INTRODUCTION TO MICROLOCAL ANALYSIS WITH APPLICATIONS

MIHAJLO CEKIĆ

Microlocal analysis studies singularities of distributions in phase space, by describing the behaviour of the singularity in both position and direction. It is a part of the field of partial differential equations, created by Hörmander, Kohn, Nirenberg and others in 1960s and 1970s, and is used to study questions such as solvability, regularity and propagation of singularities of solutions of PDEs. To name a few other classical applications, it can be used to study asymptotics of eigenfunctions for elliptic operators, trace formulas and inverse problems.

There have been recent exciting advances in the field and many applications to geometry and dynamical systems. These include dynamical zeta functions and injectivity properties of X-ray (geodesic) transforms with applications to rigidity questions in geometry. The study was originally designed for linear PDEs, but there are more recent techniques for studying nonlinear problems through the paradifferential calculus.

In this course, we study the fundamentals of microlocal analysis. We aim to develop main tools and provide some of the recent striking applications. A sketch plan of the course is given below.

Contents

1. Distributions and Fourier transform (recap). Symbol classes and oscillatory integrals. Fourier integral operators. Non-stationary and stationary phase lemma. (2-3 lectures)
2. Pseudodifferential operators (PDO). Compositions, changes of coordinates, calculus of PDOs. PDOs on manifolds. (2 lectures)
3. Elliptic regularity. L^2-continuity. Sobolev spaces and PDOs. (1-2 lectures)
4. Wavefront set. Products, pullbacks of distributions. Propagation of singularities. (2 lectures)
5. Possible applications: Egorov's theorem. Weyl's law and Quantum ergodicity. Existence of resonances for uniformly hyperbolic (Anosov) diffeomorphisms/flows via adapted anisotropic Sobolev spaces. Inverse problems. (5-6 lectures)

Note below: the applications part of the course will change according to the time constraints.

Pre-requisites

Elementary theory of distributions and Fourier transforms (these will be recalled briefly). Basics of functional analysis and differential geometry.

Literature

There will be lecture notes provided by the lecturer. However, the following books provide with complementary reading:

1. F.G. Friedlander, *Introduction to the theory of distributions*, Second edition. With additional material by M. Joshi, Cambridge University Press, Cambridge, 1998.
2. A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators. An introduction*, London Mathematical Society Lecture Note Series, 196. Cambridge University Press, Cambridge, 1994.

3. M. A. Shubin, *Pseudodifferential operators and spectral theory*, Translated from the 1978 Russian original by Stig I. Andersson, Second edition, Springer-Verlag, Berlin, 2001.
4. M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, 138. American Mathematical Society, Providence, RI, 2012.

There are also a few relevant papers and surveys:

1. F. Faure, J. Sjstrand, *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. in Math. Physics, vol. 308 (2011), 325-364.
2. S. Zelditch, *Recent Developments in Mathematical Quantum Chaos*, Current developments in mathematics, 2009, 115204, Int. Press, Somerville, MA, 2010.