A MORSE-BOTT APPROACH TO CONTACT HOMOLOGY

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Abstract. Contact homology was introduced by Eliashberg, Givental and Hofer; this contact invariant is based on J-holomorphic curves in the symplectization of a contact manifold. We expose an extension of contact homology to Morse-Bott situations, in which closed Reeb orbits form submanifolds of the contact manifold. We then illustrate how to use this to compute contact homology with several examples.

1. Introduction

Contact homology [3] and Symplectic Field Theory [4] were introduced by Eliashberg, Givental and Hofer. It is a contact invariant that is based on rational curves with one positive punctures and several negative punctures in the symplectization of a contact manifold. The usefulness of contact homology has already been demonstrated by several computations for certain contact manifolds. Unfortunately, these computations are limited and difficult, because of an important assumption in the theory : the closed Reeb orbits must be nondegenerate (and, in particular, isolated). Therefore, when the contact manifold admits a natural and very symmetric contact form, this contact form has to be perturbed before starting the computation. As a consequence of this, the Reeb flow becomes rather chaotic and hard to study. But the worst part comes from the Cauchy-Riemann equation, which becomes perturbed as well. It is then nearly impossible to compute the moduli spaces of holomorphic curves. To avoid these difficulties, one would like to extend the theory to a larger set of admissible contact forms. This paper is an announcement for the PhD thesis [1] of the author, developing computational techniques for contact homology.

Contact homology can be thought of as some sort of Morse theory for the action functional for loops γ in M : $\mathcal{A}(\gamma) = \int_\gamma \alpha$. The critical points of $\mathcal{A}$ are the closed orbits under the Reeb flow φ_t and the corresponding critical values are the periods of these orbits. The set of critical values of $\mathcal{A}$ is called the action spectrum and will be denoted by $\sigma(\alpha)$. If the contact form is very symmetric, the closed Reeb orbits will not be isolated, so we have to think of $\mathcal{A}$ as a Morse-Bott functional. These considerations motivate the following definition.

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Definition 1. *A contact form α on M is said to be of Morse-Bott type if the action spectrum $\sigma(\alpha)$ of α is discrete and if, for every $T \in \sigma(\alpha)$, $N_T = \{p \in M \mid \varphi_T(p) = p\}$ is a closed, smooth submanifold of M, such that rank $d\alpha|_{N_T}$ is locally constant and $T_pN_T = \ker(\varphi_{T*} - I)_p$.*

This paper is organized as follows : in section 2, we generalize the construction of the moduli spaces of holomorphic curves to the Morse-Bott setting. In section 3, we define a Morse-Bott chain complex for contact homology. This will be our main tool for applications. Finally, in section 4, we use the Morse-Bott methods on several examples to illustrate their effectiveness.

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2. Construction of the moduli spaces

2.1. **Holomorphic curves in a symplectization.** Let (M, α) be a compact, $2n-1$-dimensional contact manifold. We denote the Reeb vector field associated to α by R_α. We are interested in the periodic orbits γ of R_α, i.e. curves $\gamma : [0, T] \to M$ such that $\frac{d\gamma}{dt} = R_\alpha$ and $\gamma(0) = \gamma(T)$. The period T of γ is also called action and can be computed using the action functional $\int_\gamma \alpha$.

If α is not a generic contact form for $\xi = \ker \alpha$ but has some symmetries, then the closed Reeb orbits are not isolated but come in families. Let $N_T = \{p \in M \mid \varphi_T(p) = p\}$, where φ_t is the flow of R_α. We assume that α is of Morse-Bott type (see definition 1), so that N_T is a smooth submanifold of M. The Reeb flow on M induces an S^1 action on N_T. Denote the quotient N_T/S^1 by S_T; this is an orbifold with singularity groups $\mathbb{Z}_k$. The singularities correspond to Reeb orbits with period T/k, covered k times. Since M is compact, there will be countably many such orbit spaces S_T. We will denote by S_i the connected components of the orbit spaces $(i = 1, 2, \dots)$.

The contact distribution ξ is equipped with a symplectic form $d\alpha$. Let $\mathcal{J}$ be the set of almost complex structures on ξ, compatible with $d\alpha$. This set is nonempty and contractible. Note that $\mathcal{J}$ is independent of the choice of contact form α for ξ (for a given coorientation of ξ), because the conformal class of $d\alpha$ is fixed. Let $\tilde{J} \in \mathcal{J}$; we can extend J to an almost complex structure J on the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$, where t denotes the coordinate of $\mathbb{R}$, by $J|_\xi = \tilde{J}$ and $J\frac{\partial}{\partial t} = R_\alpha$. Note that if we replace α with $f\alpha$, where

f is a positive function on M, we can keep the same $\tilde{J}$, but the extension to $\mathbb{R} \times M$ is modified.

Let (Σ, j) be a compact Riemann surface, and let $x_1^+, \ldots, x_{s^+}^+, x_1^-, \ldots, x_{s^-}^- \in \Sigma$ be a set of punctures. We are interested in J-holomorphic curves

$$\tilde{u} = (a, u) : (\Sigma \setminus \{x_1^+, \ldots, x_{s^+}^+, x_1^-, \ldots, x_{s^-}^-\}, j) \to (\mathbb{R} \times M, J)$$

which have the following behavior near the punctures : $\lim_{p \to x_i^\pm} a(p) = \pm\infty$ and the map u converges, near a puncture, to a closed Reeb orbit. We say that x_i^+ $(i = 1, \ldots, s^+)$ are positive punctures and x_j^- $(j = 1, \ldots, s^-)$ are negative punctures. Hofer showed that such J-holomorphic maps are characterized by $E(\tilde{u}) < \infty$. The Hofer energy is defined as follows : let $\mathcal{C} = \{\phi \in C^0(\mathbb{R}, [0,1]) \,|\, \phi' \geq 0\}$; then $E(\tilde{u}) = \sup_{\phi \in \mathcal{C}} \int_\Sigma \tilde{u}^* d(\phi\alpha)$. We will study these asymptotic properties with more details in section 2.2.

We want to associate a homology class to a holomorphic map. In order to do this, we need to fix some additional data. Choose a base point in each orbit space S_T and, for the corresponding Reeb orbit, choose a capping disk in M (if the Reeb orbit is not contractible, we can modify this discussion as in [4]). Then, given a holomorphic map with asymptotic Reeb orbits $\gamma_1^+, \ldots, \gamma_{s^+}^+, \gamma_1^-, \ldots, \gamma_{s^-}^-$, we join each asymptotic Reeb orbit $\gamma_i^\pm$ to the base point of the corresponding orbit space. Gluing the holomorphic curve, the cylinders lying above the paths and the capping disks, we obtain a homology class in $H_2(M, \mathbb{Z})$.
However, the result depends on the homotopy class of the chosen path in S_T. Clearly, the homology class is well-defined modulo $\mathcal{R} = \text{Image } (i_T \circ \pi_T^{-1} : H_1(S_T, \mathbb{Z}) \to H_2(M, \mathbb{Z}))$, where $i_T : N_T \to M$ is the embedding of N_T into M and $\pi_T : N_T \to S_T$ is the quotient under the Reeb flow. The elements of $\mathcal{R}$ are analogous to the rim tori of Ionel and Parker [8]. Note that $c_1(\xi)$ vanishes on $\mathcal{R}$, because ξ restricted to a torus lying above a loop in S_T is the pullback of a vector bundle over that loop. Hence, the quotient of the Novikov ring of $H_2(M, \mathbb{Z})$ by $\mathcal{R}$ is well-defined and we can choose to work with these somewhat less precise coefficients.
Note that it would be possible to recover more information on the homology class, using a topological construction as in [8], but this would be very impractical for computations. Therefore, we prefer to content ourselves with $H_2(M, \mathbb{Z})/\mathcal{R}$.

The moduli spaces of such J-holomorphic curves are defined under the following equivalence relation :

$$(\Sigma \setminus \{x_1^+, \ldots, x_{s^+}^+, x_1^-, \ldots, x_{s^-}^-\}, j, \tilde{u}) \sim (\Sigma' \setminus \{x'^+_1, \ldots, x'^+_{s^+}, x'^-_1, \ldots, x'^-_{s^-}, j', \tilde{u}')$$

if there exists a biholomorphism

$$h : (\Sigma \setminus \{x_1^+, \ldots, x_{s^+}^+, x_1^-, \ldots, x_{s^-}^-\}, j) \to (\Sigma' \setminus \{x'^+_1, \ldots, x'^+_{s^+}, x'^-_1, \ldots, x'^-_{s^-}\}, j')$$

such that $h(x_i^\pm) = x'^\pm_i$ and $\tilde{u} = \tilde{u'} \circ h$.

We will denote the moduli space of J-holomorphic maps of genus g, of homology class $A \in H_2(M)/\mathcal{R}$, with s^+ positive punctures and asymptotic Reeb orbits in $S_1^+, \ldots, S_{S^+}^+$, with s^- negative punctures and asymptotic Reeb orbits in $S_1^-, \ldots, S_{s^-}^-$ by

$$\mathcal{M}_g^A(S_1^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-)$$

This moduli space $\mathcal{M}$ will be equipped with evaluation maps $ev_i^+ : \mathcal{M} \to S_i^+$ ($i = 1, \ldots, s^+$) and $ev_j^- : \mathcal{M} \to S_j^-$ ($j = 1, \ldots, s^-$).

In order to construct contact homology, we just consider moduli spaces with genus $g = 0$ and one positive puncture : $s^+ = 1$. However, we will construct these moduli spaces in full generality, since that does not really require more work.

2.2. **Asymptotic behavior of holomorphic curves.** The asymptotic behavior of holomorphic curves in the symplectization of a contact 3-manifold was studied by Hofer, Wysocki and Zehnder [7] when the closed Reeb orbits come in smooth 1-parameter families. They show that holomorphic curves converge at exponential speed to a fixed closed Reeb orbit in the 1-parameter family.

