TD 1 : Covering Lemmas

Some notations : $U(x,R)$ is the open ball of center x and radius R, $B(x,R)$ is the closed ball of center x and radius R, $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.

Exercise 1.— *Almost everywhere Vitali's theorem.*

Theorem. *Let $E \subset \mathbb{R}^n$ be a Borel set and $\mathcal{F} \subset \{$ closed balls $\}$ be a Vitali covering of E by closed balls, that is, for all $x \in E$,*

$$\inf\{\operatorname{diam}(B) \,:\, B \in \mathcal{F} \text{ and } x \in B\} = 0.$$

Then there exists a countable family of two by two disjoint closed balls $\mathcal{G} = \{B_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $E \setminus \sqcup_k B_k$ is Lebesgue–negligible.

1. Let $W \subset \mathbb{R}^n$ be an open set such that $E \subset W$. Check that

$$\forall x \in E, \quad \inf\{\operatorname{diam}(B) \,:\, B \in \mathcal{F} \text{ and } x \in B \text{ and } B \subset W\} = 0.$$

2. Let $1 - (1/2)5^{-n} < \theta < 1$. We assume $\mathcal{L}^n(E) < +\infty$.
   (a) Show that there exists a countable family of two by two disjoint balls $\{B_i\}_{i\in\mathbb{N}}$ such that for all i, $B_i \subset W$ and
   $$\mathcal{L}^n\left(E - \cup_i B_i\right) \le \left(1 - (1/2)5^{-n}\right)\mathcal{L}^n(E).$$
   (b) Infer that there exists a finite family of two by two disjoint balls $\{B_i\}_{i=1}^N$ such that for all i, $B_i \subset W$ and
   $$\mathcal{L}^n\left(E - \cup_{i=1}^N B_i\right) \le \theta\mathcal{L}^n(E).$$
   (c) Show that there exist a countable family of two by two disjoint balls $\mathcal{G} = \{D_i\}_{i\in\mathbb{N}}$ and an increasing sequence of positive integers $(N_k)_{k\in\mathbb{N}^*}$ such that for all $k \ge 1$,
   $$\mathcal{L}^n\left(E - \cup_{i=1}^{N_k} D_i\right) \le \theta^k\mathcal{L}^n(E).$$
   (d) Conclude the case $\mathcal{L}^n(E) < +\infty$.

3. Prove the theorem (without assuming $\mathcal{L}^n(E) < +\infty$ any more).

   Notice that the same proof can be done with the exterior Lebesgue measure $\mathcal{L}^{n*}$ (since we did not use the additivity but only the subadditivity) and thus obtain the result for any $E \subset \mathbb{R}^n$ without assuming that E is Borel or measurable.

4. Show that the theorem remains true if we replace $\mathcal{L}^n$ by a doubling Radon measure μ : $\exists C \ge 1$, $\forall x \in \mathbb{R}^n$, $\forall r > 0$,
$$\mu(B(x,2r)) \le C\mu(B(x,r)).$$

Exercise 2.— *A weak version of Sard's Lemma*
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function. We consider

$$Z = \{x \in \mathbb{R}^n \,;\, Df(x) \text{ is not invertible}\}.$$

The aim of this exercise is to prove that $f(Z)$ is Lebesgue negligible.

1. Let $x \in Z$ be fixed. Show that there exists $C_x > 0$ such that for all $0 < \delta \leq 1$ we can find $\rho_{x,\delta} > 0$ such that if $r \leq \rho_{x,\delta}$ then $f(B(x,r))$ is included in a set of the form
$$[-\delta r, \delta r] \times [-C_x r, C_x r]^{n-1} ,$$
up to translation and rotation.

2. Let $\eta > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded open set and $x \in Z \cap \Omega$. Show that we can find $0 < r_x (= r_{x,\eta}) \leq 1$ such that $B(x, r_x) \subset \Omega$ and
$$\mathcal{L}^n(f(B(x, r_x))) \leq \eta \mathcal{L}^n(B(x, r_x)) .$$

3. Cover $Z \cap U(0,R)$ with balls $B(x, r_x)$ (or $B(x, r_x/5)$) as in the previous question. Applying Besicovitch covering lemma (or Vitali covering lemma) conclude that $f(Z)$ is negligible.

4. Prove the result without using any covering lemma under the stronger asumption f of class C^1.

Exercise 3.— *Lebesgue's Theorem for the Differentiability of Monotone Functions*
The aim is to prove the following result:

Theorem (Lebesgue's Theorem)**.** *Let $f :]a,b[\subset \mathbb{R} \to \mathbb{R}$ be non-decreasing (monotone), then f is $\mathcal{L}^1$–almost everywhere differentiable in $]a,b[$.*

1. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be (strictly) increasing, $p > 0$ and define
$$E_p = \left\{ x \in]a,b[: \liminf_{h \to 0} \frac{f(x+h) - f(x)}{h} < p \right\} .$$
Let us prove that if $F_p \subset E_p$ then $\mathcal{L}^{1^*}(f(F_p)) \leq p \mathcal{L}^{1^*}(F_p)$, where $\mathcal{L}^{1^*}$ is the outer Lebesgue measure in $\mathbb{R}$. We introduce the following notations, for $x \in \mathbb{R}, h \in \mathbb{R}^*$,
$I_h(x) = \begin{cases} [x, x+h] & \text{if} \quad h > 0 \\ [x+h, x] & \text{if} \quad h < 0 \end{cases}$ and (as f is increasing) $J_h(x) = \begin{cases} [f(x), f(x+h)] & \text{if} \quad h > 0 \\ [f(x+h), f(x)] & \text{if} \quad h < 0 \end{cases}$.
   (a) Let $U \subset]a,b[$ be an open set such that $F_p \subset U$, show that
$$\inf\{\mathrm{diam}(J_r(x)) \, : \, \mathcal{L}^1(J_r(x)) < p\mathcal{L}^1(I_r(x)), \, I_r(x) \subset U\} = 0 .$$
   (b) Infer that $\mathcal{L}^{1*}(f(F_p)) \leq p \mathcal{L}^{1*}(F_p)$.

2. For $q > 0$, we similarly define $E^q = \left\{ x \in]a,b[: \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h} > q \right\}$. Show that if $F^q \subset E^q$ then $\mathcal{L}^{1^*}(f(F^q)) \geq q \mathcal{L}^{1^*}(F^q)$.

3. (a) Show that the set of points $x \in]a,b[$ such that $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = +\infty$ is negligible (for $\mathcal{L}^1$).
   (b) For $0 < p < q$, we define
$$E_p^q = \left\{ x \in]a,b[: \liminf_{h \to 0} \frac{f(x+h) - f(x)}{h} < p < q < \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h} \right\} .$$
   Show that $\mathcal{L}^{1^*}(E_p^q) = 0$.
   (c) Show that the set of points where f is not differentiable is negligible for Lebesgue measure.

   If we only assume f non-decreasing, we can consider $f(x) + x$ which is then increasing.