AN EXPOSITION OF JORDAN'S ORIGINAL PROOF OF HIS THEOREM ON FINITE SUBGROUPS OF $\mathrm{GL}_n(\mathbb{C})$.

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Abstract. We discuss Jordan's theorem on finite subgroups of invertible matrices and give an account of his original proof.

1. introduction

In 1878 Camille Jordan [12] proved the following theorem:

Theorem 1.1 (Jordan's theorem)**.** *Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$, then there is a normal abelian subgroup A in G of index bounded by a constant $f(n)$ depending on n only.*

Although several different proofs of Jordan's theorem have appeared since 1878, no other exposition of Jordan's original proof seems to be available in the literature apart from Jordan's original memoir. It is the purpose of these notes to provide such an exposition.

Following prior work of Fuchs and Klein, Jordan's original motivation came from his study of linear differential equations of order n with rational functions as coefficients and with algebraic solutions: then finite subgroups of GL_n arise naturally as monodromy groups and information such as Theorem 1.1 on the monodromy group translates immediately into structural properties for the solutions of the equation.

After Jordan memoir, several authors gave new proofs of his theorem. The first of these appears to be Blichfeldt, who studied the p-Sylow subgroups giving explicit bounds on their size in terms of p and n (see [3, 14]), and subsequently Bieberbach and Frobenius [2, 9] came up with a very different geometric argument which is much slicker, and is in fact the proof of Jordan's theorem that can be found in most textbooks that treat the question (such as [8, chap. V], [15, chap. 8]). Basically, their proof starts with what people refer nowadays as Weyl's unitary trick, or rather the easy part of Weyl's unitary trick (i.e. the observation that a finite (or compact) subgroup G of $\mathrm{GL}_n(\mathbb{C})$ can be conjugated inside the compact subgroup $\mathbb{U}_n(\mathbb{C})$ by averaging a hermitian product over G). Then one makes use of a volume packing argument in combination with the *commutator shrinking property* of Lie groups, i.e., the fact that commutators of elements close to the identity in $\mathbb{U}_n(\mathbb{C})$ are themselves close, and in fact much closer, to the identity. This commutator shrinking property has inspired several other authors ([17], [4] ,[10], [1]) and is nowadays a crucial tool in the study of discrete subgroups of Lie groups and in modern Riemannian geometry. We refer the reader to [6, §2] for the proof of Jordan's theorem via this argument.

Jordan's original proof on the other hand was based on a completely different idea, which should be traced back to the classification of finite subgroups of rotations of the 2-sphere and of Plato's solids as in Klein's famous book on the icosahedron [11]. Basically, one enumerates the elements of G according to the shape and size of their centralizers and one can thus write a class equation involving the order of G and of the centralizers of its elements. Using induction on the dimension one can then analyse this equation and discuss the various cases that may arise. Although more cumbersome, this method gives potentially much more information on the finite subgroup G. For example, Jordan used it to list all finite subgroups of $\mathrm{GL}_3(\mathbb{C})$, giving an explicit set of generators for each group, after examining some 47 different cases[1].

In the last twenty or so years, a lot of attention has been devoted to Jordan's theorem by various authors, in particular to the question of its extension to characteristic p ([5], [16]). Using the classification of finite simple groups, B. Weisfeiler [16] and more recently M. Collins [7] have found very tight bounds in this direction. For example, Collins proved that if $n \geqslant 71$, then $f(n)$ can be taken to be $(n+1)!$. This is tight, i.e., $(n+1)!$ is always a lower bound for $f(n)$, because the symmetric group on $n+1$ letters acts irreducibly on the hyperplane $\sum_{i=1}^{n+1} x_i = 0$ by permuting the $n+1$ coordinates.

In another direction, albeit with weaker bounds, Larsen and Pink [13] gave a softer classification-free proof of Weisfeiler's results. Interestingly enough, their proof is very much akin to Jordan's original argument.

