THE BASS–QUILLEN PHENOMENON FOR REDUCTIVE GROUP TORSORS

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Abstract. We complete the proof of the Nisnevich conjecture in equal characteristic: for a smooth variety X over a field, a smooth divisor $D \subset X$, and a reductive X-group G that is totally isotropic along D, we show that each generically trivial G-torsor over $X \backslash D$ trivializes Zariski (semi)locally on X. We then use it to give a new proof for the generalization of the Bass–Quillen conjecture to torsors under reductive groups in equal characteristic: for a regular ring R that contains a field and a totally isotropic reductive R-group scheme G, we show that each generically trivial G-torsor over the affine space $\mathbb{A}^d_R$ descends to R, equivalently, that $H^1_{\mathrm{Nis}}(R, G) \overset{\sim}{\longrightarrow} H^1_{\mathrm{Nis}}(\mathbb{A}^d_R, G)$. We base our arguments on an extension of the Gabber–Quillen presentation lemma, on the relative version of the Grothendieck–Serre conjecture, and on an extension result for G-torsors over a smooth relative curve.

1. The equicharacteristic case of the generalized Bass–Quillen conjecture

In the 1970's, Quillen and Suslin (independently) resolved Serre's problem about vector bundles on affine spaces: in [Qui76], [Sus76], they showed that, for a field k, every vector bundle on the affine space $\mathbb{A}^d_k$ is trivial. The subsequent Bass–Quillen conjecture predicts more: for a regular ring R, every vector bundle on $\mathbb{A}^d_R$ ought to descend to R, see [Bas73, Section 4.1] and [Qui76, Comment (1) on page 170]. This is known when R contains a field or, more generally, when all the local rings of R are unramified, but remains open in general, see [Čes22b, Section 2.1] for an overview.

We give a new proof for the equal characteristic case of the generalization of the Bass–Quillen conjecture in which vector bundles are replaced by torsors under a reductive R-group scheme G.

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Theorem 1.1 (§7.2). *For a regular ring R containing a field and a totally isotropic reductive R-group scheme G, every generically trivial G-torsor over $\mathbb{A}^d_R$ descends to a G-torsor over R, equivalently,*

$$H^1_{\mathrm{Zar}}(R,G) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\mathbb{A}^d_R,G) \quad \textit{or, if one prefers,} \quad H^1_{\mathrm{Nis}}(R,G) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(\mathbb{A}^d_R,G).$$

The equivalence of the three formulations follows from the Grothendieck–Serre conjecture [FP15], [Pan20]: more precisely, by Theorem 5.5 below, a G-torsor over $\mathbb{A}^d_R$ is generically trivial, if and only if it is Zariski locally trivial, if and only if it is Nisnevich locally trivial. The total isotropicity assumption means that each simple factor of G^{ad} contains a parabolic subgroup as follows.

Definition 1.2 ([Čes22a, Definition 8.1]). Let S be a scheme and let G be a reductive S-group scheme. We say that G is *totally isotropic* at a point $s \in S$ if in the canonical decomposition

$$G^{\mathrm{ad}} \cong \textstyle\prod_i \mathrm{Res}_{R_i/\mathscr{O}_{S,\,s}}(G_i) \tag{1.2.1}$$

in which i runs over the types of nonempty connected Dynkin diagrams, R_i is a finite étale $\mathscr{O}_{S,\,s}$-algebra, and G_i is an adjoint semisimple R_i-group scheme with simple geometric R_i-fibers of type i (see [SGA $3_{\text{III new}}$, Exposé XXIV, Proposition 5.10 (i)]), each G_i contains $\mathbb{G}_{m,\,R_i}$ as an R_i-subgroup, equivalently, each G_i has a proper parabolic R_i-subgroup (see [SGA $3_{\text{III new}}$, Exposé XXVI, Corollaire 6.12]). We say that G is *totally isotropic* along a subset $Z \subset S$ if it is so at all $s \in Z$.

For example, a quasi-split, so also a split, reductive group scheme is totally isotropic, as is any torus.

The generic triviality assumption is needed in Theorem 1.1 because, for instance, for every separably closed field k that is not algebraically closed, there are nontrivial PGL_n-torsors over $\mathbb{A}^1_k$, see [CTS21, Theorem 5.6.1 (vi)]. The total isotropicity assumption is needed in Theorem 1.1 because of [BS17, Proposition 4.9], where Balwe and Sawant show that a Bass–Quillen statement cannot hold beyond totally isotropic G, see also [Par78] and [Fed16, Remark 2.6] for earlier counterexamples to generalizations of the Bass–Quillen conjecture beyond totally isotropic reductive groups.

Theorem 1.1 was established by Stavrova in [Sta22, Corollary 5.5] by a different method. Prior to that, the case when R is smooth over a field k and G is defined and totally isotropic over k was settled by Asok–Hoyois–Wendt: they used methods of $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky to verify axioms of Colliot-Thélène–Ojanguren [CTO92] that were known to imply the statement, see [AHW18, Theorem 3.3.7] for infinite k and [AHW20, Theorem 2.4] for finite k. As was explained in [Li21], one could also check these axioms directly, without $\mathbb{A}^1$-homotopy theory. In mixed characteristic, Theorem 1.1 is only known in sporadic cases, for instance, when G is a torus, see [CTS87, Lemma 2.4], as well as [Čes22b, Section 3.6.4] for an overview.

We approach Theorem 1.1 by reducing it to the equicharacteristic case of the Nisnevich conjecture [Nis89, Conjecture 1.3] and then settling the latter. The geometric case of this conjecture is as follows.

Theorem 1.3. *For a smooth scheme X of dimension $d > 0$ over a field k, a k-smooth divisor $D \subset X$, and a reductive X-group scheme G that is totally isotropic along D, every generically trivial G-torsor E on $X \backslash D$ is trivial Zariski semilocally on X, that is, for every $x_1, \ldots, x_m \in X$ that lie in a single affine open, there is an affine open $U \subset X$ containing all the x_i such that $E|_{U\backslash D}$ is trivial.*

Example 1.4. For instance, for $G = \mathrm{GL}_n$, Theorem 1.3 says that every vector bundle on $X\backslash D$ is trivial Zariski semilocally on X. With 'semilocally' weakened to 'locally,' this case has been settled by Bhatwadekar–Rao [BR83, Theorem 2.5] and Popescu [SP, Theorem 07GC], and had been conjectured by Quillen in [Qui76, (2) on page 170]. In contrast, even in the case when $G = \mathrm{GL}_n$ and X is affine, not every G-torsor over $X\backslash D$ extends to a G-torsor over X, see [Swa78, Section 2];

such extendibility had been a question posed by Quillen in [Qui76, (3) on page 170]. Likewise, the smoothness of D (and even more so of X) cannot be dropped, see [Lam06, pages 34–35].

The slightly more general algebraic case of the Nisnevich conjecture in equicharacteristic is as follows.

Theorem 1.5 (§7.1). *For a regular semilocal ring R containing a field, an $r \in R$ that is a regular parameter in the sense that $r \notin \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m} \subset R$, and a reductive R-group G that is totally isotropic along $\{r = 0\} \subset \mathrm{Spec}(R)$, every generically trivial G-torsor over $R[\frac{1}{r}]$ is trivial.*

The case $r \in R^\times$ recovers the Grothendieck–Serre conjecture in equal characteristic settled in [FP15], [Pan20], so Theorem 1.5 may be viewed as an extension result: every generically trivial G-torsor over $R[\frac{1}{r}]$ extends to a G-torsor over R. Theorem 1.3 follows by applying Theorem 1.5 to the semilocal ring of X at $x_1, \dots, x_m$ (which one builds via prime avoidance, see [SP, Lemma 00DS]). In [Fed21b], Fedorov settled Theorem 1.5 in the case when R contains an infinite field and showed that its total isotropicity assumption cannot be dropped. A key novelty that allows us to bypass difficulties with finite residue fields is the following extension result for G-torsors over smooth relative curves.

Theorem 1.6.

(a) (*Theorem* 6.3). *For a regular semilocal ring R containing a field, a reductive R-group G, a smooth affine R-scheme C of pure relative dimension 1, and an R-(finite étale) closed subscheme $Y \subset C$ such that G_Y is totally isotropic, every G-torsor E over $C \backslash Y$ that is trivial away from some R-finite closed subscheme $Z \subset C$ extends to a G-torsor over C.*

(b) (*Theorem* 6.4). *In (a), if $C = \mathbb{A}^1_R$ and G is totally isotropic, then E is even trivial.*

Roughly speaking, extending a G-torsor to all of C in Theorem 1.6 corresponds to extending a G-torsor in Theorem 1.5 to all of R, in effect, to reducing the Nisnevich conjecture to the Grothendieck–Serre conjecture—this is why Theorem 1.6 is crucial for our purposes. Conversely, to reduce Theorem 1.5 to Theorem 1.6 we use a presentation lemma that extends its variants due to Quillen and Gabber: we first use Popescu theorem to pass to the geometric setting of Theorem 1.3 and then show in §2 that, up to replacing X by an affine open neighborhood of $x_1, \dots, x_m$, we can express X as a smooth relative curve over some affine open of $\mathbb{A}^{d-1}_k$ in such a way that D is relatively finite étale and our generically trivial G-torsor over X is trivial away from a relatively finite closed subscheme. With this relative curve setting in hand, we carry out the aforementioned reduction in §3, see Proposition 3.4.