This result was extended for any dimension of M, and for more general degeneracies of the Reeb flow [1]. The Morse-Bott condition on α is crucial, because we need to work with nice coordinates in a tubular neighborhood of a closed Reeb orbit. The existence of these coordinates relies strongly on the Morse-Bott assumption.

After carefully computing the Cauchy-Riemann operator in the coordinates and estimating the decaying rate of the holomorphic curves, we obtain

Theorem 2. **([1], Chap. 3)** *Let (M, α) be a contact manifold with contact form α of Morse-Bott type. Let $\tilde{u} = (a, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ be a holomorphic map satisfying $E(\tilde{u}) < \infty$. Then there exists a closed Reeb orbit γ such that*

$$\lim_{s \to \infty} u(s,t) = \gamma(Tt) \qquad \text{in } C^\infty(S^1)$$

Moreover, there exists $r > 0$ and $a_0, \vartheta_0 \in \mathbb{R}$ such that

$$\begin{aligned} |\partial^I(a(s,t) - Ts - a_0)| &\leq C_I e^{-rs} \\ |\partial^I(\vartheta(s,t) - t - \vartheta_0)| &\leq C_I e^{-rs} \\ |\partial^I z(s,t)| &\leq C_I e^{-rs} \end{aligned}$$

for all multi-indices I, for some constants C_I.

This result shows that holomorphic maps converge to closed Reeb orbits, which is essential in the philosophy of contact homology. Moreover, the exponential convergence is important in several steps of the construction of the moduli spaces.

2.3. **Compactness.** The proof of Gromov-Hofer compactness for holomorphic curves in an exact symplectic cobordism of the form $(\mathbb{R} \times M, J)$ involves the local bubbling-off analysis and the Gromov-Schwarz lemma.

In order to compactify the moduli space of holomorphic curves, we have to consider nodal curves. Near a node, the map can converge to a point or a periodic Reeb orbit. This leads to the following definition.

Definition 3. *A level k holomorphic map $\tilde{u}$ from (Σ, j) to $(\mathbb{R} \times M, J)$ consists of the following data :*

(i) *A labeling of the connected components of $\Sigma^* = \Sigma \setminus \{nodes\}$ by integers in $\{1, \dots, k\}$, called levels, such that two components sharing a node have levels differing by at most 1. We denote by Σ_i the union of connected components of level i.*

(ii) *Holomorphic maps $\tilde{u}_i : (\Sigma_i, j) \to (\mathbb{R} \times M, J)$ with $E(\tilde{u}_i) < \infty$, $i = 1, \dots, k$, such that*
  - *each node shared by Σ_i and Σ_{i+1}, is a positive puncture for $\tilde{u}_i$, asymptotic to some periodic Reeb orbit γ and is a negative puncture for $\tilde{u}_{i+1}$, asymptotic to the same periodic Reeb orbit γ*
  - *$\tilde{u}_i$ extends continuously across each node within Σ_i.*

As in Gromov-Witten theory, we work with stable curves only. We need an appropriate definition in our setting.

Definition 4. *A level k holomorphic map $(\Sigma, j, \tilde{u})$ to $(\mathbb{R} \times M, J)$ is stable if, for every $i = 1, \dots, k$, either $\int_{\Sigma_i} \tilde{u}_i^* d\alpha > 0$ or Σ_i has a negative Euler characteristic (after removing marked points).*

Next we define the notion of convergence for stable curves.

Definition 5. *We say that a sequence of stable level k holomorphic maps $(S_n, j_n, \tilde{u}_n)$ converges to a stable level k' ($k' \geq k$) holomorphic map $(S, j, \tilde{u})$ if there exist a sequence of maps $\phi_n : S_n \to S$ and sequences $t_n^{(i)} \in \mathbb{R}$ ($i = 1, \dots, k'$), such that*

(i) *the maps ϕ_n are diffeomorphisms, except that they may collapse a circle in S_n to a node in S, and $\phi_{n*} j_n \to j$ away from the nodes of S.*

(ii) *the sequences of maps* $(t_n^{(i)} + a_n \circ \phi_n^{-1}, u_n \circ \phi_n^{-1}) : S_i \to \mathbb{R} \times M$ *converge in the* C^∞ *topology to* $\tilde{u}_i : S_i \to \mathbb{R} \times M$ *on every compact subset of* S_i*, for* $i = 1, \dots, k'$.

With this definition, we can state the compactness theorem.

Theorem 6. **([1], Chap. 4)** *Let* $\tilde{u}_n : (\Sigma_n, j_n) \to (\mathbb{R} \times M, J)$ *be a sequence of stable level* k *holomorphic maps of same genus and same asymptotics such that* $E(\tilde{u}_n) < C$*. Then there exists a subsequence that converges to a stable level* k' $(k' \geq k)$ *holomorphic map* $(\Sigma, j, \tilde{u})$*. Moreover,* $E(\tilde{u}) = \lim_{n\to\infty} E(\tilde{u}_n)$.

Note that, even though closed Reeb orbits exist in continuous families, the 2 parts of the split holomorphic curves have to converge to the same closed Reeb orbit in the middle. The proof of this fact relies on the analysis of the asymptotic behavior of holomorphic curves.

2.4. **Fredholm theory with degeneracies and gluing.** In this section, we consider a linear operator $\overline{\partial}$ acting on sections of a vector bundle E over the Riemann surface $\Sigma \setminus \{x_1^+, \dots, x_{s^+}^+, x_1^-, \dots, x_{s^-}^-\}$. In a trivialization of E in the interior of the base, the operator $\overline{\partial}$ looks like

$$\Big(\frac{\partial}{\partial x} + J(z)\frac{\partial}{\partial y} + S(z)\Big)(dx - idy)$$

In an appropriate trivialization of E near puncture $x_i^\pm$, $\overline{\partial}$ looks like

$$\Big(\frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t} + S_i^\pm(s,t)\Big)(ds - idt)$$

in cylindrical coordinates (s,t) such that $z = e^{\mp(s+2\pi i t)}$ vanishes at the puncture $x_i^\pm$. We assume that $S_i^\pm(t) = \lim_{s\to\pm\infty} S_i^\pm(s,t)$ is a path of symmetric matrices. Let $\Psi_i^\pm(t)$ be the path of symplectic matrices characterized by $\Psi_i^\pm(0) = I$ and $\frac{d\Psi_i^\pm}{dt}(t) = J_0 S_i^\pm(t)\Psi_i^\pm(t)$. We will view the operator $\overline{\partial}$ as the linearized Cauchy-Riemann operator of some holomorphic map $\tilde{u}$. Since the Reeb orbits are not isolated, we do not assume that $\det(\Psi_i^\pm(1) - I) \neq 0$.

2.4.1. *Fredholm property.* Let $p > 2$ and $k \geq 1$; since the asymptotics of the linear operator $\overline{\partial}$ are degenerate, it is certainly not true that

$$\overline{\partial} : L_k^p(E) \to L_{k-1}^p(\Lambda^{0,1}(E))$$

is Fredholm. Indeed, if that operator were Fredholm, it would follow that the Fredholm index of linear Cauchy-Riemann operators with nondegenerate asymptotics would be independent of the Conley-Zehnder index of the asymptotics, by continuity of the Fredholm index.

Therefore, we have to modify the Banach spaces by introducing some weights near the punctures. For $d > 0$, let

$$L_k^{p,d} = \{f(s,t) \mid f(s,t)e^{d|s|/p} \in L_k^p\}$$

We define a Banach norm on $L_k^{p,d}$ by multiplying the measure $ds\,dt$ by $e^{d|s|/p}$ near a puncture. The use of such exponential weights near the punctures is justified by theorem 2, when $0 < d < r$.

Note that the functions in $L_k^{p,d}$ vanish at infinity, even though holomorphic maps could a priori slide along the orbit spaces. Hence, we have to add to the domain functions of the form $\rho_i^{\pm}(s)v_j^{(\pm,i)}(t)$, where $\rho_i^{\pm}$ is a bump function with support in a neighborhood of the puncture $x_i^{\pm}$ and $v_j^{(\pm,i)}$, $(j = 1, \dots, \dim S_i^{\pm} + 2)$ form a basis of solutions for $\frac{dv}{dt}(t) = J_0 S_i^{\pm}(t)v(t)$, $v(0) = v(1)$. The term 2 in the number of independent solutions accounts for the Reeb field and the vector field $\frac{\partial}{\partial t}$. We therefore add a finite dimensional summand to the domain, of dimension $N = \sum_{i=1}^{s^+}(\dim S_i^+ + 2) + \sum_{i=1}^{s^-}(\dim S_i^- + 2)$.

Proposition 7. **([1], Chap. 5)** *The linear operator*

$$\overline{\partial} : \mathbb{R}^N \oplus L_k^{p,d}(E) \to L_{k-1}^{p,d}(\Lambda^{0,1}(E))$$

is Fredholm.

2.4.2. *Fredholm index.* The Fredholm index of the $\overline{\partial}$ operator is usually computed in terms of the Conley-Zehnder index μ_{CZ} corresponding to the asymptotic conditions. Here however, those asymptotics are degenerate, so the Conley-Zehnder index is not defined. Robbin and Salamon [12] introduced a Maslov index for general paths of symplectic matrices. Let $\Psi(t)$ be a path of symplectic matrices such that $\Psi(0) = I$; assume that there are a finite number of values of t $(0 < t < 1)$, $t_1, \dots, t_l$, called crossings, such that $V_t = \ker(\Psi(t) - I) \neq 0$, and that $J_0 \frac{d}{dt}\Psi(t)$, the crossing form, is nondegenerate on V_t. Denote the signature of that symmetric form by $\sigma(t)$. Then, the Maslov index $\mu(\Psi)$ can be defined by :

$$\mu(\Psi) = \frac{1}{2}\sigma(0) + \sum_{i=1}^{l} \sigma(t_i) + \frac{1}{2}\sigma(1)$$

where $\sigma(1)$ is defined to be zero if $\Psi(1) - I$ is invertible. Then, the Maslov index is half-integer valued, invariant under homotopy with fixed ends, additive under catenation of paths, and $\mu(\Psi) + \frac{1}{2}\dim V_1 \in \mathbb{Z}$.