2. A reformulation of Theorem 1.1

As we will see below, Jordan's argument uses nothing about the field $\mathbb{C}$ and in fact his proof carries over to an arbitrary field provided we assume that every elment of G is semisimple, i.e., diagonalizable in some field extension. So we let K be an arbitrary field, which we assume algebraically closed without loss of generality.

Let us first reformulate Theorem 1.1 in the form orginally proved by Jordan. For this we need to introduce a couple of definitions.

Definition 2.1. *By a* root torus*, we mean subgroup of* $\mathrm{GL}_n(K)$ *which is conjugate to a subgroup of the diagonal matrices defined by a set of equalities between the diagonal entries.*

For example, the subgroup of diagonal matrices $\{g = diag(a_1, ..., a_6) | a_i \in K^*, a_1 = a_2, a_5 = a_6\}$ is a root torus of $\mathrm{GL}_6(K)$.

Definition 2.2. *Let G be a finite subgroup of* $\mathrm{GL}_n(K)$*. Given $M \geqslant 2$, we say that a subgroup F of G is an* M-beam *if it is conjugate to a subgroup of the diagonal matrices $\{g = diag(a_1, ..., a_n), a_i \in K^*\}$ such that for every pair of indices i, j the set of ratios $a_i(g)/a_j(g)$ is either reduced to $\{1\}$ or achieves at least M distinct values as g varies in F.*

The terminology *beam* is a liberal translation of Jordan's *faisceau*[2]. Note that the subgroup Φ of all scalar matrices in G is clearly an M-beam.

[1]In fact Jordan missed some groups, see [3]

[2]This way we avoid an obvious conflict with the topological notion of *sheaf*, another translation of the French *faisceau*.

Note that every M-beam is contained in a unique minimal root torus $S_{\mathcal{F}}$ defined by the same equalities between diagonal elements $a_i = a_j$ that hold in $\mathcal{F}$.

We can now formulate an alternative, slightly more precise, version of Theorem 1.1 above:

Theorem 2.3 (Jordan's theorem, second form)**.** *Given $n \in \mathbb{N}$, there are constants $M = M(n), N = N(n) \geqslant 1$ such that the following holds. Let K be an algebraically closed field, and let G be a finite subgroup of $\mathrm{GL}_n(K)$, such that every element of G is diagonalizable. Then G contains a unique maximal M-beam. Call it $\mathcal{F}$. We have $[G : \mathcal{F}] \leqslant N$.*

The proof of Theorem 2.3 will span over the next two sections. Before we start, a number of simple observations are in order:

1) Since $\mathcal{F}$ is unique, it must be normal in G, so $\mathcal{F}$ is a normal abelian subgroup of G.

2) To see that Theorem 2.3 implies Theorem 1.1, one only has to check that if $K = \mathbb{C}$, then every element of G be diagonalizable: this is indeed the case, because G being finite, every element of G has finite order and is thus diagonalisable $\mathbb{C}$.

3) If $g \in GL_n(K)$ normalizes $\mathcal{F}$, then it must normalize the root torus $S_{\mathcal{F}}$ too. This implies that G is contained in the normalizer of a root torus $S_{\mathcal{F}}$ such that $[G : G \cap S_{\mathcal{F}}] \leqslant N$.

4) Since $S_{\mathcal{F}}$ is normalized by G, G must permute the eigenspaces of $S_{\mathcal{F}}$. So if G acts primitively on K^n (i.e. does not permute the components of any non-trivial direct sum decomposition of K^n), then S must be reduced to scalar matrices and those have bounded index in G.

5) The proof of Theorem 2.3 will proceed by induction on the dimension.

As seen in item 2) above, Theorem 2.3 implies Theorem 1.1. It turns out that one can also derive Theorem 2.3 from Theorem 1.1 directly and we sketch this in the paragraph below. However Jordan's proof goes through proving Theorem 2.3 first, because its formulation is more adequate for the induction scheme.

