∞-Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 3. Monoidal ∞-categories

Last week, we introduced ∞-categories and defined colimits (and limits) in this context. To state Lurie's higher categorical Barr-Beck theorem, we will also need the theory of monads (and their algebras) in this setting, which in turn relies on the theory of monoidal (and tensored) ∞-categories – these will be the topic of today.

3.1. **CoCartesian fibrations.** To examine ∞-categories in families, we will need:

Definition 3.1 (coCartesian lifts)**.** Given a map of simplicial sets $p : \mathcal{C} \to S$ and an edge $f : x \to y$ in S, an edge

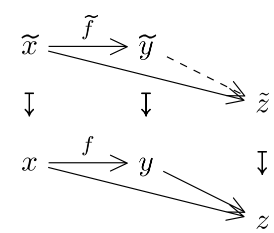
$$\widetilde{f} : \widetilde{x} \to \widetilde{y}$$

in $\mathcal{C}$ is said to be a *p-coCartesian lift* of f if

a) The edge $\widetilde{f}$ lifts f, which means that $p(\widetilde{f}) = f$.
b) The map $\mathcal{C}_{\widetilde{f}} \to \mathcal{C}_{\widetilde{x}/} \times_{S_{x/}} S_{f/}$ is a trivial Kan fibration of simplicial sets.

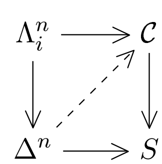
Condition b) says that in the diagram below, specifying the upper triangle, an element of $\mathcal{C}_{\widetilde{f}}$, is equivalent to compatibly specifying $(\widetilde{x} \to \widetilde{z}) \in \mathcal{C}_{\widetilde{x}/}$ and the lower triangle, an element of $S_{f/}$.

(1)



Definition 3.2 (CoCartesian fibration)**.** A map $\mathcal{C} \xrightarrow{p} S$ in **sSet** is a *coCartesian fibration* if

(1) p is an *inner fibration*, i.e. it satisfies the right lifting property for all inner horns:



(2) Given $x \xrightarrow{f} y$ in S and $\widetilde{x} \in \mathcal{C}$ with $p(\widetilde{x}) = x$, there is a p-coCartesian lift $\widetilde{x} \xrightarrow{\widetilde{f}} \widetilde{y}$ of f.

As a heuristic, it might be helpful to think of coCartesian fibrations as bundles with flat connection; in this picture, coCartesian lifts correspond to paths along the connection.

3.2. **Unstraightening.** One can show that coCartesian fibrations over S are equivalent to functors from S into the ∞-category Cat_∞ introduced last lecture. The proof of this result is challenging, and we refer to [Lur09, Section 3.2] for a more comprehensive treatment.

We will content ourselves with constructing coCartesian fibrations for *certain* functors to Cat_∞. More precisely, let J be an ordinary category and fix a functor

$$F : J \to \mathbf{sSet}\,.$$

Definition 3.3 (Relative nerve)**.** The *relative nerve* $\mathrm{N}_F(J)$ is the simplicial set over $\mathrm{N}(J)$ with

$$\mathrm{N}_F(J)_0 = \{(j_0 \in \mathrm{N}(J)_0\ ,\ x_0 \in F(j_0)\}$$

$$\mathrm{N}_F(J)_1 = \{(j_0 \to j_1) \in \mathrm{N}(J)_1\ ,\ {\scriptstyle x_0 \in F(j_0) \atop x_1 \in F(j_1)}\ ,\ F(j_0 \to j_1)(x_0) \to x_1\}$$

$$\mathrm{N}_F(J)_2 = \left\{ \begin{array}{ccc} j_0 & \longrightarrow & j_1 \\ & \searrow & \downarrow \\ & & j_2 \end{array}\ ,\ \begin{array}{c} x_0 \in F(j_0) \\ x_1 \in F(j_1) \\ x_2 \in F(j_2) \end{array}\ ,\ \begin{array}{c} F(j_0 \to j_1)(x_0) \to x_1 \\ F(j_0 \to j_2)(x_0) \to x_2 \\ F(j_1 \to j_2)(x_1) \to x_2 \end{array}\ ,\ \begin{array}{ccc} F(j_0 \to j_2)(x_0) & \longrightarrow & F(j_1 \to j_2)(x_1) \\ & \searrow & \downarrow \\ & & x_2 \end{array} \right\}$$

