∞-Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 2. Higher Categorical Background

Last week, we discussed the Barr-Beck theorem, which specifies conditions under which an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces an equivalence $\mathcal{D} \xrightarrow{\simeq} \mathrm{Alg}_T(\mathcal{C})$. Here, $T = GF$ is the monad associated with $F \dashv G$, and $\mathrm{Alg}_T(\mathcal{C})$ is the category of T-algebras $(X \in \mathcal{C}, T(X) \xrightarrow{\alpha} X)$.

As a toy application, we proved that if R is a ring and $Q \in \mathrm{Mod}_R^{\heartsuit}$ is a finite projective generator, then R and $S = \mathrm{End}_R(Q)^{op}$ have equivalent categories of (left) modules, via the functor $M \mapsto \mathrm{Map}_R(Q, M)$. In fact, any equivalence of module categories arises in this way.

From Morita to Koszul. The basic setup for Koszul duality is a field k and an augmented associative k-algebra R. Note that this gives k the structure of an R-module.

Taking inspiration from Morita theory, we may ask:

Question 2.1. Is the functor $G : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathrm{Hom}_R(k,k)^{op}}$, $M \mapsto \mathrm{Map}_R(k, M)$ an equivalence?

The answer is a resounding *"no"*, as is manifest from the following simple example:

Example 2.2. For $R = k[\epsilon]/\epsilon^2$, we have $\mathrm{Hom}_R(k,k)^{op} = k$, and the functor G sends $M \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ to $\ker(\epsilon : M \to M)$. Hence G is far from an equivalence.

However, a more sophisticated variant of this construction will give an interesting functor.

Indeed, we can refine the functor $(M \mapsto \mathrm{Map}_R(k, M))$ using homological algebra. If M and N are left modules over a ring R, then $\mathrm{Hom}_R(M, N)$ is only a fragment of a more refined construction called $\mathbb{R}\mathrm{Hom}_R(M, N)$, which is a *complex* of R-modules. It can be computed by chosing a projective resolution $\ldots \to P_2 \to P_1 \to P_0 \to M$ of M and setting

$$\mathbb{R}\mathrm{Hom}_R(M,N) \;:=\; (\ldots \to \; 0 \; \to \; \mathrm{Hom}_R(P_0, N) \; \to \; \mathrm{Hom}_R(P_1, N) \quad \to \ldots)\ .$$

This complex depends on the chosen projective resolution $P_\bullet$, but different resolutions give quasi-isomorphic complexes. We will provide a clean formulation of this phenomenon in the language of ∞-categories in later lectures.

Notation 2.3. We will now adjust our notation to stress that chain complexes are henceforth the basic objects of interest. If R is a ring, we will define an ∞-category whose objects are chain complexes of left R-modules, and we write Mod_R for this enhancement of the classical triangulated category $D(R)$. We will, from now on, write $\mathrm{Mod}_R^{\heartsuit}$ for the ordinary category of ordinary left R-modules.

Notation 2.4. Given a chain complex $M \in \mathrm{Mod}_R$, write $\pi_*(M)$ for its homology groups. Concretely, if $M = (\ldots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \ldots)$, then $\pi_i(M) = \ker(d_i)/\mathrm{im}(d_{i+1})$.

Remark 2.5. Note that $\pi_*(\mathbb{R}\mathrm{Hom}_R(M,N)) \cong \mathrm{Ext}_R^{-*}(M,N)$ recovers the usual Ext-groups.

We will soon define the structure of a differential graded algebra on $\mathbb{R}\mathrm{Hom}_R(k,k)^{op}$, which will allow us to modify the question raised above:

Question 2.6. Is the functor $G\colon \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{R}\,\mathrm{Hom}_R(k,k)^{op}}$, $M \mapsto \mathbb{R}\,\mathrm{Map}_R(k,M)$ an equivalence?

The answer is also "no", but we will see that G sometimes restricts to an interesting equivalence on a subcategory of Mod_R. To formulate and prove this equivalence, we will make use of Lurie's higher categorical generalisation of the classical Barr-Beck theorem.