Proof of the equivalence of Theorems 1.1 and 2.3. Theorem 1.1 is clearly a consequence of Theorem 2.3 as is explained in item 2) above. To see the opposite implication, note first that since every element of G is diagonalizable and A is abelian, A is simultaneously diagonalizable and K^n decomposes as a direct sum of eigenspaces for A; since A is normal in G, these eigenspaces are permuted by G and thus G lies in the normalizer $N(S)$ of the root torus S that acts on K^n by a scalar multiple on each one of the weight spaces of A. Moreover, if F is an M-beam with $M \geqslant 2N$ and $m := [F : F \cap A] \leqslant [G : A] \leqslant N$, $f^m \in A$ for all $f \in F$, and thus $F \cap A \subset S$ is an M/m-beam lying in S; since $M/m \geqslant 2$ this implies that F itself lies in S, thus completing the claims of Theorem 2.3.

3. Jordan's fundamental equation

In this section we begin the proof of Theorem 2.3 and obtain Jordan's fundamental equation (3.2) below, which expresses an enumeration of the elements of G into various classes which we are about to describe. The proof of Theorem 2.3 will be completed in the next section after a discussion of the fundamental equation.

The proof of Theorem 2.3 proceeds by induction on the dimension n.

If $n = 1$, then $\mathrm{GL}_1(K) = K^*$ is abelian and there is nothing to prove. We assume the theorem proven for all dimensions $< n$.

If G perserves a direct sum decomposition $K^n = K^r \oplus K^{n-r}$, with $1 < r < n$, then we may use the induction hypothesis in the obvious way applying it to the projections $\pi_r(G)$ and $\pi_{n-r}(G)$ to $\mathrm{GL}_r(K)$ and $\mathrm{GL}_{n-r}(K)$ respectively. The result follows as soon as $N \geqslant \max_{0<r<n} N_r N_{n-r}$.

We will use repeatedly the last observation for subgroups of G that perserve such a decomposition. If $g \in G$ is not a scalar matrix, then the centralizer $C_G(g)$ preserves the eigenspace decomposition of g on K^n. We can therefore apply this observation to $C_G(g)$ and conclude from the induction hypothesis that $C_G(g)$ contains a unique maximal M-beam (for all M larger than a number depending on n only). That is

Lemma 3.1. *If $g \in G$ is not a scalar matrix, then the centralizer $C_G(g)$ contains a unique maximal M-beam.*

We can thus set the following definition:

Definition 3.2. *An element g is said to be* associated with *an M-beam F if F lies in the centralizer $C_G(g)$ and is the unique maximal M-beam of $C_G(g)$.*

We denote by F_g the M-beam associated with g. This definition makes sense (so far thanks to the induction hypothesis) as soon as g is not a scalar matrix in $\mathrm{GL}_n(K)$ by the remarks above the definition. Note that, by maximality, F must contain the subgroup Φ of G of all scalar matrices in G. Moreover, it follows from the induction hypothesis and from Observation (2) following Theorem 2.3 that

$$[C_G(g) : F_g] \leqslant NM^{n-1}. \tag{3.1}$$

These remarks also have the following three consequences:

Lemma 3.3. *If F is an M-beam of G not entirely made of scalar matrices, then F is contained in a unique maximal M-beam $\mathcal{F}$ of G.*

Proof. Indeed, an M-beam F_1 containing F must commute with all elements of F and thus lie in the centralizer $C_G(f)$ of some non-scalar element $f \in F$. Therefore F_1 lies in the unique maximal M-beam of $C_G(f)$. □

Let F be an M-beam of G not contained in the scalar matrices Φ and let $\mathcal{F}$ be the maximal M-beam of G containing F. Since $\mathcal{F}$ is contained in the centralizer $C_G(F)$, it must be the maximal M-beam there too and, by the induction hypothesis and observation (2) after Theorem 2.3, we must have $[C_G(F) : \mathcal{F}] \leqslant NM^{n-1}$.