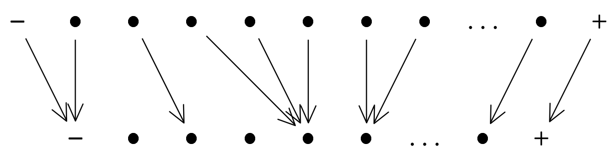
Exercise 3.4. a) Write down $N_F(J)_n$ for all n and check that it is a simplicial set.
b) Show that if $F(j)$ is an ∞-category for all j, then $\mathrm{N}_F(J) \xrightarrow{p} \mathrm{N}(J)$ is a coCartesian fibration.

3.3. **Monoidal ∞-categories.** We are finally in a position to define monoidal ∞-categories.

But first, we observe that the category Δ^{op} from Lecture 2 admits an alternative description. Indeed, the objects of Δ^{op} can be written as

$$[0] = (-\ \ +),\quad [1] = (-\ \bullet\ +),\quad [2] = (-\ \bullet\ \bullet\ +),\quad [3] = (-\ \bullet\ \bullet\ \bullet\ +),\quad \ldots$$

Morphisms from $[n]$ to $[m]$ are maps which preserve the order and send $-$ to $-$ and $+$ to $+$:

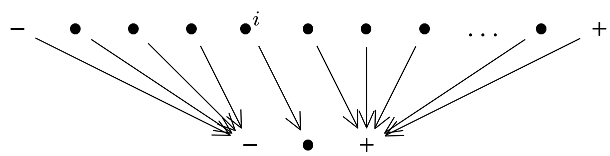


Exercise 3.5. Show that the category defined in this way indeed agrees with the opposite of the usual simplex category Δ.

Informally, we think of the bullets as placeholders of potential elements in a monoidal category. The symbols $+$ and $-$ will act as “trashcans”; arrows will parametrise multiplications.

We give a name to the morphisms which “throw away” all but one element:

Definition 3.6. Given $n \geq 0$ and $1 \leq i \leq n$, we write $\rho_i^n : [n] \to [1]$ for the morphism



This motivates the following definition:

Definition 3.7 (Monoidal ∞-categories)**.** A monoidal ∞-category is a coCartesian fibration $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ such that for all n, the following morphism is an equivalence:

$$\mathcal{C}^{\circledast}_{[n]} \xrightarrow{\prod_{i=1}^{n} (\rho_i^n)_!} \prod_{i=1}^{n} \mathcal{C}^{\circledast}_{[1]} \qquad \text{(Segal condition)}$$

Here $\mathcal{C}^{\circledast}_{[n]}$ denotes the fibre of p over $[n]$, and $\mathcal{C}^{\circledast}_{[n]} \xrightarrow{(\rho_i^n)_!} \mathcal{C}^{\circledast}_{[1]}$ is the functor associated with ρ_i^n.

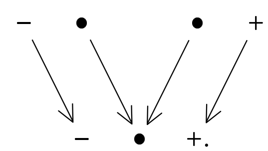
Informally, we simply say that $\mathcal{C} \simeq \mathcal{C}^{\circledast}_{[1]}$ is equipped with a monoidal structure.

Remark 3.8. The functor $(\rho_i^n)_!$ sends $x \in \mathcal{C}^{\circledast}_{[n]}$ to the endpoint of a coCartesian lift of ρ_i^n starting at $\tilde{x}$. For a complete definition, we refer to [Lur09, Section 2.2.1].

The monoidal product $\circ$ is determined, up to equivalence, by the following composite:

$$\mathcal{C}^{\circledast}_{[1]} \times \mathcal{C}^{\circledast}_{[1]} \xleftarrow{\simeq} \mathcal{C}^{\circledast}_{[2]} \xrightarrow{m_!} \mathcal{C}^{\circledast}_{[1]},$$

where $m : [2] \to [1]$ is the morphism in Δ^{op} represented by the diagram



Exercise 3.9. Define the monoidal unit $\mathbf{1}$ of a monoidal ∞-category $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.

Notation 3.10. We will often say "let $(\mathcal{C}, \circ, \mathbf{1})$ be a monoidal ∞-category" instead of "let $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category with $\mathcal{C}^{\circledast}_{[1]} \simeq \mathcal{C}$, multiplication $\circ$, and unit $\mathbf{1}$".