With this definition in mind, we can now compute the Fredholm index of $\overline{\partial}$:

Proposition 8. ([1], **Chap. 5**) *The Fredholm index of the linear operator*

$$\overline{\partial} : \mathbb{R}^N \oplus L_k^{p,d}(E) \to L_{k-1}^{p,d}(\Lambda^{0,1}(E))$$

is given by the formula

$$n(2-2g-s^+-s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu(\Psi_i^+) - \sum_{j=1}^{s^-} \mu(\Psi_j^-) + \frac{1}{2}N$$

If we add to this the dimension of the conformal space for genus g and $s^+ + s^-$ punctures, we obtain the virtual dimension of the moduli space :

$$(n-3)(2-2g-s^+-s^-) + 2c_1(A) + \sum_{i=1}^{s+} \left(\mu(\Psi_i^+) + \frac{1}{2}\dim S_i^+\right) - \sum_{j=1}^{s^-} \left(\mu(\Psi_j^-) - \frac{1}{2}\dim S_j^-\right)$$

For contact homology, we work only with rational curves having one positive puncture and r negative punctures, so we obtain :

$$(n-3)(1-r) + 2c_1(A) + \mu(\Psi^+) + \frac{1}{2}\dim S^+ - \sum_{j=1}^{r} \left(\mu(\Psi_j^-) - \frac{1}{2}\dim S_j^-\right)$$

2.4.3. *Gluing.* In order to prove that all level k holomorphic curves (with $k > 1$) are in the boundary of the moduli space, we have to generalize the gluing theorem to this degenerate Fredholm setup.

Let $\tilde{u} : (\Sigma_{\tilde{u}}, j_{\tilde{u}}) \to (\mathbb{R} \times M, J)$ and $\tilde{v} : (\Sigma_{\tilde{v}}, j_{\tilde{v}}) \to (\mathbb{R} \times M, J)$ be 2 holomorphic maps, such that some positive Reeb orbits of $\tilde{u}$ coincide with some negative Reeb orbits of $\tilde{v}$.

For large $R \in \mathbb{R}^+$, we define a pre-glued map $\tilde{u}\sharp_R\tilde{v} : (\Sigma_{\tilde{u}}\sharp_R\Sigma_{\tilde{v}}, j) \to (\mathbb{R} \times M, J)$ which is approximately holomorphic. Roughly speaking, we cut the cylindrical ends of $\tilde{u}$ above height R and translate down the remaining part by R, we cut the cylindrical ends of $\tilde{v}$ below height $-R$ and translate up the remaining part by R; then we glue the 2 fragments using some smooth cutoff.

We choose a finite dimensional vector space W of smooth sections of $E = \tilde{u}^*(\mathbb{R} \times M)$ with compact support in the punctured Riemann surface $\Sigma_{\tilde{u}}$, and we enlarge the domain of $\overline{\partial}_{\tilde{u}}$ with a summand W. We choose W so that the restriction of $\overline{\partial}_{\tilde{u}}$ to $W \oplus L_k^{p,d}(E)$, called $\overline{\partial}_{\tilde{u}}^W$, is surjective. We choose a vector space W' for $\overline{\partial}_{\tilde{v}}$ in a similar way.

We then have to generalize the estimates on $\overline{\partial}_{\tilde{u}\sharp_R\tilde{v}}$ to the Banach structures $L_k^{p,d}$ in order to obtain a right inverse.

Proposition 9. ([1], **Chap. 5**) *The operator* $\overline{\partial}_R = \overline{\partial}_{\tilde{u}\sharp_R\tilde{v}}^{W\oplus W'}$ *has a uniformly bounded right inverse* Q_R*, if* R *is sufficiently large.*

From this point on, the remaining steps that are necessary to find a smooth holomorphic map nearby a split holomorphic map are almost identical to the standard arguments without exponential weights.

2.5. **Transversality and fibered products.** In order to realize our moduli spaces as nice geometric objects with the virtual dimension predicted by the Fredholm index, we have to make sure that the Fredholm operators obtained by linearizing the Cauchy-Riemann equation are surjective.

In order to construct the moduli spaces with degenerate asymptotics, we choose to keep the almost complex structure J fixed and perturb the right hand side of the Cauchy-Riemann equation. Indeed, the greatest benefit of this Morse-Bott setup is to work with symmetric Reeb dynamics and symmetric almost complex structures. Therefore, we prefer to keep this symmetry during all steps of the construction of the moduli spaces : it is probably much easier to solve the Cauchy-Riemann equations for a natural J, and then understand the obstruction bundle in this natural setup, than to solve those equations for generic J.
Moreover, a generic J is generally not enough in contact homology to guarantee transversality, because of multiply covered cylinders, for example. The almost complex structure would have to depend on the points of the Riemann surface as well, which would make things even harder for computations.

The construction of the virtual cycle in contact homology is a mixture of the existing work in Gromov-Witten theory (see for example [9]) and in Floer homology (see for example [10]). Indeed, we are dealing with holomorphic curves having various topologies, and with codimension 1 degeneration.

A special feature of the Morse-Bott setup is the compactification of the moduli spaces. Indeed, level 2 holomorphic curves correspond to the fibered product of moduli spaces over a product of orbit spaces :

$$\mathcal{M}(S_1^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-) \times_{S^-_{s^- - t+1} \times \ldots \times S^-_{s^-}} \mathcal{M}({S'}_1^+, \ldots, {S'}_{s'^+}^+; {S'}_1^-, \ldots, {S'}_{s'^-}^-)$$

where $S^-_{s^- - i+1} = {S'}_i^+$ for $i = 1, \ldots, t$.

Recall that, given $f : A \to C$ and $g : B \to C$, the fibered product of A and B over C is defined as $A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$. When A and B are moduli spaces, the maps f and g are evaluation maps to a space S_i of closed Reeb orbits.

Since these fibered products correspond to substrata of the moduli spaces, we want them to be as regular as the moduli spaces themselves. Therefore, we need to make sure that all evaluation maps are transversal to each other. We can achieve this ([1], Chap. 6) by choosing the vector spaces W of section 2.4.3 sufficiently large, so that the evaluation maps are transverse on the virtual neighborhood of the set of holomorphic maps. If we then choose generic multi-sections of the obstruction bumdle, we keep that transversality property on the virtual cycle.

After this perturbation, our moduli space becomes a branched, labeled pseudo-manifold with corners (see [11]). The codimension 1 boundary (which may have corners, at codimension ≥ 2 strata) corresponds to level 2 holomorphic curves.

2.6. **Coherent orientations.** The construction of a set of coherent orientations on the moduli spaces in Symplectic Field theory has been carried out in a joint work with Klaus Mohnke [2]. We now explain how to generalize this construction to the Morse-Bott case.

First, the definition of coherent orientations requires that the orbit spaces S_i are orientable. Indeed, in order to induce an orientation on $A \times_S B$ from orientations on A and B, we also need an orientation on S. Then, we define an orientation of $A \times_S B$ so that the isomorphism

$$T_{(a,b)}(A \times_S B) \oplus T_s S \simeq T_a A \oplus T_b B$$

changes the orientations by a sign $(-1)^{\dim B \dim S}$. This sign is necessary to make the fibered product associative.

Then, note that the moduli spaces are not always orientable. Indeed, when the asymptotic expression of the linearized Cauchy-Riemann operator is not fixed, theorem 2 of [5] shows that the determinant line bundle over the space of Cauchy-Riemann operators is not trivial. Therefore, a non contractible loop in N_T may induce a "disorienting loop" of asymptotic linearized Cauchy-Riemann operators that makes the determinant line bundle non orientable.
If the projection of that disorienting loop to S_T is contractible, then the original loop in N_T is homotopic to a closed Reeb orbit with period dividing T. That Reeb orbit is then *bad* in the following sense :

Definition 10. *A Reeb orbit γ is said to be bad if it is the $2m$-cover of a simple orbit $\gamma' \in S_T$ and if $(\mu(S_{2T}) \pm \frac{1}{2}\dim S_{2T}) - (\mu(S_T) \pm \frac{1}{2}\dim S_T)$ is odd. If a Reeb orbit γ is not bad, then we say it is good.*

This definition extends the definition of bad orbits in the non-degenerate case that was formulated in [2].

Note that there are no bad orbits if and only if there are no orbits $\gamma \in S_T$ so that

$(\mu(S_{2T}) - \frac{1}{2}\dim S_{2T}) - (\mu(S_T) - \frac{1}{2}\dim S_T)$ is odd and if $\dim S_{2T} - \dim S_T$ is even. If $\dim S_{2T} - \dim S_T$ is odd, then the Poincaré return map of a Reeb orbit contained in N_T is orientation reversing in N_{2T}. This implies that N_{2T} is not orientable.

Assume that there are no bad orbits. Then a disorienting loop in N_T for the determinant line bundle of the linearized Cauchy-Riemann operator projects to a noncontractible loop in S_T. Therefore, in order to guarantee that the moduli spaces are orientable, we also have to assume that $\pi_1(S_T)$ has no disorienting loops.

We now assume that the moduli spaces are orientable. Then, using the gluing map described in section 2.3, we can construct, as in [2], a set of coherent orientations on the moduli spaces.
Coherent orientations satisfy the following axioms :

(i) The coherent orientation of $\mathcal{M}(S_1^+, \dots, S_k^+, S_{k+1}^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ and the coherent orientation of $\mathcal{M}(S_1^+, \dots, S_{k+1}^+, S_k^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ coincide up to a factor $(-1)^{|S_k^+|.|S_{k+1}^+|}$, where $|S_i^\pm| = \mu(S_i^\pm) \pm \frac{1}{2}\dim S_i^\pm + n - 3$.
A similar statement holds for reordering of negative punctures.