Lemma 3.4. *Suppose $\Phi \subsetneqq F \subsetneqq \mathcal{F}$. Then the number n_F of elements of G associated with F is divisible by $|\mathcal{F}|$ and $\frac{n_F}{|\mathcal{F}|} \leqslant NM^{n-1}$.*

Proof. If $n_F = 0$ there is nothing to prove, so we assume $n_F \geqslant 1$. Every element associated with F lies in the centralizer $C_G(F)$. Moreover, if $g \in C_G(F)$ is associated with F and $f \in \mathcal{F}$, then gf is also associated with F, i.e. $F_{gf} = F_g = F$. Indeed, since $\Phi \subsetneqq F \subsetneqq \mathcal{F}$ we must have $gf \notin \Phi$ and by Lemma 3.1 there is a unique maximal M-beam F_1 in $C_G(gf)$. Since $F \subset C_G(gf)$ we have $F \subset F_1 \subset C_G(gf)$.

Moreover F_1 is contained in $\mathcal{F}$ and must therefore commute with f, and hence also with g. It follows that $F \subset F_1 \subset C_G(g)$ and $F = F_1$ by maximality of F.

Consequently, the set of elements of G associated with F is a union of cosets of $\mathcal{F}$ all lying in $C_G(F)$. Since $C_G(F)$ contains $\mathcal{F}$ as a subgroup of index at most NM^{n-1} the result follows. □

For maximal beams the corresponding result is as follows:

Lemma 3.5. *Let $\mathcal{F}$ be a maximal M-beam in G. Then the number $n_{\mathcal{F}}$ of non-scalar elements g in G which are associated with $\mathcal{F}$ is $n_{\mathcal{F}} = \rho|\mathcal{F}| - |\Phi|$, where ρ is a positive integer of size at most NM^{n-1}.*

Proof. Indeed the union of Φ and the subset of non-scalar elements of G that are associated with $\mathcal{F}$ is a union of cosets of $\mathcal{F}$ inside $C_G(\mathcal{F})$ for the same reason as in the proof of Lemma 3.4. Since $\frac{n_{\mathcal{F}}}{|\mathcal{F}|} \leqslant NM^{n-1}$, the result follows. □

The strategy of Jordan's proof consists in enumerating the elements of G according to their associated M-beam. Let Φ be the scalar matrices in G. We may decompose G as the *disjoint* union

$$G = \Phi \cup_F \{g,\, g \text{ associated with } F\}$$

where the union is taken over beams arising as maximal M-beams of centralizers of non scalar elements of G. We will split this union into four disjoint parts,

$$G = \Phi \cup G_1 \cup G_2 \cup G_3,$$

where G_1 is the subset of g's not in Φ such that $F_g = \Phi$, G_2 is the subset of g's not in Φ such that F_g contains Φ strictly but is not the maximal M-beam $\mathcal{F}_g$ which contains it by Lemma 3.3, and finally G_3 is the remaining subset of those g's not in Φ for which F_g is not Φ and is maximal in G. We now consider each subset G_i one after the other.

1) We first enumerate the elements of G_1, that is the g's outside Φ which are associated with Φ. This subset is invariant under conjugation by G. Also it is clearly a union of cosets of Φ, for if $\phi \in \Phi$, then $C_G(g\phi) = C_G(g)$ and thus $g\phi$ is also associated with Φ. It follows that conjugation by G permutes those Φ-cosets.

The stabiliser $N_G(g\Phi)$ of a Φ-coset $g\Phi$ under the G-action by conjugation must have index at most n in the $C_G(g)$. Indeed, if $h \in G$ has $hg\Phi h^{-1} = g\Phi$, then $hgh^{-1} = g\phi$ for some $\phi \in \Phi$. It follows that $\det(\phi) = 1$ and thus ϕ is an n-th root of unity.

Therefore the number of elements in the G-conjugacy class of the coset $g\Phi$ equals

$$\frac{|G|}{N_G(g\Phi)}|\Phi| = |G|\frac{1}{[N_G(g\Phi) : C_G(g)]}\frac{1}{[C_G(g) : \Phi]} = |G|\frac{1}{\lambda}.$$

Enumerating all such conjugacy classes, we find:

$$|G_1| = |G|\left(\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_{k_1}}\right).$$

where each λ_i is a positive integer of size at most nNM^{n-1} by (3.1) and the remark above.