As for Theorem 1.6, in §6 we present a series of excision and patching dévissages to reduce to when $C = \mathbb{A}^1_R$ and $C \backslash Y$ descends to a smooth curve defined over a field k contained in R. In this "constant" case, we argue that our G-torsor over $C \backslash Y$ is even trivial thanks to the "relative Grothendieck–Serre" theorem of Fedorov from [Fed21a] (with an earlier version due to Panin–Stavrova–Vavilov [PSV15]), according to which, for any k-algebra W, no nontrivial G-torsor over $R \otimes_k W$ trivializes over $\mathrm{Frac}(R) \otimes_k W$. Indeed, by choosing W to be the coordinate ring of a descent of $C \backslash Y$ to k, we reduce the constant case of Theorem 1.6 to the relatively straight-forward case when R is a field and $C = \mathbb{A}^1_R$. This is a similar overall strategy to what Fedorov used in [Fed21b], but it is important to focus on Theorem 1.6 rather than on its special case $C = \mathbb{A}^1_R$: this gives additional flexibility required for excision and patching that allows us to bypass difficulties with finite fields. From a more practical vantage point, we overcome these difficulties with a novel version of Panin's "finite field tricks" presented in Lemma 4.2 and a novel version of the Lindel style embedding Lemma 4.4, see §4.

In §5, we include a quick reproof of the relative Grothendieck–Serre conjecture in equal characteristic that is key for §6. This makes the article more self-contained and gives us an opportunity to remove several unnecessary hypotheses from the results of [Fed21a]. Ultimately, §5 is based on the geometry of the affine Grassmannian, although the latter largely remains behind the curtain of inputs from the survey article [Česnavičius22b] (that mildly generalized the corresponding results of Fedorov from [Fed21a]).

1.7. Notation and conventions. All the rings we consider are commutative and unital. For a point s of a scheme (resp., for a prime ideal $\mathfrak{p}$ of a ring), we let k_s (resp., $k_{\mathfrak{p}}$) denote its residue field.

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2. A generalization of the Gabber–Quillen presentation lemma

The argument for the equal characteristic case of the Nisnevich conjecture, that is, for Theorem 1.5, begins with the following geometric presentation theorem that is in the spirit of Gabber's refinement [Gab94, Lemma 3.1] of the Quillen presentation lemma [Qui73, Section 7, Lemma 5.12] (which itself is a variant of the Noether normalization theorem). This is a purely geometric step in which reductive groups or their torsors play no role: its purpose is to express a given smooth affine variety as a smooth relative curve in such a way that a fixed smooth divisor and a fixed closed subvariety become relatively finite étale and relatively finite, respectively.

Theorem 2.1. *For a smooth, affine, irreducible scheme X of dimension $d > 0$ over a field k that is either finite or of characteristic 0,[1] points $x_1, \dots, x_m \in X$, a proper closed subscheme $Z \subset X$, and a k-smooth divisor $D \subset X$, there are an affine open $X' \subset X$ containing $x_1, \dots, x_m$, an affine open $S \subset \mathbb{A}^{d-1}_k$, and a smooth morphism*

$$f \colon X' \to S$$

of relative dimension 1 such that

$$X' \cap Z = f^{-1}(S) \cap Z \quad \textit{is } S\textit{-finite and} \quad X' \cap D = f^{-1}(S) \cap D \quad \textit{is } S\textit{-(finite étale)}.$$

Proof. In the case $d = 1$, we may choose $X' = X$ and $S = \operatorname{Spec}(k)$, so we assume that $d > 1$. We also replace each x_i by a specialization to reduce to x_i being a closed point (see [SP, Lemma 02J6]), and in this case we will force each $f(x_i)$ to be the origin of $\mathbb{A}^{d-1}_k$. We embed X into some projective space $\mathbb{P}^N_k$ and then form closures to arrange that X is an open of a projective $\overline{X} \subset \mathbb{P}^N_k$ of dimension d with $\overline{X} \backslash X$ of dimension $\leqslant d-1$ and that there are

- a projective $\overline{D} \subset \overline{X}$ of dimension $d-1$ with $\overline{D} \backslash D$ of dimension $\leqslant d-2$, and
- a projective $\overline{Z} \subset \overline{X}$ of dimension $\leqslant d-1$ with $\overline{Z} \backslash Z$ of dimension $\leqslant d-2$.

We use the avoidance lemma [GLL15, Theorem 5.1] and postcompose with a Veronese embedding to build a hyperplane H_0 not containing any x_i such that $(\overline{X} \backslash X) \cap H_0$ is of dimension $\leqslant d-2$ (to force the dimension drop, choose appropriate auxiliary closed points and require H_0 to not contain them). By the Bertini theorem [Poo04, Theorem 1.3] of Poonen if k is finite and by the Bertini theorem of [Česnavičius22a, second paragraph of the proof of Lemma 3.2] applied both to X and to D in place of X if k is of characteristic 0, there is a hypersurface $H_1 \subset \mathbb{P}^N_k$ such that

- H_1 contains $x_1, \dots, x_m$;

[1] The assumption on k is likely not optimal but it will suffice and we do not wish to further complicate the proof.

- $X \cap H_1$ (resp., $D \cap H_1$) is k-smooth of dimension $d-1$ (resp., $d-2$);
- $Z \cap H_1$ is (resp., $(\overline{D} \backslash D) \cap H_1$ and $(\overline{Z} \backslash Z) \cap H_1$ are) of dimension $\leqslant d-2$ (resp., $\leqslant d-3$);
- $(\overline{X} \backslash X) \cap H_0 \cap H_1$ is of dimension $\leqslant d-2$.

In particular, by passing to intersections with H_1, we are left with an analogous situation with d replaced by $d-1$. Therefore, by iteratively applying the Bertini theorem in this way, we build hypersurfaces $H_1, \ldots, H_{d-1}$ such that

(i) the $x_1, \ldots, x_m$ lie in $H_1 \cap \ldots \cap H_{d-1}$ but not in H_0;

(ii) $X \cap H_1 \cap \ldots \cap H_{d-1}$ (resp., $D \cap H_1 \cap \ldots \cap H_{d-1}$) is k-smooth of dimension 1 (resp., k-étale);

(iii) $(\overline{D} \backslash D) \cap H_1 \cap \ldots \cap H_{d-1} = (\overline{Z} \backslash Z) \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

(iv) $(\overline{X} \backslash X) \cap H_0 \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

By letting $1, w_1, \ldots, w_{d-1}$ be the degrees of the hypersurfaces $H_0, H_1, \ldots, H_{d-1}$ and choosing defining equations h_i of the H_i, we determine a projective morphism $\widetilde{f} \colon \widetilde{X} \to \mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ from the weighted blowup $\widetilde{X} := \mathrm{Bl}(h_0, \ldots, h_{d-1})$ to the weighted projective space such that the diagram

$$\begin{array}{ccccc} \overline{X} \backslash H_0 & \hookrightarrow & \overline{X} \backslash (H_0 \cap \ldots \cap H_{d-1}) & \hookrightarrow & \widetilde{X} \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \mathbb{A}_k^{d-1} & \hookrightarrow & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) & = & \mathbb{P}_k(1, w_1, \ldots, w_{d-1}) \end{array}$$

commutes, where the bottom left arrow is the inclusion of the open locus where the first standard coordinate of $\mathbb{P}_k(1, w_1, \ldots, w_{d-1})$ does not vanish, see [Česnavičius22a, Sections 3.4 and 3.5]. By (i), each $f(x_i)$ is the origin of $\mathbb{A}_k^{d-1}$. By (ii) and the dimensional flatness criterion [EGA IV$_2$, Proposition 6.1.5], at every point of the fiber above the origin of $\mathbb{A}_k^{d-1}$, the map f is smooth of relative dimension 1 and its restriction to D is étale. Since $\widetilde{f}$ is projective, (iii)–(iv) and the openness of the quasi-finite locus [SP, Lemma 01TI] ensure that for some affine open neighborhood of the origin $S \subset \mathbb{A}_k^{d-1}$ both $f^{-1}(S) \cap Z$ and $f^{-1}(S) \cap D$ are S-finite (see also [SP, Lemma 02OG]). In conclusion, any affine open of $f^{-1}(S)$ that contains all the x_i and all the points of Z and D that lie above the origin of $\mathbb{A}_k^{d-1}$ becomes a sought X' after possibly shrinking S further. □

3. A reduction to studying G-torsors over a relative curve C

The geometric presentation lemma obtained in §2 will transform the Nisnevich conjecture as in Theorem 1.5 to a problem about studying G-torsors on a smooth relative curve C over R, see Proposition 3.4. For this, we need Lemma 3.3 about equating reductive groups, which is a variant of [PSV15, Theorem 3.6] of Panin–Stavrova–Vavilov. To obtain this variant, we combine ideas from [Česnavičius22a, Lemma 5.1] with some aspects from the survey [Česnavičius22b, Chapter 6].

Definition 3.1 ([Čes22b, ($\star$) in the beginning of Section 6.2])**.** For a ring A and an ideal $I \subset A$, we consider the following property of a set-valued functor $\mathscr{F}$ defined on the category of A-algebras:

$$\text{for every } x \in \mathscr{F}(A/I), \text{ there are a faithfully flat, finite, étale } A\text{-algebra } \widetilde{A}, \\ \text{an } A/I\text{-point } a \colon \widetilde{A} \twoheadrightarrow A/I, \text{ and an } \widetilde{x} \in \mathscr{F}(\widetilde{A}) \text{ whose } a\text{-pullback is } x. \qquad (\bigstar)$$

Remark 3.2. Let $f \colon \mathscr{F} \to \mathscr{F}'$ be a map of functors on the category of A-algebras and, for a $y \in \mathscr{F}'(A)$, let $\mathscr{F}_y \subset \mathscr{F}$ denote the f-fiber of y. If $\mathscr{F}'$ has property ($\bigstar$) with respect to $I \subset A$ and, for every faithfully flat, finite, étale A-algebra $\widetilde{A}$ and every $y \in \mathscr{F}'(\widetilde{A})$, the fiber $(\mathscr{F}|_{\widetilde{A}})_y$ has property $(\star)$ with respect to any ideal $\widetilde{I} \subset \widetilde{A}$ with $\widetilde{A}/\widetilde{I} \cong A/I$, then $\mathscr{F}$ itself has property $(\star)$ with respect to $I \subset A$. This straight-forward dévissage is useful in practice for dealing with short exact sequences.

