

THE MANIN–STEVENS CONSTANT IN THE SEMISTABLE CASE

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ABSTRACT. Stevens conjectured that for every optimal parametrization $\phi: X_1(n) \rightarrow E$ of an elliptic curve E over \mathbb{Q} of conductor n , the pullback of some Néron differential on E is the differential associated to the normalized new eigenform that corresponds to the isogeny class of E . We prove this conjecture under the assumption that E is semistable, the key novelty lying in the 2-primary analysis when n is even. For this analysis, we first relate the general case of the conjecture to a divisibility relation between $\deg \phi$ and a certain congruence number and then reduce the semistable case to a question of exhibiting enough suitably constrained oldforms. Our methods also apply to parametrizations by $X_0(n)$ and prove new cases of the Manin conjecture.

1. INTRODUCTION

With the purpose of relating the arithmetic of an elliptic curve E over \mathbb{Q} of conductor n to the arithmetic of the modular curve $X_1(n)$ via a given modular parametrization

$$\phi: X_1(n) \rightarrow E,$$

one normalizes by arranging that the cusp “0” $\in X_1(n)(\mathbb{Q})$ maps to $0 \in E(\mathbb{Q})$ and, at the cost of replacing E by an isogenous curve, that the induced quotient map

$$\pi: J_1(n) \rightarrow E$$

from the Jacobian is *optimal* in the sense that its kernel is connected. For such a ϕ (equivalently, π), one seeks to understand the differential aspect of the modularity relationship captured by the equality

$$\pi^*(\omega_E) = c_\pi \cdot f_E \quad \text{for some } c_\pi \in \mathbb{Q}^\times, \quad (\star)$$

where $\omega_E \in H^0(E, \Omega^1)$ is a Néron differential and $f_E \in H^0(X_1(n), \Omega^1) \cong H^0(J_1(n), \Omega^1)$ is the differential form associated to the normalized new eigenform that corresponds to the isogeny class of E . Since π is *new*, that is, factors through the new quotient of $J_1(n)$, the multiplicity one principle supplies (\star) and it remains to understand the appearing *Manin–Stevens constant* c_π .

Conjecture 1.1 (Stevens, [Ste89, Conj. I’ (a)]). *For a new elliptic optimal quotient $\pi: J_1(n) \rightarrow E$,*

$$c_\pi = \pm 1.$$

We settle Conjecture 1.1 for semistable E and, more generally, settle its p -primary part for primes p with $\text{ord}_p(n) \leq 1$. For this, existing techniques suffice if p is odd, so the key new case is $p = 2$.

Theorem 1.2 (§4.1 and §4.5). *For a new elliptic optimal quotient $\pi: J_1(n) \rightarrow E$ and a prime p ,*

$$\text{if } \text{ord}_p(n) \leq 1, \quad \text{then } \text{ord}_p(c_\pi) = 0.$$

In particular, if E is semistable (that is, if n is squarefree), then $c_\pi = \pm 1$.

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Date: October 14, 2018.

2010 *Mathematics Subject Classification.* Primary 11G05; Secondary 11G18, 11F33, 14G35.

Key words and phrases. Elliptic curve, Manin constant, modular curve, modular parametrization, oldform.

Conjecture 1.1 of Stevens is a variant of an earlier conjecture of Manin on parametrizations by $X_0(n)$ (equivalently, by its Jacobian $J_0(n)$), for which the analogous $c_\pi \in \mathbb{Q}^\times$ is the *Manin constant*.

Conjecture 1.3 (Manin, [Man71, 10.3]). *For a new elliptic optimal quotient $\pi: J_0(n) \twoheadrightarrow E$,*

$$c_\pi = \pm 1.$$

One knows that Conjecture 1.3 implies Conjecture 1.1 (see Lemma 4.4 (b)). Cremona has proved Conjecture 1.3 for all E of conductor at most 130000, see [ARS06, Thm. 2.6] (see also [Cre16] for the same result up to conductor 370000).

One typically studies the Manin constant through its p -adic valuations. The following theorem summarizes the known cases of the p -primary part of Conjecture 1.3 at the semistable primes p .

Theorem 1.4. *For a new elliptic optimal quotient $\pi: J_0(n) \twoheadrightarrow E$ and a prime p ,*

if $\text{ord}_p(n) \leq 1$, then $\text{ord}_p(c_\pi) = 0$ in any of the following cases:

- (i) (Mazur, [Maz78, Cor. 4.1]; see also Remark 3.7 and Proposition 4.3). *If p is odd;*
- (ii) (Abbes–Ullmo, [AU96, Thm. A]; see also Theorem 2.5). *If $p = 2$ and $\text{ord}_2(n) = 0$;*
- (iii) (Raynaud, [AU96, (ii) on p. 270]). *If $p = 2$ and $\text{ord}_2(\Delta_E)$ is odd, where $\Delta_E \in \mathbb{Q}^\times$ is the discriminant of a Weierstrass equation for E ;*
- (iv) (Agashe–Ribet–Stein, [ARS06, Thm. 2.7]; see also Remark 2.9). *If $p = 2$ and the degree of the composite $X_0(n) \hookrightarrow J_0(n) \xrightarrow{\pi} E$ obtained by choosing a point in $X_0(n)(\mathbb{Q})$ is odd.*

In addition, for a π as in Theorem 1.4, one knows that $0 \leq \text{ord}_2(c_\pi) \leq 1$ when $\text{ord}_2(n) \leq 1$ thanks to a result of Mazur–Raynaud, [AU96, Prop. 3.1], based on exactness properties of semiabelian Néron models. The techniques of the proof of Theorem 1.2 reprove this result in Remark 3.15 without using such exactness properties. Remark 3.7 achieves the same for Theorem 1.4 (i).

Beyond the semistable p , for a π as in Theorem 1.4, Edixhoven proved in [Edi91, Thm. 3] that $\text{ord}_p(c_\pi) = 0$ in the case when $p > 7$ and $E_{\mathbb{Q}_p}$ does not have potentially ordinary reduction of Kodaira type II, III, or IV. Further cases may be supplied by the unfinished manuscript [Edi01].

In addition to streamlined reproofs of Theorem 1.4 (i), (ii), and (iv), our methods also lead to the following new cases of the 2-primary part of the Manin conjecture.¹

Theorem 1.5. *For a new elliptic optimal quotient $\pi: J_0(n) \twoheadrightarrow E$,*

if $\text{ord}_2(n) \leq 1$, then $\text{ord}_2(c_\pi) = 0$ in any of the following cases:

- (i) (§4.2). *If n has a prime factor q with $q \equiv 3 \pmod{4}$;*
- (ii) (§4.2). *If $n = 2p$ for some prime p ;*
- (iii) (§4.6). *If $E(\mathbb{Q})[2] = 0$.*

For further input which would prove that $\text{ord}_2(c_\pi) = 0$ whenever $\text{ord}_2(n) \leq 1$, see Remark 3.19.

The conditions (i)–(iii) in Theorem 1.5 are global, so the combination of Theorems 1.4 and 1.5 covers significantly more new elliptic optimal quotients than Theorem 1.4 alone.

¹The semistable case of the Manin conjecture has now been settled in full in [Čes18] by a different method.

