Infering varifold structure from the data

Charly Boricaud surpervised by Blanche Buet

September 2021

I share my gratitude to Blanche for giving me her human kindness and rigor to achieve this memoir. For my family, friends and teachers, heartfelt thanks.

Contents

1 Introduction

Continuous definitions (such as those of surface, regularity, dimension, curvatures ...) generally cannot be readily given a discrete counterpart. Moreover, this discrete counterpart is generally not unique and highly scale-dependent. There are multiple ways of developing a theory for discrete surfaces and the choice of an appropriate framework is directly related to the kind of discrete data we aim to process, and for which purpose i.e. the type of surfaces we try to model. Regarding the kind of discrete data, two different situations occur: either they have been collected in an external context and come in some given form one has to deal with, or one has the freedom to decide which discretization is best suited for this issue. Let us mention some examples of discrete representations: triangulated surfaces, digital shapes, graph representations, level sets and diffuse interfaces etc. In this memoir, we propose to focus on point cloud data (on the discrete side). On the continuous side, the denomination "surface" encompasses a wide variety of objects ranging from usual 2–dimensional surfaces embedded in $\mathbb{R}^3$ to any dimension and co-dimension submanifold, abstract Riemannian and sub-Riemannian manifolds, rectifiable sets, tree-like and graphs structures etc.

Geometric measure theory actually shows a major advantage: both discrete and continuous surfaces can be associated with a natural measure and thus naturally lie in the same space. It is in particular possible to say that a point cloud is close to a surface in the sense that it is close to a measure supported by the surface, with different possible choices of distances between measures to quantify the closeness. In geometric inference, such a closeness in terms of measures is a classical assumption in order to establish convergence of geometric estimators (tangent, curvature, second fundamental form, Laplace-Beltrami operator, see for instance [4] [5] [11]). However, such an assumption is not meaningless and essentially implies that the set of point is uniformly distributed along the underlying continuous surface (which is rarely true) or that it is possible to weight points to rectify the sampling.

In most cases we are only provided with sets of points in $\mathbb{R}^n$ and we need to infer weights and tangents (not to mention dimension) in order to infer the order 1 ("varifold") structure. In this memoir, we focused on infering the weights with a statistical perspective. More precisely we assume that a continuous object S is given through a probability measure μ supported in S and our data are obtained by sampling μ with N points: $(X_1, \ldots, X_N) \sim \mu$ is an i.i.d. sample and our data is an instance of the empirical measure

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}$$

The specific and important case where S is d–dimensional submanifold of $\mathbb{R}^n$ and μ is the volume form in S (possibly weighted by some density) has been investigated with varying degrees of formality and different assumptions on the regularity of the manifold. On the other hand, we aim at studying he case where S is less regular, focusing on rectifiable sets. Depending on the nature of the collected data, the measure μ is not necessarily uniformly distributed in S which consequently is also reflected by μ_N. In such a case, it is important to decouple the geometric information contained in S from the whole information encoded by μ. More explicitly, assume that $\mu = \theta \mathcal{H}^d_{|S}$ S for some positive density function θ that we want to recover. In practice, a common approach consists in computing local averages of the following form

$$\theta_i = \frac{1}{N} \sum_{j=1}^{N} \eta \left(\frac{|x_i - x_j|}{\delta} \right).$$

We address here the following question that is the core of this memoir: **Is it possible to recover the density θ relying on the knowledge of the empirical measure μ_N?**

Before giving our answer to this issue, we recall some bases of measure theory in Section 3:more precisely we recall the definitions of Radon measure, weak-$*$ convergence and bounded Lipschitz distance. In Section 4 We then focus on Ahlfors regular measures because they reflect the notion of being a d-dimensional measure in a flexible way: the mass of a ball with a center in our surface is comparable to its d-volume (Lebesgue measure).

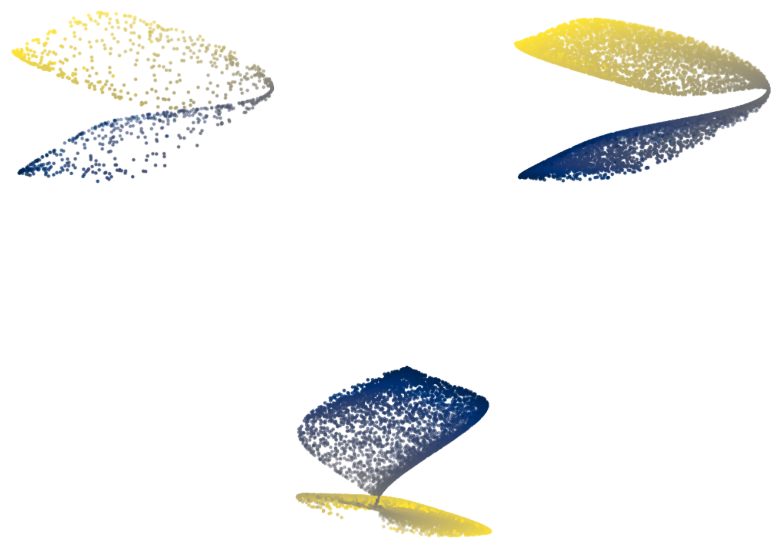

Figure 1: Examples of point clouds

Thereupon, in Section 5 we introduce some statistical tools with a particular emphasis on the empirical measure μ_N,$N \in \mathbb{N}^*$ sampled from our Ahlfors regular measure μ. In order to recover the density θ of $\mu = \theta \mathcal{H}^d_{|S}$ from the empirical measure μ_N, a key point is to quantify the distance (more precisely the bounded Lipschitz distance) between μ and μ_N. Following the original result of Dudley we were able to adapt the proof so as to obtain a result (see Theorem 5.10) involving a "local" bounded Lipschitz distance: loosely speaking we compare μ and μ_N in an open bounded open subset T, such as a small ball and we evidence that the rate of convergence is preserved up to dividing by the mas $\mu(T)$. Note that the clever proof of Dudley switch from a thin partition to a rougher one so as to take advantage of better compensation arising then as stated in Lemma 5.9.

Theorem 6.10 at hand, we introduce in Section 6 the classical notion of rectifiable set in order to prove that for a d-Ahlfors and d-rectifiable measure μ with a density θ it is possible to approximate θ by convolution of the empirical measure μ_N as stated in Theorem 6.10. This part is an utmost step to recover the so called varifold structure from a point cloud. We then present in Section 7 the bases of varifold theory because it is a gateway to have estimates on the convergence rates for differential operators and geometrical features such as mean curvature or Laplace Beltrami operator.

2 Notations

- S: a bounded set of $\mathbb{R}^n$ representing our surface
- $Bor(\mathbb{R}^n)$: the Borel sets of $\mathbb{R}^n$
- $\mathcal{P}(S)$: the set of Borel probability measures on S
- $(\Omega, \mathcal{F}, \mathbb{P})$: the probability space
- μ: a Borel probability measure representing our surface
- $\mu_N = \frac{1}{N}\sum_{q=1}^{N} \delta_{X_q}$: the empirical measure of N points i.i.d $X_1, \dots, X_N$ with the law μ
- $B(x,r)$: is the open ball with centre $x \in \mathbb{R}^n$ and radius $r > 0$
- V(A,δ): the smallest number of balls of radius δ needed to cover A
- $P(A,\delta)$: the packing number of a bounded set A is the maximum of disjoint balls of diameter δ we can have with center in A
- B^δ: for $B \subset \mathbb{R}^n$, $\delta > 0$ is the δ-thickening of B, $B^\delta = \{x \in \mathbb{R}^n \text{ s.t } d(x,B) < \delta\}$
- $BL(S,d)$: the Banach space of all bounded Lipschitz real-valued functions f on S with the norm $||f||_{BL} = ||f||_\infty + ||f||_L$ where $||f||_L = \underset{x \neq y}{sup} \frac{|f(x)-f(y)|}{|x-y|}$
- $||\mu||_{BL}^* = sup\{\int f d\mu : ||f||_{BL} \leq 1\}$ the weak-$*$ norm linked to BL
- $\beta(\mu,\nu) = ||\mu - \nu||_{BL}^*$, a distance for the weak-$*$ topology on $\mathcal{P}(S)$

3 Basic notions of measure theory

In this section we introduce a locally compact and separable metric space (X,d) and for the next sections we deal with $X = \mathbb{R}^n$, equipped with the euclidean norm $||.||$. Note that when working with varifolds, we will also use the case $X = \mathbb{R}^n \times G_{d,n}$ equipped with a product distance. We follow [10] and [8] in this section.

3.1 Measure properties

We recall some basic definitions and fix associated notations. For $x \in X$ and $0 < r < \infty$

$$B(x,r) = \{y \in X : d(x,y) \leq r\}$$

is the open ball of center x and radius x. The diameter of a non-empty subset $A \subset X$ is

$$diam(A) = \sup\{d(x,y) : x, y \in A\}.$$

We agree that $diam(\emptyset) = 0$. The distance of a point $x \in X$ to a non empty set A $\subset X$ is

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

If B is also a non empty subset of X then the distance between A and B is

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

For $\delta > 0$, the open δ-neighbourhood(δ-thickening) of A is

$$A^\delta = \{x \in X : d(x,A) < \delta\} = \bigcup_{x \in A} B(x,\delta). \quad (1)$$

A measure is a non-negative, monotonic, subadditive set function vanishing for the empty set.

Definition 3.1. *A set function* $\mu : \{A : A \subset X\} \leftarrow [0, \infty]$ *is called a (outer) measure if*

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$ *whenever* $A \subset B \subset X$ *(monotony)*
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ *whenever* $A_1, A_2 \dots \subset X$ *(subadditivity)*

Remark 3.2. *We note that the usual terminology for* μ *satisfying definition 3.1 is "outer measure", however, we follow [8] in that respect and recover the additivity property when restricting* μ *to so called measurable sets.*

An example of measure is the very important for our project is the Dirac mass defined as follows.

Definition 3.3 (Dirac mass)**.** *The Dirac mass associated to a point* $x \in \mathbb{R}^n$ *is the measure defined as follows*

$$\delta_x(A) = \mathbb{1}_A(x) \qquad \forall A \subset \mathbb{R}^n.$$

Definition 3.4. *A set* $A \subset X$ *is* μ *measurable if:*

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \text{ for all } E \subset X$$

Definition 3.5 (σ-algebra)**.** $\mathcal{A}$ *is a* σ*-algebra on* X *if:*

1. $\mathcal{A} \neq \emptyset$
2. $\forall B \in \mathcal{A}$, $B \subset X$ *and its complementary* $X \setminus B$ *is in* $\mathcal{A}$,
3. $\mathcal{A}$ *is stable by countable union.*

Usually in measure theory a measure is defined on some σ-algebra of subsets of X, which does not need to be the whole power set $\{A : A \subset X\}$. However Definition 3.1 permits us to extend any μ defined on the σ-algebra $\mathcal{A}$ to an outer measure μ^* on X by:

$$\mu^*(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{A}\}$$

From the previous definitions arise those wonderful properties that permit us to focus on nice measurable sets to deduce properties on irregular ones.