Lemma 3.3. *For a Noetherian semilocal ring A whose local rings are geometrically unibranch, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibers have the same type, maximal A-tori $T \subset G$ and $T' \subset G'$, and an A/I-group isomorphism*

$$\iota \colon G_{A/I} \xrightarrow{\sim} G'_{A/I} \quad \text{such that} \quad \iota(T_{A/I}) = T'_{A/I},$$

there are a faithfully flat, finite, étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a \colon \widetilde{A} \twoheadrightarrow A/I$ and an $\widetilde{A}$-group isomorphism $\widetilde{\iota} \colon G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ whose a-pullback is ι and such that $\widetilde{\iota}(T_{\widetilde{A}}) = T'_{\widetilde{A}}$.

Proof. The claim is that the functor

$$X := \underline{\mathrm{Isom}}_{\mathrm{gp}}((G, T), (G', T'))$$

that parametrizes those group scheme isomorphisms between base changes of G and G' that bring T to T' has property $(\star)$ with respect to $I \subset A$. By [SGA $3_{\mathrm{III\ new}}$, Exposé XXIV, Corollaires 1.10 et 2.2 (i)], the normalizer $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ of the A-torus $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$ induced by T acts freely on X and, thanks to the assumption about the geometric fibers of G and G', the quotient $\overline{X} := X/N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ is a faithfully flat A-scheme that becomes constant étale locally on A. The geometric unibranchedness assumption then allows us to apply [Česnavičius22b, Example 6.2.1] to conclude that $\overline{X}$ has property $(\star)$ with respect to $I \subset A$. By Remark 3.2, we may therefore replace A by a finite étale cover to reduce to showing that every $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$-torsor has property $(\star)$. However, $N_{G^{\mathrm{ad}}}(T^{\mathrm{ad}})$ is an extension of a finite étale A-group scheme by T^{ad} (see, for instance, [Čes22b, Section 1.3.2]), so we may repeat the same reduction based on Remark 3.2 and be left with showing that every T^{ad}-torsor has property $(\star)$ with respect to $I \subset A$. This, however, is a special case of [Čes22b, Corollary 6.3.2] (that is based on building an equivariant projective compactification of the A-torus T^{ad} using toric geometry). ☐

We now combine the results of §§2–3 to obtain the following intermediate result towards Theorem 1.5, that is, towards the equicharacteristic case of the Nisnevich conjecture. In essence, we show that our torsor can be lifted to a smooth relative curve equipped with a section. By changing this curve while preserving the overall structure present in Proposition 3.4, we will eventually reduce to the case when $Y = \varnothing$, which amounts to the Grothendieck–Serre case. We directly state Proposition 3.4 in its "relative version" that includes an auxiliary W, but the main case of interest is $W = \mathrm{Spec}(k)$.

Proposition 3.4. *For a regular semilocal ring R containing a field k, an $r \in R$ with $r \notin \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m} \subset R$, a reductive R-group G, a quasi-compact and quasi-separated k-scheme W, and a G-torsor E over $W \otimes_k R[\frac{1}{r}]$ that is trivial over $W \otimes_k \mathrm{Frac}(R)$, there are*

(i) *a smooth affine R-scheme C of pure relative dimension* 1;

(ii) *an R-finite closed subscheme $Z \subset C$ and an R-(finite étale) closed subscheme $Y \subset Z$;*

(iii) *a section $s \in Z(R) \subset C(R)$ such that $s|_{R[\frac{1}{r}]}$ factors through $C \backslash Y$; and*

(iv) *a G-torsor $\mathscr{E}$ over $W \times_k (C \backslash Y)$ that is trivial over $W \times_k (C \backslash Z)$ such that*

$$(\mathrm{id}_W \times_k s|_{R[\frac{1}{r}]})^*(\mathscr{E}) \cong E \quad \text{as } G\text{-torsors over } W \otimes_k R[\tfrac{1}{r}];$$

moreover, if $r \in R^\times$, then we may arrange that, in addition, $Y = \varnothing$.

Proof. Let $\mathbb{F} \subset k$ be the prime subfield and consider the k-algebra $k \otimes_{\mathbb{F}} R$. The composition $R \xrightarrow{a} k \otimes_{\mathbb{F}} R \xrightarrow{b} R$, in which the second map uses the k-algebra structure of R, is the identity. The base change of E along $\mathrm{id}_W \otimes_k a[\frac{1}{r}]$ is a G-torsor E' over $W \otimes_{\mathbb{F}} R[\frac{1}{r}]$. Thus, if we settle the claim with $\mathbb{F}$ in place of k, in particular, build a G-torsor $\mathscr{E}'$ over $W \times_{\mathbb{F}} (C \backslash Y)$ as in (iv), then base change along b will settle the claim over the original k. This reduces us to the case when $k = \mathbb{F}$.

Since k is now perfect, Popescu theorem [SP, Theorem 07GC] expresses R as a filtered direct limit of smooth k-algebras. Thus, by passing to connected components of $\mathrm{Spec}(R)$ and doing a limit argument, we may assume that R is a semilocal ring of a smooth, affine, irreducible k-scheme X of dimension $d \geqslant 0$, that r is a global section of X (resp., a global unit of X if $r \in R^\times$) that cuts out a divisor $D \subset X$, and that G (resp., E) is defined over all of X (resp., all of $W \times_k (X \backslash D)$). Since k is perfect and D is regular at its points in $\mathrm{Spec}(R)$, we may shrink X (using [SP, Lemma 00DS]) to make D be k-smooth. Since E is trivial over $W \otimes_k \mathrm{Frac}(X)$, there is a proper closed subscheme $\mathscr{Z} \subset X$ with $D \subset \mathscr{Z}$ such that E trivializes over $W \times_k (X \backslash \mathscr{Z})$. If $d = 0$, then E is trivial and we may choose $C = \mathbb{A}^1_R$, so we assume that X is of dimension $d > 0$. Finally, we use [SGA 3_{II}, Exposé XIV, Corollaire 3.20] to shrink X more to make G have a maximal torus T defined over all of X.

With these preparations, Theorem 2.1 allows us to shrink X around $\mathrm{Spec}(R)$ to arrange that there exist an affine open $S \subset \mathbb{A}^{d-1}_k$ and a smooth morphism $f\colon X \to S$ of relative dimension 1 such that $\mathscr{Z}$ is S-finite and D is S-(finite étale). We base change f along the map $\mathrm{Spec}(R) \to S$ to obtain

- a smooth affine R-scheme C of pure relative dimension 1 (the base change of X);
- an R-finite closed subscheme $Z \subset C$ (the base change of $\mathscr{Z}$);
- an R-(finite étale) closed subscheme $Y \subset Z$ that is empty if $r \in R^\times$ (the base change of D);
- a section $s \in C(R)$ (induced by the "diagonal" section) such that $s|_{R[\frac{1}{r}]}$ factors through $C \backslash Y$;
- a reductive C-group scheme $\mathscr{G}$ with $s^*(\mathscr{G}) \cong G$ (the base change of G);
- a maximal C-torus $\mathscr{T} \subset \mathscr{G}$ (the base change of T) with $s^*(\mathscr{T}) \cong T$; and
- a $\mathscr{G}$-torsor $\mathscr{E}$ over $W \times_k (C \backslash Y)$ (the base change of E) that is trivial over $W \times_k (C \backslash Z)$ with
$$(\mathrm{id}_W \times_k s|_{R[\frac{1}{r}]})^*(\mathscr{E}) \cong E \quad \text{as } G\text{-torsors over } W \otimes_k R[\tfrac{1}{r}].$$

We may replace Z by $Z \cup s$ if needed to arrange that $s \in Z(R)$, and it then only remains to make $\mathscr{G}$ equal to G_C. By Lemma 3.3 and spreading out, there is a finite étale cover $\widetilde{C}$ of some affine open neighborhood of Z in C such that s lifts to some $\widetilde{s} \in \widetilde{C}(R)$ and $\mathscr{G}_{\widetilde{C}} \cong G_{\widetilde{C}}$, compatibly with an already fixed such isomorphism after pullback along $\widetilde{s}$. Thus, it remains to replace C and s by $\widetilde{C}$ and $\widetilde{s}$ and to replace Y, Z, $\mathscr{G}$, $\mathscr{E}$ by their corresponding base changes. □

4. Replacing C by the relative affine line $\mathbb{A}^1_R$

The goal of this section is to reduce Proposition 3.4 to the case when $C = \mathbb{A}^1_R$ there, see Proposition 4.7 below. This reduction is based on the following lemmas that are generalizations of their counterparts from [Česnavičius22a, Section 6] (which, in turn, extended earlier versions from works of Panin and Fedorov). The new aspects of these lemmas will be crucial in §6, so we discuss them in detail.

Definition 4.1. For a semilocal ring A, a finite scheme Z, and an A-scheme C, there is no *finite field obstruction* to embedding Z into C if for each maximal ideal $\mathfrak{m} \subset A$ with $k_{\mathfrak{m}}$ finite we have
$$\{z \in Z_{k_{\mathfrak{m}}} \,|\, [k_z : k_{\mathfrak{m}}] = d\} \leqslant \{z \in C_{k_{\mathfrak{m}}} \,|\, [k_z : k_{\mathfrak{m}}] = d\} \quad \text{for every} \quad d \geqslant 1. \qquad (\dagger)$$

The following variant of [Čes22a, Lemma 6.1] that is built on "finite field tricks" of Panin will allow us to arrange that there be no finite field obstruction to embedding Z into $\mathbb{A}^d_A$.

