilon,\mathcal{T}_2}(t)) \leq \left[\Delta(V_0, W_0) + c_{7,M}\,t\,\delta\,\varepsilon^{-n-11}\right]\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right).$$

Proof. We start by setting $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$. Let $V_{\varepsilon,\mathcal{T}_3}(t)$ (resp. $W_{\varepsilon,\mathcal{T}_3}(t)$) be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_3$ and starting from V_0 (resp. W_0). By Lemma 3.17 we infer that: for all $t \in [0, 1]$,

$$\max\{\Delta(V_{\varepsilon,\mathcal{T}_1}(t), V_{\varepsilon,\mathcal{T}_3}(t)), \Delta(W_{\varepsilon,\mathcal{T}_2}(t), W_{\varepsilon,\mathcal{T}_3}(t))\} \leq c_{7,M}\,t\,\delta\,\varepsilon^{-n-11}\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right),$$

and, by Proposition 3.11, we obtain $\Delta(V_{\varepsilon,\mathcal{T}_3}(t), W_{\varepsilon,\mathcal{T}_3}(t)) \leq \Delta(V_0, W_0)\exp\left(c_{5,M}\, t\, \varepsilon^{-n-7}\right)$, and we conclude the proof of Corollary 3.18 using the triangular inequality. □

4. Existence, uniqueness and properties of a limit approximate flow

For any given varifold V_0 of finite mass, $\varepsilon \in (0,1)$ and $\mathcal{T}$ a subdivision of $[0,1]$, we constructed a time-discrete approximate mean curvature flow, denoted $(V_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ (see Definition 3.6). The goal of this section is to prove that, as the subdivision is refined—regardless of how the successive finer subdivisions are chosen—there is convergence to a unique limit flow (Theorem 4.1) that we call the approximate mean curvature flow starting from V_0 and that we denote $(V_\varepsilon(t))_{t\in[0,1]}$. We will exhibit some properties of this limit flow, namely, the stability with respect to initial data (Proposition 4.3) and the mass decay (Remark 4.7). In Proposition 4.6, we will prove that $V_\varepsilon(t)$ satisfies a Brakke-type inequality (in reference to inequality (2)) depending on its approximate mean curvature.

4.1. Existence and uniqueness of a limit approximate flow.

In the following theorem, we state that the time-discrete approximate mean curvature flows starting from a given varifold V_0 of compact support converge, as the subdivision time step converges to 0, to a unique limiting flow. The proof is based on the uniform boundedness of the masses (see (50)) and the stability result with respect to the subdivision of Proposition 3.13.

Theorem 4.1 (Convergence). *Let $\varepsilon \in (0,1)$, $M \geq 1$ and let $V_0 \in V_d(\mathbb{R}^n)$ be a varifold of compact support and satisfying $\|V_0\|(\mathbb{R}^n) \leq M$. For each $j \in \mathbb{N}$,*

- *let $\mathcal{T}_j^D = \{k\ 2^{-j}\}_{k=0,1,\dots,2^j}$ be the dyadic subdivision of the interval $[0,1]$ of size $\delta(\mathcal{T}_j^D) = 2^{-j} \xrightarrow[j\to\infty]{} 0$,*
- *let $V_{\varepsilon,\mathcal{T}_j^D}(t)_{t\in[0,1]}$ be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_j^D$ starting from V_0. Note that according to condition (49) in Definition 3.6, such a flow is well-defined for j large enough so that $c_3 2^{-j} \leq (M+1)^{-3}\varepsilon^8$.*

Then,

(i) there exists a family $(V_\varepsilon(t))_{t\in[0,1]}$ in $V_d(\mathbb{R}^n)$ such that for any $t \in [0,1]$:

1. *$\|V_\varepsilon(t)\|(\mathbb{R}^n) \leq M+1$,*
2. *$V_{\varepsilon,\mathcal{T}_j^D}(t) \overset{*}{\rightharpoonup} V_\varepsilon(t)$,*
3. *$\Delta\left(V_{\varepsilon,\mathcal{T}_j^D}(t), V_\varepsilon(t)\right) \to 0$ as $j \to +\infty$.*

(ii) If $(\mathcal{T}_j)_{j\in\mathbb{N}}$ is any other sequence of subdivisions of size $\delta(\mathcal{T}_j)$ tending to 0, then $V_{\varepsilon,\mathcal{T}_j}(t)_{t\in[0,1]}$ (the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_j$ starting from V_0) converges to the same family $(V_\varepsilon(t))_{t\in[0,1]}$ as for the dyadic subdivisions: for any $t \in [0,1]$,

$$V_{\varepsilon,\mathcal{T}_j}(t) \overset{*}{\rightharpoonup} V_\varepsilon(t) \quad \text{and} \quad \Delta\left(V_{\varepsilon,\mathcal{T}_j}(t), V_\varepsilon(t)\right) \to 0 \quad \text{as } j \to +\infty\,.$$

In other words, there exists a unique limit flow $(V_\varepsilon(t))_{t\in[0,1]}$ starting from V_0, that we call the approximate mean curvature flow of V_0.

Proof. Let $\varepsilon \in (0,1)$, $M \geq 1$ and let $V_0 \in V_d(\mathbb{R}^n)$ be a varifold with compact support and satisfying $\|V_0\|(\mathbb{R}^n) \leq M$.
We start with the proof of (i). For $j \in \mathbb{N}$, let $\mathcal{T}_j^D$ and $V_{\varepsilon,\mathcal{T}_j^D}$ be as in the statement of the theorem. Let $t \in [0,1]$. By construction, we know that for any $t \in [0,1]$:

$$\|V_{\varepsilon,\mathcal{T}_j^D}(t)\|(\mathbb{R}^n) \leq M+1.$$

Thus, by Banach-Alaoglu's theorem, there exists a subsequence $a_t(j)$ (depending on t) for which the sequence $V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t)$ converges weakly-* to a certain limit denoted by $V_\varepsilon(t)$. Note that up to this point, such a limit could depend on the extraction a_t and on the specific choice of the dyadic subdivisions $(\mathcal{T}^D_j)_j$. We first show that the whole sequence $V_{\varepsilon,\mathcal{T}^D_j}(t)$ (and not only the extracted one) converges to $V_\varepsilon(t)$ as $j \to \infty$.
As V_0 has compact support, there exists $R_0 > 0$ such that $\operatorname{spt} V_0 \subset B_{R_0}(0) \times G_{d,n}$. Then, thanks to Remark 3.7, all the varifolds we are considering hereafter are supported in the common bounded set $B\left(0, R_0 + c_1(M+1)\varepsilon^{-2}\right) \times G_{d,n}$. Applying Proposition 2.6, we can deduce that,

$$\Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t)) \xrightarrow[j\to\infty]{} 0\,.$$

Note that $a_t(j) \geq j$ and therefore the dyadic subdivision $\mathcal{T}^D_{a_t(j)}$ is finer than $\mathcal{T}^D_j$. For j large enough so that $c_3 2^{-j} \leq (M+1)^{-3}\varepsilon^8$, we can apply Lemma 3.17 with $\mathcal{T}^D_j \subset \mathcal{T}^D_{a_t(j)}$ and obtain

$$\Delta(V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t), V_{\varepsilon,\mathcal{T}^D_j}(t)) \leq c_{7,M}\, t\, \delta_j\, \varepsilon^{-n-11} \exp\left(c_{5,M}\, t\, \varepsilon^{-n-7}\right) \quad \text{with } \delta_j = 2^{-j}\,.$$

This implies

$$\Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}^D_j}(t)) \leq \Delta(V_\varepsilon(t), V_{\mathcal{T}^D_{a_t(j)}}(t)) + \Delta(V_{\varepsilon,\mathcal{T}^D_{a_t(j)}}(t), V_{\varepsilon,\mathcal{T}^D_j}(t)) \xrightarrow[j\to\infty]{} 0.$$

Thus, the full sequence $V_{\varepsilon,\mathcal{T}^D_j}(t)$ converges to $V_\varepsilon(t)$ for each $t \in [0,1]$ in the bounded Lipschitz topology and thus in the weakly-* topology (again thanks to Proposition 2.6).
We now prove (ii). Let $\left(\mathcal{T}_j\right)_{j\in\mathbb{N}}$ be a sequence of subdivisions of size $\delta\left(\mathcal{T}_j\right)$ tending to 0. For j large enough so that $c_3\delta(\mathcal{T}_j) \leq (M+1)^{-3}\varepsilon^8$, let $V_{\varepsilon,\mathcal{T}_j}(t)_{t\in[0,1]}$ be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_j$ starting from V_0. Let $t \in [0,1]$ and set $\tilde{\delta}_j = \max\{\delta(\mathcal{T}_j), \delta(\mathcal{T}^D_j)\}$; we apply Proposition 3.13 and obtain

$$\Delta(V_{\varepsilon,\mathcal{T}^D_j}(t), V_{\varepsilon,\mathcal{T}_j}(t)) \leq c_{7,M}\, t\, \tilde{\delta}_j\, \varepsilon^{-n-11} \exp\left(c_{5,M}\, t\, \varepsilon^{-n-7}\right) \xrightarrow[j\to\infty]{} 0.$$

This implies that for any $t \in [0,1]$, $V_{\varepsilon,\mathcal{T}_j}(t)$ converges to $V_\varepsilon(t)$ both in the bounded Lipschitz topology and thus in the weak-* topology (again thanks to Proposition 2.6).
We conclude that independently of how the time step goes to 0, the limit flow exists and is *unique*, we call it the approximate mean curvature flow and we will denote it by $(V_\varepsilon(t))_t$. □

Given $\varepsilon \in (0,1)$ and a subdivision $\mathcal{T}$ of $[0,1]$, we proposed in Remark 3.8 an alternative definition $V^{pc}_{\varepsilon,\mathcal{T}}$ of time-discrete approximate mean curvature flow: we recall that the difference with $V_{\varepsilon,\mathcal{T}}$ lies in the way the flow is extended from the points $t_0, t_1, \ldots, t_m \in \mathcal{T}$ of the subdivision to any $t \in [0,1]$. While $V_{\varepsilon,\mathcal{T}}$ is defined through a kind of linear interpolation between the flow at time t_i and t_{i+1}, $V^{pc}_{\varepsilon,\mathcal{T}}$ is set to be constant in between such subdivision times. In the following proposition, we derive an error term estimate between both extensions and infer that they lead to the same definition of limit flow $(V_\varepsilon(t))_{t\in[0,1]}$.

Proposition 4.2. *Let $\varepsilon \in (0,1)$. Let $\mathcal{T} = \{t_i\}_{i=0}^m$ be a subdivision of $[0,1]$ satisfying (49). Let $V_0 \in V_d(\mathbb{R}^n)$ of compact support and $V_{\varepsilon,\mathcal{T}}(t)$ the time-discrete approximate mean curvature flow with respect to $\mathcal{T}$ starting from V_0. Let $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ be the associated piecewise constant flow with respect to $\mathcal{T}$ (Remark 3.8). Then,*

$$\Delta(V_{\varepsilon,\mathcal{T}}(t), V^{pc}_{\varepsilon,\mathcal{T}}(t)) \leq c_8\,(M+1)^2\,\delta(\mathcal{T})\,\varepsilon^{-4}, \quad \forall t \in [0,1],$$

where $c_8 > 0$ is a constant that depends only on n. As a consequence, when the step of the subdivision goes to 0, $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ converges to $V_\varepsilon(t)$ (defined in Theorem 4.1): for any $t \in [0,1]$,

$$V^{pc}_{\varepsilon,\mathcal{T}_j}(t) \text{ converges weakly-* to } V_\varepsilon(t) \quad \text{and} \quad \Delta\left(V^{pc}_{\varepsilon,\mathcal{T}_j}(t), V_\varepsilon(t)\right) \to 0 \quad \text{as } j \to +\infty\,. \tag{109}$$

Proof. As ε and $\mathcal{T}$ are fixed, we denote $V_{\varepsilon,\mathcal{T}}(t)$ and $V^{pc}_{\varepsilon,\mathcal{T}}(t)$ by $V(t)$ and $V(t)^{pc}$ throughout the proof. Let $i \in \{0,\dots,m-1\}$ be such that $t \in [t_i, t_{i+1})$. Introducing $f = \mathrm{id} + (t-t_i)h_\varepsilon(\cdot, V(t_i))$ and $g = \mathrm{id}$, we have

$$\Delta(V(t), V^{pc}(t)) = \Delta(V(t), V(t_i)) = \Delta\Big(f_\# V(t_i), g_\# V(t_i))\Big). \tag{110}$$

Using (31) we can check that (noting that $\|V(t_i)\| \le M+1$)

$$\|Df - Dg\|_\infty = (t-t_i)\|Dh_\varepsilon(\cdot, V(t_i))\|_\infty \le 2c_1(t-t_i)(M+1)\varepsilon^{-4} \le 2c_1\,\delta(\mathcal{T})(M+1)\varepsilon^{-4} \le \frac{1}{2}$$

and we then can apply (56) with $V = V(t_i)$ so that

$$\Delta\Big(f_\# V(t_i), g_\# V(t_i)\Big) \le (M+1)\left(c_5(t-t_i)\|h_\varepsilon(\cdot, V(t_i))\|_{C^1} + \|J.f - 1\|_\infty\right). \tag{111}$$

We conclude the proof of Proposition 4.2 thanks to (110), (111), Proposition 3.3 and (39). □

Thereafter $(V_\varepsilon(t))_{t\in[0,1]}$ denotes the approximate mean curvature flow starting from V_0 as defined in Theorem 4.1. We now investigate the properties of this flow, starting with the stability with respect to the initial varifold.

Proposition 4.3. *Let $\varepsilon \in (0,1)$, $M \ge 1$. Let V_0, W_0 be two varifolds in $V_d(\mathbb{R}^n)$ satisfying $\|V_0\|(\mathbb{R}^n) \le M$ and $\|W_0\|(\mathbb{R}^n) \le M$, both compactly supported. Then, for all $t \in [0,1]$,*

$$\Delta(V_\varepsilon(t), W_\varepsilon(t)) \le \Delta(V_0, W_0)\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right),$$

where V_ε (resp. W_ε) is the approximate mean curvature flow starting from V_0 (resp. W_0).