(ii) The disjoint union map u

$$\begin{aligned}&\mathcal{M}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times \mathcal{M}(S'^+_1, \dots, S'^+_{s'^+}; S'^-_1, \dots, S'^-_{s'^-})\\ &\quad\to \mathcal{M}(S_1^+, \dots, S_{s^+}^+, S'^+_1, \dots, S'^+_{s'^+}; S_1^-, \dots, S_{s^-}^-, S'^-_1, \dots, S'^-_{s'^-})\end{aligned}$$

preserves coherent orientations up to a factor

$$(-1)^{(|S_1^-|+\dots+|S_{s^-}^-|)(|S'^+_1|+\dots+|S'^+_{s'^+}|)}$$

(iii) The gluing map ϕ

$$\begin{aligned}&\mathcal{M}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times_{S^-_{s^--t+1}\times\dots\times S^-_{s^-}} \mathcal{M}(S'^+_1, \dots, S'^+_{s'^+}; S'^-_1, \dots, S'^-_{s'^-})\\ &\quad\to \mathcal{M}(S_1^+, \dots, S_{s^+}^+, S'^+_{t+1}, \dots, S'^+_{s'^+}; S_1^-, \dots, S_{s^--t}^-, S'^-_1, \dots, S'^-_{s'^-})\end{aligned}$$

that is defined when $S^-_{s^-+1-i} = S'^+_i$ for $i = 1, \dots, t$, preserves coherent orientations up to a factor

$$(-1)^{(|S'^+_{t+1}|+\dots+|S'^+_{s'^+}|)(|S_1^-|+\dots+|S^-_{s^--t}|)}\ (-1)^{\sum_{i=1}^t(\dim S^-_{s^--t+i}\sum_{j=i+1}^t|S^-_{s^-t+j}|)}$$

Summing up, we have

Lemma 11. **([1], Chap. 7)** *Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all elements of S_T are good. Then the moduli spaces of holomorphic maps are orientable and can be equipped with coherent orientations.*

3. Construction of Morse-Bott contact homology

The general strategy of this section is to slightly perturb contact form α in a tubular neighborhood of the submanifolds N_T, so that only a finite number of closed Reeb orbit remain. Then we can define contact homology using the construction of Eliashberg, Givental and Hofer [4]. Next we let the perturbation vanish, and we rewrite the chain complex for contact homology using geometric data from the Morse-Bott setup.

3.1. **Perturbation of contact form.** Let us construct a function $\bar{f}_T$ with support in a small neighborhood of $\cup_{T' \leq T} N_{T'}$ and such that $d\bar{f}_T(R_\alpha) = 0$ on $N_{T'}$. In particular, $\bar{f}_T$ will descend to a differentiable function f_T on the orbifold S_T. We will choose $\bar{f}_T$ generically, so that that it induces a Morse function f_T on S_T.

We proceed by induction on T. For the smallest $T \in \sigma(\alpha)$, the orbit space S_T is a smooth manifold. Pick any Morse function f_T on it.
For larger values of $T \in \sigma(\alpha)$, S_T will be an orbifold having as singularities the orbit spaces $S_{T'}$ such that T' divides T. We extend the functions $f_{T'}$ to a function f_T on S_T, so that the Hessian of f_T restricted to the normal bundle to $S_{T'}$ is positive definite.
Finally, we extend f_T to a tubular neighborhood of N_T so that it is constant on the fibers of the normal bundle of N_T (for some metric invariant under the Reeb flow). We then use cut off depending on the distance from N_T.

Consider the perturbed contact form $\alpha_\lambda = (1+\lambda\bar{f}_T)\alpha$, where λ is a small positive constant.

Lemma 12. *For all T, we can choose $\lambda > 0$ small enough so that the periodic Reeb orbits of R_{α_λ} in M of action $T' \leq T$ are nondegenerate and correspond to the critical points of $f_{T'}$.*

Proof. The new Reeb vector field $R_{\alpha_\lambda} = R_\alpha + X$ where X is characterized by

$$\begin{aligned} i(X)d\alpha &= \lambda \frac{d\bar{f}_T}{(1+\lambda\bar{f}_T)^2} \\ \alpha(X) &= -\lambda \frac{\bar{f}_T}{1+\lambda\bar{f}_T} \end{aligned}$$

The first equation describes the transversal deformations of the Reeb orbits. These vanish when $df_T = 0$, that is at critical points of f_T. On the other hand, if λ is small enough, the perturbation cannot create new periodic orbits, for a fixed action range, because we have an upper bound on the deformation of the flow for the corresponding range of time. The surviving periodic orbits are nondegenerate, because the Hessian at a critical point is nondegenerate. This corresponds to first order variations of X, that is of the linearized Reeb flow. $\square$

Let $p \in S_{T'}$ be a simple Reeb orbit that is a critical point of $f_{T'}$. Then we will denote the closed orbit corresponding to $p \in S_{kT'}$ by $\gamma^p_{kT'}$ $(k = 1, 2, \dots)$.

We can compute the Conley-Zehnder index of these closed Reeb orbits for a small value of λ.

Lemma 13. *If λ is as in lemma 12 and $kT' \leq T$, then*

$$\mu_{CZ}(\gamma^p_{kT'}) = \mu(S_{kT'}) - \frac{1}{2}\dim S_{kT'} + \mathrm{index}_p(f_{kT'})$$

Proof. Let H be the Hessian of f_T at critical point p. Then, the ξ-component of X is given by $-\lambda JHx$, where x is a local coordinate in a uniformization chart near p. The linearized Reeb flow now has a new crossing at $t = 0$, with crossing form $-\lambda H$. Its signature is $\sigma(0) = \mathrm{index}_p(f_{kT}) - (\dim S_{kT} - \mathrm{index}_p(f_{kT}))$. Half of this must be added to $\mu(S_{kT})$ to obtain the Conley-Zehnder index of the nondegenerate orbit. □

Using a few additional assumptions (see the statements of the main theorems 18 and 19), we can check that all closed orbits with a very large period also have a very large grading. Therefore, these will not contribute to contact homology, and we can ignore them.
Next, we need to determine whether the perturbed orbits $\gamma^p_{kT'}$ are good or bad. This is the reason we chose to extend the Morse functions f_T using a positive definite Hessian on the normal bundle of S_T.

Lemma 14. *Under the assumptions of lemma 11, all perturbed Reeb orbits $\gamma^p_{kT'}$ are good.*

Proof. The orbit $\gamma^p_{kT'}$ is bad if and only if k is even and $\mu_{CZ}(\gamma^p_{2T'}) - \mu_{CZ}(\gamma^p_{T'})$ is odd. By lemma 13, the last condition reads : $(\mu(S_{2T'}) - \frac{1}{2}\dim S_{2T'} + \mathrm{index}_p(f_{2T'})) - (\mu(S_{T'}) - \frac{1}{2}\dim S_{T'} + \mathrm{index}_p(f_{T'}))$ is odd. But $\mathrm{index}_p(f_{2T'}) = \mathrm{index}_p(f_{T'})$, since the normal bundle to $S_{T'}$ in $S_{2T'}$ does not contribute to the Morse index of $f_{2T'}$. Hence, $\gamma^p_{kT'}$ is bad if and only if the non perturbed orbit $p \in S_{kT'}$ is bad. There are no such orbits under the assumptions of lemma 11. □

Since the Morse index at p does not depend on the Morse function $f_{kT'}$, we can denote it simply by $\mathrm{index}(p)$.

3.2. **Degeneracy of holomorphic curves.** Let $\mathcal{M}_{(0,\lambda_0]}(\gamma^{p_1^+}, \dots, \gamma^{p_{s^+}^+}; \gamma^{p_1^-}, \dots, \gamma^{p_{s^-}^-})$ be the moduli space of J_λ- holomorphic curves with fixed asymptotics, for all $\lambda \in (0, \lambda_0]$. We would like to understand the behavior of these holomorphic curves when we let $\lambda \to 0$. For this, we have to generalize the compactness theorem from section 2.3 to this present situation. Indeed, the almost complex structure J_λ corresponding to α_λ satisfies $J_\lambda \frac{\partial}{\partial t} = R_{\alpha_\lambda}$, so the complex structure is modified in the sequence, including near the ends of the symplectization.

From now on, we assume that the almost complex structure J is invariant under the Reeb flows, on all submanifolds N_T. Note that complex structures on ξ satisfying this property always exist, and that many examples of contact forms of Morse-Bott type are naturally equipped with such an almost complex structure.
As a consequence of this, the gradient vector of $\bar{f}_T$, with respect to the metric $d(e^t\alpha)(\cdot, J\cdot)$, descends to the orbit spaces S_T, and we can talk about the corresponding gradient flow trajectories in S_T.

In order to describe the compactification of the moduli space, we need the following definition.