Lemma 4.2. *Let A be a semilocal ring, let $Z \cong \mathrm{Spec}(B)$ be an A-finite scheme, let $Y \subset Z$ be an A-(finite étale) closed subscheme, and let $T \subset \mathbb{A}^d_A$ be an A-finite closed subscheme.*

(a) *There is a finite étale surjection $\widetilde{Z} \twoheadrightarrow Z$ such that there is no finite field obstruction to embedding $\widetilde{Z}$ into $\mathbb{A}^d_A \backslash T$, moreover, for every large $N > 0$ we may find such a $\widetilde{Z}$ of the form $\widetilde{Z}_0 \sqcup \widetilde{Z}_1$ with $\widetilde{Z}_i \cong \mathrm{Spec}(B[t]/(f_i(t))) \twoheadrightarrow Z$ surjective and f_i monic of constant degree $N + i$.*

(b) *For a fixed $s \in Z(A)$, if there is no finite field obstruction to embedding $\mathrm{Spec}(A)$ into $\mathbb{A}^d_A \backslash T$, then (a) holds with the requirement that there exist an $\widetilde{s} \in \widetilde{Z}(A)$ lifting s.*

(c) *Fix an $n \geqslant 0$, suppose that A contains a field k, that $Y = Y_0 \sqcup Y_1$, and that there is no finite field obstruction to embedding Y_0 into $\mathbb{A}^d_A \backslash T$. Then (a) holds with the requirement that*

$$\widetilde{Y} := Y \times_C \widetilde{C} \quad \text{be a disjoint union} \quad \widetilde{Y} = \widetilde{Y}_0 \sqcup \widetilde{Y}_1$$

with $\widetilde{Y}_0 \xrightarrow{\sim} Y_0$ and each connected component of $\widetilde{Y}_1$ an algebra over some field k' with

$$\#k' > n \cdot \deg(\widetilde{Z}/Z).$$

Part (c) is a critical statement that we will use in §6 to bypass finite field difficulties of [Fed21b]. To be clear, in (c) the extension k' depends on the connected component of $\widetilde{Y}_1$ in question.

Proof. In (a), we let $N > 2$ be sufficiently large and choose the following monic polynomials: for each closed point $z \in Z$ with k_z finite (resp., infinite), a monic $f_z(t) \in k_z[t]$ that is irreducible of degree N (resp., that is the product of N distinct monic linear polynomials). We let $f_0(t) \in B[t]$ be a monic polynomial that simultaneously lifts all the $f_z(t)$, and we define a monic $f_1(t) \in B[t]$ analogously with N replaced by $N + 1$. Granted that N is large enough, the resulting $\widetilde{Z}_i$ settle (a).

In (b), we choose the same $f_1(t)$ but we modify the construction of $f_0(t)$. More precisely, for the closed points $z \in Z$ not in s, we choose the same monic $f_z(t) \in k_z[t]$ of degree $N > 2$ as before. However, for the closed points $z \in Z$ in s, we let $f_z(t) \in k_z[t]$ be the product of t with a monic irreducible polynomial of degree $N - 1$ if k_z is finite (resp., with the product of $N - 1$ distinct monic linear polynomials that have nonzero constant terms if k_z is infinite). Viewing A as the coordinate ring of s, we let $f_s(t) \in A[t]$ be the product of t with a monic polynomial of degree $N - 1$ that reduces to $f_z(t)$ for each closed point $z \in Z$ that lies in s. The union of s and the closed points of Z not in s is a closed subscheme of Z, so there is a monic $f_0(t) \in B[t]$ that simultaneously lifts $f_s(t)$ and all the $f_z(t)$. Granted that N is large enough, the resulting $\widetilde{Z}_i$ settle (b).

In (c), if k is infinite, then the desired aspect about k' is automatic with $k' = k$ and it suffices to let $f_0(t)$ be a product of t with $N - 1$ distinct monic linear polynomials that have nonzero constant terms in k and to let $f_1(t)$ be a product of $N + 1$ distinct monic linear polynomials with coefficients in k. Thus, we focus on the key case when k is finite with $N > 2$ and begin by choosing

- an $f_{Y_0}(t) \in k[t]$ that is the product of t and a monic irreducible polynomial of degree $N - 1$;
- an $f_{Y_1}(t) \in k[t]$ that is monic irreducible of degree N;
- for each closed point $z \in Z$ not in Y with k_z finite (resp., infinite), an $f_z(t) \in k_z[t]$ that is irreducible of degree N (resp., that is the product of N distinct monic linear polynomials).

We write $Y_i = \mathrm{Spec}(A_i)$ and use the k-algebra structure of A_i to view $f_{Y_i}(t)$ as an element of $A_i[t]$. Since Y and the closed points of Z not in Y form a closed subscheme of Z, there is a monic $f_0(t) \in B[t]$ whose image in $A_i[t]$ (resp., in $k_z[t]$ for each closed point $z \in Z$ not in Y) is $f_{Y_i}(t)$ (resp., $f_z(t)$). With the resulting $\widetilde{Z}_0$ in hand, let $\widetilde{Y}_0$ be component of $Y_0 \times_Z \widetilde{Z}_0$ cut out by the factor t of $f_{Y_0}(t)$ to arrange that $\widetilde{Y}_0 \xrightarrow{\sim} Y_0$. By the choice of the $f_{Y_i}(t)$, each connected component of the complement of $\widetilde{Y}_0$ in $Y \times_Z \widetilde{Z}_0$ is an algebra over a field extension k'/k of degree either $N-1$ or N.

We repeat the construction with N replaced by $N+1$, except that now we let $f_{Y_0}(t) \in k[t]$ be monic irreducible of degree $N+1$, to build a monic $f_1(t) \in B[t]$ of degree $N+1$. For the resulting $\widetilde{Z}_1$, by construction, each connected component of $Y \times_Z \widetilde{Z}_1$ is an algebra over a field extension of k'/k of degree $N+1$. Overall, with the resulting $\widetilde{Y}_1$ as in (c), every connected component of $\widetilde{Y}_1$ is an algebra over a field extension k'/k of degree $N-1$, N, or $N+1$. For large N, any such k' will satisfy

$$\# k' > nN(N+1) = n \cdot \deg(\widetilde{Z}/Z).$$

By construction, the number of closed points of $\widetilde{Z}$ not in $\widetilde{Y}_0$ with finite residue fields is uniformly bounded as N grows and the degree over k of the residue field of every such closed point is $\geqslant \varepsilon N$ for some $\varepsilon > 0$ (that depends on the degrees $[k_z : k]$ for closed points $z \in Z$ with k_z finite but not on N). In particular, for large N, there is no finite field obstruction to embedding $\widetilde{Z}$ into $\mathbb{A}^d_A \backslash T$. □

Remark 4.3. In practice, the A-finite Z that we wish to modify as in Lemma 4.2 so that there be no finite field obstruction to embedding it into $\mathbb{A}^d_A \backslash T$ occurs as a closed subscheme of a smooth affine A-scheme C, and it is useful to lift $\widetilde{Z} \twoheadrightarrow Z$ to a finite étale cover $\widetilde{C} \twoheadrightarrow D$ of some affine open neighborhood $D \subset C$ of Z. Due to the explicit nature of $\widetilde{Z}_i$, this is possible to arrange: it suffices to lift each $f_i(T)$ to a monic polynomial with coefficients in the coordinate ring of the semilocalization of C at the closed points of Z (built using prime avoidance [SP, Lemma 00DS]) and to spread out.

By the following refinement of [Čes22a, Lemma 6.3] and in the spirit of Remark 4.3, if an A-finite Z is a closed subscheme of *some* smooth affine A-scheme of relative dimension d and there is no finite field obstruction to embedding it into $\mathbb{A}^d_A \backslash T$, then such an embedding indeed exists and may even be chosen to be excisive as follows.

Lemma 4.4. *For a semilocal ring A and an A-finite closed subscheme $T \subset \mathbb{A}^d_A$, an A-finite scheme Z may be embedded into $\mathbb{A}^d_A \backslash T$ if and only if it is a closed subscheme $Z \subset C$ of some smooth affine A-scheme C of pure relative dimension d and there is no finite field obstruction to embedding Z into $\mathbb{A}^d_A \backslash T$; if this is the case, then one may even embed Z into $\mathbb{A}^d_A \backslash T$ excisively: then there are an affine open $D \subset C$ containing Z and an étale A-morphism $f \colon D \to \mathbb{A}^d_A \backslash T$ that fits into a Cartesian square*

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \hookrightarrow & \mathbb{A}^d_A \backslash T, \end{array} \qquad (4.4.1)$$

in particular, such that f embeds Z as a closed subscheme $Z' \subset \mathbb{A}^d_A \backslash T$; in fact, for every closed subscheme $Y \subset Z$ and an embedding $\iota \colon Y \hookrightarrow \mathbb{A}^d_A \backslash T$, there are D and f as above with $f|_Y = \iota$.

Proof. The 'only if' is clear, so we focus on the 'if' and fix an embedding $Z \subset C$ as in the statement. Throughout the proof, especially to handle the aspect about Y, we will use the fact that the schematic union of two closed subschemes $\mathrm{Spec}(B/I)$ and $\mathrm{Spec}(B/J)$ of an affine scheme $\mathrm{Spec}(B)$ is the closed subscheme $\mathrm{Spec}(B/(I \cap J)) \subset \mathrm{Spec}(B)$, as follows from the isomorphism

$$B/(I \cap J) \xrightarrow{\sim} B/I \times_{B/(I+J)} B/J. \tag{4.4.2}$$

To begin with, we claim that it suffices to build an A-morphism $\widetilde{f}\colon C \to \mathbb{A}^d_A$ that agrees with ι on Y, restricts to a closed immersion on Z, is étale at the closed points of Z, and satisfies $\widetilde{f}(Z) \cap T = \varnothing$. Indeed, then there will be an affine open neighborhood $D \subset C$ of Z such that $f := \widetilde{f}|_D$ is étale, such that f factors through $\mathbb{A}^d_A \backslash T$, and, since a section of a separated étale morphism, such as $f^{-1}(f(Z)) \to Z$, is an inclusion of a clopen subset, even such that $Z = f^{-1}(f(Z))$, as desired.