Using the relative nerve from Definition 3.3, we can now equip ∞-categories of endofunctors $\mathcal{C} = \mathrm{End}(\mathcal{D})$ with monoidal structures:

Definition 3.11 (Endomorphism ∞-categories)**.** Given an ∞-category $\mathcal{D}$, we equip

$$\mathcal{C} = \mathrm{End}(\mathcal{D}) \coloneqq \mathcal{D}^{\mathcal{D}}$$

(cf. Definition 2.7 in Lecture 2) with the structure of a monoidal ∞-category as follows.

First, use that $\mathcal{C}$ is a simplicial monoid (under composition) to construct a diagram

$$\cdots \qquad \mathcal{C} \times \mathcal{C} \quad \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \quad \mathcal{C} \quad \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} [0]$$

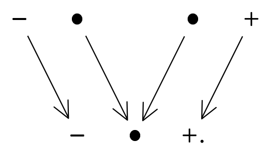
Second, apply the relative nerve (cf. Definition 3.3) to obtain a coCartesian fibration

$$\mathrm{End}(\mathcal{D})^{\circledast} \to \mathrm{N}(\Delta^{op}).$$

Exercise 3.12. Check that $\mathrm{End}(\mathcal{D})^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a monoidal ∞-category.

3.4. **Algebra objects.** To generalise the notion of a monad to a higher categorical context, we first need to define what we mean by an algebra A in a monoidal ∞-category $(\mathcal{C}, \circ, \mathbf{1})$.

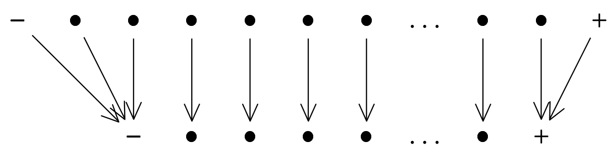
We certainly want to specify a multiplication map $A \circ A \to A$, which, by diagram (1), is equivalent to lifting the morphism $m : [2] \to [1]$ in Δ^{op} drawn below along $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.



We can also specify higher compositions (e.g. $A \circ A \circ A \xrightarrow{m \circ \mathrm{id}} A \circ A$) as lifts of corresponding maps in Δ^{op}. One might hope that algebra objects are simply sections of $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$.

This is almost true, but we need to make sure that certain dull morphisms have dull lifts:

Definition 3.13 (Inert morphism)**.** A morphism $f : [n] \to [m]$ in Δ^{op} is *inert* if every bullet $\bullet$ in $[m]$ has a unique preimage in $[n]$:



Definition 3.14 (Algebras)**.** An *algebra* in a monoidal ∞-category $p : \mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p sending inert morphisms to p-coCartesian morphisms.

Exercise 3.15. Show that if $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ specifies an algebra, then $s([2])$ corresponds to the pair $(s([1]), s([1]))$ under the equivalence $\mathcal{C}^{\circledast}_{[2]} \simeq \mathcal{C} \times \mathcal{C}$.

Finally, we can generalise Definition 1.30 from Lecture 1 to the setting of ∞-categories:

Definition 3.16 (Monads)**.** A monad on an ∞-category $\mathcal{C}$ is an algebra object in $\mathrm{End}(\mathcal{C})$.

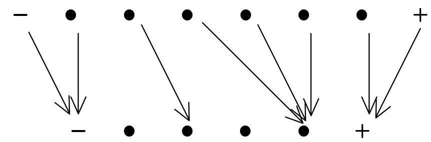
3.5. **Algebras over monads.** To state the monadicity theorem, we will need to define what we mean by algebras over a monad. We will use the setup of tensored ∞-categories. Let $\mathcal{C}^{\circledast} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category, written informally as $(\mathcal{C}, \otimes, 1)$.

Definition 3.17 (Tensored ∞-categories)**.** A $\mathcal{C}$*-tensored* ∞*-category* is given by a diagram of ∞-categories $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ satisfying the following conditions:

a) $p \circ q : \mathcal{M}^{\circledast} \to \mathrm{N}(\Delta^{op})$ is a coCartesian fibration;
b) $q : \mathcal{M}^{\circledast} \to \mathcal{C}^{\circledast}$ is a categorical fibration sending $(p \circ q)$-coCartesian to p-coCartesian edges;
c) For all n, the inclusion $\{n\} \subset [n]$ induces an equivalence $\mathcal{M}^{\circledast}_{[n]} \xrightarrow{\simeq} \mathcal{C}^{\circledast}_{[n]} \times \mathcal{M}^{\circledast}_{\{n\}}$.

