LECTURE NOTES: INTRODUCTION TO MICROLOCAL ANALYSIS WITH APPLICATIONS

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Course Summary

Microlocal analysis studies singularities of distributions in phase space, by describing the behaviour of the singularity in both position and direction. It is a part of the field of partial differential equations, created by Hörmander, Kohn, Nirenberg and others in 1960s and 1970s, and is used to study questions such as solvability, regularity and propagation of singularities of solutions of PDEs. To name a few other classical applications, it can be used to study asymptotics of eigenfunctions for elliptic operators, trace formulas and inverse problems.

There have been recent exciting advances in the field and many applications to geometry and dynamical systems. These include dynamical zeta functions and injectivity properties of X-ray (geodesic) transforms with applications to rigidity questions in geometry. The study was originally designed for linear PDEs, but there are more recent techniques for studying nonlinear problems through the paradifferential calculus.

In this course, we study the fundamentals of microlocal analysis. We aim to develop main tools and provide some of the recent striking applications. A sketch plan of the course is given below.

Contents

1. Distributions and Fourier transform (recap). Symbol classes and oscillatory integrals. Fourier integral operators. Non-stationary and stationary phase lemma. (2-3 lectures)
2. Pseudodifferential operators (PDO). Compositions, changes of coordinates, calculus of PDOs. PDOs on manifolds. (2 lectures)
3. Elliptic regularity. L^2-continuity. Sobolev spaces and PDOs. (1-2 lectures)
4. Wavefront set. Products, pullbacks of distributions. Propagation of singularities. (2 lectures)
5. Possible applications: Egorov's theorem. Weyl's law and Quantum ergodicity. Existence of resonances for uniformly hyperbolic (Anosov) diffeomorphisms/flows via adapted anisotropic Sobolev spaces. Inverse problems. (5-6 lectures)

Note below: the applications part of the course will change according to the time constraints.

Pre-requisites

Elementary theory of distributions and Fourier transforms (these will be recalled briefly). Basics of functional analysis and differential geometry.

Literature

There will be lecture notes provided by the lecturer. However, the following books provide with complementary reading:

1. F.G. Friedlander, *Introduction to the theory of distributions*, Second edition. With additional material by M. Joshi, Cambridge University Press, Cambridge, 1998.
2. A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators. An introduction*, London Mathematical Society Lecture Note Series, 196. Cambridge University Press, Cambridge, 1994.

3. M. A. Shubin, *Pseudodifferential operators and spectral theory*, Translated from the 1978 Russian original by Stig I. Andersson, Second edition, Springer-Verlag, Berlin, 2001.
4. M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI, 2012.

There are also a few relevant papers and surveys:

1. F. Faure, J. Sjstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. in Math. Physics, vol. 308 (2011), 325-364.
2. S. Zelditch, *Recent Developments in Mathematical Quantum Chaos*, Current developments in mathematics, 2009, 115204, Int. Press, Somerville, MA, 2010.

Notation

For a given open set $\Omega \subset \mathbb{R}^n$, we denote the set of compactly supported smooth functions by $C_0^\infty(\Omega)$.

For an integer $N \geq 0$, the space of N times differentiable function with support in $K \subset \mathbb{R}^n$ is denoted by $C_0^N(K)$.

1. Recap: LCS topologies, distributions, Fourier transforms

Distribution theory makes sense of the ad-hoc constructions coming from physics and provides a framework to study PDE. For example, the Dirac delta $\delta(x)$ is the distribution that makes sense of the following, for all $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \delta(x) f(x) dx = f(0) \tag{1.1}$$

Aim: define the space of distributions as a dual space to $C_0^\infty(\mathbb{R}^n)$ with suitable topology, with operation properties defined by duality.

1.1. **Topological vector spaces.**

Let us recall some definitions. A *topological vector space* X is a complex or real vector space equipped with a topology in which multiplication with a scalar $\mathbb{C} \times X \to X$ and addition $X \times X \to X$ are continuous operations. In fact, since translation is continuous, it is enough to specify a base of neighbourhoods at zero.

Moreover, a topological vector space is called *locally convex* if there is a base of neighbourhoods at zero consisting of sets which are convex, balanced and absorbing. A set U is called *convex* if $tx + (1-t)y \in U$ for all x, y and $0 \leq t \leq 1$; *absorbing*, if for all $x \in X$, there is a $t > 0$ with $x \in tU = \{ty \mid u \in U\}$ and finally U is *balanced* if $cx \in U$ for all $x \in U$ and $|t| \leq 1$. All the topologies we will consider will be locally convex.

A *seminorm* p on a vector space X is a function $p : X \to \mathbb{R}$ homogeneous in the sense

$$p(cx) = |c|p(x) \text{ for all } x \in X \text{ and } c \in \mathbb{C} \tag{1.2}$$

and satisfies the triangle inequality

$$p(x+y) \leq p(x) + p(y) \text{ for all } x, y \in X \tag{1.3}$$

A family of seminorms P on a vector space X is called *separating* if for all $x \in X$, there is a $p \in P$ with $p(x) \neq 0$. Given a separating family P of seminorms on X, we may induce a locally convex topology, by declaring the base of neighbourhoods at zero consisting of, for every $\varepsilon > 0$ and $p \in P$

$$U(\varepsilon, p) = \{x \mid p(x) < \varepsilon\} \tag{1.4}$$

and taking finite intersections of such sets. We call this the topology generated by the seminorms in P.

Conversely, let X be a locally convex topological vector space. If U is a convex, balanced and absorbing neighbourhood of zero, we may introduce the *Minkowski functional* of U

$$p_U(x) = \inf\{t > 0 \mid x \in tU\} \tag{1.5}$$

It is an exercise to check that this is a seminorm, and to show that the topology on X is generated by these seminorms.

An important special case happens when P is countable, say $P = \{p_1, p_2, \dots\}$. Then it can be shown that

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1 + p_k(x-y)} \tag{1.6}$$

gives a metrization of the topology on X generated by P. Conversely, one can show that if a loclly convex toplogical vector space is metrizable, then it can be generated by a countable family of seminorms.

A *Fréchet space* is a locally convex topological vector space which is metrizable with a complete metric.

Example 1.1. Given a compact set $K \subset \mathbb{R}^n$, we may introduce the norms p_N for $N \in \mathbb{N}_0$ on $C_0^N(K)$ as follows

$$p_{N,K}(\phi) = \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi(x)| \tag{1.7}$$

It is an exercise to show that $C_0^N(K)$ with norm $p_{N,K}$ is a Banach space. Similarly, we may topologise $C_0^\infty(K)$ by using the seminorms in the set $P = \{p_{N,K} \mid N \geq 0\}$. The obtained LCS topology is a Fréchet topology, since $C_0^N(K)$ is a Banach space with the norm $p_{N,K}$.

If $\Omega \subset \mathbb{R}^n$ is an open set, and $K_j \subset \Omega$ is an increasing, exhausting sequence of compact sets, then we may equip $C^\infty(\Omega)$ with the locally convex topology coming from the family $P = \{p_{N,K_j} \mid N, j \in \mathbb{N}_0\}$. It is an exercise to show $C^\infty(\Omega)$ is a Fréchet space with this topology.

We finish the section with a criterion for continuity of maps between two LCS.

Proposition 1.2. *Let X and Y be two LCS and $f : X \to Y$ a linear map between them. Assume P and Q are two families of seminorms generating the topologies on X and Y, respectively. Then f is continuous if and only if, for each $q \in Q$, there exists $p_1, \dots, p_N \in P$ and a constant $C \geq 0$, such that for all $x \in X$:*

$$q\big(f(x)\big) \leq C \max\big(p_1(x), \dots, p_N(x)\big) \tag{1.8}$$

Proof. See the proof of Proposition A.1. [3]. □

1.2. **Space of distributions.** We consider an open $\Omega \subset \mathbb{R}^n$ and the space $C_0^\infty(\Omega)$. We will define a locally convex version of the inductive limit topology on $C_0^\infty(\Omega)$.

This is done by specifying a base of open sets $\mathcal{B}$ at zero. We will say $X \in \mathcal{B}$ if X is convex and balanced, and additionally we ask that $X \cap C_0^\infty(K)$ is open in the uniform topology as in Example 1.1.

The sets $X \in \mathcal{B}$ are absorbing since when intersected with $C_0^\infty(K)$, they are open and the associated topology of $C_0^\infty(K)$ is LCS. Therefore we obtain a LCS topology on $C_0^\infty(\Omega)$, which we call the *inductive limit* topology. Next, we consider the dual of $C_0^\infty(\Omega)$ and denote it by

$$\mathcal{D}'(\Omega) = \Big(C_0^\infty(\Omega)\Big)' \tag{1.9}$$

We call the space $\mathcal{D}'(\Omega)$ equipped with the weak*-topology, the space of *distributions* on Ω. A few remarks are in place. The inclusions $C_0^\infty(K) \subset C_0^\infty(\Omega)$ are continuous by definition; so if $u \in \mathcal{D}'(\Omega)$, then $u|_{C_0^\infty(K)}$ is continuous for all compact K.

Proposition 1.3. *A linear functional u on $C_0^\infty(\Omega)$ is in $\mathcal{D}'(\Omega)$ if and only if for all compact $K \subset \Omega$, there is an integer $N \geq 0$ and $C \geq 0$ depending on K, such that for all $\phi \in C_0^\infty(K)$*

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi| \tag{1.10}$$

Proof. If $u \in \mathcal{D}'(\Omega)$, then the restriction to $C_0^\infty(K)$ is continuous, so the inequality (1.10) follows from Proposition 1.2.

Conversely, if u is a linear functional satisfying equation (1.10) for all K, the by Proposition 1.2 it is continuous on $C_0^\infty(K)$. If we then take a ε-ball $0 \in B_\varepsilon$, we easily see that $u^{-1}(B_\varepsilon)$

is convex and balanced, and intersects each $C_0^\infty(K)$ in an open set. Thus by definition it is open; by translation invariance we see u is continuous. □

Remark 1.4. It can be shown that $C_0^\infty(\Omega)$ does not have a countable basis at zero; therefore it is not metrizable.

Proposition 1.5. *We have $\phi_j \to 0$ in $C_0^\infty(\Omega)$ if and only if there exists a compact $K \subset \Omega$ with $\operatorname{supp}(\phi_j) \subset K$ and $\phi_j \to 0$ uniformly on K.*

Proof. Given the uniform convergence on compact set K, we immediately see the convergence in the LCS topology.

Conversely, suppose for the sake of contradiction that supports of ϕ_j are contained in compacts $K_j \nearrow \Omega$. Find points $x_1 \in K_1$, $x_2 \in K_2 \setminus K_1$ for $j = 1, 2, \dots$, such that $\phi_j(x_j) \neq 0$. Introduce, for $\phi \in C_0^\infty(\Omega)$

$$p(\phi) = \sum_{j=1}^{\infty} \sup \left\{ \left| \frac{\phi(x)}{\phi_j(x_j)} \right| : x \in K_j \setminus K_{j-1} \right\} \tag{1.11}$$

Therefore, p is seminorm on $C_0^\infty(\Omega)$. By applying Proposition 1.2 and observing there is a constant $C_K \geq 0$ with $p(\phi) \leq C_K \sup |\phi|$ for $\phi \in C_0^\infty(K)$, we deduce p is continuous on $C_0^\infty(K)$, hence it is continuous on $C_0^\infty(\Omega)$. Hence $p(\phi_j) \to 0$ as $j \to \infty$ by assumption; but by construction $p(\phi_j) \geq 1$, contradiction.

Therefore there is a K with $\operatorname{supp}(\phi_j) \subset K$ for all j, which finishes the proof. □

Now given a partial derivative ∂^α, we may extend its the action to distributions via duality

$$\langle \partial^\alpha u, \phi \rangle := \langle u, (-1)^{|\alpha|} \partial^\alpha \phi \rangle \tag{1.12}$$

Similarly we extend the multiplication by a function f as

$$\langle fu, \phi \rangle := \langle u, f\phi \rangle \tag{1.13}$$

So given a linear partial differential operator $P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ of order m, it acts via

$$\langle Pu, \phi \rangle := \langle u, P^t \phi \rangle \tag{1.14}$$

Here ${}^tP$ is the *adjoint* of P, given by

$${}^tP\phi = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \phi) \tag{1.15}$$

It is an exercise to show $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is a continuous linear map and that ${}^t({}^tP) = P$.

Note we may restrict the distribution to any open subset $U \subset \Omega$ and obtain a distribution in $\mathcal{D}'(U)$. We define the *support*, written $\operatorname{supp}(u)$, of the distribution u as a complement of the set

$$\{x \in \Omega \mid u = 0 \text{ in a neighbourhood of } x\} \tag{1.16}$$

We say that u has a compact support if $\operatorname{supp}(u)$ and write $u \in \mathcal{E}'(X)$.

Proposition 1.6. *The distributions in $\mathcal{E}'(X) \subset \mathcal{D}'(X)$ may be extended to continuous functionals on $C^\infty(X)$; moreover, we have identification $\mathcal{E}'(X) = \big(C^\infty(X)\big)'$.*

Proof. If $u \in \big(C^\infty(\Omega)\big)$, we may restrict to $C_0^\infty(\Omega)$ and obtain continuous functional since restrictions to $C_0^\infty(K)$ for compact K are.

Conversely, given $u \in \mathcal{E}'(\Omega)$, we need to show there exists a unique element of $\big(C^\infty(\Omega)\big)'$ whose restriction to $C_0^\infty(\Omega)$ is equal to u. This is left as an exercise to the reader. □

We can also make sense of when is a distribution $u \in \mathcal{D}'(\Omega)$.

1.3. **Schwartz kernel theorem.** Here we consider integral transforms of the form

$$f \mapsto kf = \int_Y k(x,y)f(y)dy \tag{1.17}$$

where k is called the *kernel of the transform* is a function on $X \times Y$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Clearly k maps functions on Y to functions on X.

We will write $\phi \otimes \psi \in C^\infty(X \times Y)$ for the tensor product of functions $\phi \in C^\infty(X)$ and $\psi \in C^\infty(Y)$. Then given $k \in \mathcal{D}'(X \times Y)$, we may consider the bilinear form for $\phi \in C_0^\infty(X)$ and $\psi \in C^\infty(Y)$:

$$(\phi, \psi) \mapsto \langle k, \phi \otimes \psi \rangle \tag{1.18}$$

For fixed ϕ, we get a linear form on Y and for fixed ψ, we get a linear form on X; they are both distributions (exercise). We write $k\psi$ for the linear form $\phi \mapsto \langle k, \phi \otimes \psi \rangle$ and so we have the sequentially continuous maps $\psi \mapsto k\psi$

$$k : C_0^\infty(Y) \to \mathcal{D}'(X) \tag{1.19}$$

We will say that the map in (1.19) is generated by a *distribution kernel* or a *Schwartz kernel* k. By transposing x and y, we obtain the *transposed kernel* ${}^t k \in \mathcal{Y} \times \mathcal{X}$, by simply requiring that

$$\langle {}^t k, \chi \rangle = \langle k, {}^t \chi \rangle \tag{1.20}$$

where $\psi \in C_0^\infty(Y \times X)$, $\chi \in C_0^\infty(X \times Y)$ and ${}^t\chi(x,y) = \chi(x,y)$.

The main point of this section is the following theorem, known as the *Schwartz kernel theorem*, which associates a kernel to a sequentially continuous map in (1.19):

Theorem 1.7 (Schwartz kernel theorem)**.** *A linear map $\mu : C_0^\infty(Y) \to \mathcal{D}'(X)$ is sequentially continuous if and only if it is generated by a Schwartz kernel $k \in \mathcal{D}'(X \times Y)$, such that for all $\phi \in C_0^\infty(X)$ and $\psi \in C_0^\infty(Y)$ we have:*

$$\langle \mu\psi, \phi \rangle = \langle k, \phi \otimes \psi \rangle \tag{1.21}$$

Moreover, the kernel k is uniquly determined by μ.

Proof. See the proof of Theorem 6.1.1. [3]. □

We will call a kernel $k \in \mathcal{D}'(X \times Y)$ a *regular kernel* if it has the following properties:

$$k : C_0^\infty(Y) \to C^\infty(X) \text{ and } {}^t k : C_0^\infty(X) \to C^\infty(Y) \tag{1.22}$$

We can then extend k and ${}^t k$ to continous maps $k : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ and ${}^t k : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ by duality:

$$\langle ku, \phi \rangle = \langle u, {}^t k \phi \rangle \text{ and } \langle {}^t k v, \psi \rangle = \langle v, k\psi \rangle \tag{1.23}$$

for distributions $u \in \mathcal{E}'(Y)$ and $v \in \mathcal{E}'(X)$. It is an exercise to show continuity holds. We will often meet with regular kernels in the course.

Example 1.8 (Differential operators)**.** *Consider $P : C_0^\infty(X) \to C_0^\infty(X)$ is a differential operator with C^∞ coefficients. Then the Schwartz kernel of this operator is given by*

$$k_P(x,y) = {}^t P(y, \partial_y) \delta(x - y)$$

Here $\delta(x-y) \in \mathcal{D}'(X \times Y)$ is the Schwartz kernel of the identity map Id, defined via $\langle \delta(x-y), \chi \rangle = \int_X \chi(x,x)dx$. This follows from

$$\langle {}^tP(y,\partial_y)\delta(x-y), \phi \otimes \psi \rangle = \langle \delta(x-y), \phi \otimes P\psi \rangle = \int_X \phi(x) P\psi(x) dx$$

1.4. **Fourier transform.** In this section we recall the definition and basic properties of the Fourier transform.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by, for $\xi \in \mathbb{R}^n$:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx \tag{1.24}$$

Here $x \cdot \xi = \sum_{j=1}^n x_k \xi_j$. We note that by Dominated convergence theorem that $f \in C^0(\mathbb{R}^n)$ for $f \in L^1(\mathbb{R}^n)$.

Note that we cannot use duality directly extend the Fourier transform to $\mathcal{D}'(\mathbb{R}^n)$, since $\mathcal{F}$ does not map $C_0^\infty(\mathbb{R}^n)$ to itself. We therefore introduce the space of *rapidly decreasing functions* or *Schwartz functions*, and call a function $\phi \in C^\infty(\mathbb{R}^n)$ this way if

$$\|\phi\|_{\alpha,\beta} = \sup |x^\alpha D^\beta \phi| < \infty \tag{1.25}$$

for all pairs of multindices α and β. We denote the space of such function by $\mathcal{S}(\mathbb{R}^n)$. The definition (1.25) means that every derivative $D^\beta \phi$ is faster than polynomial decreasing. In fact, the seminorms $\|\cdot\|_{\alpha,\beta}$ above generate a Fréchet topology on $\mathcal{S}(\mathbb{R}^n)$.

We collect a few basic properties of Fourier transform in one claim:

Proposition 1.9. *Let $f, g \in L^1(\mathbb{R}^n)$. Then:*

1. $|\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R}^n)}$ and $\hat{f}(\eta) \to 0$ *as* $\eta \to \infty$
2. $\displaystyle\int f(x)\hat{g}(x)dx = \int \hat{f}(x) g(x) dx$
3. $\widehat{f * g} = \hat{f}\hat{g}$
4. $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ *continuously.*
5. $\widehat{D^\alpha \phi}(\xi) = \xi^\alpha \hat{\phi}(\xi)$ *and* $\widehat{x^\alpha \phi} = (-1)^{|\alpha|} D^\alpha \hat{\phi}(\xi)$ *for* $\phi \in \mathcal{S}(\mathbb{R}^n)$
6. $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is a continuous isomorphism and the inverse is given by*

$$\mathcal{F}^{-1}\phi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(\xi) e^{ix \cdot \xi} d\xi$$

Proof. See [3, Chapter 8] for details. □

As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ continuously, we may extend the isomorphism to $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

2. Symbol classes, oscillatory integrals and Fourier Integral Operators

In this lecture, we introduce symbol classes, following the first chapters of [4, 5]. We fix $X \subset \mathbb{R}^n$ an open set and real numbers $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$. Also, we fix $m \in \mathbb{R}$ and $N \in \mathbb{Z}_{\geq 1}$. Our aim is to make sense of distributions kernels

$$I(a, \varphi) = \int_{\mathbb{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) d\theta \in \mathcal{D}'(X \times Y)$$

for suitable choices of a phase function φ and a symbol a. For example, we have seen that the distribution kernel of $(1 - \Delta)^{-1}$ has $a(x, y, \theta) = \frac{1}{1+|\xi|^2}$ and $\varphi(x, y, \theta) = (x - y) \cdot \theta$.

2.1. Symbol classes.

Definition 2.1 (Symbol classes). *We introduce $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ as the space of all $a \in C^\infty(X \times \mathbb{R}^N)$, such that for all compact $K \subset X$ and all multiindices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}^N$, there is a constant $C = C_{K,\alpha,\beta}(a)$ such that, for all $(x, \theta) \in K \times \mathbb{R}^n$*

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \leq C(1 + |\theta|)^{m-\rho|\beta|+\delta|\alpha|} \tag{2.1}$$

We say that $S^m_{\rho,\delta}$ is the space of all symbols of order m and of type (ρ, δ).

We equip the space $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ with the LCS topology given by the seminorms, for each compact $K \subset X$ and multiindices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}^N$:

$$P_{K,\alpha,\beta}(a) = \sup_{(x,\theta) \in K \times \mathbb{R}^N} \frac{|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)|}{(1 + |\theta|)^{m-\rho|\beta|+\delta|\alpha|}} \tag{2.2}$$

A countable family of seminorms is given by taking a nested sequence of compact sets $K_j \nearrow X$ and seminorms $P_{K_j,\alpha,\beta}$. It is an exercise to check the induced topology is Fréchet. Moreover, we clearly have $\partial_x^\alpha \partial_\theta^\beta : S^m_{\rho,\delta}(X \times \mathbb{R}^N) \to S^{m-|\beta|\rho+|\alpha|\delta}_{\rho,\delta}(X \times \mathbb{R}^N)$ continuous.

We also observe that $S^m_{\rho,\delta}(X \times \mathbb{R}^N) \subset S^{m'}_{\rho',\delta'}(X \times \mathbb{R}^N)$ if $m \leq m'$, $\delta \leq \delta'$ and $\rho \geq \rho'$. The space of symbols of order $-\infty$ is given by

$$S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(X \times \mathbb{R}^N) \tag{2.3}$$

Equivalently, symbols of order $-\infty$ are such that for all compact K, multiindices α and β, and $M \in \mathbb{R}$, we have for $(x, \theta) \in K \times \mathbb{R}^N$:

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \leq C(1 + |\theta|)^{-M} \tag{2.4}$$

The space $S^{-\infty}$ also carries the natural Fréchet topology, by choosing "the best" constants in (2.4). From now on we fix the notation S^m for $S^m_{1,0}$ and any m.

Remark 2.2. There is no point in introducing $S^m_{\rho,\delta}$ for $\rho > 1$ or $\delta < 0$. For example, if $a \in S^m_{\rho,\delta}$ for $m < 0$ and $\rho > 1$, the equation (2.1) yields that after applying ∂_r many times and then integrating, that there is a gain in the decay in ∂_θ derivative; so $a \in S^{-\infty}$.