Proof. We fix a sequence of subdivisions $(\mathcal{T}_j)_{j\in\mathbb{N}}$ with time step $\delta_j \to 0$ as $j \to \infty$ (we can take the dyadic subdivisions for instance). Let $\left(V_{\varepsilon,\mathcal{T}_j}(t)\right)_{t\in[0,1]}$ (resp. $\left(W_{\varepsilon,\mathcal{T}_j}(t)\right)_{t\in[0,1]}$) be the time-discrete approximate mean curvature flow with respect to $\mathcal{T}_j$ starting from V_0 (resp. W_0). Let j be large enough so that (49) holds: $c_3\delta_j \le (M+1)^{-3}\varepsilon^8$, then we can simply apply Proposition 3.11 and obtain for all $t \in [0,1]$,

$$\Delta(V_{\varepsilon,\mathcal{T}_j}(t), W_{\varepsilon,\mathcal{T}_j}(t)) \le \Delta(V_0, W_0)\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right),$$

and therefore, by the triangle inequality and Theorem 4.1, letting j tend to ∞, we can conclude that

$$\begin{aligned}\Delta(V_\varepsilon(t), W_\varepsilon(t)) &\le \Delta(V_\varepsilon(t), V_{\varepsilon,\mathcal{T}_j}(t)) + \Delta(V_{\varepsilon,\mathcal{T}_j}(t), W_{\varepsilon,\mathcal{T}_j}(t)) + \Delta(W_{\varepsilon,\mathcal{T}_j}(t), W_\varepsilon(t))\\ &\le \Delta(V_0, W_0)\exp\left(c_{5,M}\,t\,\varepsilon^{-n-7}\right).\end{aligned}$$

□

Remark 4.4. Note that if we reformulate Proposition 4.3 in the case where a compactly supported varifold V_0 is approximated by a sequence of varifolds $W_k \xrightarrow[k\to\infty]{*} V_0$, W_k being successive discretizations of V_0 for instance. Then, considering

- $(V_\varepsilon(t))_{t\in[0,1]}$, the approximate mean curvature flow starting from V_0,
- $\forall k$, $((W_k)_\varepsilon(t))_{t\in[0,1]}$, the approximate mean curvature flow starting from W_k,

we have that

$$\Delta(V_\varepsilon(t), (W_k)_\varepsilon(t)) \le \Delta(V_0, W_k)\exp\left(tc_{5,M}\varepsilon^{-n-7}\right) \xrightarrow[k\to\infty]{} 0\,,$$

provided that $\operatorname{spt} W_k$ are contained in a common compact set and $\sup \|W_k\|(\mathbb{R}^n) < \infty$. In order to state some convergence result, one can choose $\varepsilon_k \to 0$ such that $\Delta(V_0, W_k) \exp\big(t c_{5,M} \varepsilon^{-n-7}\big) \to 0$, however, we would need a convergence property of $V_\varepsilon(t)$ as $\varepsilon \to 0$ to some limit flow $V(t)$. This issue is addressed in Section 5.

4.2. **Equality à la Brakke.** In Proposition 4.6, we show a Brakke-type equality for the approximate flow $(V_\varepsilon(t))_{t\in[0,1]}$ with respect to its approximate mean curvature. The proof consists of taking the limit in inequality (117) which results from the expansion of the push-forward varifold formula (40) and Theorem 4.1. We conclude the section with the decay property of mass $t \mapsto \|V_\varepsilon(t)\|(\mathbb{R}^n)$, which follows directly from (116) and (41).
We first introduce the following lemma on the regularity of the weighted first variation with respect to the varifold.

Lemma 4.5. *Let $\varphi \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, $X \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and let V, $W \in V_d(\mathbb{R}^n)$ be two varifolds of finite mass. Then*

$$|\delta(V,\varphi)(X) - \delta(W,\varphi)(X)| \leq n\|\varphi\|_{\mathrm{C}^2}\|X\|_{\mathrm{C}^2}\Delta(V,W)\,.$$

Proof. Let $\varphi \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}_+)$, $X \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R}^n)$ and set $\Theta(x,S) := \varphi(x)\operatorname{div}_S X(x) + \nabla\varphi(x)\cdot X(x)$, we recall that $\operatorname{div}_S X = \operatorname{tr}(S \circ DX)$. From Definition (12) one has

$$|\delta(V,\varphi)(X) - \delta(W,\varphi)(X)| \leq \max\{\|\Theta\|_\infty, \operatorname{Lip}(\Theta)\}\Delta(V,W). \tag{112}$$

First note that for $A, B \in \mathcal{M}_n$, one has

$$|\operatorname{tr}(A \circ B)| \leq n|A \circ B|_\infty \leq n\|A \circ B\| \leq \|A\|\|B\|\,.$$

In particular, for $x \in \mathbb{R}^n$, $S \in G_{d,n}$, one has

$$|\operatorname{div}_S X(x)| = |\operatorname{tr}(S \circ DX(x))| \leq n\|DX(x)\|. \tag{113}$$

and consequently,

$$\|\Theta\|_\infty \leq n\|\varphi\|_\infty\|DX\|_\infty + \|\nabla\varphi\|_\infty\|X\|_\infty \leq n\|X\|_{\mathrm{C}^1}\|\varphi\|_{\mathrm{C}^1}\,. \tag{114}$$

We are left with estimating $\operatorname{Lip}(\Theta)$.
Let $x, y \in \mathbb{R}^n$ and $(S,T) \in G_{d,n}$, thanks to (113), we have

$$\begin{aligned}
&|\varphi(x)\operatorname{div}_S X(x) - \varphi(y)\operatorname{div}_T X(y)| \\
&\quad\leq |\varphi(x)-\varphi(y)||\operatorname{div}_S X(x)| + |\varphi(y)||\operatorname{div}_S X(x) - \operatorname{div}_S X(y)| + |\varphi(y)||\operatorname{div}_S X(y) - \operatorname{div}_T X(y)| \\
&\leq n\|\nabla\varphi\|_\infty\|DX\|_\infty|x-y| + n\|\varphi\|_\infty\|DX(x) - DX(y)\| + n\|\varphi\|_\infty\|DX\|_\infty\|S-T\| \\
&\leq n\left(\|\nabla\varphi\|_\infty\|DX\|_\infty + \|\varphi\|_\infty\|D^2X\|_\infty\right)|x-y| + n\|\varphi\|_\infty\|DX\|_\infty\|S-T\|\,.
\end{aligned}$$

Furthermore, $\operatorname{Lip}(\nabla\varphi\cdot X) \leq \|\nabla^2\varphi\|_\infty\|X\|_\infty + \|\nabla\varphi\|_\infty\|DX\|_\infty$. Therefore,

$$|\Theta(x,S) - \Theta(t,y)| \leq n\|\varphi\|_{\mathrm{C}^2}\|X\|_{\mathrm{C}^2}\,; \tag{115}$$

We conclude the proof of Lemma 4.5 thanks to (112), (114) and (115). □

Proposition 4.6. *Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V_0 in $V_d(\mathbb{R}^n)$ be a varifold with compact support such that $\|V_0\|(\mathbb{R}^n) \leq M$. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate mean curvature flow starting from V_0. For any $\varphi \in \mathrm{C}^1([0,1]\times\mathbb{R}^n, \mathbb{R}_+)$ and $0 \leq a \leq b \leq 1$ we have*

$$\|V_\varepsilon(b)\|(\varphi(\cdot,b)) - \|V_\varepsilon(a)\|(\varphi(\cdot,a)) = \int_a^b \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt + \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\,dt. \tag{116}$$

Proof. Let $\varepsilon \in (0,1)$ and $M \geq 1$. Let V_0 in $V_d(\mathbb{R}^n)$ be a varifold with compact support such that $\|V_0\|(\mathbb{R}^n) \leq M$. Let $\mathcal{T} = \{t_i\}_{i=1}^m$ be a uniform subdivision of $[0,1]$ of time step $\Delta t = \frac{1}{m}$ satisfying (49). Let $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ be the piecewise constant approximate mean curvature flow with respect to to $\mathcal{T}$ starting from V_0. We first prove that (116) holds for $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ up to an error term of order Δt. More precisely, we prove in Steps 1 and 2 that $\exists C > 0$ (only depending on n and M) such that for any $\varphi \in \mathrm{C}^2([0,1]\times\mathbb{R}^n, \mathbb{R}_+)$ and $0 \leq a \leq b \leq 1$ we have

$$\left| \|V^{pc}_{\varepsilon,\mathcal{T}}(b)\|(\varphi(\cdot,b)) - \|V^{pc}_{\varepsilon,\mathcal{T}}(a)\|(\varphi(\cdot,a)) - \int_a^b \delta(V^{pc}_{\varepsilon,\mathcal{T}}(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V^{pc}_{\varepsilon,\mathcal{T}}(t)))\,dt \right.$$
$$\left. - \int_a^b \int_{\mathbb{R}^n} \partial_t\varphi(\cdot,t)\,d\|V^{pc}_{\varepsilon,\mathcal{T}}(t)\|dt \right| \leq C\|\varphi\|_{\mathrm{C}^2}\Delta t\varepsilon^{-8}\,. \quad (117)$$

In Step 3, recalling that $(V^{pc}_{\varepsilon,\mathcal{T}}(t))_{t\in[0,1]}$ converges to $(V_\varepsilon(t))_{t\in[0,1]}$ when considering subdivisions $\mathcal{T}$ whose size tends to 0, we take the limit in (117) and establish (116) for φ of regularity C^2. We conclude the proof of Proposition 4.6 by density of $\mathrm{C}^2([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ in $\mathrm{C}^1([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ in Step 4. Remark 3.7 states that $\bigcup_{t\in[0,1]} \operatorname{spt} V^{pc}_{\varepsilon,\mathcal{T}}(t)$ is contained in a compact set $K = K_\varepsilon$ that does not depend on the subdivision $\mathcal{T}$, hence $\bigcup_{t\in[0,1]} \operatorname{spt} V_\varepsilon(t)$ is contained in K as well. In the proof, the C^k-norms ($k \in \{1,2\}$) of the test functions ψ and φ are implicitly taken with respect to the set K (in particular they are finite). In both Steps 1 and 2, $\mathcal{T}$ and ε are fixed and we denote for simplicity $V(t) := V^{pc}_{\varepsilon,\mathcal{T}}(t)$. Throughout the proof, $C > 0$ is a generic constant that depends only on n and M and may change from line to line.

Step 1: We prove the inequality (117) for $a, b \in \mathcal{T}$: let $\ell \in \{0,1,\dots,m-1\}$ and $\varphi \in \mathrm{C}^2([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$.

We can apply (40) in Proposition 3.4 to a spatial test function $\psi \in \mathrm{C}^2(\mathbb{R}^n,\mathbb{R}_+)$. We recall that the piecewise constant mean curvature flow coincides with the time discrete approximate mean curvature flow at the points of the subdivision and furthermore, $f_\# V(t_\ell) = V(t_{\ell+1})$ for $f = \mathrm{id} + \Delta t h_\varepsilon(\cdot, V(t_\ell))$ as in (40). Therefore,

$$\left| \|V(t_{\ell+1})\|(\psi) - \|V(t_\ell)\|(\psi) - \Delta t\ \delta(V(t_\ell),\psi)(h_\varepsilon(\cdot,V(t_\ell))) \right| \leq c_3(M+1)^3\|\psi\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8}$$
$$\leq C\|\psi\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8}\,.$$

We now recall that $V(t)$ is piecewise constant and thus, for all $t \in (t_\ell, t_{\ell+1})$, $V(t) = V(t_\ell)$ and

$$\int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\psi)(h_\varepsilon(\cdot,V(t)))\,dt = \Delta t\,\delta(V(t_\ell),\psi)(h_\varepsilon(\cdot,V(t_\ell)))$$

so that taking $\psi = \varphi(\cdot,t_{\ell+1})$, we obtain

$$\left| \|V(t_{\ell+1})\|(\varphi(\cdot,t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot,t_{\ell+1})) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t),\varphi(\cdot,t_{\ell+1}))(h_\varepsilon(\cdot,V(t)))\,dt \right|$$
$$\leq C\|\varphi(\cdot,t_{\ell+1})\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8} \leq C\|\varphi\|_{\mathrm{C}^2}(\Delta t)^2\varepsilon^{-8}\,. \quad (118)$$

Applying the mean value theorem to φ and $\nabla\varphi$:

$$\int_{t_\ell}^{t_{\ell+1}} \|\varphi(\cdot,t) - \varphi(\cdot,t_{\ell+1})\|_{\mathrm{C}^1}\,dt \leq \int_{t_\ell}^{t_{\ell+1}} \|\varphi\|_{\mathrm{C}^2}|t - t_{\ell+1}|\,dt \leq \|\varphi\|_{\mathrm{C}^2}(\Delta t)^2\,,$$

and then using Remark 2.13 and Proposition 3.3, we have

$$\begin{aligned}
&\Big| \int_{t_\ell}^{t_{\ell+1}} \delta(V(t), \varphi(\cdot, t_{\ell+1}))(h_\varepsilon(\cdot, V(t)))\, dt - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t), \varphi(\cdot, t))(h_\varepsilon(\cdot, V(t)))\, dt \Big| \\
\leq &\int_{t_\ell}^{t_{\ell+1}} |\delta(V(t), \varphi(\cdot, t) - \varphi(\cdot, t_\ell))(h_\varepsilon(\cdot, V(t)))|\, dt \\
\leq &\int_{t_\ell}^{t_{\ell+1}} nM \|h_\varepsilon(\cdot, V(t))\|_{\mathrm{C}^1} \|\varphi(\cdot, t) - \varphi(\cdot, t_\ell)\|_{\mathrm{C}^1}\, dt \\
\leq &C\|\varphi\|_{\mathrm{C}^2} (\Delta t)^2 \varepsilon^{-4}\,. \qquad (119)
\end{aligned}$$

Writing for $x \in \mathbb{R}^n$, $\varphi(x, t_{\ell+1}) - \varphi(t, x_\ell) = \int_{t_\ell}^{t_{\ell+1}} \partial_t \varphi(t, x)\, dt$ and integrating with respect to $\|V(t_\ell)\| = \|V(t)\|$ for all $t \in (t_\ell, t_{\ell+1})$,

$$\begin{aligned}
\|V(t_\ell)\|(\varphi(\cdot, t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot, t_\ell)) &= \int_{x\in\mathbb{R}^n} \int_{t_\ell}^{t_{\ell+1}} \partial_t \varphi(x, t)\, dt\, d\|V(t_\ell)\|(x) \\
&= \int_{t_\ell}^{t_{\ell+1}} \int_{x\in\mathbb{R}^n} \partial_t \varphi(x, t)\, d\|V(t_\ell)\|(x)\, dt = \int_{t_\ell}^{t_{\ell+1}} \partial_t \varphi(\cdot, t)\, d\|V(t)\|\, dt\,. \qquad (120)
\end{aligned}$$

We can now combine (118), (119) and (120) to obtain that for $\ell \in \{0, 1, \dots, m-1\}$,

$$\begin{aligned}
&\Big| \|V(t_{\ell+1})\|(\varphi(\cdot, t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot, t_\ell)) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t), \varphi(\cdot, t))(h_\varepsilon(\cdot, V(t)))\, dt \\
&\qquad - \int_{t_\ell}^{t_{\ell+1}} \|V(t)\|(\partial_t \varphi(\cdot, t))\, dt \Big| \\
= &\Big| \|V(t_{\ell+1})\|(\varphi(\cdot, t_{\ell+1})) - \|V(t_\ell)\|(\varphi(\cdot, t_{\ell+1})) - \int_{t_\ell}^{t_{\ell+1}} \delta(V(t), \varphi(\cdot, t))(h_\varepsilon(\cdot, V(t)))\, dt \Big| \\
&\leq C\|\varphi\|_{\mathrm{C}^2} (\Delta t)^2\, \varepsilon^{-8}\,,
\end{aligned}$$

which implies (117) for $a = t_\ell$ and $b = t_{\ell+1}$ since $\Delta t \leq 1$. Let now $a = t_p \leq t_q = b$, note that if $p = q$, (117) is trivial and otherwise, summing up the previous estimates for $\ell \in \{p, \dots, q-1\}$ and using $(q-p)\Delta t = t_q - t_p \leq 1$ leads to the inequality (117), which concludes the proof of Step 1 (case where $a, b \in \mathcal{T}$).