Property 3.6. *Let* μ *be a measure on* X *and let* $\mathcal{M}$ *be the family of all* μ *measurable subsets of* X.

1. $\mathcal{M}$ *is a* σ*-algebra.*
2. *If* $\mu(A) = 0$*, then* $A \in \mathcal{M}$.
3. *If* $A_1, A_2, \dots \in \mathcal{M}$ *are pairwise disjoint, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

4. *If* $A_1, A_2, \dots \in \mathcal{M}$*, then*
   - $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ *provided* $A_1 \subset A_2 \subset \dots$,
   - $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$ *provided* $A_1 \supset A_2 \supset \dots$ *and* $\mu(A_1) < \infty$.

The smallest σ-algebra containing the open subsets of X is called the $\sigma-$algebra of Borel sets. It is well defined since the notion of $\sigma-$algebra is stable by intersection. For our study in $\mathbb{R}^n$ it coincide with the σ-algebra generated by the open balls. We concentrate on them because some useful measures are very well known on them. Also the previous properties permit to control the measure of sets only by knowing on the balls if we are in the framework of the following definition.

Definition 3.7. *Let μ be a measure on X.*

1. *μ is locally finite if for every $x \in X$ there is $r > 0$ such that $\mu(B(x,r)) < \infty$.*
2. *μ is a Borel measure if all Borel sets are μ measurable*
3. *μ is Borel regular if it is a Borel measure and if for every $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$*
4. *μ is a Radon measure if it is a Borel measure and*
   - *μ is locally finite,*
   - *$\mu(V) = \sup\{\mu(K) : K \subset V,\ K$ compact $\}$ for open sets $V \subset X$,*
   - *$\mu(A) = \inf\{\mu(V) : A \subset V,\ V$ open$\}$ for $A \subset X$.*

Theorem 3.8 (Caratheodory's Criterion)**.** *Let μ be a measure on X, then μ is a Borel measure if and only if μ is a metric measure:*

$$\mu(A \cup B) = \mu(A) + \mu(B) \text{ whenever } A, B \subset X \text{ with } d(A,B) > 0. \tag{2}$$

Proof. Let $A, B \subset X$ such that $d(A,B) > 0$ and

- Suppose that μ is a Borel measure. From the subadditivity of μ, we have $\mu(A \cup B) \leq \mu(A) + \mu(B)$. Let us prove the converse inequality. Let $0 < \delta < d(A,B)$, $A^\delta \cap B = \emptyset$, A^δ is open so measurable. We use then the Definition 3.4:

  $$\mu(A \cup B) = \mu((A \cup B) \cap A^\delta)) + \mu((A \cup B) \setminus A^\delta) \geq \mu(A) + \mu(B \setminus A^\delta) \geq \mu(A) + \mu(B)$$

  Conclusion μ is a Borel measure $\implies$ μ is a metric measure.

- Now, suppose that μ is a metric measure. We want to prove that it is a Borel measure. Given $O \subset X$ an open set and $A \subset X$, let us prove that O is measurable (Definition 3.4). From the subadditivity we have that $\mu(A) \leq \mu(A \cap O) + \mu(A \setminus O)$.

  For the converse inequality, if $\mu(A) = +\infty$ then the inequality holds. We hence assume that $\mu(A) < +\infty$. We define for $k \in \mathbb{N}^*$ $R_k = \left\{x \in A : \frac{1}{k+1} \leq d(x,O) < \frac{1}{k}\right\}$, $A \cap (O^1 \setminus O) = \bigsqcup_{k \in \mathbb{N}^*} R_k$ is a decomposition of $A \cap (O^1 \setminus O)$ according to disjoint rings reminding that $O^{\frac{1}{k}}$ is the $\frac{1}{k}$-thickening of O(see (1). The aim is to use the positive distance between non consecutive rings to smartly apply the metric hypothesis.

  First, we slightly enlarge O into its $\frac{1}{k}$-neighbourhood so that $d(A \setminus O^{\frac{1}{k}}, A \cap O) \geq \frac{1}{k} > 0$ and by equation (2):

  $$\mu(A \setminus O^{\frac{1}{k}}) + \mu(A \cap O) = \mu((A \setminus O^{\frac{1}{k}}) \cup (A \cap O)) \leq \mu(A) \tag{3}$$

  It remain to prove that $\mu(A \setminus O^{\frac{1}{k}}) \xrightarrow[k \to \infty]{} \mu(A \setminus O)$.

  Applying the subadditivity of μ with

  $$A \setminus O = (A \setminus O^{\frac{1}{k}}) \cup \left(\bigcup_{i=k}^{\infty} R_i\right),$$

We infer then

$$\mu(A \setminus O^{\frac{1}{k}}) \underset{O \subset O^{\frac{1}{k}}}{\leq} \mu(A \setminus O) \leq \mu(A \setminus O^{\frac{1}{k}}) + \mu\left(\bigcup_{i=k}^{\infty} R_i\right) \leq \mu(A \setminus O^{\frac{1}{k}}) + \sum_{i=k}^{\infty} \mu(R_i). \tag{4}$$

For all $i, j \in \mathbb{N}^*$ $i \geq j+2$, $d(R_i, R_j) > 0$, and applying the subadditivity and the metric hypothesis it follows:

$$\sum_{i=1}^{\infty} \mu(R_i) \leq \sum_{i=1}^{\infty} \mu(R_{2i}) + \sum_{i=0}^{\infty} \mu(R_{2i+1}) \leq 2\mu\left(\bigcup_{i=1}^{\infty} R_i\right) \leq 2\mu(A) < +\infty$$

As $\sum\limits_{i \geq 1} \mu(R_i)$ converges, the remainder tends to 0. From (4) we have

$$\mu(A \setminus O) - \underbrace{\sum_{i=k}^{\infty} \mu(R_i)}_{\xrightarrow[k\to\infty]{} 0} \leq \mu(A \setminus O^{\frac{1}{k}}) \leq \mu(A \setminus O)$$

Then $\mu(A \setminus O^{\frac{1}{k}}) \xrightarrow[k\to\infty]{} \mu(A \setminus O)$.

Passing to the limit in (3), we finally have:

$$\mu(A) \geq \mu(A \setminus O) + \mu(A \cup O)$$

O is measurable then all open are measurable and we conclude that μ is a Borel measure.

Conclusion in a metric space: μ is a Borel measure $\iff$ μ is a metric measure. ☐

Definition 3.9. *We define the restriction of μ to a set $S \subset X$ as follows, for all $A \subset X$,*

$$\mu_{|S}(A) := \mu(A \cap S)$$

.

Theorem 3.10. *Let μ be a Borel regular measure on $\mathbb{R}^n$. Suppose $S \subset X$ is μ-measurable and $\mu(S) < \infty$. Then $\mu_{|S}$ is a Radon measure.*

Proof. Let $\nu = \mu_{|S}$. For each compact set K, $\nu(K) \leq \mu(S) < +\infty$. Every $\mu-$measurable set A $\mu-measurable$ is ν-measurable because $A \cap S$ is the intersection of two measurable sets hence ν is a Borel measure. with the remark above.

Let us show that ν is Borel regular. Since μ is Borel regular, there exists a Borel set B such that $S \subset B$ and $\mu(S) = \mu(B) < +\infty$. Then, since S is $\mu-$measurable,

$$\mu(B) = \mu(B \cap S) + \mu(B \subset S) \text{ ie. } \mu(B \setminus S) = \mu(B) - \mu(S) = 0.$$

Let $C \subset X$, using aain that S is μ-measurable:

$$\begin{aligned}
\mu_{|B}(C) =& \mu(C \cap B) \\
=& \mu(\underbrace{C \cap B \cap S}_{C \cap S}) + \underbrace{\mu((C \cap B) \setminus S)}_{\leq \mu(B \setminus S) = 0} \\
=& \mu_{|S}(C)
\end{aligned}$$

Thus $\mu_{|B} = \mu_{|S}$. Let E be Borel set such that $C \cap S \subset E$ and $\mu(E) = \mu(C \cap S)$. Let $D := E \cup (X \setminus S)$. Since S and E are Borel sets, so is D. Moreover, $C \subset (C \cap S) \cup (X \setminus S) \subset D$. Finally, since $D \cap S = E \cap A$,

$$\nu(D) = \mu(D \cap S) = \mu(E \cap S) \leq \mu(E) = \mu(C \cap S) = \nu(C)$$

We conclude that $D \subset X$ is a Borel set such that $C \subset D$ and $\nu(C) = \nu(D)$, therefore ν is Borel regular. ☐

3.2 Weak-star convergence notions

Theorem 3.11. *Let $\mu, (\mu_k)_{k\in\mathbb{N}}$ be Radon measures on X. The following statements are equivalent:*

1. $\lim\limits_{k\to\infty} \int_X f d\mu_k = \int_X f d\mu$ *for all* $f \in C_c(X)$

2. $\limsup\limits_{k\to\infty} \mu_k(K) \leq \mu(K)$ *for each compact set* $K \in X$ *and* $\mu(U) \leq \liminf\limits_{k\to\infty} \mu_k(U)$ *for each open set* $U \subset X$.

3. $\lim\limits_{k\to\infty} \mu_k(B) = \mu(B)$ *for each bounded Borel set* $B \subset X$ *with* $\mu(\partial B) = 0$.

Definition 3.12. *If $\mu, (\mu_k)_{k\in\mathbb{N}}$ verify those statements then we say that $(\mu_k)_{k\in\mathbb{N}}$ converge weakly-* to μ and we denote it:*

$$\mu_k \overset{*}{\rightharpoonup} \mu$$

Proof. 1. Assume (1) holds and fix $\epsilon > 0$. Let $U \subset X$ be open, we first check $\mu(U) \leq \liminf\limits_{k\to+\infty} \mu_k(U)$ we choose any compact set $K \subset U$ and take $f \in C_c(X)$ such that $0 \leq f \leq 1, supp(f) \subset U$, $f \equiv 1$ on K. For instance $f(x) := \frac{d(x,K)}{d(x,K)+d(x,X\setminus U)}$, then

$$\mu(K) \leq \int_X f d\mu = \lim_{k\to+\infty} \int_X f d\mu_k \leq \liminf_{k\to+\infty} \mu_k(U).$$

Thus by regularity of μ,

$$\mu(U) = \sup\{\mu(K),\ \text{K compact},\ K \subset U\} \leq \liminf_{k\to+\infty} \mu_k(U).$$

A similar argument gives the first part of (2) and it concludes (1) $\implies$ (2).