For building $\widetilde{f}$, we now argue that we may base change to the union of the closed points of $\mathrm{Spec}(A)$. For this, note that giving a map to $\mathbb{A}^d_A$ amounts to giving d global sections, so, once we construct $\widetilde{f}$ over the closed points of $\mathrm{Spec}(A)$, we will be able to use (4.4.2) to lift it to a map over A that is compatible with ι. Moreover, since Z is finite, any such lift $\widetilde{f}$ will satisfy $\widetilde{f}(Z) \cap T = \varnothing$ and the Nakayama lemma [SP, Lemma 00DV] will ensure that $\widetilde{f}|_Z$ is a closed immersion. Finally, $\widetilde{f}$ will be étale at the closed points of Z thanks to the fibral criterion of flatness [EGA IV_3, Corollaire 11.3.11].

In conclusion, it suffices to build the desired $\widetilde{f}$ in the case when A is a field, in which Z is a finite union of possibly nonreduced points. In this case, by [Česnavičius22b, Proposition 4.1.4] (whose proof uses a presentation theorem similar to Theorem 2.1), every closed point of C may be embedded as a closed point of $\mathbb{A}^d_A$. Thus, since there is no finite field obstruction to embedding Z into $\mathbb{A}^d_A \backslash T$, we may embed the union of all the points z of Z into $\mathbb{A}^d_A \backslash T$ over A compatibly with the fixed embedding $\iota\colon Y \hookrightarrow \mathbb{A}^d_A \backslash T$. By the Cohen structure theorem [Mat89, Theorem 29.7], the first infinitesimal neighborhood ε_z of z in its A-fiber of C is isomorphic to $\mathrm{Spec}(k_z[x_1, \dots, x_d]/(x_1^2, \dots, x_d^2))$. Consequently, we may extend the fixed embedding of all the z into $\mathbb{A}^d_A \backslash T$ to an embedding of all the ε_z into $\mathbb{A}^d_A \backslash T$ compatibly with ι. The union of Y and all the ε_z is a closed subscheme of C, so we may choose global sections $s_1, \dots, s_d \in \Gamma(C, \mathscr{O}_C)$ such that the restriction of s_i to Y and to each ε_z is the image of the i-th standard coordinate of $\mathbb{A}^d_A$ under the fixed embeddings into $\mathbb{A}^d_A \backslash T$. By mapping this coordinate to s_i, we obtain a morphism

$$\widetilde{f}\colon C \to \mathbb{A}^d_A$$

that, by construction and the Nakayama lemma [SP, Lemma 00DV], agrees with ι on Y, restricts to a closed immersion on Z, and satisfies $\widetilde{f}(Z) \cap T = \varnothing$. By construction, the *a priori* open locus of C where $\widetilde{f}$ is quasi-finite (see [SP, Lemma 01TI]) contains the points of Z. Due to this quasi-finiteness, the flatness criteria [EGA IV_2, Proposition 6.1.5] and [EGA IV_3, Corollaire 11.3.11] ensure that $\widetilde{f}$ is flat at the points of Z and, by construction, $\widetilde{f}$ is then even étale at the points of Z, as desired. □

The following corollary is useful for embedding a finite *étale* Z into $\mathbb{A}^1_A \backslash T$ without an ambient curve C.

Corollary 4.5. *For a semilocal ring A, a $d \geqslant 1$, and an A-finite closed subscheme $T \subset \mathbb{A}^d_A$, an A-(finite étale) scheme Z may be embedded into $\mathbb{A}^d_A \backslash T$ iff there is no finite field obstruction to it.*

Proof. The 'only if' is clear. For the 'if,' by Lemma 4.4, it is enough to embed Z into $\mathbb{A}^d_A$, so we may assume that $T = \varnothing$. It then suffices to show that $Z \cong \mathrm{Spec}(B)$ with a B that may be generated by d elements as an A-algebra. Thus, since B is a finite A-module and A is semilocal, the Nakayama lemma [SP, Lemma 00DV] allows us to replace A by the product of the residue fields of its maximal ideals, so we may assume that A is a field k. In this case, Z is a disjoint union of spectra of finite separable field extensions k and, since there is no finite field obstruction to embedding Z into $\mathbb{A}^d_k$, such an embedding exists by the primitive element theorem. □

An excisive square like (4.4.1) may be used to glue torsors as follows. This glueing statement is due to Moret-Bailly [MB96], with a more restrictive version implicit already in [FR70, Proposition 4.2].

Lemma 4.6 ([Čes22b, Proposition 4.2.1]). *For a scheme S, a closed subscheme $Z \subset S$ that is locally cut out by a finitely generated ideal, a Cartesian square*

$$\begin{array}{ccc} Z & \hookrightarrow & S \\ \sim\downarrow & & \downarrow \\ Z' & \hookrightarrow & S' \end{array}$$

in which the map $S \to S'$ is affine, flat, and embeds Z as a closed subscheme $Z' \subset S'$ (for which $Z \cong Z' \times_{S'} S$ by the Cartesianess requirement), and a quasi-affine, flat, finitely presented S'-group scheme G, base change induces an equivalence of categories

$$\{G\text{-torsors over } S'\} \xrightarrow{\sim} \{G\text{-torsors over } S\} \times_{\{G\text{-torsors over } S\backslash Z\}} \{G\text{-torsors over } S'\backslash Z'\},$$

that is, giving a G-torsor over S' amounts to giving a G-torsor over S, a G-torsor over $S'\backslash Z'$, and an isomorphism between their base changes to $S\backslash Z$ (and likewise for torsor isomorphisms). ☐

We are ready for the promised improvement of Proposition 3.4 in which we replace C by $\mathbb{A}^1_R$.

Proposition 4.7. *Proposition 3.4 holds with the requirements that $C = \mathbb{A}^1_R$ and $(\mathbb{A}^1_R\backslash Z)(R) \neq \varnothing$.*

Proof. Fix the notation of Proposition 3.4 and let $Y \subset Z \subset C$ with $s \in Z(R)$ and a G-torsor $\mathscr{E}$ on $W \times_k (C\backslash Y)$ be as provided by that result. By Lemma 4.2 (b) and Remark 4.3, we may replace C by a finite étale cover $\widetilde{C}$ of some affine open neighborhood of Z in C such that s lifts to an $\widetilde{s} \in \widetilde{C}(R)$, replace Y and Z by their preimages in $\widetilde{C}$, and replace s by $\widetilde{s}$ to reduce to the case when there is no finite field obstruction to embedding Z into $\mathbb{A}^1_R$ or even into $\mathbb{G}_{m,R}$. In particular, Lemma 4.4 applies and gives an affine open $D \subset C$ containing Z and a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ \sim\downarrow & & \downarrow f \\ Z' & \hookrightarrow & \mathbb{G}_{m,R} \end{array}$$

in which the map f is étale and embeds Z as a closed subscheme $Z' \subset \mathbb{G}_{m,R}$. These properties of the square persist after forming its fiber product with W over k. Thus, by the glueing Lemma 4.6, we may descend $\mathscr{E}|_{W\times_k(D\backslash Y)}$ to a G-torsor $\mathscr{E}'$ over $W \times_k (\mathbb{A}^1_R\backslash Y)$ that is trivial over $W \times_k (\mathbb{A}^1_R\backslash Z')$ with $(\mathbb{A}^1_R\backslash Z')(R) \neq \varnothing$. It remains to replace $Z \subset C$ and $\mathscr{E}$ by $Z' \subset \mathbb{A}^1_R$ and $\mathscr{E}'$, as desired. ☐

5. The relative Grothendieck–Serre conjecture in equal characteristic

The total isotropicity assumption that is critical for the results announced in the introduction played no role in §§2–4, but it will be important for progressing further. More precisely, we need it for the relative Grothendieck–Serre conjecture established by Fedorov in equal characteristic in [Fed21a, Theorem 1] (with an earlier more restrictive case due to Panin–Stavrova–Vavilov [PSV15, Theorem 1.1]) and stated as Theorem 5.5 below. The latter will be crucial in §6, so we review its argument, especially, since this is an opportunity to improve some intermediate results from [Fed21a]. The latter are special cases of the following general conjecture of Horrocks type.

Conjecture 5.1 ([Čes22b, Conjecture 3.5.1]). *For a ring A and a totally isotropic reductive A-group scheme G, every G-torsor over $\mathbb{A}^d_A$ that is trivial away from an A-finite closed subscheme is trivial.*

We recall from [Čess22b, Section 3.5.2] that Conjecture 5.1 holds for any A when G is either semisimple simply connected, or split, or a torus. For a general G, we show that the torsor in question trivializes after pulling back along any map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \to t^d$ for a sufficiently divisible d (Lemma 5.3 (b)) and also after pulling back by an A-point of $\mathbb{A}^1_A$ (Proposition 5.4 (i)). In Proposition 5.8, we also establish Conjecture 5.1 in the case when A is regular and contains a field.

Under additional assumptions, the triviality after pulling back by an A-point is [Fed21a, Theorem 5], and we will follow Fedorov's strategy that is based on the geometry of the affine Grassmannian. The latter will enter through (self-contained) citations to the survey article [Čes22b] that mildly generalized some of Fedorov's key steps. We will also use the following general form of Quillen patching.