2) We now pass to the subset G_2. Clearly G_2 is stable under conjugation by G. Let F be an M-beam of G with maximal M-beam $\mathcal{F}$ such that $\Phi \subsetneq F \subsetneq \mathcal{F}$. Let n_F be the number of g's which are associated with F. By Lemma 3.4 $n_F/|\mathcal{F}|$ is an integer of size at most NM^{n-1}.

Grouping together the beams that are conjugate to F, we obtain $\frac{|G|}{|N_G(F)|}$ different beams, where $N_G(F)$ is the normalizer of F in G. Note that $[N_G(F) : C_G(F)] \leqslant n!$ since $N_G(F)$ permutes the weight spaces of F and hence a subgroup of index at most $n!$ will preserve them and thus commute with F.

It follows that the number of elements that are associated with a beam lying in the G-conjugacy class of F equals

$$n_F \frac{|G|}{|N_G(F)|} = |G| \frac{1}{[N_G(F) : C_G(F)]} \frac{n_F/|\mathcal{F}|}{[C_G(F) : \mathcal{F}]} = |G| \frac{\nu}{\mu},$$

and thus enumerating the different conjugacy classes

$$|G_2| = |G| \left(\frac{\nu_1}{\mu_1} + ... + \frac{\nu_{k_2}}{\mu_{k_2}} \right),$$

where the ν_i, μ_i 's are positive integers of size at most $n!NM^{n-1}$.

3) Finally we consider the subset G_3 of those non-scalar g's such that F_g is maximal in $C_G(g)$ and different from Φ. Clearly this set is invariant under conjugation by G. Given a maximal M-beam $\mathcal{F}$, the number $n_{\mathcal{F}}$ of elements of G which are associated with $\mathcal{F}$ equals $\rho|\mathcal{F}| - |\Phi|$ according to Lemma 3.5.

Setting $q = \frac{|\mathcal{F}|}{|\Phi|}$, we get $n_{\mathcal{F}} = |\mathcal{F}|(\rho - \frac{1}{q})$. Therefore the number of elements that are associated with a maximal beam conjugate to $\mathcal{F}$ is

$$n_{\mathcal{F}} \frac{|G|}{N_G(\mathcal{F})} = |G| \frac{1}{[N_G(\mathcal{F}) : \mathcal{F}]} \left(\rho - \frac{1}{q} \right) = |G| \frac{1}{\eta} \left(\rho - \frac{1}{q} \right),$$

where η is a positive integer of size at most $n!NM^{n-1}$, ρ is a positive integer of size at most NM^{n-1}.

Summing over the conjugacy classes, we get:

$$|G_3| = |G| \left(\frac{1}{\eta_1} \left(\rho_1 - \frac{1}{q_1} \right) + ... + \frac{1}{\eta_{k_3}} \left(\rho_{k_3} - \frac{1}{q_{k_3}} \right) \right).$$

Combining all three cases we have thus completed our enumeration of G and we have obtained:

Proposition 3.6 (Jordan's fundamental equation)**.** *Let G be a finite subgroup of $\mathrm{GL}_n(K)$. Then there are positive integers q_i such that*

$$|G| = |\Phi| + |G| \sum_{i=1}^{k_1} \frac{1}{\lambda_i} + |G| \sum_{i=1}^{k_2} \frac{\nu_i}{\mu_i} + |G| \sum_{i=1}^{k_3} \frac{1}{\eta_i} \left(\rho_i - \frac{1}{q_i} \right), \tag{3.2}$$

where k_i, λ_i, ν_i, μ_i, ρ_i and η_i are positive integers of size at most $2n!NM^{n-1}$. In particular, setting $n_i := \eta_i q_i = [N_G(\mathcal{F}_i) : \Phi]$ and $g := \frac{|G|}{|\Phi|}$, we obtain:

$$\frac{1}{g} = \frac{1}{n_1} + ... + \frac{1}{n_{k_3}} - \frac{b}{a}, \tag{3.3}$$

where $\frac{b}{a}$ is an irreducible fraction whose numerator and denominator are bounded in terms of n only.