2. Suppose now that (2) holds and let $B \subset X$ is a bounded Borel set such that $\mu(\partial B) = 0$. Then

$$\begin{aligned}\mu(B) = \mu(\overset{o}{B}) \leq& \liminf_{k\to+\infty} \mu_k(\overset{o}{B})\\ \leq& \limsup_{k\to+\infty} \mu_k(\overline{B})\\ \leq& \mu(\overline{B}) = \mu(B).\end{aligned}$$

Then (2) $\Longrightarrow$ (3)

3. To conclude, assume that (3) holds. By linearity of the integral we just need to prove (1) for $f \in C_c^+(X)$ and f not the zero function, because it is similar for $f \in C_c^-(X)$. Let $\epsilon, r > 0$ such that $supp(f) \subset B(0,r)$ there is $R > r$ such that $\mu(\partial B(0,R)) = 0$. It is possible because the function

$$\begin{aligned} g : [r, 2r] &\to \mathbb{R} \\ x &\mapsto \mu(B(0,x)) \end{aligned}$$

is monotonous then, it has a countable number of discontinuities thanks to Darboux–Froda's theorem (see[6]). For $r < x < 2r$, $g(x^+) - g(x^-) = \mu(\partial B(0,x))$ thanks to monotony properties from Property 3.6 for graded countable Borel rings subfamilies of $\{B(0,x+\epsilon)\setminus B(0,x)\}_{0<\epsilon}$ and $\{B(0,x)\setminus B(0,x-\epsilon)\}_{0<\epsilon<x}$. Then there is $R > r$ such that $\mu(\partial B(0,R))) = 0$.

We choose $0 = t_0 < t_1 < \cdots < t_N$ such that $t_N \equiv 2||f||_{L^\infty}$, $0 < t_i - t_{i-1} \leq \epsilon$, and $\mu(f^{-1}(\{t_i\})) = 0$ for $i = 1, \ldots, N$. It is possible because f is continuous with compact support so $f^{-1}([0, t_N])$ is a Borel set and carry out a similar argument with the functions h_r defined for each $r > 0$ as

$$\begin{aligned} h_r : [r, r+\epsilon] &\to \mathbb{R} \\ x &\mapsto \mu(f^{-1}([0, x[)). \end{aligned}$$

We just have to replace r by t_{i+1} each time we find t_i. Set $B_i = f^{-1}((t_{i-1}, t_i])$; then $\mu(\partial B_i) = 0$ for $i \geq 2$. Now we decompose the integral and minor (resp major) like stairs:

$$\int_X f d\mu_k = \sum_{i=1}^{N} \int_{B_i} f d\mu_k + \int_{\partial B_1} \underbrace{f}_{=0} d\mu_k$$

therefore

$$\sum_{i=2}^{N} t_{i-1}\mu_k(B_i) \leq \int_X f d\mu_k \leq \sum_{i=2}^{N} t_i \mu_k(B_i) + t_1 \mu_k(B(0,R))$$

and similarly

$$\sum_{i=2}^{N} t_{i-1}\mu(B_i) \leq \int_X f d\mu \leq \sum_{i=2}^{N} t_i \mu(B_i) + t_1 \mu(B(0,R)).$$

For each $i \in [|1, N|]$:

$$\sum_{i=2}^{N} t_{i-1}\mu_k(B_i) - t_i\mu(B_i) = \sum_{i=2}^{N} t_{i-1} \underbrace{(\mu_k(B_i) - \mu(B_i))}_{\to 0} + \underbrace{(t_{i-1} - t_i)}_{\leq \epsilon} \mu(B_i)$$

and

$$\sum_{i=2}^{N} t_{i-1}\mu(B_i) - t_i\mu_k(B_i) = \sum_{i=2}^{N} t_{i-1} \underbrace{(\mu(B_i) - \mu_k(B_i))}_{\to 0} + \underbrace{(t_{i-1} - t_i)}_{\leq \epsilon} \mu_k(B_i).$$

Therefore, Theorem 3.11(3), additivity of μ on Borel sets and $t_1 \leq \epsilon$ imply that

$$\limsup_{k \to +\infty} \left| \int_X f d\mu_k - \int_X f d\mu \right| \leq 2\epsilon\mu(B(0,R)).$$

□

The notion of weak-$*$ convergence is important. Usually it is better suited to obtain compactness.In our case, we will use it to characterise our convergence of discrete measures sampled to a measure representing a d-dimensional surface.

Definition 3.13 (Bounded Lipschitz distance)**.** *For μ, ν two Radon measure in X, the bounded Lipschitz distance is defined by:*

$$\beta(\mu, \nu) = \sup \left\{ \left| \int_X f d(\mu - \nu) \right| : f \in C_c(X, \mathbb{R}), ||f||_\infty \leq 1, ||f||_L \leq 1, \right\}$$

The Lipschitz functions are crucial because the variations of the function is controlled by a linear variation of the space. Then we can dominate the integral by the measure of sets with a certain diameter.

4 Ahlfors regular measure

We recall that our aim is to estimate the density θ of a given measure $\mu = \theta \mathcal{H}^d_{|S}$ relying on the associated empirical measure μ_N. To this end, it is necessary to assume some additional regularity and instead of requiring strong smoothness of the set S we rather transfer regularity assumption on the measure μ.

4.1 General properties

Definition 4.1 (Ahlfors regular measure)**.** *Let μ be a Radon measure in $\mathbb{R}^n$, for $d > 0$, we say that μ is a d-Ahlfors regular measure if $\exists C_0 > 0$ such that $\forall x \in supp(\mu) = S$, $\forall r \in]0, 1]$:*

$$\frac{1}{C_0} r^d \leq \mu(B(x, r)) \leq C_0 r^d.$$

A first example of d-Ahlfors regular measure is the Hausdorff measure of dimension d restricted to a not too bad d-dimensional measurable set like the sphere or the Koch snowflake. Our main concern is to rebuild those objects with point clouds.

Definition 4.2. *We define the d dimensional Hausdorff measure as follows: For each set $A \subset \mathbb{R}^n$*

$$\mathcal{H}^d(A) = \inf_{\delta \to 0} \left\{ \sum_{i \in \mathbb{N}} diam(E_i)^d : A \subset \bigcup_{i \in \mathbb{N}} E_i, E_i \subset \mathbb{R}^n \text{ and } diam(E_i) \leq \delta \right\}$$

Property 4.3. *Let $S \subset \mathbb{R}^n$ be $\mathcal{H}^d$ measurable set with $\mathcal{H}^d(S) < +\infty$, then $\mathcal{H}^d_{|S}$ is:*

1. *An outer measure.*
2. *Borel regular.*
3. *A Radon measure.*

4.2 Packing number of an Ahlfors regular measure

In this subsection we express an essential control over the number of balls with small diameter δ needed to cover a set A. Such estimates will be crucial in this proof of Theorem 5.10

Lemma 4.4. *Let $\delta > 0$ and $A \subset \mathbb{R}^n$. The packing number $k := P(A, \delta)$, is the greatest number of disjoint balls with center in A of radius δ and $V(A, \delta)$ is the smallest number of balls with radius δ needed to cover A. We have:*

$$V(A, 2\delta) \leq P(A, \delta) \leq V(A, \delta/2)$$

Proof.

- Let $A \subset \mathbb{R}^n$, Let $B(x_1, \delta), \dots, B(x_k, \delta)$ be disjoint balls with centers $x_1, \dots, x_k \in A$.

  Suppose that $V(A, 2\delta) > k$ then $\bigcup_{i=1}^k B(x_i, 2\delta)$ does not cover A thus there exists $x \in A \backslash \bigcup_{i=1}^k B(x_i, 2\delta)$. Then $B(x, \delta), B(x_1, \delta), \dots, B(x_k, \delta)$ are k+1 disjoint balls of radius δ with center in A, that is impossible by definition of k. We conclude that $V(A, 2\delta) \leq P(A, \delta)$.

- Let $k = P(A, \delta)$ and $x_1, \dots x_k$ as previously.

  Let $v = V(A, \delta/2)$ and $y_1, \dots, y_v$ such that $B(y_1, \delta/2), \dots, B(y_v, \delta/2)$ cover A. Suppose that $v < k$, the points $x_1, \dots, x_k$ are in A thus there are two points x_i, x_j in the same ball $B(y_z, \delta/2)$ for some $z \in \{1, \dots v\}$. Then

  $$||x_i - x_j|| \leq ||x_i - y_z|| + ||x_j - y_z|| < \delta$$

  . And it is impossible because $B(x_i, \delta) \cap B(x_j, \delta) = \emptyset$ meaning that $||x_i - x_j|| \geq \delta$. We conclude that $P(A, \delta) = k \leq v = V(A, \delta/2)$.

□

Lemma 4.5. *We define $m(A, \delta)$ as the minimal number of sets of diameter smaller than δ needed to form a partition of A then:*

$$V(A, \delta) \leq m(A, \delta) \leq V(A, \delta/2)$$

Proof.

- We easily have $V(A,\delta) \leq m(A,\delta)$ because a partition of A is also covering A.
- Let $k = V(A,\delta/2)$ and $B_1,\dots,B_k$ balls of radius $\delta/2$ such that $A \subset \bigcup_{j=1}^{k} B_j$. We take $A_1 = A \cap B_1$, $A_2 = A \cap B_2 \setminus A_1, \dots,$ $A_k = (A \cap B_k) \setminus \bigcup_{j=1}^{k-1} A_j$ Then $A_1,\dots,A_k$ form a partition of A with sets of diameter smaller than δ.

□

Theorem 4.6. *Let μ be a Radon measure in $\mathbb{R}^n$, $S = supp(\mu)$. Assume that there exists $d > 0, C_0 \geq 1$ such that $\forall x \in S$, $\forall 0 < r \leq 1$,*

$$\mu(B(x,r)) \geq C_0^{-1} r^d \tag{5}$$

Then, for all bounded set $B \subset \mathbb{R}^n$

$$m(B \cap S,\delta) \leq 4^d C_0 \delta^{-d} \mu(B^{\delta/4}).$$

Proof. Let $k = P(B \cap S,\delta)$ and $B_1,\dots,B_k$ be disjoint balls of radius δ and center in $B \cap S$. Then for $i = 1,\dots,k$ since B_i has center on S,(5) gives

$$\mu(B_i) \geq C_0^{-1}\delta^d.$$

Therefore

$$kC_0^{-1}\delta^d \leq \sum_{i=1}^{k} \mu(B_i) = \mu(\bigsqcup_{i=1}^{k} B_i) \leq \mu(B^\delta)$$

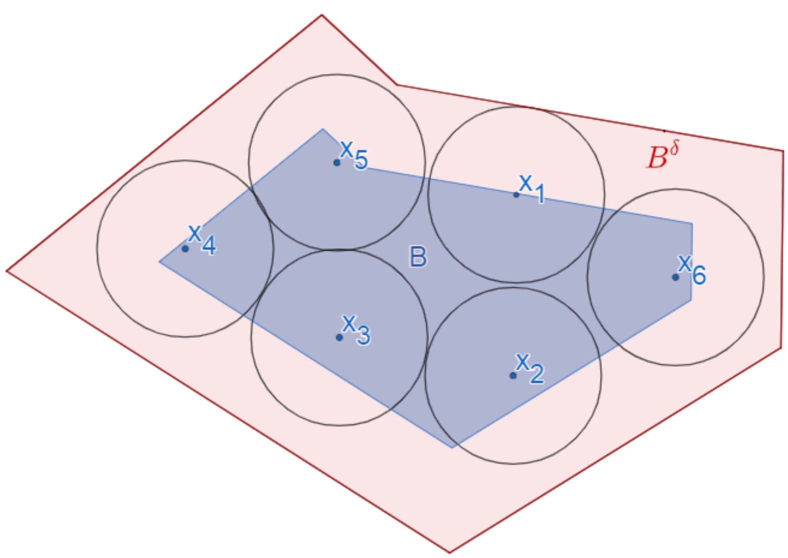

Figure 2: δ-Thickening view with the packing number

Then we deduce

$$P(B \cap S,\delta) = k \leq C_0 \mu(B^\delta)\delta^{-d}$$

We conclude applying Lemma 4.5

$$m(B\cap S,\delta)\le V(B\cap S,\delta/2)\le P(B\cap S,\delta/4)\le 4^dC_0\delta^{-d}\mu(B^{\delta/4}).$$

□

5 Mean rates convergence of empirical measure toward an Ahlfors regular measure

In this section we want to estimate the density bias of a surface S represented by a Radon measure in the form $\mu=\theta\mathcal{H}^d_{|S}$ where $\theta>0$ is the density bias. The main proof is adapted from the article [7]. Our adaptation rely on a clever use of packing number for an Ahlfors regular measure that is a beautiful technique of Geometric Measure Theory.