Theorem 5.2 (Gabber, [Čes22b, Corollary 5.1.5 (b)]). *For a ring A and a locally finitely presented A-group algebraic space G, a G-torsor (for the fppf topology) over $\mathbb{A}^d_A$ descends to a G-torsor over A if and only if it does so Zariski locally on* $\mathrm{Spec}(A)$. □

Lemma 5.3. *Let A be a ring, let G be a totally isotropic reductive A-group scheme, and let E be a G-torsor over $\mathbb{A}^1_A$ that is trivial away from some A-finite closed subscheme $Z \subset \mathbb{A}^1_A$.*

(a) *If, for some extension of E to a G-torsor $\widetilde{E}$ over $\mathbb{P}^1_A$ obtained by glueing E with the trivial torsor over $\mathbb{P}^1_A \backslash Z$ and for every prime ideal $\mathfrak{p} \subset A$, the G^{ad}-torsor over $\mathbb{P}^1_{k_\mathfrak{p}}$ induced by $\widetilde{E}$ lifts to a generically trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{k_\mathfrak{p}}$, then E is trivial.*

(b) *For any $d > 0$ divisible by the A-fibral degrees of the isogeny $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$, the pullback of E along any finite flat map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ of degree d that extends to a map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ is trivial.*

Proof. In (b), we extend E to a G-torsor $\widetilde{E}$ over $\mathbb{P}^1_A$ as in (a) and consider the pullback of this extension under our map $\mathbb{P}^1_A \to \mathbb{P}^1_A$. By [Čes22b, Lemma 5.3.5] (or [Fed21a, Proposition 2.3]), the choice of d ensures that the fibral condition of (a) holds for this pullback, so (b) follows from (a).

In (a), it suffices to show that both E and the restriction of $\widetilde{E}$ to the complementary affine line $\mathbb{P}^1_A \backslash \{t = 0\}$ descend to G-torsors over A: both of these descents will agree with the restriction of $\widetilde{E}$ to $t = 1$, which will agree with the restriction of $\widetilde{E}$ to $t = \infty$ and hence be trivial, and then E will also be trivial. By Quillen patching of Theorem 5.2, for the descent claim we may replace A by its localization at a maximal ideal to reduce to the case of a local A.

Once A is local, we will directly show that both E and the restriction of $\widetilde{E}$ to $\mathbb{P}^1_A \backslash \{t = 0\}$ are trivial. For this, we first show that we may modify Z so that it does not meet $t = 0$. Namely, if the residue field k of A is infinite, then there is some $s \in (\mathbb{A}^1_A \backslash (Z \cup \{t = 0\}))(A)$ and, by [Čes22b, Proposition 5.3.6] (which uses the total isotropicity assumption, the fibral assumption on $\widetilde{E}$, and is based on geometric input about the affine Grassmannian in the style of [Fed21a, Theorem 6]), the restriction of $\widetilde{E}$ to $\mathbb{P}^1_A \backslash s$ is a trivial G-torsor, so that we may replace Z by s to arrange the desired $Z \cap \{t = 0\} = \varnothing$. In contrast, if the residue field k of A is finite, then there is some large n such that $\mathbb{A}^1_k \backslash (Z_k \cup \{t = 0\})$ contains a finite étale subscheme y that is the union of a point valued in the field extension of k of degree n and a point valued in the field extension of k of degree $n + 1$. Both of these components of y are cut out by separable monic polynomials with coefficients in k, so y lifts to an A-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_A \backslash (Z \cup \{t = 0\})$ that is a disjoint union of an A-(finite étale) closed subscheme of degree n and an A-(finite étale) closed subscheme of degree $n + 1$. In particular, both $\mathscr{O}(n)$ and $\mathscr{O}(n + 1)$ restrict to trivial line bundles on $\mathbb{P}^1_A \backslash Y$, and hence so does $\mathscr{O}(1)$. Thus, by [Čes22b, Proposition 5.3.6] once more, $\widetilde{E}$ is trivial on $\mathbb{P}^1_A \backslash Y$, to the effect that in the case when k is finite we may replace Z by Y to again arrange that $Z \cap \{t = 0\} = \varnothing$.

Once our $Z \subset \mathbb{A}^1_A$ does not meet $\{t = 0\}$, it suffices to apply [Česnavičius22b, Proposition 5.3.6] twice to conclude that $\widetilde{E}$ restricts to the trivial torsor both on $\mathbb{P}^1_A \backslash \{t = \infty\}$ and on $\mathbb{P}^1_A \backslash \{t = 0\}$, as desired. ☐

Proposition 5.4. *For a ring A and a reductive A-group scheme G, every G-torsor E over $\mathbb{A}^d_A$ that is trivial away from an A-finite closed subscheme has a trivial pullback along each $s \in \mathbb{A}^d_A(A)$ if either*

(i) *G is totally isotropic*;

(ii) *A is semilocal.*

Parts (i) and (ii) are generalizations of [Fed21a, Theorem 5] and [Fed21a, Theorem 4], respectively.

Proof. Any A-point of $\mathbb{A}^d_A$ factors through some $\mathbb{A}^{d-1}_A$-point, so we lose no generality by replacing A by $A[t_1, \dots, t_{d-1}]$ to reduce to the case when $d = 1$. We then change the variable of $\mathbb{A}^1_A$ to transform s into the section $t = 0$. This ensures that s lifts to an A-point along any map $\mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $t \mapsto t^d$. Consequently, in (i) we may pull back along such a map and then conclude from Lemma 5.3.

In the remaining case (ii) with $d = 1$, we fix an A-finite closed subscheme $Z \subset \mathbb{A}^1_A$ such that E is trivial over $\mathbb{A}^1_A \backslash Z$, and we extend E to a G-torsor $\widetilde{E}$ over $\mathbb{P}^1_A$ by glueing E with the trivial torsor over $\mathbb{P}^1_A \backslash Z$. We then let d be the least common multiple of the A-fibral degrees of the isogeny $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$ and, as in the proof of Lemma 5.3 (b), replace $\widetilde{E}$ by its pullback along the map $\mathbb{P}^1_A \to \mathbb{P}^1_A$ given by $[x : y] \mapsto [x^d : y^d]$ to arrange that, for every maximal ideal $\mathfrak{m} \subset A$, the G^{ad}-torsor over $\mathbb{P}^1_{k_\mathfrak{m}}$ induced by $\widetilde{E}$ lifts to a generically trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{k_\mathfrak{m}}$ (see [Česnavičius22b, Lemma 5.3.5]).

In (ii), we will eventually obtain the conclusion from [Česnavičius22b, Proposition 5.3.6]. To prepare for applying it, consider the canonical decomposition as in (1.2.1):

$$G^{\mathrm{ad}} \cong \textstyle\prod_i H_i \quad \text{with} \quad H_i \cong \mathrm{Res}_{A_i/A}(G_i),$$

where A_i is a finite étale A-algebra and G_i is an adjoint A_i-group scheme with simple geometric fibers. For each i, consider the projective, smooth A-scheme X_i parametrizing parabolic subgroups of H_i (see [SGA $3_{\mathrm{III\ new}}$, Exposé XXVI, Corollaire 3.5]). For each i, consider the closed subscheme $\mathrm{Spec}(A/I_i) \subset \mathrm{Spec}(A)$ that is the disjoint union of those maximal ideals $\mathfrak{m} \subset A$ such that $(H_i)_{k_\mathfrak{m}}$ is isotropic, in other words, such that $(H_i)_{k_\mathfrak{m}}$ has a proper parabolic subgroup (see [SGA $3_{\mathrm{III\ new}}$, Exposé XXVI, Corollaire 6.12]), and fix such parabolic subgroups to obtain an $x_i \in X_i(A/I_i)$. By [Česnavičius22b, Lemma 6.2.2] (which is based on Bertini theorem), there are a faithfully flat, finite, étale A-scheme Y_i equipped with an A/I_i-point $y_i \in Y_i(A/I_i)$ and an A-morphism $Y_i \to X_i$ that maps y_i to x_i, so that, in particular, $(H_i)_{Y_i}$ is totally isotropic for every i.

By Lemma 4.2 (a), there is a finite étale cover $\pi\colon \widetilde{Y} \twoheadrightarrow \bigsqcup_i Y_i$ such that there is no finite field obstruction to embedding $\widetilde{Y}$ into $\mathbb{A}^1 \backslash (Z \cup s)$ and $\widetilde{Y} = \widetilde{Y}' \sqcup \widetilde{Y}''$ with $\widetilde{Y}'$ (resp., $\widetilde{Y}''$) surjective over $\bigsqcup_i Y_i$ of constant degree N (resp., $N + 1$) for some $N > 0$. By Corollary 4.5, we may therefore find an embedding $\widetilde{Y} \hookrightarrow \mathbb{A}^1_A$ whose image does not meet Z or s. By construction, for each i and each maximal ideal $\mathfrak{m} \subset A$ such that $(H_i)_{k_\mathfrak{m}}$ is isotropic, Y_i has a $k_\mathfrak{m}$-point, and so the $k_\mathfrak{m}$-fiber of the preimage $\widetilde{Y}_i := \pi^{-1}(Y_i)$ has two disjoint clopens that have degrees N and $N + 1$ over $k_\mathfrak{m}$. Consequently, for each such i and $\mathfrak{m}$, the line bundle $\mathscr{O}(1)$ is trivial over $(\mathbb{P}^1_A \backslash \widetilde{Y}_i)_{k_\mathfrak{m}}$. Thus, since $(\widetilde{Y} \cup \{t = \infty\}) \cap Z = \varnothing$, we may apply [Česnavičius22b, Proposition 5.3.6] to conclude that E is trivial over $\mathbb{A}^1_A \backslash \widetilde{Y}$. In particular, since $\widetilde{Y}$ is disjoint from s, the pullback $s^*(E)$ is also trivial, as desired. ☐

We are ready for the relative Grothendieck–Serre conjecture in equal characteristic, which is the following mild improvement to [Fed21a, Theorem 1].