Example 1.6. To get a sense of the scope of Theorem 1.5 we used the website [LMFDB] to inspect all elliptic curves over \mathbb{Q} of conductor ≤ 200 that are optimal with respect to $X_0(n)$ and semistable at 2; we found 205 such curves. For 143 of them Theorem 1.4 proves that $\text{ord}_2(c_\pi) = 0$. Theorem 1.5 (i) then proves that $\text{ord}_2(c_\pi) = 0$ for 47 of the remaining 62 curves 30.a8, 34.a4, \dots , 198.d4, 198.e3, leaving 15 curves: 34.a4, 58.a1, \dots , 178.b2, 194.a2. Theorem 1.5 (ii) proves that $\text{ord}_2(c_\pi) = 0$ for 10 of these, leaving 5 curves: 130.a2, 130.b4, 130.c1, 170.a2, 170.b1, all of which have $E(\mathbb{Q})[2] \neq 0$.

Theorem 1.5 (iii) does provide new information for some curves, for instance, for 530.a1, which has $E(\mathbb{Q})[2] = 0$ but for which neither Theorem 1.4 (ii)–(iv) nor Theorem 1.5 (i)–(ii) apply.

1.7. The overview of the proofs. The first step of the proofs of Theorems 1.2 and 1.5 is a reduction, not specific to semistable p , to a divisibility relation between $\deg \phi$ and a certain congruence number (see Proposition 2.3 (c)). However, the required divisibility differs from the ones available in the literature because we measure congruences between weight 2 cusp forms with respect to the cotangent space at the identity of the Néron model of $J_1(n)$ (resp., of $J_0(n)$) rather than with respect to q -expansions at “ ∞ .” The proofs proceed to isolate a module that controls the difference between the two types of congruences and, under a semistability assumption, the problem becomes that of exhibiting its vanishing (see Theorem 2.10). For this, it suffices to show that oldforms offset the difference between two integral structures on the \mathbb{Q} -vector space of weight 2 cusp forms (see the introduction of §3). The technical heart of the argument lies in exhibiting suitable oldforms in §3.²

Ultimately, the sought oldforms come from the analysis of the degeneracy maps

$$\pi_{\text{forg}}, \pi_{\text{quot}} : X_1(n) \rightarrow X_1\left(\frac{n}{2}\right) \quad \text{over } \mathbb{F}_2 \quad \text{for an } n \quad \text{with } \text{ord}_2(n) = 1$$

(and their analogues for $X_0(n)$), but at the cost of several complications. Firstly, to exploit the moduli interpretations and to overcome the failure of (S_2) of $\Omega_{X_1(n)/\mathbb{Z}_{(2)}}^1$, we are forced to work with the line bundle $\omega^{\otimes 2}(-\text{cusps})$ of weight 2 cusp forms, and hence also with the $\Gamma_1(n)$ -level (resp., $\Gamma_0(n)$ -level) modular stack $\mathcal{X}_1(n)$ (resp., $\mathcal{X}_0(n)$) in place of its coarse moduli scheme (albeit the difference only manifests itself for $\Gamma_0(n)$). The passage to stacks is facilitated by a certain comparison result overviewed and proved in Appendix A. Secondly, several key arguments rest on intersection theory for $\mathcal{X}_0(n)$ and $\mathcal{X}_1(n)$, so we crucially use the regularity of these stacks (which may fail for coarse spaces). At multiple places of the overall proof, the moduli interpretations and the analysis of $\mathcal{X}_0(n)$ and $\mathcal{X}_1(n)$ presented in [Čes17] come in handy—although we primarily work over $\mathbb{Z}_{(2)}$, we cannot ignore the subtleties of the moduli interpretation of $\mathcal{X}_0(n)_{\mathbb{Z}_{(2)}}$ at the cusps caused by the fact that n may be divisible by the square of an odd prime.

In the case of Theorem 1.2, the resulting proof is *a posteriori* carried out entirely with schemes because the relevant stacks $\mathcal{X}_1(n)_{\mathbb{Z}_{(2)}}$ and $\mathcal{X}_1\left(\frac{n}{2}\right)_{\mathbb{Z}_{(2)}}$ identify with their coarse spaces (see (3.1.1)). In contrast, we do not know how to carry out the proof of Theorem 1.5 (i)–(ii) without resorting to stacks. Theorem 1.5 (iii) is based on a direct reduction to Theorem 1.2.

1.8. Notation. The following notation will be in place throughout the paper (see also §1.9):

- For an open subgroup $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$, we let \mathcal{X}_H denote the level H modular \mathbb{Z} -stack defined in [DR73, IV.3.3] via normalization (so \mathcal{X}_H is always Deligne–Mumford and is a scheme for “small enough” H ; see [Čes17, §4.1] for a review of basic properties of \mathcal{X}_H);

²The reading of [Edi06, §2] was beneficial for the genesis of §3.

- We let X_H denote the coarse moduli space of \mathcal{X}_H , so X_H is the “usual” projective modular curve over³ \mathbb{Z} of level H (see [Ces17, 6.1–6.3] for a review of basic properties of X_H);
- For an $n \in \mathbb{Z}_{\geq 1}$, we let $\Gamma_0(n) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ (resp., $\Gamma_1(n) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$) be the preimage of the subgroup $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ (resp., of the subgroup $\left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$);
- We write $\mathcal{X}(1)$ for $\mathcal{X}_{\mathrm{GL}_2(\widehat{\mathbb{Z}})}$ and sometimes write $\mathcal{X}_0(n)$ and $\mathcal{X}_1(n)$ (resp., $X_0(n)$ and $X_1(n)$) for $\mathcal{X}_{\Gamma_0(n)}$ and $\mathcal{X}_{\Gamma_1(n)}$ (resp., for $X_{\Gamma_0(n)}$ and $X_{\Gamma_1(n)}$);
- We let $J_0(n) := \mathrm{Pic}_{X_0(n)_{\mathbb{Q}}/\mathbb{Q}}^0$ and $J_1(n) := \mathrm{Pic}_{X_1(n)_{\mathbb{Q}}/\mathbb{Q}}^0$ be the Jacobian varieties of $X_0(n)_{\mathbb{Q}}$ and $X_1(n)_{\mathbb{Q}}$, respectively (so $J_0(n)$ and $J_1(n)$ are abelian varieties over \mathbb{Q});
- For a Cohen–Macaulay morphism $\mathcal{X} \rightarrow S$ (see §1.9) of some pure relative dimension from a Deligne–Mumford stack \mathcal{X} to a scheme S , we let $\Omega_{\mathcal{X}/S}$ (or simply Ω) denote the “relative dualizing” quasi-coherent $\mathcal{O}_{\mathcal{X}}$ -module discussed in §A.1. (We likewise shorten $\Omega_{\mathcal{X}/S}^1$ to Ω^1 .)

1.9. Conventions. A morphism $\mathcal{X} \rightarrow S$ from a Deligne–Mumford stack (or a scheme) \mathcal{X} towards a scheme S is *Cohen–Macaulay* if it is flat, locally of finite presentation, and its fibers are Cohen–Macaulay. We write $\mathcal{O}_{\mathcal{X},x}^{\mathrm{sh}}$ for the strict Henselization of \mathcal{X} at a geometric point x .