5.1 Statistical framework

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability triplet that is respectively a set called the universe, a $\sigma-$Algebra and a measure $\mathbb{P}$ on $\mathcal{A}$. We assume that our Radon measure $\mu=\theta\mathcal{H}^d_{|S}$ is a probability measure:$\int_S\theta d\mathcal{H}^d_{|S}$.

Definition 5.1 (Random Variable)**.** *For $k\in\mathbb{N}^*$, a random variable $X:\Omega\to\mathbb{R}^k$ is a measurable function relatively to the σ-algebra $(\Omega,\mathcal{A})$ and $(\mathbb{R}^k,Bor(\mathbb{R}^k))$. We say that X follows the law μ if for all $T\in Bor(\mathbb{R}^k)$:*

$$\mathbb{P}(X\in T)=\mu(T)$$

Remark 5.2. *If k=1 then it is a real random variable.*

Definition 5.3. *Let X,Y two random variables in the same space. We say that X and Y are independent if for all $x,y\in\mathbb{R}^k$*

$$\mathbb{P}(X=x,Y=y)=\mathbb{P}(X=x)\mathbb{P}(Y=y)$$

Definition 5.4 (Expectation)**.** *A random variable X has an expectation $\mathbb{E}[X]$ if the following integral is absolutely convergent:*

$$\mathbb{E}[X]:=\int_{\omega\in\Omega}X(\omega)d\mathbb{P}(\omega).$$

Definition 5.5 (Variance)**.** *A real random variable X has a variance $Var(X)$ if X has an expectation and the expression below is defined:*

$$Var(X):=\mathbb{E}((X-\mathbb{E}(X))^2)$$

Property 5.6.

1. *The expectation is linear*
2. *For X,Y two independent real random variables with variance and $\lambda\in\mathbb{R}$ we have*

$$Var(X+\lambda Y)=Var(X)+\lambda^2Var(Y)$$

Definition 5.7 (Empirical measure)**.** *Let $N\in\mathbb{N}^*$, $X_1,\dots,X_N$ be N random independent variables with the same law μ, the associated empirical measure μ_N is defined as:*

$$\mu_N=\frac{1}{N}\sum_{i=1}^N\delta_{X_i}$$

Remark 5.8. *The empirical measure permits to sample a measure.*

5.2 The mean speed convergence of empirical measure in terms of the bounded Lipschitz distance

Lemma 5.9. *[7] Let $T \subset \mathbb{R}^n$ be a Borel set and let S_j, $j = 1, \dots, m$ be disjoint Borel sets with union T then:*

$$\mathbb{E}\left[\sum_{j=1}^{m} |(\mu_N - \mu)(S_j)|\right] \leq (m\mu(T)/N)^{\frac{1}{2}}$$

Proof. Let $q, j \in [|1, N|]$ and $\omega \in \Omega$. Notice that

$$\delta_{X_q(\omega)}(T) = \begin{cases} 1 & if \quad X_q(\omega) \in T \\ 0 & otherwise \end{cases}$$

hence,

$$\mathbb{E}\left[\ \delta_{X_q}(T)\right] = \int_{\omega\in\Omega} \mathbb{1}_{\{X_j(\omega)\in T\}} d\mathbb{P}(\omega) = \mathbb{P}(X_j \in T) = \mu(T) \tag{6}$$

Similarly

$$\mathbb{E}\left[\ \delta_{X_q}(T)\delta_{X_j}(T)\right] = \begin{cases} \mathbb{P}(X_j \in T) & if \quad q = j \\ \mathbb{P}(X_q \in T \text{ and } X_j \in T) & otherwise. \end{cases}$$

And because of the independence of the variables we have

$$\mathbb{E}\left[\ \delta_{X_q}(T)\delta_{X_j}(T)\right] = \begin{cases} \mu(T) & if \quad q = j \\ \mu(T)^2 & otherwise. \end{cases}$$

In particular,

$$Var(\delta_{X_q}(T)) = \mu(T) - \mu(T)^2. \tag{7}$$

Then we use the linearity of the expectation

$$\mathbb{E}[\mu_N(T)] = \frac{1}{N}\sum_{q=1}^{N} \mathbb{E}[\delta_{X_q}(T)] = \mu(T),$$

Analogously with the independence of the variables $X_1, \dots, X_N$, we infer the independence of $\delta_{X_1}(T), \dots, \delta_{X_N}(T)$. Thus by the Poperty 5.6 and (7)

$$Var(\mu_N(T)) = \frac{1}{N^2}\sum_{q=1}^{N} Var(\delta_{X_q(\omega)}(T)) = \frac{\mu(T) - \mu(T)^2}{N}.$$

Secondly we did it for every measurable set so that for the disjoints measurable sets (S_j), we have for every $j \in [|1, m|]$

$$\mathbb{E}\left[(\mu_N - \mu)(S_j)^2\right] = \mathbb{E}\left[(\mu_N)(S_j)^2\right] - 2\mu(T)\mathbb{E}\left[(\mu_N)(S_j)\right] + \mu(S_j)^2 = Var(\mu(S_j))$$

and thus

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^{m}(\mu_N - \mu)(S_j)^2\right] &= \sum_{j=1}^{m} Var(\mu_N(S_j))^2 \\ &= \sum_{j=1}^{m}(\mu(S_j) - \mu^2(S_j))/N \\ &= \left(\mu(T) - \sum_{j=1}^{m}\mu(S_j)^2\right)/N \leq \mu(T)/N. \end{aligned} \tag{8}$$

With Cauchy-Schwarz inequality for the finite sum:

$$\mathbb{E}\left[\sum_{j=1}^{m}|(\mu_N-\mu)(S_j)|\right] \leq \mathbb{E}\left[\left(\sum_{j=1}^{m}(\mu_N-\mu)(S_j)^2\right)^{\frac{1}{2}}\underbrace{\left(\sum_{j=1}^{m}1^2\right)^{\frac{1}{2}}}_{=\sqrt{m}}\right] \tag{9}$$

And then we use again Cauchy-Schwartz inequality for the expectation together with (8) and (9)

$$\mathbb{E}\left[\sum_{j=1}^{m}|(\mu_N-\mu)(S_j)|\right] \leq (m\mu(T)/N)^{\frac{1}{2}} \tag{10}$$

□

The following theorem is adapted from [7] and gives the speed of convergence and the ratio between the number of points and the scale of observation to obtain the convergence of the empirical measure to the measure related to S even with a certain type of noise. More precisely, in [7], the author prove that $\mathbb{E}[\beta(\mu_N,\mu)]$ tends to 0 with a speed of convergence at least $N^{-\frac{1}{d}}$. In order to obtain pointwise convergence of our density estimator as stated in Theorem 6.10, we need a similar result but with an additional localization. For $B\subset\mathbb{R}^n$ bounded open set, we define the localized bounded Lipschitz distance in B

$$\beta_B(\mu,\nu)=\sup\left\{\left|\int_X fd(\mu-\nu)\right| : f\in C_c(X,\mathbb{R}), ||f||_\infty\leq 1, ||f||_L\leq 1, supp(f)\subset B\right\}.$$

Adapting proof of Theorem 3 in [7], we were able to obtain Theorem 5.10.

Theorem 5.10. *[7] Suppose that μ is d-Ahlfors regular(see Definition 4.1) for some real number $d>2$ with regularity constant $C_0>0$. Then, there exists an $M=M(d,C_0)>0$ such that for all $N\in\mathbb{N}^*$ and $T\subset\mathbb{R}^n$ bounded open set,*

$$\mathbb{E}[\beta_T(\mu_N,\mu)]\leq MN^{-1/d}\mu(T^{\gamma_N}) \qquad \textit{with } \gamma_N=N^{-\frac{d-2}{d^2}}\xrightarrow[N\to+\infty]{} 0.$$

Proof. Let $T\subset\mathbb{R}^n$ be a bounded open set. For each positive integer r, we can write T as the disjoint union of Borel sets, S_j^r, $j=1,\ldots,m_r$, of diameter at most $\varepsilon_r:=3^{-r+1}$, we know thanks to Theorem 4.6 that we can chose the partition so that

$$m_r\leq 4^dC_0\varepsilon_r^{-d}\mu(B^{\varepsilon_r/4})$$

.

Given a positive integer N let $\epsilon=N^{-\frac{1}{d}}$ and let $0\leq s\leq t$ be integers such that:

$$\begin{aligned}3^{-t} <&\epsilon\leq 3^{-t+1}=:\varepsilon_t\\ 3^{-s} <&\epsilon^{(d-2)/d}\leq 3^{-s+1}=:\varepsilon_s.\end{aligned} \tag{11}$$

Step 1:. Construction of nested partitions of T. for each integer $s\leq r\leq t$, we start with a partition $\{S_j^r\}_{j=1,\ldots,m_r}$ of T. We define sets A_j^{t-u}, $j=1,\ldots,m_{t-u}$, inductively on $u=0,\ldots,t-s$, as follows. Let $A_j^t=S_j^t$. Let $u\in[|0,t-s-1|]$ and assume that $\{A_j^{t-u}\}_{j=1,\ldots,m_{t-u}}$. Each A_j^{t-u} intersects at least one S_q^{t-u-1} for some $q\geq 1$, and we choose such a $q=q(t-u,j)\in[|1,m_{t-u-1}|]$ to classify the sets. Then, for $z\in[|1,m_{t-u-1}|]$ we define

$$A_z^{t-u-1}=\bigcup_{j:q(t-u,j)=z}A_j^{t-u} \tag{12}$$

We can visualize A_z^{t-u-1} as the tills capturing the set of diameter less than 3^{-t+u} intersecting possibly up to some choice $S_{t-u-1,z}$ Then for each z, we have for diameters

$$diam(A_z^{t-u-1}) \leq 2\max_j diam(A_j^{t-u}) + \underbrace{3^{u-t}}_{\geq diam(S_{t-u,z})}$$

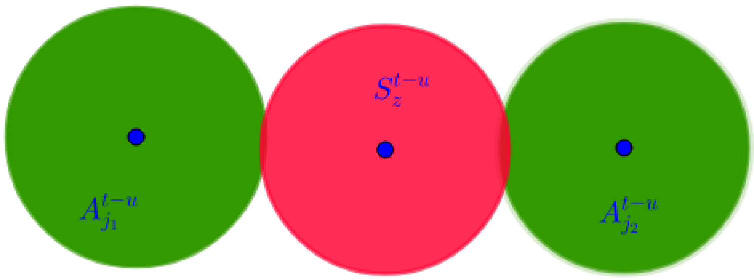

Figure 3: Worst case

Thus by induction on u, the diameter of each A_j^r is at most $3\varepsilon_r = 3^{-r}$.