Theorem 5.5. *For a regular semilocal ring R containing a field k, a reductive R-group G, and an affine k-scheme W, no nontrivial G-torsor over $W \otimes_k R$ trivializes over $W \otimes_k \mathrm{Frac}(R)$ if either*

(i) *G is totally isotropic;*

(ii) *$W \otimes_k R$ is semilocal, for instance, if $W = k$.*

Proof. Let E be a G-torsor over $W \otimes_k R$ that trivializes over $W \otimes_k \mathrm{Frac}(R)$. By Proposition 4.7, there are an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$, an $s \in Z(R)$, and a G-torsor $\mathscr{E}$ over $W \times_k \mathbb{A}^1_R$ trivial over $W \times_k (\mathbb{A}^1_R \backslash Z)$ such that

$$(\mathrm{id}_W \times_k s)^*(\mathscr{E}) \cong E \quad \text{as } G\text{-torsors over } W \otimes_k R.$$

Let A be the coordinate ring of $W \otimes_k R$ and view $\mathscr{E}$ as a G-torsor over $\mathbb{A}^1_A$ that is trivial away from an A-finite closed subscheme $\mathscr{Z} := W \times_k Z \subset \mathbb{A}^1_A$. We need to show that the pullback of $\mathscr{E}$ along a given A-point $z := \mathrm{id}_W \times_k s$ of $\mathscr{Z}$ is trivial. This, however, is a special case of Proposition 5.4. □

We close the section with the promised case of Conjecture 5.1 in which the ring A is regular and contains a field. This uses the following lemmas, the first of which is a variant of Quillen patching that reduces Conjecture 5.1 to the case when A is local.

Lemma 5.6 ([Čes22b, Corollary 5.1.9])**.** *For a ring A and a locally finitely presented A-group scheme G, every G-torsor over $\mathbb{A}^1_A$ that is trivial away from an A-finite closed subscheme is trivial as soon as the same holds with A replaced by its localization $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset A$.* □

Lemma 5.7 ([Gil02, Corollaire 3.10])**.** *For a reductive group G over a field k and an open $U \subset \mathbb{P}^1_k$, each generically trivial G-torsor E over U reduces to a torsor under a maximal k-split subtorus of G; in particular, if $U \subset \mathbb{A}^1_k$, then, since U has no nontrivial line bundles, E is a trivial G-torsor.* □

Proposition 5.8. *Conjecture 5.1 holds in the case when the ring A is regular and contains a field, in other words, for a totally isotropic reductive group scheme G over a regular ring A that contains a field, every G-torsor over $\mathbb{A}^d_A$ that is trivial away from an A-finite closed subscheme is trivial.*

Proof. We may replace A by $A[t_1, \dots, t_{d-1}]$ to reduce to the case when $d = 1$. We may then apply Lemma 5.6 to reduce further to the case when our regular ring A is local. At this point we may apply the relative Grothendieck–Serre conjecture, more precisely, Theorem 5.5 to replace A by its fraction field. Once A is a field with $d = 1$, the claim becomes a special case of Lemma 5.7. □

6. Extending a G-torsor over $C \backslash Y$ to a G-torsor over C

The main step of the overall proof of Theorem 1.5 is to reduce the intermediate Proposition 3.4 further to the case when the R-(finite étale) closed subscheme $Y \subset C$ there is empty, that is, to when the G-torsor $\mathscr{E}$ is defined over all of C. In terms of analogies, this reduces the Nisnevich conjecture to the Grothendieck–Serre conjecture, which is known in equal characteristic. In addition to relative Grothendieck–Serre reviewed in §5, a key mechanism for shrinking Y like this is the following lemma.

Lemma 6.1. *Let R be a ring, let C be an R-scheme, let $Y \subset C$ be an R-(separated étale) closed subscheme locally cut out by a finitely generated ideal, and consider the decomposition*

$$Y \times_R Y = \Delta \sqcup Y'$$

in which $\Delta \subset Y \times_R Y$ is the diagonal copy of Y. For a quasi-affine, flat, finitely presented R-group scheme G, a G-torsor E over $C \backslash Y$ extends to a G-torsor over C if and only if the base change of E to $(C \backslash Y)_Y$ extends to a G-torsor over $C_Y \backslash Y'$.

Proof. The claimed decomposition $Y \times_R Y = \Delta \sqcup Y'$ exists because any section of a separated étale morphism, such as the projection $Y \times_R Y \to Y$, is both a closed immersion and an open immersion. The étale map $C_Y \backslash Y' \to C$ induces an isomorphism $\Delta \xrightarrow{\sim} Y$ and fits into a Cartesian square

$$\begin{array}{ccc} \Delta & \hookrightarrow & C_Y \backslash Y' \\ \sim \downarrow & & \downarrow \\ Y & \hookrightarrow & C. \end{array}$$

Thus, the desired claim is an immediate consequence of patching, more precisely, of Lemma 4.6. ☐

We are ready for the following proposition that is essentially due to Fedorov [Fed21b, Proposition 2.6]. We will improve it in Theorem 6.4: there we will remove the condition on the base field k.

Proposition 6.2. *For a regular semilocal ring R containing a field k, a totally isotropic reductive R-group scheme G, an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$, and an R-(finite étale) closed subscheme $Y \subset Z$ of degree $\leqslant \#k$ over R, no nontrivial G-torsor over $\mathbb{A}^1_R \backslash Y$ becomes trivial over $\mathbb{A}^1_R \backslash Z$, that is,*

$$\mathrm{Ker}(H^1(\mathbb{A}^1_R \backslash Y, G) \to H^1(\mathbb{A}^1_R \backslash Z, G)) = \{*\}.$$

Proof. We follow the argument of *loc. cit.* Our goal is to show that every G-torsor E over $\mathbb{A}^1_R \backslash Y$ that becomes trivial over $\mathbb{A}^1_R \backslash Z$ is trivial. If $Y = \varnothing$, more generally, if Y is constant in the sense that $Y \simeq Y_0 \times_k R$ for a k-subscheme $Y_0 \subset \mathbb{A}^1_k$, then, by the relative Grothendieck–Serre conjecture of Theorem 5.5, we may first replace R by its total ring of fractions $\mathrm{Frac}(R)$ to reduce to R being a field and then conclude by Lemma 5.7. In general, we reduce to the case of a constant Y as follows.

By the settled case $Y = \varnothing$, it suffices to extend E to a G-torsor over $\mathbb{A}^1_R$, that is, to extend E over Y. For this, by passing to connected components, we may assume that $\mathrm{Spec}(R)$ is connected and we will argue by descending induction on the number of disjoint copies of $\mathrm{Spec}(R)$ contained in Y. Thus, we begin with the base case when $Y \simeq \bigsqcup \mathrm{Spec}(R)$. By our assumption on k, the number of copies of $\mathrm{Spec}(R)$ in this disjoint union does not exceed $\#k$. Thus, there is a closed immersion

$$\iota \colon Y \hookrightarrow \mathbb{A}^1_R$$

that maps the copies of $\mathrm{Spec}(R)$ comprising Y to distinct k-rational points, in particular, such that $\iota(Y) = Y_0 \times_k \mathrm{Spec}(R)$ for a closed k-subscheme $Y_0 \subset \mathbb{A}^1_k$. By Lemma 4.4, there are an affine open $C' \subset \mathbb{A}^1_R$ containing Z and an étale map $f \colon C' \to \mathbb{A}^1_R$ with $f|_Y = \iota$ that fits into a diagram

$$\begin{array}{ccccc} Y & \hookrightarrow & Z & \hookrightarrow & C' \\ \sim \downarrow & & \sim \downarrow & & f \downarrow \\ Y_0 \times_R k & \hookrightarrow & Z' & \hookrightarrow & \mathbb{A}^1_R \end{array}$$

with Cartesian squares. By patching of Lemma 4.6, we may therefore descend $E|_{C'}$ to a G-torsor E' over $\mathbb{A}^1_R \backslash (Y_0 \times_R k)$ that becomes trivial over $\mathbb{A}^1_R \backslash Z'$. Extending E over Y amounts to extending E' over $Y_0 \times_R k$, to the effect that we have reduced to the already settled case when Y is constant.

For the inductive step, suppose that Y has a connected component Y' that does not map isomorphically to $\mathrm{Spec}(R)$, so that Y' is of degree $\geqslant 2$ over R. Since $Y' \times_R Y'$ contains the diagonal copy of Y' as a clopen (compare with Lemma 6.1), the Y'-(finite étale) closed subscheme $Y \times_R Y' \subset \mathbb{A}^1_{Y'}$, contains strictly more disjoint copies of Y' than Y contained disjoint copies of $\mathrm{Spec}(R)$. Thus, by the inductive hypothesis, the pullback of E to a G-torsor over $\mathbb{A}^1_{Y'}\backslash(Y \times_R Y')$ extends over $Y \times_R Y'$. By Lemma 6.1, this implies that E extends over Y', that is, that E extends to a G-torsor over $\mathbb{A}^1_R\backslash(Y\backslash Y')$. By repeating this argument for each possible Y', we effectively shrink Y and reduce to the already settled case when Y is a disjoint union of copies of $\mathrm{Spec}(R)$. ☐

The key step of the proof of Proposition 6.2, namely, extending E over Y, generalizes as follows.