Example 2.3. We give a few examples of symbols:

1. Assume $n = N$. Then $a(x, \theta) = \sum_{|\alpha| \leq m} a_\alpha \theta^\alpha$ belongs to the symbols class $S^m_{1,0}(X \times \mathbb{R}^N)$.
2. More generally, let $a \in C^\infty(X \times \mathbb{R}^N)$ be positively homogeneous of order m for $|\theta| \geq 1$, i.e. $a(x, \lambda\theta) = \lambda^m a(x, \theta)$ for $\lambda \geq 1$ and $|\theta| \geq 1$. Then $a \in S^m_{1,0}(X \times \mathbb{R}^n)$.

3. Assume $N = n$, so that $e^{-|\xi|^2} \in S^{-\infty}(X \times \mathbb{R}^n)$.
4. We have $e^{ix\cdot\xi} \in S^0_{0,1}$.
5. Exotic: assume $f \in C^\infty(X \times \mathbb{R}^N; [0,\infty))$ and is positively homogeneous of degree 1 for $|\theta| \geq 1$. Then $e^{-f} \in S^0_{\frac{1}{2},\frac{1}{2}}(X \times \mathbb{R}^N)$.

We now prove a few properties about the topologies on the spaces of symbols, with the aim to make sense of an asymptotic formula $a \sim \sum_{j=0}^\infty a_j$ where $a_j \in S^{m_k}_{\rho,\delta}(X \times \mathbb{R}^N)$ for $m_j \to -\infty$ as $j \to \infty$.

Proposition 2.4. *Assume $m_1, m_2 \in \mathbb{R}$. The map $(a,b) \mapsto ab$ on the sets*

$$S^{m_1}_{\rho,\delta}(X \times \mathbb{R}^N) \times S^{m_2}_{\rho,\delta}(X \times \mathbb{R}^N) \to S^{m_1+m_2}_{\rho,\delta}(X \times \mathbb{R}^N) \tag{2.5}$$

is well-defined and continuous.

Proof. If we apply the Leibniz' formula to $\partial_x^\alpha \partial_\theta^\beta a$, we get

$$\partial_x^\alpha \partial_\theta^\beta (ab) = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} C(\alpha_1,\alpha_2,\beta_1,\beta_2) \partial_x^{\alpha_1}\partial_\theta^{\beta_1} a \cdot \partial_x^{\alpha_2}\partial_\theta^{\beta_2} b \tag{2.6}$$

for suitable multinomial coefficients $C(\alpha_1,\alpha_2,\beta_1,\beta_2)$. Well-definedness follows from the estimates (2.1) applied to each summand on a set $K \times \mathbb{R}^N$; continuity follows from estimating the RHS and applying Proposition 1.2. ☐

Proposition 2.5. *Let $(a_j)_{j=1}^\infty$ be a bounded sequence*[1] *in $S^m_{\rho,\delta}$, which converges pointwise for each $(x,\theta) \in X \times \mathbb{R}^N$. Then the pointwise limit a belongs to $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and for every $m' > m$, we have $a_j \to a$ as $j \to \infty$ in the topology of $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$.*

Proof. By assumption, the $P_{K,\alpha,\beta}(a_j)$ are bounded uniformly on j. So by Arzela-Ascoli on compact subsets, there exists a convergent subsequence; by uniqueness of limits this is a and so a is smooth.[2] By the same argument, moreover $a_j \to a$ uniformly in $C^\infty(X \times \mathbb{R}^N)$.

In order to prove convergence in $S^{m'}_{\rho,\delta}$, consider compact $K \subset X$ and multiindices α, β. Consider for each $(x,\theta) \in K \times \mathbb{R}^N$:

$$k_j(x,\theta) = \frac{|\partial_x^\alpha \partial_\theta^\beta (a - a_j)|}{(1+|\theta|)^{m'-\rho|\beta|+\delta|\alpha|}} = \frac{1}{(1+\theta)^{m'-m}} \frac{|\partial_x^\alpha \partial_\theta^\beta (a - a_j)|}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}} \tag{2.7}$$

We analyse this factorisation and aim to prove $P_{K,\alpha,\beta}(a - a_j) \to 0$. First, observe that in the regime where $|\theta| \geq L$ where L is large enough, by boundedness of the second factor, we have $k_j(x,\theta) < \varepsilon$.

In the regime where $|\theta| \leq L$, by uniform convergence in C^∞ we have $k_j(x,\theta) \to 0$ uniformly on $K \times B(0,L)$. Therefore, by taking ε small enough, we see that

$$\lim_{j\to\infty} \sup_{(x,\theta)\in K\times\mathbb{R}^N} k_j(x,\theta) = 0 \tag{2.8}$$

This finishes the proof. ☐

[1] In the LCS topology.

[2] Alternatively, conclude that a is continuous by Arzela-Ascoli. Then apply the inequality $|f'(0)| \leq C_\varepsilon(\|f\|^{\frac{1}{2}}_{L^\infty}\|f''\|^{\frac{1}{2}}_{L^\infty} + \|f\|^{\frac{1}{2}}_{L^\infty})$, where $f \in C^2([-\varepsilon,\varepsilon])$. Then argue that $(a_j)_{j=1}^\infty$ is Cauchy in C^1 by applying this inequality to terms $(a_i - a_j)$; iterate to get Cauchy in C^k for each k.

Proposition 2.6. *If $m' > m$, then $S^{-\infty}(X \times \mathbb{R}^N)$ is dense in $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ in the topology of $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$.*

Proof. Let $\chi \in C_0^\infty(X \times \mathbb{R}^N)$ with $\chi = 0$ for $|\theta| \leq 1$ and $|\theta| \geq 2$. Then $\chi_j(\theta) = \chi(\frac{\theta}{j})$ for $j = 1, 2, \dots$ is a bounded sequence in $S^0_{1,0}(X \times \mathbb{R}^N)$. The amounts to checking that

$$\partial_\theta^\alpha \chi_j(\theta) = j^{-|\alpha|} \partial_\theta^\alpha \chi\Big(\frac{\theta}{j}\Big) = O\big((1 + |\theta|)^{-|\alpha|}\big) \tag{2.9}$$

uniformly in j, θ, according to (2.1); this follows since χ is supported on $1 \leq |\theta| \leq 2$. To finish the proof, it suffices to put $a_j(x, \theta) = \chi(\frac{\theta}{j}) a(x, \theta)$ and apply Propositions 2.4, 2.5 and the inclusion $S^0_{1,0} \subset S^0_{\rho,\delta}$. □

We now come to the main point: we may sum the symbols asymptotically.

Proposition 2.7. *Let $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^N)$ for $j = 0, 1, \dots$ with $m_j \searrow -\infty$ as $j \to \infty$. Then there exists $a \in S^{m_0}(X \times \mathbb{R}^N)$, unique modulo $S^{-\infty}(X \times \mathbb{R}^N)$, such that for every $k = 0, 1, \dots$*

$$a - \sum_{0 \leq j < k} a_j \in S^{m_k}_{\rho,\delta}(X \times \mathbb{R}^N) \tag{2.10}$$

Proof. Uniqueness part follows from the fact that, if a' also satisfies (2.10), then $(a - a') \in S^{m_k}_{\rho,\delta}$ for all k.

For existence, assume first w.l.o.g. m_j is strictly decreasing by regrouping the summands according to order. Let $P_{j,0}, P_{j,1}, \dots$ be an enumeration of the seminorms defining the topology of $S^{m_j}_{\rho,\delta}$. Then Proposition 2.6 implies that, for every j, there is $b_j \in S^{-\infty}$ such that for all $0 \leq \nu, \mu \leq j - 1$

$$P_{\nu,\mu}(a_j - b_j) \leq 2^{-j} \tag{2.11}$$

Therefore, the sum $\sum_{j \geq k}(a_j - b_j)$ converges in $S^{m_k}_{\rho,\delta}$ for all k, and if we put $a := \sum_{j=0}^\infty (a_j - b_j) \in S^{m_0}_{\rho,\delta}$, then

$$a - \sum_{j<k} a_j = -\sum_{j<k} b_j + \sum_{j=k}^\infty (a_j - b_j) \in S^{m_k}_{\rho,\delta} \tag{2.12}$$

□

Remark 2.8. We want to make few remarks. First, the argument in the previous proposition uses a form of Cantor diagonal argument in the choice of approximating b_j. Secondly, the proposition is reminiscent of a statement known as Borel's lemma. This lemma states that we may construct smooth functions with prescribed jets; i.e. for any given a_α, there exists $f \in C^\infty(\mathbb{R}^n)$ with $\partial^\alpha f(0) = a_\alpha \alpha!$. Finally, the form of the a we constructed can be seen as follows:

$$a = \sum_{j=0}^\infty \Big(1 - \chi\Big(\frac{\theta}{t_j}\Big)\Big) a_j(x, \theta)$$

where $t_j \to \infty$ are chosen to grow sufficiently fast so that the sum and the derivatives converge.

2.2. **Oscillatory integrals.** We start with a definition of a phase function

Definition 2.9. *A function* $\varphi(x,\theta) \in C^\infty(X \times (\mathbb{R}^N \setminus 0))$ *is called a* phase function *if for all* $(x,\theta) \in X \times (\mathbb{R}^N \setminus 0)$

1. $\operatorname{Im} \varphi \geq 0$
2. $\varphi(x, \lambda\theta) = \lambda\varphi(x,\theta)$ *for* $\lambda > 0$ *(positive homogeneity)*
3. $d_{x,\theta}\varphi \neq 0$

Recall that we aim to make sense of the integrals of the form, for $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and φ a phase function

$$I(a,\varphi) = \int e^{i\varphi(x,\theta)} a(x,\theta) d\theta \tag{2.13}$$

We will call integrals of this form *oscillatory integrals*. Observe that if $m < -N - k$ the above integral converges and $I(a,\varphi) \in C^k(X)$. To make sense of it we integrate by parts many times; e.g. if $N = n$ and $\varphi(x,\theta) = x \cdot \theta$, we trade x derivatives of u for inverse powers of θ.

Lemma 2.10. *Let* φ *be a phase function. There exists* $a_j \in S^0(X \times \mathbb{R}^N)$, $b_k, c \in S^{-1}(X \times \mathbb{R}^N)$ *for* $j = 1, \dots, N$ *and* $k = 0, \dots, n$, *such that*

$$L = \sum_{j=1}^N a_j \frac{\partial}{\partial \theta_j} + \sum_{k=1}^n b_j \frac{\partial}{\partial x_j} + c \tag{2.14}$$

satisfies ${}^tL e^{i\varphi} = e^{i\varphi}$. *Here* ${}^tL$ *is the formal adjoint*[3] *and is given by*

$${}^tLu = -\sum_{j=1}^N \frac{\partial(a_j u)}{\partial \theta_j} - \sum_{k=1}^n \frac{\partial(b_k u)}{\partial x_j} + cu \tag{2.15}$$

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end of Lecture 3, 24.10.2018.

Proof. Introduce

$$\Phi(x,\theta) = \sum_{k=1}^n \Big|\frac{\partial \varphi}{\partial x_k}\Big|^2 + |\theta|^2 \sum_{j=1}^N \Big|\frac{\partial \varphi}{\partial \theta_j}\Big|^2 \tag{2.16}$$

The following properties hold true: $\Phi(x,\theta) \neq 0$ for $\theta \neq 0$ by properties of φ, $\Phi(x,\lambda\theta) = \lambda^2\Phi(x,\theta)$ for $\lambda > 0$ and $\theta \neq 0$. Denote by $\chi \in C_0^\infty(\mathbb{R}^N)$ a bump function such that $\chi = 1$ near zero. We let

$${}^tL = \frac{1-\chi(\theta)}{i\Phi(x,\theta)}\Big(\sum_{j=1}^N |\theta|^2 \overline{\frac{\partial\varphi}{\partial\theta_j}} \frac{\partial}{\partial\theta_j} + \sum_{k=1}^n \overline{\frac{\partial\varphi}{\partial x_k}} \frac{\partial}{\partial x_k}\Big) + \chi(\theta) \tag{2.17}$$

$$= \sum_{j=1}^N a'_j \frac{\partial}{\partial\theta_j} + \sum_{k=1}^n b'_k \frac{\partial}{\partial x_k} + c' \tag{2.18}$$

It is easy to check that the coefficients $a'_j \in S^0$, $b'_k \in S^{-1}$ and $c \in S^{-\infty}$. Then one sees that ${}^tLe^{i\varphi} = e^{i\varphi}$ and the formal adjoint $L = {}^t({}^tL)$ satisfies the conditions of the lemma. □

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[3] I.e. $\int\int Lfg = \int\int f\,{}^tLg$ for all $f, g \in C_0^\infty(X \times \mathbb{R}^N)$.

Theorem 2.11. *Assume $\rho > 0$ and $\delta < 1$ and that ϕ is a phase function. Then there exists a unique way to extend $I(a, \varphi)$ from $S^{-\infty}(X \times \mathbb{R}^N)$ to $S^{\infty}_{\rho,\delta}(X \times \mathbb{R}^N) = \cup_{m\in\mathbb{R}} S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, such that*

$$S^m_{\rho,\delta}(X \times \mathbb{R}^N) \ni a \mapsto I(a, \varphi) \in \mathcal{D}'(X) \tag{2.19}$$

is continuous. [4]

Proof. Uniqueness is clear, as we have $S^{-\infty}$ dense in $S^m_{\rho,\delta}$ in any topology $S^{m'}_{\rho,\delta}$ for $m' > m$ by Proposition 2.6. For existence, we integrate by parts formally. For $a \in S^{-\infty}$ and $u \in C_0^\infty(X)$:

$$\langle I(a, \varphi), u\rangle := \int\int e^{i\varphi(x,\theta)} a(x, \theta)u(x)dxd\theta \tag{2.20}$$

Consider now L given by Lemma 2.10. Let $t := \min(\rho, 1 - \delta)$. For general $a \in S^m_{\rho,\delta}$, observe that $L^k\big(a(x, \theta)u(x)\big) \in S^{m-kt}_{\rho,\delta}$ by Proposition 2.4. So choose k with $m - kt < -N$ and define $I_k(a, \varphi)$ by the following expression:

$$\langle I_k(a, \varphi), u\rangle = \int\int e^{i\varphi(x,\theta)} L^k\big(a(x, \theta)u(x)\big)dxd\theta \tag{2.21}$$

Observe we have the following estimate, for $K \Subset \Omega$:

$$\sup_{K\times\mathbb{R}^N} |L^k(au)|(1 + |\theta|)^{-m+kt} \leq f_{k,K}(a) \cdot \sum_{|\alpha|\leq k} \sup_K |\partial^\alpha u(x)| \tag{2.22}$$

where $f_{k,K}(a)$ is a suitable seminorm on $S^m_{\rho,\delta}$. This implies that $I_k(a, \varphi) \in \mathcal{D}'^{(k)}(X)$ and that $S^m_{\rho,\delta} \ni a \mapsto I_k(a, \varphi) \in \mathcal{D}'^{(k)}(X)$ is continuous.

It is left to notice: for any choice of k' with $m - k't < -N$, we $I_{k'}(a, \varphi) = I_k(a, \varphi)$ and $I(a, \varphi) = I_k(a, \varphi)$ for any $a \in S^{-\infty}$. Therefore, the extension is well-defined. □

Next, we prove simple properties of oscillatory integrals $I(a, \varphi)$ and give the definition of a Fourier Integral Operator (FIO in short). We specialise this definition to pseudodifferential operators.

Given a phase function φ on $X \times \big(\mathbb{R}^N \setminus 0\big)$, define the *critical set* of φ as

$$C_\varphi = \{(x, \theta) \in X \times \big(\mathbb{R}^N \setminus 0\big) \mid \partial_\theta\varphi(x, \theta) = 0\} \tag{2.23}$$

The next claim is that the singularities of $I(a, \varphi)$ are determined by the behaviour of a and φ near C_φ. We let $\Pi : X \times \mathbb{R}^N \to X$ be the projection to the first coordinate.

Proposition 2.12. *The following two claims hold, if $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$:*

1. $\operatorname{sing\,supp} I(a, \varphi) \subset \Pi(C_\varphi)$
2. *If a vanishes on a conical neighbourhood of C_φ, then $I(a, \varphi) \in C^\infty(X)$.*

Proof. Let $R_\varphi = X \setminus \Pi(C_\varphi)$. The first part will follow if we show

$$I(a, \varphi)(x) = \int e^{i\varphi(x,\theta)} a(x, \theta)d\theta \tag{2.24}$$

is smooth on R_φ. But this integral is itself an oscillatory integral depending on a parameter $x \in R_\varphi$. Differentiating under the integral sign gives integrals of the same type.

[4]Moreover, if $k \in \mathbb{N}$ and $m - k\min(\rho, 1 - \delta) < -N$, then $S^m_{\rho,\delta} \ni a \mapsto I(a, \varphi) \in \mathcal{D}'^{(k)}$ is continous. Here, by $\mathcal{D}'^{(k)}(X)$ we denote the space of distributions of order $\leq k$. By definition, this is the dual space of $C^k_0(X)$.

For the second part, we will follow the proof of Lemma 2.10. Introduce

$$ {}^tLe^{i\varphi} = \frac{1-\chi(\theta)}{i\widetilde{\Phi}}|\theta|^2 \sum_{j=1}^{N} \overline{\frac{\partial \varphi}{\partial \theta_j}} \frac{\partial}{\partial \theta_j} + \chi_0 \tag{2.25} $$

Here $\chi_0 \in C_0^\infty(X \times \mathbb{R}^N)$ is such that $\chi_0 = 1$ near zero and zero for $|\theta| \geq 1$. Then, χ is such that $\chi = 1$ on C_φ and $\operatorname{supp}\chi \subset (\operatorname{supp} a)^c \cup B_1$, but $\operatorname{supp}(\chi - \chi_0) \subset (\operatorname{supp} a)^c$ (draw a picture). Then one computes ${}^tLe^{i\varphi} = (1 - \chi + \chi_0)e^{i\varphi}$.

We see that the coefficients of $L = {}^t({}^tL) = \sum a_j(x,\theta)\frac{\partial}{\partial\theta_j} + c(x,\theta)$ satisfy $a \in S^0$ and $c \in S^{-1}$, as before. Therefore, we have $I(a,\varphi) = I(L^k(a),\varphi)$, but since $L^k(a) \in S^{m-kt}_{\rho,\delta}$, we have $I(a,\varphi) \in C^l(X)$ for any l, which finishes the proof. ☐

As an example of previous theory, we consider $X \times Y \times \mathbb{R}^N$, where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. An operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ of the following form, for $a \in S^m_{\rho,\delta}(X \times Y \times \mathbb{R}^N)$ is called a *Fourier Integral Operator* (FIO)

$$ Au(x) = \int\int e^{i\varphi(x,y,\theta)} a(x,y,\theta)u(y)dyd\theta \tag{2.26} $$

Here $u \in C_0^\infty(Y)$ and we think of $K_A = I(a,\varphi) \in \mathcal{D}'(X \times Y)$ as a distribution kernel (c.f. Schwartz kernel theorem 1.7) generating operator A via

$$ \langle Au, v\rangle_X = \langle K_A, v \otimes u\rangle_{X\times Y} \tag{2.27} $$

Here $v \in C_0^\infty(X)$. If $X = Y$, $N = n$, $\varphi(x,y,\xi) = (x-y)\cdot\xi$ and $K_A = (2\pi)^{-n}I(a,\varphi)$,[5] then A is called a *pseudodifferential operator* (PDO). More precisely, for PDOs we have

$$ Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta} a(x,y,\theta)u(y)dyd\theta \tag{2.28} $$

We denote the space of such operators $\Psi^m_{\rho,\delta}(X)$ and say that $A \in \Psi^m_{\rho,\delta}(X)$ is of order $\leq m$ and of type (ρ,δ). We write just $\Psi^{-\infty}(X)$ to denote the set of operators of order $-\infty$.

We now prove first properties of PDOs. We denote by Δ the diagonal in $X \times X$.

Proposition 2.13. *Let $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ and A is the associated PDO given by equation (2.26), then:*

1. *A is a continuous map $A : C_0^\infty(X) \to C^\infty(X)$.*
2. *A has a unique continuous extension $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$.*
3. $K_A \in C^\infty(X \times X \setminus \Delta)$.
4. $\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)$ *(pseudolocality)*

Proof. For the first point, note that $d_{y,\theta}\varphi = (-\theta, x-y) \neq 0$ for all points $(x,y,\theta) \in X \times X \times \mathbb{R}^n$. By Lemma 2.10, there is an $L \in S^0_{1,0}\frac{\partial}{\partial\theta} \oplus S^{-1}_{1,0}\frac{\partial}{\partial y} \oplus S^{-1}_{1,0}$, depending smoothly in x, such that ${}^tL(e^{i\varphi}) = e^{i\varphi}$. Then for $u, v \in C_0^\infty(X)$ and $k \in \mathbb{N}$ with $m - kt < -N$:

$$ \langle Au, v\rangle = \langle K_A, v\otimes u\rangle = \int\int\int e^{i(x-y)\cdot\theta} v(x) L^k\big(a(x,y,\theta)u(y)\big)dxdyd\theta $$

Therefore, $Au(x)$ is the smooth function given by

$$ Au(x) = (2\pi)^{-n}\int\int e^{i(x-y)\cdot\theta} L^k\big(a(x,y,\theta)u(y)\big)dyd\theta $$

[5]Note the extra $(2\pi)^{-n}$ factor.

The continuity of $A : C_0^\infty(X) \to C^\infty(X)$ follows from this formula. For the second part, in view of Schwartz kernel Theorem 1.7, the transpose of A is given by ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$ by exchanging the roles of x and y, i.e. for $u, v \in C_0^\infty(X)$

$$\langle u, {}^tAv\rangle = \langle Au, v\rangle = \langle K_A, v \otimes u\rangle$$

Note that ${}^tA$ is generated by the symbol $a(y, x, \theta)$; since $d_{x,\theta}\varphi = (\theta, x - y) \neq 0$ as above, we get ${}^tA : C_0^\infty(X) \to C^\infty(X)$. Therefore, there is a unique extension given by, for $u \in \mathcal{E}'(X)$ and $v \in C_0^\infty(X)$:

$$\langle Au, v\rangle = \langle u, {}^tAv\rangle$$

The third point follows from Proposition 2.12 and as $C_\varphi = \Delta \times \big(\mathbb{R}^n \setminus 0\big)$.