In this lecture, we will introduce the very basics of the theory of ∞-categories. Due to our time restrictions, we can only scratch the surface – for a more comprehensive treatment, we recommend [Lur09] or the online resource Karedon.

2.1. **Simplicial sets.** While ∞-categories might look scary (possibly due to symbol "∞"), it can be helpful to remember that they are just simplicial sets satisfying a certain property.

To fix notation, we briefly recall the basic setup of simplicial sets.

Write Δ for the simplex category; its objects are the nonempty finite linearly ordered sets

$$[0] = \{0\}\ ,\quad [1] = \{0 < 1\}\ ,\quad [2] = \{0 < 1 < 2\}\ ,\quad \dots\ ,$$

and morphisms are order-preserving maps.

Definition 2.7 (Simplicial sets)**.** A *simplicial set* is a functor $\Delta^{op} \to \mathrm{Set}$. Write **sSet** for the resulting (ordinary) category of simplicial sets.

Notation 2.8 (Simplices and horns)**.** Fix integers $n \geq 0$ and $0 \leq i \leq n$.

(1) Write $\Delta^n = \mathrm{Map}_{\Delta}(-,[n]) : \Delta^{op} \to \mathrm{Set}$ for the simplicial set represented by $[n] \in \Delta$.
(2) Let Λ^n_i be the simplicial set sending $[k] \in \Delta$ to $\{f : [k] \to [n] \text{ s.t. } [n]\backslash\{i\} \notin f([k])\}$.

We refer to Δ^n as the simplicial n-simplex, and call Λ^n_i the i^{th} horn of Δ^n.

Definition 2.9 (Mapping objects)**.** Given $X, Y \in \mathbf{sSet}$, define $Y^X \in \mathbf{sSet}$ by $(Y^X)_n = \mathrm{Map}_{\mathbf{sSet}}(\Delta^n \times X, Y)$; the simplicial structure maps are induced by the Yoneda embedding.

The category of simplicial sets is therefore enriched in **sSet**.

Simplicial sets are closely related to **Top**, the category of (compactly generated) spaces. To make this statement precise, we will need, for every $n \geq 0$, the topological n-simplex

$$\boldsymbol{\Delta}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 + \dots + x_n = 1,\ x_i \geq 0\}.$$

$$\text{(1)} \quad \text{Any } [n] \xrightarrow{f} [m] \text{ induces } \boldsymbol{\Delta}^n \xrightarrow{f_*} \boldsymbol{\Delta}^m \text{ with } f_*(s_0, \dots, s_n) = (t_0, \dots, t_m),\ t_j = \sum_{f(i)=j} s_i.$$

We can build spaces from simplicial sets:

Definition 2.10 (Geometric realisation)**.** The *geometric realisation* of a simplicial set X is given by $|X| = \mathrm{colim}_{\Delta^n \to X}(\boldsymbol{\Delta}^n)$; this colimit is computed in the ordinary category **Top**.

We call a simplicial set X weakly contractible if $|X|$ has this property.

Exercise 2.11.
a) Reformulate Definition 2.10 both as a left Kan extension and as a coend.
b) Give an explicit formula for $|X|$ as a quotient of a coproduct by an equivalence relation.
c) Describe the spaces $|\Delta^n|$ and $|\Lambda^n_i|$ (cf. Notation 2.8).

We can also go into the reverse direction and attach simplicial sets to spaces:

Definition 2.12 (Singular chains)**.** Given $X \in \mathbf{Top}$, the simplicial set $\mathrm{Sing}(X)$ satisfies

$$\mathrm{Sing}(X)_n = \mathrm{Map}_{\mathbf{Top}}(\mathbf{\Delta}^n, X),$$

Given a map $[n] \xrightarrow{f} [m]$ in Δ, the corresponding structure map $\mathrm{Sing}(X)_m \to \mathrm{Sing}(X)_n$ is obtained by precomposing with the map $f^* : \mathbf{\Delta}^n \xrightarrow{f_*} \mathbf{\Delta}^m$ from (1) above.

Exercise 2.13. Show that the singular chains functor Sing is right adjoint to the geometric realisation functor $|-|$ from Definition 2.10.