We check by induction that Each A_q^{r-1} is the disjoint union of those A_j^r. The sets $(S_j^t)_{j=1}^{m_t}$ are disjoint and using that

$$\forall j \in [|1, m_t|], \quad A_j^t = S_j^t$$

then for all $j_1, j_2 \in [|1, m_t|]$, $j_1 \neq j_2$ we get

$$A_{j_1}^t \cap A_{j_2}^t = \emptyset$$

Let $u \in [|0, t-s-1|]$ and $j_1, j_2 \in [|1, m_{t-u-1}|]$. If $A_{j_1}^{t-u+1} \cap A_{j_2}^{t-u+1} \neq \emptyset$ then using (12), $\exists z_1, z_2$ such that $q(t-u, z_1) = j_1$, $q(t-u, z_2) = j_2$ and $A_{z_1}^{t-u} \cap A_{z_2}^{t-u} \neq \emptyset$. It follows from the induction property that $z_1 = z_2$ we conclude that $q(t-u, z_1) = j_1 = j_2$

We infer that, for each $r = s+1, \ldots, t$, we have

$$\bigsqcup_{j=1}^{m_r} A_j^r = \bigsqcup_{q=1}^{m_{r-1}} \bigsqcup_{j:q(t,j)=q} A_j^r = \bigsqcup_{q=1}^{m_{r-1}} A_q^{r-1}. \tag{13}$$

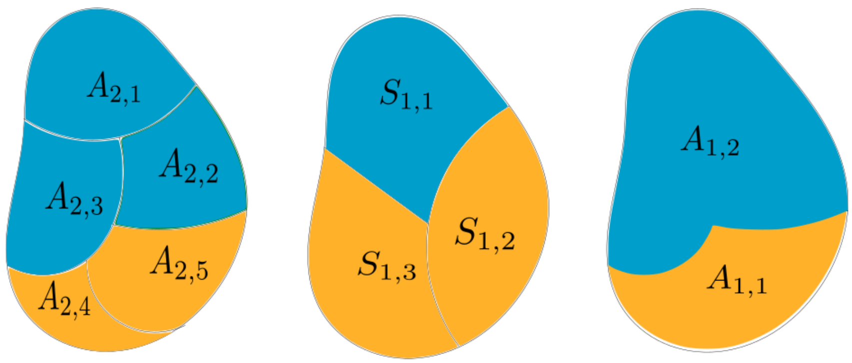

Figure 4: Illustration of a step

Step 2:. For $s \leq r \leq t$, we introduce $M_r := \sum_{j=1}^{m_r} |(\mu_N - \mu)(A_j^r)|$. The main idea of this proof is to link two partitions of the set T: we start from the thinner partition to get a union-related rough one with less pieces and slightly higher diameter. The reason is that the regularity of the distribution of mass gives a nice control on M_r in the sense that the mass is linked to a power of the diameter.

Applying Lemma 5.9 and Theorem 4.6 we have the following estimate:

$$\mathbb{E}[M_r] \leq (N^{-1}\mu(T)m_r)^{1/2} \leq \frac{2^d C_0^{1/2}}{N^{1/2}} \varepsilon_r^{-d/2}(\mu(T)\mu(T^{\varepsilon_r/4}))^{1/2} \leq 2^d C_0^{1/2} \frac{1}{\sqrt{N}} \varepsilon_r^{-d/2} \mu(T^{\varepsilon_r/4}). \tag{14}$$

Let $f \in BL(S,d), ||f||_{BL} \leq 1$ For each $r = s, \dots, t$ and $j = 1, \dots, m_r$ we choose $x_j^r \in A_j^r$ and let $f(x_j^r) = f_j^r$.

Step 3:. We introduce $I_r := \left| \sum_{j=1}^{m_r} f_j^r(\mu_N(A_j^r) - \mu(A_j^r)) \right|$. Using that $(A_j^t)_{j=1}^{m_t}$ is a partition of T we got the following estimate

$$\begin{aligned}
\left| \int_T f d(\mu_N - \mu) \right| \leq & \left| \sum_{j=1}^{m_t} \int_{A_j^t} f(x) d(\mu_N - \mu)(x) \right| \\
\leq & \left| \sum_{j=1}^{m_t} \int_{A_j^t} f(x) - f_j^t + f_j^t d(\mu_N - \mu)(x) \right| \\
\leq & I_t + \left| \sum_{j=1}^{m_t} \int_{A_j^t} \underbrace{(f(x) - f_j^t)}_{|.| \leq diam(A_j^t) \leq 3\varepsilon_t} d(\mu_N - \mu)(x) \right| \\
\leq & I_t + 3\varepsilon_t |\mu_N - \mu|(T) \\
\leq & I_t + 3\varepsilon_t (\mu_N + \mu)(T).
\end{aligned}$$

Step 4:. We pass from the point x_j^t to x_j^{t-1} to switch to the previous generation and then we use the correspondence between each partition (13) and the fact that for $r = s+1, \dots, t$,

$$|f_j^r - f_{q(r,j)}^{r-1}| \le ||f||_L diam(A_q^{r-1}) \le 3^{1-r} \tag{15}$$

This inequality puts emphasis on the control of the variation of f from the generation r to r-1.

$$\begin{aligned}
I_t =& \left| \sum_{q=1}^{m_{t-1}} \sum_{j:q(t,j)=q} (f_j^t - f_q^{t-1} + f_q^{t-1})(\mu_N - \mu)(A_j^t) \right| \\
\le& \underbrace{3^{1-t}}_{(15)} M_t + \left| \sum_{q=1}^{m_{t-1}} \sum_{j:q(t,j)=q} f_q^{t-1}(\mu_N - \mu)(A_j^t) \right| \\
\le& \varepsilon_t M_t + \underbrace{\left| \sum_{q=1}^{m_{t-1}} f_q^{t-1}(\mu_N - \mu)(A_q^{t-1}) \right|}_{I_{t-1}}.
\end{aligned}$$

We conclude that

$$I_t \le I_{t-1} + \varepsilon_t M_t.$$

Using this method inductively thanks to the control (15) from r=t down to r=s+1 and that , we obtain

$$\beta_T(\mu_N, \mu) \le 3\varepsilon_t(\mu_N(T) + \mu(T)) + I_s + \sum_{r=s+1}^{t} \varepsilon_r M_r \tag{16}$$

Thus by (14), (6) and that $I_s \le M_s$, we have

$$\mathbb{E}[\beta_T(\mu_N, \mu)] \le 6\varepsilon_t \mu(T) + \frac{2^d C_0^{1/2}}{N^{1/2}} \varepsilon_s^{-d/2} \mu(T^{\varepsilon_s/4}) + 2^d C_0^{1/2} \sum_{r=s+1}^{t} \frac{1}{N^{1/2}} \varepsilon_r^{-d/2+1} \mu(T^{\varepsilon_r/4}).$$

We use the Ahlfors regularity of μ and remind that for $0 \le s \le r$, $0 < \varepsilon_r \le \varepsilon_s$ hence $\mu(T^{\varepsilon_r/4}) \le \mu(T^{\varepsilon_s/4})$:

$$\mathbb{E}[\beta_T(\mu_N, \mu)] \le \frac{2^d C_0^{1/2} \mu(T^{\varepsilon_s/4})}{N^{1/2}} \left(\varepsilon_s^{-d/2} + \sum_{r=s+1}^{t} \varepsilon_r^{-d/2+1} \right) + 6\varepsilon_t \mu(T). \tag{17}$$

Note that:

$$\varepsilon_r^{-d/2+1} = (3^{-r+1})^{-d/2+1} = 3^{-d/2+1} \times (3^{d/2-1})^r$$

For $d > 2$, $3^{d/2-1} > 1$, we estimate the geometric sum as follows:

$$\begin{aligned}
\sum_{r=s+1}^{t} \varepsilon_r^{d/2+1} =& 3^{-d/2+1} \sum_{r=s+1}^{t} (3^{d/2-1})^r \\
=& 3^{-d/2+1} \frac{(3^{d/2-1})^{s+1} - (3^{d/2-1})^{t+1}}{1 - 3^{d/2-1}} \le 3^{-d/2+1} \frac{(3^{d/2-1})^{t+1}}{3^{d/2-1} - 1} \\
\le& 3^{d/2-1} \frac{(3^{d/2-1})^{t-1}}{3^{d/2-1} - 1} = 3^{d/2-1} \frac{\varepsilon_t^{d/2-1}}{3^{d/2-1} - 1} \\
\sum_{r=s+1}^{t} \varepsilon_r^{d/2+1} \le& \frac{3^{d/2-1}}{3^{d/2-1} - 1} \varepsilon_t^{d/2-1}.
\end{aligned}$$

Now we take (17) and we deduce for $d > 2$

$$\mathbb{E}[\beta_T(\mu_N,\mu)] \leq \frac{2^d C_0^{1/2}\mu(T^{\varepsilon_s/4})}{N^{1/2}}\left(\varepsilon_s^{-d/2} + \frac{3^{d/2-1}}{3^{d/2-1}-1}\varepsilon_t^{d/2-1}\right) + 6\varepsilon_t\mu(T).$$

Thence there is a constant $M > 0$ depending only on d and C_0 such that

$$\mathbb{E}[\beta_T(\mu_N,\mu)] \leq M\frac{\mu(T^{\varepsilon_s/4})}{N^{1/2}}\left(\varepsilon_s^{-d/2} + \varepsilon_t^{d/2-1}\right) + \varepsilon_t\mu(T).$$

Using (11)

$$\begin{aligned}\mathbb{E}[\beta_T(\mu_N,\mu)] \leq & M\frac{\mu(T^{\varepsilon_s/4})}{N^{1/2}}\left(\epsilon^{-d/2\times\frac{d-2}{d}} + \epsilon^{-d/2+1}\right) + \epsilon\mu(T)\\ \leq & M\frac{\mu(T^{\varepsilon_s/4})}{N^{1/2}}\epsilon^{-d/2+1} + \epsilon\mu(T).\end{aligned}$$