Step 2: We now recover the approximate Brakke-type equality (117) for any arbitrary a, b. Let $0 \leq a < b \leq 1$ and $\varphi \in \mathrm{C}^2([0,1] \times \mathbb{R}^n, \mathbb{R}_+)$. Let the points $t_p, t_q \in \mathcal{T}$ be such that $t_p \leq a < t_{p+1}$ and $t_q \leq b < t_{q+1}$. We then have $|a - t_p| < \Delta t$ and $|t_q - b| < \Delta t$, and recalling that $V(t)$ is piecewise constant on intervals of the form $[t_\ell, t_{\ell+1})$, we also have $V(a) = V(t_p)$ and $V(b) = V(t_q)$ so that

$$\begin{aligned}
&\Big| \|V(b)\|(\varphi(\cdot, b)) - \|V(a)\|(\varphi(\cdot, a)) - \|V(t_q)\|(\varphi(\cdot, t_q)) + \|V(t_p)\|(\varphi(\cdot, t_p)) \Big| \\
&\qquad = \Big| \|V(t_q)\|(\varphi(\cdot, b) - \varphi(\cdot, t_q)) - \|V(t_p)\|(\varphi(\cdot, a) - \varphi(\cdot, t_p)) \Big| \\
&\qquad \leq 2(M+1)\|\varphi\|_{\mathrm{C}^1} \Delta t \qquad (121)
\end{aligned}$$

thanks to the mean value theorem applied to φ.

Furthermore, for all $t \in [0,1]$, using Remark 2.13 and Proposition 3.3, we have

$$|\delta(V(t), \varphi(\cdot, t))(h_\varepsilon(\cdot, V(t)))| \leq n\|h_\varepsilon(\cdot, V(t)))\|_{\mathrm{C}^1}(M+1)\|\varphi(\cdot, t)\|_{\mathrm{C}^1} \leq nc_1(M+1)^2\|\varphi\|_{\mathrm{C}^1}\, \varepsilon^{-4}$$

and therefore

$$\left|\int_a^b \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt - \int_{t_p}^{t_q} \delta(V(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V(t)))\,dt\right|$$
$$\leq \underbrace{(|t_p-a|+|t_q-b|)}_{\leq 2\Delta t} \sup_{t\in[0,1]} |\delta(V(t),\varphi)(h_\varepsilon(\cdot,V(t)))| \leq C\|\varphi\|_{\mathrm{C}^1}\Delta t\,\varepsilon^{-4}\,. \quad (122)$$

We are left with estimating

$$\left|\int_a^b \|V(t)\|(\partial_t\varphi(\cdot,t))\,dt - \int_{t_p}^{t_q} \|V(t)\|(\partial_t\varphi(\cdot,t))\,dt\right| \leq 2(M+1)\|\varphi\|_{\mathrm{C}^1}\Delta t\,. \quad (123)$$

We can complete the proof of Step 2 and establish (117) by combining (121) and (122) and (123) together with Step 1.

Step 3: We show (116) restricted to C^2 test functions.

We first recall that the approximate mean curvature flow $(V_\varepsilon(t))_{t\in[0,1]}$ starting from V_0 can be obtained as the limit flow ($j\to\infty$) of any time-discrete approximate mean curvature flow $(V_{\varepsilon,\mathcal{T}_j}(t))_t$ for subdivisions $\mathcal{T}_j$ of size $\delta(\mathcal{T}_j)$ tending to 0, as stated in Theorem 4.1. We can thus choose a sequence of uniform subdivisions $\mathcal{T}_j$ of size $\Delta t_j := \delta(\mathcal{T}_j) \xrightarrow[j\to\infty]{} 0$, and we fix the subdivisions $\mathcal{T}_j = \{t_{\ell,j}\}_{\ell=0,\dots,m_j}$ hereafter, we will write t_ℓ instead of $t_{\ell,j}$ in the proof in order to lighten notations. We additionally recall that the piecewise constant flow $(V^{pc}_{\varepsilon,\mathcal{T}_j}(t))_t$ introduced in Remark 3.8 converges as well to $(V_\varepsilon(t))_t$ thanks to Proposition 4.2 and as it is more convenient in this proof, we work with $(V^{pc}_{\varepsilon,\mathcal{T}_j}(t))_t$. For the sake of lightening the notation we will denote $V_j(t) := V^{pc}_{\varepsilon,\mathcal{T}_j}(t)$ hereafter.

We carry on with the proof of Step 3, let $\varphi\in\mathrm{C}^2([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ and $0\leq a<b\leq 1$, from (117) we have for any $j\in\mathbb{N}$,

$$\left|\|V_j(b)\|(\varphi(\cdot,b)) - \|V_j(a)\|(\varphi(\cdot,a)) - \int_a^b \delta(V_j(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_j(t)))\,dt\right.$$
$$\left. - \int_a^b \|V_j(t)\|(\partial_t\varphi(\cdot,t))\,dt\right| \leq C\|\varphi\|_{\mathrm{C}^2}\Delta t_j\,\varepsilon^{-8}\,. \quad (124)$$

When j goes to ∞ we know thanks to (109) in Proposition 4.2 that for all $t\in[0,1]$,

$$V_j(t) \text{ converges weakly-* to } V_\varepsilon(t) \quad \text{and} \quad \Delta(V_j(t),V_\varepsilon(t)) \xrightarrow[j\to\infty]{} 0\,.$$

As a first consequence, we obtain that

$$\|V_j(b)\|(\varphi(\cdot,b)) \xrightarrow[j\to\infty]{} \|V_\varepsilon(b)\|(\varphi(\cdot,b)) \quad \text{and} \quad \|V_j(a)\|(\varphi(\cdot,a)) \xrightarrow[j\to\infty]{} \|V_\varepsilon(a)\|(\varphi(\cdot,a))\,. \quad (125)$$

We recall that $\|V_\varepsilon(t)\|(\mathbb{R}^n)\leq M+1$ and thanks to Proposition 3.3 and Lemma 3.9, we have for all $t\in[0,1]$,

$\|h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^2}\leq 3c_1(M+1)\varepsilon^{-6}$ and $\|h_\varepsilon(\cdot,V_j(t))-h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^1}\leq 2c_4(M+1)\varepsilon^{-n-7}\Delta(V_j(t),V_\varepsilon(t))$

and therefore, applying Remark 2.13, Lemma 4.5 and Lemma 3.9 (with $V=V_j(t)$ and $W=V_\varepsilon(t)$) we infer

$$\begin{aligned}&\left|\delta(V_j(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_j(t))) - \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\right|\\ \leq& n(M+1)\|\varphi\|_{\mathrm{C}^1}\|h_\varepsilon(\cdot,V_j(t))-h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^1} + n\|\varphi\|_{\mathrm{C}^1}\|h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^2}\Delta(V_j(t),V_\varepsilon(t))\\ \leq& C\|\varphi\|_{\mathrm{C}^1}\varepsilon^{-n-7}\Delta(V_j(t),V_\varepsilon(t))\,.\end{aligned}$$

Integrating the previous inequality, we obtain by dominated convergence, noting that for all t, $\Delta(V_j(t), V_\varepsilon(t)) \leq 2(M+1)$:

$$\left| \int_a^b \delta(V_j(t), \varphi(\cdot,t))(h_\varepsilon(\cdot, V_j(t)))\, dt - \int_a^b \delta(V_\varepsilon(t), \varphi(\cdot,t))(h_\varepsilon(\cdot, V_\varepsilon(t)))\, dt \right| \xrightarrow[j\to\infty]{} 0\,. \tag{126}$$

It remains to let j tend to ∞ in the following term

$$\lim_{j\to\infty} \int_a^b \|V_j(t)\|(\partial_t\varphi(\cdot,t))\, dt = \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\, dt \tag{127}$$

where the convergence holds by dominated convergence: for each $t \in [0,1]$ the weakly-* convergence of $V_j(t)$ to $V_\varepsilon(t)$ implies that $\lim_{j\to\infty} \|V_j(t)\|(\partial_t\varphi(\cdot,t)) = \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))$ and the integrands are uniformly bounded by the constant $\|\varphi\|_{\mathrm{C}^1}(M+1)$.
We can eventually let j tend to $+\infty$ in (124) and conclude the proof of Step 3 (i.e. (116) for all C^2 test function φ) combining the convergence of the 3 terms given by (125), (126) and (127) in the l.h.s. while the r.h.s. tends to 0.
Step 4: It remains to check that we can pass from C^2 to C^1 test functions φ in (116) to conclude the proof of Proposition 4.6.
Let $\varphi \in \mathrm{C}^1([0,1]\times\mathbb{R}^n, \mathbb{R}_+)$ and apply density of $\mathrm{C}^2([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ in $\mathrm{C}^1([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ to have a sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ such that for all k, $\varphi_k \in \mathrm{C}^2([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ and

$$\|\varphi - \varphi_k\|_{\mathrm{C}^1} \xrightarrow[k\to\infty]{} 0\,.$$

Thanks to Step 3, we know that (116) holds for C^2 functions and thus for all $k \in \mathbb{N}$,

$$\|V_\varepsilon(b)\|(\varphi_k(\cdot,b)) - \|V_\varepsilon(a)\|(\varphi_k(\cdot,a)) = \int_a^b \delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\, dt + \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t))\, dt \tag{128}$$

and it remains to check that we can let k tend to $+\infty$, which basically follows from the fact that each term involved in (116) is linearly continuous with respect to $\varphi \in \mathrm{C}^1([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ endowed with $\|\cdot\|_{\mathrm{C}^1}$. Indeed, we first note that for any $t\in[0,1]$,

$$\left| \|V_\varepsilon(t)\|(\varphi_k(\cdot,t)) - \|V_\varepsilon(t)\|(\varphi(\cdot,t)) \right| \leq (M+1)\|\varphi_k(\cdot,t) - \varphi(\cdot,t)\|_\infty \leq (M+1)\|\varphi-\varphi_k\|_{\mathrm{C}^1}$$

and consequently,

$$\lim_{k\to\infty} \|V_\varepsilon(b)\|(\varphi_k(\cdot,b)) = \|V_\varepsilon(b)\|(\varphi(\cdot,b)) \quad \text{and} \quad \lim_{k\to\infty} \|V_\varepsilon(a)\|(\varphi_k(\cdot,a)) = \|V_\varepsilon(a)\|(\varphi(\cdot,a))\,. \tag{129}$$

Similarly, for all $t\in[0,1]$,

$$\left| \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t)) - \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t)) \right| \leq (M+1)\|\partial_t\varphi_k(\cdot,t) - \partial_t\varphi(\cdot,t)\|_\infty \leq (M+1)\|\varphi-\varphi_k\|_{\mathrm{C}^1}$$

and consequently,

$$\lim_{k\to\infty} \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi_k(\cdot,t))\, dt = \int_a^b \|V_\varepsilon(t)\|(\partial_t\varphi(\cdot,t))\, dt\,. \tag{130}$$

We can apply Remark 2.13 and Proposition 3.3 to the remaining term:

$$\begin{aligned}\Big|\delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t))) - \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\Big| \\ \leq n(M+1)\,\|h_\varepsilon(\cdot,V_\varepsilon(t))\|_{\mathrm{C}^1}\,\|\varphi_k(\cdot,t)-\varphi(\cdot,t)\|_{\mathrm{C}^1} \leq C\varepsilon^{-4}\|\varphi-\varphi_k\|_{\mathrm{C}^1}\end{aligned}$$

and consequently,

$$\lim_{k\to\infty}\int_a^b \delta(V_\varepsilon(t),\varphi_k(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt = \int_a^b \delta(V_\varepsilon(t),\varphi(\cdot,t))(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt\,. \tag{131}$$

We eventually conclude the proof of Step 4, hence of the Proposition 4.6 thanks to (128), (129), (130) and (131). □

We conclude this section by noting a straightforward though important consequence of the Brakke-type equality we established in Proposition 4.6: the mass $t \mapsto \|V_\varepsilon(t)\|(\mathbb{R}^n)$ is non-increasing along the flow.

Remark 4.7 (Mass decay)**.** Let $\varepsilon \in (0,1)$ and let $V_0 \in V_d(\mathbb{R}^n)$ be a compactly supported varifold satisfying $\|V_0\|(\mathbb{R}^n) \leq M$. Let $(V_\varepsilon(t))_{t\in[0,1]}$ be the approximate mean curvature flow starting from V_0. Then, for all $0 \leq a < b \leq 1$,

$$\|V_\varepsilon(b)\|(\mathbb{R}^n) - \|V_\varepsilon(a)\|(\mathbb{R}^n) = -\int_a^b \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y)+\varepsilon}\,dy\,dt \leq 0\,.$$

In particular,

$$\|V_\varepsilon(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n),\ \forall t \in [0,1], \tag{132}$$

and

$$\int_0^1 \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y)+\varepsilon}\,dy\,dt \leq \|V_0\|(\mathbb{R}^n).$$

Indeed, let $0 \leq a < b \leq 1$, applying Proposition 4.6 with the constant test function $\varphi : (t,y) \mapsto 1 \in \mathrm{C}^1([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ we have $\partial_t\varphi = 0$, $\delta(V_\varepsilon(t),\varphi(\cdot,t)) = \delta V_\varepsilon(t)$ (see (12)) and then using (41) in Proposition 3.4, we infer

$$\|V_\varepsilon(b)\|(\mathbb{R}^n) - \|V_\varepsilon(a)\|(\mathbb{R}^n) = \int_a^b \delta V_\varepsilon(t)\,(h_\varepsilon(\cdot,V_\varepsilon(t)))\,dt = -\int_a^b \int_{\mathbb{R}^n} \frac{|(\Phi_\varepsilon * \delta V_\varepsilon(t))(y)|^2}{(\Phi_\varepsilon * \|V_\varepsilon(t)\|)(y)+\varepsilon}\,dy\,dt.$$

5. Convergence of approximate mean curvature flows to spacetime Brakke flows

Up to this point, given a compactly supported varifold $V_0 \in V_d(\mathbb{R}^n)$, we constructed an approximate mean curvature flow $(V_\varepsilon(t))_{t\in[0,1]}$ (Theorem 4.1) obtained as the limit of time-discrete approximate mean curvature flows (Definition 3.6) when the subdivision time step converges to 0, the approximation scale ε being fixed. In this section, we investigate the behavior of $(V_\varepsilon(t))_{t\in[0,1]}$ in the limit $\varepsilon \to 0$.

Following the works of Brakke and Kim & Tonegawa [12, 28], one can prove the convergence of $(\|V_\varepsilon(t)\|)_{t\in[0,1]}$ to a limit measure $\mu(t)$ for all $t \in [0,1]$, up to an extraction that is independent of t. In [12, 28], the convergence is first established for a common sequence $(\varepsilon_j)_j$ for all dyadic numbers of $[0,1]$ thanks to the uniform boundedness of the mass and a diagonal extraction argument. Then, it is extended to all $t \in [0,1]$ using the continuity property of $t \mapsto \|V_\varepsilon(t)\|$. The previous procedure does not work for the measures $(V_\varepsilon(t))_{t\in[0,1]}$ because of the lack of a continuity property. This issue is very common when studying the compactness of Brakke-type flows (cf. for instance [41, Theorem 3.7]). To circumvent this issue, we consider the tensor product measure $dt \otimes V_\varepsilon(t)$.