Definition 15. *A generalized level* 1 *holomorphic map* $\tilde{u}$ *from* (Σ, j) *to* $(\mathbb{R} \times M, J)$ *with Morse functions* f_T *consists of the following data :*

(i) *A labeling of the connected components of* $\Sigma^* = \Sigma \setminus \{nodes\}$ *by integers in* $\{1, \dots, l\}$*, called sublevels, such that two components sharing a node have sublevels differing by at most 1. We denote by* Σ_i *the union of connected components of sublevel* i*.*
(ii) *Positive numbers* t_i*,* $i = 1, \dots, l-1$*.*
(iii) *Holomorphic maps* $\tilde{u}_i : (\Sigma_i, j) \to (\mathbb{R} \times M, J)$ *with* $E(\tilde{u}_i) < \infty$*,* $i = 1, \dots, l$*, such that*
   - *each node shared by* Σ_i *and* Σ_{i+1}*, is a positive puncture for* $\tilde{u}_i$*, asymptotic to some periodic Reeb orbit* $\gamma \in S_T$ *and is a negative puncture for* $\tilde{u}_{i+1}$*, asymptotic to a periodic Reeb orbit* $\delta \in S_T$*, such that* $\varphi_{t_i}^{f_T}(\gamma) = \delta$*, where* $\varphi_t^{f_T}$ *is the gradient flow of* f_T*.*
   - $\tilde{u}_i$ *extends continuously across each node within* Σ_i*.*

We then extend this definition to a generalized level k holomorphic map as in definition 3. Here is the generalization of the compactness theorem :

Proposition 16. *Let* $\tilde{u}_n : (\Sigma_n, j_n) \to (\mathbb{R} \times M, J_{\lambda_n})$ *be a sequence of holomorphic curves of fixed genus and asymptotics, such that* $\lim_{n\to\infty} \lambda_n = 0$ *and* $E(\tilde{u}_n) < C$*. Then there exists a subsequence that converges to a generalized holomorphic map* $\tilde{u}$ *with Morse functions* f_T*, such that* $E(\tilde{u}) = \lim_{n\to\infty} E(\tilde{u}_n)$*.*

Sketch of proof. The convergence away from the asymptotic Reeb orbits is proved exactly as before. We then write the Cauchy-Riemann equations in local coordinates near a periodic orbit; there is an additional term due to the perturbation. We can modify the estimates for the exponential convergence in this situation and prove that, after rescaling the cylinders in the thin part, the sequence converges to a gradient flow trajectory of f_T in S_T. □

On the other hand, we can generalize the estimates of section 2.4.3 in order to prove that we can glue a generalized holomorphic curve to produce a 1-parameter family of J_λ-holomorphic curves, with small λ.

Let $\mathcal{M}^{f_T}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ be the moduli space of generalized J-holomorphic curves with Morse functions f_T, with asymptotics in fixed orbit spaces. This moduli space can be constructed from the moduli spaces $\mathcal{M}(S'_1, \dots, S'_s; S''_1, \dots, S''_r)$ of ordinary holomorphic curves, using the gradient flow of f_T and fibered products. If the virtual cycle is constructed generically, we obtain a weighted sum of smooth manifolds with corners.

The compactification of $\mathcal{M}_{(0,\lambda_0]}(\gamma^{p_1^+}, \dots, \gamma^{p_{s^+}^+}; \gamma^{p_1^-}, \dots, \gamma^{p_{s^-}^-})$ at $\lambda = 0$ is given by

$$W^u(p_1^+) \times_{S_1^+} \dots W^u(p_{s^+}^+) \times_{S_{s^+}^+} \mathcal{M}^{f_T}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times_{S_1^-} W^s(p_1^-) \dots \times_{S_{s^-}^-} W^s(p_{s^-}^-)$$

where $W^u(p)$ (resp. $W^s(p)$) denotes the unstable (resp. stable) manifold of p in its orbit space S.

The moduli space $\mathcal{M}_{(0,\lambda_0]}(\gamma^{p_1^+}, \dots, \gamma^{p_{s^+}^+}; \gamma^{p_1^-}, \dots, \gamma^{p_{s^-}^-})$ can also have boundary components for λ in the interior of $(0, \lambda_0]$. If our path of almost complex structures J_λ is chosen generically, holomorphic curves can degenerate into a level k holomorphic curve involving at least one nongeneric component : the dimension of the corresponding moduli space will be one more than the predicted dimension. Since the moduli spaces of nongeneric holomorphic curves are compact, there exists $\lambda_1 \in (0, \lambda_0]$ such that no such curve appears in the subinterval $(0, \lambda_1]$.

In particular, for moduli spaces of dimension 1, we obtain a 1-dimensional cobordism between rigid curves for $\lambda = 0$ and $\lambda = \lambda_1$. We want to construct a suitable orientation on the compactification at $\lambda = 0$ so that the algebraic number of J_{λ_1}-holomorphic curves coincides with the algebraic number of generalized J_0-holomorphic curves.

Lemma 17. **([1], Chap. 7)** *A coherent set of orientations on $\mathcal{M}$, as in section 2.6, induces a coherent set of orientations for the moduli spaces with non-degenerate asymptotics by*

$$W^u(p_1^+) \times_{S_1^+} \dots W^u(p_{s^+}^+) \times_{S_{s^+}^+} \mathcal{M}^{f_T}(S_1^+, \dots S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times_{S_1^-} W^s(p_1^-) \dots \times_{S_{s^-}^-} W^s(p_{s^-}^-)$$

multiplied with the sign $(-1)^{\delta^+ + \delta^-}$*, where*

$$\begin{aligned}
\delta^+ &= \sum_{i=1}^{s^+} \Big((\operatorname{index}(p_i^+) + \dim S_i^+) \sum_{j=1}^{i-1} |S_j^+| \Big) \\
\delta^- &= \sum_{i=1}^{s^-} \Big(\operatorname{index}(p_i^-) \sum_{j=i+1}^{s^-} |S_j^-| \Big)
\end{aligned}$$

3.3. **Morse-Bott chain complex.** The Morse-Bott chain complex C_* is the unitary supercommutative algebra generated by the critical points p of all Morse functions f_T, or equivalently by the nondegenerate periodic orbits γ_{kT}^p, with grading $\mu(S_{kT}) - \frac{1}{2}\dim S_{kT} + \operatorname{index}(p) + n - 3$.

Using the results of the last two sections, we can rewrite the differential d for contact homology with nondegenerate Reeb orbits using moduli spaces of generalized J-holomorphic curves instead.

First, let us consider rigid J_λ-holomorphic curves converging for $\lambda \to 0$ to a generalized holomorphic curve containing no nontrivial J_0-holomorphic curve. In other words, the limit will be a gradient flow trajectory for f_T. The terms of the differential d involving these curves coincide exactly with the differential ∂ of the Morse-Witten complex of f_T.

Next, let us consider rigid J_λ-holomorphic curves such that their limit for $\lambda \to 0$ contains a non-trivial J-holomorphic curve. We can count these curves using generalized holomorphic curves, as in the original definition of contact homology.

Therefore, the differential d for contact homology is characterized by its value on a critical point $p \in S_T$:

$$dp = \partial p + \sum n_{1,\dots,s} \frac{p_1^{i_1}}{i_1!} \cdots \frac{p_s^{i_s}}{i_s!}$$

where we sum over all unordered monomials $p_1^{i_1} \dots p_s^{i_s}$ and the coefficient $n_{1,\dots,s}$ is the algebraic number of elements in the fibered product

$$(-1)^{\delta^-} \Big(W^u(p) \times_S \mathcal{M}^{f_T}(S; S_1, \dots, S_s) \times_{S_1} W^s(p_1) \dots \times_{S_s} W^s(p_s) \Big) / \mathbb{R}$$

if it is zero dimensional, or 0 otherwise.

Since the resulting chain complex is identical to the chain complex constructed in [4], homology of our Morse-Bott chain complex will be the contact homology of (M, ξ). We are now in position to state the main result of [1].

Theorem 18. *Let α be a contact form of Morse-Bott type for a contact structure ξ on on M satisfying $c_1(\xi) = 0$.*
Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all Reeb orbits in S_T are good. Assume that the almost complex structure J is invariant under the Reeb flow on all submanifolds N_T. Assume that there exists $c > 0, c'$ such that $|\mu(S_T)| \geq cT + c'$ for all $T \in \sigma(\alpha)$, and that there exists $\Delta T < \infty$ such that, for every Reeb trajectory leaving a small tubular neighborhood U_T of N_T at p, we have $\varphi_t(p) \in U_T$ for some $0 < t < \Delta T$.
Then the homology $H_*(C_*, d)$ of the Morse-Bott chain complex (C_*, d) of (M, α) is isomorphic to the contact homology $HC_*(M, \xi)$ of $(M, \xi = \ker \alpha)$ with coefficients in the Novikov ring of $H_2(M, \mathbb{Z})/\mathcal{R}$.

It is sometimes better to consider instead cylindrical homology, for which we count cylindrical curves only. The Morse-Bott chain complex $C_*^{\bar{a}}$ is the graded vector space generated by the nondegenerate periodic orbits γ_{kT}^p, in homotopy class $\bar{a}$ with grading $\mu(S_{kT}) - \frac{1}{2}\dim S_{kT} + \mathrm{index}(p) + n - 3$.

The differential d for cylindrical homology is given by :

$$dp = \partial p + \sum_q n_q\, q$$

where the coefficient n_q is the algebraic number of elements in the fibered product

$$(W^u(p) \times_S \mathcal{M}^{f_T}(S; S') \times_{S'} W^s(q))/\mathbb{R}$$

if it is zero dimensional, or 0 otherwise.

As before, homology of our Morse-Bott chain complex will be the cylindrical homology of (M, ξ).

Theorem 19. *Let α be a contact form of Morse-Bott type for a contact structure ξ on M satisfying $c_1(\xi) = 0$.*
Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all Reeb orbits in S_T are good. Assume that the almost complex structure J is invariant under the Reeb flow on all submanifolds N_T. Assume that cylindrical homology is well defined : $C_k^0 = 0$ for $k = -1, 0, +1$. Assume that there exists $c > 0, c'$ such that $|\mu(S_T)| \geq cT + c'$ for all orbit spaces S_T of contractible periodic orbits, and that there exists $\Delta T < \infty$ such that, for every Reeb trajectory leaving a small tubular neighborhood U_T of N_T at p, we have $\varphi_t(p) \in U_T$ for some $0 < t < \Delta T$.

Then the homology $H_*(C_*^{\bar{a}}, d)$ of the Morse-Bott chain complex $(C_*^{\bar{a}}, d)$ of (M, α) is isomorphic to the cylindrical homology $HF_*^{\bar{a}}(M, \xi)$ of $(M, \xi = \ker \alpha)$ with coefficients in the Novikov ring of $H_2(M, \mathbb{Z})/\mathcal{R}$.

