TD 7 : Whitney curve and extension

The purpose of this exercise is to show the existence of a (not rectifiable) curve Γ in $\mathbb{R}^2$ and a function $f \in C^1(\mathbb{R}^2)$ being not constant along Γ, but such that Γ lies in the set of critical points for f.

Construction of Whitney curve. Let Q be the unit square in $\mathbb{R}^2$. Denote then by Q_0, Q_1, Q_2 and Q_3 four squares of side $1/3$ lying inside Q in a cyclical order, at a distance $1/12$ from the boundary of Q as indicated on Figure 1.

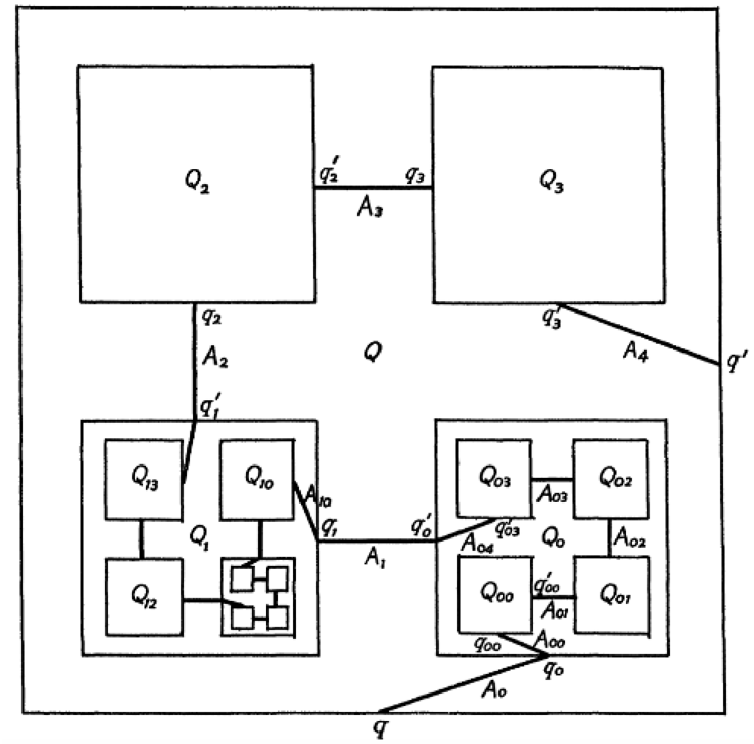

Figure 1: Construction of the Whitney curve

Let q and q' be the mid-points of the sides of Q lying along Q_0, Q_1 and Q_0, Q_3 respectively. Let also, for $i \in \{0, 1, 2, 3\}$, q_i and q_i' be the centers of two adjacent sides of Q_i numbered in such a way that q_{i-1}' faces q_i ($i \in \{1, 2, 3\}$) and such that q_0 and q_3' are near q and q' respectively. Denote then by A_0 the line segment joining q and q_0, by A_i ($i \in \{1, 2, 3\}$) the line segment joining q_{i-1}' and q_i, and by A_4 the line segment joining q_3' and q'.

Assume now that squares $Q_{i_1,\dots,i_k}$, points $q_{i_1,\dots,i_k}$, $q_{i_1,\dots,i_k}'$ and lines $A_{j_1,\dots,j_k}$ have been constructed for $k < n$ and $i_1, \dots, i_k \in \{0, 1, 2, 3\}$, $j_1, \dots, j_{k-1} \in \{0, 1, 2, 3\}$ and $j_k \in \{0, 1, 2, 3, 4\}$. Shrinking $Q_{i_1,\dots,i_{n-2}}$ by a factor $1/3$, turning and flipping it if necessary, we may place it inside $Q_{i_1,\dots,i_{n-1}}$ in such a way that $q_{i_1,\dots,i_{n-2}}$ and $q_{i_1,\dots,i_{n-2}}'$ are sent to $q_{i_1,\dots,i_{n-1}}$ and $q_{i_1,\dots,i_{n-1}}'$ respectively allowing us, by repeating the pattern inside $Q_{i_1,\dots,i_{n-2}}$, to define squares $Q_{i_1,\dots,i_n}$, points $q_{i_1,\dots,i_n}$ and $q_{i_1,\dots,i_n}'$ and lines $A_{i_1,\dots,i_{n-1},j_n}$ for $i_n \in \{0, 1, 2, 3\}$ and $j_n \in \{0, 1, 2, 3, 4\}$ (see Figure 1 for the first iterations of this construction). Denote finally, given a sequence $\iota = (i_k)_{k\in\mathbb{N}^*} \in \{0, 1, 2, 3\}^{\mathbb{N}^*}$, by Q_ι the unique point satisfying:

$$\{Q_\iota\} = \bigcap_{k\in\mathbb{N}^*} Q_{i_1,\dots,i_k}.$$

One lets Γ denote the union of all line segments $A_{j_1,\dots,j_n}$ ($n \in \mathbb{N}^*$, $j_k \in \{0,1,2,3\}$, $1 \leqslant k \leqslant n-1$, $j_n \in \{0,1,2,3,4\}$) together with all points of the form q_ι, $\iota \in \{0,1,2,3\}^{\mathbb{N}^*}$. One can parametrize Γ by a continuous homeomorphism $\gamma : [0,1] \to \Gamma$ by letting γ send the interval $I_{j_1,\dots,j_n}$ onto $A_{j_1,\dots,j_n}$ for all $j_1,\dots,j_{n-1} \in \{0,1,2,3\}$ and $j_n \in \{0,1,2,3,4\}$, where one defines:

$$I_{j_1,\dots,j_k} := \left[\left(\sum_{k=1}^{n-1} \frac{2j_k+1}{9^k}\right) + \frac{2j_n}{9^n}, \sum_{k=1}^{n} \frac{2j_k+1}{9^k}\right],$$

and by setting, for $\iota = (i_k)_{k\in\mathbb{N}^*} \in \{0,1,2,3\}^{\mathbb{N}^*}$:

$$\gamma\left(\sum_{k=1}^{\infty} \frac{2i_k+1}{9^k}\right) := Q_\iota.$$

Exercise 1.—

We now define a function $f : \Gamma \to \mathbb{R}$ by letting:

$$f(x,y) := \begin{cases} \sum_{k=1}^{n} \frac{j_k}{4^k} & \text{if } (x,y) \in A_{j_1,\dots,j_n} \text{ for some } j_1,\dots,j_{n-1} \in \{0,1,2,3\} \text{ and } j_n \in \{0,1,2,3,4\}, \\ \sum_{k=1}^{\infty} \frac{i_k}{4^k} & \text{if } (x,y) = Q_\iota \text{ for some } \iota = (i_k)_{k\in\mathbb{N}^*} \in \{0,1,2,3\}^{\mathbb{N}^*}. \end{cases}$$

1. Show that if (x,y) and (x',y') are two points in Γ located inside $Q_{i_1,\dots,i_n}$ for some $i_1,\dots,i_n \in \{0,1,2,3\}$, then one has:
$$|f(x,y) - f(x',y')| \leqslant \frac{1}{4^n}.$$

2. Show that if (x,y) and (x',y') are two distinct points in Γ separated by some point Q_ι for a $\iota = (i_k)_{k\in\mathbb{N}^*} \in \{0,1,2,3\}^{\mathbb{N}^*}$, and if $Q_{i_1,\dots,i_n}$ is the smallest square containing both of them, then one has:
$$|(x,y) - (x',y')| > \frac{1}{12} \cdot \frac{1}{3^{n+1}}.$$

3. Conclude that f is α–Hölder and explicit $\alpha > 1$.

4. Conclude, using Whitney's extension theorem, that there exists a C^1 function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfying $g_{|\Gamma} = f$ and, for all $(x,y) \in \Gamma$:
$$\nabla g(x,y) = 0.$$