Simplicial sets arising as the singular chains of a topological space have a special property:

Definition 2.14 (Kan complexes)**.** A simplicial set X is called a *Kan complex* if it satisfies the right lifting property for all horns. Concretely, this means that for all n and any $0 \leq i \leq n$, every map $f_0 : \Lambda^n_i \to X$ extends to a map $\overline{f} : \Delta^n \to X$ from the n-simplex:

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & X \\ \downarrow & \nearrow_{\overline{f}} & \\ \Delta^n & & \end{array}$$

We can also attach simplicial sets to ordinary categories. For this, we identify the linearly ordered set $[n]$ with the category $(0 \to 1 \to \ldots \to n)$ and define:

Definition 2.15. The *nerve* of a category $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C})$ with $\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}_{\mathrm{Cat}}([n], \mathcal{C})$; The structure maps are induced by pullback along maps $[n] \to [m]$ in Δ^{op}.

Remark 2.16. The observant reader might object that if $\mathcal{C}$ is not small, then $\mathrm{N}(\mathcal{C})$ is too large to be a set. This technical difficulty can be handled rigorously using Grothendieck universes; we refer to [Lur09, Section 1.2.15] for a discussion. In these expository lectures, we will confidently sweep size issues of this kind under the rug.

Simplicial sets which arise as nerves of ordinary categories share a special property:

Exercise 2.17. Show that a simplicial set $X \in \mathbf{sSet}$ is the nerve $\mathrm{N}(\mathcal{C})$ of a category $\mathcal{C}$ if and only if for all $0 < i < n$ and each map $f : \Lambda^n_i \to X$, there is a *unique* extension to Δ^n.

2.2. **Higher categories.** To define ∞-categories, also known as quasi-categories or weak Kan complexes, we relax the uniqueness assertion in Exercise 2.17:

Definition 2.18 (Boardman-Vogt)**.** An ∞-category is a simplicial set $\mathcal{D} \in \mathbf{sSet}$ such that for all $0 < i < n$ and each map $f : \Lambda^n_i \to \mathcal{D}$, there is a (not necessarily unique) extension

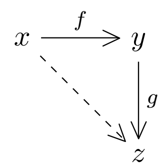
$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & \mathcal{D} \\ \downarrow & \nearrow_{\overline{f}} & \\ \Delta^n & & . \end{array}$$

A functor between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map of simplicial sets.

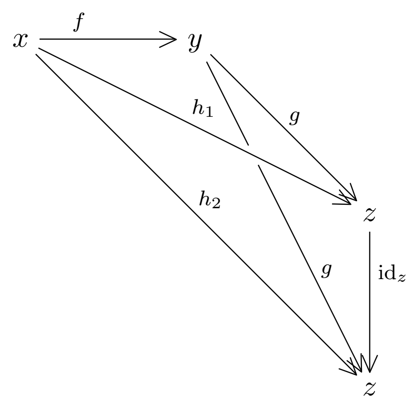
The 0-simplices of an ∞-category $\mathcal{C}$ are its *objects*; the 1-simplices are the *morphisms*.

In an ordinary category, we can compose morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} z$ and obtain a third morphism $x \xrightarrow{g \circ f} z$. This is reflected in the fact that any $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ admits a *unique* filler.

In an ∞-category $\mathcal{D}$, the composite of morphisms $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ is no longer defined uniquely. Instead, there could be many 2-simplices $\Delta^2 \to \mathcal{D}$ of the form



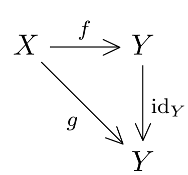
For any two such fillers $\Delta^2 \to \mathcal{D}$ with $\{0,2\}$-edges $h_1, h_2 \in \mathcal{D}_1$, which we think of as two composites of f and g, we obtain a morphism $\Lambda^3_1 \to \mathcal{D}$ depicted below:



By the inner horn filling condition in Definition 2.18, we can again extend this to a map $\Delta^3 \to \mathcal{D}$, which we think of as an identification between h_1 and h_2. There could of course be many such 3-simplices, but any two can be "identified" by a 4-simplex, and so on.