On a modular curve over a subfield of \mathbb{C} , we identify a weight two cusp form with its corresponding Kähler differential. We use the j -invariant to identify $X_{\mathrm{GL}_2(\widehat{\mathbb{Z}})}$ with $\mathbb{P}_{\mathbb{Z}}^1$ (see [DR73, VI.1.1 and VI.1.3]). We use ‘new’ and ‘optimal’ in the sense of the beginning of the introduction.

For a proper smooth geometrically connected curve X over a field k , we make the identification

$$H^0(X, \Omega^1) \cong H^0(\mathrm{Pic}_{X/k}^0, \Omega^1) \quad (1.9.1)$$

supplied by the combination of Grothendieck–Serre duality and the deformation-theoretic identification $H^1(X, \mathcal{O}_X) \cong \mathrm{Lie}(\mathrm{Pic}_{X/k}^0)$, and, whenever we choose an $x_0 \in X(k)$, we freely use the alternative description of the identification (1.9.1) as pullback of Kähler differentials along the “ $x \mapsto \mathcal{O}(x) \otimes \mathcal{O}(x_0)^{-1}$ ” closed immersion $X \hookrightarrow \mathrm{Pic}_{X/k}^0$ (see [Con00, Thm. B.4.1]).

An element of a torsion free module over a Dedekind domain is *primitive* if the quotient by the submodule that it generates is torsion free. For a prime p , we let ord_p denote the p -adic valuation with $\mathrm{ord}_p(p) = 1$ and let $(-)_{(p)}$ denote localization at p . For an $n \in \mathbb{Z}_{\geq 1}$, we set $\mu_n := \mathrm{Ker}(\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m)$, let $\mathbb{Q}(\zeta_n)$ denote the n^{th} cyclotomic field, let $\mathbb{Z}[\zeta_n]$ denote its ring of integers, and let $\mathbb{Z}[\zeta_n]^+$ denote the ring of integers of the maximal totally real subfield of $\mathbb{Q}(\zeta_n)$. A dual abelian variety, a Cartier dual commutative finite locally free group scheme, or a dual homomorphism is denoted by $(-)^{\vee}$.

Acknowledgements. I thank Brian Conrad for several extended discussions. I thank Kevin Buzzard, Frank Calegari, John Cremona, Bas Edixhoven, Haruzo Hida, Erick Knight, Martin Olsson, Michel Raynaud, Ken Ribet, Sug Woo Shin, Yunqing Tang, Xinyi Yuan, and David Zureick-Brown for helpful conversations or correspondence. I thank the referee for helpful comments. I thank the MathOverflow community—the reading of several anonymous discussions was useful while working on this paper. I thank the Miller Institute for Basic Research in Science at the University of California Berkeley for its support.

³Unlike in the introduction, where $X_0(n)$ and $X_1(n)$ were curves over \mathbb{Q} , all the modular curves and stacks in the rest of the paper are assumed to be over \mathbb{Z} and we use base change notation $X_0(n)_{\mathbb{Q}}$, etc., to denote their \mathbb{Q} -fibers.

Our approach to the conjectures of Manin and Stevens rests on Proposition 2.3 (c), which relates them to a comparison between the modular degree and a certain congruence number. Our first task is to define the latter in §2.2 after introducing the relevant setup in §2.1.

2.1. The setup. Throughout §2 we supplement the notation of §1.8 with the following:

- We let Γ denote either $\Gamma_0(n)$ or $\Gamma_1(n)$ for a fixed $n \in \mathbb{Z}_{\geq 1}$;
- We let X denote X_Γ , i.e., either $X = X_0(n)$ or $X = X_1(n)$;
- We let J denote the Jacobian $\text{Pic}_{X_{\mathbb{Q}}/\mathbb{Q}}^0$, i.e., either $J = J_0(n)$ or $J = J_1(n)$;
- We let $\pi: J \rightarrow E$ be a new elliptic optimal quotient (so E is an elliptic curve over \mathbb{Q} and $\text{Ker } \pi$ is an abelian variety);
- We let $\pi: \mathcal{J} \rightarrow \mathcal{E}$ denote the extension to Néron models over \mathbb{Z} ;
- We let $f \in H^0(X_{\mathbb{Q}}, \Omega^1) \stackrel{(1.9.1)}{\cong} H^0(J, \Omega^1)$ denote the normalized new eigenform corresponding to π ('normalized' means that the q -expansion $(\sum_{n \geq 1} a_n q^n) \frac{dq}{q}$ of f at the cusp⁴ “ ∞ ” has $a_1 = 1$);
- We let $c_\pi \in \mathbb{Q}^\times$ denote the Manin(–Stevens) constant of E , i.e., $\pi^*(\omega_{\mathcal{E}}) = c_\pi \cdot f$ in $H^0(J, \Omega^1)$, where $\omega_{\mathcal{E}} \in H^0(E, \Omega^1)$ is a generator of $H^0(\mathcal{E}, \Omega^1)$ (so c_π is only well-defined up to ± 1).

One knows that $c_\pi \in \mathbb{Z}$ (see [Edi91, Prop. 2] and [Ste89, Thm. 1.6]), and the conjectures of Manin and Stevens predict that $c_\pi = \pm 1$.

2.2. Congruence with respect to the lattice $H^0(\mathcal{J}, \Omega^1)$. The relevant “congruence module” is

$$\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f) + H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp}, \quad (2.2.1)$$

where the orthogonal complement is taken in $H^0(X_{\mathbb{Q}}, \Omega^1)$ with respect to the Petersson inner product. This \mathbb{Z} -module is finite and cyclic (because $\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \simeq \mathbb{Z}$), and we denote its order by

$$\text{cong}_{f, \mathcal{J}} := \# \left(\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f) + H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \right).$$

Proposition 2.3. *Let $\phi: X_{\mathbb{Q}} \rightarrow E$ denote the composition of $\pi: J \rightarrow E$ with the immersion $i_P: X_{\mathbb{Q}} \hookrightarrow J$ obtained by choosing a base point $P \in X(\mathbb{Q})$ (for instance, a rational cusp).*

- (a) *The composition $\pi \circ \pi^\vee: E \rightarrow J \rightarrow E$ is multiplication by $\deg \phi$ (which is independent of P).*
- (b) *With the notation of §2.2,*

$$\text{cong}_{f, \mathcal{J}} \mid \deg \phi.$$

- (c) *If p is a prime such that $f \in H^0(\mathcal{J}_{\mathbb{Z}_{(p)}}, \Omega^1)$, then*

$$\text{ord}_p(c_\pi) \leq \text{ord}_p\left(\frac{\deg \phi}{\text{cong}_{f, \mathcal{J}}}\right).$$

Proof.

⁴We form q -expansions after identifying $(X_\Gamma)_{\mathbb{C}}$ with the quotient of the upper half-plane, so over \mathbb{C} . When $\Gamma = \Gamma_1(n)$, the cusp ∞ does not descend to a \mathbb{Q} -point of X , and the q -expansion of f need not have rational coefficients.