We remind that $\epsilon = N^{-1/d}$ then we have

$$N^{-\frac{1}{2}}\epsilon^{-d/2+1} = N^{-1/d}$$

and consequently, using that $\mu(T) \leq \mu(T^{\frac{\varepsilon_s}{4}})$,

$$\begin{aligned}\mathbb{E}[\beta_T(\mu_N,\mu)] \leq & M\cdot N^{-\frac{1}{d}}\mu(T^{\frac{\varepsilon_s}{4}})\\ \leq & M\cdot N^{-\frac{1}{d}}\mu(T^{\gamma_N}) \text{ since } \frac{\varepsilon_s}{4} \leq \epsilon^{\frac{d-2}{2}} = N^{-\frac{d-2}{d^2}} = \gamma_N\end{aligned}$$

Conclusion for $d > 2$ there is $M(C_0, d) > 0$ such that

$$\mathbb{E}[\beta_T(\mu_N,\mu)] \leq MN^{-1/d}\mu(T^{\gamma_N}) \text{ with } \gamma_N = N^{-\frac{d-2}{d^2}} \xrightarrow[N\to\infty]{} 0.$$

$\square$

Proposition 5.11. *Assume that μ is a 2-Ahlfors regular measure in $\mathbb{R}^n$ with regularity cnstant $C_0 > 0$. Then there exists $M > 0$ such that for $T \subset \mathbb{R}^n$ open bounded set with diameter $diam(T) < 1$,*

$$\mathbb{E}[\beta_T(\mu_N,\mu)] \leq MN^{-\frac{1}{d}}|ln(N^{-\frac{1}{2}})|\mu(T)$$

Proof. We follow the proof of Theorem 5.10 introducing the following change in the definition of ε_s in (11). Let $0 < \alpha < 1$ to be set later and let $0 \leq s \leq t$ such that

$$3^{-s} \leq \epsilon^{\alpha} \leq 3^{-s+1} =: \varepsilon_s.$$

Note that for $d > 2$,$\alpha = \frac{d-2}{d}$ was a suitable choice and proof of Theorem 5.10 is unchanged up to estimate (16) ie

$$\beta_T(\mu_N,\mu) \leq 3\varepsilon_t(\mu_N(T) + \mu(T)) + I_s + \sum_{r=s+1}^{t}\varepsilon_r M_r.$$

First, we remark that $I_s \leq ||f||_\infty \cdot M_s$ and $f \in C_c(\mathbb{R}^n$ is 1-Lipschitz with *supp* $\subset T$ implies that $||f||_\infty \leq$ *diam*(T), therefore,

$$\mathbb{E}[\beta_T(\mu_N,\mu)] \leq 6\varepsilon_t\mu(T) + diam(T)\cdot\mathbb{E}[M_s] + \sum_{r=s+1}^{t}\varepsilon_r\mathbb{E}[M_r].$$

Moreover, thanks to 14,

$$\begin{aligned}\varepsilon_r \mathbb{E}[M_r] \leq & 4C_0 \frac{1}{N^{\frac{1}{2}}} \varepsilon_r^{-1} \mu(T^{\varepsilon_r/4}) \varepsilon_r \\ \leq & 4C_0 \frac{1}{N^{\frac{1}{2}}} \mu(T^{\varepsilon_s/4})\end{aligned}$$

and then

$$\begin{aligned}\sum_{r=s+1}^{t} \varepsilon_r \mathbb{E}[M_r] \leq & 4C_0 |t-s| \frac{1}{N^{\frac{1}{2}}} \mu(T^{\varepsilon_s/4}) \\ \lesssim & t \cdot N^{-\frac{1}{2}} \mu(T^{\varepsilon_s/4}) \qquad \text{and } \frac{\epsilon}{3} \leq 3^{-t} \Longleftrightarrow ln\epsilon - ln3 \leq ln3 \Longleftrightarrow t \leq -\frac{ln\epsilon}{ln3} + 1 \\ \lesssim & |ln(N^{-\frac{1}{2}})| N^{-\frac{1}{2}} \mu(T^{\varepsilon_s/4}).\end{aligned}$$

Furthermore, reminding that $\epsilon = N^{-\frac{1}{2}}$ and $\epsilon \geq \epsilon^{\alpha}$, we have

$$\begin{aligned}diam(T) \cdot \mathbb{E}[M_s] \leq & 4C_0^{\frac{1}{2}} \frac{1}{N^{\frac{1}{2}}} \varepsilon_s^{-1} \mu(T^{\varepsilon_s/4}) \cdot diam(T) \\ \leq & 4C_0^{\frac{1}{2}} \epsilon^{1-\alpha} diam(T) \mu(T^{\varepsilon_s/4}), \\ \lesssim & \epsilon^{1-\alpha} diam(T) \cdot (diam(T) + \epsilon^{\alpha})^2 \text{ by 2- Ahlfors regularity of } \mu \\ \lesssim & \epsilon^{1+\alpha} diam(T) \left(\frac{diam(T)}{\epsilon^{\alpha}} + 1 \right)^2.\end{aligned}$$

For N large enough, $\epsilon = N^{-\frac{1}{2}} < diam(T) < 1$ and it is possible to choose $0 < \alpha < 1$ such that $diam(T) = \epsilon^{\alpha}$. Then $diam(T).\mathbb{E}[M_s] \lesssim \epsilon^{1+\alpha} diam(T) = \epsilon diam(T)^2 \lesssim \epsilon \mu(T)$ again by 2-Ahlfors regularity. We eventually conclude that

$$\mathbb{E}[\beta_T(\mu_N, \mu)] \lesssim \epsilon |ln\epsilon| \mu(T) = \frac{1}{2} N^{-\frac{1}{2}} ln(N) \mu(T).$$

$\square$

Remark 5.12. *If we directly estimate $\mathbb{E}[\beta_T(\mu_N, \mu)]$ with $I_t \leq M_t$ at the end of step 3 we would rather obtain*

$$\mathbb{E}[\beta_T(\mu_N, \mu)] \leq \mathbb{E}[M_t] + 6\varepsilon_t \mu(T)$$

And then by (14)

$$\begin{aligned}\mathbb{E}[M_t] \lesssim & \frac{1}{N^{\frac{1}{2}}} \varepsilon_t^{-\frac{d}{2}} \mu(T^{\varepsilon_t/4}) \\ \lesssim & \frac{1}{N^{\frac{1}{2}}} \epsilon^{-\frac{d}{2}} \mu(T^{\epsilon}) \\ \lesssim & \mu(T^{\epsilon}).\end{aligned}$$

On the other hand, $s \leq t$ has been chosen to ensure that

$$\begin{aligned}\mathbb{E}[M_s] \lesssim & \frac{1}{N^{\frac{1}{2}}} \varepsilon_s^{-\frac{d}{2}} \mu(T^{\varepsilon_s/4}) \\ \lesssim & \frac{1}{N^{\frac{1}{2}}} \left(\epsilon^{-\frac{d-2}{d}} \right)^{-\frac{d}{2}} \mu(T^{\varepsilon_s/4}) \\ \lesssim & \underbrace{\frac{1}{N^{\frac{1}{2}}} \epsilon^{-\frac{d-2}{2}}}_{=N^{-\frac{1}{d}}} \mu(T^{\varepsilon_s/4}).\end{aligned}$$

Step 4 takes advantage of compensation effects at larger scale ε_s rather than ε_t in order to gain the factor $N^{-\frac{1}{d}}$.

6 Rectifiability

A first interesting result combining statistic and geometric measure theory gives a pointwise convergent density estimator for regular enough measures. Such a result is stated in Theorem 6.10. As we rely on Theorem 5.10, we naturally assume Ahlfors regularity of the measure μ. Moreover, we require now some "order 1" additional regularity and ask the measure μ to be rectifiable as defined below.

6.1 Rectifiability and approximate tangent plane

We first recall some classical definitions and results on rectifiability.

Definition 6.1 (d-rectifiable set)**.** *A set S is d-rectifiable if there exists a countable family $(f_i)_{i\in\mathbb{N}}$ of Lipschitz maps from $\mathbb{R}^d$ to $\mathbb{R}^n$ such that*

$$\mathcal{H}^d\left(S\setminus\bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^d)\right)=0$$

Remark 6.2. *It means that the set S is $\mathcal{H}^d$-almost included in a d-dimensional Lipschitz graphs. As Lipschitz functions are a.e. differentiable, it is natural to try to define tangent planes a.e on rectifiable sets.*

The following definitions can be found in [3]

Definition 6.3 (Rectifiable measures)**.** *Let μ be a Radon measure in $\mathbb{R}^n$.We say that μ is d-rectifiable if there exist a d-rectifiable set S and a Borel function $\theta : S\mapsto\mathbb{R}_+$ such that $\mu=\theta\mathcal{H}^d_{|S}$.*

Now we want to analyse the local behaviour of our Radon measure μ around $x\in\mathbb{R}^n$, we use the rescaled measures for $r>0$:

$$\mu_{x,r}(B):=\mu(x+rB)\qquad\text{for } B\in Bor(\mathbb{R}^n)$$

We study then the behaviour of $r^{-d}\mu_{x,r}$ when $r\to 0$

Definition 6.4 (Approximate tangent plane)**.** *Let μ be a Radon measure and let $x\in\mathbb{R}^n$. We say that μ has approximate tangent space $P\in G_{d,n}$ with multiplicity $\theta\in\mathbb{R}_+$ at x, if $r^{-d}\mu_{x,r}$ locally weakly-$*$ converge to $\theta\mathcal{H}^d_{|P}$ in $\mathbb{R}^n$ as $r\to 0_+$. That is:*

$$\lim_{r\to 0_+} r^{-d}\int_{\mathbb{R}^n}\phi\left(\frac{y-x}{r}\right)d\mu(y)=\theta\int_P\phi(y)d\mathcal{H}^d(y)\qquad\forall\phi\in C_c(\mathbb{R}^n).$$

The measure $\theta\mathcal{H}d_{|P}$ related to the approximate tangent plan is a particular case of tangent measures as well exposed in [9].