Theorem 6.3 (Theorem 1.6)**.** *For a regular semilocal ring R containing a field k, a smooth affine R-scheme C of pure relative dimension 1, an R-(finite étale) closed subscheme $Y \subset C$, an open $C' \subset C$ containing Y whose complement $Y' := C\backslash C'$ is R-(finite étale), and a reductive R-group scheme G such that G_Y is totally isotropic, every G-torsor E over $C'\backslash Y$ that is trivial over $C'\backslash Z$ for some R-finite closed subscheme $Z \subset C$ containing Y extends to a G-torsor over C', that is:*

$$\mathrm{Ker}(H^1(C', G) \to H^1(C'\backslash Z, G)) \twoheadrightarrow \mathrm{Ker}(H^1(C'\backslash Y, G) \to H^1(C'\backslash Z, G)).$$

In practice we will have $C = C'$, but for the proof it is convenient to allow C' to not contain Z.

Proof. We enlarge Z so that $Y' \subset Z$ and decompose R to reduce to the case when $\mathrm{Spec}(R)$ is connected (so that $\deg(Y/R)$ is a well-defined integer). By Lemma 6.1, we may base change along $Y \to \mathrm{Spec}(R)$ and then shrink the base changed C' (by removing the off-diagonal part of $Y \times_R Y$) to reduce to the case when $Y \cong \mathrm{Spec}(R)$, so that, in particular, G is totally isotropic. We then combine Lemma 4.2 (c) with Remark 4.3 to find an affine open $D \subset C$ containing Z as well as a finite étale $\widetilde{C} \twoheadrightarrow D$ such that

$$\widetilde{Y} := Y \times_C \widetilde{C} \quad \text{decomposes as} \quad \widetilde{Y} = \widetilde{Y}_0 \sqcup \widetilde{Y}_1 \quad \text{where} \quad \widetilde{Y}_0 \xrightarrow{\sim} \mathrm{Spec}(R),$$

each connected component of $\widetilde{Y}_1$ or $\widetilde{Y}' := Y' \times_C \widetilde{C}$ is an algebra over a field k' with

$$\#k' > \deg(\widetilde{Y} \cup \widetilde{Y}'/R)$$

(choose $n := \deg(Y \sqcup Y'/R)$ in Lemma 4.2 (c)), and there is no finite field obstruction to embedding $\widetilde{Z} := Z \times_C \widetilde{C}$ into $\mathbb{A}^1_R$. By construction, with $\widetilde{C}' := (C' \cap D) \times_C \widetilde{C}$, the following square is Cartesian:

$$\begin{array}{ccc} \widetilde{Y}_0 & \hookrightarrow & \widetilde{C}'\backslash\widetilde{Y}_1 \\ \sim\downarrow & & \downarrow \\ Y & \hookrightarrow & C' \cap D. \end{array}$$

Thus, by patching of Lemma 4.6, it suffices to extend the base change of E to $\widetilde{C}'\backslash\widetilde{Y}_1$ to a G-torsor over $\widetilde{C}'$. In other words, since $D = (C' \cap D) \cup Y'$, we may replace $Y \subset C' \subset C$ by $\widetilde{Y}_0 \subset \widetilde{C}'\backslash\widetilde{Y}_1 \subset \widetilde{C}$ and E by its base change to $\widetilde{C}'\backslash\widetilde{Y}_1$ to reduce to the case when each connected component of Y' is an algebra over a field k' with $\#k' > \deg(Y \cup Y'/R)$ and there is no finite field obstruction to embedding Z into $\mathbb{A}^1_R$. By Lemma 4.4, such an embedding then exists, more precisely, then there are an affine open $D \subset C$ containing Z and a Cartesian square

$$\begin{array}{ccc} Z & \hookrightarrow & D \\ \sim\downarrow & & \downarrow f \\ Z' & \hookrightarrow & \mathbb{A}^1_R \end{array}$$

in which the map f is étale and embeds Z as a closed subscheme $Z' \subset \mathbb{A}^1_R$. By passing to the complements of the R-(finite étale) $Y \cup Y'$ (viewed also inside Z'), we find a Cartesian square

$$\begin{array}{ccc} Z \cap (C'\backslash Y) & \hookrightarrow & (C'\backslash Y) \cap D \\ \sim \downarrow & & \downarrow f|_{(C'\backslash Y)\cap D} \\ Z'\backslash (Y \cup Y') & \hookrightarrow & \mathbb{A}^1_R \backslash (Y \cup Y'). \end{array}$$

Since our G-torsor E over $C'\backslash Y$ is trivial over $C'\backslash Z$, we may use patching of Lemma 4.6 to build a G-torsor E' over $\mathbb{A}^1_R\backslash(Y \cup Y')$ that is trivial over $\mathbb{A}^1_R\backslash Z'$ and whose base change to $(C'\backslash Y) \cap D$ agrees with the corresponding base change of E. Since $Y \subset C' \cap D$, if this E' extends to a G-torsor over $\mathbb{A}^1_R$, then E extends to a G-torsor over $C' \cap D$ and so, by Zariski patching, then E also extends to a G-torsor over C'. In conclusion, we reduced to the case when $C' = C = \mathbb{A}^1_R$ and $Y \cong \mathrm{Spec}(R) \sqcup y$ with each connected component y' of y an algebra over a field k' with $\deg(Y/R) \leqslant \#k'$.

Our next step is to extend E over y, equivalently, to extend E over each connected component y' of y. For this, by Lemma 6.1, it suffices to do the same after base change to y'. The conditions on y' and k' then allow us to apply Proposition 6.2, which gives us the desired extension. By repeating this, we eliminate all the connected components y' one by one and are left with the case when $Y \cong \mathrm{Spec}(R)$ and $C' = C = \mathbb{A}^1_R$. This final case, however, also follows from Proposition 6.2. ☐

We are ready to remove the assumption on the base field from Proposition 6.2 as follows.

Theorem 6.4. *For a regular semilocal ring R containing a field, a totally isotropic reductive R-group scheme G, and an R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$, no nontrivial G-torsor over $\mathbb{A}^1_R\backslash Y$ becomes trivial over $\mathbb{A}^1_R\backslash Z$ for some R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ containing Y, that is,*

$$\mathrm{Ker}(H^1(\mathbb{A}^1_R\backslash Y, G) \to H^1(\mathbb{A}^1_R\backslash Z, G)) = \{*\}.$$

Proof. By Theorem 6.3, every G-torsor over $\mathbb{A}^1_R\backslash Y$ that is trivial over $\mathbb{A}^1_R\backslash Z$ extends to a G-torsor over $\mathbb{A}^1_R$, that is, we may assume that $Y = \varnothing$. This case, however, is covered by Proposition 6.2. ☐

7. The Nisnevich and the generalized Bass–Quillen conjectures over a field

With the work of the previous sections in hand, we are ready for our main results. We begin with the equicharacteristic case of the Nisnevich conjecture, that is, with Theorem 1.5, and then use it to deduce the equicharacteristic generalized Bass–Quillen conjecture of Theorem 1.1.

7.1. Proof of Theorem 1.5. We have a regular semilocal ring R containing a field, an $r \in R$ with $r \notin \mathfrak{m}^2$ for each maximal ideal $\mathfrak{m} \subset R$, a reductive R-group scheme G that is totally isotropic along $\{r = 0\} \subset \mathrm{Spec}(R)$, and a generically trivial G-torsor E over $R[\frac{1}{r}]$. We need to show that E is trivial. Equivalently, by the Grothendieck–Serre conjecture, more precisely, by the final aspect of Theorem 5.5, we need to extend E to a G-torsor $\widetilde{E}$ over R. For this, by patching supplied by Lemma 4.6 and a limit argument, we may semilocalize R along the union of those maximal ideals $\mathfrak{m} \subset R$ that contain r and reduce ourselves to the case when G is totally isotropic.

We apply Proposition 3.4 to get a smooth affine R-scheme C of pure relative dimension 1, an R-finite closed subscheme $Z \subset C$, an R-(finite étale) closed subscheme $Y \subset Z$, a section $s \in C(R)$ such that $s|_{R[\frac{1}{r}]}$ factors through $C\backslash Y$, and a G-torsor $\mathscr{E}$ on $C\backslash Y$ that is trivial over $C\backslash Z$ such that

$$(s|_{R[\frac{1}{r}]})^*(\mathscr{E}) \cong E \quad \text{as } G\text{-torsors over } R[\tfrac{1}{r}].$$

By Theorem 6.3, this G-torsor $\mathscr{E}$ extends to a G-torsor $\widetilde{\mathscr{E}}$ over C. Thus, by pulling back along s, our G-torsor E extends to a desired G-torsor $\widetilde{E}$ over R. ☐

7.2. Proof of Theorem 1.1. We have a regular ring R containing a field, a totally isotropic reductive R-group G, and a generically trivial G-torsor E over $\mathbb{A}^d_R$. We need to show that E descends to a G-torsor over R. For this, by induction on d, we may assume that $d = 1$. By Quillen patching of Theorem 5.2, we may assume that R is local. In this key local case, we will show that E is trivial.

For this, by Theorem 6.4, it suffices to show that E is trivial on $\mathbb{A}^1_R \backslash Z$ for some R-finite closed subscheme $Z \subset \mathbb{A}^1_R$. By a limit argument, it therefore suffices to show that E becomes trivial over the localization of $R[t]$ obtained by inverting all the monic polynomials. By the change of variables $x := t^{-1}$, this localization is the localization of $\mathbb{P}^1_R$ along the section ∞, and hence is isomorphic to

$$(R[x]_{1+xR[x]})[\tfrac{1}{x}].$$

The ring $R' := R[x]_{1+xR[x]}$ is regular, local, and shares its fraction field with $\mathbb{A}^1_R$. In particular, the base change of E to R' is generically trivial. Thus, since x is a regular parameter of R', Theorem 1.5 implies that this base change of E is trivial, as desired. ☐

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