A stroll through some important tools of model theory illustrated by the example of the Mordell-Lang conjecture

Colloquium Logicum, Hamburg, September 2016

Elisabeth Bouscaren

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In the present: Talk inspired by a series of papers joint work with Franck Benoist and Anand Pillay on the model theory of semiabelian varieties although this won't be visible here.

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The notions

- induced structure
- enriched structure
- orthogonality

- Characterizing classical algebraic structures abstractly

or reconstructing algebraic structures (groups, fields) from abstract or combinatorial data .

In the spirit of the very old classical theorem in geometry which says that a Desarguesian projective geometry of dimension at least 3 is the projective geometry over a division ring.

Mordell-Lang

Theorem *Function field Mordell-Lang, a special case* (and a little cheating especially in char. p)
$k \subset K$ two algebraically closed fields, G abelian variety over K, X irreducible subvariety of G, $\Gamma \subset G(K)$ finitely generated subgroup.
Then ,
– either G has a sub-abelian variety H "defined" over k, which contains a translate of $\Gamma \cap X$,
– or $X \cap \Gamma = a_1 + (H_1 \cap \Gamma) \cup \ldots \cup a_n + (H_n \cap \gamma)$, where for each i, H_i is a sub-abelian variety of G.

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Explaining the statement and the objects

We have $k \subset K$ algebraically closed fields .
Recall :
Definition A field K is algebraically closed if every polynomial $P(X)$ in one variable in $K[X]$, of degree ≥ 1, has a solution in K

Ex: $\mathbb{C}$ the complex numbers , but not the reals $\mathbb{R}$.
There is a theory T_{acf} (a set of sentences) such that a field is algebraically closed iff it is a model of T_{acf}:
for every $n > 1$

$$\forall y_0 \dots \forall y_{n-1} \exists x \left(x^n + \Sigma_{i=0}^{n-1} y_i . x^i \right) = 0.$$

Any two algebraically closed fields of same characteristic are elementarily equivalent i.e. they satisfy exactly the same statements in the language of rings.

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Explaining Mordell-Lang

For a field K as a first-order model theory structure, we consider the basic ring operations: $(K, +, ., -, 0, 1)$ addition $x + y$, multiplication $x.y$, the inverse for addition $-x$, and the two constants 0 and 1.
And we consider first order formulas in this language
But we are really interested in the sets we can define with them. A set $D \subset K^n$ is definable if there is a formula $\phi(x)$ such that $D = \{a \in K^n ; K \models \phi(a)\}$. Then we write $D = \phi(K)$.
– Atomic sets in K^n = solutions of one polynomial equation , $\{a \in K^n ; P(a) = 0\}$ for some polynomial P in $K[X_1, \ldots, X_n]$. For each n, finite intersections of these (finite conjonctions of polynomial equations) form a topology on K^n the Zariski topology.
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Definable sets in ACF

– quantifier free formulas $\longrightarrow$ finite boolean combinations of atomic sets = finite boolean combinations of closed sets = constructible sets in the sens of the topology.
– to get the definable sets: should close also under projection or existential quantifier for the corresponding formula .
But :

Algebraically closed fields

Theorem(Tarski, Chevalley) Algebraically closed fields admit quantifier elimination.
or definable sets = constructible sets.

Theorem (Macintyre, 70's) If in an infinite field K every definable set is constructible, then the field is algebraically closed.

Modern model theory : Get algebraic information from abstract data.

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Algebraic groups

Recall a group $(G, .)$ is definable in K if G is a definable subset of some K^n and the multiplication and inverse maps are definable (ie their graphs are definable sets.)

Obvious definable groups in K : the additive group, the multiplicative group, the affine groups, that is, closed subgroups of $GL_n(K)$ (definable in $K^n x K^n$).

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Mordell-Lang continued, the definable objects

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Mordell-Lang continued, the definable objects

G is an Abelian variety: *Complete* connected algebraic group.
Ex: Elliptic curves, Jacobians of curves, never affine
For us they are definable groups with certain properties
$(G, +)$ is commutative, has finite n-torsion for every n, G is divisible.
But the torsion is infinite.
G as an algebraic group has an induced topology, its Zariski topology.
And X is a closed irreducible subset of G in this sense . In particular X is definable.
A subabelian variety of G is an irreducible closed subgroup (so also definable) .
H is "defined" over k: defined with parameters in k or more accurately here, isomorphic to some algebraic group defined over k, that is H "descends " to k.

The non definable objects

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Γ is not definable or algebraic !
Γ is just a finitely generated subgroup of $G(K)$. Even if Γ is generated by one element g_0 ,$x \in \Gamma$ iff
$x = g_0 \vee x = g_0 + g_0 \vee x = g_0 + g_0 + g_0 \ldots$

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The Dichotomy

The dichotomy :
in the second case , note that the conclusion talks about the topology induced on Γ by the topology of G, it is induced by the closed subgroups only:
the closed subsets of Γ are the sets of the form $X \cap \Gamma$ for X closed in G, and it says they are just given by translates of subgroups.

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This says either we can descend the situation to the smaller field k, or ....

k is not a definable subfield, how to make this dichotomy into a "model theory concept" ?