Definition 6.5 (Tangent measure)**.** *For $\alpha\geq 0, x\in\mathbb{R}^n,\mu$ a Radon measure the α-tangent measure set of μ is:*

$$Tan_\alpha(\mu,x)=\left\{\nu\ s.t\ \frac{\mu_{x,r_i}}{r_i^\alpha}\overset{*}{\rightharpoonup}\nu, r_i\to 0^+,(r_i)_{i\in\mathbb{N}}\subset(\mathbb{R}_+^*)^{\mathbb{N}}\right\}$$

Definition 6.6 (d-dimensional density)**.** *Let μ be a Radon measure in a $\mathbb{R}^n$. The lower and upper d-dimensional densities of μ at $x\in\mathbb{R}^n$ are respectively*

$$\Theta^d_*(\mu,x)=\liminf_{\substack{r_i\to 0^+\\ r_i>0}}\frac{\mu(B(x,r_i))}{r_i^d}$$

$$\Theta^{*d}(\mu,x)=\limsup_{\substack{r_i\to 0^+\\ r_i>0}}\frac{\mu(B(x,r_i))}{r_i^d}$$

Remark 6.7. *Those densities express locally the mass behaviour around x. If they are equal we denote them $\Theta^d(\mu,x)$. There is a beautiful theorem of Marstrand proving that if they are equal then d is an integer.*

Theorem 6.8 (Rectifiable criterion for measures)**.** *Let μ be a positive Radon measure in $\mathbb{R}^n$.*

1. *If $\mu = \theta \mathcal{H}^d_{|S}$ and S is d-rectifiable, then μ admits an approximate tangent space with multiplicity $\theta(x)$ for $\mathcal{H}^d$-almost every $x \in S$. In particular $\theta(x) = \Theta^d(\mu, x)$ for $\mathcal{H}^d$-almost every $x \in S$.*

2. *If μ is concentrated on a Borel set S and admits an approximate tangent space with multiplicity $\theta(x) > 0$ for μ-almost every $x \in S$, then S is d-rectifiable and $\mu = \theta \mathcal{H}_{|S}$. In particular*

$$\exists Tan^d(\mu, x) \text{ for } \mu - a.e. x \in \Omega \implies \mu \text{ is d-rectifiable.}$$

Remark 6.9. *The Cantor "four corners" is 1-Ahlfors regular but not 1-rectifiable, even worse it is, purely unrectifiable. Properly speaking we cannot build a linear approximation around any points.*

6.2 Pointwise convergent density estimator

We have now all the ingredients needed to state a pointwise convergence result: we are able to recover the density θ a.e. in S from the knowledge of the empirical measures μ_N, $N \in \mathbb{N}^*$ assuming both d-rectifiability and d-Ahlfors regularity of $\mu = \theta \mathcal{H}^d_{|S}$.

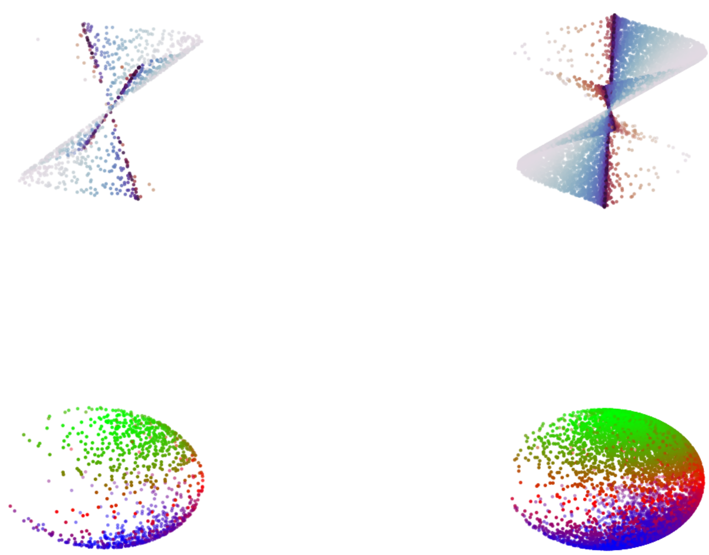

Figure 5: Examples of non uniformly sampled point clouds

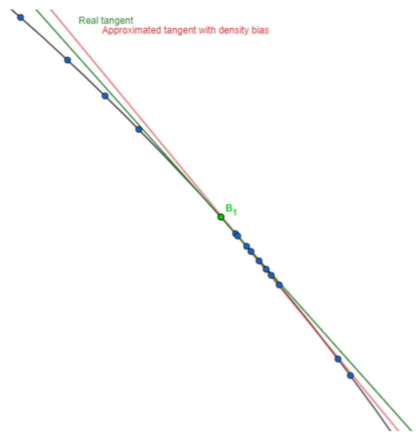

Figure 6: Density bias effect on tangent approximation

For instance, we mention that in order to rebuild the tangent planes it is useful to find the density bias. As the figure above illustrates, if one gives to all the points the same weight then the tangent planes will not be consistent with the surface.

Theorem 6.10. *Let $d \in \mathbb{N}, d > 2$ and let μ be a d-Ahlfors regular and d-rectifiable measure $\mu = \theta \mathcal{H}^d_{|S}$ where $\theta : S \mapsto \mathbb{R}^*_+$ is a Borel function such that $\int_S \theta \mathcal{H} d_{|S}(x) = 1$. Let μ_N be the empirical measure associated with μ. Let $\eta : \mathbb{R} \mapsto \mathbb{R}_+$ be a Lipschitz even function such that $supp(\eta) \subset [-1, 1]$ and $||\eta||_{BL} \leq 1$. Denote $c_\eta = d\omega_d \int_{r=0}^1 \eta(r) r^{d-1} dr$ where $\omega_d = \mathcal{L}^d(B(0,1))$ is the volume of the unite ball in $\mathbb{R}^d$. Then for $\mathcal{H}^d-$ a.e. $y \in S$, for all $\delta_N \geq N^{-\frac{(d-2)}{d^2}}$*

$$\mathbb{E}\left[\left|\frac{1}{c_\eta \delta^d}\int_{\mathbb{R}^n} \eta\left(\frac{|x-y|}{\delta}\right) d\mu_N(x) - \theta(y)\right|\right] \lesssim \underbrace{\frac{N^{-1/d}}{\delta_N} + \left|\frac{1}{c_\eta \delta_N^d}\int_{\mathbb{R}^n} \eta\left(\frac{|x-y|}{\delta}\right) d\mu(x) - \theta(y)\right|}_{\xrightarrow[N\to+\infty]{} 0 \text{ as soon as } \delta_N \xrightarrow[N\to+\infty]{} 0} \tag{18}$$

Proof.

Step 1: Let $y \in S$ and $0 < \delta < 1$, then

$$\left\|\eta\left(\frac{|\cdot - y|}{\delta}\right)\right\|_L \leq \delta^{-1}||\eta||_L$$

We can therefore apply Theorem (5.10) to $\delta\eta(\frac{\cdot - y}{\delta})$ in $T = B(y, \delta) : \exists M > 0$ such that

$$\mathbb{E}\left[\left|\int_{B(y,\delta)} \delta\eta\left(\frac{|x-y|}{\delta}\right) d(\mu_N - \mu)(x)\right|\right] \leq M\mu(B(y, \delta + \gamma_N))N^{-\frac{1}{d}}$$

Then we use the d-Ahlfors regularity of μ

$$\leq MC_0(\delta + \gamma_N)^d N^{-\frac{1}{d}}$$

We recall that $\gamma_N = N^{-\frac{d-2}{d^2}}$ and we obtain

$$\mathbb{E}\left[\frac{1}{\delta^d}\left|\int_{B(y,\delta)} \eta\left(\frac{|x-y|}{\delta}\right) d(\mu_N - \mu)(x)\right|\right] \leq MC_0(\delta + N^{-\frac{d-2}{d^2}})^d \frac{1}{\delta^{d+1}} N^{-1/d}$$

$$= MC_0 \frac{N^{-\frac{1}{d}}}{\delta}\left(1 + \frac{N^{-\frac{1}{d}}}{\delta} \cdot N^{\frac{2}{d^2}}\right)^d$$

If we want that the expectation tends to 0 when $N \to +\infty$, we have to choose a good $\delta = \delta_N$ depending on N that will converge to 0 to ensure that

$$\epsilon_N = \max\left(1, \frac{N^{-\frac{1}{d}}}{\delta_N} \cdot N^{\frac{2}{d^2}}\right)^d \frac{N^{-\frac{1}{d}}}{\delta_N} \xrightarrow[N\to+\infty]{} 0.$$

- If $\frac{N^{-\frac{1}{d}}}{\delta_N} \cdot N^{\frac{2}{d^2}} \le 1 \Longleftrightarrow \delta_N \ge N^{-\frac{1}{d}} N^{\frac{2}{d^2}}$ then $\epsilon_N \le \frac{N^{-\frac{1}{d}}}{\delta_N} \le N^{-\frac{2}{d^2}} \xrightarrow[N\to+\infty]{} 0$.
- Otherwise, $\delta_N \le N^{-\frac{1}{d}} N^{\frac{2}{d^2}}$ and $\epsilon_N = \frac{N^{-\frac{1}{d}}}{\delta_N} \cdot \frac{N^{-1} N^{\frac{2}{d}}}{\delta_N^d} = \frac{N^{-1+\frac{1}{d}}}{\delta_N^{d+1}}$. Therefore, $\epsilon_N \xrightarrow[N\to+\infty]{} 0 \Longleftrightarrow \frac{N^{-1+\frac{1}{d}}}{\delta_N^{d+1}} \xrightarrow[N\to+\infty]{} 0$. Note that $\delta_N \le N^{-\frac{1}{d}} \cdot N^{\frac{2}{d^2}}$ implies that $\delta_N^{d+1} \le N^{-\frac{d-2}{d^2}(d+1)} \implies \epsilon_N \ge N^{-1+\frac{1}{d}} \cdot N^{\frac{(d-2)(d+1)}{d^2}} = N^{-\frac{2}{d^2}}$. Choosing $\delta_N \le N^{-\frac{1}{d}} \cdot N^{\frac{2}{d^2}}$ hence leads to a slower convergence of ϵ_N.

<u>**Step 2:**</u> We now use the rectifiability of μ implying that $\mathcal{H}^d$ a.e. in S, μ has an approximate tangent plane. Let $y \in S$ be such a point, then (see Definition 6.4)

$$\forall \phi \in C_c(\mathbb{R}^n), \quad \frac{1}{\delta^d} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\delta}\right) d\mu(x) \xrightarrow[\delta\to 0]{} \theta(y) \int_{T_y S} \phi(x) d\mathcal{H}^d(x).$$

In particular, $\frac{1}{\delta^d} \int_{\mathbb{R}^n} \eta\left(\frac{|x-y|}{\delta}\right) d\mu(x) \xrightarrow[\delta\to 0]{} \theta(y) \int_{T_y S} \eta(|x|) d\mathcal{H}^d(x)$. Moreover we integrate along the spheres $\mathbb{S}(0,r), 0 < r < 1$:

$$\begin{aligned} \int_{T_y S} \eta(|x|) d\mathcal{H}^d &= \int_{r=0}^1 \int_{x\in\mathbb{S}(0,r)} \eta(|x|) d\mathcal{H}^{d-1}(x) dr \\ &= \int_{r=0}^1 \eta(r) \mathcal{H}^{d-1}(\mathbb{S}(0,1)) r^{d-1} dr \\ &= d\omega_d \int_0^1 \eta(r) r^{d-1} dr = C_\eta, \qquad \text{because } \mathcal{H}^{d-1}(\mathbb{S}(0,1)) = d\mathcal{L}^d(B(0,1)). \end{aligned}$$

We conclude the proof applying Step 1 and Step 2.

