TD 3 : Sobolev spaces

*$B(x,r)$ is the **open** ball of radius r and center x.*
We recall the following *Localization principle.*

Proposition. *Let $U \subset \mathbb{R}^n$ be an open set and let $v, w \in \mathrm{L}^1_{loc}(U)$. Then,*

$$u = v \text{ as distributions} \quad \Leftrightarrow \quad u = v \text{ a.e. in } U\,.$$

Exercice 1.— *Pointwise discontinuity.*
Let $\Omega \subset \mathbb{R}^n$ be an open set containing 0. Let $u : \Omega \to \mathbb{R}$. We assume that $u \in \mathrm{C}^1(\Omega \setminus \{0\})$ and we denote by $\nabla_p u$ the pointwise gradient of u (defined in $\Omega \setminus \{0\}$ and thus almost everywhere in Ω), whereas we denote by ∇u the distributional gradient of u (defined if $u \in \mathrm{L}^1_{loc}$).

1. Show that if $u \in \mathrm{L}^1_{loc}(\Omega)$ and $\nabla u \in \mathrm{L}^1_{loc}(\Omega)$ then $\nabla_p u \in \mathrm{L}^1_{loc}(\Omega)$.
2. In the case $n = 1$, give an example of function u defined in $\mathbb{R} \setminus \{0\}$ (for instance) such that $u \in \mathrm{C}^1(\mathbb{R} \setminus \{0\})$ and whose pointwise derivative is in $\mathrm{L}^1_{loc}(\mathbb{R})$ while its distributional derivative $u' \notin \mathrm{L}^1_{loc}$.

   We now assume that $n \geq 2$ and $\nabla_p u \in \mathrm{L}^1_{loc}(\Omega)$. The purpose of the exercise is to show that $u \in \mathrm{L}^1_{loc}$, $\nabla u \in \mathrm{L}^1_{loc}(\Omega)$ and that $\nabla_p u$ and ∇u coincide almost everywhere.
3. Let $\epsilon > 0$ such that $B(0,\epsilon) \subset \Omega$. Show that $\displaystyle\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} |u| \, d\mathcal{H}^{n-1} = 0$.

   Indication : We recall that for f positive measurable or L^1, $\displaystyle\int_{\mathbb{R}^n} f(x)\,dx = \int_{r=0}^{\infty} \int_{\partial B(0,r)} f \, d\mathcal{H}^{n-1}\,dr$, and if $f \in \mathrm{L}^1\left(\partial B(0,r), \mathcal{H}^{n-1}\right)$ then $\displaystyle\int_{\partial B(0,r)} u(x)\,d\mathcal{H}^{n-1}(x) = r^{n-1} \int_{\partial B(0,1)} u(ry)\,d\mathcal{H}^{n-1}(y)$.
4. Infer that $u \in \mathrm{L}^1_{loc}(\Omega)$.
5. Show that the distributional gradient of u is L^1_{loc} and coincide almost everywhere with the pointwise gradient $\nabla_p u$.
6. Let $1 \leq p < +\infty$ and $\alpha \in \mathbb{R}$. Let $\Omega = B(0,1)$, for which values of α, $u_\alpha : x \mapsto |x|^{-\alpha} \in \mathrm{W}^{1,p}$?
   Indication : We recall that $u_\alpha \in \mathrm{L}^1_{loc}(\mathbb{R}^n)$ if and only if $\alpha < n$.

Further extensions. $k \in \mathbb{N}$, $k \leq n-2$

- Let P be a k–dimensional subspace of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in \mathrm{C}^1(\Omega \setminus P)$ be such that $\nabla_p u \in \mathrm{L}^1_{loc}(\Omega)$. Then $u \in \mathrm{L}^1_{loc}(\Omega)$, $\nabla u \in \mathrm{L}^1_{loc}(\Omega)$ and $\nabla u = \nabla_p u$ a.e. in Ω.
- Let Σ be a k–dimensional subspace of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in \mathrm{C}^1(\Omega \setminus \Sigma)$ be such that $\nabla_p u \in \mathrm{L}^1_{loc}(\Omega)$. Then $u \in \mathrm{L}^1_{loc}(\Omega)$, $\nabla u \in \mathrm{L}^1_{loc}(\Omega)$ and $\nabla u = \nabla_p u$ a.e. in Ω.

Exercice 2.— *Product differentiation.*
Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq +\infty$.

1. Let $f \in \mathrm{W}^{1,p}(\Omega)$ and $a \in \mathrm{C}^1(\Omega)$ be bounded and with bounded order 1 partial derivatives. Check that $af \in \mathrm{W}^{1,p}$ and $\nabla(af) = a\nabla f + f\nabla a$.
2. Let $f, g \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Show that $fg \in \mathrm{W}^{1,p}(\Omega)$ and $\nabla(fg) = f\nabla g + g\nabla f$.