The Zilber principle of dichotomy, or in fact trichotomy

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Strong minimality

A definable subset D in K^n is strongly minimal if for any definable $E \in K^n$, $D \cap E$ is finite or its complement in D is finite .
The algebraically closed field K itself is strongly minimal: any definable set in one variable is a boolean combination of sets which are the solution set of a polynomial equation in one variable .

Examples

1. Infinite set with no structure (only equality in the language)
2. An infinite vector spaces over a fixed division ring. It has the property that any definable subset is a translate of a subgroup
3. Algebraically closed fields

or “avatars“ of these.

Zilber trichotomy principle

Conjecture proposed by Boris Zilber in the 1980's : every strongly minimal theory "is" of one of these forms.

Disproved by Hrushovski in 90'S.

But proved (Hrushovski-Zilber, 93) to hold for a class of strongly minimal sets with extra properties, the Zariski structures.

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Zariski Structures and Trichotomy

A Zariski structure is a strongly minimal structure M where the atomic sets form a noetherian topology on each M^n , the definable sets are the constructible sets of the topology and the dimension (given by the noetherian topology) satisfies some properties (the dimension theorem)

Then TRICHOTOMY :

1. either M has "no structure"

or

2. there is a commutative group definable in M^n , and M is basically a vector space over a division ring (the ring of definable quasi-endomorphisms of the group)

or

3. in M there is an algebraically closed field K which is definable in some M^n , and M is homeomorphic to a projective curve over K , or "nearly" so .

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Zilber trichotomy principle

Very successfull case of getting algebraic structure from abstract conditions , but very difficult proof.

Used by Hrushovski for Mordell-Lang (and other applications since). But difficulty means it makes the proof and what exactly it is saying in this particular context a little difficult to understand.

Enriching structures

One way to make Γ definable is by brute force:

Just add a predicate to the language for Γ then Mordell Lang indeed says that in the new structure $(K, +, ., 0, 1, G, \Gamma)$; Γ is a locally modular group: i.e. every definable subset of Γ is a finite boolean combination of translates of definable subgroups. A concept that had been indeed introduced in model theory some years before, a generalization of the case of vector space for stronly minimal ....
BUT it is not easier to prove this than the original statement .
So, find another way to make Γ definable, so that one could use the Zilber trichotomy principle: enrich to a structure we can study and which has good properties.

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Adding a derivation

We will also manage to make the small field k definable.
Add more definable sets by adding a derivation on the field K,
a map δ from K to K such that
- $\delta(x + y) = \delta(x) + \delta(y)$
- $\delta(x.y) = x.\delta(y) + y.\delta(y)$

in characteristic 0 .

A little more complicated in Char. p, must add a family of strange derivations (Hasse derivations).

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DCF_0

Can do this so that k becomes the field of constants of δ so definable as $\{a \in K; \delta(a) = 0\}$.
Pass to a differentially closed field: we can suppose that K is differentally closed (=existentally closed) and k is the field of constants in K.
The theory of differentially closed fields of char. 0, DCF_0 is richer than ACF_0 but still good from model theoretic point of view.
we know a lot about definable groups, and *the strongly minimal subsets are Zariski structures !*

Working in DCF_0

Replace Γ by : $G^\sharp$ which is the smallest definable subgroup of G which is zariski dense in G or also the smallest definable subgroup containing the torsion of G.

$G^\sharp$ is nice, it is a group with finite Morley rank , but it is not strongly minimal .

Saved!

Any abelian variety G is a sum of simple abelian varieties pairwise not homomorphic, it follows that $G^\sharp = J_1 + \ldots + J_n$ a sum of (almost)strongly minimal groups which are pairwise orthogonal. J_1 and J_2 are orthogonal means that they really are "independent": any definable subset of $J_1 x J_2$ is of the form $D_1 x D_2$ where D_1 is definable in J_1 and D_2 is definable in J_2.
And in DCF_0 by the Zariski structures dichotomy:
an almost strongly minimal group J is either locally modular : any definable subset is boolean combiantion of translates of subgroups or it is isomorphic to a group defined over the field of constants (so k), J descends to the constants. From this, get the two cases in the conclusion of Mordell-Lang: recall

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Had to “enrich”: in K as a pure field (in the ring language,) any two definable groups are non orthogonal (true in any uncountably categorical theory) !

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Semiabelian varieties and model theory

In fact Mordell-Lang is about semiabelian varieties, more complicated algebraic groups of the form

G is a Semiabelian variety : $G \in Ext(A, T)$ i.e.
$0 \to T \to G \to A \to 0$,
with $T = \mathbb{G}_m^r$ torus and A abelian variety .

Examples : $T \times A$, or semi-split (G isogenous to $T \times A$)
But also non split complicated examples.
Have examples where the induced sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is not exact.
So one cannot deduce good properties for $G^\sharp$ from the same properties for $A^\sharp$ and $T^\sharp$.

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But model theory of finite rank groups shows that still $G^\sharp$ is a sum of pairwise orthogonal minimal groups

and hence allows to reduce the question to the good cases, when $G = TxA$.

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