For the fourth point, let $u \in \mathcal{E}'(X)$ and take $x_0 \in X \setminus \operatorname{sing\,supp}(u)$. Take $\varphi, \psi \in C_0^\infty(X)$ with $\operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi) = \emptyset$, with $\varphi = 1$ near x_0 and $\psi = 1$ near $\operatorname{sing\,supp}(u)$. Then by 1. above we have $A\big((1-\psi)u\big) \in C^\infty(X)$. Furthermore, since by 3. above $\varphi(x)K_A(x, y)\psi(y) \in C^\infty(X \times X)$, which is the kernel of the operator $\varphi A\psi$, we have

$$\varphi A(\psi u) \in C^\infty(X) \tag{2.29}$$

Combining the above two facts, we have $\varphi Au \in C^\infty(X)$, which by definition implies that Au is C^∞ outside $\operatorname{sing\,supp}(u)$. □

Remark 2.14. We give a few remarks about the proof of the previous proposition. First, in proving the smoothness in (2.29) we used a fact from the theory of distributions, i.e. whenever we have a continuous operator $B : C_0^\infty(Y) \to \mathcal{D}'(X)$ with distribution kernel $K_B \in \mathcal{D}'(X \times Y)$ (c.f. Theorem 1.7), then the following two statements are equivalent:

1. B extends to a continuous operator $B : \mathcal{E}'(Y) \to C^\infty(X)$.
2. $K_B \in C^\infty(X \times Y)$.

If either of two statements hold, we say B is *smoothing*.

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end of Lecture 4, 31.10.2018.

We note that the set of smoothing operators for which $X = Y$ coincides with the set $\Psi^{-\infty}(X)$ of pseudodifferential operators. To see this, assume first $A \in \Psi^{-\infty}(X)$. Then the integral kernel $K_A(x, y) \in C^\infty(X \times X)$ (as $I(a, \varphi)$ is smooth, where $a \in S^{-\infty}$ is the corresponding symbol). Conversely if A is smoothing, then $K_A(x, y)$ is smooth. So pick $a(x, y, \theta) = K_A(x, y)e^{-i(x-y)\cdot\theta}\chi(\theta) \in S^{-\infty}$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\int \chi(\theta)d\theta = 1$. It is easy to see that A is of form (2.28). Hence the conclusion.

3. Method of stationary phase

In this chapter we consider integrals of the form, for $\varphi \in C^\infty(X; \mathbb{R})$, $u \in C_0^\infty(X)$ and $\lambda \in \mathbb{R}$:

$$I(\lambda) = \int_X e^{i\lambda\varphi(x)}u(x)dx \tag{3.1}$$

We are interested in the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$.

If we assume $d\varphi \neq 0$ on X and take ${}^tL = \frac{1}{i\lambda|d\varphi|^2}\sum_{j=1}^n \frac{\partial\varphi}{\partial x_j}\frac{\partial}{\partial x_j}$ (c.f. Lemma 2.10), so that ${}^tLe^{i\lambda\varphi} = e^{i\lambda\varphi}$, then by repeated integrations by parts, we get

$$I(\lambda) = O(\lambda^{-\infty})$$

by which we mean, for each $N > 0$ there exists C_N such that

$$|I(\lambda)| \leq C_N \lambda^{-N}$$

We arrive at the following natural question.

Question 3.1. *What is the asymptotic behaviour of $I(\lambda)$ as $\lambda \to \infty$ if φ has critical points?*

The idea is that $I(\lambda)$ will asymptotically be affected by values of u and φ near the critical points of φ.

Recall that $x_0 \in X$ a non-degenerate critical point of φ if $d\varphi(x_0) = 0$ and $\varphi''(x_0) \neq 0$. Here $\varphi''(x) = \det\big(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\big)(x)$ denotes the Hessian. By Morse lemma, near such a point x_0, there exists a local diffeomorphism $\mathcal{H} : x_0 \in U \to 0 \in V$ with $\mathcal{H}(x_0) = 0$, such that[6]

$$\varphi \circ \mathcal{H}^{-1}(x_1, \dots, x_n) = \varphi(x_0) + \frac{1}{2}(x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots x_n^2)$$

Here r is the maximal dimension of a subspace on which the matrix $\varphi''(x_0)$ is positive; we denote by $\operatorname{sgn}\varphi''(x_0) = r - (n - r)$ the signature. Morse lemma reduces Question 3.1 to considering a symmetric, quadratic form Q with signature $\operatorname{sgn} Q = r - (n - r)$.

Recall the following identity:

$$\mathcal{F}\big(e^{\frac{1}{2}i\langle x, Qx\rangle}\big) = (2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\operatorname{sgn} Q} |\det Q|^{-\frac{1}{2}} e^{-i\langle \xi, Q^{-1}\xi\rangle/2} \tag{3.2}$$

Now by Parseval's identity (c.f. Proposition 1.9 3.), we obtain:

$$\int_X e^{i\lambda\langle x, Qx\rangle/2} u(x)dx = (2\pi)^{-\frac{n}{2}} e^{i\frac{\pi}{4}\operatorname{sgn} Q} |\det Q|^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, Q^{-1}\xi\rangle/(2\lambda)} \widehat{u}(\xi) d\xi$$

Now an application of Taylor's formula gives that $\big|e^{it} - \sum_{k=0}^{N-1} \frac{(it)^k}{k!}\big| \leq \frac{|t|^N}{N!}$ for $t \in \mathbb{R}$. Therefore, we obtain:

$$e^{-i\langle \xi, Q^{-1}\xi\rangle/(2\lambda)} = \sum_{k=0}^{N-1} \frac{1}{k!}\Big(\frac{1}{2\lambda i}\langle \xi, Q^{-1}\xi\rangle\Big)^k + R_N(\xi, \lambda) \tag{3.3}$$

where we have the remainder estimate $|R_N(\xi, \lambda)| \leq (2\lambda)^{-N} |\langle \xi, Q^{-1}\xi\rangle|^N / N!$.

Also, by the properties of the Fourier transform (c.f. Proposition 1.9 5.), we obtain

$$\int \langle \xi, Q^{-1}\xi\rangle^k \widehat{u}(\xi) d\xi = (2\pi)^n \big(\langle D_x, Q^{-1} D_x\rangle^k u\big)(0)$$

Here $\langle D_x, Q^{-1} D_x\rangle$ is the second order operator $-Q_{ij}^{-1}\partial_i\partial_j$ associated to Q^{-1}. Combining the results above, we obtain the expansion

$$\int e^{i\lambda\langle x, Qx\rangle/2} u(x)dx = \sum_{k=0}^{N-1} \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\operatorname{sgn} Q}}{k! |\det Q|^{\frac{1}{2}} \lambda^{k+\frac{n}{2}}} \Big(\frac{1}{2i}\langle D_x, Q^{-1} D_x\rangle\Big)^k u(0) + S_N(u, \lambda) \tag{3.4}$$

Here we have remainder estimate of the form $S_N(u, \lambda) = O_u(\lambda^{-\frac{n}{2} - N})$ (here O_u indicates that the constant depends on u).

[6] For a proof of this fact, see [4, Lemma 2.1.].

We now specialise to $Q = \begin{pmatrix} 0 & -Id \\ -Id & 0 \end{pmatrix}$, where each block is an n by n matrix, and switch to $(x, y) \in \mathbb{R}^{2n}$ variables, so that $\frac{1}{2}\langle (x, y), Q(x, y)\rangle = -x \cdot y$. Then by equation (3.4)

$$\begin{aligned} \Big(\frac{\lambda}{2\pi}\Big)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\lambda x\cdot y} u(x,y)dxdy &= \sum_{k=0}^{N-1} \frac{1}{k!\lambda^k}\Big(\frac{1}{i}\sum_{j=1}^n \partial_{x_j}\partial_{y_j}\Big)^k u(0,0) + S_N(u,\lambda) \\ &= \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!\lambda^{|\alpha|} i^{|\alpha|}} \partial_x^\alpha \partial_y^\alpha u(0,0) + S_N(u,\lambda) \end{aligned} \tag{3.5}$$

We record this expansion for subsequent use. We also have the remainder estimate

$$|S_N(u,\lambda)| \leq C\lambda^{-N} \sum_{|\alpha+\beta|\leq 2n+1} \|\partial_x^\alpha \partial_y^\beta (\partial_x\cdot\partial_y)^N u\|_{L^1} \tag{3.6}$$

Remark 3.2. More generally, assume φ has a single non-degenerate critical point x_0 (if there are more, we just sum over these). By Morse lemma and using expansion (3.4), one has

$$\int_X e^{i\lambda\varphi(x)} u(x)dx \sim \sum_{k=0}^\infty \lambda^{-\frac{n}{2}-k} c_k$$

Here c_k are coefficients determined by applying a differential operator A_{2k} of order $2k$ to u at x_0; the coefficients of A_{2k} depend on derivatives of φ at x_0. The expansion is in powers of λ^{-1}. For instance, we have

$$c_0 = \frac{(2\pi)^{\frac{n}{2}} e^{i\frac{\pi}{4}\operatorname{sgn}\varphi''(x_0)}}{|\det\varphi''(x_0)|^{\frac{1}{2}}}$$

Remark 3.3. This method is useful in various areas of mathematics and physics: in PDEs, proving various formulas, Chern-Simons theory (infinite dimensional setting) etc.

4. Algebra of pseudodifferential operators

The aim for this section is to prove: left symbol quantisation for PDOs, composition properties of PDOs, invariance under coordinate changes and to extend this theory to compact manifolds.

To this end, say $C \subset X \times Y$ is *proper* if the projections $\Pi_x : X \times Y \to X$ and $\Pi_y : X \times Y \to Y$ are proper maps. Recall a map $f : V \to W$ is proper is $f^{-1}(K) \subset V$ is compact for all compact $K \subset W$. An operator $A : C_0^\infty(Y) \to C^\infty(X)$ is *properly supported* if $\operatorname{supp} K_A \subset X \times Y$ is proper. Given $K \subset Y$, $L \subset X$ and $C \subset X \times Y$, write

$$C(K) = \{x \in X \mid \exists y \in K \text{ s.t. } (x,y) \in C\} = \Pi_x(C \cap \Pi_y^{-1} K)$$

and

$$C^{-1}(L) = \{y \in Y \mid \exists x \in L \text{ s.t. } (x,y) \in C\} = \Pi_y(C \cap \Pi_x^{-1} L)$$

Observe that for $u \in C_0^\infty(X)$ and for a properly supported $A \in \Psi^m_{\rho,\delta}(X)$, we have

$$\operatorname{supp}(Au) \subset \operatorname{supp} K_A(\operatorname{supp} u) \tag{4.1}$$

Since A is properly supported, we have $A : C_0^\infty(X) \to C_0^\infty(X)$ is continuous. Analogous argument applies to ${}^tA$ and so by duality A extends to a map $A : \mathcal{D}'(X) \to \mathcal{D}'(X)$. Moreover, notice that (4.1) holds for $u \in \mathcal{E}'(X)$ (c.f. Proposition 2.13), so we have $A : \mathcal{E}'(X) \to \mathcal{E}'(X)$; as before, by duality we have A extends to a map $A : C^\infty(X) \to C^\infty(X)$.

Note also that if A properly supported PDO and B an arbitrary PDO, the compositions $A \circ B, B \circ A$ are well defined maps $C_0^\infty(X) \to C^\infty(X)$, by above properties.

Proposition 4.1. *Any PDO $A \in \Psi^m_{\rho,\delta}(X)$ can be written as $A = A_0 + A_1$, where $A_1 \in \Psi^{-\infty}(X)$ is smoothing and $A_0 \in \Psi^m_{\rho,\delta}(X)$ is properly supported.*

Proof. Let $\chi \in C^\infty(X \times X)$ be such that $\chi = 1$ near the diagonal Δ and $\operatorname{supp} \chi \subset X \times X$ proper (exercise: prove the existence of such a function). We put

$$a_0(x, y, \theta) = \chi(x, y)a(x, y, \theta) \quad \text{and} \quad a_1(x, y, \theta) = (1 - \chi(x, y))a(x, y, \theta)$$

Consider the operators A_0 and A_1 associated to symbols a_0 and a_1, respectively. Then $K_{A_1}(x, y) \in C^\infty(X \times X)$ by Proposition 2.13 and $K_{A_0}(x, y)$ is properly supported by the properties of χ. Therefore, we have $A = A_0 + A_1$ with desired properties. □

The next result establishes the left quantisation, i.e. eliminates the dependence on y in the symbols $a(x, y, \theta)$.

Theorem 4.2. *Let $A \in \Psi^m_{\rho,\delta}$ be properly supported and assume $\rho > \delta$. Then $b(x, \xi) := e^{-ix\cdot\xi}A(e^{i(\cdot)\cdot\xi})$ belongs to $S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ and has an asymptotic development*

$$b(x, \xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{i^{-|\alpha|}}{\alpha!}\big(\partial_\xi^\alpha \partial_y^\alpha a(x, y, \xi)\big)\big|_{y=x} \tag{4.2}$$

Moreover, $Au(x) = (2\pi)^{-n} \int e^{ix\xi}b(x, \xi)\widehat{u}(\xi)d\xi$ for $u \in C_0^\infty(X)$. We call $\sigma_A(x, \xi) := b(x, \xi)$ the complete symbol of A.

Proof. By multiplying $a(x, y, \theta)$ by a cut-off $\chi \in C^\infty(X \times X)$ with $\chi = 1$ near $\operatorname{supp} K_A$ and with $\operatorname{supp} \chi$ proper, we may assume that $\Pi_{x,y}a = \{(x, y) \in X \times X \mid \exists \theta \in \mathbb{R}^n \text{ s.t. } a(x, y, \theta) \neq 0\}$ is proper. Furthermore, for simplicity from now on assume $\rho = 1$ and $\delta = 0$. Then we write

$$e^{-ix\xi}Ae^{i(\cdot)\cdot\xi} = (2\pi)^{-n} \int\int a(x, y, \theta)e^{i(x-y)\cdot(\theta-\xi)}dyd\theta =: b(x, \xi) \tag{4.3}$$

The integral is interpreted as an iterated one and note that the phase $(x - y) \cdot (\theta - \xi)$ has critical points at $\theta = \xi$, $x = y$.

Let $\chi \in C_0^\infty([0, \infty); [0, 1])$ be such that $\chi = 1$ on $[0, \frac{1}{3}]$ and $\operatorname{supp} \chi \subset [0, \frac{1}{2})$. For $|\xi| \geq 2$ and $x \in K \subset X$ compact, put

$$b_2(x, \xi) = (2\pi)^{-n} \int\int a(x, y, \theta)\Big(1 - \chi\Big(\frac{|\theta - \xi|}{|\xi|}\Big)\Big)e^{i(x-y)\cdot(\theta-\xi)}dyd\theta \tag{4.4}$$

In this integral, we integrate by parts with $L = \frac{1}{|\theta-\xi|^2}\sum_{j=1}^n(\xi_j - \theta_j)D_{y_j}$; observe that ${}^tL = -L$ and $Le^{i(x-y)\cdot(\theta-\xi)} = e^{i(x-y)\cdot(\theta-\xi)}$. Observe also that on the support of $1 - \chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)$, we have $|\theta - \xi| \sim 1 + |\theta| + |\xi|$.[7] Then

$$L^k\big(a(1 - \chi)\big) = O_k(1)\frac{(1 + |\theta|)^m}{(1 + |\theta| + |\xi|)^k} = O_k(1)(1 + |\theta| + |\xi|)^{m-k} = O_k(1)\frac{(1 + |\xi|)^{m+n+1-k}}{(1 + |\theta|)^{n+1}}$$

Here we write $O_k(1)$ for a term uniformly bounded in (ξ, θ), $k \in \mathbb{N}$ and we also assume $m + n + 1 - k \leq 0$. Now for such $k \in \mathbb{N}$, from (4.4) we have $b_2(x, \xi) = O_k(1)(1 + |\xi|)^{m+n+1-k}$.

[7] Here we write $a \sim b$ if there is a $C > 0$ such that $\frac{a}{C} \leq b \leq Ca$.

Similar estimates for derivatives of b_2 yield $b_2 \in S^{-\infty}$.

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end of Lecture 5, 7.11.2018.

It remains to study for $|\xi| \geq 1$ and $x \in K \subset X$ compact,

$$b_1(x,\xi) = (2\pi)^{-n} \int\int a(x,y,\theta)\chi\Big(\frac{|\theta-\xi|}{|\xi|}\Big)e^{i(x-y)\cdot(\theta-\xi)}dyd\theta \tag{4.5}$$

$$= (2\pi)^{-n} \int\int a(x,x+s,\xi+\sigma)\chi\Big(\frac{|\sigma|}{|\xi|}\Big)e^{-is\sigma}dsd\sigma \tag{4.6}$$

$$= \Big(\frac{\lambda}{2\pi}\Big)^n \int\int a(x,x+s,\lambda(\omega+\sigma))\chi(|\sigma|)e^{-i\lambda s\sigma}dsd\sigma \tag{4.7}$$

Here, in the second line we used the substitutions $y = x+s$ and $\theta = \xi+\sigma$; in the third line, we put $\xi = \lambda\omega$, where $\lambda = |\xi|$ and changed the variables by writing $\sigma = \lambda\widetilde{\sigma}$. We want to apply the method of stationary phase from Section 3. More precisely, we use the expansion in (3.5) to write

$$b_1(x,\lambda\omega) = \sum_{|\alpha|\leq N-1} \lambda^{-|\alpha|}i^{-|\alpha|}\partial_s^\alpha\partial_\sigma^\alpha\Big(a(x,x+s,\lambda(\omega+\sigma))\Big)\Big|_{s=\sigma=0} + S_N(\lambda)$$

$$= \sum_{|\alpha|\leq N-1} \frac{i^{-|\alpha|}}{\alpha!}\Big(\partial_y^\alpha\partial_\xi^\alpha a(x,y,\xi)\Big)\Big|_{y=x} + S_N(\lambda)$$

The remainder term is estimated using (3.6)

$$|S_N(\lambda)| \leq C_{N,K}\lambda^{-N}\sum_{|\alpha+\beta|\leq 2n+1}\sup_{s,\sigma}\big|\partial_s^\alpha\partial_\xi^\beta(\partial_s\cdot\partial_\sigma)^N\big(a(x,x+s,\lambda(\omega+\sigma))\chi(|\sigma|)\big)\big|$$

Now a typical term of the operator $(\partial_s\cdot\partial_\sigma)^N$ acting on $a\chi$ is of the form $\lambda^{|\beta'|}(\partial_y^{\alpha'}\partial_\theta^{\beta'}a(x,x+s,\lambda(\omega+\sigma))\partial_\sigma^{\beta''}\chi(|\sigma|)$ with $\beta'+\beta''=\alpha'$ with $|\alpha'| = N$. If we apply $\partial_s^\alpha\partial_\sigma^\beta$ with $|\alpha+\beta|\leq 2n+1$ to such a term, we get an upper bound of $C\lambda^m$. Thus $|S_N(\lambda)| \leq C_N|\xi|^{m-N}$ for $x\in K$ and $|\xi|\geq 1$.

A similar argument applies to provide estimates on derivatives of $b_1(x,\xi)$ and this shows directly that $b_1(x,\xi)\in S^m$ and that b_1 has the good asymptotic expansion. Furthermore, since $b = b_1+b_2$ and $b_2\in S^{-\infty}$, we deduce the properties of b.

Finally, if $u\in C_0^\infty(X)$, we write $u(x) = (2\pi)^{-n}\int e^{ix\xi}\widehat{u}(\xi)d\xi$ and approximate this integral by Riemann sums

$$u_\varepsilon(x) = \Big(\frac{\varepsilon}{2\pi}\Big)^n \sum_{\nu\in(\varepsilon\mathbb{Z})^n, |\nu|\leq\frac{1}{\varepsilon}} e^{ix\cdot\nu}\widehat{u}(\nu)$$

Then $u_\varepsilon \to u$ in $C^\infty(X)$. Since $A: C^\infty(X)\to C^\infty(X)$ is continuous, we get

$$Au(x) = (2\pi)^{-n}\int A(e^{ix\cdot\xi})\widehat{u}(\xi)d\xi = (2\pi)^{-n}\int e^{ix\cdot\xi}b(x,\xi)\widehat{u}(\xi)d\xi$$

This finishes the proof. $\square$

Recall now that by Proposition 4.1, any PDO $A\in\Psi^m_{\rho,\delta}(X)$ can be decomposed as $A = A_0 + A_1$, where A_1 is smoothing and A_0 properly supported. We define the complete symbol of A, $\sigma_A \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)/S^{-\infty}$ as the class of σ_{A_0} (check this is well-defined). This way, we obtain a bijective map

$$A \mapsto \sigma_A : \Psi^m_{\rho,\delta}(X)/\Psi^{-\infty} \to S^m_{\rho,\delta}(X \times \mathbb{R}^n)/S^{-\infty}$$

Equip now $L^2(X)$ with the standard inner product $(u, v) := \int u(x)\overline{v}(x)dx$. Now if $A : C_0^\infty(X) \to \mathcal{D}'(X)$ is continuous, we define the complex adjoint $A^* : C_0^\infty(X) \to \mathcal{D}'(X)$ via $(Au, v) = (u, A^*v)$. Thus $K_{A^*}(x, y) = \overline{K_A(y, x)}$.

Theorem 4.3. *Assume $\rho > \delta$. Then*

1. *Let $A \in \Psi^m_{\rho,\delta}(X)$. Then $A^* \in \Psi^m_{\rho,\delta}(X)$ and*

$$\sigma_{A^*}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\sigma_A(x, \xi)} \tag{4.8}$$

2. *Let $A \in \Psi^{m'}_{\rho,\delta}(X)$ and $B \in \Psi^{m''}_{\rho,\delta}(X)$, with at least one of A, B properly supported. Then the composition $AB \in \Psi^{m'+m''}_{\rho,\delta}(X)$ and*

$$\sigma_{AB}(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_A(x, \xi) D_x^\alpha \sigma_B(x, \xi) \tag{4.9}$$

Proof. One checks directly that the symbol of A^* is given by $a^*(x, y, \xi) = \overline{a(y, x, \xi)}$ and so $A^* \in \Psi^m_{\rho,\delta}(X)$. Now recall that we may take $a(x, y, \xi) = \sigma_A(x, \xi)$ modulo $S^{-\infty}$, so Theorem 4.2 gives the expansion (4.8).

For the second part, assume both A and B are properly supported, by using Proposition 4.1 and observing that the result is unaffected by changes modulo $\Psi^{-\infty}$. By Theorem 4.2 we may assume

$$Au(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} a(x, \xi)u(y)dyd\xi \tag{4.10}$$

$$Bu(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} b(x, \xi)\widehat{u}(\xi)d\xi \tag{4.11}$$

Here $a = \sigma_A$, $b = \sigma_B$ and $u \in C_0^\infty(X)$. Approximate the integral defining Bu by Riemann sums converging in $C^\infty(X)$ (c.f. proof of Theorem 4.2), to obtain

$$(2\pi)^{-n} A \int e^{iy\cdot\xi} b(y, \xi)\widehat{u}(\xi)d\xi = (2\pi)^{-n} \int e^{ix\cdot\xi} \underbrace{e^{-ix\cdot\xi} A\big(e^{i(\cdot)\cdot\xi} b(\cdot, \xi)\big)}_{c(x,\xi):=} \widehat{u}(\xi)d\xi$$

By the proof of Theorem 4.2 we find $c \in S^{m'+m''}_{\rho,\delta}(X)$ and

$$c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha \big(a(x, \xi)b(y, \theta)\big)\big|_{y=x,\theta=\xi} = \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_A(x, \xi) D_x^\alpha \sigma_B(x, \xi)$$

□

Remark 4.4. Alternatively, for 2. in the previous theorem, argue as follows. Show that for B properly supported and $u \in C_0^\infty(X)$, we have $\widehat{Bu}(\xi) = \int e^{-iy\cdot\xi}\sigma_B(y, -\xi)u(y)dy$. Therefore

$$ABu(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi}\sigma_A(x, \xi)\sigma_B(y, -\xi)u(y)dyd\xi$$

Then apply Theorem 4.2 directly and manipulate the expansion.