3.4. **Marked points on holomorphic curves.** We now explain how to generalize the above results to holomorphic curves with marked points. The corresponding moduli space will be denoted by

$$\mathcal{M}_{g,k}^A(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$$

where k is the number of marked points. This moduli space modulo rigid vertical translations, $\mathcal{M}/\mathbb{R}$ will be equipped with additional evaluation maps $ev_i : \mathcal{M}/\mathbb{R} \to M$ $(i = 1, \dots, k)$. We can use those maps to pullback cycles from M. This is especially useful in very symmetric situations, in which there are no rigid curves. In that case, we can produce isolated curves satisfying extra conditions at the marked points, and define finer invariants of (M, ξ).

However, since the moduli spaces have a codimension 1 boundary, we have to be careful with the way we pullback cycles. Let $\theta_1, \dots, \theta_m$ be a set of cycles in M such that their homology classes form a basis for $H_*(M)$. Let $t_1, \dots, t_m$ be variables associated to these cycles, with grading $|t_i| = \dim \theta_i - 2$. Then, in the definition of the differential for the Morse-Bott chain complex, replace coefficient

$$\#\Big(W^u(p) \times_S \mathcal{M}^{f_T}(S; S_1, \dots, S_s) \times_{S_1} W^s(p_1) \dots \times_{S_s} W^s(p_s)\Big)/\mathbb{R}$$

with

$$\#\Big(W^u(p) \times_S (\mathcal{M}_{0,k}^{f_T}(S; S_1, \dots, S_s) \times_{M^k} \big(\sum_{i=1}^m (t_i\theta_i)^k\big) \times_{S_1} W^s(p_1) \dots \times_{S_s} W^s(p_s)\Big)/\mathbb{R}$$

where the extra fibered products are defined using the evaluations maps ev_i $(i = 1, \dots, k)$. After perturbation by the Morse functions f_T, we obtain the same definition of contact homology with marked points as in [4]. In that paper, it was shown that the resulting homology is independent of the choice of the cycles $\theta_1, \dots, \theta_m$.

4. Examples and applications

In this section, we illustrate how our Morse-Bott formalism can be used to compute contact homology, with several explicit examples. These examples come from 3 important families :

1. Complex line bundles over a symplectic manifold. In this case, the Reeb flow is completely periodic and every simple Reeb orbit has the same action.

2. Complex line bundles over a symplectic orbifold. The Reeb flow is still periodic, but the simple Reeb orbit can have different actions.
3. Unit cotangent bundle of (product of) Riemannian manifold. The closed Reeb orbits are the closed geodesics of the Riemannian manifold. If this manifold is a product (such as a torus), we obtain infinitely many closed Reeb orbits for any pair of closed geodesics.

4.1. **Complex line bundle over a symplectic manifold.** Let (M, ω) be a compact symplectic manifold of dimension $2n-2$, and assume that $[\omega] \in H^2(M, \mathbb{Z})$. Let $\pi : L \to M$ be the complex line bundle over M with $c_1(L) = [\omega]$.

For any choice of hermitian metric on L, the unit circle bundle $\pi : V \to M$ is a contact manifold. A contact form is obtained by choosing a connection form $i\alpha$ on V so that $\frac{1}{2\pi} d\alpha = \pi^* \omega$. For such a choice of α, the Reeb field R_α is tangent to the S^1 fibers of V. Therefore, every Reeb orbit is closed, and the space of Reeb orbits in every multiplicity $k = 1, 2, \ldots$ is naturally identified with M.

The symplectization of (V, α) is, as a manifold, the line bundle L with its zero section removed; we will denote it by L^*. An almost complex structure $\tilde{J}$ on $\xi = \ker \alpha$ compatible with $d\alpha$ induces an almost complex structure $\pi_* \tilde{J}$ on M compatible with ω. The extension J of $\tilde{J}$ on L^* is compatible with the standard complex structure on the fibers of L.

Let $\Delta_1, \ldots, \Delta_r$ be a basis of $H^*(M)$. Pick a basis of $H^*(V)$ of the form $\pi^*\Delta_{i_1}, \ldots, \pi^*\Delta_{i_{r'}}$, $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_s$ where the elements $\pi^*\Delta_{i_1}, \ldots, \pi^*\Delta_{i_{r'}}$ span $\pi^* H^*(M)$. Introduce variables $t_{i_1}, \ldots, t_{i_{r'}}, \tilde{t}_1, \ldots, \tilde{t}_s$ corresponding to these basis elements of $H^*(V)$, and introduce variables $p_{k,i}$ and $q_{k,i}$ $(i = 1, \ldots, r)$ corresponding to the base elements of $H^*(M)$, for every positive integer k.

Let $\beta_1, \ldots, \beta_u$ be a basis of $H_2(M, \mathbb{Z})$, so that $\omega(\beta_2) = \ldots = \omega(\beta_u) = 0$ and $l = \omega(\beta_1) > 0$. Introduce variables $z_1, \ldots, z_u$ corresponding to these basis elements. Let $\tilde{z}_i$ $(i = 2, \ldots, u)$ be the variable corresponding to the image of β_i in $H_2(L)$ under the inclusion of M into L as the zero section. Those homology classes generate exactly $H_2(V)/\mathcal{R}$.

The grading of these variables is defined as follows :

$$\begin{array}{ll} |t_i| = \deg(\Delta_i) - 2 & |\tilde{t}_j| = \deg(\widetilde{\Delta}_j) - 2 \\ |p_{k,i}| = \deg(\Delta_i) - 2 - 2\frac{c}{l}k & |q_{i,k}| = \deg(\Delta_i) - 2 + 2\frac{c}{l}k \\ |\tilde{z}_i| = -2c_1(TM)[\beta_i] & \end{array}$$

where $c = c_1(TM)[\beta_1]$. Note that this grading is fractional if $l \neq 1$, because in that case $H_1(L^*)$ contains torsion elements.

Define

$$\bar{u} = \sum_{j=1}^{r'} t_{i_j} \Delta_{i_j} + \epsilon \sum_{i=1}^{s} \tilde{t}_i \pi_* \widetilde{\Delta}_i + \sum_{k=1}^{\infty} \left(\bar{p}_k e^{ikx} + \bar{q}_k e^{-ikx} \right)$$

where ϵ is an odd variable, π_* is the integration along the fiber of V, $\bar{p}_k = \sum_{i=1}^{r} p_{k,i} \Delta_i$ and $\bar{q}_k = \sum_{i=1}^{r} q_{k,i} \Delta_i$.

Let

$$F(\bar{v}, z) = \sum_{d} \sum_{n=0}^{\infty} \frac{z_1^{d_1} \ldots z_u^{d_u}}{n!} < \bar{v}, \ldots, \bar{v} >_{0,n,d}$$

be the Gromov-Witten potential (for genus 0) of (M, ω).

Proposition 20. *Assume that M admits a perfect Morse function and that only one of the $\tilde{t}$ variables is nonzero and has odd parity. Then contact homology $HC_*(V, \xi)$ is the homology of the chain complex generated by infinitely many copies of $H_*(M, \mathbb{R})$, with degree shifts $2\frac{c}{l}k - 2$, $k = 1, 2, \ldots$ and with differential d, given by*

$$dq_{k,i} = k \sum_{j=1}^{r} (g^{-1})_{ij} \frac{\partial}{\partial p_{k,j}} H(p, q, t, \tilde{t}, \tilde{z})|_{p=0}$$

where

$$H(p, q, t, \tilde{t}, \tilde{z}) = \int d\epsilon \frac{1}{2\pi} \oint dx F(\bar{u}(x), \tilde{z} e^{-i<c_1(L),\beta>x})$$

and where $g_{ij} = \int_M \Delta_i \cup \Delta_j$.

Recall that integrating with respect to an odd variable ϵ has for effect to pick the coefficient B of ϵ in the integrand $A + B\epsilon$.

Sketch of proof. Since the projection p is holomorphic, it is clear that holomorphic curves in L^* are equivalent to the data of a closed holomorphic sphere C in M, with a holomorphic section of L over C. The zeroes and poles of that section correspond to the positive and negative punctures in L^* respectively, and their multiplicities match. The corresponding projection from the moduli space in L^* to the moduli space in M is a fibration with fiber S^1. Indeed, once the position and multiplicities of zeroes and poles of a section have been chosen, the only remaining degree of freedom is the phase of the section. Since M admits a perfect Morse function, the chain complex for contact homology involves directly homology for the orbit spaces, all diffeomorphic to M. Pulling back a single class $\widetilde{\Delta}_j$ to the moduli space in L^* corresponds to pulling back the class $\pi_* \widetilde{\Delta}_j$ to the moduli space in M and fixing the phase of the section. This explains the relationship between d and F. Note that if we pull back a second $\widetilde{\Delta}_{j'}$ class, we cannot interpret this in terms of Gromov-Witten invariants, since the S^1 degree of freedom was already fixed. This is why only one $\tilde{t}$ variable may occur in each term of d (and therefore this variable must be odd, so that $\tilde{t}^2 = 0$). In this case, the generalized holomorphic curves (including pieces of gradient flow trajectories between several components) do not appear in the differential, because the unique $\tilde{t}$ variable can kill the S^1 degree of freedom for only one component of the generalized holomorphic curve. Therefore, the differential of the Morse-Bott chain complex is given by the above formula.