To prove Proposition 3.6 it remains to show the bound on the number k_i of elements in each sum and then derive (3.3). But this follows from the equation (3.2) and from the bounds previously obtained, because $\rho_i - \frac{1}{q_i} \geqslant \frac{1}{2}$ for each $i = 1, ..., k_3$ and thus each term in the above sums contributes at least $|G|/2n!NM^{n-1}$ forcing $k_1 + k_2 + k_3 \leqslant 2n!NM^{n-1}$.

Showing (3.3) is a simple matter of rearranging (3.2):

$$\frac{|G|}{|\Phi|}\left(\sum_{i=1}^{k_1}\frac{1}{\lambda_i}+\sum_{i=1}^{k_2}\frac{\nu_i}{\mu_i}+\sum_{i=1}^{k_3}\frac{\rho_i}{\eta_i}-1\right)=\sum_{i=1}^{k_3}\frac{|G|}{|\Phi|}\frac{1}{\eta_i q_i}-1. \tag{3.4}$$

Then letting $g := \frac{|G|}{|\Phi|}$, $n_i := \eta_i q_i$, and $\frac{b}{a} := \sum_{i=1}^{k_1}\frac{1}{\lambda_i} + \sum_{i=1}^{k_2}\frac{\nu_i}{\mu_i} + \sum_{i=1}^{k_3}\frac{\rho_i}{\eta_i} - 1$, where $\frac{b}{a}$ is an irreducible fraction. We thus get (3.3).

Note further that a is bounded in terms of n only, indeed it cannot exceed the least common multiple of at most $2n!NM^{n-1}$ integers of size at most $n!NM^{n-1}$. A similar bound holds for b. This completes the proof of Proposition 3.6.

4. Proof of Theorem 2.3

It remains to discuss the fundamental equation (3.2) according to the possible values of the integers λ_i, ν_i, μ_i, ρ_i and η_i.

The proof will rest on the following elementary lemma about fractions:

Lemma 4.1. *Consider the following equation, where all variables are positive integers:*

$$\frac{1}{g}=\frac{1}{n_1}+...+\frac{1}{n_k}-\frac{b}{a}.$$

Suppose that $n_i \leqslant g$ for all i, then $g \leqslant f(k,a)$, where $f(k,a)$ is a function of k and a only. One may take $f(k,a) = (k!a)^{2^k}$.

Proof. The proof proceeds by induction on k. If $k = 1$, then $\frac{1}{n_1} \geqslant \frac{b}{a}$ implies $n_1 \leqslant a$ and $\frac{1}{g} \geqslant \frac{1}{n_1 a} \geqslant \frac{1}{a^2}$, so $g \leqslant a^2 =: f(1,a)$.

Suppose the lemma proven for all indices $\leqslant k-1$. Without loss of generality, we may assume that $\frac{1}{n_1} \leqslant ... \leqslant \frac{1}{n_k}$. Then $\frac{1}{n_2} + ... + \frac{1}{n_k} < \frac{b}{a}$, for otherwise $\frac{1}{g} \geqslant \frac{1}{n_1}$, and thus $n_1 = g$ forcing $b = 0$, which is a contradiction to our standing assumptions.

It follows that $\frac{b}{a} > \frac{1}{n_2} + ... + \frac{1}{n_k} \geqslant \frac{1}{n_k}$. We may thus write $\frac{c}{d} = \frac{b}{a} - \frac{1}{n_k}$, where c, d are positive integers and $\frac{c}{d}$ is an irreducible fraction.