4.1. **Changes of variables (Kuranishi trick).** We discuss transformations of PDOs under changes of variables. Let $\kappa : X \to \widetilde{X}$ be a C^∞ diffeomorphism. This defines the pullback $\kappa^* : C^\infty(\widetilde{X}) \to C^\infty(X)$ by $\kappa^* u = u \circ \kappa$. If $\widetilde{A} \in \Psi^m_{\rho,\delta}(\widetilde{X})$ is a PDO on $\widetilde{X}$, we want to study $A = \kappa^* \circ \widetilde{A} \circ (\kappa^*)^{-1} : C_0^\infty(X) \to C^\infty(X)$. We may write

$$Au(x) = \widetilde{A}(u \circ \kappa^{-1})\big(\kappa(x)\big) = (2\pi)^{-n} \int\int e^{i(\kappa(x)-\widetilde{y})\cdot\widetilde{\theta}} \widetilde{a}\big(\kappa(x), \widetilde{y}, \widetilde{\theta}\big) u(\kappa^{-1}(\widetilde{y})) d\widetilde{y} d\widetilde{\theta} \tag{4.12}$$

If we make a change of variables $\widetilde{y} = \kappa(y)$, we obtain

$$(2\pi)^{-n} \int\int e^{i\overbrace{(\kappa(x)-\kappa(y))\cdot\widetilde{\theta}}^{\Phi(x,y,\widetilde{\theta}):=}} \underbrace{\widetilde{a}(\kappa(x), \kappa(y), \widetilde{\theta}) u(y) \Big| \det \frac{\partial \widetilde{y}}{\partial y}\Big|}_{\widetilde{b}(x,y,\widetilde{\theta}):=} dy d\widetilde{\theta} \tag{4.13}$$

Here $\Big|\det \frac{\partial \widetilde{y}}{\partial y}\Big|$ denotes the Jacobian. Note that $\widetilde{b} \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ and that $\Phi(x, y, \widetilde{\theta})$ is a phase function, so A is an FIO. By a change of variables in the fibres, usually referred to in literature as the Kuranishi trick, we would like to show $A \in \Psi^m_{\rho,\delta}(X)$. By Proposition 2.12 1., we have that the kernel $K_A = (2\pi)^{-n} I(\widetilde{b}, \Phi)$ is smooth away from the diagonal Δ. Therefore, we may restrict attention to a neighbourhood Ω of Δ. Then for $(x, y) \in \Omega$ we may write $\Phi(x, y, \widetilde{\theta}) = \langle F(x, y)(x - y), \widetilde{\theta}\rangle = \langle x - y, G(x, y)\widetilde{\theta}\rangle$, where $G(x, y) = {}^tF(x, y)$ and[8]

$$F(x, y) = \int_0^1 \frac{\partial \kappa(tx + (1 - t)y)}{\partial x} dt \tag{4.14}$$

Then F and G are smooth in Ω and $F(x, x) = \frac{\partial \kappa}{\partial x}(x)$. We choose Ω small enough such that F and G are invertible on Ω.

After a change of variables in the fibres depending on $(x, y) \in \Omega$, $\theta = G(x, y)\widetilde{\theta}$, we may introduce

$$a(x, y, \theta) = \widetilde{a}\big(\kappa(x), \kappa(y), G^{-1}(x, y)\theta\big) \frac{\big|\det \frac{\partial \kappa(y)}{\partial y}\big|}{\big|\det G(x, y)\big|} \tag{4.15}$$

We obtain on Ω that $K_A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\theta} a(x, y, \theta) d\theta$, so it remains to determine the symbol class of a. On a compact set $K \subset \Omega$, we check that for each i, since $\widetilde{a} \in S^m_{\rho,\delta}$

$$|a(x, y, \theta)| \leq C(1 + |G^{-1}(x, y)\theta|)^m \leq C_K(1 + |\theta|)^m \tag{4.16}$$

$$|\partial_{x_i} a(x, y, \theta)| \leq C_K(1 + |\theta|)^{m+\delta} + C_K(1 + |\theta|)^{m-\rho+1} \tag{4.17}$$

Here we used the chain rule in the second estimate. We conclude that we must have $\rho+\delta = 1$ in order to get the estimate (2.1), which we assume further on. We use the notation $S^m_\rho := S^m_{\rho,1-\rho}$ and $\Psi^m_\rho := \Psi_{\rho,1-\rho}$. Iterating, we then get $a \in S^m_\rho(\Omega \times \mathbb{R}^n)$ and so $A \in \Psi^m_\rho(X)$. Assuming $\rho > \delta$, i.e. $\rho > \frac{1}{2}$, by Theorem 4.2 we get:

$$\sigma_A(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha \Big(\sigma_{\widetilde{A}}(\kappa(x), G(x, y)^{-1}\xi) \frac{|\det \frac{\partial \kappa(y)}{\partial y}|}{|\det G(x, y)|}\Big)\Big|_{y=x} \tag{4.18}$$

[8]Here $\langle a, b\rangle = \sum_i a_i b_i$ is the usual inner product on $\mathbb{R}^n$.

It is possible to show directly that σ_A does not depend on the choice made. In particular, by identifying the leading order terms in (4.18)

$$\sigma_A(x,\xi) = \sigma_{\widetilde{A}}(\kappa(x), ({}^t\kappa'(x))^{-1}\xi) \mod S_\rho^{m-(2\rho-1)} \tag{4.19}$$

Here $\kappa'(x)$ denotes the differential of κ.

end of Lecture 6, 14.11.2018.

We now introduce the principal symbol of a PDO.

Definition 4.5. *Let $A \in \Psi_\rho^m(X)$, for $\rho > \frac{1}{2}$. The* principal symbol *of A is defined as the image of σ_A in $\big(S_\rho^m / S_\rho^{m-(2\rho-1)}\big)(X \times \mathbb{R}^n)$.*

Observe that the principal symbols gives rise to a bijection

$$\Psi_\rho^m / \Psi_\rho^{m-(2\rho-1)} \to S_\rho^m / S_\rho^{m-(2\rho-1)}$$

Coming back to changes of variables, we conclude that if a and $\widetilde{a}$ denote the principal symbols of A and $\widetilde{A}$ respectively, then

$$a(x, {}^t\kappa'(x)\widetilde{\xi}) = \widetilde{a}(\kappa(x), \widetilde{\xi}) \tag{4.20}$$

Finally, we introduce a special class of symbols, called *classical symbols*, denoted by $S_{cl}^m(X \times \mathbb{R}^n) \subset S^m(X \times \mathbb{R}^n)$, and defining $a(x,\theta) \in S_{cl}^m$ by asking

$$a(x,\theta) \sim \sum_{j=0}^{\infty} (1 - \chi(\theta)) a_{m-j}(x,\theta) \tag{4.21}$$

Here $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi = 1$ close to 0, and $a_k(x,\theta) \in C^\infty(X \times \mathbb{R}^n)$ are positively homogeneous of degree k in θ (c.f. Example 2.3 2.). We let $\Psi_{cl}^m(X) \subset \Psi^m(X)$ be the class of PDOs A for which $\sigma_A \in S_{cl}^m$; we say A is classical. For such an operator we may identify the principal symbol in S_{cl}^m / S_{cl}^{m-1} with the positively homogeneous function $a_m(x,\xi)$ if the expansion (4.21) holds for $\sigma_A(x,\xi)$.

Remark 4.6. Note that the class of classical PDOs is closed under taking compositions in the sense of and by Theorem 4.3. It is also closed under changes of variables by the expansion (4.18). Note also that differential operators are classical PDOs.

5. Elliptic operators and L^2-continuity

From now on, we assume $\rho = 1$ and $\delta = 0$. Let $P \in \Psi^m(X)$ with symbol $p(x,\xi)$. We say that P is *elliptic* or *non-characteristic* at $(x_0,\xi_0) \in X \times \big(\mathbb{R}^n \setminus 0\big)$ if there exits a conical neighbourhood V of (x_0,ξ_0) in $X \times \big(\mathbb{R}^n \setminus 0\big)$ and $C > 0$ such that

$$|p(x,\xi)| \geq \frac{1}{C}(1 + |\xi|)^m \tag{5.1}$$

for $(x,\xi) \in V$ with $|\xi| \geq C$. We say that P is *elliptic at* x_0 if P is elliptic at (x_0,ξ) for all $\xi \in \mathbb{R}^n \setminus 0$. Furthermore, we say P is elliptic on $X' \subset X$ if it is so for all $x \in X'$.

In what follows we denote the symbols by lower letters.

Theorem 5.1. *If $P \in \Psi^m(X)$ is elliptic, there exists $Q \in \Psi^{-m}(X)$ properly supported, such that*[9]

$$P \circ Q \equiv Q \circ P \equiv Id \mod \Psi^{-\infty} \tag{5.2}$$

[9]We write $A \equiv B \mod \Psi^k$ for some k and PDOs A and B if $A - B \in \Psi^k(X)$.

Moreover, Q is unique modulo $\Psi^{-\infty}(X)$.

Proof. Using a partition of unity, we may find $q_0 \in C^\infty(X \times \mathbb{R}^n)$ such that for every $K \subset X$ compact, there exists $C_K > 0$ with

$$q_0(x, \xi) = \frac{1}{p(x, \xi)}, \quad \text{for } x \in K, |\xi| \geq C_K$$

Lemma 5.2. *We have $q_0(x, \xi) \in S^{-m}(X \times \mathbb{R}^n)$.*

Proof. Let $x \in K$ and $|\xi| \geq C_K'$, such that $|q_0(x, \xi)| \leq C(1 + |\xi|)^{-m}$. By induction, we may assume for $x \in K$ and $|\alpha| + |\beta| < N$

$$|\partial_x^\alpha \partial_\xi^\beta q_0(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{-m-|\beta|} \tag{5.3}$$

Now let $|\alpha| + |\beta| = N$. Differentiating $q_0 p = 1$, we get

$$p\partial_x^\alpha \partial_\xi^\beta q_0 = \sum_{\substack{\alpha'+\alpha''=\alpha, \alpha'\neq 0 \\ \beta'+\beta''=\beta, \beta'\neq 0}} C(\alpha', \beta', \alpha'', \beta'')\big(\partial_x^{\alpha'} \partial_\xi^{\beta'} p\big)\big(\partial_x^{\alpha''} \partial_\xi^{\beta''} q_0\big) \leq C(1 + |\xi|)^{-|\beta|} \tag{5.4}$$

Here we used the symbol estimates for p and the induction hypothesis. Therefore, we conclude:

$$|\partial_x^\alpha \partial_\xi^\beta q_0| \leq C(1 + |\xi|)^{-m-|\beta|} \tag{5.5}$$

□

Take now a properly supported operator $Q_0 \in \Psi^{-m}(X)$ with symbol q_0 modulo $S^{-\infty}$. By Theorem 4.3 we see $Q_0 \circ P - Id \in \Psi^{m-1}(X)$. Now let Q_1 be a PDO such that

$$(Q_0 + Q_1) \circ P - Id \in \Psi^{m-2}(X) \tag{5.6}$$

For this, by Theorem 4.3 we see it sufices to take $q_1 = -\frac{\sigma_{Q_0 P - Id}}{p}$. By Lemma 5.2, we see $q_1 \in S^{-m-1}$ and so $Q_1 \in \Psi^{-m-1}(X)$. Iterating, we obtain $Q_k \in \Psi^{-m-k}(X)$ for all $k \in \mathbb{N}_0$, such that

$$(Q_0 + Q_1 + \ldots + Q_k) \circ P - Id \in \Psi^{m-k-1}(X) \tag{5.7}$$

Taking the operator Q_L to be associated to the asymptotic expansion $\sum_{j\geq 0} q_j$ gives $Q_L \circ P - Id \in \Psi^{-\infty}$. Similarly, there is a properly supported Q_R with $P \circ Q_R - Id \in \Psi^{-\infty}$.

Applying Q_R to the right of $Q_L P \equiv Id \mod \Psi^{-\infty}$ gives $Q := Q_L \equiv Q_R \mod \Psi^{-\infty}$. Uniqueness of Q is now easy to see. □

Such Q in Theorem 5.1 is called a *parametrix*.

Corollary 5.3. *Assume $P \in \Psi^m(X)$ is elliptic and properly supported. Then*

1. *$P : \mathcal{D}'(X)/C^\infty(X) \to \mathcal{D}'(X)/C^\infty(X)$ is a bijection.*
2. *For $u \in \mathcal{D}'(X)$, we have $\operatorname{sing\,supp}(Pu) = \operatorname{sing\,supp}(u)$.*

Proof. Note that P is well-defined on $\mathcal{D}'(X)$ since it is properly supported. For 1., the inverse of P is given by the parametrix from Theorem 5.1.

For 2., it suffices to apply pseudolocality of PDOs from Proposition 2.13 4. to P and its parametrix. □

Remark 5.4. 1. If $\mathcal{L}$ is an elliptic differential operator, such that $\mathcal{L}u \in C^\infty(X)$ for $u \in \mathcal{D}'(X)$, then $u \in C^\infty(X)$. Thus we re-obtain a result about elliptic regularity.

2. More generally, a PDO P is *hypoellipic* if $Pu \in C^\infty$ implies $u \in C^\infty$. It is possible to show, for example, that the *heat operator* $\partial_t - \Delta$ is hypoelliptic.
3. Later, we will be able to show a more precise regularity result using *wavefront* sets.

5.1. **L^2-continuity.** We focus on mapping properties of PDOs in $\Psi^0(X)$ between L^2 spaces at first, and then generalise to mappings on Sobolev spaces.

Theorem 5.5. *Let $A \in \Psi^0(X)$ with $K_A \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$. Define*

$$M := \limsup_{|\xi|\to\infty} \sup_{x\in\mathbb{R}^n} |\sigma_A(x,\xi)| \tag{5.8}$$

Then $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous and for every $\varepsilon > 0$, we can find a decomposition $A = A_\varepsilon + K_\varepsilon$, where $\|A_\varepsilon\| \leq M + \varepsilon$ and $K_\varepsilon \in \Psi^{-\infty}(\mathbb{R}^n)$. Also, we have $\operatorname{supp} K_{A_\varepsilon} \subset \operatorname{supp} K_A + \{0\} \times B(0,\varepsilon)$.

Proof. Since $\operatorname{supp} K_A$ is compact, we may assume $\sigma_A(x,\xi)$ vanishes for x outside a compact set. The idea is to show that $A^*A \leq (M+\varepsilon)^2 Id$ in the sense of self-adjoint operators by constructing a square root of $(M+\varepsilon)^2 Id - A^*A$.

Lemma 5.6. *For all $\varepsilon > 0$, there exists $B \in \Psi^0(\mathbb{R}^n)$ with $K_{(M+\varepsilon)Id-B} \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$ and such that*

$$(M+\varepsilon)^2 Id = A^*A + B^*B + K$$

where $K \in \Psi^{-\infty}(\mathbb{R}^n)$.

Proof. Consider the formally self adjoint operator $C = (M+\varepsilon)^2 Id - A^*A$. Then

$$|\sigma_C(x,\xi)| \geq \text{const.} > 0 \tag{5.9}$$

for $|\xi|$ large. Also, we have $\sigma_C - \overline{\sigma_C} \in S^{-1}$.

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We claim we can find $b_0 \in S^0$, with $b_0 = M + \varepsilon$ for x outside a compact set, such that

$$\sigma_C - \overline{b_0} b_0 \in S^{-1}(\mathbb{R}^n) \tag{5.10}$$

Existence of such b_0 can be shown by setting $b_0 = \sqrt{(M+\varepsilon)^2 - |\sigma_A|^2}$ for large $|\xi|$ and $b_0 = M + \varepsilon$ for x outside a compact set; one checks $b_0 \in S^0$ similar to the proof of Lemma 5.2. Then we may write

$$B_0 = (M+\varepsilon)Id + D$$

Here B_0 has the symbol b_0 modulo $S^{-\infty}$ and $\operatorname{supp} K_D$ is compact. Thus we have

$$B_0^* B_0 = (M+\varepsilon)^2 Id + (M+\varepsilon)(D + D^*) + DD^* = C + C_1 \tag{5.11}$$

Here $C_1 \in \Psi^{-1}$ is implicitly defined, formally self-adjoint and has $\operatorname{supp} K_{C_1}$ compact. We think of B_0 as a zeroth order approximation to the square root, as $C - B_0^* B_0 \in \Psi^{-1}$. We seek for the next approximation $B_0 + B_1$, where $B_1 \in \Psi^{-1}$:

$$C - (B_0^* + B_1^*)(B_0 + B_1) = (C - B_0^* B_0) - (B_0^* B_1 + B_1^* B_0) - B_1^* B_1 \tag{5.12}$$

To reduce the order of the operator on the right hand side, we take B_1 to be properly supported with symbol $b_1 \in S^{-1}$ such that for large $|\xi|$

$$2 b_1(x,\xi) b_0(x,\xi) = \sigma_{C - B_0^* B_0}(x,\xi) \tag{5.13}$$

Now b_0 is bounded from below by (5.9) and (5.10) and so by Lemma 5.2, $b_0^{-1} \in S^0$ and we define $b_1 \in S^{-1}$ by the previous relation.

We iterate this procedure and obtain $B_k \in \Psi^{-k}(\mathbb{R}^n)$ with symbols b_k, with

$$C - (B_0 + \ldots + B_k)^*(B_0 + \ldots + B_k) \in \Psi^{-k-1}(\mathbb{R}^n) \tag{5.14}$$

The operator B is constructed by taking the properly supported operator associated to $\sum_{k=0}^{\infty} b_k$, or in other words take $B \sim \sum_{k\geq 0} B_k$.[10] □

For $u \in C_0^\infty(\mathbb{R}^n)$, we have $(B^*Bu, u) = \|Bu\|^2 \geq 0$. Thus

$$\|Au\|^2 \leq (M+\varepsilon)^2\|u\|^2 + |(Ku, u)| \leq \text{const.}\|u\|^2 \tag{5.15}$$

Here we used the classical fact that $K \in \Psi^{-\infty}$ is a continuous map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, since $K_K \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.[11] This proves the first part of the statement.

We sketch the second part. Let $\psi \in C_0^\infty\big(B(0, \frac{1}{2})\big)$ with $\int \psi(x)dx = 1$ and $\psi \geq 0$. Then also $0 \leq |\widehat{\psi}| \leq 1$ and $\widehat{\psi}(0) = 1$. Put $\chi = \psi * \check{\psi} \in C_0^\infty\big(B(0,1)\big)$, where $\check{\psi}(x) = \psi(-x)$. Then $\widehat{\chi} = |\widehat{\psi}|^2$ (c.f. Proposition 1.9 3.), so also $0 \leq \widehat{\chi} \leq 1$ and $\widehat{\chi}(0) = 1$. Put $\chi_\varepsilon(x) = \varepsilon^{-n}\chi\big(\frac{x}{\varepsilon}\big)$ and define $P_\varepsilon u = u - \chi_\varepsilon * u$.

Now by Fourier and Plancherel theorems (c.f. Proposition 1.9), $\|P_\varepsilon u\| \leq \|u\|$. Then define $A_\varepsilon := AP_\varepsilon$, so $Au = A_\varepsilon u + A(\chi_\varepsilon * u)$. Now apply the inequality (5.15) to A_ε to conclude the result. □

5.2. **Sobolev spaces and PDOs.** Our aim is now to extend the L^2-continuity properties of PDOs to properties about mappings between Sobolev spaces. We will recall some properties of Sobolev spaces first, but will not prove them.

Recall that for each $s \in \mathbb{R}$, we may define the Sobolev space H^s as

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \in L^2(\mathbb{R}^n)\} \tag{5.16}$$

We equip $H^s(\mathbb{R}^n)$ with the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} := (2\pi)^{-n}\int_{\mathbb{R}^n} (1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi \tag{5.17}$$

and consider the associated inner product, given explicitly by

$$(u,v)_{H^s(\mathbb{R}^n)} := (2\pi)^{-n}\int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi \tag{5.18}$$

We will sometimes use just the sub-indices H^s or s to denote the Sobolev norms. We collect the basic properties of Sobolev spaces in one Proposition:

Proposition 5.7 (Properties of Sobolev spaces)**.**

1. *$H^s(\mathbb{R}^n)$ is a Hilbert space with the inner product given by* (5.18) *and $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ is dense.*
2. *$H^s(\mathbb{R}^n) \subset H^t(\mathbb{R}^n)$ for $s > t$.*
3. *There exists a non-degenerate continuous pairing, continuously extending the one on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$*

$$H^s(\mathbb{R}^n) \times H^{-s}(\mathbb{R}^n) \to \mathbb{C}, \quad (u,v) \mapsto \langle u, v\rangle = \int_{\mathbb{R}^n} u(x)v(x)dx$$

[10] Alternatively, set $B_1 = B_0 - \frac{1}{2}(B_0^{-1})^*C_1$ and iterate this.

[11] Exercise: prove this elementary statement directly!

This gives a canonical isometric isomorphism $H^{-s}(\mathbb{R}^n) = \big(H^s(\mathbb{R}^n)\big)'$, the dual space of $H^s(\mathbb{R}^n)$.

4. *We have an alternative definition of Sobolev spaces for $m \in \mathbb{N}$:*

$$H^m(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) \mid D^\alpha u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| \leq m\}$$

The norm $\|u\|_{H^s}$ is equivalent to $\sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^2(\mathbb{R}^n)}$.

5. *The map $M_\varphi : u \mapsto \varphi u$, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, is continuous $M_\varphi : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$. Therefore a differential operator P of order m with coefficients in $\mathcal{S}(\mathbb{R}^n)$ is continuous $P : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$.*
6. *Sobolev embedding theorem holds: for any $k \in \mathbb{N}$, if $s > \frac{n}{2}+k$, then $H^s(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n)$ continuously.*[12]

Proof. The proofs can be found in [3, Chapter 9.3.]. To give an taste of the proofs, we prove 6. Consider the case $k = 0$ and note that $\xi \mapsto (1 + |\xi|^2)^{-s} \in L^1(\mathbb{R}^n)$ for $s > \frac{n}{2}$. Let $u \in H^s(\mathbb{R}^n)$ and put $f(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{u}(\xi)$. Then $f \in L^2(\mathbb{R}^n)$ and $\|f\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}} \|u\|_{H^s(\mathbb{R}^n)}$. So by Cauchy-Schwarz

$$\|\widehat{u}\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)} \Big(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-s} d\xi \Big)^{\frac{1}{2}} = C' \|u\|_{H^s}$$

By the inverse Fourier transform (c.f. Proposition 1.9), we have $u \in C(\mathbb{R}^n)$ and $|u(x)| \leq C\|u\|_{H^s}$ for any x, which proves the claim.