A rigorous proof must include a comparison of the virtual cycles in L^* and in M showing that the above correspondence persists after perturbation. □

4.2. **Standard contact sphere.** We can apply the results of the previous section to compute explicitly contact homology of the standard contact 3-sphere. In this case, the base M is $\mathbb{C}P^1$, and we obtain variables $q_{k,0}$ and $q_{k,1}$ ($k = 1, 2, \dots$) corresponding to the generators of $H^0(\mathbb{C}P^1)$ and $H^2(\mathbb{C}P^1)$ respectively. It is convenient to reindex these variables in the following way :

$$\begin{cases} q_{2i} &= q_{i,1} \\ q_{2i-1} &= q_{i,0} \end{cases} \text{and} \begin{cases} p_{2i} &= p_{i,0} \\ p_{2i-1} &= p_{i,1} \end{cases}$$

Proposition 21. *Contact homology $HC_*(S^3, \xi_0)$ of the standard contact 3-sphere is isomorphic to the free unitary supercommutative algebra generated by t_0 and q_i, ($i = 2, 3, \dots$), where $|t_0| = -2$ and $|q_i| = 2i$.*

Proof. In this case, $M = \mathbb{C}P^1$ with its standard Kähler structure. Its Gromov-Witten potential is given by

$$F(v, z) = \frac{1}{2} v_0^2 v_1 + z \sum_{n=0}^{\infty} \frac{v_1^n}{n!}$$

where v_0 generates $H^0(\mathbb{C}P^1)$, v_1 generates $H^2(\mathbb{C}P^1)$ and z generates $H_2(\mathbb{C}P^1)$ so that $\omega(z) = 1$. Using proposition 20, we obtain

$$H = \frac{1}{2} t_0^2 \tilde{t} + \tilde{t} \sum_{i=1}^{\infty} p_{2i} q_{2i-1} + \tilde{t} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sum_{l=1}^{n} j_l = i} p_{2i+1} q_{2j_1} \dots q_{2j_n} + O(p^2)$$

where $\tilde{t}$ is the variable corresponding to the volume form on S^3. From this we deduce the formula for the differential :

$$dq_{2i} = i \tilde{t} q_{2i-1}$$

and

$$dq_{2i+1} = (i+1) \tilde{t} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sum_{l=1}^{n} j_l = i} q_{2j_1} \dots q_{2j_n}$$

Claim 1. Every expression containing a $\tilde{t}$ factor is exact.

Let us prove this by induction on the largest index of the q present in the expression, and on the exponent of that variable. First note that $\tilde{t}q_1^k = \frac{1}{k+1}d(q_1^{k+1})$. Then, let us assume that the expression has the form $\tilde{t}q_n^kF$, where F involves only variables t_0 and q_i $(i = 1, \dots, n-1)$. By the induction hypothesis, $\tilde{t}F$ has a primitive B involving variables t_0 and q_i $(i = 1, \dots, n-1)$ only as well. We have

$$d(q_n^kB) = \tilde{t}q_n^kF + kq_n^{k-1}dq_nB$$

Since dq_n is an expression containing only variables with index lower than n, by the induction hypothesis $kq_n^{k-1}dq_nB$ is exact.

Claim 2. For every monomial q_n^k $(n \geq 2)$, there exists an expression C containing only variables t_0 and q_i $(i = 1, \dots, n-1)$ and q_n up to power $k-1$, such that $q_n^k + C$ is closed.

Such a C would have to satisfy

$$dC = -kq_n^{k-1}dq_n$$

But since dq_n contains a factor $\tilde{t}$, the right hand size is exact and we can find a solution C.

Note that the above claim is not true for $n = 1$, since $dq_1 = \tilde{t}$. Moreover, an expression without a $\tilde{t}$ factor cannot be exact. The proposition now clearly follows. □

4.3. **Brieskorn spheres.** We now turn to a more general example involving the Brieskorn spheres. Let $\Sigma(a) = \Sigma(a_0, \dots, a_n) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0\} \cap S^{2n+1}$.

Theorem 22. (Brieskorn) *When $n = 2m+1$ and $p = \pm 1 (mod\, 8)$, $a_0 = p, a_1 = 2, \dots, a_n = 2$, then $\Sigma(a)$ is diffeomorphic to S^{4m+1}.*

On $\mathbb{C}^{n+1}$, consider the 1-form $\alpha_p = \frac{i}{8}\sum_{j=0}^n a_j(z_j d\overline{z}_j - \overline{z}_j dz_j)$. Its restriction to $\Sigma(a)$ is a contact form, with Reeb field $R_{\alpha_p} = 4i(\frac{z_0}{a_0}, \dots, \frac{z_n}{a_n})$. Denote the corresponding contact structure by ξ_p. These are distinguished by contact homology. This result is originally due to Ustilovsky [14], and was proved by perturbing contact form α_p in order to have non-degenerate closed Reeb orbits.

Note that the quotient of $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ by the flow of R_{α_p} is a weighted projective space $\mathbb{C}P^n_w$, i.e. an orbifold. The quotient of $\Sigma(a) = S^{2n-1}$ by this Reeb flow is the zero locus of the polynomial $z_0^p + z_1^2 + \dots + z_n^2$ in $\mathbb{C}P^n_w$, i.e. a complete intersection in a toric orbifold. $\Sigma(a)$ is a principal circle orbi-bundle over this orbifold. Therefore, this example belongs to the second family discussed in section 4.

Theorem 23. **(Ustilovsky)** *Under the assumptions of theorem 22, the contact homology for cylindrical curves* $HF_k(\Sigma, \xi_p) = \mathbb{Q}^{c_k}$ *where*

$$c_k = \begin{cases} 0 & \text{if } k \text{ is odd or } k < 2n-4 \\ 2 & \text{if } k = 2\lfloor \frac{2N}{p} \rfloor + 2(N+1)(n-2), \text{ for } N \in \mathbb{Z}, N \geq 1, 2N+1 \notin p\mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$

Here, we will prove this theorem using the contact form α_p and the Morse-Bott formalism, instead of perturbing α_p to obtain nondegenerate Reeb orbits.

Let us first study the periodic orbits of R_{α_p} and their Maslov indices. The Reeb flow is given by

$$\varphi_t(z_0, \ldots, z_n) = (e^{4it/p} z_0, e^{2it} z_1, \ldots, e^{2it} z_n)$$

Hence, all Reeb orbits are periodic, and there are 2 values of the action for simple orbits :

(i) Action $= \pi$ (when $z_0 = 0$).
In that case, the orbit space is

$$S_\pi = \{[z_1, \ldots, z_n] \in \mathbb{C}P^{n-1} \mid z_1^2 + \ldots + z_n^2 = 0\}$$

i.e. the nondegenerate quadric Q_{n-2} in $\mathbb{C}P^{n-1}$.

Lemma 24. *If n is odd, then $H_*(Q_{n-2}) \simeq H_*(\mathbb{C}P^{n-2})$.*

Proof. Note that Q_{n-2} is the Grassmannian of oriented 2-planes in $\mathbb{R}^n$. Indeed, the manifold $N_\pi = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \ldots, z_n^2 = 0\}$ is the unit tangent bundle of the sphere S^{n-1}, and the Reeb flow coincides with the geodesic flow on S^{n-1}. The computation of the homology is then standard and gives, for n odd, the announced result. □

Let us compute the Maslov index of these periodic orbits. The linearized Reeb flow splits into the tangential and normal bundles to S_π. For the tangential part, the linearized flow is $e^{2it} I_{n-2}$ for $0 \leq t \leq \pi$, so we obtain contribution $2(n-2)N$, where N is the multiplicity of the orbit, and for the normal part, the linearized flow is just multiplication by $e^{4it/p}$, so we obtain contribution $1 + 2\lfloor \frac{2N}{p} \rfloor$. Hence, the Maslov index is :

$$\mu = 2N(n-2) + 1 + 2\lfloor \frac{2N}{p} \rfloor$$

(ii) Action $= p\pi$ (when $z_0 \neq 0$).
In that case, the orbit space incorporates the p-covered orbits of case (i) as a singularity with group $\mathbb{Z}_p$..

Lemma 25. *$S_{p\pi}$ is homeomorphic to $\mathbb{C}P^{n-1}$.*

Proof. We follow the arguments of [13]. Consider the projection $\phi : \Sigma(a) \to S^{2n-1}$: $(z_0, \dots, z_n) \to \frac{(z_1,\dots,z_n)}{\|(z_1,\dots,z_n)\|}$. Clearly, this map is surjective, and equivariant with respect to the Reeb flow on $\Sigma(a)$ and multiplication by a phase on S^{2n-1}. Moreover, any two points in $\Sigma(a)$ projecting to the same point in S^{2n-1} lie on the same Reeb orbit. Hence, the orbit spaces are homeomorphic. But the one of S^{2n-1} is clearly $\mathbb{C}P^{n-1}$. □

The Maslov index is very easy to compute, since the Reeb flow is now completely periodic. For the tangential part to S_π, we obtain p times the previous result, and for the normal part, we obtain 2 (one complete turn). Hence

$$\mu = 2N((n-2)p + 2)$$

Proof of Theorem 23. Note that all holomorphic cylinders come in S^1 families, since they can be pushed along the Reeb field. Therefore, the differential coincides with the Morse-Witten differential of the orbit spaces. Hence, cylindrical homology is just the direct sum of the homology of all orbit spaces, with the appropriate degree shiftings.

The grading corresponding to the homology classes in $S_{N\pi}$, for $N \notin p\mathbb{Z}$, is given by :

$$2N(n-2) + 2\lfloor \frac{2N}{p} \rfloor + 2k \qquad k = 0, \dots n-2$$

Hence, we obtain one generator in each even degree, starting at degree $2n-4$ corresponding to $N = 1$ and $k = 0$. Moreover, there is an overlap between N ($k = n-2$) and $N+1$ ($k = 0$) at

$$2(N+1)(n-2) + 2\lfloor \frac{2N}{p} \rfloor$$

when the integral part of $\frac{2N}{p}$ does not jump between N and $N+1$. We get exactly two generators for these degrees. However, there will be a jump when $N+1 \in p\mathbb{Z}$ or $2N+1 \in p\mathbb{Z}$. In the first case, $N+1 = mp$, and we actually have to use the generators of case (ii) above. The degrees of the generators corresponding to the homology classes in $S_{mp\pi}$ are given by

$$2mp(n-2) + 4m - 2 + 2k \qquad k = 0, \dots, n-1$$

For $N = mp - 1$ and $k = n-2$, we obtain a generator in degree

$$2(mp-1)(n-2) + 2(2m-1)$$

But the generator for mp and $k = 0$ has degree

$$2mp(n-2) + 4m - 2 - 2(n-2)$$

So we still have 2 generators in that degree, despite the jump. However, when $2N+1 \in p\mathbb{Z}$, there is nothing to compensate for the jump, and we do not have an overlap. Therefore, we obtain exactly the ranks given in theorem 23. $\square$

4.4. **Unit cotangent bundle of the torus.** Let $M = ST^*T^n$ be the unit cotangent bundle of T^n, with respect to the standard flat metric. M is equipped with a natural contact form α, which is obtained by restricting the Liouville 1-form $\theta = \sum_{i=1}^n p_i dq_i$ on M.