- (a) We compute the effect of $\pi \circ \pi^\vee$ on a variable point $Q \in E(\overline{\mathbb{Q}})$. The canonical principal polarization of E sends Q to $\mathcal{O}_{E_{\overline{\mathbb{Q}}}}([-Q] - [0])$, which $\text{Pic}^0(\phi) = \text{Pic}^0(i_P) \circ \pi^\vee$ then sends to $\mathcal{O}_{X_{\overline{\mathbb{Q}}}}([\phi^{-1}(-Q)] - [\phi^{-1}(0)])$. Thus, since $\text{Pic}^0(i_P)$ is the negative of the inverse of the canonical principal polarization of J (see, for instance, [Mil86, 6.9]), the overall effect of $\pi \circ \pi^\vee$ is to send Q to the negative of the sum under the group law of $E_{\overline{\mathbb{Q}}}$ of the ϕ -image of the divisor $[\phi^{-1}(-Q)] - [\phi^{-1}(0)]$ on $X_{\overline{\mathbb{Q}}}$, i.e., to $\deg \phi \cdot Q$.
- (b) Due to the optimality of π , the dual $\pi^\vee : E \rightarrow J$ is a closed immersion. Since π^\vee is Hecke equivariant (see [ARS12, pp. 24–25]), it induces the injection

$$\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \hookrightarrow H^0(\mathcal{E}, \Omega^1). \quad (2.3.1)$$

Due to the Hecke equivariance of π , we have $\pi^*(H^0(\mathcal{E}, \Omega^1)) \subset H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)$. Moreover, the \mathbb{Z} -line $H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)$ maps injectively into the source of (2.3.1), so from (a) and (2.3.1) we get the injection

$$\frac{H^0(\mathcal{J}, \Omega^1)}{\pi^*(H^0(\mathcal{E}, \Omega^1)) + H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \hookrightarrow \frac{H^0(\mathcal{E}, \Omega^1)}{(\deg \phi) \cdot H^0(\mathcal{E}, \Omega^1)} \quad (2.3.2)$$

that exhibits the ‘‘congruence module’’ of (2.2.1) as a subquotient of $\mathbb{Z}/(\deg \phi)\mathbb{Z}$. It remains to observe that the order of every subquotient of $\mathbb{Z}/(\deg \phi)\mathbb{Z}$ divides $\deg \phi$.

- (c) By quantifying at a prime p the extent to which the inclusion (2.3.2) fails to be an isomorphism between $\frac{H^0(\mathcal{E}, \Omega^1)}{(\deg \phi) \cdot H^0(\mathcal{E}, \Omega^1)}$ and the ‘‘congruence module’’ of (2.2.1), we arrive at the equality

$$\text{ord}_p \left(\frac{\deg \phi}{\text{cong}_{f, \mathcal{J}}} \right) = \text{ord}_p \left(\# \left(\frac{H^0(\mathcal{E}, \Omega^1)}{\text{Im}((\pi^\vee)^* : H^0(\mathcal{J}, \Omega^1) \rightarrow H^0(\mathcal{E}, \Omega^1))} \right) \right) + \text{ord}_p \left(\# \left(\frac{H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)}{\pi^*(H^0(\mathcal{E}, \Omega^1))} \right) \right).$$

Since $\mathbb{Z}_{(p)} \cdot f \subset H^0(\mathcal{J}_{\mathbb{Z}_{(p)}}, \Omega^1) \cap (\mathbb{Q} \cdot f)$ and $\pi^*(H^0(\mathcal{E}, \Omega^1)) = \mathbb{Z} \cdot c_\pi f$ with $c_\pi \in \mathbb{Z}$, the last summand is at least $\text{ord}_p(c_\pi)$, and the sought inequality follows. \square

Applying Proposition 2.3 (c) to study the conjectures of Manin and Stevens at a prime p essentially amounts to establishing the p -part of the divisibility converse to the divisibility $\text{cong}_{f, \mathcal{J}} \mid \deg \phi$ supplied by Proposition 2.3 (b). A result of Ribet, [ARS12, Thm. 3.6 (a)] (see also [AU96, Lem. 3.2] and [CK04, Thm. 1.1] for other expositions in the case when $\Gamma = \Gamma_0(n)$), supplies the sought converse divisibility, but with the caveat that the congruences be considered with respect to another lattice $S_2(\Gamma, \mathbb{Z}) \subset H^0(X_{\mathbb{Q}}, \Omega^1)$ in place of $H^0(\mathcal{J}, \Omega^1)$. Therefore, our task is to relate the two types of congruences. For this, we work under the assumption that $\text{ord}_p(n) \leq 1$ and focus on the key case $p = 2$ (although, as we point out along the way, for $\Gamma_0(n)$ most of the arguments also work for odd p). In this setting, we relate the two types of congruences in the proofs of Theorems 2.5 and 2.10.

In the focal case $p = 2$ with $\text{ord}_2(n) \leq 1$, we begin with the simpler possibility $\text{ord}_2(n) = 0$.

2.4. The structure of $X_{\mathbb{Z}_{(p)}}$ when $\text{ord}_p(n) = 0$. If $p \nmid n$, then the ‘‘level’’ of Γ is prime to p , so the proper $\mathbb{Z}_{(p)}$ -curve $X_{\mathbb{Z}_{(p)}}$ is smooth (see [DR73, VI.6.7], possibly also [Čes17, 6.4 (a)]). Moreover, its geometric fibers are irreducible by [DR73, IV.5.6]. Due to these properties, $\text{Pic}_{X_{\mathbb{Z}_{(p)}}/\mathbb{Z}_{(p)}}^0$ is an abelian $\mathbb{Z}_{(p)}$ -scheme (see [BLR90, 9.4/4]), and hence identifies with $\mathcal{J}_{\mathbb{Z}_{(p)}}$. In particular,

$$H^0(X_{\mathbb{Z}_{(p)}}, \Omega^1) = H^0(\mathcal{J}_{\mathbb{Z}_{(p)}}, \Omega^1) \quad \text{inside} \quad H^0(X_{\mathbb{Q}}, \Omega^1). \quad (2.4.1)$$

The method of proof of the following theorem in essence amounts to the method used in [AU96, proof of Thm. A] in the setting of $\Gamma_0(n)$. At least when $\Gamma = \Gamma_0(n)$, the method is not specific to $p = 2$.

Theorem 2.5. *If $\text{ord}_2(n) = 0$, then, in the setting of §§2.1–2.2,*

$$\text{ord}_2(\text{cong}_{f, \mathcal{J}}) = \text{ord}_2(\deg \phi) \quad \text{and} \quad \text{ord}_2(c_\pi) = 0.$$

Proof. For a subring $R \subset \mathbb{C}$, let $S_2(\Gamma, R)$ denote the R -module of those weight 2 cusp forms of level Γ whose Fourier expansion at the cusp “ ∞ ” has coefficients in R . As described in [DI95, §12.3],

$$S_2(\Gamma, R) \cong S_2(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} R \tag{2.5.1}$$

(both sides of this identification are defined in terms of $(X_\Gamma)_\mathbb{C}$, so for arguing it in the $\Gamma = \Gamma_1(n)$ case one is free to use the “ μ -model” of the modular curve to descend the cusp “ ∞ ” to a \mathbb{Z} -point).