For $k > 0$, apply the first part of the claim to $D^\alpha u$ for $|\alpha| \leq k$; note that $D^\alpha u \in H^{s-k}(\mathbb{R}^n)$ for such range of α. □

We next introduce a few versions of Sobolev spaces for an open set $X \subset \mathbb{R}^n$. The space of distributions that locally live in a Sobolev space is given by

$$H^s_{\text{loc}}(X) = \{u \in \mathcal{D}'(X) \mid \phi u \in H^s(\mathbb{R}^n) \text{ for all } \phi \in C^\infty_0(X)\} \tag{5.19}$$

We give $H^s_{\text{loc}}(X)$ the topology of a locally convex space, by introducing the space of seminorms $\mathcal{P} = \{p_\varphi(u) = \|\varphi u\|_{H^s} \mid \varphi \in C^\infty_0(X)\}$. It suffices to take a countable, exhausting family of φ, and then $H^s_{\text{loc}}(X)$ becomes a Fréchet space (exercise). Next, let $K \subset X$ be compact. Then we define

$$H^s(K) = \{u \in H^s(\mathbb{R}^n) \mid \text{supp}(u) \subset K\} \subset H^s(\mathbb{R}^n) \tag{5.20}$$

The space $H^s(K)$ inherits the subspace topology, making it into a Hilbert space. Finally, the Sobolev space of distributions with compact support is introduced by

$$H^s_{\text{comp}}(X) = \bigcup_{K \subset X} H^s(K)$$

The union is over all $K \subset X$ compact. We topologise $H^s_{\text{comp}}(X)$ with the locally convex inductive limit topology (c.f. the topology on $C^\infty_0(X)$ in Section 1.2). Then one may show that the pairing $H^s_{\text{loc}}(X) \times H^{-s}_{\text{comp}}(X)$ is non-degenerate (exercise, c.f. Propositon 5.7 3.)

We are ready to present the main theorem of this subsection:

[12] We have more: $H^s(\mathbb{R}^n) \hookrightarrow C^k_0(\mathbb{R}^n)$ continuously, where the sub-index zero denotes the subspace of $f \in C^k(\mathbb{R}^n)$, such that $D^\alpha f(x) \to 0$ as $x \to \infty$ for $|\alpha| \leq m$.

Theorem 5.8. *Let $A \in \Psi^m(X)$ properly supported. Then for all $s \in \mathbb{R}$, A is continious*[13]

$$A : H^s_{\mathrm{loc}}(X) \to H^{s-m}_{\mathrm{loc}}(X), \quad A : H^s_{\mathrm{comp}}(X) \to H^{s-m}_{\mathrm{comp}}(X) \tag{5.21}$$

If A is elliptic, then for all $u \in \mathcal{D}'(X)$, we have the qualitative elliptic regularity

$$u \in H^s_{loc}(X) \iff Au \in H^{s-m}_{\mathrm{loc}}(X) \tag{5.22}$$

Proof. We show only the argument for H^s_{loc} spaces, the other case follows similarly. We first give a reduction argument to $X = \mathbb{R}^n$. Let $\varphi \in C_0^\infty(X)$. Then in suffices to show $\varphi A : H^s_{\mathrm{loc}}(X) \to H^{s-m}_{\mathrm{loc}}(\mathbb{R}^n)$ is continuous.

Now let $\psi \in C_0^\infty(X)$ be equal to one near the compact set $C^{-1}\big(\operatorname{supp}(\varphi)\big)$, where $C = \operatorname{supp} K_A$ (c.f. start of Section 4). Then $\varphi A = \varphi A \psi$ and it suffices to prove $\widetilde{A} = \varphi A \psi : H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$. Here $\widetilde{A} \in \Psi^m(\mathbb{R}^n)$ and $\operatorname{supp} K_{\widetilde{A}}$ compact.

Claim. For every $l \in \mathbb{R}$, there is an elliptic, properly supported $\Lambda_l \in \Psi^l(\mathbb{R}^n)$, such that for all $s \in \mathbb{R}$ we have isomorphisms

$$\Lambda_l : H^s(\mathbb{R}^n) \to H^{s-l}(\mathbb{R}^n) \tag{5.23}$$

We assume the claim for a moment and give the proof of the main result. Note that $H^s(\mathbb{R}^n) = \Lambda_{-s} L^2(\mathbb{R}^n)$ by the claim. So it suffices to prove that $B = \Lambda_{s-m}\widetilde{A}\Lambda_{-s} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous. But this follows from Theorem 5.5, as $B \in \Psi^0(\mathbb{R}^n)$ and $\operatorname{supp} K_B$ compact.

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To prove the claim, we first note that $(1+|D|^2)^{\frac{l}{2}} = (1-\Delta)^{\frac{l}{2}}$ lies in $\Psi^s(\mathbb{R}^n)$; its symbol is defined as $(1+|\xi|^2)^{\frac{l}{2}} \in S^l(\mathbb{R}^n)$. Note also that by definition of a PDO

$$(1+|D|^2)^{\frac{l}{2}} u = \mathcal{F}^{-1}\big((1+|\xi|^2)^{\frac{l}{2}}\widehat{u}\big) \tag{5.24}$$

This means that $(1+|D|^2)^{\frac{l}{2}} : H^s(\mathbb{R}^n) \to H^{s-l}(\mathbb{R}^n)$ is an isometric isomorphim (c.f. definition of Sobolev norms (5.17)).

We now modify $(1+|D|^2)^{\frac{l}{2}}$ into a properly supported PDO $\Lambda_l \Psi^l$ that satisfies the required properties. First observe that we may write $(1+|D|^2)^{\frac{l}{2}} u = v_l * u$, for $u \in C_0^\infty(X)$, where

$$v_l(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{l}{2}} d\xi \quad \in \mathcal{D}'(X)$$

is defined as an oscillatory integral. Now by Proposition 2.12 we have v_l smooth outside $x = 0$. In fact, x is of Schwartz class $\mathcal{S}$ for $x \neq 0$, as for any $k \in \mathbb{N}$:

$$v_l(x) = (2\pi)^{-n}\frac{1}{|x|^{2k}}\int e^{ix\cdot\xi}(-\Delta_\xi)^k(1+|\xi|^2)^{\frac{l}{2}} d\xi$$

Then take $\chi_l \in C_0^\infty(\mathbb{R}^n)$ with $\chi_l = 1$ near zero and define $w_l = \chi_l v_l$. Then, as $(1-\chi_l)v_l \in \mathcal{S}(\mathbb{R}^n)$

$$\widehat{w_l}(\xi) - (1+|\xi|^2)^{\frac{l}{2}} = \mathcal{F}\big((\chi_l - 1)v_l\big) \quad \in \mathcal{S}(\mathbb{R}^n)$$

Also, for $\chi_l = 1$ on a large enough set, we see that $\widehat{w_l}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. Define finally $\Lambda_l u := w_l * u = \mathcal{F}^{-1}\big(\widehat{w_l}\widehat{u}\big)$. Then clearly $\Lambda_l \in \Psi^l$ and $\Lambda_l : H^s(\mathbb{R}^n) \to H^{s-l}(\mathbb{R}^n)$ is continuous as $\widehat{w_l} - (1+|\xi|^2)^{\frac{l}{2}} \in \mathcal{S}(\mathbb{R}^n)$; Λ_l properly supported as its kernel is $w_l(x-y)$; its an isomorphism as $\frac{1}{\widehat{w_l}} \in S^{-l}$ (here we use $\widehat{w_l} \neq 0$) and satisfies properties similar to $\widehat{w_l}$. □

[13]If $A \in \Psi^m(X)$ is not assumed properly supported, then it can be shown that $A : H^s_{\mathrm{comp}}(X) \to H^s_{\mathrm{loc}}(X)$. Compare with the mapping properties of PDOs in Proposition 2.13; i.e. we need compact support of functions in the domain of a general PDO.

Corollary 5.9 (Quantitative elliptic estimates)**.** *Let A be an elliptic differential operator of order m with smooth coefficients. Then for $K \subset X$ compact, $s \in \mathbb{R}$ and $N \in \mathbb{Z}$, there exists $C = C_{K,s,N}$ such that*

$$\|u\|_{H^{s+m}} \leq C(\|Au\|_{H^s} + \|u\|_{H^{-N}}), \quad \text{for } u \in H^{s+m}(K) \tag{5.25}$$

Proof. Let $B \in \Psi^{-m}(X)$ be the parametrix of A. Then $u = BAu + Ru$, where $R \in \Psi^{-\infty}$ is smoothing. The estimate follows by applying Theorem 5.8. □

6. The wavefront set of a distribution

From now on, we will use the notation $T^*X = X \times \mathbb{R}^n$ for the cotangent space of some open $X \subset \mathbb{R}^n$. All PDOs in this section are assumed to be properly supported, unless otherwise stated.

Recall that if $u \in \mathcal{D}'(X)$, then $x_0 \notin \operatorname{sing\,supp}(u)$ if and only if there is a $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$, such that $\varphi u \in C^\infty(X)$.

Idea. By using pseudodifferential operators instead of cut-offs, we will get a refinement of the singular support, called the *wavefront set* $WF(u) \subset T^*X \setminus 0$.

Definition 6.1. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \in T^*X \setminus 0$. We say that $u \in C^\infty$* microlocally near *(x_0, ξ_0) if there exists $A \in \Psi^m(X)$ non-characteristic at (x_0, ξ_0), such that $Au \in C^\infty(X)$. Then*

$$WF(u) = \{(x, \xi) \in T^*X \setminus 0 \mid u \text{ is not } C^\infty \text{ near } (x, \xi)\} \tag{6.1}$$

We make a couple of initial observations:

1. The set of points in $T^*X \setminus 0$ where u is C^∞ is an open conic set, so $WF(u)$ is a closed conic set in $T^*X \setminus 0$.
2. We introduce "symbols in a cone". Let $V \subset T^*X \setminus 0$ be any subset. Then $a \in S^m_{\rho,\delta}(V)$ if $a \in C^\infty(V)$, and for all $\varepsilon > 0$, $\alpha, \beta \in \mathbb{N}^n$ and $W \Subset V$, there exists $C = C_{\varepsilon,\alpha,\beta,W} > 0$ such that $|\partial_x^\alpha \partial_\xi^\beta a| \leq C(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}$ for $(x, \xi) \in W$ and $|\xi| \geq \varepsilon$. The space $S^{-\infty}(V)$ is defined similarly. The general results for symbols from Section 2 remain valid for $S^m_{\rho,\delta}(V)$.

Definition 6.2. *Given a PDO $A \in \Psi^m(X)$, we define its wavefront set as:*

$$WF(A) := \text{smallest closed cone } \Gamma \subset T^*X \setminus 0 \text{ s.t. } \sigma_A|_{\Gamma^c} \in S^{-\infty}(\Gamma^c) \tag{6.2}$$

We make a couple of observations:

1. Given two PDOs A and B, we have that

$$WF(A \circ B) \subset WF(A) \cap WF(B) \tag{6.3}$$

This is because at least one of σ_A and σ_B is $S^{-\infty}$ on $WF(A)^c \cup WF(B)^c$; then so is $\sigma_{A \circ B}$ by the composition formula from Theorem 4.3.

2. Moreover, we have that $WF(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(X)$. This follows from the definitions.

We prove now that the wavefront set behaves well under the action of PDOs.

Lemma 6.3. *If $u \in \mathcal{D}'(X)$ and $A \in \Psi^m(X)$, then*

$$WF(Au) \subset WF(A) \cap WF(u) \tag{6.4}$$

Proof. Let $(x_0, \xi_0) \in \Big(T^*X \setminus 0\Big) \setminus \Big(WF(A) \cap WF(u)\Big)$. We consider two cases:

Case 1: $(x_0, \xi_0) \notin WF(A)$. Then if $B \in \Psi^0(X)$ non-characteristic at (x_0, ξ_0), with $WF(B) \cap WF(A) = \emptyset$, we have $BA \in \Psi^{-\infty}(X)$ and so $BAu \in C^\infty(X)$. This means that $(x_0, \xi_0) \notin WF(Au)$. To construct B, take a positive homogeneous of order one symbol, supported close to ξ_0, multiplied with an appropriate cut-off.

Case 2: $(x_0, \xi_0) \notin WF(u)$. Let $B \in \Psi^m(X)$ be non-characteristic at (x_0, ξ_0), such that $Bu \in C^\infty(X)$. Let V be an open conic neighbourhood of (x_0, ξ_0), such that B non-characteristic at every point of V. Then we can construct $c \in S^{-m}(V)$ such that

$$c\#\sigma_B \equiv 1 \mod S^{-\infty}(V)$$

This can be done by following the parametrix construction in Theorem 5.1. Here we introduce the notation for expressions appearing in the symbol of a composition by (c.f. Theorem 4.3)

$$a\#b := \sum_{\alpha \geq 0} \frac{\partial_x^\alpha a D_\xi^\alpha b}{\alpha!}$$

Let $\chi(x, \xi) \in C^\infty(T^*X)$, such that $\chi = 0$ for small $|\xi|$, χ positive homogeneous of degree zero for $|\xi| \geq 1$, $\chi = 1$ on an open conic neighbourhood W, where $(x_0, \xi_0) \in W \Subset V$ and $\operatorname{supp}(\chi) \Subset V' \subset V$, where V' some cone.

Define $C_1 \in \Psi^{-m}(X)$ with symbol $c_1 = \chi c \in S^{-m}(X)$; then $D := C_1 \circ B \in \Psi^0(X)$ satisfies $\sigma_D \equiv \chi c\#\sigma_B \mod S^{-\infty}$. Thus $Bu \in C^\infty(X)$ implies $Du \in C^\infty(X)$. On the other hand, $\sigma_D \equiv 1 \mod S^{-\infty}$ in W by the construction.

Take $E \in \Psi^0(X)$ non-characteristic at (x_0, ξ_0) with $WF(E) \subset W$. Then by (6.3) we get $WF\big(EA(Id - D)\big) \subset W \cap W^c = \emptyset$ and so $EA \equiv EAD \mod \Psi^{-\infty}$. So

$$EAu \equiv EA(Du) \equiv 0 \mod C^\infty(X)$$

Then by definition, $(x_0, \xi_0) \notin WF(Au)$. □

The next claim rigorously relates the notions of the wavefront set and the singular support of a distribution.

Proposition 6.4. *Let $\Pi : T^*X \to X$ be the natural projection. Then $\Pi\big(WF(u)\big) = \operatorname{sing\,supp}(u)$.*

Proof. We show the inclusions from both sides and separate two cases:

Step 1: $\Pi\big(WF(u)\big) \subset \operatorname{sing\,supp}(u)$. Let $x_0 \in X \setminus \operatorname{sing\,supp}(u)$. By definition, there exists $\chi \in C_0^\infty(X)$ such that $\chi u \in C^\infty(X)$. Multiplication with χ is a PDO with symbol $\chi \in S^0(X)$, which is non-characteristic at every (x_0, χ_0), for $\chi_0 \neq 0$. This ends the first part of the proof.

end of Lecture 9, 12.12.2018.

Step 2: $\operatorname{sing\,supp}(u) \subset \Pi\big(WF(u)\big)$. Let $x_0 \in \Pi\big(WF(u)\big)$. By definition, for every $\xi \in S^{n-1}$ there exists $A_\xi \in \Psi^0(X)$,[14] non-characteristic at (x_0, ξ) such that $A_\xi u \in C^\infty(X)$. By compactness, we may find finitely many $A_1, \dots, A_N \in \Psi^0(X)$ such that for every $\xi \in S^{n-1}$, at least one of A_j is non-characteristic at (x_0, ξ). Then

$$A := \sum_{j=1}^N A_j^* A_j \quad \in \Psi^0(X) \tag{6.5}$$

[14]Note that we may always assume $m = 0$ in the definition 6.1, by post-composing with a parametrix type operator C_1 constructed as in the second step of the proof of Lemma 6.3.

is non-characteristic at every point (x_0, ξ), $\xi \neq 0$. So by the parametrix construction (c.f. Theorem 5.1), we have that $Au \in C^\infty(X)$ implies $u \in C^\infty(X)$ near x_0. This finishes the proof. ☐

Finally, we give an alternative definition of the wavefront set, in terms of directions in which the Fourier transform is rapidly decaying.

Proposition 6.5. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \in T^*X \setminus 0$. Then we have $(x_0, \xi_0) \notin WF(u)$ if and only if there exists $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ and a conical neighbourhood Γ of ξ_0 in $\mathbb{R}^n \setminus 0$, such that for all $N > 0$*

$$|\widehat{\varphi u}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \quad \text{for } \xi \in \Gamma \tag{6.6}$$

Proof. First assume that φ, u, Γ satisfy (6.6). Let $\chi \in S^0(\mathbb{R}^n)$ such that $\mathrm{supp}(\chi) \subset \Gamma$, χ positive homogeneous for $|\xi| \geq 1$ of degree zero and $\chi(t\xi_0) = 1$ for large t. Then define, for $u \in C_0^\infty(X)$

$$Au(x) := (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} \varphi(x)\varphi(y)\chi(\xi)u(y)dyd\xi \tag{6.7}$$

Then $a(x, y, \xi) = \varphi(x)\varphi(y)\chi(\xi) \in S^0$ and so $A \in \Psi^0(X)$. Also, we have

$$Au(x) = \varphi(x)\mathcal{F}^{-1}\big(\chi(\xi)\widehat{\varphi u}(\xi)\big) \tag{6.8}$$

Now we know that $\chi(\xi)\widehat{\varphi u}$ is rapidly decaying and so $\mathcal{F}^{-1}\big(\chi(\xi)\widehat{\varphi u}(\xi)\big) \in H^N(\mathbb{R}^n)$ for all N. By Sobolev embedding (Proposition 5.7 6.) we obtain $Au \in C^\infty(X)$. Now A is non-characteristic at (x_0, ξ_0) by construction, so $(x_0, \xi_0) \notin WF(u)$.

For the other direction, assume $(x_0, \xi_0) \notin WF(u)$, so there is a properly supported $A \in \Psi^0(X)$ non-characteristic at (x_0, ξ_0) with $Au \in C^\infty(X)$. Let now $\chi(\xi) \in S^0(\mathbb{R}^n)$ be a zero order symbol, positively homogeneous of order zero for $|\xi| \geq 1$ and $\chi(t\xi_0) = 1$ for $t \gg 1$ and $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ and sufficiently small support. By the parametrix construction (c.f. Theorem 5.1), there is a $B \in \Psi^0(X)$ and $R \in \Psi^{-\infty}(X)$ properly supported with

$$\chi(D)\varphi = B \circ A + R \tag{6.9}$$

More precisely, we first construct $B_0 \in \Psi^0(X)$ with $\sigma_{B_0} = \frac{\chi(\xi)\varphi(x)}{\sigma_A(x,\xi)}$, so equation (6.9) holds modulo S^{-1} (c.f. Lemma 5.2). We then iterate this to obtain $B_1 \in \Psi^{-1}$, so (6.9) holds for $B = B_0 + B_1$ modulo S^{-2} and so on. Note that φ and χ are such that σ_A is non-characteristic near $\mathrm{supp}(\chi\varphi)$. Write $v := \varphi u \in \mathcal{E}'(X)$. Thus we have $\chi(D)v \in C^\infty(X)$; by pseudolocality $\chi(D)v \in C^\infty(\mathbb{R}^n)$.

Claim. We have $\chi(D)v \in \mathcal{S}(\mathbb{R}^n)$.

To prove the claim, we note that we may write $\chi(D) = k*$, where we have

$$k(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\chi(\xi)d\xi \quad \in \mathcal{D}'(\mathbb{R}^n)$$

As in the proof of Theorem 5.8, by integrating by parts using $(-\Delta_\xi)^N$ we get that $k \in \mathcal{S}(\mathbb{R}^n)$ outside zero. Thus for large x (outside $\mathrm{supp}\, v$), the integral

$$\chi(D)v(x) = \int_{\mathbb{R}^n} k(x - y)v(y)dy \tag{6.10}$$

converges absolutely. Since $|x| \lesssim |x-y|$ for $y \in \mathrm{supp}\, v$ and x large, repeatedly differentiating, this directly implies that $\chi(D)v \in \mathcal{S}(\mathbb{R}^n)$ and proves the claim.

Given the claim, we have $\chi(\xi)\widehat{\varphi u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, which shows the decay condition (6.6). ☐

Remark 6.6. Note that for $\varphi \in C_0^\infty(X)$ and $u \in \mathcal{D}'(X)$, the Fourier transform $\widehat{\varphi u}(\xi)$ belongs to $C^\infty(X)$, and is moreover analytic. Also, we have $\widehat{\varphi u}(\xi) = \langle u(x), \varphi(x)e^{-ix\cdot\xi}\rangle$ (see [3, Theorem 8.4.1.] for a proof). If N is the order of u, by distribution estimates this implies for some $C > 0$ (c.f. estimate (1.10) or [3, Lemma 8.4.1.])

$$|\widehat{\varphi u}(\xi)| \leq C(1 + |\xi|)^N$$

For the Fourier-Laplace transform and more precise relations between $u \in \mathcal{E}'(\mathbb{R}^n)$ and the decay of $\widehat{u}(\xi)$ as $\xi \to \infty$, see the celebrated Paley-Wiener theorem [3, Chapter 10].

Example 6.7. Here are some simple examples, where we will use Proposition 6.5:

1. $WF(\delta) = \{(0, \xi) \mid \xi \neq 0\}$. Here $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ is the Dirac delta. This follows since $\widehat{\varphi\delta}(\xi) = \varphi(0)$ for $\varphi \in C_0^\infty(X)$.
2. $WF(\delta \otimes f) = \{(0, x'', \xi', 0) \mid x'' \in \operatorname{supp}(f)\}$. Here $u = \delta(x') \otimes f(x'')$, for $x' \in \mathbb{R}^{n'}$, $x'' \in \mathbb{R}^{n''}$ and $f \in C^\infty(\mathbb{R}^{n''})$.

   We first assume $f(x'') \neq 0$ for all x''. Then $WF(\delta\otimes f) = WF(\delta\otimes 1)$, by Lemma 6.3. Secondly, if $\psi \in C_0^\infty(\mathbb{R}^{n'+n''})$, then $\widehat{\psi\delta \otimes 1}(\xi', \xi'') = \widehat{\psi_0}(\xi'')$, where $\psi_0(x'') = \psi(0, x'')$. If $x' \neq 0$, then $u = 0$; if $\xi'' \neq 0$, then $\widehat{\psi_0}$ is of rapid decay near ξ'', so $WF(u) \subset \{(0, x'', \xi', 0)\}$. Assume now $\psi(0, x'') \neq 0$. The other inclusion follows by observing that for $\widehat{\psi_0}(\xi'')$ to be of rapid decay near $(\xi', 0)$, $\widehat{\psi_0}(\xi'') \equiv 0$ for small ξ'', so $\psi_0 \equiv 0$ by analyticity, contradicting the assumption.