Exercice 3.— *A characterization of* $\mathrm{W}^{1,\infty}(\Omega)$ *functions.*

Let $\Omega \subset \mathbb{R}^n$ be an open set. We want to prove the following characterizations:

$$f \in \mathrm{W}^{1,\infty}_{loc}(\Omega) \quad \Leftrightarrow \quad f \text{ is locally Lipschitz}$$

and

$$f \in \mathrm{W}^{1,\infty}(\Omega) \quad \Leftrightarrow \quad f \in \mathrm{L}^\infty(\Omega) \text{ and } \exists C > 0,\ \forall x, y \in \Omega \text{ such that } [x,y] \subset \Omega,\ |f(x) - f(y)| \leq C|x-y|$$

Notice that in this case, we will check that we can take $C = \|\nabla f\|_{\mathrm{L}^\infty(\Omega)}$. *Pay attention to the fact that those characterizations are to be understood as* f *is a.e. equal to a function satisfying ...*

We recall that if $(\rho_\epsilon)_{\epsilon>0}$ is a mollifier and $f \in \mathrm{L}^1_{loc}(\Omega)$ then

(*i*) the convolution $f_\epsilon := f * \rho_\epsilon$ is well-defined and of class C^∞ in $\Omega_\epsilon = \{x \in \Omega \,:\, \mathrm{dist}(x, \mathbb{R}^n - \Omega) > \epsilon\}$ and converges almost everywhere to f when $\epsilon \to 0$.

(*ii*) if $f \in \mathrm{W}^{1,p}_{loc}(\Omega)$, $1 \leq p \leq \infty$, then $\nabla f_\epsilon = \nabla f * \rho_\epsilon$ in Ω_ϵ.

If $\omega \subset \mathbb{R}^n$ *is an open set,* $\omega \subset\subset \Omega$ *stands for* $\overline{\omega}$ *is compact and* $\overline{\omega} \subset \Omega$.

1. Lesgue's number lemma: let $K \subset \mathbb{R}^n$ and $\{U\}_{U \in \mathcal{U}}$ be a covering of K with open sets. Then, there exists $\delta > 0$ such that for every set $X \subset K$, if $\operatorname{diam} X \leq \delta$ then X is contained in one open set of the covering.

   True for (K, d) *compact metric space.*

2. We first assume that $f \in \mathrm{W}^{1,\infty}_{loc}(\Omega)$, that is, $f \in \mathrm{W}^{1,\infty}(\omega)$ for all open set $\omega \subset\subset \Omega$. Let f_ϵ be defined as above.

   (a) Show that $(f_\epsilon)_\epsilon$ is equi-Lipschitz on every compact set.

   (b) Show that $(f_\epsilon)_\epsilon$ converges uniformly on compact sets to a continuous function $g \in \mathrm{C}(\Omega)$, and that f and g coincide a.e., we identify f and g hereafter.

   (c) Conclude that f is locally Lipschitz in the sense that for all compact set $K \subset \Omega$, $f_{|K}$ is Lipschitz.

   (d) Adapt the previous arguments to prove that if $\mathrm{W}^{1,\infty}(\Omega)$ then

   $$|f(x) - f(y)| \leq \|\nabla f\|_{\mathrm{L}^\infty(\Omega)}|x-y|,\ \forall x, y \in \Omega \text{ such that } [x,y] \subset \Omega\,.$$

3. We conversely assume that f is locally Lipschitz. Let $\omega \subset\subset \Omega$ be an open set and $\delta = \mathrm{dist}(\overline{\omega}, \mathbb{R}^n - \Omega) > 0$ and define $K = \{x \in \Omega \,:\, \mathrm{dist}(x, \overline{\omega}) \leq \delta/2\} \subset \Omega$. Let $\phi \in \mathrm{C}^\infty_c(\omega)$.

   (a) Let $t \in \mathbb{R}$, $|t| \leq \delta/2$ and let $(e_1, \ldots, e_n)$ bet the canonical basis of $\mathbb{R}^n$. Prove that for $i \in \{1, \ldots, n\}$,

   $$\left| \int_\Omega f(x) \frac{\phi(x - te_i) - \phi(x)}{|t|} \right| \leq \mathrm{Lip}(f_{|K})\|\phi\|_{\mathrm{L}^1(\omega)}\,.$$

   (b) Infer that $\nabla f \in \mathrm{L}^\infty(\omega)$ and conclude.

   (c) Notice that $[x, x + \delta/2 e_i] \subset B(x, \delta/2)$ and conclude the proof of the second characterization.

4. Let $\Omega = \{x \in \mathbb{R}^2 \setminus \mathbb{R}_- \times \{0\} \,:\, 1 < |x| < 2\}$ and let f be defined in Ω by $f(re^{i\theta}) = \theta$ ($1 < r < 2$ and $\theta \in]-\pi, \pi[$). Check that $f \in \mathrm{W}^{1,\infty}(\Omega)$ but is not Lipschitz in Ω.

   Actually, in such a case or more generally in a connected open set, f *is Lipschitz but with respect to the geodesic ditance that is*

   $$|f(x) - f(y)| \leq \|\nabla f\|_{\mathrm{L}^\infty(\Omega)} d_\Omega(x,y) \quad \textit{with} \quad d_\Omega(x,y) = \inf\left\{\mathrm{length}(\Gamma) \,:\, {\scriptstyle \Gamma \text{ polygonal line connecting} \atop x \text{ and } y,\, \Gamma \subset \Omega}\right\}$$

   and note that in a Lipschitz bounded connected open set, euclidean and geodesic distance are equivalent.