Definition 2.19. Given two parallel morphisms $f, g : X \to Y$ in some ∞-category $\mathcal{C}$, a homotopy from f to g is a 2-simplex $\Delta^2 \to \mathcal{C}$ such that

$$d_0(\sigma) = \mathrm{id}_Y \quad d_1(\sigma) = g \quad d_2(\sigma) = f$$



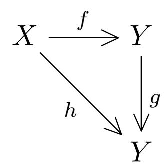
If two morphisms are homotopic, we write $f \simeq g$. Note that this is an equivalence relation, and we write $[f]$ for the set of all morphisms homotopic to f.

Given an ∞-category, we can define a 1-category by identifying homotopic morphisms:

Definition 2.20 (The homotopy category)**.** The *homotopy category* $h\mathcal{C}$ of an ∞-category $\mathcal{C}$ has objects $\mathcal{C}_0$. Given two objects $x, y \in h\mathcal{C}$, the set of morphisms is given by

$$\mathrm{Map}_{h\mathcal{C}}(x,y) = \{f : x \to y\}/\simeq.$$

We define the identity morphism on an object X to be $[id_X]$, and define the composition of $[f] : x \to y$ and $[g] : y \to z$ as $[h] : x \to z$, where $h = d_1(\sigma)$ for any diagram $\sigma : \Delta^2 \to \mathcal{C}$ with $d_2(\sigma) = f$ and $d_0(\sigma) = g$.



Exercise 2.21. Prove that this construction of $h\mathcal{C}$ is well-defined and satisfies all the axioms of a category.

Remark 2.22. We will later characterise the homotopy category $h\mathcal{C}$ by a universal property.

We will now specify four important examples of ∞-categories:

Example 2.23.

a) For any ordinary category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category.
b) For any Given $X \in \mathbf{Top}$, the simplicial set $\mathrm{Sing}(X)$ defines an ∞-category.
c) The ∞-category of (compactly generated Hausdorff) spaces is defined in several steps. Let $\mathbf{Kan} \subset \mathbf{sSet}$ be the full subcategory spanned by all Kan complexes (cf. Definition 2.14). By Definition 2.9, $\mathbf{Kan}$ is in fact a simplicial category (i.e. enriched in simplicial sets). For each n, we define a simplicial category $\mathrm{Path}[n]$ with objects $0, 1, \ldots, n$, and where $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j)$ is given by the nerve of the opposite of the poset

$$\{S \mid \{i,j\} \subset S \subset \{i, i+1, i+2, \ldots, j-1, j\}\}$$

. We then define the ∞-category $\mathcal{S}$ of spaces using Cordier's simplicial nerve, i.e. set

$$\mathcal{S}_n = \mathrm{Map}_{\mathrm{sCat}}(\mathrm{Path}[n], \mathbf{Kan}),$$

where sCat is the category of simplicial categories. We leave it as an exercise to define the simplicial structure maps, and to verify that $\mathcal{S}$ satisfies the inner horn filling axiom.
d) The ∞-category Cat_∞ of ∞-categories is defined by a very similar procedure.
We start with $\mathbf{Cat}_\infty^\Delta$, the simplicial category whose objects are (small) ∞-categories and where $\mathrm{Map}_{\mathrm{Cat}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ is the largest Kan complex contained in $\mathcal{D}^\mathcal{C}$ (cf. Definition 2.9). We then define the ∞-category Cat_∞ of small ∞-categories by

$$(\mathrm{Cat}_\infty)_n = \mathrm{Map}_{\mathrm{sCat}}(\mathrm{Path}[n], \mathbf{Cat}_\infty^\Delta).$$

Again, we leave the definition of the simplicial structure maps as an exercise.
Note that in Cat_∞, we have not captured noninvertible natural transformations.

2.3. **Colimits.** In ordinary category theory, colimits are defined as initial objects in the category of cones over a given diagram. To generalise this definition to ∞-categories, we will first need to discuss the notion of an initial object. For this, we need to consider the space of maps between two objects in an ∞-category:

Definition 2.24 (Mapping space)**.** Given objects x, y in an ∞-category $\mathcal{C}$, we define the *space of right morphisms* $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ as the simplicial set with

$$(\mathrm{Hom}^R_{\mathcal{C}}(x,y))_n = \{z : \Delta^{n+1} \to \mathcal{C} \mid z|_{\Delta^{0,\ldots,n}} = \mathrm{id}_x,\ z(n+1) = y\}$$

Exercise 2.25. Define the simplicial structure maps in Definition 2.24 and prove that $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ is a Kan complex for all x, y.