   The case when f is allowed to vanish follows similarly.
3. $WF(H(x_1) \otimes 1(x'')) = \{(0, x'', \xi_1, 0)\}$. Here $H(x_1)$ is the 1D Heaviside step function, $H(x_1) = 1$ for $x_1 \geq 0$ and zero otherwise; $x'' \in \mathbb{R}^{n-1}$.

   Denote $F(x_1, x'') = F(x) = H(x_1) \otimes 1(x'')$. First observe that $\frac{\partial F}{\partial x_1} = \delta(x_1) \otimes 1(x'')$, so by Lemma 6.3 and point 1. above, $\{0, x'', \xi_1, 0\} \subset WF(F)$.

   To show this is all, consider $(x_1, x'', \xi_0, \xi_0'')$ and note if $x_1 \neq 0$, then F smooth; if $x_1 = 0$ and $\xi_0'' \neq 0$ we let $\varphi \in C_0^\infty(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R}^{n-1})$. Then $\widehat{\varphi H} \otimes \widehat{\psi}(\xi_1, \xi'')$ and consider $\varepsilon > 0$, such that the cone $V = \{|\xi_1| < \varepsilon|\xi''|\}$ contains (ξ_0, ξ_0''). Since $\widehat{\psi}(\xi'')$ is Schwartz, we see that in V, $\widehat{\varphi H}\otimes\widehat{\psi}(\xi_1, \xi'')$ is of rapid decay, so $(0, x'', \xi_0, \xi_0'') \notin WF(F)$.
4. Let φ be a real-valued phase function on $X \times \mathbb{R}^n \setminus 0$ and let $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ with $\delta < 1$, $\rho > 0$. We put:

$$\Lambda_\varphi = \{(x, \varphi_x'(x, \theta)) \mid \varphi_\theta'(x, \theta) = 0, \theta \neq 0\} \tag{6.11}$$

   which is a closed conic set in $T^*X \setminus 0$. Then we claim that

$$WF(I(a, \varphi)) \subset \Lambda_\varphi$$

   To prove this, we first recall the critical set C_φ given by $C_\varphi = \{(x, \theta) \in X \times \mathbb{R}^N \setminus 0 \mid \varphi_\theta'(x, \theta) = 0\}$. Then by Proposition 2.12 we may assume that a is supported in a small conic closed neighbourhood Γ of C_φ without affecting the wavefront set.

   Then, choose $\psi \in C_0^\infty(X)$ supported near x_0, such that $\xi_0 \neq \varphi_x'(x, \theta) \neq 0$ for all $(x, \theta) \in \Big(\operatorname{supp}\psi \times \mathbb{R}^N \setminus 0\Big) \cap \Gamma$ (we can do so as φ a phase function and by the defining property of C_φ).

   Now, by homogeneity there is a $C > 0$ such that

$$\Big|\frac{\partial}{\partial x}(\varphi(x, \theta) - x \cdot \xi)\Big| \geq \frac{1}{C}(|\theta| + |\xi|) \tag{6.12}$$

for $(x, \theta) \in \Big(\operatorname{supp}\psi \times \mathbb{R}^N \setminus 0\Big) \cap \Gamma$ and for ξ in a small conic neighbourhood V of ξ_0. Here it is important that V and the cone determined by $\varphi'_x(x,\theta)$ are disjoint.

Now define ${}^tL = \frac{1}{|\varphi'_x(x,\theta)-\xi|^2}\sum_{j=1}^n \big(\frac{\partial\varphi}{\partial x_j} - \xi_j\big) D_{x_j}$ and note that ${}^tLe^{i\varphi} = e^{i\varphi}$. Then we write

$$J(\xi) = \widehat{\psi I(a,\varphi)}(\xi) = \int\int e^{i(\varphi(x,\theta)-x\cdot\xi)}a(x,\theta)\psi(x)dxd\theta$$

We integrate by parts using L many times and use the inequality (6.12) to see that the coefficients of L are in S^{-1} in the cone V. Thus $J(\xi)$ is of rapid decay for ξ in V and so $(x_0, \xi_0) \notin WF(I(a,\varphi))$, which finishes the proof.

We collect a few remarks about what we discussed so far and state a few influental results.

Remark 6.8 (Propagation of singularities)**.** Let $P \in \Psi^m_{cl}(X)$ and assume $Pu \in C^\infty(X)$ for $u \in \mathcal{D}'(X)$. By definition, then $WF(u) \in p^{-1}(0) \subset T^*X \setminus 0$, where $p \in C^\infty(T^*X \setminus 0)$ is the principal symbol of P, positively homogeneous of degree m.

Question. Which subsets of $p^{-1}(0)$ can be achieved as $WF(u)$, for some u as above?

Partial answer. Necessary conditions are given by **propagation of singularities results**. To formulate this result, we need some notation. We say P is of real principal type if p is real-valued and $dp(x,\xi)$ and $\sum \xi_j dx_j$ are linearly independent for $\xi \neq 0$ and all x.

Next, we consider the canonical symplectic structure on T^*X, given by a closed, non-degenerate differential two form ω. It defines a Hamiltonian vector field H_p by $\omega(H_p, \cdot) = dp(\cdot)$. Note that H_p keeps $p^{-1}(0)$ invariant.

Assume P is of real principal type and let $\gamma : [a,b] \to T^*X \setminus 0$ be an integral curve of H_p in $p^{-1}(0)$. The statement is that then either $\gamma([a,b]) \subset WF(u)$ or $\gamma([a,b]) \cap WF(u) = \emptyset$. See [4, Chapter 8] for more details. Note that the theorem implies

$$WF(u) = \cup_{\gamma\in S}\gamma \subset T^*X \setminus 0$$

where S is a suitable subset of the set of all integral curves of H_p. One can formulate a version of the theorem for $Pu \in \mathcal{D}'(X)$ as well.

For a version of this theorem formulated for the wave operator $\mathbb{R} \times \mathbb{R}^3$, see [3, Section 11.5.].

6.1. **Operations with distributions: pullbacks and products.** Here we outline some remarks about pullbacks and products of distributions in slightly more detail than at the lectures. It will be convenient to introduce a notation for distributions with a wavefront set condition, so for $\Gamma \subset T^*X \setminus 0$ be a closed conic subset:

$$\mathcal{D}'_\Gamma(X) := \{u \in \mathcal{D}'(X) \mid WF(u) \subset \Gamma\}, \quad \mathcal{E}'_\Gamma(X) := \mathcal{E}'(X) \cap \mathcal{D}'_\Gamma(X) \tag{6.13}$$

We equip $\mathcal{D}'_\Gamma(X)$ with the LCS topology given by seminorms $P_\varphi(u) = |\langle\varphi, u\rangle|$, $\varphi \in C_0^\infty(X)$, as well as seminorms

$$P_{\varphi,V,N}(u) = \sup_{\xi\in V}|\widehat{\varphi u}(\xi)|(1+|\xi|)^N \tag{6.14}$$

where $N \geq 0$, $V \subset \mathbb{R}^n$ a closed cone with $(\operatorname{supp}\varphi \times V) \cap \Gamma = \emptyset$. Note that $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ implies $u_j \to u$ weakly; also, it is true that $C^\infty(X) \subset \mathcal{D}'_\Gamma(X)$ is dense [4, Proposition 7.5.]. However, for the remainder of this section we ignore the topological issues and point to references for such details.

We start with a couple of basic propositions; here $Y \subset \mathbb{R}^m$ an open set.

Proposition 6.9. *Let $\Gamma_1 \subset T^*X \setminus 0$ and $\Gamma_2 \subset T^*Y \setminus 0$ be two closed cones. Then the map*

$$(u, v) \mapsto u \otimes v, \quad \mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(Y) \to \mathcal{D}'_{\Gamma_1 \times \Gamma_2 \cup \Gamma_1 \times O_Y \cup O_X \times \Gamma_2}(X \times Y) \tag{6.15}$$

is continuous for sequences. Here $O_X = X \times \{0\}$ and $O_Y = Y \times \{0\}$.

Proof. We just prove the upper bound on the wavefront set, whereas the continuity statement is left as an exercise. We will use Remark 6.6 below.

Let $(x_0, y_0, \xi_0, \eta_0) \in T^*(X \times Y) \setminus 0$, s.t. both $\xi_0 \neq 0$ and $\eta_0 \neq 0$. Take cutoffs $\varphi \in C_0^\infty(X)$ and $\psi \in C_0^\infty(Y)$. Then if $\widehat{\varphi u}(\xi)$ is of rapid decrease near ξ_0 or $\widehat{\varphi v}(\eta)$ is of rapid decrease near η_0, then so is $\widehat{\varphi u} \otimes \widehat{\psi v}(\xi, \eta)$ near (ξ_0, η_0). So for this range of cotangent vectors, $WF(u \otimes v) \subset \Gamma_1 \times \Gamma_2$.

Assume now $\eta_0 = 0$. So $\widehat{\varphi u} \otimes \widehat{\psi v}(\xi, \eta)$ will be rapidly decreasing near $(\xi_0, 0)$ if $\widehat{\varphi u}(\xi)$ is rapidly decreasing near ξ_0 or $\psi v \equiv 0$. Similarly for $\xi_0 = 0$, and so we have for this range of cotangent vectors $WF(u \otimes v) \subset \Gamma_1 \times O_Y \cup O_X \times \Gamma_2$[15], which finishes the proof. □

Recall our big statement about Sobolev spaces, Proposition 5.7 and the point 3.: one can extend the pairing $(u, v) \mapsto \int uv dx$ to distributions, as long as Sobolev indices of u and v add up to a non-negative number.

For a set $V \subset T^*X$, we define $-V := \{(x, -\xi) \mid (x, \xi) \in V\}$. We then have the following:

Proposition 6.10. *Let $\Gamma_1, \Gamma_2 \subset T^*X \setminus 0$ be closed cones with $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Then the map*

$$(u, v) \mapsto \langle u, v \rangle = \int u(x)v(x)dx, \quad C^\infty(X) \times C_0^\infty(X) \to \mathbb{C} \tag{6.16}$$

has a unique, sequentially continuous extension to $\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{E}'_{\Gamma_2}(X) \to \mathbb{C}$.

Proof. As before, we only prove the existence of such a map; for continuity see [4, Proposition 7.6.]. Uniqueness then follows by density of $C^\infty(X)$ in $\mathcal{D}'(X)$.

Let $u \in \mathcal{D}'_{\Gamma_1}(X)$ and $v \in \mathcal{E}'_{\Gamma_2}(X)$. We first make a local construction: let $x_0 \in X$. Take $\varphi \in C_0^\infty(X)$ with $\varphi(x_0) \neq 0$ such that $V_1 \cap -V_2 = \emptyset$, where $V_j = \{\xi \in \mathbb{R}^n \setminus 0 \mid \exists x \in \operatorname{supp} \varphi,\ (x, \xi) \in \Gamma_j\}$ be the union of all directions in Γ_j near x_0. Then define $\langle \varphi u, \varphi v \rangle$ by Parseval's formula (c.f. Proposition 1.9 2.):

$$\langle \varphi u, \varphi v \rangle = (2\pi)^{-n} \int \widehat{\varphi u}(\xi) \widehat{\varphi v}(-\xi) d\xi \tag{6.17}$$

The integral above converges, since $\widehat{\varphi u}(\xi)$ and $\widehat{\varphi v}(-\xi)$ are of rapid decrease outside any conic neighbourhood of V_1 and V_2, respectively (and by Remark 6.6).

For a global definition, let $\{\varphi_j^2\}_j$ be a locally finite partition of unity, s.t. $\sum \varphi_j^2 = 1$ with each φ_j as above. Define $\langle u, v \rangle := \sum \langle \varphi_j u, \varphi_j v \rangle$ with each summand defined as above. □

We now discuss mapping properties of wavefront sets under integral transforms. Given an operator $K : C_0^\infty(Y) \to \mathcal{D}'(X)$, we consider its kernel $K \in \mathcal{D}'(X \times Y)$ by the same letter and denote:

$$WF'(K) := \{(x, \xi; y, -\eta) \in T^*(X \times Y) \setminus 0 \mid (x, \xi; y, \eta) \in WF(K)\}$$
$$WF'_X(K) := \{(x, \xi) \in T^*X \setminus 0 \mid \exists y \in Y \text{ with } (x, \xi; y, 0) \in WF'(K)\}$$
$$WF'_Y(K) := \{(y, \eta) \in T^*Y \setminus 0 \mid \exists x \in X \text{ with } (x, 0; y, \eta) \in WF'(K)\}$$

[15]This actually proves $WF(u \otimes v) \subset \Gamma_1 \times (\operatorname{supp} v \times \{0\}) \cup (\operatorname{supp} u \times \{0\}) \times \Gamma_2$.

If $WF'(K)$ is considered as a (set-theoretic) relation $T^*Y \to T^*X$, then $WF'_X(K)$ is the image of O_Y and $WF'_Y(K)$ is the inverse image of O_X. The elements of $T^*(X\times Y)$ will be denoted by $(x,\xi;y,\eta)$. We now state a theorem about the action of K.

Let $\Gamma \subset T^*Y\setminus 0$ be a closed cone with $WF'_Y(K)\cap\Gamma = \emptyset$ and let $\widetilde{\Gamma} = WF'(K)(\Gamma)\cup WF'_X(K)$. Then K has a unique, sequentially continuous extension

$$K : \mathcal{D}'_\Gamma(Y) \to \mathcal{D}'_{\widetilde{\Gamma}}(X) \tag{6.18}$$

We give no proof of this fact: for details, see [4, Theorem 7.8.].

We now specialise to pullbacks and consider a C^∞ map $\mathcal{H} : X \to Y$ and the operator $\mathcal{H}^*$ acting on $C_0^\infty(Y)$ by $\mathcal{H}^*(u)(x) = u(\mathcal{H}(x))$, so we can write:

$$\mathcal{H}^*u(x) = (2\pi)^{-n}\int\int e^{i(\mathcal{H}(x)-y)\cdot\eta}u(y)dyd\eta$$

Similarly to Section 4.1, $\mathcal{H}^*$ is a FIO with phase $(\mathcal{H}(x)-y)\cdot\eta$, and by Example 6.7 4., we have for $A = \mathcal{H}^*$

$$WF'(A) \subset \{(x,\xi;y,\eta) \mid y = \mathcal{H}(x) \text{ and } \xi = {}^t\mathcal{H}'(x)\eta\}$$

By definition it follows that $WF'_X(A) = \emptyset$ and that $WF'_Y(A) \subset \{(y,\eta) \mid \exists x\in X \text{ with } y = \mathcal{H}(x),\ {}^t\mathcal{H}'(x)\eta = 0\}$. By putting $K = \mathcal{H}^* = A$ in equation (6.18), we have:

Lemma 6.11. *If $u \in \mathcal{D}'(Y)$ and $WF(u)\cap WF'_Y(\mathcal{H}^*) = \emptyset$, then $\mathcal{H}^*u$ is well-defined in $\mathcal{D}'(X)$ and*

$$WF(\mathcal{H}^*u) \subset \{(x,\xi)\in T^*X\setminus 0 \mid \exists(y,\eta)\in WF(u),\ y = \mathcal{H}(x),\ \xi = {}^t\mathcal{H}'(x)\eta\}$$

We are finally ready to make a statement about products of distributions.

Theorem 6.12. *Let $u_1, u_2 \in \mathcal{D}'(X)$ satisfy $WF(u_1)\cap -WF(u_2) = \emptyset$. Then u_1u_2 is well defined in $\mathcal{D}'(X)$ and*

$$WF(u_1u_2) \subset \{(x,\xi_1+\xi_2) \mid (x,\xi_j)\in(\operatorname{supp} u_j\times\{0\})\cup WF(u_j)\} \tag{6.19}$$

Proof. Let $\mathcal{H} : X \ni x \mapsto (x,x) \in X\times X$. Then we define $u_1u_2 := \mathcal{H}^*(u_1\otimes u_2)$. We may do so by Proposition 6.9 and Lemma 6.11; note that ${}^t\mathcal{H}'(x)(\xi_1,\xi_2) = \xi_1+\xi_2$, where $(x,x,\xi_1,\xi_2)\in T^*(X\times X)$. The upper bound on the wavefront of u_1u_2 is also a consequence of the lemma above. □

Remark 6.13. We make a couple of remarks on the wavefront set under coordinate changes.

1. Let $\mathcal{H} : X \to Y$ be a diffeomorphism and $u \in \mathcal{D}'(Y)$. Then by Lemma 6.11

   $$WF(\mathcal{H}^*u) = \{(x, {}^t\mathcal{H}'(x)\xi) \mid (\mathcal{H}(x),\xi)\in WF(u)\}$$

   This implies that the wavefront set transforms as a subset of the cotangent space and thus can be defined for distributions on manifolds.
2. In particular, if $S\subset\mathbb{R}^n$ is a hypersurface, we may define δ_S to be the associated delta distribution, given by $\delta_S : \varphi \mapsto \int_S \varphi dvol_S$, where $dvol_S$ is the volume element of S. The point 1. above and the point 2. in Example 6.7 show $WF(\delta_S) = N^*S\setminus 0$, where $N^*S = \{(x,\xi)\in T^*\mathbb{R}^n \mid x\in S,\ \langle\xi,v\rangle_x = 0 \text{ for all } v\in T_xS\}$ is the conormal bundle.
3. Let $\mathcal{H} : S \hookrightarrow \mathbb{R}^n$ is the inclusion of a hypersurface and assume $u\in\mathcal{D}'(\mathbb{R}^n)$. By Lemma 6.11, we may form the *restriction* $\mathcal{H}^*u$ if

   $$WF(u)\cap N^*S = \emptyset$$

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end of Lecture 10, 9.1.2019.

7. Manifolds

Here we outline how to build the theory of distributions, calculus of pseudodifferential operators and Sobolev spaces on compact manifolds. For this purpose, let M be a compact smooth manifold. We will follow mostly [7, Chapter 14].

We denote by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M and by $T^*M = \cup_{x\in M} T^*_xM$ the cotangent bundle, where T^*_xM denotes the dual vector space to T_xM. Recall that a vector bundle $\pi : V \to M$ of rank N is determined by a system of trivialisations $\{(U_i, \psi_i)\}_{i\in I}$, where I an indexing set, $U_i \subset M$ open sets and $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^N$ an ismorphism. Furthermore, it is enough to give *transition maps* for $i, j \in I$

$$\gamma_{ij} = \psi_i \circ \psi_j^{-1} \in C^\infty(U_i \cap U_j, GL(N, \mathbb{C}))$$

satisfying the natural *cocycle conditions*, to define uniquely V. We now discuss several relevant constructions to transfer the calculus in $\mathbb{R}^n$ to manifolds.

s-density bundles $\Omega^s(M)$. Let $\{(U_i, \varphi_i)\}$ be a set of coordinate charts of M, where $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ diffeomorphisms. Define $\Omega^s(M)$ for $0 \leq s \leq 1$ to be a line bundle (rank 1) with transition functions given by, for $x \in U_i \cap U_j$

$$\gamma_{ij}(x) = |\det(d(\varphi_j \circ \varphi_i^{-1}))|^s \circ \varphi_i(x)$$

Now given a section $u \in C^\infty(M; \Omega^1(M))$,[16] one may define invariantly the integral[17]

$$\int_M u$$

Riemannian manifolds. A Riemannian manifold (M, g) is a manifold M equipped with a smooth inner product g_x on fibers T_xM; in other words $g \in C^\infty(M; T^*M \otimes T^*M)$. Given a pair (M, g), there is a canonical 1-density given in local coordinates $(x_1, x_2, \dots, x_n)$ by

$$\omega_g := \sqrt{g}|dx_1 \wedge \dots dx_n|$$

Here $\sqrt{g}$ is a shorthand for $\sqrt{\det g}$. Here $|dx_1 \wedge \dots dx_n|$ denotes 1-density induced by the coordinate system. It is an exercise to check that ω_g is well-defined independently of coordinates. We will sometimes denote ω_g simply by dx.

Musical isomorphism. Given a metric g on M, there is an identification (isomorphism) between TM and T^*M. Explicitly, it is given by

$$T^*_xM \ni \xi \mapsto v \in T_xM, \quad \text{where} \quad \xi(u) = g_x(u, v) \text{ for all } u \in T_xM$$

An inner product is then also induced on fibers T^*_xM by asking $g_x(\xi, \xi) := g_x(v, v)$.

Symplectic structure on T^*M. We define the *canonical* 1*-form* on T^*M by setting for each $(x, \xi) \in T^*M$ and $\eta \in T_{x,\xi}T^*M$

$$\theta_{(x,\xi)}(\eta) := \eta(d\pi\xi)$$

Here $\pi : T^*M \to M$ is the canonical projection. Now define the *symplectic* 2*-form* by $\omega := d\theta$. Thus ω is exact. By going to local coordinates we see $\theta = \sum_{i=1}^n \xi_i dx_i$, so $\omega = \sum_{i=1}^n d\xi_i \wedge dx_i$ is seen to be non-degenerate.

We may thus define the *symplectic volume form* by $\omega^n = \omega \wedge \dots \wedge \omega$ (we wedge n times), which defines a 1-density on T^*M and so a way to integrate functions.

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[16]A C^∞ map $u : M \to \Omega^1(M)$ such that $\pi \circ u(x) = x$ for all $x \in M$.

[17]Hint: take a partition of unity $\{\rho_i\}$ subordinate to a cover by coordinate charts U_i. In a local trivialisation define the integral simply by integrating with respect to Lebesgue measure. Check that the integral doesn't depend on any choices made.

Hamiltonian dynamics. Given a real-valued function $p \in C^\infty(T^*M)$, or in other words a classical observable, we may define the *Hamiltonian vector field* H_p on T^*M via

$$\omega(H_p, v) = dp(v) \quad \text{for all} \quad v \in TM \tag{7.1}$$

This is well-defined as ω non-degenerate. The flow ψ^t defined by H_p is called the *Hamiltonian flow* of p.

We claim two things: ψ^t preserves the symplectic volume and the level sets of p. The proof is simple and we give it here for completeness. First note that $\mathcal{L}_{H_p}\omega = d\iota_{H_p}\omega = ddp = 0$ and so $\mathcal{L}_{H_p}\omega^n = 0$ by Leibniz's rule.[18] Therefore ψ^t is volume-preserving. Next, note that $\frac{\partial p \circ \psi^t}{\partial t}|_{t=0} = H_p p = \omega(H_p, H_p) = 0$, so p is indeed constant along the flow ψ^t.

Function spaces on manifolds. We start by carrying over the notion of a distribution to a manifold.