If $\Gamma = \Gamma_0(n)$, then (2.4.1) and [Edi06, 2.2 and 2.5] ensure that

$$H^0(\mathcal{J}_{\mathbb{Z}(2)}, \Omega^1) = S_2(\Gamma, \mathbb{Z}_{(2)}) \quad \text{inside} \quad H^0(X_\mathbb{Q}, \Omega^1) = S_2(\Gamma, \mathbb{Q}).$$

Thus, in the $\Gamma = \Gamma_0(n)$ case,

$$\left(\frac{S_2(\Gamma, \mathbb{Z})}{S_2(\Gamma, \mathbb{Z}) \cap \mathbb{Q} \cdot f + S_2(\Gamma, \mathbb{Z}) \cap (\mathbb{Q} \cdot f)^\perp} \right) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \cong \left(\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap \mathbb{Q} \cdot f + H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \right) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}. \tag{2.5.2}$$

By [ARS12, Thm. 3.6 (a)], the order of the left side of (2.5.2) is divisible by $2^{\text{ord}_2(\deg \phi)}$, so $\text{ord}_2(\deg \phi) \leq \text{ord}_2(\text{cong}_{f, \mathcal{J}})$. Due to the converse inequality supplied by Proposition 2.3 (b), equality must hold. Proposition 2.3 (c) then settles the $\Gamma = \Gamma_0(n)$ case because $c_\pi \in \mathbb{Z}$ and the equality $H^0(\mathcal{J}_{\mathbb{Z}(2)}, \Omega^1) = S_2(\Gamma, \mathbb{Z}_{(2)})$ also provides the containment $f \in H^0(\mathcal{J}_{\mathbb{Z}(2)}, \Omega^1)$.

For the remainder of the proof we assume that $\Gamma = \Gamma_1(n)$. One special feature of this case is that $X_{\mathbb{Z}(2)} = \mathcal{X}_{\mathbb{Z}(2)}$ due to the triviality of the automorphism functors of the geometric points of $\mathcal{X}_{\mathbb{Z}(2)}$ forced by the inequality $n \geq 5$ resulting from the existence of f (see [Čes17, 4.1.4, 4.4.4 (c)] and [KM85, 2.7.4]). By pulling back f along the $\mathbb{Z}_{(2)}$ -base change of the forgetful map $X_1(n) \rightarrow X_0(n)$, we see with the help of (2.4.1) that the containment $f \in H^0(\mathcal{J}_{\mathbb{Z}(2)}, \Omega^1)$ continues to hold.

The cusp “ ∞ ” arises from a $\mathbb{Z}[\zeta_n]$ -point (even a $\mathbb{Z}[\zeta_n]^+$ -point) of X via an embedding $\mathbb{Q}(\zeta_n) \subset \mathbb{C}$ whose choice we fix, and the completion of $X_{\mathbb{Z}[\zeta_n]_{(2)}}$ along the resulting $\mathbb{Z}[\zeta_n]_{(2)}$ -point is isomorphic to $\mathbb{Z}[\zeta_n]_{(2)}[[q]]$ and is described by a Tate curve (combine [DR73, VII.2.1] and [KM85, 1.12.9]; see also [Con07, 4.3.7]). Therefore, [Edi06, proof of 2.2 and top of p. 6] provide the identification

$$H^0(X_{\mathbb{Q}(\zeta_n)}, \Omega^1) = S_2(\Gamma, \mathbb{Q}(\zeta_n)) \quad \text{under which} \quad H^0(X_{\mathbb{Z}[\zeta_n]_{(2)}}, \Omega^1) = S_2(\Gamma, \mathbb{Z}[\zeta_n]_{(2)}).$$

Therefore, with the help of (2.4.1) and (2.5.1) we obtain the following analogue of (2.5.2):

$$\left(\frac{S_2(\Gamma, \mathbb{Z})}{S_2(\Gamma, \mathbb{Z}) \cap \mathbb{Q} \cdot f + S_2(\Gamma, \mathbb{Z}) \cap (\mathbb{Q} \cdot f)^\perp} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n]_{(2)} \cong \left(\frac{H^0(\mathcal{J}, \Omega^1)}{H^0(\mathcal{J}, \Omega^1) \cap \mathbb{Q} \cdot f + H^0(\mathcal{J}, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n]_{(2)}. \tag{2.5.3}$$

By [ARS12, Thm. 3.6 (a)], the exponent of the left side of (2.5.3) is divisible by $\text{ord}_2(\deg \phi)$. Since the exponent of the right side equals $\text{ord}_2(\text{cong}_{f, \mathcal{J}})$, the resulting inequality

$$\text{ord}_2(\deg \phi) \leq \text{ord}_2(\text{cong}_{f, \mathcal{J}})$$

combines with Proposition 2.3 (b)–(c) to conclude the proof as in the case when $\Gamma = \Gamma_0(n)$. \square

We turn our attention to the more complex possibility $\text{ord}_2(n) = 1$.

2.6. The structure of $X_{\mathbb{Z}(2)}$ when $\text{ord}_2(n) = 1$. The $\mathbb{Z}_{(2)}$ -curve $X_{\mathbb{Z}(2)}$ is always normal, and proper and flat over $\mathbb{Z}_{(2)}$ (these are general properties of modular curves, see, for instance, [Čes17, 6.1]). Moreover, the $\mathbb{Z}_{(2)}$ -fibers of $X_{\mathbb{Z}(2)}$ are geometrically connected by [DR73, IV.5.5]. However, if

$\text{ord}_2(n) = 1$, as we assume from now on, then the \mathbb{F}_2 -fiber is singular. Nevertheless, if we set $\Gamma' := \Gamma_0(\frac{n}{2})$ when $\Gamma = \Gamma_0(n)$ and $\Gamma' := \Gamma_1(\frac{n}{2})$ when $\Gamma = \Gamma_1(n)$, then we have

$$\Gamma = \Gamma_0(2) \cap \Gamma'. \quad (2.6.1)$$

Therefore, by [DR73, VI.6.9], $X_{\mathbb{F}_2}$ is semistable and has two irreducible components that meet precisely at the supersingular points. Both components are isomorphic to the proper, smooth, geometrically connected \mathbb{F}_2 -curve $(X_{\Gamma'})_{\mathbb{F}_2}$, so the semistable $\mathbb{Z}_{(2)}$ -curve $X_{\mathbb{Z}_{(2)}}$ is smooth away from the supersingular points on its special fiber.

The locus of $X_{\mathbb{F}_2}$ that corresponds to ordinary elliptic curves is a disjoint union of two affine connected curves: the open whose geometric points correspond to Γ -level structures with a connected $\Gamma_0(2)$ -part, and the open for which this $\Gamma_0(2)$ -part is étale. We let

$$X_{\mathbb{F}_2}^\mu \quad (\text{resp.}, X_{\mathbb{F}_2}^{\text{ét}})$$

denote the irreducible component of $X_{\mathbb{F}_2}$ that contains the former (resp., the latter) open, and we define the $\mathbb{Z}_{(2)}$ -smooth open $U^\mu \subset X_{\mathbb{Z}_{(2)}}$ by

$$U^\mu := X_{\mathbb{Z}_{(2)}} \setminus X_{\mathbb{F}_2}^{\text{ét}}.$$

The existence of f ensures that $\frac{n}{2} \geq 5$, so, as in the proof of Theorem 2.5,

$$X_{\mathbb{Z}_{(2)}} = \mathcal{X}_{\mathbb{Z}_{(2)}} \quad \text{if} \quad \Gamma = \Gamma_1(n).$$