In section 5.1, we introduce the notions of spacetime mean curvature and spacetime Brakke flow, and we list some of their properties. In section 5.2, we prove that, up to an extraction, the measure $dt \otimes V_\varepsilon(t)$ converges, thanks to the uniform boundedness of the mass in ε, to a limit measure, denoted by λ. We prove that λ has a bounded spacetime mean curvature in L^2, and that it is a spacetime Brakke flow provided its $\mathbb{R}^n \times G_{d,n}$-component is rectifiable.

We introduce after [28] classes of test functions and vector fields that are suitable for studying the behaviour of the approximate mean curvature flows. For $j \in \mathbb{N}$ we define

$$\mathcal{A}_j := \{\varphi \in C^2(\mathbb{R}^n;\mathbb{R}_+) : \varphi(x) \le 1, |\nabla\varphi(x)| \le j\varphi(x), \|\nabla^2\varphi(x)\| \le j\varphi(x) \text{ for all } x \in \mathbb{R}^n\},$$
$$\mathcal{B}_j := \{g \in C^2(\mathbb{R}^n;\mathbb{R}^n) : |g(x)| \le j, \|\nabla g(x)\| \le j, \|\nabla^2 g(x)\| \le j \text{ for all } x \in \mathbb{R}^n \text{ and } \|g\|_{L^2(\mathbb{R}^n)} \le j\}.$$

5.1. **Spacetime mean curvature and spacetime Brakke flows.** As we explained, our purpose is to investigate the limit of $\lambda_\varepsilon = dt \otimes V_\varepsilon(t)$ when ε tends to 0. Let us start by recalling the framework that allows to consider such a generalized tensor product of measures. Following [5, Section 2.5], we say that *a family of d–varifolds $(W(t))_{t\in[0,1]}$ is dt–measurable* if for any Borel set $B \subset \mathbb{R}^n \times G_{d,n}$, the map $t \in [0,1] \mapsto W(t)(B) \in \mathbb{R}_+$ is dt–measurable. In this case, the measure $dt \otimes W(t)$ is well-defined as follows:

$$\forall B \subset [0,1] \times \mathbb{R}^n \times G_{d,n} \text{ Borel set}, \quad [dt \otimes W(t)](B) = \int_{[0,1]} \left(\int_{\mathbb{R}^n \times G_{d,n}} \mathbf{1}_B \, dW(t) \right) dt$$

and satisfies that for any Borel function $f : [0,1] \times \mathbb{R}^n \times G_{d,n} \to \mathbb{R}_+$,

$$\int f \, d[dt \otimes W(t)] = \int_{t\in[0,1]} \left(\int_{(y,S)\in\mathbb{R}^n\times G_{d,n}} f(t,y,S) \, dW(t) \right) dt \, .$$

Note that the above assertion then holds for any $f \in L^1(dt \otimes W(t))$. We point out that in such a case the measure $dt \otimes W(t)$ is not a Radon measure in general. Indeed, given a compact set $K \subset \mathbb{R}^n$, $K' = [0,1] \times K \times G_{d,n}$ is compact and

$$[dt \otimes W(t)](K') = \int_0^1 \|W(t)\|(K) \, dt \, .$$

As any compact set of $[0,1]\times\mathbb{R}^n\times G_{d,n}$ can be included in a compact set of the form K' we conclude that $dt \otimes W(t)$ is a Radon measure if and only if for all compact sets $K \subset \mathbb{R}^n$,

$$\int_0^1 \|W(t)\|(K) \, dt < \infty \, .$$

We now introduce the *spacetime mass* and *first variation* of a Radon measure β defined in $[0,1] \times \mathbb{R}^n \times G_{d,n}$. Such definitions naturally arise when requiring that they are consistent with the usual mass and first variation when $\beta = dt \otimes W(t)$ (see Remark 5.2).

Definition 5.1 (Spacetime mean curvature)**.** *Let β be a Radon measure on $[0,1] \times \mathbb{R}^n \times G_{d,n}$.*

- *We introduce the* spacetime mass $\|\beta\|$ *of β: $\|\beta\| := \Pi_\# \beta$ is a Radon measure in $[0,1]\times\mathbb{R}^n$, where $\Pi : [0,1] \times \mathbb{R}^n \times G_{d,n} \to [0,1] \times \mathbb{R}^n$ is the canonical projection.*
- *We define the* first variation of β*: for any vector field $X \in \mathrm{C}^1_c([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$,*

$$\delta\beta(X) := \int_0^1 \int_{\mathbb{R}^n \times G_{d,n}} \mathrm{div}_S \, X(t,y) d\beta(t,y,S).$$

- *If in addition, $\delta\beta$ is bounded in the $\mathrm{C}_c([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$–topology, we say that β has locally bounded first variation, and then, by the Riesz representation theorem, $\delta\beta$ identifies with a vector-valued Radon measure (also denoted by $\delta\beta$)*

$$\delta\beta(X) = \int_{[0,1]\times\mathbb{R}^n} X \cdot \, d\delta\beta \quad \forall X \in \mathrm{C}_c([0,1] \times \mathbb{R}^n, \mathbb{R}^n) \, .$$

It follows from the Radon–Nikodym decomposition theorem that there exist a Radon measure $(\delta\beta)_s$ singular with respect to $\|\beta\|$, and $h = h(\cdot,\cdot,\beta) \in L^1([0,1] \times \mathbb{R}^n, \mathbb{R}^n, \|\beta\|)$ that we call the

spacetime mean curvature, *satisfying* $\delta\beta = -h\|\beta\| + (\delta\beta)_s$, *i.e.:*

$$\delta\beta(X) = -\int_0^1 \int_{\mathbb{R}^n} h(t,y,\beta)\cdot X(t,y) d\|\beta\|(t,y) + (\delta\beta)_s(X), \tag{133}$$

for any $X \in \mathrm{C}_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$.

Remark 5.2. Let $(W(t))_{t\in[0,1]}$ be a family of d–varifolds that is dt–measurable and such that $\beta = dt\otimes W(t)$ is a Radon measure. Then, $dt\otimes\|W(t)\|$ is a well-defined Radon measure and $\|\beta\| = dt\otimes\|W(t)\|$; and for any $X\in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$,

$$\delta\beta(X) = \int_0^1 \delta W(t)(X)\, dt\,.$$

Assume moreover that for a.e $t\in[0,1]$, $W(t)$ has locally bounded first variation and $\delta W(t) \ll \|W(t)\|$:

$$\delta W(t) = -h(\cdot,W(t))\|W(t)\| \text{ with } h(\cdot,W(t))\in L^1_{loc}(\mathbb{R}^n,\|W(t)\|)\,.$$

Then, β has locally bounded first variation in the sense of Definition 5.1 if and only if

$$\int_{[0,1]\times\mathbb{R}^n} |h(y,W(t))|\, dW(t)\, dt < \infty$$

and in this case,

$$\delta\beta = -dt\otimes h(\cdot,W(t))\, dW(t) \quad \text{that is}$$
$$\delta\beta \ll \|\beta\| \quad \text{and} \quad h(t,y,\beta) = h(y,W(t)) \quad \text{for } \|\beta\|\text{–a.e } (t,y)\in[0,1]\times\mathbb{R}^n. \tag{134}$$

Indeed, for $X\in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$

$$\begin{aligned}\delta\beta(X) &= \int_{t=0}^1 \left(\int_{\mathbb{R}^n\times G_{d,n}} \mathrm{div}_S\, X(t,y)\, dW(t)(y,S)\right) dt\\ &= -\int_0^1\int_{\mathbb{R}^n} h(\cdot,W(t))\cdot X(t,\cdot)\, d\|W(t)\|\, dt\\ &= -\int_{[0,1]\times\mathbb{R}^n} h(y,W(t))\cdot X(t,y) d\|\beta\|(t,y)\,,\end{aligned}$$

and we infer (134).

We now give the definition of spacetime Brakke flows.

Definition 5.3. *(Spacetime Brakke flows) Let* λ *be a Radon measure on* $[0,1]\times\mathbb{R}^n\times G_{d,n}$. *The measure* λ *is called a spacetime Brakke flow if:*

- (i) *there exists a family of* d*–varifolds* $(V(t))_{t\in[0,1]}$ *such that* $\lambda = dt\otimes V(t)$,
- (ii) λ *has locally bounded first variation,* $\delta\lambda \ll \|\lambda\|$, *and there exists* $h = h(\cdot,\cdot,\lambda)\in L^2_{loc}([0,1]\times\mathbb{R}^n,\mathbb{R}^n,\|\lambda\|)$ *such that*
$$\delta\lambda = -h\,\|\lambda\|\,.$$
- (iii) *(Integral Brakke inequality). For any* $\varphi\in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ *and for all* $0\le t_1\le t_2\le 1$ *we have*

$$\begin{aligned}\|V(t_2)\|(\varphi(t_2,\cdot)) - \|V(t_1)\|(\varphi(t_1,\cdot)) \le &-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi(t,\cdot)|h(t,\cdot,\lambda)|^2\, d\|V(t)\|\, dt\\ &+\int_{t_1}^{t_2}\int_{\mathbb{R}^n\times G_{d,n}} S^\perp(\nabla_y\varphi(t,y))\cdot h(t,y,\lambda)\, d\lambda(t,y,S) + \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\partial_t\varphi(t,\cdot)\, d\|V(t)\|\, dt\end{aligned} \tag{135}$$

We furthermore say that λ starts from $V(0)$.

Remark 5.4. We give two important consequences of the definition of the spacetime Brakke flow in the case where $h \in L^1(\|\lambda\|)$:

(i) *Mass decay.* For all $0 \leq t_1 \leq t_2 \leq 1$,

$$\|V(t_2)\|(\mathbb{R}^n) \leq \|V(t_1)\|(\mathbb{R}^n) \leq \|V(0)\|(\mathbb{R}^n)\,.$$

(ii) *L^2 bound on the mean curvature.* If $V(0)$ has finite mass $\|V(0)\| < \infty$ then $h \in L^2(\|\lambda\|)$ and more precisely,

$$\|h\|_{L^2(\|\lambda\|)} = \int_0^1 \int_{\mathbb{R}^n} |h(t,y,\lambda)|^2 \, d\|V(t)\|(y)\, dt \leq \|V(0)\|(\mathbb{R}^n)\,.$$

Proof of (i) and (ii). Given a cutoff function $\chi \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}_+)$ satisfying $0 \leq \xi \leq 1$, $\chi = 1$ in $B_1(0)$ and $\operatorname{spt}\chi \subset B_3(0)$, we introduce for $r > 0$, $\varphi_r \in \mathrm{C}^1_c([0,1] \times \mathbb{R}^n, \mathbb{R}_+)$ as:

$$\forall (t,x) \in [0,1] \times \mathbb{R}^n,\ \ \varphi_r(t,x) = \chi\left(\frac{x}{r}\right)\,.$$

Note that $0 \leq \varphi_r \leq 1$ satisfies $\operatorname{spt}\varphi_r \subset [0,1] \times B_{3/r}(0)$, $\partial_t \varphi = 0$ and $\|\nabla_y \varphi_r\|_\infty \leq r^{-1}$. Plugging φ_r in (135) we obtain (denoting $h := h(\cdot,\cdot,\lambda)$ for simplicity)

$$\begin{aligned}\|V(t_2)\|(\varphi_r) &\leq \|V(t_1)\|(\varphi_r) - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \varphi_r |h|^2 \, d\mu(t)dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla\varphi_r)\cdot h \, d\lambda \\ &\leq \|V(t_1)\|(\mathbb{R}^n) - \int_{t_1}^{t_2}\int_{B_{1/r}(0)} |h|^2 \, d\|V\|(t)\, dt + \frac{1}{r}\|h\|_{L^1(\|\lambda\|)}.\end{aligned}$$

Since $\|V(t_2)\|(\varphi_r) \geq \|V(t_2)\|(B_{\frac{1}{r}}(0)) \xrightarrow[r\to 0]{} \|V(t_2)\|(\mathbb{R}^n)$ and $h \in L^1(\|\lambda\|)$ we let $r \to +\infty$ and obtain

$$\|V(t_2)\|(\mathbb{R}^n) + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} |h|^2 \, d\|V\|(t)\, dt \leq \|V(t_1)\|(\mathbb{R}^n)$$

thus proving (i). Then specifying $t_1 = 0$ and $t_2 = 1$ allows to conclude the proof of (ii). □

Remark 5.5 (Brakke flows and spacetime Brakke flows)**.** Let $(V(t))_{t\in[0,1]}$ be a Brakke flow (see Definition 1.1), then $\lambda = dt \otimes V(t)$ is a spacetime Brakke flow in the sense of Definition 5.3.

5.2. **Convergence of approximate mean curvature flows, properties of the limit.** We now study the convergence of the approximate mean curvature flow $(V_\varepsilon(t))_{t\in[0,1]}$ introduced in Section 4. As previously mentioned, it is possible to find an extracted sequence $(\varepsilon_j)_j$ that does not depend on $t \in [0,1]$ and a family of Radon measures $(\mu(t))_{t\in[0,1]}$ such that for all $t \in [0,1]$, $\|V_{\varepsilon_j}\|$ converges to $\mu(t)$, as shown in Proposition 5.9. However, we are not able to obtain a similar result concerning the whole varifold structure: it is possible to have an extracted sequence $(\varepsilon_j^t)_j$ such that for each time $t \in [0,1]$, $V_{\varepsilon_j^t}(t)$ converges to some limit varifold, but we are not able to remove the time dependency of the extracted sequence $(\varepsilon_j^t)_j$. Therefore, in Theorem 5.6, we investigate the limit of the measures $dt \otimes V_\varepsilon(t)$ when ε tends to 0 instead.

Theorem 5.6. *(Convergence) Let $V_0 \in V_d(\mathbb{R}^n)$ of compact support and finite mass. Given $\varepsilon \in (0,1)$, $(V_\varepsilon(t))_{t\in[0,1]}$ denotes the approximate mean curvature flow starting from V_0. We have:*

(i) For $\varepsilon \in (0,1)$, the measure $dt \otimes V_\varepsilon(t)$ is well-defined and is a Radon measure in $[0,1] \times \mathbb{R}^n \times G_{d,n}$.

(ii) There exists a sequence $(\varepsilon_j)_j \xrightarrow[j\to\infty]{} 0$ such that

$$dt \otimes V_{\varepsilon_j}(t) \xrightarrow[j\to\infty]{} \lambda = dt \otimes V(t) \quad \text{and} \quad \forall t \in [0,1],\ \|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{} \|V(t)\|,$$

where $(V(t))_{t\in[0,1]}$ is a family of d–varifolds.

(iii) *$\delta\lambda$ is locally bounded, $\delta\lambda = -h(\cdot,\cdot,\lambda)\,\|\lambda\|$ and $\|h(\cdot,\cdot,\lambda)\|^2_{L^2(\|\lambda\|)} \leq \|V_0\|(\mathbb{R}^n)$.*

(iv) *If we assume that $V(t)$ is a rectifiable varifold for a.e $t \in [0,1]$ then λ is a spacetime Brakke flow.*

We emphasize that, in this theorem, we have the convergence of the mass $\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{} \|V(t)\|$ for all $t \in [0,1]$ for a sequence $\varepsilon_j \to 0$ that does not depend on t, however, we do not know whether $V_{\varepsilon_j}(t)$ converges to $V(t)$.