Definition 7.1. *A linear map $u : C^\infty(M) \to \mathbb{C}$ is a* distribution *on M if there is a finite set of charts $\{(U_i, \varphi_i)\}_{i=1}^N$ covering M, such that $\varphi_{i*}u$ is a distribution on $\varphi(U_i) \subset \mathbb{R}^n$ for each i, where*

$$\langle \varphi_{i*}u, \psi\rangle := \langle u, \varphi_i^*\psi\rangle, \quad \text{for all } \psi \in C_0^\infty(\varphi_i(U_i)) \tag{7.2}$$

If u is a distribution on M, we write $u \in \mathcal{D}'(M)$.

Remark 7.2. Firstly, one can check that this definition is coordinate invariant. Secondly, we may define a metric topology on $C^\infty(M)$ coming from charts $\varphi_i(U_i)$ in such a way that $\mathcal{D}'(M) = \big(C^\infty(M)\big)'$. Thirdly and finally, the notion of the wavefront set of a distribution $WF(u) \subset T^*M \setminus 0$ can be defined similarly by going to local charts and using Remark 6.13 to check coordinate independence.

Definition 7.3. *The following generalises L^2-Sobolev space to manifolds*

1. $L^2(M) := \{u : M \to \mathbb{C} \text{ measurable } \mid \|u\|^2_{L^2(M)} := \int_M |u|^2 dx < \infty\}$.
2. *Let $s \in \mathbb{R}$. Let $\{(U_i, \varphi_i)\}_{i=1}^N$ be a cover of M by coordinate charts. Then we say $u \in \mathcal{D}'(M)$ lies in $H^s(M)$, if $\varphi_{i*}u \in H^s(\varphi_i(U_i))$ for all $i = 1, \dots, N$.*

   Take now a finite partition of unity $\{\rho_i\}_{i=1}^N$ subordinate to U_i. Then we define for $u \in H^s(M)$

$$\|u\|_{H^s(M)} := \sum_{i=1}^N \|\varphi_{i*}(\rho_i u)\|_{H^s(\varphi_i(U_i))}$$

Remark 7.4. It is an exercise to show that: $H^s(M)$ is a Hilbert space equipped with the norm above and that all possible norms defined in this way are equivalent. It is also an exercise to show $\mathcal{D}'(M) = \cup_{s\in\mathbb{R}} H^s(M)$ (hint: recall the proof that all distributions in $\mathcal{E}'(\mathbb{R}^n)$ are of finite order).

Pseudodifferential operators on manifolds. For a guideline on how to prove what follows, see [4, Exercise 3.4.].

Definition 7.5. *Let $P : C^\infty(M) \to \mathcal{D}'(M)$ be a linear operator. We say $P \in \Psi^m(M)$, the space of pseudodifferential operators on M of order m, if the following two condition hold:*

1. *There is a covering set of charts $\{(U_i, \varphi_i)\}_{i=1}^N$ and $P_i \in \Psi^m(\varphi_i(U_i))$ such that for all $i = 1, \dots, N$ and all $u \in C_0^\infty(U_i)$*

$$Pu \circ \varphi_i^{-1} = P_i(u \circ \varphi_i^{-1})$$

[18] We used Cartan's magic formula for the Lie derivative $\mathcal{L}_X = \iota_X d + d\iota_X$.

2. *For every* $\chi_1, \chi_2 \in C^\infty(M)$ *with* $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$*, we have* $\chi_1 P \chi_2$ *extending to a map*
$$\chi_1 P \chi_2 : \mathcal{D}'(M) \to C^\infty(M)$$

Remark 7.6. The following points hold:

1. It can be straightforwardly checked from the definition that $P : C^\infty(M) \to C^\infty(M)$.
2. We define the set of *smoothing operators* as $\Psi^{-\infty}(M) := \cap_{s\in\mathbb{R}} \Psi^s(M)$.
3. We may define the spaces $S^m(T^*M)$ as a suitable subspace of $C^\infty(T^*M)$, by asking that a belongs to $S^m(T^*M)$ if locally in a system of charts, we have a belonging to local S^m spaces. To see this is invariant of any choice of coordinates use results proved in Section 4.1, which also explains why the cotangent space is natural to consider.
4. Similarly to the previous definitions, the space $\Psi^m_{cl}(M)$ is defined and so are the spaces of symbols $S^m_{cl}(T^*M)$.
5. There is a suitable principal symbol map, $\sigma : \Psi^m(M) \to S^m(T^*M)/S^{m-1}(T^*M)$. Exercise: define σ locally and use (4.20) to show well-defined. If $A \in \Psi^m_{cl}$, then $\sigma(A) \in S^m_{cl}(T^*M)$ is a well-defined positive homogeneous function of order m in ξ.
6. An operator $P \in \Psi^m(M)$ if it is said to be *elliptic* if its local representatives P_i are elliptic. Equivalently, P is elliptic if $\sigma(P) \in S^m(T^*M)/S^{m-1}(T^*M)$ has a suitable asymptotic growth in the ξ variable.

Theorem 7.7. *The following holds true for PDOs on manifolds*

1. *There is a (non-canonical) quantisation map* $Op : S^m(T^*M) \to \Psi^m(M)$*, such that the following is a short exact sequence*[19]
$$0 \to \Psi^{m-1}(M) \overset{\iota}{\hookrightarrow} \Psi^m(M) \overset{\sigma}{\to} S^m(T^*M)/S^{m-1}(T^*M) \to 0 \tag{7.3}$$
The first map is inclusion, while the second is the principal symbol map. We also have $\sigma(AB) = \sigma(A)\sigma(B)$ *for any two PDOs* A *and* B.
2. *Given* $A \in \Psi^m(M)$*, we have for all* s *that* $A : H^s(M) \to H^{s-m}(M)$*. If* $A \in \Psi^{-\infty}(M)$*, then* $A : H^{s_1}(M) \to H^{s_2}(M)$ *is compact for any* $s_1, s_2 \in \mathbb{R}$.

Remark 7.8. It is important to note that proving this theorem is a great exercise of definitions involved.

Also, given $A \in \Psi^m(M)$, from this theorem we are able to deduce the existence of a parametrix $B \in \Psi^{-m}(M)$ with $AB - Id = K_1 \in \Psi^{-\infty}(M)$ and $BA - Id = K_2 \in \Psi^{-\infty}(M)$ (c.f. the iterative procedure in Theorem 5.1).

8. Quantum ergodicity

Let (M, g) be a compact Riemannian manifold. We consider the Laplace-Beltrami operator $-\Delta_g$, defined locally in $(x_1, \dots, x_n)$ coordinates as
$$-\Delta_g = -\frac{1}{\sqrt{\det g}}\partial_i\big(\sqrt{\det g}\, g^{ij}\partial_j\big) \tag{8.1}$$
One can check $-\Delta_g : C^\infty(M) \to C^\infty(M)$ is a well-defined differential operator. A coordinate invariant way to define it is $-\Delta_g = d^*d$, where d is the exterior derivative and d^* its formal adjoint.

It follows from (8.1) that $\sigma(-\Delta_g)(x,\xi) = |\xi|_g^2$ for $(x,\xi) \in T^*M$ and so $-\Delta_g$ is elliptic. Therefore, there exists a countably infinite spectrum $\lambda_0^2 = 0 < \lambda_1^2 \le \lambda_2^2 \le \ldots \to \infty$ (with

[19]This means that ι is injective, σ is surjective and $\sigma \circ \iota = 0$.

multiplicity), and an orthonormal basis of (smooth) eigenfunctions $\{\varphi_j\}_{j=1}^\infty \subset L^2(M)$, such that[20]

$$-\Delta_g \varphi_j = \lambda_j^2 \varphi_j$$

Question. How does the measure $|\varphi_j|^2 dx$ distribute when $j \to \infty$? We say a sequence of measures μ_n on M converges to μ weakly if for all $f \in C^\infty(M)$

$$\int_M f d\mu_n \xrightarrow{n\to\infty} \int_M f d\mu$$

Example 8.1. Here we answer the above question in a few simple cases.

1. $(M, g) = (S^1, \mu_{Leb})$. Here μ_{Leb} is the Lebesgue measure. The eigenfunctions are given by the Fourier basis: $\sqrt{2}\sin(2\pi nx), \sqrt{2}\cos(2\pi nx)$ for $n \in \mathbb{N}_0$. Then if we take $f \in C^\infty(S^1)$, as the coefficients of Fourier series converge to zero, we have
$$\int_0^1 2\sin^2(2\pi nx) f(x) dx = \int_0^1 f(x)(1-\cos(4\pi nx))dx \xrightarrow{n\to\infty} \int_0^1 f dx$$
Thus $|\varphi_j|^2 dx \to dx$, i.e. we have *equidistribution* as $j \to \infty$.
2. $(M, g) = ([0,1]^2, g_{Eucl})$. Consider the sequence of Dirichlet eigenfunctions
$$u_{jk}(x, y) = 2\sin(j\pi x)\sin(k\pi y)$$
with eigenvalues $\lambda_{jk}^2 = \pi^2(j^2 + k^2)$. From the previous example it follows
   a. $|u_{jj}|^2 dxdy \xrightarrow{j\to\infty} dxdy$, i.e. we have equidistribution.
   b. $|u_{j1}|^2 dxdy \xrightarrow{j\to\infty} 2\sin^2(\pi y) dxdy$, i.e. there is *no* equidisitrubution.
3. $(M, g) = (S^2, g_{round})$. Exercise: using the expression for the Laplacian in spherical coordinates, show that homogeneous, harmonic polynomials v in $\mathbb{R}^3$ of degree m, restrict to eigenfunctions $u = v|_{S^2}$ of $-\Delta_{S^2}$ with eigenvalue $m(m+1)$. Moreover, all eigenfunctions on S^2 arise in this way.
   Take now $v_m = (x_1 + ix_2)^m$ and set $u_m = c_m v_m|_{S^2}$. Here c_m is a constant such that $\|u_m\|_{L^2(S^2)} = 1$. Then $|u_m|^2 dx \to 2\delta_{equator}$ as $m \to \infty$, where $\delta_{equator}$ is the Dirac measure on the equator $\{x_3 = 0\} \cap S^2$.

Remark 8.2. Recall that from Quantum Mechanics, the integrals $\int_A |\varphi_j|^2 dx$ have the interpretation of being the probability of finding a particle in stationary state φ_j, in the subset $A \subset M$. On the other hand, in Classical Mechanics the trajectory of a free particle on (M, g) lies on geodesics; denote the geodesic flow by φ^t. Our aim: show that if φ^t is chaotic (ergodic), then $|\varphi_j|^2 dx$ equidistributes in the high energy limit.

8.1. **Ergodic theory.** We recall a few notions from ergodic theory, which studies statistical properties of dynamical systems (continuous or discrete) in the long time limit.

Recall. Given a point $x \in M$ and a vector $v \in T_xM$, there exists a unique *geodesic* $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$, $\dot\gamma(0) = v$. A geodesic satisfies a system of second order ordinary

[20]Exercise: prove this by using the tools developed in this course, together with the spectral theorem for compact operators. Hint: consider the formally self-adjoint $1 - \Delta_g : H^2(M) \to L^2(M)$, prove it is an isomorphism and use $H^2(M) \hookrightarrow L^2(M)$ compact.

Also, if M has a boundary, we may take the *Dirichlet conditions*, i.e. $\varphi_j|_{\partial M} = 0$. Then one has the same basic result about the spectrum.

differential equations locally, which can be seen as either Euler-Lagrange equations for the functional

$$E(\gamma) = \int_a^b |\dot\gamma(t)|^2 dt$$

or more invariantly, as $\nabla_{\dot\gamma}\dot\gamma = 0$. Here ∇ is the Levi-Civita connection. Locally, geodesics minimise distance. It can be seen that geodesics have constant speed, i.e. $|\dot\gamma(t)| = |\dot\gamma(0)|$ for all time t.

Consider now the unit sphere bundle $SM = \{(x, v) \in TM \mid |v| = 1\}$; similarly we introduce the unit cosphere bundle $S^*M = \{(x, \xi) \in T^*M \mid |\xi| = 1\}$.

We define the *geodesic flow* $\varphi^t : SM \to SM$ by asking

$$\varphi^t(x, v) = (\gamma(t), \dot\gamma(t))$$

where γ is the unique geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. One then sees that $\varphi^0 = Id$ and $\varphi^{s+t} = \varphi^s \circ \varphi^t$ for all $s, t \in \mathbb{R}$. By the musical isomorphism, we obtain a flow on S^*M that we still denote by φ^t.

Define $X := \frac{\partial\varphi^t}{\partial t}|_{t=0}$ to be the *geodesic vector field*. Note X is a vector field on SM.

Liouville measure. We equip each hypersurface $\{|\xi| = c\} \subset T^*M$ by a measure μ_c induced by the property

$$\int\int_{a\leq|\xi|\leq b} f dx d\xi = \int_a^b \int_{|\xi|=c} f d\mu_c dc \tag{8.2}$$

for all $f \in C(\{a \leq |\xi| \leq b\})$. Here $\omega^n = dxd\xi$ is the symplectic volume. It can be checked that the geodesic flow is the Hamiltonian flow on T^*M for the Hamiltonian function $p(x, \xi) = |\xi|_x^2$.[21] One then sees that $d\mu_c$ is invariant by the geodesic flow, i.e. $\varphi^{t*}\mu_L = \mu_L$ for all time t, by using the fact that Hamiltonian flows preserve the symplectic volume and the level sets (c.f. lines around 7.1). The probability measure $\mu_L := \mu_1$ will be called the Liouville measure.

Definition 8.3. *We say that the geodesic flow is ergodic on SM (S^*M) or just that (M, g) is ergodic if for any $A \subset SM$ (S^*M) that satisfies $\varphi^t(A) = A$ for all $t \in \mathbb{R}$, we have $\mu_L(A) \in \{0, 1\}$.*

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end of Lecture 11, 16.1.2019.

Remark 8.4. Here are some important points about ergodicity:

1. (S^2, g_{round}) is *not ergodic*. Take a small conical neighbourhood N of (x, v), where x lies on the equator and v in its direction. Iterate N by φ^t and obtain an invariant set that has measure strictly between 0 and 1.
2. In the sense of measure theory, ergodicity of (SM, φ^t) is same as *irreducibility*, i.e. it means that SM cannot be split into two non-trivial, invariant pieces.
3. It is well known that if (M, g) has negative sectional curvature, then φ^t is ergodic.

Theorem 8.5 (L^2-von Neumann mean ergodic theorem)**.** *Assume φ^t is ergodic. Then for $a \in L^2(S^*M) = L^2(S^*M, \mu_L)$, we have*

$$\langle a\rangle_T := \frac{1}{T}\int_0^T a \circ \varphi^t dt \xrightarrow{T\to\infty} \int_{S^*M} a d\mu_L, \quad \text{in } L^2(S^*M) \tag{8.3}$$

[21] Exercise: prove this!

Proof. Consider $H_1 := \{h \in L^2(S^*M) \mid \varphi^{t*}h = h\} \subset L^2(S^*M)$. Then clearly (8.3) holds for all $a \in H_1$. Now define $H_2 := \{\varphi^{t*}h - h \mid h \in L^2(S^*M)\}$.

One then observes $H_2 \perp H_1$ and H_1 closed. Now if $f = \varphi^{t*}g - g$, then

$$\langle f\rangle_T = \frac{1}{T}\int_0^T (\varphi^{t*}g - g)dt = \frac{1}{T}\int_0^T (g \circ \varphi^{2t} - g \circ \varphi^t)dt = \langle g\rangle_{2T} - \langle g\rangle_T$$

Therefore $\langle f\rangle_T \to 0$ as $T \to \infty$ as $\lim_{T\to\infty}\langle g\rangle_T$ exists. Thus (8.3) holds for $a \in H_1 + H_2 =: S$.

One now proves (8.3) holds for $a \in \overline{S}$ by a limiting procedure.

To show $\overline{S} = L^2(S^*M)$, take $h \in \overline{S}^\perp$. Note that $\varphi^{t*}h - h, h - \varphi^{-t*}h \in H_2$. Thus

$$\langle \varphi^{t*}h - h, \varphi^{t*}h - h\rangle_{L^2} = \langle \varphi^{t*}h - h, \varphi^{t*}h\rangle - \langle \varphi^{t*}h - h, h\rangle = \langle h - \varphi^{-t*}h, h\rangle = 0$$

Therefore $h \circ \varphi^t \equiv h$. Consider the sets $S_c = \{h \leq c\}$. They are invariant under φ^t and thus have measure zero or one. Thus $h = 0$, which finishes the proof. □

8.2. **Statement and ingredients.** We are now able to formulate precisely the main result of this section, which is the Quantum Ergodicity theorem. We will roughly follow the presentation given in [6].

Theorem 8.6 (Schnirelman, Zelditch, Colin de Verdiére)**.** *Assume (M, g) is ergodic. Then the eigenfunctions φ_j equidistribute in phase space, i.e. there exists a subset $S = \{i_k\}_{k=1}^\infty \subset \mathbb{N}$ of density one,*[22] *such that for any $A \in \Psi^0_{cl}(M)$*

$$\langle A\varphi_{i_k}, \varphi_{i_k}\rangle \xrightarrow{k\to\infty} \int_{S^*M} \sigma_A d\mu_L \tag{8.4}$$

Remark 8.7. A few points about the theorem are in place.

1. The role of pseudodifferential operators A is to microlocalise the eigenfunctions φ_{i_k} in phase space (T^*M), i.e. there is a localisation in velocities as well as physical space.
2. If $a = a(x)$ is a function on the base M, we obtain equidistribution in space as considered in Example 8.1.
3. There is an example (by A. Hassell) where we cannot have (8.4) for $S = \mathbb{N}$: one considers a rectangle in the plane with two half-discs added to the vertical sides, called the Bunimovich stadium.
4. It is a famous conjecture by Rudnick and Sarnak that if (M, g) has negative sectional curvature, then *unique quantum ergodicity* holds, i.e. we may take $S = \mathbb{N}$ in (8.4).

For the proof of this theorem we will need two auxiliary results, both interesting on their own: Egorov's theorem and the local Weyl's law. We begin with the former.

Let us introduce *the wave operator* $U^t := e^{it\sqrt{-\Delta_g}}$, which is a quantisation of the geodesic flow in some sense. Note that $\sqrt{-\Delta_g} \in \Psi^1(M)$ is defined by the spectral theorem and acts diagonally on eigenfunctions. The operator U^t is an FIO. Thus we have $U^t\varphi_k = e^{it\lambda_k}\varphi_k$ for all k and $U^{t*} = U^{-t}$, i.e. U^t is unitary on $L^2(M)$.

Now the evolution of the quantum observable A (a PDO) in the "Heisenberg picture" is given by

$$\alpha_t(A) = U^tAU^{-t}$$

[22]This means that $\lim_{N\to\infty} \frac{|\{1,\dots,N\}\cap S|}{N} = 1$.

It is known that there is a correspondence with the evolution of classical observables

$$\varphi^{t*}a = a \circ \varphi^t, \quad a \in C^\infty(S^*M)$$

The relation is made rigorous by the following

Theorem 8.8 (Egorov's theorem)**.** *For $A \in \Psi^m_{cl}(M)$, we have $\alpha_t(A) \in \Psi^m_{cl}(A)$. Furthermore, we have for all $(x, \xi) \in T^*M \setminus 0$*

$$\sigma_{\alpha_t(A)}(x, \xi) = \sigma_A \circ \varphi^t(x, \xi) = \varphi^{t*}\sigma_A(x, \xi) \tag{8.5}$$

We now switch to spectral asymptotics. We will write for the spectral counting function

$$N(\lambda) = \#\{j \mid \lambda_j \leq \lambda\}$$

A classical result is the Weyl's theorem (or law), which says as $\lambda \to \infty$

$$N(\lambda) = \frac{|B_n|}{(2\pi)^n} \operatorname{Vol}(M, g)\lambda^n + O(\lambda^{n-1}) \tag{8.6}$$

Here $|B_n|$ is the Euclidean volume of the unit ball and $\operatorname{Vol}(M, g)$ is the volume of M with respect to g. In other words,

$$\operatorname{tr} E_\lambda \sim \frac{\operatorname{Vol}(|\xi| \leq \lambda)}{(2\pi)^n}$$

Here E_λ is the spectral projection to $\oplus_{\lambda_j \leq \lambda} E_j$ and E_j is the eigenspace with eigenvalue λ_j; the volume is the symplectic volume. An important generalisation of (8.6) is the *local Weyl law* or *LWL*, concerning traces $\operatorname{tr} AE_\lambda$ for a PDO A.

Theorem 8.9 (Local Weyl's law)**.** *Given $A \in \Psi^0_{cl}(M)$, we have the asymptotic expansion as $\lambda \to \infty$*

$$\sum_{\lambda_j \leq \lambda} \langle A\varphi_j, \varphi_j \rangle_{L^2} = (2\pi)^{-n}\lambda^n \int_{B^*M} \sigma_A dx d\xi + O(\lambda^{n-1}) \tag{8.7}$$

We postpone the discussion of the proofs of Theorems 8.8 and 8.9 until the last section.

Semiclassical defect measures (or quantum limits). Given a function $a \in C^\infty(S^*M)$, we define the expected value of $A = Op(a) \in \Psi^0_{cl}(M)$ in the state φ_k by setting

$$\rho_k(a) := \langle Op(a)\varphi_k, \varphi_k \rangle \tag{8.8}$$

Note that ρ_k is a linear map on $C^\infty(S^*M)$. We introduce the set of *quantum limits* or *semiclassical defect measures* of the sequence $\{\varphi_k\}_{k=1}^\infty$ as

$$\mathcal{Q} := \text{weak}^* \text{ limits of } \rho_k \tag{8.9}$$

Now consider a compact operator $K : L^2(M) \to L^2(M)$. We claim $\langle K\varphi_k, \varphi_k \rangle_{L^2} \to 0$ as $k \to \infty$ and it suffices to prove that any subsequence has a further subsequence along which the convergence holds. Note that the set $\{K\varphi_k\}_{k=1}^\infty$ is relatively compact by definition. Take a convergent subsequence $K\varphi_{j_k} \to \varphi$ in $L^2(M)$. Then $|\langle \varphi - K\varphi_{j_k}, \varphi_{j_k} \rangle| \leq \|\varphi - K\varphi_{j_k}\|_{L^2} \to 0$ as $k \to \infty$. Now we just observe that $\langle K\varphi, \varphi_k \rangle_{L^2} \to 0$ as $k \to \infty$ since there is an expansion in L^2 of $K\varphi$ in terms of the basis φ_k; therefore, $\langle K\varphi_{j_k}, \varphi_{j_k} \rangle_{L^2} \to 0$. In particular, since any $A \in \Psi^{-1}(M)$ is compact $L^2(M) \to L^2(M)$, this shows that the the set $\mathcal{Q}$ in (8.9) is well-defined regardless of the choice of the quantisation Op.

