TD 4 : Hausdorff Measure and Cantor Sets

We recall the definition of the d–dimensional Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$: let $E \subset \mathbb{R}^n$, then for any $\delta > 0$ we define

$$\mathcal{H}^d_\delta(E) = \inf \left\{ \sum_{i\in I} (\mathrm{diam}(E_i))^d \ : \ \mathrm{diam}(E_i) \leq \delta,\ \{E_i\}_{i\in I} \textit{ is a countable (or finite) cover of } E \right\}$$

and then $\mathcal{H}^d(E) = \lim_{\delta\downarrow 0} \mathcal{H}^d_\delta(E) = \sup_{\delta>0} \mathcal{H}^d_\delta(E)$ *since* $\delta \mapsto \mathcal{H}^d_\delta(E)$ *is non increasing.*

Exercise 1.— Let μ be a positive Radon measure in $\mathbb{R}^n$, $0 < t < +\infty$ and $A \subset \mathbb{R}^n$ be a Borel set. Then, prove the following facts:

$$\theta^*_d(\mu, x) \geq t,\ \forall x \in A \quad \Rightarrow \quad \mu(A) \geq t\mathcal{H}^d(A)$$
$$\theta^*_d(\mu, x) \leq t,\ \forall x \in A \quad \Rightarrow \quad \mu(A) \leq 2^d t\mathcal{H}^d(A),$$

with $\theta^*_d(\mu, x) = \limsup_{r\to 0_+} \dfrac{\mu(B(x, r))}{(2r)^d}$.

Exercise 2.— *Some Cantor sets.*

We fix $0 < \lambda < \lambda' < \frac{1}{2}$ and $d \in]0, 1[$ such that $2\lambda^d = 1$.
The aim of the first part of this exercise(up to question 4) is to define a Cantor set $K \subset [0, 1]$ such that

$$0 < \mathcal{H}^d(K) < +\infty \quad \text{and} \quad \forall x \in K,\ \theta^d_*(K, x) = \liminf_{r\to 0_+} \frac{\mathcal{H}^d(K \cap B(x, r))}{(2r)^d} = 0\,.$$

Let us provide the construction of the Cantor set K associated with the sequence $(r_n)_n$ defined in question 1. The process is based on the following fundamental operation : given a segment $[a, b] \subset [0, 1]$ and a length ℓ satisfying $0 < 2\ell < b - a$, we devide it into two segments of equal length ℓ by removing a segment of length $b - a - 2\ell$ in the middle

$$\mathcal{C}([a, b], \ell) := \{[a, a + \ell], [b - \ell, b]\}$$

We construct iteratively a family of sub-segments in $[0, 1]$ starting with the family $\mathcal{C}_0 = \{[0, 1]\}$ ($[0, 1]$ of length $r_0 = 1$) and for $n \in \mathbb{N}$,

$$\mathcal{C}_{n+1} = \bigcup_{I\in\mathcal{C}_n} \mathcal{C}(I, r_{n+1}) \qquad \text{(requires } r_n - 2r_{n+1} > 0)\,.$$

In words : each segment from generation n produces 2 *segments of equal lengths* $\ell = r_{n+1}$. We can now define the compact set K_n as the union of the 2^n segments in $\mathcal{C}_n$. The resulting sequence $(K_n)_n$ is non increasing and its intersection is the compact set K :

$$\forall n \in \mathbb{N},\ K_n = \bigcup_{I\in\mathcal{C}_n} I \quad \text{et} \quad K = \bigcap_{n\in\mathbb{N}} K_n\,.$$

Notation : it will be convenient to enumerate the segments in each $\mathcal{C}_n$ at some point, hence we introduce the following notation, for $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n - 1\}$,

$$I_{n,k} \in \mathcal{C}_n \quad \text{is the } k^{th}\text{segment, when ordered from 0 to 1 so that } I_{n+1,2k}\,,\ I_{n+1,2k+1} \subset I_n\,.$$

Remark : taking $r_n = 3^{-n}$ produces the "usual" ternary Cantor set.

1. Define a sequence $(r_n)_n$ satisfying the following properties :

$$r_0 = 1, \text{ and } \forall n \in \mathbb{N}, \begin{cases} r_{n+1} & \leq \lambda' r_n \\ r_n & \geq \lambda^n \end{cases} \tag{1}$$

and moreover, there exists a subsequence $(n_j)_j$ such that

$$\lim_{j\to\infty} \frac{r_{n_j}}{\lambda^{n_j}} = 1, \tag{2}$$

$$\lim_{j\to\infty} \frac{r_{n_j}}{r_{n_j-1}} = 0. \tag{3}$$

We can rather work with $a_n = \ln(\lambda^{-n} r_n)$.

2. Construct a probability measure μ defined on $[0,1]$ such that for all $n \in \mathbb{N}$ and $k = 0 \dots 2^n - 1$, $\mu(I_{n,k}) = 2^{-n}$.

3. We now show that the measures μ and $\mathcal{H}^d_{|K}$ are equivalent and in particular, $0 < \mathcal{H}^d(K) < \infty$.

   (a) Let I be a segment such that $\overset{\circ}{I} \cap K \neq \emptyset$, check that there exist $n \in \mathbb{N}$, $k \in \{0, \dots, 2^n - 1\}$ and 4 "consecutive" intervals such that

   $$I_{n,k} \subset I \quad \text{et} \quad I \cap K \subset I_{n,k_1} \cup I_{n,k_2} \cup I_{n,k_3} \cup I_{n,k_4}.$$

   Let $x \in K$, $r > 0$ and $I = [x - r, x + r]$ then check that we have in addition $r_n \leq 2r$.

   (b) Show that for all n, k, $\mathcal{H}^d(K \cap I_{n,k}) \leq \mu(I_{n,k})$ (and thus $\mathcal{H}^d(K) < +\infty$).

   (c) Show that $\mu \leq C\mathcal{H}^d_{|K}$ applying exercise 1 (and in particular $\mathcal{H}^d(K) > 0$).

   (d) Conclude.

4. We have proved that K has Hausdorff dimension d. Let us show that its lower d–dimensional density $\theta^d_*(K, x) = 0$ at any point x of K. Let

$$v(r) = \frac{1}{r^d} \sup \{\mu(I) \ : \ |I| = r, \ I \text{ closed interval } \} ,$$

using (3) show that $\liminf_{r\to 0_+} v(r) = 0$ and conclude.

5. As previously, we define two Cantor sets K and K' associated with two sequences $(r_n)_n$ and $(s_n)_n$ such that $0 < \mathcal{H}^d(K) < +\infty$ and $0 < \mathcal{H}^d(K') < +\infty$. We want to choose them so that

$$\mathcal{H}^{2d}(K \times K') = +\infty.$$

   (a) Exhibit two sequences $(r_n)_n$ and $(s_n)_n$ as in first point and such that in addition, if $a_n = \ln(\lambda^{-n} r_n)$ and $b_n = \ln(\lambda^{-n} s_n)$, then

   $$c_n := \max(a_n, b_n) \xrightarrow[n\to\infty]{} +\infty.$$

   (b) Let μ' denote the natural probability measure associated with K'. For C closed square of side length r such that $\overset{\circ}{C} \cap K \times K' \neq \emptyset$, show that

   $$\lim_{r\to 0_+} \frac{1}{r^{2d}} (\mu \times \mu')(C) = 0.$$

   (c) Infer that $\mathcal{H}^{2d}(K \times K') = +\infty$.

6. What is the Hausdorff dimension of $K \times K \subset \mathbb{R}^2$?