Remark 5.7. The proof of Theorem 5.6 is crucially based on Propositions 5.4 and 5.5 in [28]. Note however that they are stated in [28] for codimension 1 varifolds, while we work with varifolds of arbitrary codimension in $\mathbb{R}^n$. Actually, the proofs of Propositions 5.4 and 5.5 in [28] do not require that the varifolds have codimension 1 because they are based on results due to Brakke, which are valid for general varifolds.

For simplicity, we split the proof of Theorem 5.6 into several steps: it will follow from Propositions 5.8 to 5.12. The next proposition combines the generalized estimates (generalized in terms of codimension) of [28, Proposition 5.4] and [28, Proposition 5.5].

Proposition 5.8. *There exists some $\varepsilon_* = \varepsilon_*(n,M) > 0$ (depending only on n and M) such that for any varifold $W \in V_d(\mathbb{R}^n)$ such that $\|W\|(\mathbb{R}^n) \leq M$; if $0 < \varepsilon \leq \varepsilon_*$ and $\varepsilon^{-\frac{1}{6}} \geq 2m$, we have*

- *for any $\psi \in \mathcal{A}_m$*

$$\left|\delta W\left(\psi h_\varepsilon(\cdot,W)\right) + \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\,dx\right| \leq \varepsilon^{\frac{1}{4}}\left(1 + \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\,dx\right) \tag{136}$$

and

$$\int_{\mathbb{R}^n} \psi|h_\varepsilon(\cdot,W)|^2\,d\|W\| \leq \int_{\mathbb{R}^n} \frac{\psi|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}(1+\varepsilon^{\frac{1}{4}})\,dx + \varepsilon^{\frac{1}{4}}, \tag{137}$$

- *for any $X \in \mathcal{B}_m$*

$$\left|\int_{\mathbb{R}^n} h_\varepsilon \cdot X\,d\|W\| + \delta W(X)\right| \leq \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}}\left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta W|^2}{\Phi_\varepsilon * \|W\| + \varepsilon}\,dx\right)^{\frac{1}{2}}. \tag{138}$$

Proposition 5.9 (Existence of a limit for the mass measure)**.** *Let $V_0 \in V_d(\mathbb{R}^n)$ of bounded support, for any $\varepsilon \in (0,1)$, let $V_\varepsilon(t)$ be the approximate mean curvature flow starting from V_0.*
There exists a sequence $(\varepsilon_j)_{j=1}^\infty$ (not depending on t) converging to 0 as $j \to \infty$, and a family of Radon measures $(\mu(t))_{t\in[0,1]}$ on $\mathbb{R}^n$ such that:

$$\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{*} \mu(t) \tag{139}$$

for <u>all</u> $t \in [0,1]$.

Proof. Proposition 5.9 states that, up to an extraction independent of t, the mass measure $\|V_\varepsilon(t)\|$ converges as ε goes to 0 to a limit time-dependent measure $\mu(t)$. The proof is a direct adaptation of the proof of [28, Proposition 6.4 (1)], which is itself based on the results of Section 5 in [28] and two estimates [28, (6.3)], [28, (6.5)]. In short, the proof of [28] is based on the following arguments: a limit measure $\mu(t)$ is defined for a countable, dense collection D of times using an extraction argument, the extension of $\mu(t)$ to almost all times, more precisely the complement of a countable set of times, follows from a continuity property of $\mu(t)$ on D, the convergence of the subsequence $\|V_\varepsilon(t)\|$ to $\mu(t)$ for all these times t follows from an approximate continuity property satisfied by $\|V_\varepsilon(t)\|$, and a last extraction is used to recover $\mu(t)$ for all t. The adaptation of the proof to our framework uses the following arguments:

- $V_0 \in V_d(\mathbb{R}^n)$ is not necessarily an open partition in our case, but the proof in [28] remains valid in this case;
- the proof of [28] is based on the results of [28, Section 5] which are valid for varifolds of any codimension (and not only for codimension 1 varifolds which are the subject of [28, Proposition 6.4(1)]);
- the weight function Ω is identically equal to 1 in our case because we work with varifolds of finite mass;
- estimate [28, Inequality (6.3)] is replaced by the decay property of the mass stated in Remark 4.7;
- estimate [28, Inequality (6.5)] is replaced by (116) (with test functions depending only on the space variable).

Step 1: Let $\varepsilon \in (0,1)$, $V \in V_d(\mathbb{R}^n)$. We prove the following inequality

$$\delta(V,\varphi)(h_\varepsilon(\cdot,V)) \leq 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\|,$$

for any $\varphi \in \mathcal{A}_j$ and $2j \leq \varepsilon^{-\frac{1}{6}}$.

Indeed, let $\varphi \in \mathcal{A}_j$ with $2j \leq \varepsilon^{-\frac{1}{6}}$ and set for simplicity $b := \displaystyle\int_{\mathbb{R}^n} \frac{\varphi|\Phi_\varepsilon * \delta V|^2}{\Phi_\varepsilon * \|V\| + \varepsilon} dx$, we have

$$\begin{aligned}
&\delta\left(V,\varphi\right)\left(h_\varepsilon(\cdot,V)\right) = \delta V(\varphi h_\varepsilon) + \int_{\mathbb{R}^n} S^\perp(\nabla\varphi)\cdot h_\varepsilon(\cdot,V)\, dV(x,S)\\
&\leq -b + \varepsilon^{\frac{1}{4}} b + \varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \varphi|h_\varepsilon(\cdot,V)|^2\, d\|V\| + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\| \text{ by (136)}\\
&\leq -b + \varepsilon^{\frac{1}{4}} b + \varepsilon^{\frac{1}{4}} + \frac{1}{2} b(1+\varepsilon^{\frac{1}{4}}) + \frac{1}{2}\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\| \text{ by (137)}\\
&\leq \frac{1}{2}(-1+3\varepsilon^{\frac{1}{4}})b + 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\|\\
&\leq 2\varepsilon^{\frac{1}{4}} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla\varphi|^2}{\varphi}\, d\|V\|.
\end{aligned}$$

Step 2: We define the limit measure $\mu(t)$ for a.e $t \in [0,1]$.

Let $D \cap [0,1]$ be the set of dyadic numbers in $[0,1]$. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence converging to 0 such that, $\|V_{\varepsilon_j}(t)\|$ converges for any $t \in D$, denote the limit $\mu(t)$. The previous claim stems from the Banach-Alaoglu theorem and the uniform boundedness of the mass (132).

Let $Z := (\varphi_q)_{q\in\mathbb{N}}$ be a countable subset of $C_c^2(\mathbb{R}^n,\mathbb{R}_+)$ which is dense in $C_c(\mathbb{R}^n,\mathbb{R}_+)$. Take $\varphi_q \in Z$ and assume without loss of generality that $\varphi_q < 1$. Then, for any $i \in \mathbb{N}$ large enough, we have $\varphi_q + i^{-1} \in \mathcal{A}_m$ for any $m \geq m_0$, where m_0 depends on i and φ_q. We apply (116) with $\varphi(\cdot,t) = \varphi_q$, together with Step 1 to obtain

$$\|V_{\varepsilon_j}(b)\|(\varphi_q + i^{-1}) - \|V_{\varepsilon_j}(a)\|(\varphi_q + i^{-1}) \leq (b-a)2\varepsilon_j^{\frac{1}{4}} + \frac{1}{2}\int_a^b \int_{\mathbb{R}^n} \frac{|\nabla(\varphi_q + i^{-1})|^2}{\varphi_q + i^{-1}}\, d\|V_{\varepsilon_j}(t)\| dt,$$

for any $a,b \in [0,1], a \leq b$ and $2m_0 \leq \varepsilon_j^{-\frac{1}{6}}$. We obtain from [41, Lemma 3.1]

$$\frac{|\nabla(\varphi_q + i^{-1})|^2}{\varphi_q + i^{-1}} \leq \frac{|\nabla\varphi_q|^2}{\varphi_q} \leq 2\|\nabla^2\varphi_q\|_\infty.$$

Therefore, for any $a, b \in [0,1], a \leq b$

$$\|V_{\varepsilon_j}(b)\|(\varphi_q + i^{-1}) - \|V_{\varepsilon_j}(a)\|(\varphi_q + i^{-1}) \leq (b-a)2\varepsilon_j^{\frac{1}{4}} + (b-a)\|\nabla^2\varphi_q\|_\infty \|V_0\|(\mathbb{R}^n). \tag{140}$$

We let $j \to \infty$, we deduce for $a, b \in D, a \leq b$ that

$$\mu(b)(\varphi_q + i^{-1}) - \mu(a)(\varphi_q + i^{-1}) \leq (b-a)\|\nabla^2\varphi_q\|_\infty \|V_0\|(\mathbb{R}^n).$$

We let $i \to \infty$, using the uniform boundedness of the mass (132), we deduce for $a, b \in D, a \leq b$ that

$$\mu(b)(\varphi_q) - \mu(a)(\varphi_q) \leq (b-a)\|\nabla^2\varphi_q\|_\infty \|V_0\|(\mathbb{R}^n).$$

The previous inequality tells us that the map $g_q : t \mapsto \mu(t)(\varphi_q) - t(b-a)\|\nabla^2\varphi_q\|_\infty \|V_0\|(\mathbb{R}^n)$ is nonincreasing for $t \in D$. Define

$$\mathcal{C} := \{t \in [0,1], \text{for some } q \in \mathbb{N} \lim_{s \to t^-} g_q(s) > \lim_{s \to t^+} g_q(s)\}.$$

By the monotonicity property of g_q, $\mathcal{C}$ is a countable set in $[0,1]$, and $\mu(t)(\varphi_q)$ may be defined continuously on the complement of $\mathcal{C}$ uniquely from the values on D; then, one can define the measure $\mu(t)$ for a.e $t \in [0,1]$ by density of Z in $C_c(\mathbb{R}^n, \mathbb{R}_+)$.

Step 3: We prove that for any $t \in [0,1] \setminus \mathcal{C}$

$$\|V_{\varepsilon_j}(t)\| \overset{*}{\rightharpoonup} \mu(t).$$

Let $t \in [0,1] \setminus \mathcal{C}$ and $s \in D, t < s$. From (140) we have

$$\|V_{\varepsilon_j}(s)\|(\varphi_q + i^{-1}) \leq \|V_{\varepsilon_j}(t)\|(\varphi_q + i^{-1}) + O(s-t).$$

We let $j \to$ so that

$$\mu(s)(\varphi_q + i^{-1}) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q + i^{-1}) + O(s-t).$$

Then we take the limit in i to obtain

$$\mu(s)(\varphi_q) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q) + O(s-t).$$

We now let $s \to t^-$ and use the continuity of g_q at t so that

$$\mu(t)(\varphi_q) \leq \liminf_j \|V_{\varepsilon_j}(t)\|(\varphi_q).$$

The same reasoning for $s < t$ gives $\mu(t)(\varphi_q) \geq \limsup_j \|V_{\varepsilon_j}(t)\|(\varphi_q)$; hence

$$\lim_j \|V_{\varepsilon_j}(t)\|(\varphi_q) = \mu(t)(\varphi_q), \quad \forall \varphi_q \in Z \text{ and } \forall t \in [0,1] \setminus \mathcal{C};$$

we conclude the proof of step 3 by density of Z in $C_c(\mathbb{R}^n, \mathbb{R}_+)$.

The set $\mathcal{C}$ is countable, hence, by further extraction of the sequence $(\varepsilon_j)_j$, we can define $\mu(t)$ on $[0,1]$ entirely and ensure the convergence for <u>all</u> $t \in [0,1]$. □

Proposition 5.10 (Limiting flow). *Let $V_0 \in V_d(\mathbb{R}^n)$ be of compact support.*

(i) For any $\varepsilon \in (0,1)$, let $V_\varepsilon(t)$ be the approximate mean curvature flow starting from V_0. Then $(V_\varepsilon(t))_{t \in [0,1]}$ is dt–measurable and $\lambda_\varepsilon = dt \otimes V_\varepsilon(t)$ is a Radon measure, satisfying for all $\varphi \in \mathrm{C}_c([0,1] \times \mathbb{R}^n \times G_{d,n}, \mathbb{R})$,

$$\int \varphi \, d\lambda_\varepsilon = \int_0^1 \left(\int_{(x,S) \in \mathbb{R}^n \times G_{d,n}} \varphi(t,x,S) \, dV_\varepsilon(t) \right) dt.$$

(ii) There exists a sequence $(\varepsilon_j)_j \to 0$ and a Radon measure λ in $[0,1]\times\mathbb{R}^n\times G_{d,n}$ such that

$$\lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{*} \lambda \tag{141}$$

and furthermore, $\lambda = dt \otimes V(t)$ where $(V(t))_{t\in[0,1]}$ is a family of d–varifolds.

(iii) The mass of the varifolds converges: for all $t\in[0,1]$, $\|V_{\varepsilon_j}(t)\| \xrightarrow[j\to\infty]{*} \|V(t)\|$.

Proof. Let $B\subset\mathbb{R}^n\times G_{d,n}$ be a Borel set and $f:[0,1]\to\mathbb{R}_+$, $t\mapsto V_\varepsilon(t)(B)$. We check that f is dt–measurable. According to [5, Prop 2.26], it is sufficient to check that f is measurable in the case where B is an open set, if so,

$$\begin{aligned} f(t) = V_\varepsilon(t)(B) &= \sup\left\{\int \varphi\, dV_\varepsilon(t) \;:\; \varphi\in \mathrm{C}_c(B,\mathbb{R}_+),\, 0\le\varphi\le 1\right\} \\ &= \sup\left\{\int \varphi\, dV_\varepsilon(t) \;:\; \varphi\in \mathcal{D},\, 0\le\varphi\le 1\right\} \end{aligned}$$

where $\mathcal{D}$ is countable and dense in $\mathrm{C}_c(B,\mathbb{R}_+)$ and the measurability of f reduces to the measurability of $g: t\mapsto \int\varphi\, dV_\varepsilon(t)$ given any $\varphi\in\mathcal{D}\subset\mathrm{C}_c(B,\mathbb{R}_+)$. Recall that for any $t\in[0,1]$, $V_\varepsilon(t)$ is the weak star limit of $V^{pc}_{\varepsilon,\mathcal{T}_j^D}(t)$ (where $\mathcal{T}_j^D$ refers to dyadic subdivisions, see Theorem 4.1 and (109)) so that

$$g(t) = \int \varphi\, dV_\varepsilon(t) = \lim_{j\to\infty} g_j(t) \quad\text{with}\quad g_j(t) = \int \varphi\, dV^{pc}_{\varepsilon,\mathcal{T}_j^D}(t)\,.$$

The functions g_j are measurable as they are piecewise constant since $t\mapsto V^{pc}_{\varepsilon,\mathcal{T}_j^D}(t)$ is piecewise constant by definition (see Remark 3.8), and therefore f is measurable and the family $(V_\varepsilon(t))_{t\in[0,1]}$ is dt–measurable. Moreover, thanks to (132)

$$\lambda_\varepsilon([0,1]\times\mathbb{R}^n\times G_{d,n}) = \int_0^1 \|V_\varepsilon(t)\|(\mathbb{R}^n)\, dt \le \|V_0\|(\mathbb{R}^n) < \infty \tag{142}$$

and λ_ε is a finite Radon measure, which concludes the proof of (i).