The Reeb flow on M coincides with the geodesic flow on T^n, so we obtain closed Reeb orbits when the coordinates p_i $(i = 1, \dots, n)$ are rationally dependent 2 by 2. Each connected component of N_T corresponds to a nonzero element $\bar{a} = (a_1, \dots, a_n) \in \pi_1(T^n) = \mathbb{Z}^n$, where $T = \sqrt{a_1^2 + \ldots + a_n^2} = \|\bar{a}\|$, and is a copy of the torus T^n.

The symplectization of M is isomorphic to T^*T^n minus its zero section. The symplectic form is the standard $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ if we substitute $e^t = r = \sqrt{\sum_{i=1}^n p_i^2}$. We can equip the symplectization with almost complex structure J, preserving ξ, defined by $J\frac{\partial}{\partial p_i} = \frac{1}{r}\frac{\partial}{\partial q_i}$. Note that this almost complex structure is not integrable.

The linearized Reeb flow along a periodic Reeb orbit is given, in the q_i, p_j coordinates, by

$$\Psi(t) = \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix}$$

Therefore, all the periodic orbits have Maslov index $\frac{n-1}{2}$, since we have to restrict ourselves to the unit cotangent bundle. Subtracting half the dimension of the orbit space and adding $n-3$, we get grading $n-3$.

Since there are no contractible periodic orbits, cylindrical homology is well defined, and we can even restrict ourselves to a fixed homotopy class $\bar{a}$ of closed Reeb orbits.
On the other hand, holomorphic cylinders have zero energy, since the period of a Reeb orbit depends only on its homotopy class. Hence, all holomorphic cylinders are vertical cylinders over a Reeb orbit. Therefore, the differential d of our chain complex coincides exactly with the Morse-Witten complex of the orbit space T^{n-1} in homotopy class $\bar{a}$. Gathering our results, we have shown

Proposition 26. *Cylindrical homology $HF_*^{\bar{a}}(ST^*T^n, \xi)$ in homotopy class $\bar{a}$ is isomorphic to the standard homology $H_{*+n-3}(T^{n-1})$ of T^{n-1}, shifted by degree $n-3$.*

As a corollary of this result, we can reprove a theorem originally due to Giroux [6]. On T^3, let $\alpha_k = \cos 2\pi kz\, dx + \sin 2\pi kz\, dy$ and denote the corresponding contact structure by

ξ_k. Then ξ_1 is the contact structure considered above, when $n = 2$. The contact structure ξ_k is obtained from ξ_1 by a k-fold covering of T^3.

Corollary 27. (Giroux) *Contact structures ξ_k on T^3 are pairwise non isomorphic.*

Proof. The computation of $HF_*^{\bar{a}}(T^3, \xi_k)$ is analogous to the above computation, except that we now have k copies of the orbit space S^1 in homotopy class $\bar{a}$. Therefore, cylindrical homology is the direct sum of k copies of $H_{*-1}(S^1)$. In particular, we obtain different results for different values of k. $\square$

The proof of corollary 27 using cylindrical homology was already mentioned in [4], but its proof relies on the techniques developed in this paper.

4.5. **Unit cotangent bundle of the Klein bottle.** This last example is a little more exotic. It will turn out that theorem 19 does not apply to this case. However, we will see that our Morse-Bott techniques still allow us to compute cylindrical homology without working out an explicit perturbation of the contact form.

As in our previous example, the contact form is the Liouville 1-form restricted to the cotangent bundle of K^2. The Reeb flow on ST^*K^2 coincides with the geodesic flow. We choose of course to work with the flat metric of K^2.

We see the Klein bottle K^2 as the quotient of $\mathbb{R}^2$ under the discrete group generated by $(x, y) \to (x + 1, 1 - y)$ and $(x, y) \to (x, y + 1)$. The homotopy class $\bar{a} = (a_1, a_2)$ of loops in K^2 contains the projection of the line $y = \frac{a_2}{a_1}x$ in $\mathbb{R}^2$.

Let us determine the orbit spaces in homotopy class (a_1, a_2) :

(i) $a_1 \neq 0$, a_2 odd.
There are no periodic orbits, because the projection of the line $y = \frac{a_2}{a_1}x$ to K^2 closes with an angle.

(ii) $a_1 \neq 0$, a_2 even.
This time, the projection of the line $y = \frac{a_2}{a_1}x$ to K^2 closes smoothly. Therefore, the closed orbits foliate a torus, and the orbit space is S^1.

(iii) $a_1 = 0$, a_2 odd.
The projection of the line $y = y_0$ to K^2 is closed if and only if $y_0 \in \frac{1}{2}\mathbb{Z}$. Therefore, there are exactly 2 closed orbits.

(iv) $a_1 = 0$, $a_2 \neq 0$ even.
This time, the projection of the line $y = y_0$ is always closed. Therefore, the closed orbits foliate K^2, and the orbit space is a closed interval. The endpoints are a_2-covers of the 2 simple orbits in homotopy class $(0, 1)$.

As in the previous example, the period of a closed Reeb orbit in homotopy class $\bar{a} = (a_1, a_2)$ is given by $T = \sqrt{a_1^2 + a_2^2}$.

When a_2 is even, the Reeb dynamics are identical to the case of ST^*T^2, therefore the corresponding closed Reeb orbits have grading $n - 3 = -1$.
When $a_1 = 0$ and a_2 is odd, the pull-back of the contact distribution to a closed Reeb orbit is not trivial. Therefore, we have to trivialize ξ along a double cover of that orbit and use fractional grading as explained in [4].

On the other hand, for $a_1 = 0$, $a_2 \neq 0$ even, the submanifold N_T is not orientable, so lemma 14 does not apply and we have to check for bad orbits. Use a Morse function f_T on the closed interval with 2 minima at the endpoints and a maximum in the middle. Clearly, the perturbed Reeb orbits at the maximum is good, because its index is independent of the multiplicity.

Claim. The perturbed Reeb orbits corresponding to the endpoints are bad.

Consider a linear Cauchy-Riemann operator on a rank 2 vector bundle E over a 1-punctured sphere, with the asymptotics of those perturbed Reeb orbits. Since the asymptotics of that operator will be invariant under rotation, we can choose the linear operator to be invariant under rotation as well. The change of trivialization of such a double Reeb orbit corresponds to a $\mathbb{Z}_2$ action on $E : (z, x_1, x_2) \to (-z, -x_1, -x_2)$.
Clearly, $(z, x_1, x_2) \to (-z, x_1, x_2)$ induces the identity on the kernel and cokernel, because that induced map is homotopic to the identity via $(z, x_1, x_2) \to (e^{i\theta} z, x_1, x_2)$, $0 \leq \theta \leq \pi$.
On the other hand, $(z, x_1, x_2) \to (z, -x_1, -x_2)$ clearly induces $-I$ on the kernel and cokernel. Since the index of the Cauchy-Riemann operator is odd (it is -1, see above), this action reverses the orientation of the determinant line. Therefore, the corresponding perturbed Reeb orbits are bad.

Summing up, we have

Proposition 28. *Cylindrical homology $HF_k^{\bar{a}}(ST^*K^2, \xi) = \mathbb{Q}^{c_k}$, where the non-vanishing ranks c_k are given by*

$$\begin{cases} c_{-1} = 1, c_0 = 1 & \textit{if } a_1 \neq 0, a_2 \textit{ even} \\ c_{-1} = 2 & \textit{if } a_1 = 0, a_2 \textit{ odd} \\ c_0 = 1 & \textit{if } a_1 = 0, a_2 \neq 0 \textit{ even} \end{cases}$$

References

[1] F. Bourgeois. *A Morse-Bott approach to contact homology.* PhD thesis, Stanford University, 2002.

[2] F. Bourgeois and K. Mohnke. Coherent orientations in Symplectic Field Theory. ArXiv preprint, math.SG/0102095, 2001.

[3] Y. Eliashberg. Invariants in contact topology. *Doc. Math. J. DMV*, Extra Volume ICM 1998, 1998.

[4] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to Symplectic Field Theory. *Geom. Funct. Anal.*, Special Volume, Part II:560–673, 2000.

[5] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math.Zeitschr.*, 212:13–38, 1993.

[6] E. Giroux. Une structure de contact, même tendue est plus ou moins tordue. *Ann. Scient. Ec. Norm. Sup.*, 27:697–705, 1994.

[7] H. Hofer, K. Wysocki, and E. Zehnder. Pseudoholomorphic curves in symplectizations IV : Asymptotics with degeneracies. *Publ. Newton Inst.*, 8:78–117, 1996.

[8] E. Ionel and T. Parker. Relative Gromov-Witten invariants. arXiv preprint math.SG/9907155, 1999.

[9] J. Li and G. Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic 4-manifolds*, volume I of *First Int. Press Lect. Ser.*, pages 47–83. Internat. Press, 1998.

[10] G. Liu and G. Tian. Floer homology and Arnold conjecture. *J. Diff. Geom.*, 49:1–74, 1998.

[11] D. McDuff. The virtual moduli cycle. *Amer. Math. Soc. Transl. Ser. 2*, 196:73–102, 1999.

[12] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[13] C. B. Thomas. Almost regular contact manifolds. *J. Diff. Geom.*, 11:521–533, 1976.

[14] I. Ustilovsky. Infinitely many contact structures on S^{4m+1}. *Int. Math. Res. Notices*, 14:781–791, 1999.

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