We say that the ∞-category $\mathcal{M} := \mathcal{M}^{\circledast}_{[0]}$ is equipped with a $\mathcal{C}$-tensored structure, written $\otimes$.

Informally, elements of $\mathcal{M}^{\circledast}_{[n]}$ correspond to tuples $(c_1, c_2, \ldots, c_n, m)$ with $c_i \in \mathcal{C}$, $m \in \mathcal{M}$; we think of the c_i's as labels of the bullets and m as a label of the $+$. The $(p \circ q)$-coCartesian lifts tensor according to the arrows; for example, the coCartesian lift of the morphism

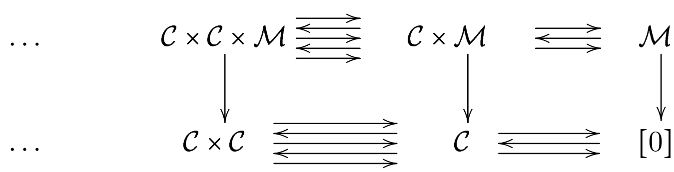


starting at a tuple $(c_1, c_2, c_3, c_4, c_5, c_6, m)$ ends at the tuple $(1, c_2, 1, c_3 \otimes c_4 \otimes c_5, c_6 \otimes m)$.

Example 3.18. Any ∞-category $\mathcal{M} = \mathcal{D}$ is naturally tensored over the monoidal ∞-category $\mathcal{C} = \mathrm{End}(\mathcal{D})$, where the tensoring evaluates functors on objects.

To formally construct this tensored structure, observe that the simplicial set $\mathcal{M} = \mathcal{D}$ is equipped with an action by the simplicial monoid $\mathcal{C} = \mathrm{End}(\mathcal{D})$.

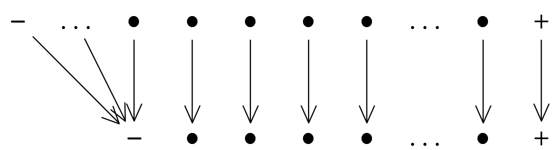
We obtain the diagram $\mathrm{N}(\Delta^{op}) \times \Delta^1 \to \mathbf{sSet}$ drawn below.



Exercise. Applying the relative nerve construction to this diagram gives rise to an $\mathcal{C} = \mathrm{End}(\mathcal{D})$-tensored structure on $\mathcal{M} = \mathcal{C}$.

Let $\mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category and $\mathcal{M}^{\circledast} \xrightarrow{q} \mathcal{C}^{\circledast} \xrightarrow{p} \mathrm{N}(\Delta^{op})$ be a $\mathcal{C}$-tensored ∞-category. Fix an algebra object A in $\mathcal{C}$, parametrised by a section $s : \mathrm{N}(\Delta^{op}) \to \mathcal{C}^{\circledast}$ of p.

Definition 3.19 (Modules)**.** An A-module M in $\mathcal{M}$ consists of a section $s' : \mathrm{N}(\Delta^{op}) \to \mathcal{M}^{\circledast}$ with $q \circ s' = s$ and such that all morphisms drawn below are sent to $(p \circ q)$-coCartesian edges:



Informally, an A-module is an element $M \in \mathcal{M}$ with a multiplication map $A \otimes M \to M$ which is unital and associative up to coherent homotopy.

Definition 3.20 (Algebras over monads)**.** Given a monad T on an ∞-category $\mathcal{D}$, i.e. an algebra object in the monoidal ∞-category $\mathrm{End}(\mathcal{D})$, a T-algebra is simply a T-module object in the $\mathrm{End}(\mathcal{D})$-tensored ∞-category $\mathcal{D}$.

Remark 3.21. One could argue that T-algebras should be called T-modules instead, and this notational convention is indeed implemented in [Lur07]. However, we decided against this for higher consistency with the 1-categorical literature on monads.

References

[Lur07] Jacob Lurie, *Derived algebraic geometry II: Noncommutative algebra*, Preprint from the author's web page (2007).

[Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659