We carry on with the proof of (ii) and (iii). We know by Proposition 5.9 that there exists a sequence $(\varepsilon_j)_j\to 0$ for which $\|V_{\varepsilon_j}(t)\|$ converges to a limit measure $\mu(t)$ for all $t\in[0,1]$. Using (142), we can assert by Banach-Alaoglu's compactness theorem that, up to a further extraction, $\lambda_{\varepsilon_j}\overset{*}{\rightharpoonup}\lambda$, where λ is a finite Radon measure on $[0,1]\times\mathbb{R}^n\times G_{d,n}$.
Denoting by Π the canonical projection $(t,x,S)\in[0,1]\times\mathbb{R}^n\times G_{d,n}\mapsto(t,x)\in[0,1]\times\mathbb{R}^n$ we now show that $\Pi_\#\lambda = dt\otimes\mu(t)$. Indeed, on the one hand, as a consequence of (141), we have

$$\Pi_\#\lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{*} \Pi_\#\lambda\,. \tag{143}$$

On the other hand, by definition of push-forward measure, we have for $\varphi\in\mathrm{C}_c([0,1]\times\mathbb{R}^n,\mathbb{R})$,

$$\begin{aligned} \Pi_\#\lambda_{\varepsilon_j}(\varphi) &= \int_{[0,1]\times\mathbb{R}^n\times G_{d,n}} \varphi\circ\Pi\, d\lambda_{\varepsilon_j} = \int_{(t,x,S)\in[0,1]\times\mathbb{R}^n\times G_{d,n}} \varphi(\Pi(t,x,S))\, dV_{\varepsilon_j}(t)\, dt \\ &= \int_{t=0}^1\int_{x\in\mathbb{R}^n} \varphi(t,x)\, d\|V_{\varepsilon_j}(t)\|\, dt \\ &\xrightarrow[j\to\infty]{} \int_{t=0}^1\int_{x\in\mathbb{R}^n} \varphi(t,x)\, d\mu(t)\, dt. \end{aligned} \tag{144}$$

where the convergence follows from dominated convergence and for all $t \in [0,1]$, $\|V_{\varepsilon_j}\| \xrightarrow[j\to\infty]{*} \mu(t)$. From (143) and (144) we obtain

$$\Pi_\# \lambda = dt \otimes \mu(t)\,.$$

It follows by Young's disintegration theorem [5, Theorem 2.28] that there exists a family of probability measures $\big(\nu_{(t,x)}\big)_{(t,x)}$ on $G_{d,n}$ (defined up to a $dt\otimes\mu(t)$–null set), such that:

$$\lambda = \Pi_\#\lambda \otimes \nu_{(t,x)} = dt \otimes \mu(t) \otimes \nu_{(t,x)},$$

which completes the proof (denoting $V(t) = \mu(t)\otimes\nu_{(t,x)}$). □

Proposition 5.11. *Let $(\varepsilon_j)_j$ be the extracted sequence introduced in Proposition 5.10. Let $\lambda_{\varepsilon_j} = dt \otimes V_{\varepsilon_j}(t)$ and λ be the limit measure defined in Proposition 5.10. We have:*

$$\forall X \in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n), \quad \delta\lambda_{\varepsilon_j}(X) \xrightarrow[j\to\infty]{} \delta\lambda(X)\,. \tag{145}$$

Moreover, λ has bounded first variation (i.e. $\delta\lambda$ is a finite Radon measure) and

$$\delta\lambda = -h(\cdot,\cdot,\lambda)\|\lambda\| = -h(t,\cdot,\lambda)\,dt\otimes\|V(t)\| \quad \textit{and} \quad \|h\|^2_{L^2(\|\lambda\|)} \le \|V_0\|(\mathbb{R}^n) < \infty\,, \tag{146}$$

Furthermore, for all bounded $\psi \in \mathrm{C}([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$, for all $0 \le t_1 < t_2 \le 1$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \psi(t,y)|h(t,y,\lambda)|^2\, d\|V(t)\|\,dt \le \liminf_{j\to\infty} \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \psi(t,y)\frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\,dy\,dt. \tag{147}$$

Proof. Let us first check that (145) is a consequence of the convergence $\lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{*} \lambda$. Indeed, let $X \in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$, then $g : (t,y,S) \mapsto \mathrm{div}_S\, X(y) \in \mathrm{C}_c([0,1]\times\mathbb{R}^n\times G_{d,n})$ and thus

$$\delta\lambda_{\varepsilon_j}(X) = \int g\, d\lambda_{\varepsilon_j} \xrightarrow[j\to\infty]{} \int g\, d\lambda = \delta\lambda(X)\,.$$

Let us now consider the sequences $(\mu_j)_{j\in\mathbb{N}}$ of Radon measures in $[0,1]\times\mathbb{R}^n$ and $(f_j)_{j\in\mathbb{N}}$ of functions in $\mathrm{C}^\infty([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$ defined as

$$\mu_j = dt\otimes(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)\mathrm{d}y \quad \text{and} \quad f_j(t,\cdot) = \frac{\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)} \quad \text{for all } j\in\mathbb{N}.$$

Let $\varphi \in \mathrm{C}_c([0,1]\times\mathbb{R}^n)$. First note that by definition,

$$\int_{[0,1]\times\mathbb{R}^n} \varphi f_j\, d\mu_j = \int_0^1\int_{\mathbb{R}^n} \varphi(t,y)\left(\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right)dy\,dt \quad \Longrightarrow \quad f_j\mu_j = dt\otimes(\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t))\mathrm{d}y\,.$$

We obtain by standard arguments (bicontinuity of distributional bracket to be more specific) that μ_j converge to $\|\lambda\| = dt\otimes\|V(t)\|$ as Radon measures and $f_j\mu_j$ converges to $\delta\lambda$ as distributions of order 1. Indeed, as Φ_ε is a mollifier, we recall that for all $t\in[0,1]$, $\|\varphi(t,\cdot)*\Phi_{\varepsilon_j} - \varphi(t,\cdot)\|_\infty \xrightarrow[j\to\infty]{} 0$, and $\|\varphi(t,\cdot)*\Phi_{\varepsilon_j} - \varphi(t,\cdot)\|_{\mathrm{C}^1} \xrightarrow[j\to\infty]{} 0$ if φ is additionally C^1. Therefore, by dominated convergence and $\|V_{\varepsilon_j}\|(\mathbb{R}^n) \le \|V_0\|(\mathbb{R}^n)$,

$$\int_0^1 \left|\int_{\mathbb{R}^n} \varphi\,\left(d\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}\|(t) - d\|V_{\varepsilon_j}\|(t)\right)\right| dt \le \|V_0\|(\mathbb{R}^n)\int_0^1 \|\varphi(t,\cdot)*\Phi_{\varepsilon_j} - \varphi(t,\cdot)\|_\infty\, dt \xrightarrow[j\to\infty]{} 0$$

so that recalling that $\|\lambda_{\varepsilon_j}\| = dt\otimes\|V_{\varepsilon_j}(t)\|$ converges to $\|\lambda\|$,

$$\left|\int \varphi\, d\mu_j - \int \varphi\, d\|\lambda\|\right| \le \int_0^1 \left|\int_{\mathbb{R}^n} \varphi\,\left(d\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}\|(t) - d\|V_{\varepsilon_j}\|(t)\right)\right| dt$$

$$+\left|\int \varphi\ \left(d\|\lambda_{\varepsilon_j}\| - d\|\lambda\|\right)\right| + \varepsilon_j \left|\int \varphi\, dy\, dt\right| \xrightarrow[j\to\infty]{} 0\,.$$

We hence checked that $\mu_j \xrightarrow[j\to\infty]{*} \|\lambda\|$. In a very similar way, we can check that for all $X \in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$, $\int X\cdot f_j\, d\mu_j \xrightarrow[j\to\infty]{} \delta\lambda(X)$ since

$$\begin{aligned}\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)(X) - \delta V_{\varepsilon_j}(t)(X)\right| &= \left|\delta V_{\varepsilon_j}(t)(\Phi_{\varepsilon_j} * X) - \delta V_{\varepsilon_j}(t)(X)\right| \le \|V_{\varepsilon_j}(t)\|(\mathbb{R}^n)\|\Phi_{\varepsilon_j} * X - X\|_{\mathrm{C}^1}\\ &\le \|V_0\|(\mathbb{R}^n)\|\Phi_{\varepsilon_j} * X - X\|_{\mathrm{C}^1} \xrightarrow[j\to\infty]{} 0\end{aligned}$$

where we used Remark 4.7, recalling (145) we obtain the desired distributional convergence

$$\begin{aligned}\left|\int X\cdot f_j\, d\mu_j - \delta\lambda(X)\right| &\le \int_0^1 \left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)(X) - \delta V_{\varepsilon_j}(t)(X)\right|\, dt + \left|\delta(V_{\varepsilon_j}(t)\otimes dt)(X) - \delta\lambda(X)\right|\\ &\xrightarrow[j\to\infty]{} 0\,.\end{aligned}$$

Let $\psi \in \mathrm{C}([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ be bounded and consider $F : ((t,y),q) \mapsto \psi(t,y)|q|^2$, then F is non-negative continuous, and with respect to q, it is convex and has superlinear growth, hence satisfying the assumptions of [25] 4.1.2. We additionally have by Remark 4.7 that for all j,

$$\begin{aligned}\int_{[0,1]\times\mathbb{R}^n} F((t,y),f_j(t,y))\, d\mu_j(t,y) &= \int_{[0,1]\times\mathbb{R}^n} \psi(t,y)\frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)}\, dy\, dt\\ &\le \|\psi\|_\infty \|V_{\varepsilon_j}(0)\|(\mathbb{R}^n) = \|\psi\|_\infty\|V_0\|(\mathbb{R}^n) < \infty\end{aligned}$$

and we can apply [25] 4.4.2(i) and (ii) (see also 2.36 in [5]): there exists $f \in L^1([0,1]\times\mathbb{R}^n,\mathbb{R}^n,\|\lambda\|)$ such that, up to extraction, the sequence of vector measures $f_j\mu_j$ converge to $f\|\lambda\|$ and

$$\begin{aligned}\int_0^1\int_{\mathbb{R}^n} \psi(t,y)|f(t,y)|^2\, d\|\lambda\| &= \int F((t,y),f(t,y))\, d\|\lambda\|\\ &\le \liminf_{j\to\infty} \int F((t,y),f_j(t,y))\, d\mu_j(t,y)\\ &\le \liminf_{j\to\infty} \int_{[0,1]\times\mathbb{R}^n} \psi(t,y)\frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)}\, dy\, dt \le \|\psi\|_\infty\|V_0\|(\mathbb{R}^n)\,.\end{aligned} \tag{148}$$

Thanks to the uniqueness of the distributional limit: $f\|\lambda\| = \delta\lambda$ so that $(\delta\lambda)_s = 0$ and $f = -h(\cdot,\cdot,\lambda)$, and we obtain (146) plugging $\psi = 1$ in (148).
We are left with proving (147) for $0 \le t_1 \le t_2 \le 1$, and we can take an affine cut–off approximating $\mathbf{1}_{[t_1,t_2]}$ from below: for k large enough with respect to $t_2 - t_1$, let χ_k be a continuous piecewise-affine function satisfying $\mathbf{1}_{[t_1+\frac{1}{k},t_2-\frac{1}{k}]} \le \chi_k \le \mathbf{1}_{[t_1,t_2]}$, then applying (148) to $\chi_k\psi$ gives

$$\int_0^1\int_{\mathbb{R}^n} \chi_k\psi|h|^2\, d\|\lambda\| \le \liminf_{j\to\infty} \int_{[0,1]\times\mathbb{R}^n} \chi_k\psi\frac{\left|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)\right|^2}{(\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j)}\, dy\, dt$$

and we can take the limit $k\to\infty$ in the l.h.s. by dominated convergence while we use $\chi_k \le \mathbf{1}_{[t_1,t_2]}$ in the r.h.s. to conclude the proof of (147), and hence the current proof. □

Proposition 5.12 (Spacetime Brakke inequality for the limit flow). *Let $\lambda = dt\otimes V(t)$ be the limit measure defined in Proposition 5.10. We assume that $V(t)$ is a rectifiable d–varifold for a.e $t\in[0,1]$:*

$$V(t) = \theta_t\mathcal{H}^d_{|\mathcal{M}_t}\otimes\delta_{T.\mathcal{M}_t} = \|V(t)\|\otimes\delta_{T.\mathcal{M}_t}\,.$$

Then, for any $\varphi \in \mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$ *and* $0\le t_1\le t_2\le 1$,

$$\|V(t_2)\|(\varphi(t_2,\cdot))-\|V(t_1)\|(\varphi(t_1,\cdot)) \le -\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(t,y)|h(t,y,\lambda)|^2\,d\|V(t)\|\,dt$$
$$+\int_{t_1}^{t_2}\int_{\mathbb{R}^n\times G_{d,n}} T_y\mathcal{M}_t^{\perp}(\nabla\varphi(t,y))\cdot h(t,y,\lambda)\,d\|V(t)\|\,dt+\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\partial_t\varphi(t,\cdot)\,d\|V(t)\|\,dt\,.$$

Proof. Denote $\lambda_{\varepsilon_j}=dt\otimes V_{\varepsilon_j}(t)$ and choose (as in Proposition 5.10) a sequence $(\varepsilon_j)_j$ satisfying:

$$\lim_{j\to\infty}\lambda_{\varepsilon_j}=\lambda=dt\otimes V(t)\quad\text{and}\quad\lim_{j\to\infty}\|V_{\varepsilon_j}(t)\|=\|V(t)\|\,.$$

Consider $\varphi\in\mathrm{C}^1_c([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$, t_1,t_2 such that $0\le t_1\le t_2\le 1$. The inequality we are seeking to prove is linear in φ, without loss of generality we assume $\varphi<1$, and for all sufficiently large $i\in\mathbb{N}$ we define $\varphi_i:=\varphi+i^{-1}<1$. We can plug φ_i in (116), also recalling (12), we obtain:

$$\|V_{\varepsilon_j}(t_2)\|(\varphi_i(t_2,\cdot))-\|V_{\varepsilon_j}(t_1)\|(\varphi_i(t_1,\cdot))-\int_{t_1}^{t_2}\|V_{\varepsilon_j}(t)\|(\partial_t\varphi_i(t,\cdot))\,dt$$
$$=\int_{t_1}^{t_2}\int_{\mathbb{R}^n\times G_{d,n}} S^{\perp}(\nabla_x\varphi_i)\cdot h_{\varepsilon_j}(\cdot,V_{\varepsilon_j}(t))\,dV_{\varepsilon_j}(t)\,dt+\int_{t_1}^{t_2}\delta(V_{\varepsilon_j}(t))\Big[\varphi_i(t,\cdot)h_{\varepsilon}(\cdot,V_{\varepsilon}(t))\Big]\,dt \quad (149)$$

and the proof now consists in taking the limit, first in j and then in i.