In particular, $X_{\mathbb{Z}_{(2)}}$ is regular when $\Gamma = \Gamma_1(n)$ (but need not be regular when $\Gamma = \Gamma_0(n)$), see [Čes17, 4.4.4]. The semistability of $X_{\mathbb{Z}_{(2)}}$ supplies the identification

$$\text{Pic}_{X_{\mathbb{Z}_{(2)}}/\mathbb{Z}_{(2)}}^0 \cong \mathcal{J}_{\mathbb{Z}_{(2)}}^0 \quad (2.6.2)$$

as in [BLR90, 9.7/2] and ensures that the relative dualizing sheaf Ω is a line bundle on $X_{\mathbb{Z}_{(2)}}$. In particular, (2.6.2) and Grothendieck duality as in [Con00, Cor. 5.1.3] supply the analogue of (2.4.1):

$$H^0(X_{\mathbb{Z}_{(2)}}, \Omega) = H^0(\mathcal{J}_{\mathbb{Z}_{(2)}}, \Omega^1) \quad \text{inside} \quad H^0(X_{\mathbb{Q}}, \Omega^1). \quad (2.6.3)$$

Although U^μ is not $\mathbb{Z}_{(2)}$ -proper, $H^0(U^\mu, \Omega^1)$ is a finite free $\mathbb{Z}_{(2)}$ -module that contains $H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ and identifies with a $\mathbb{Z}_{(2)}$ -lattice inside $H^0(X_{\mathbb{Q}}, \Omega^1)$, see [BDP16, Prop. B.2.1.1].

When combined with (2.6.3), the following lemma will aid our analysis of the congruence module (2.2.1) in the case when $\text{ord}_2(n) = 1$. The involution trick used in its proof may be traced back at least to [Maz78, proof of Prop. 3.1].

Lemma 2.7. *If $\text{ord}_2(n) = 1$, then, in the setting of §2.1 and §2.6,*

$$H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f) = H^0(X_{\mathbb{Z}_{(2)}}, \Omega) \cap (\mathbb{Q} \cdot f) \quad \text{inside} \quad H^0(X_{\mathbb{Q}}, \Omega^1), \quad (2.7.1)$$

and $f \in H^0(X_{\mathbb{Z}_{(2)}}, \Omega) \stackrel{(2.6.3)}{=} H^0(\mathcal{J}_{\mathbb{Z}_{(2)}}, \Omega^1)$.

Proof. Since Ω is a line bundle, the normality of $X_{\mathbb{Z}_{(2)}}$ ensures that a $g \in H^0(X_{\mathbb{Q}}, \Omega^1)$ extends to $H^0(U^\mu, \Omega^1)$ if and only if g extends to the stalk of $\Omega_{X_{\mathbb{Z}_{(2)}}/\mathbb{Z}_{(2)}}^1$ at the generic point of $U_{\mathbb{F}_2}^\mu$, in which case g extends further to $H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ if and only if it extends to the stalk of $\Omega_{X_{\mathbb{Z}_{(2)}}/\mathbb{Z}_{(2)}}^1$ at the other generic point of $X_{\mathbb{F}_2}$. Due to (2.6.1), the Atkin–Lehner involution w_2 makes sense on the elliptic curve locus of $X_{\mathbb{Z}_{(2)}}$. Moreover, it extends to $X_{\mathbb{Q}}$ and interchanges the two stalks considered above. The equality (2.7.1) follows because the effect of w_2 on the newform f is scaling by ± 1 .

Due to (2.7.1), for the rest it suffices to note that $f \in H^0(U^\mu, \Omega^1)$ in the case when $\Gamma = \Gamma_0(n)$ (see [Edi06, 2.5]), and hence also in general thanks to the forgetful map $X_{\Gamma_1(n)} \rightarrow X_{\Gamma_0(n)}$. \square

Remarks.

2.8. In the case $\Gamma = \Gamma_0(n)$, the discussion of §2.6 and the proof of Lemma 2.7 are not specific to the prime $p = 2$. In particular, they show that if $\Gamma = \Gamma_0(n)$, then for any prime p with $\text{ord}_p(n) \leq 1$ we have

$$f \in H^0(X_{\mathbb{Z}_{(p)}}, \Omega) = H^0(\mathcal{J}_{\mathbb{Z}_{(p)}}, \Omega^1).$$

Continuing to assume that p is a prime with $\text{ord}_p(n) \leq 1$, we claim that this implies that

$$f \in H^0(\mathcal{J}_{\mathbb{Z}_{(p)}}, \Omega^1) \quad \text{also when } \Gamma = \Gamma_1(n).$$

For this, we first fix an $x \in X_1(n)(\mathbb{Q})$ and consider the resulting immersions $X_1(n) \hookrightarrow J_1(n)$ and $X_0(n) \hookrightarrow J_0(n)$, which, by Albanese functoriality, induce a compatible homomorphism $J_1(n) \rightarrow J_0(n)$. To then see the claim, it remains to pullback f along the resulting map of Néron models over $\mathbb{Z}_{(p)}$ and to use the alternative description of (1.9.1) mentioned in §1.9.

2.9. In the $\text{ord}_2(n) = 1$ case, (2.6.3) and Lemma 2.7 guarantee that $f \in H^0(\mathcal{J}_{\mathbb{Z}_{(2)}}, \Omega^1)$, so Proposition 2.3 (c) supplies the inequality $\text{ord}_2(c_\pi) \leq \text{ord}_2\left(\frac{\deg \phi}{\text{cong}_{f, \mathcal{J}}}\right)$. In particular,

$$\text{if } \text{ord}_2(n) = 1 \quad \text{and} \quad 2 \nmid \deg \phi, \quad \text{then} \quad \text{ord}_2(c_\pi) = 0,$$

which for $\Gamma = \Gamma_0(n)$ recovers a result of Agashe–Ribet–Stein, [ARS06, Thm. 2.7] (see also [ARS06, Thm. 3.11] for a generalization to optimal newform quotients of arbitrary dimension).

In the $\text{ord}_2(n) = 1$ case, the main result of this section is the following outgrowth of Proposition 2.3 (c).

Theorem 2.10. *If $\text{ord}_2(n) = 1$, then, in the setting of §2.1 and §2.6, the group*

$$\frac{H^0(U^\mu, \Omega^1)}{H^0(X_{\mathbb{Z}_{(2)}}, \Omega) + H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp}$$

is finite and cyclic and the 2-adic valuation of its order bounds the 2-adic valuation of c_π :

$$\text{ord}_2(c_\pi) \leq \text{ord}_2 \left(\# \left(\frac{H^0(U^\mu, \Omega^1)}{H^0(X_{\mathbb{Z}_{(2)}}, \Omega) + H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \right) \right).$$

Proof. Due to Lemma 2.7, the pullback map

$$\frac{H^0(X_{\mathbb{Z}_{(2)}}, \Omega)}{H^0(X_{\mathbb{Z}_{(2)}}, \Omega) \cap (\mathbb{Q} \cdot f) + H^0(X_{\mathbb{Z}_{(2)}}, \Omega) \cap (\mathbb{Q} \cdot f)^\perp} \hookrightarrow \frac{H^0(U^\mu, \Omega^1)}{H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f) + H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} \quad (2.10.1)$$

is injective and its cokernel is the group in question, which therefore inherits finiteness and cyclicity from the target (compare with the discussion of finiteness and cyclicity in §2.2).

