# TD 5 : BMO

**Exercise 1.**— $\ln|x| \in \mathrm{BMO}(\mathbb{R}^n)$.
We recall the following characterization of $BMO$–functions:

**Proposition.** *Let $f \in \mathrm{L}^1_{loc}(\mathbb{R}^n)$. $f \in \mathrm{BMO}$ if and only if there exists $M > 0$, and, for every ball $B$, there exists $C_B \in \mathbb{R}$ (or $\mathbb{C}$ if $f$ is complex valued) such that*

$$\frac{1}{|B|}\int_B |f(x) - C_B|dx \leq M. \tag{1}$$

1. Let us show that $\ln|x| \in \mathrm{BMO}(\mathbb{R}^n)$. Let $B = B(a, r) \subset \mathbb{R}^n$ be an open ball.
   (a) Show that we can reduce the problem to the case where $B$ has radius 1.
   (b) Conclude *(we can for instance distinguish $|a| \leq 2$ and $|a| > 2$).*
2. Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \alpha \leq 1$, show that $|f|^\alpha \in \mathrm{BMO}(\mathbb{R}^n)$.
3. Let $f(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} \ln(x) & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$.
   Check that $f,\ g \notin \mathrm{BMO}(\mathbb{R})$.

**Exercise 2.**— BMO, *p–norms are equivalent.*
Let $1 \leq p < +\infty$, $f \in \mathrm{L}^1_{loc}(\mathbb{R}^n)$ and define

$$\|f\|_{\mathrm{BMO},p} := \sup_B \left\{ \frac{1}{|B|}\int_B |f(x) - m_B f|^p dx \right\}^{\frac{1}{p}},$$

where the sup is taken over open balls and $m_B f = \dfrac{1}{|B|}\displaystyle\int_B f$.

Show that $\|\cdot\|_{\mathrm{BMO},p}$ are equivalent.