We can then define:

Definition 2.26 (Initial objects)**.** An object x in an ∞-category $\mathcal{C}$ is said to be *initial* if for all $y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ is weakly contractible.

To define cones in ∞-categories, we will make use of the following notion:

Definition 2.27 (Join)**.** Given $X, Y \in \mathbf{sSet}$, we define a new simplicial set $X \star Y$ with

$$(X \star Y)_n = \coprod_{n=a+b} X_a \times Y_b$$

Exercise 2.28.

a) Complete Definition 2.27 by describing the structure maps of $X \star Y$.
b) Verify that $\Delta^n \star \Delta^m = \Delta^{n+m-1}$.

Definition 2.29 (∞-category of cones)**.** Let now $F : I \to \mathcal{C}$ be a functor of ∞-categories. The ∞*-category of cones* $\mathcal{C}_{F/}$ is given by

$$(\mathcal{C}_{F/})_n = \{\overline{F} : I \star \Delta^n \to \mathcal{C} \mid \overline{F}|_I = F\},$$

where we again leave the definition of the structure maps as an exercise.

Example 2.30. If $I = (\bullet \leftarrow \bullet \rightarrow \bullet)$ and $F : I \to \mathcal{C}$ picks out a diagram $(b \leftarrow a \rightarrow c)$, then $\mathcal{C}_{F/}$ is the ∞-category of all diagrams

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \dashrightarrow & d \end{array}$$

Definition 2.31 (Colimits)**.** Given a diagram $F : I \to \mathcal{C}$, a colimit of F is an initial object (cf. Definition 2.26) in the ∞-category $\mathcal{C}_{F/}$.

A result of Joyal shows that if a colimit exists, then it is unique up to a contractible space of choices (cf. [Lur09, Proposition 1.2.12.9]). Limits are defined in a dual fashion.

We will often want to talk about filtered colimits in a higher categorical setting. To this end, we generalise the notion of a filtered category from ordinary to higher categories.

Given $n \geq 0$, write $\partial\Delta^n$ for the simplicial subset of Λ^{n+1}_{n+1} spanned by all simplices not containing the vertex $n+1$.

Definition 2.32. An ∞-category I is said to be *filtered* if for all integers $n \geq 0$, any map $f : \partial\Delta^n \to I$, extends to Λ^{n+1}_{n+1}:

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & \mathcal{D} \\ \downarrow & \nearrow_{\bar{f}} & \\ \Lambda^{n+1}_{n+1} & & . \end{array}$$

The case $n = 0$ shoes that I is nonempty. For $n = 1$, we conclude that for any diagram $\partial\Delta^1 \to I$ picking out two objects x, y, we can find an object z and morphisms $x \to z$, $y \to z$.

Exercise 2.33. Show that if I is the nerve of an ordinary category, then Definition 2.32 recovers Definition 2.2 from Lecture 2.

In the ordinary category Set of sets, filtered colimits commuted with finite limits. The higher categorical analogue of this fact is given by the following result:

Proposition 2.34. Filtered colimits and finite limits commute in the ∞-category $\mathcal{S}$ of spaces.

We refer to [Lur09, Proposition 5.3.3.3] for a proof of Proposition 2.34. This result illustrates the general paradigm that $\mathcal{S}$ plays the same role for ∞-categories as Set plays for ordinary categories. We generalise Definition 2.13 from Lecture 2 to this setting:

Definition 2.35 (Compact objects)**.** An object X in an ∞-category $\mathcal{C}$ is called *compact* if the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ preserves filtered colimits.

We then have the following generalisation of Corollary 2.14 from Lecture 2, which follows from Proposition 2.34:

Corollary 2.36. Finite colimits of compact objects in an ∞-category $\mathcal{C}$ are compact.

References

[Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).

[Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659