The sought inequality follows by combining Proposition 2.3 (c), Lemma 2.7, and the following claims.

Claim 2.10.2. The 2-adic valuation of the order of the source of (2.10.1) is $\text{ord}_2(\text{cong}_{f, \mathcal{J}})$.

Claim 2.10.3. The 2-adic valuation of the order of the target of (2.10.1) is at least $\text{ord}_2(\deg \phi)$.

Proof of Claim 2.10.2. It suffices to use the identification (2.6.3). \square

Proof of Claim 2.10.3. We use the same notation $S_2(\Gamma, R)$ as in the proof of Theorem 2.5. Like there, for every subring $R \subset \mathbb{C}$ we have the identification

$$S_2(\Gamma, R) \cong S_2(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} R$$

discussed in [DI95, §12.3].

If $\Gamma = \Gamma_0(n)$, then $H^0(U^\mu, \Omega^1) = S_2(\Gamma, \mathbb{Z}_{(2)})$ (see [Edi06, 2.5]), so [ARS12, Thm. 3.6 (a)] shows that the 2-primary factor of $\deg \phi$ divides the order of the target of (2.10.1).

If $\Gamma = \Gamma_1(n)$, then we carry out the same argument after base change to $\mathbb{Z}[\zeta_n]_{(2)}$. Namely, after fixing an embedding $\mathbb{Q}(\zeta_n) \subset \mathbb{C}$, we arrive at the identification

$$H^0(U_{\mathbb{Z}[\zeta_n]_{(2)}}^\mu, \Omega^1) = S_2(\Gamma, \mathbb{Z}[\zeta_n]_{(2)}),$$

analogously to the proof of Theorem 2.5. In particular, the base change to $\mathbb{Z}[\zeta_n]_{(2)}$ of the target of (2.10.1) identifies with

$$\left(\frac{S_2(\Gamma, \mathbb{Z})}{S_2(\Gamma, \mathbb{Z}) \cap (\mathbb{Q} \cdot f) + S_2(\Gamma, \mathbb{Z}) \cap (\mathbb{Q} \cdot f)^\perp} \right) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n]_{(2)},$$

so it remains to observe that the exponent of the latter is divisible by the 2-primary factor of $\deg \phi$ due to [ARS12, Thm. 3.6 (a)]. \square \square

Remark 2.11. In the case $\Gamma = \Gamma_0(n)$, neither the statement nor the proof of Theorem 2.10 is specific to the prime $p = 2$, as one sees with the help of Remark 2.8.

3. USING OLDFORMS TO OFFSET THE DIFFERENCE BETWEEN INTEGRAL STRUCTURES

According to Theorem 2.10, to settle the $\text{ord}_2(n) = 1$ case of the 2-primary part of Conjectures 1.1 and 1.3, it suffices to show that $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ consists of images of elements of $H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp$. Since f is a newform, the latter space contains the oldforms in $H^0(U^\mu, \Omega^1)$, and our strategy is to show that under suitable assumptions these oldforms sweep out the entire $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$. The merit of this approach is that the sought statement no longer involves newforms, but instead is a generality about integral structures on the \mathbb{Q} -vector space of weight 2 cusp forms. We therefore forget about f and pursue this generality with the following setup.

3.1. The setup. Throughout this section we fix an $n \in \mathbb{Z}_{\geq 1}$ with $\text{ord}_2(n) = 1$ and

- We let Γ and Γ' denote either $\Gamma_0(n)$ and $\Gamma_0(\frac{n}{2})$, respectively, or $\Gamma_1(n)$ and $\Gamma_1(\frac{n}{2})$, respectively;
- We let \mathcal{X} (resp., \mathcal{X}') denote \mathcal{X}_Γ (resp., $\mathcal{X}_{\Gamma'}$), so that, e.g., \mathcal{X}' is either $\mathcal{X}_0(\frac{n}{2})$ or $\mathcal{X}_1(\frac{n}{2})$;
- We let X and X' denote the coarse moduli schemes of \mathcal{X} and \mathcal{X}' , i.e., $X = X_\Gamma$ and $X' = X_{\Gamma'}$;
- We set $U^\mu := X_{\mathbb{Z}_{(2)}} \setminus X_{\mathbb{F}_2}^{\text{ét}}$ and $U^{\text{ét}} := X_{\mathbb{Z}_{(2)}} \setminus X_{\mathbb{F}_2}^\mu$ (see §2.6 for the definition of $X_{\mathbb{F}_2}^\mu$ and $X_{\mathbb{F}_2}^{\text{ét}}$);
- We let $\mathcal{U}^\mu \subset \mathcal{X}_{\mathbb{Z}_{(2)}}$ (resp., $\mathcal{U}^{\text{ét}} \subset \mathcal{X}_{\mathbb{Z}_{(2)}}$) be the preimage of U^μ (resp., of $U^{\text{ét}}$).

The algebraic stacks \mathcal{X} and \mathcal{X}' are regular and have moduli interpretations in terms of generalized elliptic curves equipped with additional data, see [Čes17, §4.4, esp. 4.4.4, and Ch. 5, esp. §§5.9–5.10 and 5.13 (a)]. These moduli interpretations and [KM85, 2.7.4] show that

$$\mathcal{X}_{\mathbb{Z}_{(2)}} = X_{\mathbb{Z}_{(2)}} \quad \text{if } \Gamma = \Gamma_1(n) \quad \text{with } \frac{n}{2} \geq 5 \quad (3.1.1)$$

(see [Čes17, 4.1.4]). Even though we do not rely on (3.1.1) in what follows, its significance is that the overall proof of Theorem 1.2 is actually carried out entirely in the realm of schemes.

Using the moduli interpretations of \mathcal{X} and \mathcal{X}' , we seek to expose degeneracy maps

$$\pi_{\text{forg}}, \pi_{\text{quot}}: \mathcal{X} \rightrightarrows \mathcal{X}'$$

whose base changes to $\mathbb{Z}_{(2)}$ will be instrumental for constructing enough oldforms in $H^0(U^\mu, \Omega^1)$. The construction of π_{forg} and π_{quot} is not specific to the prime 2, so we present it in the setting of any prime p and any $N \in \mathbb{Z}_{\geq 1}$ with $\text{ord}_p(N) = 1$.

3.2. The maps π_{forg} and π_{quot} in the $\Gamma_1(N)$ case. The stack $\mathcal{X}_1(N)$ (resp., $\mathcal{X}_1(\frac{N}{p})$) parametrizes generalized elliptic curves $E \rightarrow S$ equipped with an ample Drinfeld $\mathbb{Z}/N\mathbb{Z}$ -structure (resp., $\mathbb{Z}/\frac{N}{p}\mathbb{Z}$ -structure). The map

$$\pi_{\text{forg}}: \mathcal{X}_1(N) \rightarrow \mathcal{X}_1(\frac{N}{p})$$

is defined by forgetting the p -primary part of the $\mathbb{Z}/N\mathbb{Z}$ -structure and contracting E with respect to the $\frac{N}{p}$ -primary part, whereas the map

$$\pi_{\text{quot}}: \mathcal{X}_1(N) \rightarrow \mathcal{X}_1(\frac{N}{p})$$

is defined by replacing E by the quotient by the subgroup generated by this p -primary part and equipping the quotient with the induced ample Drinfeld $\mathbb{Z}/\frac{N}{p}\mathbb{Z}$ -structure (without restricting to the elliptic curve locus the quotient (here and below) is in the sense of [Čes17, 2.2.4 and 2.2.6] and carries an induced $\mathbb{Z}/\frac{N}{p}\mathbb{Z}$ -structure by [Čes17, 4.2.9 (e)]).