□

Remark 6.11. *The convergence in 6.10 is the result of a competition between two terms, it would be important to quantify convergence of the second term in the right hand side of* (18) *in order to optimize the choice of $\delta_N \to 0$. Uniform rectifiability of μ might be a good assumption for this purpose.*

7 Introduction to varifold theory

In Theorem 6.10, we have exposed a result allowing to recover the density θ of a measure $\mu = \theta\mathcal{H}^d_{|S}$. Such a result paves the way to the question of recovering the order 1 structure of S: $\{(x, T_xS) : x \in \mathbb{S}\}$ that is exactly the support of the so called "varifold" V(S) naturally associated with S. Recovering the varifold V(S) is the next step we intend to tackle. However we will restrict ourselves to introduce basic definitions on varifolds in this last section.

Definition 7.1 (Grassmanian manifold)**.** *We denote $G_{d,n} = \{$vector subspaces of dimension d in $\mathbb{R}^n\}$ the Grassmanian manifold of d-dimensional vector subspaces of $\mathbb{R}^n$. For $T, P \in G_{d,n}$ we denote Π_T, Π_P the associated orthogonal projectors and we consider the metric $d(T,P) = |||\Pi_T - \Pi_P|||$ where $|||\cdot|||$ is operator norm of linear endomorphisms of $\mathbb{R}^n$.*

Definition 7.2 (General d-varifold)**.** *A d-varifold is a Radon measure on $\mathbb{R}^N \times G_{d,n}$.*

Definition 7.3 (Mass)**.** *The mass of a general varifold V is the Radon measure defined by $||V||(B) = V(\pi^{-1}(B))$ for every $B \subset \Omega$ Borel set, with $\pi : \Omega \times G_{d,n} \to \Omega$ defined by $\pi(x,S) = x$.*

A first example of varifold is a Dirac mass δ_{x_0,P_0} in $\mathbb{R}n \times G_{d,n}$ for a given $(x_0, P_0) \in \mathbb{R}^n \times G_{d,n}$ more generally a weighted sum of such Dirac masses that we will refer to as point cloud varifold

Definition 7.4 (Point cloud varifold)**.** *Given a finite set of points $\{x_i\}_{i=1}^N \subset \mathbb{R}^n$, masses (weights) $\{m_i\}_{i=1}^N \subset \mathbb{R}_+^*$ and directions $\{P_i\}_{i=1}^N \subset G_{d,n}$, we associate the d-varifold*

$$V = \sum_{i=1}^N m_i \delta_{(x_i,P_i)} \text{ and in this case } ||V|| = \sum_{i=1}^N m_i \delta_{x_i}.$$

Remark 7.5. *P_i can be any d-plane in $G_{d,n}$. If we sample a submanifold M, then $\{P_i\}_{i=1}^N$ can be thought as tangent planes $T_{x_i}M$.*

We can observe the point cloud varifold through its action on functions $\phi \in C_c(\mathbb{R}^n \times G_{d,n})$ as follows:

$$\int_{\mathbb{R}^n \times G_{d,n}} \phi dV = \sum_{i=1}^N m_i \phi(x_i, P_i).$$

Definition 7.6 (Rectifiable d-varifold)**.** *Let S be a d-rectifiable set in $\mathbb{R}^n$ and θ be a Borel such that $\theta > 0$ $\mathcal{H}^d_{|S}$-almost everywhere. A rectifiable d-varifold $V = v(S,\theta)$ in $\mathbb{R}^n$ is a the Radon measure $V = \theta\mathcal{H}^d_{|S} \times \delta_{T_xS}$ on $\mathbb{R}^n \times G_{d,n}$ of the form i.e.*

$$\int_{\mathbb{R}^n \times G_{d,n}} \phi(x,T) dV(x,T) = \int_S \phi(x, T_xS)\theta(x) d\mathcal{H}^d(x) \quad \forall \phi \in C_c^0(\mathbb{R}^n \times G_{d,n})$$

where T_xS is the approximate tangent space at x which exists $\mathcal{H}^d_{|S}$–almost everywhere in S. The function θ is called the multiplicity of the rectifiable varifold.

Varifolds are provided with a notion of generalized mean curvature relying on the so called first variation (see [1]) that extends the classical notion of mean curvature in a distributional way. For the purpose of defining the first variation of a d-varifold we introduce the following differential operators: let $P \in G_{d,n}$, Π_P be the orthogonal projection onto P and $(\tau_1, \ldots, \tau_d)$ be an orthonormal basis of P, let $X = (X_1, \ldots, X_n) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field of class C^1, $\phi \in C^1(\mathbb{R}^n)$ and $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$, then

$$\nabla^P \phi = \Pi_P(\nabla\phi) \text{ and } div_P X = \sum_{i=1}^n \Pi_P(\nabla X_i) \cdot e_i = \sum_{i=1}^d DX\tau_i \cdot \tau_i.$$

Now we can define the first variation.

Definition 7.7 (First variation of a varifold)**.** *[1] The first variation of a d-varifold V in $\mathbb{R}^n$ is the distribution of order 1*

$$\delta V : C_c^1(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$$
$$X \mapsto \int_{\mathbb{R}^n \times G_{d,n}} div_P X(x) dV(x,P)$$

If $S \subset \mathbb{R}^n$ is a closed C^2 d-submanifold and $V = \mathcal{H}^d_{|S} \otimes \delta_{T_x S}$ is the smooth varifold associated with S then

$$\delta V(X) = \int_{\mathbb{R}^n} div_{T_x S} X(x) d\mathcal{H}^d_{|S}$$

Definition 7.8 (Generalized mean curvature)**.** *If δV is an order 0 distribution then we can see it as a Radon measure thanks to the Riesz theorem. Moreover the Radon Nikodym decomposition of δV related to the mass $||V||$ permits to decompose δV as a vector field $\frac{\delta V}{||V||} \in L^1_{loc}(||V||)$ and a Radon measure $(\delta V)_s$ singular with respect to $||V||$ as follows:*

$$\delta V = \frac{\delta V}{||V||}(x)||V|| + (\delta V)_s.$$

The vector field $H := -\frac{\delta V}{||V||}$ is the generalized mean curvature vector. Note that for a varifold associated to a C^2 closed manifold it coincides with the classical mean curvature.

The first variation is a way of catching the "tangent space" variations. For example we can detect some corners, borders and intersections in the shape S.

Example 7.9. *Let V be the 1-varifold V in $\mathbb{R}^2$ naturally associated to the corner $[0,1] \times \{0\} \cup \{0\} \times [0,1]$ with the density $\theta((x,y)) = 1 + x$. We have a simple parametrization with the natural orthonormal basis (e_1, e_2). The point (0,0) is an intersection and the density is more important on the abscissa axis. Let $X \in C_c^1(\mathbb{R}^2, \mathbb{R})$ we have*

$$\delta V(X) = \int_0^1 \frac{d}{dt} X(te_1) \cdot e_1 \theta((t,0))dt + \int_0^1 \frac{d}{dt} X(te_2) \cdot e_2 \theta((0,t))dt.$$

We integrate by parts and therefore

$$\delta V(X) = \theta((1,0))X((1,0)).e_1 - \theta(0)X(0).e_1 + \int_0^1 X(t.e_1).e_1 \frac{d}{dt}\theta((t,0))dt + \theta((0,1))X((0,1)).e_2 - \theta(0)X(0).e_2$$

Then we do a second integration by part and replace the value of θ

$$\delta V = \underbrace{\delta_{(1,0)} \cdot e_1 + \delta_{(0,1)} \cdot e_2}_{border} - \underbrace{\delta_0 \cdot (e_2 + e_1)}_{corner} + \underbrace{e_1 \mathcal{H}^1_{|[0,1]\times\{0\}}}_{density \quad impact}$$

At first sight δV seemed to be an order 1 distribution but here it is more regular:it is a 0 distribution.

We underline that the non constant density θ induces a tangential contribution along the horizontal axis in the first variation when we would expect 0 since the mean curvature of a straight line vanishes.

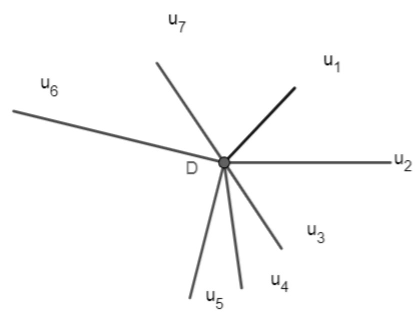

Figure 7: 6-intersections

Remark 7.10. *If we have an intersection composed of* $k \in \mathbb{N}^*$ *directions* $u_1, \ldots, u_k$ *such that* $u_1 + \cdots + u_k = 0$ *then we cannot see through the first variation because they balance, as in the case of a triple point junction.*

8 Conclusion and perspectives

In this project, we proved the mean convergence of empirical measures μ_N towards an Ahlfors regular measure μ in terms of a localized bounded Lipschitz distance (Theorem 5.10) adapting a proof from [7]. Building on this result, we obtain a pointwisely convergent density estimator in Theorem 6.10. We underline that we work in a non-smooth framework, our regularity assumptions being Ahlfors regularity and rectifiability of the measure μ.

Let us conclude with some perspectives: can we estimate $\mathcal{H}^d_{|S}$ thanks to the estimate on the pointwise density? Moreover are there insightful choices of η regarding the numerical applications? Can we prove estimates for other distances like Wasserstein distances (see [11]) and Prokhorov distance? What about estimating the varifold structure $V = \mathcal{H}^d_{|S} \times \delta_{T_x S}$ from μ_N? Alongside, discrete equivalent of differential operators like the Laplace Beltrami (see [11]) are full of potential applications and accessible from the varifold structure. In fact, those operators can deliver global features of our surface, for instance in view of classifying them. And last but not least, how robust is our model considering that point clouds can be noised through the sampling process?

References

[1] William K. Allard, On the first variation of a varifold, Annals of Mathematics **95** (1972), no. 3, 417–491.

[2] F.J. Almgren, K.A. Brakke, and J.M. Sullivan, Plateau's problem: An invitation to varifold geometry, Student mathematical library, W.A. Benjamin, 1966.

[3] L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, 2000.

[4] Blanche Buet, Gian Paolo Leonardi, and Simon Masnou, A varifold approach to surface approximation, Archive for Rational Mechanics and Analysis **226** (2017), no. 2, 639–694.

[5] Blanche Buet and Martin Rumpf, Mean curvature motion of point cloud varifolds, 37 pages, 10 figures, November 2020.

[6] Gaston Darboux, Mémoire sur les fonctions discontinues, Annales scientifiques de l'École Normale Supérieure **2e série, 4** (1875), 57–112 (fr).

[7] R. M. Dudley, The Speed of Mean Glivenko-Cantelli Convergence, The Annals of Mathematical Statistics **40** (1969), no. 1, 40 – 50.

[8] L.C. Evans and R.F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, Taylor & Francis, 1991.

[9] Camillo De Lellis, Rectifiable sets, densities and tangent measures, European Mathematical Society, 2008.

[10] Pertti Mattila, Geometry of sets and measures in euclidean spaces: Fractals and rectifiability, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[11] Nicolas Garcia Trillos, Moritz Gerlach, Matthias Hein, and Dejan Slepcev, Error estimates for spectral convergence of the graph laplacian on random geometric graphs towards the laplace–beltrami operator, 2018.