Step 1: We take the limit in the l.h.s. of (149), that is, we prove

$$\|V_{\varepsilon_j}(t_2)\|(\varphi_i(t_2,\cdot))-\|V_{\varepsilon_j}(t_1)\|(\varphi_i(t_1,\cdot))-\int_{t_1}^{t_2}\|V_{\varepsilon_j}(t)\|(\partial_t\varphi_i(t,\cdot))\,dt$$
$$\xrightarrow[i,j\to\infty]{}\|V(t_2)\|(\varphi(t_2,\cdot))-\|V(t_1)\|(\varphi(t_1,\cdot))-\int_{t_1}^{t_2}\|V(t)\|(\partial_t\varphi(t,\cdot))\,dt. \quad (150)$$

First note that $\partial_t\varphi_i=\partial_t\varphi$ and recall that for all $t\in[0,1]$,

$$\|V_{\varepsilon_j}\|(t)\xrightarrow[j\to\infty]{*}\|V(t)\|\quad\Longrightarrow\quad\begin{cases}\|V_{\varepsilon_j}(t)\|(\partial_t\varphi_i(t,\cdot))=\|V_{\varepsilon_j}(t)\|(\partial_t\varphi(t,\cdot))\xrightarrow[j\to\infty]{}\|V(t)\|(\partial_t\varphi(t,\cdot))\\ \|V_{\varepsilon_j}(t)\|(\varphi(t,\cdot))\xrightarrow[j\to\infty]{}\|V(t)\|(\varphi(t,\cdot))\end{cases}$$

and since $\|V_{\varepsilon_j}(t)\|(\partial_t\varphi(t,\cdot))\le\|\partial_t\varphi\|_\infty\|V_0\|(\mathbb{R}^n)$ by the decay of the mass (Remark 4.7), we infer by dominated convergence that for any i,

$$\int_{t_1}^{t_2}\|V_{\varepsilon_j}(t)\|(\partial_t\varphi_i(t,\cdot))\,dt\xrightarrow[j\to\infty]{}\int_{t_1}^{t_2}\|V(t)\|(\partial_t\varphi(t,\cdot))\,dt\,.$$

Using again the decay of the mass and $\varphi_i=\varphi+i^{-1}$, we obtain

$$\big|\|V_{\varepsilon_j}(t)\|(\varphi_i(t,\cdot))-\|V(t)\|(\varphi(t,\cdot))\big|\le i^{-1}\|V_0\|(\mathbb{R}^n)+\big|\|V_{\varepsilon_j}(t)\|(\varphi(t,\cdot))-\|V(t)\|(\varphi(t,\cdot))\big|\xrightarrow[i,j\to\infty]{}0$$

and with $t=t_1,t_2$ we can conclude the proof of (150) (Step 1). We now deal with the two terms involving the mean curvature.

Step 2: We now prove that

$$\limsup_{i\to\infty}\limsup_{j\to\infty}\int_{t_1}^{t_2}\delta V_{\varepsilon_j}(t)(\varphi_i h_{\varepsilon_j}(\cdot,V_{\varepsilon_j}(t)))dt\le-\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\varphi(t,y)|h(t,y,\lambda)|^2\,d\|V(t)\|(y)\,dt. \quad (151)$$

First, we note that $\varphi_i=\varphi+i^{-1}$ and then, there exists $m_{i,\varphi}\in\mathbb{N}$ (large enough, depending on i and φ: e.g. $m_{i,\varphi}\ge i\|\nabla_x\varphi\|_\infty$) such that for all $m\ge m_{i,\varphi}$, $\varphi_i\in\mathcal{A}_m$. We apply (136) with $\varepsilon=\varepsilon_j$

and $\psi = \varphi_i$ whence, for fixed φ and i, one has to take j large enough to ensure $\varepsilon_j \leq \varepsilon_*$ and $\varepsilon_j^{-\frac{1}{6}} \geq 2m_{i,\varphi}$: this is the reason why we have to take $\lim_{j\to\infty}$ before $\lim_{i\to\infty}$ hereafter. Concerning the varifold, we apply (136) with $W = V_{\varepsilon_j}(t)$ (for $t \in [0,1]$) and $M = \|V_0\|(\mathbb{R}^n)$ since we know that $\|V_{\varepsilon_j}(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) \leq M$. We obtain, for all $t \in [0,1]$ and for all j large enough,

$$\left|\delta V_{\varepsilon_j}(t)\left(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\right) + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\right| \leq \varepsilon_j^{\frac{1}{4}} \left(1 + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\right)$$

which we integrate between t_1 and t_2 so that using $0 \leq \varphi_i \leq 1$ and Remark 4.7,

$$\int_{t_1}^{t_2} \left|\delta V_{\varepsilon_j}(t)\left(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\right) + \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\right| dt$$
$$\leq \varepsilon_j^{\frac{1}{4}} \left(1 + \int_{[0,1]\times\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\, dt\right) \leq \varepsilon_j^{\frac{1}{4}} \left(1 + \|V_0\|(\mathbb{R}^n)\right) \xrightarrow[j\to\infty]{} 0\,.$$

We infer:

$$\limsup_{j\to\infty} \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(\varphi_i h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)))dt = -\liminf_{j\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{\varphi_i |\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\, dt$$
$$\leq -\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \varphi_i |h(\cdot,\cdot,\lambda)|^2 d\|V(t)\|\, dt \quad \text{by (147) in Proposition 5.11}$$

and the proof of (151) (Step 2) follows from $-\varphi_i \leq -\varphi$.

Step 3: As $\nabla_x \varphi_i = \nabla_x \varphi$, we are left with the proof of

$$\limsup_{j\to\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla_y \varphi) \cdot h_{\varepsilon_j}(y, V_{\varepsilon_j}(t)) dV_{\varepsilon_j}(t) dt \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^n \times G_{d,n}} S^\perp(\nabla_y \varphi) \cdot h(t,y,\lambda)\, d\lambda. \quad (152)$$

We fix an arbitrary function $g \in \mathrm{C}_c^2(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$, there exists $m_g \in \mathbb{N}$ such that $g(t,\cdot)) \in \mathcal{B}_m\ \forall m \geq m_g$ and $\forall t \in [0,1]$, this is due to the compactness of $[0,1]$. We apply (138) with $\varepsilon = \varepsilon_j$ and $X = g$ and we take j large enough to ensure that $\varepsilon_j \leq \varepsilon_*$ and $\varepsilon_j^{-\frac{1}{6}} \geq 2m_g$. Concerning the varifold, we apply (138) with $W = V_{\varepsilon_j}(t)$ (for $t \in [0,1]$) and $M = \|V_0\|(\mathbb{R}^n)$ since we know that $\|V_{\varepsilon_j}(t)\|(\mathbb{R}^n) \leq \|V_0\|(\mathbb{R}^n) \leq M$ by Remark 4.7. We obtain, for j large enough

$$\left|\int_{\mathbb{R}^n} h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \cdot g(t,\cdot)\, d\|V_{\varepsilon_j}(t)\| + \delta V_{\varepsilon_j}(t)(g(t,\cdot))\right| \leq \varepsilon_j^{\frac{1}{4}} + \varepsilon_j^{\frac{1}{4}} \left(\int_{\mathbb{R}^n} \frac{|\Phi_\varepsilon * \delta V_{\varepsilon_j}(t)|^2}{\Phi_\varepsilon * \|V_{\varepsilon_j}(t)\| + \varepsilon}\, dx\right)^{\frac{1}{2}}$$

which we integrate between t_1 and t_2 so that

$$\int_{t_1}^{t_2} \left|\int_{\mathbb{R}^n} h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t)) \cdot g(t,\cdot)\, d\|V_{\varepsilon_j}(t)\| + \delta V_{\varepsilon_j}(t)(g(t,\cdot))\right| dt$$
$$\leq \int_{t_1}^{t_2} \varepsilon_j^{\frac{1}{4}}\, dt + \varepsilon_j^{\frac{1}{4}} \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j} dx\right)^{\frac{1}{2}} dt$$
$$\leq \varepsilon_j^{\frac{1}{4}} + \varepsilon_j^{\frac{1}{4}} \left(\int_0^1 \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\, dt\right)^{\frac{1}{2}}$$
$$\leq \varepsilon_j^{\frac{1}{4}} \left(1 + \|V_0\|(\mathbb{R}^n)^{\frac{1}{2}}\right) \xrightarrow[j\to\infty]{} 0\,, \quad (153)$$

where we used Jensen inequality and Remark 4.7.
We now observe that the map $g : (t, y) \mapsto (T_y\mathcal{M}_t)^\perp(\nabla\varphi)$ is $dt \otimes V(t)\|$–measurable and belongs to $L^2(dt \otimes V(t)\|)$ ($\|V(t)\|$ is finite and $\varphi \in \mathrm{C}^1_c$, hence g is bounded by $\|\nabla\varphi\|$), we can assert that, for any $\eta \in (0, 1)$, there exists a map $g_\eta \in \mathrm{C}^2_c(\mathbb{R}^n \times [t_1, t_2], \mathbb{R}^n)$ such that:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left|(T_y\mathcal{M}_t)^\perp(\nabla\varphi(y)) - g_\eta(t, y)\right|^2 d\|V(t)\|(y)\, dt < \eta^2. \tag{154}$$

Now we compute

$$\begin{aligned}
\int_{t_1}^{t_2} \int_{\mathbb{R}^n\times G_{d,n}} S^\perp(\nabla\varphi) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\lambda_{\varepsilon_j} &= \int_{t_1}^{t_2} \int_{\mathbb{R}^n\times G_{d,n}} \Big(S^\perp(\nabla\varphi) - g_\eta\Big) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\lambda_{\varepsilon_j} \\
&+ \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} g_\eta \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\|V_{\varepsilon_j}(t)\| dt + \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt\right) \\
&- \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt + \delta\lambda(g_\eta) \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} h(\cdot, \cdot, \lambda) \cdot \Big(g_\eta - (T_y\mathcal{M}_t)^\perp(\nabla\varphi)\Big)\, d\|V(t)\|\, dt \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (T_y\mathcal{M}_t)^\perp(\nabla\varphi(t, y)) \cdot h(t, y, \lambda)\, d\|V(t)\|\, dt.
\end{aligned} \tag{155}$$

By the varifold convergence, (154) and (137) we have

$$\begin{aligned}
&\limsup_j \int_{t_1}^{t_2} \int_{\mathbb{R}^n\times G_{d,n}} \Big(S^\perp(\nabla\varphi) - g_\eta\Big) \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\lambda_{\varepsilon_j} \\
&\leq \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n\times G_{d,n}} |S^\perp(\nabla\varphi) - g_\eta|^2\, d\lambda\right)^{\frac{1}{2}} \left(\limsup_j \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))|^2\, d\|V_{\varepsilon_j}(t)\|\right)^{\frac{1}{2}} \\
&= \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |(T_y\mathcal{M}_t)^\perp(\nabla\varphi) - g_\eta|^2\, d\|V(t)\|\, dt\right)^{\frac{1}{2}} \left(\limsup_j \int_0^1 \int_{\mathbb{R}^n} \frac{|\Phi_{\varepsilon_j} * \delta V_{\varepsilon_j}(t)|^2}{\Phi_{\varepsilon_j} * \|V_{\varepsilon_j}(t)\| + \varepsilon_j}\, dx\, dt\right)^{\frac{1}{2}} \\
&\leq \eta\, (\|V_0\|(\mathbb{R}^n))^{\frac{1}{2}} \xrightarrow[\eta\to 0]{} 0.
\end{aligned} \tag{156}$$

By (153) we have:

$$\limsup_j \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^n} g_\eta \cdot h_{\varepsilon_j}(\cdot, V_{\varepsilon_j}(t))\, d\|V_{\varepsilon_j}\|(t) + \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt\right) = 0. \tag{157}$$

By the varifold convergence we have:

$$\limsup_j \left| \int_{t_1}^{t_2} \delta V_{\varepsilon_j}(t)(g_\eta)\, dt - \delta\lambda(g_\eta)\right| = 0, \tag{158}$$

and finally, the Cauchy-Schwarz inequality and (154) imply

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^n} h(\cdot,\cdot,\lambda)\cdot\Big(g_\eta-(T_y\mathcal{M}_t)^\perp(\nabla\varphi)\Big)\,d\|V(t)\|\,dt\\
&\leq\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^n}|h(\cdot,\cdot,\lambda)|^2\,d\|V(t)\|\,dt\right)^{\frac12}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^n}|g_\eta-(T_y\mathcal{M}_t)^\perp(\nabla\varphi)|^2\,d\|V(t)\|\,dt\right)^{\frac12} \qquad (159)\\
&\leq(\|V_0\|(\mathbb{R}^n))^{\frac12}\eta\xrightarrow[\eta\to0]{}0.
\end{aligned}$$

From (155)-(159) we deduce (152) (Step 3), this concludes the proof of Proposition 5.12. □

Remark 5.13. From the proof, writing $V(t)=\|V(t)\|\otimes\nu^{(t,x)}$, we notice that assuming that $\nu^{(t,x)}$ is a Dirac measure is sufficient, as we do not use the fact that $T.\mathcal{M}_t$ is the tangent space nor the properties of $\|V(t)\|$ as a rectifiable measure.

Remark 5.14 (Non uniqueness of the limit spacetime Brakke flows)**.** We recall that the limit measure λ in Theorem 5.6 depends on the choice of the subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$, hence is not unique (in general). This, somehow, is related to the non-uniqueness of Brakke flows, as Brakke flows themselves are spacetime Brakke flows when tensored with the measure dt.

Appendix A. Some useful lemmas in matrix algebra

We collect in this appendix several results in linear algebra used in the proofs of Section 3. The following result follows directly from the triangle inequality, but since it is used multiple times in the proof of Proposition 3.10, we think it is useful to have it as a lemma.

Lemma A.1. *Let $A,B\in\mathcal{M}_n$ and $S,T\in\mathcal{M}_{d,n}$ be such that $\|S\|=\|T\|=1$, one has*

$$\|SAS^t-TBT^t\|\leq(\|A\|+\|B\|)\,\|S-T\|+\|A-B\|. \qquad (160)$$

Proof. Using the triangle inequality, and the fact that $\|M^t\|=\|M\|$ for any matrix M we infer that

$$\begin{aligned}
\|SAS^t-TBT^t\|&\leq\|SAS^t-SAT^t\|+\|SAT^t-SBT^t\|+\|SBT^t-TBT^t\|\\
&\leq\|S\|\|A\|\|S^t-T^t\|+\|S\|\|A-B\|\|T^t\|+\|S-T\|\|B\|\|T^t\|\\
&\leq(\|A\|+\|B\|)\,\|S-T\|+\|A-B\|\,.
\end{aligned}$$

This concludes the proof of (160). □

The following lemma contains several properties on determinant's expansions, mainly used to prove Propositions 3.4 and 3.10. The matrix product is denoted by $\circ$ for clarity.