Since $\frac{1}{g} \leqslant \frac{k}{n_k} - \frac{b}{a}$, we get $n_k \leqslant ka$ and thus $d = \mathrm{lcm}(a, n_k) \leqslant ka^2$. We obtain

$$\frac{1}{g}=\frac{1}{n_1}+...+\frac{1}{n_{k-1}}-\frac{c}{d}.$$

Applying the induction hypothesis we conclude that $g \leqslant f(k-1, ka^2) =: f(k,a)$. □

We now complete the proof of Theorem 2.3. Recall from the discussion in paragraph 3) above Proposition 3.6, that $n_i = \eta_i q_i = [N_G(\mathcal{F}_i) : \Phi]$ and is thus a divisor of $g = \frac{|G|}{|\Phi|}$. Therefore the right hand side of (3.4) is a non-negative integer. If it is non zero, then $b > 0$ and we are in the situation of Lemma 4.1. We conclude that $g = \frac{|G|}{|\Phi|}$ is bounded above by a bound depending only on k_3 and a, hence on n only. In this case the root torus S is the trivial one, and there are no M-beams except the trivial ones contained in Φ as soon as M is larger than this bound.

If the right hand side of (3.4) is 0, then $k_3 = 1$, $n_1 = g$ and $G = N_G(\mathcal{F}_1)$. This means there is only one conjugacy class of maximal M-beams, i.e. that of $\mathcal{F}_1$, and in fact only one maximal M-beam since G normalizes $\mathcal{F}_1$. Clearly G normalizes the root torus $S_{\mathcal{F}_1}$ associated to $\mathcal{F}_1$ (i.e., satisfying the same equalities between diagonal coefficients as $\mathcal{F}_1$). Moreover we know from (3.1) that $[G : \mathcal{F}_1]$ is bounded in terms of n only. Theorem 2.3 is now proven in full.

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References

[1] H. Bass, *Non connected subgroups of algebraic groups*, J. of Algebra, **82**, 245-254 (1983).
[2] L. Bieberbach, *ber einen Satz des Herrn C. Jordan in der Theorie der endlichen Gruppen linearer Substitutionen*, Sitzber. Preuss. Akad. Wiss. (1911) 231240.
[3] H. F. Blichfeldt, *On the Order of Linear Homogeneous Groups I,II,III,IV*, Trans. of the AMS, Vol. 4, No. 4 (1903,1903,1905,1911).
[4] W. Boothby and H.C. Wang, *On the finite subgroups of connected Lie groups*, Comment. Math. Helv. 39 1965 281–294.
[5] R. Brauer and W. Feit, *An analogue of Jordan's theorem in characteristic p*, Annals of Math., 2nd Ser., Vol. **84**, No. 2. (Sep., 1966), pp. 119-131.
[6] E. Breuillard and B. Green, *Approximate subgroups of linear groups: the unitary case*, preprint 2010, arXiv:1006.5160.
[7] M. J. Collins, *On Jordan's theorem for complex linear groups,* J. Group Theory **10** (2007), no. 4, 411–423.
[8] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, (Interscience, New York) (1962).
[9] G. Frobenius, *Uber den von L. Bieberbach gefundenen Beweis eines Satzes von C. Jordan*, Sitzber. Preuss. Akad. Wiss. (1911), 241–248.
[10] D. A. Kazhdan and G. A. Margulis *A proof of Selberg's conjecture*, Mat. Sb. (N.S.) (1968), **75** (117) 1, 163–168.
[11] F. Klein, *Lectures on the icosahedron*, Dover Publ.
[12] C. Jordan, *Mémoire sur les équations différentielles linéaires à intégrale algébrique,* J. Reine Angew. Math. **84** (1878), 89–215.
[13] M. Larsen and R. Pink, *Finite subgroups of algebraic groups*, preprint (1995).
[14] G.A Miller, H. Blichfeldt, L.E. Dickson, *Theory and application of finite groups*, 2nd ed, Stechert New-York, 1938.
[15] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, (1972).
[16] B. Weisfeiler, *Post-classification version of Jordan's theorem on finite linear groups*, Proc. Nat. Acad. Sci. 81 (1985), 5278-5279
[17] H. Zassenhauss, *Beweis eines Satzes uber diskrete Gruppen*, Abh. Math. Sem. Hansisch. Univ. **12** 289-312 (1938).

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