We record another useful inequality for later. By generalising Theorem 5.5 to manifolds, we obtain the following inequality for $A \in \Psi^0_{cl}(M)$ for some $C = C(M) > 0$

$$\inf_{\substack{K: L^2 \to L^2 \\ \text{compact}}} \|A + K\|_{L^2 \to L^2} \leq C \sup_{(x,\xi) \in S^*M} |\sigma_A(x,\xi)| = C\|\sigma_A\|_{L^\infty} \tag{8.10}$$

Therefore, by the paragraph above, we obtain the following bound on any limit $\mu \in \mathcal{Q}$, for any $a \in C^\infty(S^*M)$:

$$|\mu(a)| \leq \limsup_{k \to \infty} |\rho_k(a)| \leq C \sup_{S^*M} |a| \tag{8.11}$$

This in particular shows that any $\mu \in \mathcal{Q}$ extends as a continuous functional on $C(S^*M)$.

Proposition 8.10. *The following properties hold for semiclassical measures*

1. *If $\mu \in \mathcal{Q}$, then μ a positive probability measure on S^*M.*
2. *If $\mu \in \mathcal{Q}$, then μ is invariant by the geodesic flow.*
3. *We have $\mathcal{Q} \neq \emptyset$.*

Proof. Claim 1. Assume without loss of generality $\rho_k \to \mu$ weakly. Take $a \in C^\infty(S^*M)$ with $a \geq 0$. We want to prove $\mu(a) \geq 0$. By the proof of Theorem 5.5, we know that for any $c > 0$ there is a $B \in \Psi^0_{cl}(M)$ with

$$Op(a + c) = Op(a) + cId + R_1 = B^*B + R_2$$

where $R_1, R_2 \in \Psi^{-\infty}(M)$. Therefore

$$\limsup_{k \to \infty} \rho_k(a + c) = \limsup_{k \to \infty} \|B\varphi_k\|^2 \geq 0$$

Thus by taking $c \to 0$, we obtain $\mu(a) \geq 0$. We also get $\mu(1) = 1$ immediately. Now apply the Riesz Representation Theorem to obtain μ a positive probability measure.

Claim 2. Assume $\rho_k \to \mu$ weakly. For $a \in C^\infty(S^*M)$, we have

$$\rho_k(a) = \langle Op(a)\varphi_k, \varphi_k\rangle = \langle AU^t\varphi_k, U^t\varphi_k\rangle = \langle \alpha_t(A)\varphi_k, \varphi_k\rangle \tag{8.12}$$

$$= \langle (Op(a \circ \varphi^t) + R)\varphi_k, \varphi_k\rangle \xrightarrow{k \to \infty} \mu(a \circ \varphi^t) \tag{8.13}$$

In the upper line we used the explicit action of U^t on eigenfunctions and in the last line we used Egorov's theorem 8.8, where $R \in \Psi^{-1}_{cl}(M)$. Therefore $\mu(a) = \mu(\varphi^{t*}a)$ for all $a \in C^\infty(S^*M)$ and we get that μ is invariant by φ^t.

Claim 3. We follow a diagonalisation procedure. Take $\{a_j\}_{j=1}^\infty \subset C^\infty(S^*M)$ a countable dense set in the supremum norm. Now the sequence $|\rho_i(a_1)| \leq \|Op(a_1)\|_{L^2 \to L^2}$ for $i \geq 1$ is bounded, so there is a convergent subsequence, converging to q_1 say. Re-label and consider a convergent subsequence of $\rho_2(a_2), \rho_3(a_2), \ldots,$ converging to q_2. Re-label and iterate this to obtain a sequence $\rho_1, \rho_2, \ldots$ such that

$$\rho_1(a_1), \rho_2(a_1), \rho_3(a_1), \ldots \xrightarrow{k \to \infty} q_1 \tag{8.14}$$

$$\rho_2(a_2), \rho_3(a_2), \ldots \xrightarrow{k \to \infty} q_2 \tag{8.15}$$

$$\rho_3(a_3), \ldots \xrightarrow{k \to \infty} q_3 \tag{8.16}$$

$$\vdots \tag{8.17}$$

Now use the density of $\{a_j\}_{j=1}^\infty$ and the bound (8.11) to define $\rho(a) := \lim_{k \to \infty} \rho_k(a)$ for any $a \in C^\infty(S^*M)$. This finishes the proof. □

Remark 8.11. Semiclassical defect measures can be considered with respect to any bounded sequence of functions $\{u_j\}_{j=1}^{\infty}$ in $L^2(M)$ by taking weak* limits of ρ_k, denoted by $\mathcal{Q}$, where $\rho_k(a) := \langle Op(a)u_j, u_j\rangle$ for $a \in C^\infty(S^*M)$. Then the points 1. and 3. of Proposition 8.10 hold: $\mathcal{Q} \neq 0$ and every $\mu \in \mathcal{Q}$ is a positive measure. Additionally, if u_j satisfies a PDE, invariance of the measure can be obtained by the flow defined by the principal symbol of the PDE. See [7, Chapter 5] for more details.

8.3. **Main proof of Quantum Ergodicity.** Here we use the results outlined above to prove Theorem 8.6.

Proof of Theorem 8.6. We divide the proof in three steps.

Step 1. Denote by $\omega(A) := \int_{S^*M} \sigma_A d\mu_L$ for $A \in \Psi^0_{cl}(M)$, a "space average" of A. We claim that

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2 = 0 \tag{8.18}$$

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Let us denote the temporal mean of an observable $A \in \Psi^0_{cl}(M)$ by

$$\langle A\rangle_T = \frac{1}{2T}\int_{-T}^{T} U^t A U^{-t} dt$$

Notice that by the local Weyl's law 8.9 we have

$$\omega(A) = \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} \langle A\varphi_j, \varphi_j\rangle \tag{8.19}$$

Now the convergence in (8.18) follows by the chain of inequalities, where $T > 0$

$$\begin{aligned}
\frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2 &= \frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} \left|\left\langle (\langle A\rangle_T - \omega(A))\varphi_j, \varphi_j\right\rangle\right|^2 \\
&\leq \frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} \left\langle (\langle A\rangle_T - \omega(A))^*(\langle A\rangle_T - \omega(A))\varphi_j, \varphi_j\right\rangle \\
&\xrightarrow{j\to\infty} \int_{S^*M} \left|\langle \sigma_A\rangle_T - \omega(A)\right|^2 d\mu_L \\
&\xrightarrow{T\to\infty} 0
\end{aligned}$$

Here we used Cauchy-Schwarz inequality, equation (8.19) and Egorov's theorem 8.8, and the mean ergodic theorem 8.5.

Step 2. We extract a density one subset $S_A \subset \mathbb{N}$ such that (8.4) holds on S_A for a fixed $A \in \Psi^0_{cl}(M)$, by using the mean convergence (8.18).

Define the following quantity, that we think of as some kind of variance

$$\varepsilon^2(\lambda) := \frac{1}{N(\lambda)} \sum_{\lambda_j\leq\lambda} |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2$$

By the previous step, we have $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$. Consider the counting (probability) measure on the finite set $I_\lambda = \{j \mid \lambda_j \leq \lambda\}$ and define the random variable X_λ on I_λ

$$X_\lambda(j) := |\langle A\varphi_j, \varphi_j\rangle - \omega(A)|^2$$

Notice that $\mathbb{E}(X_\lambda) = \varepsilon(\lambda)^2$. Now introduce the set

$$\Gamma_\lambda = \{j \in I_\lambda \mid |\langle A\varphi_j, \varphi_j\rangle|^2 \geq \varepsilon(\lambda)\} \tag{8.20}$$

By Markov's (Chebyshev's) inequality[23], we immediately get $\mathbb{P}(\Gamma_\lambda) \leq \frac{\varepsilon(\lambda)^2}{\varepsilon(\lambda)} = \varepsilon(\lambda)$ and so for $S_\lambda := I_\lambda \setminus \Gamma_\lambda$

$$\mathbb{P}(S_\lambda) = \frac{\# S_\lambda}{N(\lambda)} \geq 1 - \varepsilon(\lambda) \tag{8.21}$$

Take a sequence $\lambda_k \nearrow \infty$ with $\varepsilon(\lambda_k) \searrow 0$ as $k \to \infty$ and define $S_A := \cup_k S_{\lambda_k}$. This is of density one by (8.21) and the convergence $\langle A\varphi_j, \varphi_j\rangle \to \omega(A)$ happens for $j \in S_A$ and $j \to \infty$ by definition (8.20).

Step 3. We find a set $S \subset \mathbb{N}$ of density one such that (8.4) holds for each $A \in \Psi^0_{cl}(M)$. This is done by a diagonalisation procedure similar in spirit to Proposition 8.10 3.

Take $\{a_j\}_{j=1}^\infty \subset C^\infty(S^*M)$ be a dense countable set in the supremum norm. Now let $S_j \subset \mathbb{N}$ be a set of density one associated to the operator $Op(a_j)$ obtained in the previous step. We may assume that $S_{j+1} \subset S_j$ for all $j \geq 1$.

By the assumptions, there exist $N_j \in \mathbb{N}$ such that $N_j \nearrow \infty$ as $j \to \infty$ and for all $N \geq N_j$ we have

$$\frac{1}{N} \#\{k \in S_j \mid k \leq N\} \geq 1 - 2^{-j}$$

Now define S by the relations, for all $j \geq 1$

$$S \cap [N_j, N_{j+1}) := S_j \cap [N_j, N_{j+1}) \tag{8.22}$$

Therefore $S_j|_{[0,N_{j+1})} \subset S|_{[0,N_{j+1})}$ for all $j \in \mathbb{N}$. By equation (8.22) we have $S \subset \mathbb{N}$ of density 1, and also that for all $j \in \mathbb{N}$

$$\langle Op(a_j)\varphi_k, \varphi_k\rangle_{L^2} \xrightarrow{k \in S,\, k\to\infty} \int_{S^*M} a_j d\mu_L \tag{8.23}$$

Now use the inequality (8.11), the density of $\{a_j\}_{j=1}^\infty$ and (8.23) to finish the proof. More precisely, we take any $b \in C^\infty(S^*M)$ and a sequence $b_j \in C^\infty(S^*M)$ such that $b_j \to b$ as $j \to \infty$ in the supremum norm, and (8.4) holds for all b_j and S as above. Then we have, for all $j \in \mathbb{N}$

$$\begin{aligned}\limsup_{k\in S,\, k\to\infty} \Big| \int_{S^*M} b d\mu_L - \langle Op(b)\varphi_k, \varphi_k\rangle \Big| &\leq \limsup_{k\in S,\, k\to\infty} \Big| \int_{S^*M} (b - b_j) d\mu_L \Big| \\ &+ \limsup_{k\in S,\, k\to\infty} \Big| \int_{S^*M} b_j d\mu_L - \langle Op(b_j)\varphi_k, \varphi_k\rangle \Big| + \limsup_{k\in S,\, k\to\infty} \Big| \langle Op(b - b_j)\varphi_k, \varphi_k\rangle \Big|\end{aligned}$$

Now the second term on the right hand side vanishes, the first and the third are clearly bounded by $C\|b - b_j\|_{L^\infty(S^*M)}$. So it suffices to take j large enough to show $\langle Op(b)\varphi_k, \varphi_k\rangle \to \int_{S^*M} b d\mu_L$ as $k \to \infty$ and $k \in S$. $\square$

[23]Recall Markov's inequality says that $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$ for any $a > 0$ and random variable X.

9. Semiclassical calculus and Weyl's law

In addition to the theory developed so far, there is a parallel theory which parametrises symbols and operators by a small varying parameter $h > 0$ – *semiclassical analysis*. It is especially important in spectral theory (where $h \sim \frac{1}{\lambda}$, λ an eigenvalue), in proving Carleman estimates (these have the form $\|e^{-\frac{\varphi}{h}} P e^{\frac{\varphi}{h}} u\| \gtrsim \|u\|$, where φ a suitable weight function and P an operator), proving various estimates and properties using semiclassical defect measures (c.f. equation (8.9)) etc. In this section we briefly sketch how such a theory can be developed and prove Weyl's law. See [7] for a detailed account of the theory, or [1, Appendix E] for a concise but more optimal summary.

A *semiclassical differential operator* $P \in \mathrm{Diff}_h^k(\mathbb{R}^n)$ of order k has the form

$$P = \sum_{|\alpha| \leq k} \sum_{j=0}^{k-|\alpha|} h^j a_{\alpha j}(x)(hD_x)^\alpha$$

Note that each derivative comes with an h and the excess of h's must not in total exceed k. The *principal symbol* of P is given by

$$\sigma_h(P)(x, \xi) := \sum_{|\alpha| \leq k} a_{\alpha 0}(x) \xi^\alpha$$

Note that the map $P \mapsto \sigma_h(P)$ has the kernel equal to $h\, \mathrm{Diff}_h^{k-1}(\mathbb{R}^n)$.

Example 9.1. The following basic relations hold

1. We have $P = -h^2(\Delta - \lambda^2) \in \mathrm{Diff}_h^2(\mathbb{R}^n)$ and $\sigma_h(P)(x, \xi) = |\xi|^2$.
2. We have $P = -h^2\Delta - \lambda^2 \in \mathrm{Diff}_h^2(\mathbb{R}^n)$ and $\sigma_h(P)(x, \xi) = |\xi|^2 - \lambda^2$.
3. We have $P = -h^3\Delta \in \mathrm{Diff}_h^3(\mathbb{R}^n)$ and $\sigma_h(P)(x, \xi) = 0$.

Symbol classes. Now let $X \subset \mathbb{R}^n$ be an open set. We say $a(x, \xi; h) \in S^k_{1,0}(T^*X)$ if for all $K \Subset X$ compact and a fixed $h_0 > 0$[24]

$$\sup_{h \in [0, h_0)} \sup_{x \in K} \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)\right| (1 + |\xi|)^{|\beta| - k} < \infty \tag{9.1}$$

We may induce a Fréchet space topology on $S^k_{1,0}(T^*X)$ by taking the seminorms to be the left hand sides of (9.1) for varying $K \Subset X$ compact and multi-indices α, β.

We will write

$$a(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j a_j(x, \xi; h) \tag{9.2}$$

for some $a_j \in S^{k-j}_{1,0}(T^*X)$ if $a(x, \xi; h) - \sum_{j=0}^{N-1} h^j a_j(x, \xi; h) \in h^N S^{k-N}_{1,0}(T^*X)$ for all $N \geq 0$. Similarly to Proposition 2.7, given $a_j \in S^{k-j}_{1,0}$ as above we may always find $a \in S^k_{1,0}$ such that (9.2) holds. Note that a is unique modulo $h^\infty S^{-\infty}_{1,0}$.

If moreover $a_j \in S^k_{cl}(T^*X)$ are classical and independent of h, we say a is a *polyhomogeneous semiclassical symbol of order k* and write $a \in S^k_h(T^*X)$.

[24] Note a slight ambiguity with the standard h-independent Kohn-Nirenberg classes $S^k_{1,0} = S^k$. From now on we reserve the $S^k_{1,0}$ notation for h-dependent class and use S^k for the h-independent.

Quantisation. We quantise our symbols by introducing the *semiclassical Fourier transform* $\mathcal{F}_h$ as a rescaling of the usual Fourier transform

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h} y\cdot\xi} u(y) dy, \quad \mathcal{F}_h^{-1} u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h} x\cdot\xi} \mathcal{F}_h u(\xi) d\xi$$

Now let $a \in S_h^k(T^*X)$. We write $A = Op_h(a) \in \Psi_h^k(X)$ if (in the sense of oscillatory integrals)

$$Op_h(a)u(x) = (2\pi h)^{-n} \int\int e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi;h) u(y) dy d\xi = \mathcal{F}_h^{-1}\big(a(x,\xi;h)\mathcal{F}_h u\big) \tag{9.3}$$

The principal symbol of $A = Op_h(a)$ is given by $\sigma_h(A) = a_0$, if a admits an expansion (9.2).

Sobolev spaces. The semiclassical Sobolev spaces $H_h^s(\mathbb{R}^n)$ are equal to the usual ones $H^s(\mathbb{R}^n)$ as sets, but are equipped with different norms

$$\|u\|_{H_h^s(\mathbb{R}^n)} = \|\langle h\xi\rangle^s \widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}$$

Recall that here $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ is the Japanese bracket. Similarly to Theorem 5.8, we have an operator $P \in \Psi_h^k(X)$ is bounded uniformly in h as a map $P : H^s_{h,comp}(X) \to H^{s-k}_{h,loc}(X)$ for any $s \in \mathbb{R}$.

Manifolds. We just remark that everything developed in Section 7 for compact manifolds generalises to h-dependent theory by putting h's in right places. In particular, there is a short exact sequence for M a compact manifold (c.f. (7.3))

$$0 \to h\Psi_h^{k-1}(M) \xhookrightarrow{\iota} \Psi_h^k(M) \xrightarrow{\sigma_h} S_h^k(T^*M)/hS_h^{k-1}(T^*M) \to 0$$

Functional calculus. Before proving the Weyl's law, we need to import some technology from spectral theory. The functional calculus works well with the semiclassical operators and yields nice results. In particular, if (M,g) is a compact Riemannian manifold

Theorem 9.2. *Assume $V : M \to \mathbb{R}$ is a C^∞ potential and let $P(h) = -h^2\Delta_g + V$ be a Schrödinger operator. If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f\big(P(h)\big) \in \Psi_h^{-\infty}(M) := \bigcap_{k\in\mathbb{Z}} \Psi_h^k(M) \tag{9.4}$$

with the symbol given by

$$\sigma_h\big(f(P(h))\big) = f(|\xi|_x^2 + V(x)) \tag{9.5}$$

Proof. See [7, Theorem 14.9.] for a proof. □

Remark 9.3. Recall that for $P(h) = -h^2\Delta_g + V$, the spectral theorem has a simple form; i.e. given any $f \in L^\infty(M)$, we may define $f(P(h)) : L^2(M) \to L^2(M)$ by

$$f\big(P(h)\big) := \sum_{j=1}^{\infty} f\big(E_j(h)\big) u_j(h) \otimes u_j(h)^*$$

Here we used the convention that $P(h)u_j(h) = E_j(h)u_j(h)$, where $\{u_j\}_{j=1}^\infty \subset L^2(M)$ is an orthonormal basis of eigenvalues. If $u, v \in L^2(M)$ we write $u \otimes v^*$ for the operator $u \otimes v^*(\varphi) := u\int_M v\varphi dx$.

Then the content of Theorem 9.2 is that for sufficiently nice f, the operator $f\big(P(h)\big)$ is a semiclassical pseudodifferential operator.

Trace class operators. Let H be a (separable, complex) Hilbert space. A compact operator $A : H \to H$ is of *trace class*, written $A \in \mathcal{L}_1(H)$, if

$$\sum_{j=1}^{\infty} s_j(A) < \infty$$

Here $s_j(A)^2$ are the eigenvalues of the operator A^*A. It may be shown that, for any orthonormal basis $\{e_j\}_{j=1}^{\infty} \subset H$

1. If $A \in \mathcal{L}_1(H)$, then the *trace of* A given by $\operatorname{tr}(A) := \sum_{j=1}^{\infty} \langle Ae_j, e_j\rangle_H$ is well-defined and independent of the choice of the orthonormal basis.
2. If $A \in \Psi^{-\infty}(M)$ has a Schwartz kernel $K_A \in C^\infty(M \times M)$, then $A : L^2(M) \to L^2(M)$ is of trace class and its trace is given by the formula
$$\operatorname{tr} A = \int_\Delta K_A(x,x)dx \tag{9.6}$$
Here $\Delta = \{(x,x) \mid x \in M\} \subset M \times M$ is the diagonal.

For proofs and references, see [7, Appendix C. 3.].

9.1. **Proof of Weyl's law.** The proof will express the trace of the spectral projection by two trace formulas and we will correspondingly divide the proof into two parts. We will mostly follow the proof in [7, Theorem 14.11.]. Recall that Weyl's law reads, where (M,g) a compact Riemannian manifold

$$N(\lambda) = \frac{|B_n|}{(2\pi)^n}\operatorname{Vol}(M,g)\lambda^n + O(\lambda^{n-1}) \tag{9.7}$$

Here $N(\lambda) = \{j \mid \lambda_j \leq \lambda\}$ is the spectral counting function. Let us denote $P(h) = -h^2\Delta_g$, so that $\sigma_h(P) = |\xi|^2$. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function approximating the indicator function on $[-1,1]$, for some $\varepsilon > 0$ small

$$\chi(\lambda) = \begin{cases} 1, & |\lambda| < 1-\varepsilon \\ 0, & |\lambda| > 1+\varepsilon \end{cases}$$

Then by Theorem 9.2, $\chi(-h^2\Delta_g) \in \Psi_h^{-\infty}(M)$ and is an approximation of the spectral projection to the interval $[0, \frac{1}{h^2}]$.

Step 1. We claim that for any $A \in \Psi_h^{-\infty}(M)$

$$\operatorname{tr} A = (2\pi h)^{-n}\Big(\int_{T^*M} \sigma_h(A)dxd\xi + O(h)\Big) \tag{9.8}$$

uniformly in h. We sketch a proof of this. Note that by definition, in local coordinates on $X \subset \mathbb{R}^n$, the Schwartz kernel $K_A \in C^\infty(X \times X)$ of $A = Op_h(a)$ is given by

$$K_A(x,y;h) = (2\pi h)^{-n}\int e^{\frac{i}{h}(x-y)\cdot\xi}a(x,\xi;h)d\xi$$

Now putting $x = y$ into this formula and integrating along the diagonal, we obtain an integral over T^*X. Then use the formula (9.6) to sum over all charts and obtain (9.8).

Now apply formula (9.8) to $\chi(-h^2\Delta_g)$ to get

$$\operatorname{tr}\chi(-h^2\Delta_g) = (2\pi h)^{-n}\int_{T^*M}\chi(|\xi|^2)dxd\xi + O_\varepsilon(h^{1-n}) \tag{9.9}$$

The notation O_ε denotes that the remainder depends on ε uniformly.

Step 2. On the other hand, by the definition of trace and taking the orthonormal basis of eigenfunctions $\{\varphi_j\}_{j=1}^\infty \subset L^2(M)$

$$\operatorname{tr}\chi(-h^2\Delta_g) = \sum_{j=1}^{\infty} \chi(h^2\lambda_j^2) \tag{9.10}$$

Now take $h = \lambda^{-1}$ and observe that $\operatorname{tr}\chi(-h^2\Delta_g) = N(\lambda) + O(\varepsilon)$. Combine this with the formula (9.9) in the limit $\varepsilon \to 0$ to get

$$\begin{aligned} N(\lambda) &= (2\pi)^{-n}\lambda^n \int_{B^*M} dx d\xi + O(\lambda^{n-1}) \\ &= (2\pi)^{-n}\lambda^n |B_n| \operatorname{Vol}(M,g) + O(\lambda^{n-1}) \end{aligned}$$

This finishes the proof.

Remark 9.4. Similar argument works to prove the local Weyl's law, Theorem 8.9 – one instead considers the traces $\operatorname{tr}\big(\chi(-h^2\Delta_g)Op_h(a)\big)$. See [7, Theorem 15.3.] for a detailed proof.

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end of Lecture 13, 29.1.2019.

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