Lemma A.2. *There exists a constant $c_2\geq1$ only depending on n such that the following estimates hold:*

(1) Let $1\leq k\leq n$ and $Q\in\mathcal{M}_k$ be such that $|Q|_\infty\leq1$, then

$$|\det(I_k+Q)-\det(I_k)|\leq c_2|Q|_\infty\quad\text{and}\quad|\det(I_k+Q)-\det(I_k)-\mathrm{tr}(Q)|\leq c_2(|Q|_\infty)^2\,. \qquad (161)$$

(2) Let $1\leq d\leq n$, $L\in\mathcal{M}_{d,n}$, $R\in\mathcal{M}_n$ be such that $L\circ L^t=I_d$ and $|R|_\infty\leq1$. Then

$$((I_n+R)\circ L^t)^t\circ((I_n+R)\circ L^t)=I_d+Q\quad\text{with}\quad|Q|_\infty\leq c_2|R|_\infty\,. \qquad (162)$$

Moreover, if we assume that $c_2|R|_\infty\leq1$, then

$$\Big|\det\big(((I_n+R)\circ L^t)^t\circ((I_n+R)\circ L^t)\big)^{\frac12}-1\Big|\leq c_2|R|_\infty, \qquad (163)$$

and

$$\Big|\det\big(((I_n+R)\circ L^t)^t\circ((I_n+R)\circ L^t)\big)^{\frac12}-1-\mathrm{tr}\big(R\circ L^t\circ L\big)\Big|\leq c_2|R|_\infty^2. \qquad (164)$$

Proof. As long as c_2 only depends on n, we may increase it whenever needed throughout the proof. We consider the normed space $(\mathcal{M}_{p,q}, |\cdot|_\infty)$ and we recall that for $M \in \mathcal{M}_{p,q}$ and $N \in \mathcal{M}_{q,r}$,

$$|MN|_\infty \le q\,|M|_\infty |N|_\infty\,. \tag{165}$$

Let $Q \in \mathcal{M}_k$ be such that $|Q|_\infty \le 1$ and let $B = \{M \in \mathcal{M}_k \,:\, |I_k - M|_\infty \le 1\}$ be the closed unit ball centered at I_k. By compactness of B, we can introduce

$$c_{2,k} = \max\left\{1, \max_{M\in B} \|D\det(M)\|_\infty, \frac{1}{2}\max_{M\in B}\|D^2\det(M)\|_\infty\right\}\,,$$

where $\|\cdot\|_\infty$ and $|||\cdot|||_\infty$ denote, respectively, the linear and bilinear operator norms associated with $(\mathcal{M}_{p,q}, |\cdot|_\infty)$. Note that $c_{2,k}$ depends on k (since B depends on k) though this can be avoided by defining $c_2 = \max_{1\le k\le n} c_{2,k}$. Recall that the differential of the determinant map $\det$ at I_k is the trace map, i.e. $D\det(I_k) = \mathrm{tr}$, therefore, the application of the Taylor-Lagrange inequality for $\det$ on the line segment $[I_k, I_k + Q] \subset B$ yields (161).

Let $L \in \mathcal{M}_{d,n}$, $R \in \mathcal{M}_n$ be such that $L \circ L^t = I_d$ and $|R|_\infty \le 1$ and let us use the notation $Q = L\circ\big(R^t + R\big)\circ L^t + L\circ R^t\circ R\circ L^t \in \mathcal{M}_d$ hereafter, so that $((I_n+R)\circ L^t)^t\circ((I_n+R)\circ L^t) = I_d + Q$. First note that $|L|_\infty \le 1$, indeed, the assumption $L\circ L^t = I_d$ can be reformulated as follows: the columns of L^t (i.e. the rows of L) constitute an orthonormal family $(v_1,\dots,v_d)$ of $\mathbb{R}^n$ so that $|L|_\infty = \max_{ij}|L_{ij}| = \max_{ij}|v_i\cdot e_j| \le 1$. Using (165) and $|M^t|_\infty = |M|_\infty$, we have

$$\begin{aligned}|Q|_\infty &= \big|L\circ\big(R^t+R\big)\circ L^t + L\circ R^t\circ R\circ L^t\big|_\infty \le n^2|R+R^t|_\infty|L|_\infty^2 + n^3|L|_\infty^2|R|_\infty^2\\ &\le (2n^2 + n^3|R|_\infty)|R|_\infty \le c_2|R|_\infty\,,\end{aligned}$$

that is (162).

We now assume $c_2|R|_\infty \le 1$ and consequently $((I_n+R)\circ L^t)^t\circ((I_n+R)\circ L^t) = I_d + Q$ with $|Q|_\infty \le 1$ so that the first part of (161) gives

$$|\det(I_d+Q) - 1| \le c_2|Q|_\infty \le c_2^2|R|_\infty \le 1 \quad \text{and in particular} \quad \det(I_d+Q) \ge 0\,. \tag{166}$$

We infer (163) applying $|a-1| \le |a^2-1|$ with $a = \det(I_d+Q) \ge 0$. We are left with the proof of (164). We now apply the second inequality in (161) to obtain

$$|\det(I_d+Q) - 1 - \mathrm{tr}\,Q| \le c_2|Q|_\infty^2 \le c_2^3|R|_\infty^2\,. \tag{167}$$

Furthermore, using $\mathrm{tr}(A) = \mathrm{tr}\big(A^t\big)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ when both products make sense, we have

$$\mathrm{tr}\big(L\circ R^t\circ L^t\big) = \mathrm{tr}\big(L\circ R\circ L^t\big) = \mathrm{tr}\big(R\circ L^t\circ L\big)$$

and thus, by definition of Q and (165),

$$\begin{aligned}\big|\mathrm{tr}\,Q - 2\,\mathrm{tr}\big(R\circ L^t\circ L\big)\big| &= \big|\mathrm{tr}\big(L\circ R^t\circ R\circ L^t\big)\big| \le d|L\circ R^t\circ R\circ L^t|_\infty \le dn^3|L|_\infty^2|R|_\infty^2\\ &\le n^4|R|_\infty^2\,.\end{aligned} \tag{168}$$

From (167) and (168) we obtain

$$\big|\det(I_d+Q) - 1 - 2\,\mathrm{tr}\big(R\circ L^t\circ L\big)\big| \le (c_2^3+n^4)|R|_\infty^2 \tag{169}$$

We now apply the following inequality, valid for any $z \ge -1$,

$$\left|\sqrt{1+z} - 1 - \frac{1}{2}z\right| \le \frac{1}{2}z^2$$

with $z = \det(I_d+Q) - 1$, from (166) we know that $-1 \le z \le c_2^2|R|_\infty$, we hence obtain

$$\left|\sqrt{\det(I_d+Q)} - 1 - \mathrm{tr}\big(R\circ L^t\circ L\big)\right| \le \left|\sqrt{1+z} - 1 - \frac{1}{2}z\right| + \left|\frac{1}{2}z - \mathrm{tr}\big(R\circ L^t\circ L\big)\right|$$

$$\leq \frac{1}{2}z^2 + \frac{1}{2}(c_2^3 + n^4)|R|_\infty^2 \text{ thanks to (169)},$$
$$\leq \frac{1}{2}(c_2^4 + c_2^3 + n^4)|R|_\infty^2$$

which concludes the proof of (164). □

The following lemma stems directly from Lemma A.2.

Lemma A.3. *Let $P, N \in \mathcal{M}_d$ and assume that P is invertible, then*

$$\|P^{-1}\|\,\|P-N\| \leq 1 \quad \Rightarrow \quad |\det(P) - \det(N)| \leq c_2|\det(P)|\|P^{-1}\|\,\|P-N\|\,. \tag{170}$$

Proof. Indeed, first note that

$$|\det(P) - \det(N)| = |\det(P)|\left|1 - \det\left(P^{-1}N\right)\right| \quad \text{and} \quad P^{-1}N = I_d + P^{-1}(N-P)\,.$$

Furthermore $\left|P^{-1}(N-P)\right|_\infty \leq \|P^{-1}(N-P)\| \leq \|P^{-1}\|\,\|P-N\| \leq 1$ so that applying (161) with $k=d$ and $Q = P^{-1}(N-P)$ we can assert that

$$\left|1 - \det\left(P^{-1}N\right)\right| \leq c_2\left|P^{-1}(N-P)\right|_\infty \leq c_2\,\|P^{-1}\|\,\|N-P\|$$

which concludes the proof of (170). □

The following is a crucial step to prove Proposition 3.10.

Lemma A.4. *Let $S, T \in G_{d,n}$, there exist $\tilde{S} = (s_1|\dots|s_d)^t, \tilde{T} = (t_1|\dots|t_d)^t \in \mathcal{M}_{d,n}$ where $\{s_i\}_{i=1}^d$ and $\{t_i\}_{i=1}^d$ are two orthonormal bases of S and T such that*

$$\|\tilde{S} - \tilde{T}\| \leq 2\|S - T\|.$$

Proof. Let θ be the largest principal angle between the subspaces S and T, which can be characterized by:

$$\sin(\theta) = \max_s \min_t \sqrt{1 - \langle s,t\rangle^2},\ \ s \in S, t \in T \quad \text{and } |s| = |t| = 1.$$

We infer from [1, Proposition III.29] that $\|S - T\| = \sin(\theta)$, furthermore, there exists a rotation r of $\mathbb{R}^n$ such that $r(S) = T$, with

$$\|r - I_n\| \leq 2\sin(\theta/2).$$

Let $\tilde{S} = (s_1|\dots|s_d)^t \in \mathcal{M}_{d,n}$, with $\{s_i\}_{i=1}^d$ being an orthonormal basis of S, the matrix $\tilde{T} = r \circ \tilde{S} \in \mathcal{M}_{d,n}$ can be written as $(t_1|\dots|t_d)^t$ where $\{t_i\}_{i=1}^d$ is an orthonormal basis of and T. We have then, using that $\|\tilde{S}\| = 1$

$$\|\tilde{S} - \tilde{T}\| = \|\tilde{S} - r \circ \tilde{S}\| \leq \|I_n - r\|\|\tilde{S}\| \leq 2\sin(\theta/2);$$

the result follows from noting that $2\sin(\theta/2) \leq 2\sin(\theta)$ as $\theta \in [0, \pi/2]$. □

References

[1] E. Aamari. *Vitesses de convergence en inférence géométrique*. PhD thesis, Université Paris-Saclay (ComUE), 2017. 54

[2] S. M. Allen and J. W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta metallurgica*, 27(6):1085–1095, 1979. 2

[3] F. Almgren, J. E. Taylor, and L. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993. 2

[4] L. Ambrosio. Geometric evolution problems, distance function and viscosity solutions. In *Calculus of variations and partial differential equations (Pisa, 1996)*, pages 5–93. Springer, Berlin, 2000. 2

[5] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. 6, 40, 46, 47, 48

[6] L. Ambrosio and C. Mantegazza. Curvature and distance function from a manifold. volume 8, pages 723–748. 1998. Dedicated to the memory of Fred Almgren. 2

[7] L. Ambrosio and H. M. Soner. Level set approach to mean curvature flow in arbitrary codimension. *Journal of differential geometry*, 43(4):693–737, 1996. 2

[8] K. Atkinson, W. Han, and D. E. Stewart. *Numerical solution of ordinary differential equations*. John Wiley and Sons, 2011. 24

[9] G. Bellettini. *Lecture notes on mean curvature flow, barriers and singular perturbations*, volume 12 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2013. 2

[10] G. Bellettini and M. Novaga. A result on motion by mean curvature in arbitrary codimension. *Differential Geom. Appl.*, 11(3):205–220, 1999. 2

[11] G. Bellettini and M. Paolini. Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 19:43–67, 1995. 2

[12] K. A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1978. 2, 3, 5, 6, 9, 39

[13] B. Buet, G. P. Leonardi, and S. Masnou. A varifold approach to surface approximation. *Archive for Rational Mechanics and Analysis*, 226:639–694, 2017. 9

[14] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991. 2

[15] E. De Giorgi. Barriere, frontiere, e movimenti di varietà. In *Conferenza tenuta al Dipartimento di Matematica dell'Università di Pavia*, volume 18, 1994. 2

[16] K. Ecker. *Regularity theory for mean curvature flow*, volume 57 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004. 2

[17] S. D. Eidelman. *Parabolic systems*. North-Holland Publishing Co., Amsterdam-London; Wolters-Noordhoff Publishing, Groningen, 1969. Translated from the Russian by Scripta Technica, London. 2

[18] L. C. Evans and J. Spruck. Motion of level sets by mean curvature I. In *Fundamental Contributions to the Continuum Theory of Evolving Phase Interfaces in Solids: A Collection of Reprints of 14 Seminal Papers*, pages 328–374. Springer, 1991. 2

[19] L. C. Evans and J. Spruck. Motion of level sets by mean curvature II. *Transactions of the american mathematical society*, 330(1):321–332, 1992. 2

[20] L. C. Evans and J. Spruck. Motion of level sets by mean curvature III. *The Journal of Geometric Analysis*, 2(2):121–150, 1992. 2

[21] L. C. Evans and J. Spruck. Motion of level sets by mean curvature IV. *The Journal of Geometric Analysis*, 5(1):77–114, 1995. 2

[22] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *Journal of Differential geometry*, 26(2):285–314, 1987. 2

[23] S. Hensel and T. Laux. A new varifold solution concept for mean curvature flow: convergence of the Allen-Cahn equation and weak-strong uniqueness. *J. Differential Geom.*, 130(1):209–268, 2025. 2

[24] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *Journal of Differential Geometry*, 20(1):237–266, 1984. 2

[25] J. E. Hutchinson. Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana Univ. Math. J.*, 35:45–71, 1986. 48

[26] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *Journal of Differential Geometry*, 38(2):417–461, 1993. 2

[27] T. Ilmanen. *Elliptic regularization and partial regularity for motion by mean curvature*, volume 520. American Mathematical Soc., 1994. 2

[28] L. Kim and Y. Tonegawa. On the mean curvature flow of grain boundaries. *Ann. Inst. Fourier (Grenoble)*, 67(1):43–142, 2017. 2, 3, 5, 9, 10, 12, 39, 40, 43, 44

[29] T. Laux and F. Otto. The thresholding scheme for mean curvature flow and de Giorgi's ideas for minimizing movements. In *The role of metrics in the theory of partial differential equations*, volume 85 of *Adv. Stud. Pure Math.*, pages 63–93. Math. Soc. Japan, Tokyo, [2020] ©2020. 2

[30] S. Luckhaus and T. Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calculus of variations and partial differential equations*, 3(2):253–271, 1995. 2

[31] C. Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011. 2

[32] U. Menne. The concept of varifold. *Notices Amer. Math. Soc.*, 64(10):1148–1152, 2017. 6

[33] B. Merriman, J. K. Bence, and S. J. Osher. Diffusion generated motion by mean curvature. In J. E. Taylor, editor, *Computational Crystal Growers Workshop*, Selected Lectures in Math. Amer. Math. Soc., Providence, Rhode Island, 1992. 2
[34] R. S. Palais. When proper maps are closed. *Proceedings of the American Mathematical Society*, 24(4):835–836, 1970. 15
[35] A. Sagueni. On the avoidance principle for codimension 1 space-time Brakke flows. *arXiv e-prints*, arXiv.2509.06441, 2025 4
[36] F. Santambrogio. Optimal transport for applied mathematicians. *Birkäuser, NY*, 55(58-63):94, 2015. 7
[37] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983. 3, 6, 8
[38] K. Smoczyk. Mean curvature flow in higher codimension: introduction and survey. In *Global differential geometry*, volume 17 of *Springer Proc. Math.*, pages 231–274. Springer, Heidelberg, 2012. 2
[39] G. Strang. *Linear Algebra and its Applications*, volume 14. Academic Press, New York, 1980. 7
[40] S.Stuvard and Y. Tonegawa. *On the existence of canonical multi-phase Brakke flows*, Adv. Calc. Var., 17(1):33–78, 2024. 2
[41] Y. Tonegawa. *Brakke's mean curvature flow*. Springer, Singapore, 2019. 2, 3, 7, 39, 44

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