The maps π_{forg} and π_{quot} are representable because, due to the representability of the forgetful contraction $\mathcal{X}_1(N) \rightarrow \mathcal{X}(1)$, they do not identify any distinct automorphisms of any geometric point of $\mathcal{X}_1(N)$ (see [Čes17, 3.2.2 (b) and proof of Lemma 4.7.1]). They inherit properness from $\mathcal{X}_1(N) \rightarrow \text{Spec } \mathbb{Z}$ and, due to their moduli interpretation, quasi-finiteness from $\mathcal{X}_1(N) \rightarrow \mathcal{X}(1)$ (the quasi-finiteness of the latter ensures that over a fixed algebraically closed field there are only finitely many isomorphism classes of degenerate generalized elliptic curves equipped with an ample Drinfeld $\mathbb{Z}/N\mathbb{Z}$ -structure). Therefore, π_{forg} and π_{quot} are finite (see [LMB00, A.2] and [EGA IV₄, 18.12.4]), and hence even locally free due to the miracle flatness theorem (see [EGA IV₂, 6.1.5]).

The same argument will show that the maps π_{forg} and π_{quot} are also representable and finite locally free in the $\Gamma_0(N)$ case because π_{forg} (resp., π_{quot}) will still send underlying generalized elliptic curves to their suitable contractions (resp., quotients).

3.3. The maps π_{forg} and π_{quot} in the $\Gamma_0(N)$ case. The stack $\mathcal{X}_0(N)$ (resp., $\mathcal{X}_0(\frac{N}{p})$) parametrizes generalized elliptic curves $E \rightarrow S$ equipped with a “ $\Gamma_0(N)$ -structure” (resp., a “ $\Gamma_0(\frac{N}{p})$ -structure”), which on the elliptic curve locus, and for squarefree N also on the entire S , is an ample S -subgroup $G \subset E^{\text{sm}}$ of order N (resp., $\frac{N}{p}$) that is cyclic in the sense of Drinfeld. For general N , part of the data of a $\Gamma_0(N)$ -structure is a certain open cover $\{S_{(m)}\}_{m|N}$ of S over which G is required to live inside suitable “universal decontractions” of E , see [Čes17, §5.9] (however, since $p^2 \nmid N$, the p -primary part $G[p]$ always lives inside E itself, see [Čes17, 5.9.4]). On the elliptic curve locus, the map

$$\pi_{\text{forg}}: \mathcal{X}_0(N) \rightarrow \mathcal{X}_0(\frac{N}{p}) \quad (\text{resp., } \pi_{\text{quot}}: \mathcal{X}_0(N) \rightarrow \mathcal{X}_0(\frac{N}{p}))$$

is defined by replacing G by $G[\frac{N}{p}]$ (resp., by replacing E by $E/G[p]$ endowed with $G/G[p]$). Granted that for π_{forg} one contracts E with respect to $G[\frac{N}{p}]$, for squarefree N the same definition also works over the entire $\mathcal{X}_0(N)$, and our task is to explain how to naturally extend it to the entire $\mathcal{X}_0(N)$ for general N . For this, the following lemma suffices because the open substacks $\mathcal{X}_0(N)_{(m)} \subset \mathcal{X}_0(N)$ cut out by the $S_{(m)}$ for $m | N$ cover $\mathcal{X}_0(N)$ and pairwise intersect in the elliptic curve locus, to the

effect that we only need to define each $(\pi_{\text{forg}})|_{\mathcal{X}_0(N)_{(m)}}$ (resp., $(\pi_{\text{quot}})|_{\mathcal{X}_0(N)_{(m)}}$) compatibly with the already given definition on the elliptic curve locus.

For brevity, in the lemma if $E \rightarrow S$ is a generalized elliptic curve with d -gon degenerate geometric fibers for some $d \in \mathbb{Z}_{\geq 1}$, then the *prime to p contraction* of E , denoted by E' , is the contraction of E with respect to $E^{\text{sm}}[\frac{d}{d^{\text{ord}_p(d)}}]$ (or with respect to any other finite locally free S -subgroup that meets the same irreducible components of the geometric fibers of $E \rightarrow S$ as $E^{\text{sm}}[\frac{d}{d^{\text{ord}_p(d)}}]$, see [Čes17, 3.2.1]).

Lemma 3.3.1. *In the setting of a prime p and an $N \in \mathbb{Z}_{\geq 1}$ with $\text{ord}_p(N) = 1$, fix an $m \mid N$, let $E \rightarrow S$ be a generalized elliptic curve equipped with a $\Gamma_0(N)$ -structure for which $S_{(m)} = S$, and let $G_{(m)} \subset E^{\text{sm}}$ be the resulting ample cyclic S -subgroup of order $\frac{N}{\text{gcd}(m, \frac{N}{m})}$ (see [Čes17, 5.9.4]).*

- (a) *There is a unique $\Gamma_0(\frac{N}{p})$ -structure on E' such that for every fppf S -scheme \tilde{S} endowed with a generalized elliptic curve $\tilde{E} \rightarrow \tilde{S}$ that has m -gon degenerate geometric fibers and is equipped with an isomorphism between its contraction and $E_{\tilde{S}}$, the ample cyclic S -subgroups of \tilde{E}' of order $\frac{N}{p}$ determined by the $\Gamma_0(\frac{N}{p})$ -structure on E' and by the $\Gamma_0(N)$ -structure on E agree.*
- (b) *There is a unique $\Gamma_0(\frac{N}{p})$ -structure on $E/(G_{(m)}[p])$ such that for every fppf S -scheme \tilde{S} endowed with an $\tilde{E} \rightarrow \tilde{S}$ as in (a), the ample cyclic S -subgroup $\tilde{G} \subset \tilde{E}^{\text{sm}}$ of order N determined by the $\Gamma_0(N)$ -structure on E is such that $\tilde{G}/\tilde{G}[p] \subset (\tilde{E}/\tilde{G}[p])^{\text{sm}}$ agrees with the S -subgroup determined by the $\Gamma_0(\frac{N}{p})$ -structure on $E/(G_{(m)}[p])$.*

Proof. An fppf $\tilde{S} \rightarrow S$ endowed with a required $\tilde{E} \rightarrow \tilde{S}$ always exists, see [Čes17, 3.2.6].

- (a) The uniqueness aspect allows us to work fppf locally on S , so we assume that $S = \tilde{S}$ and let $\tilde{G} \subset \tilde{E}^{\text{sm}}$ be the S -subgroup determined by the $\Gamma_0(N)$ -structure on E . As in [Čes17, §5.11], the ample S -subgroup $\tilde{G}[\frac{N}{p}] \subset (\tilde{E}')^{\text{sm}}$ of order $\frac{N}{p}$ uniquely extends to a $\Gamma_0(\frac{N}{p})$ -structure on E' . It remains to note that this unique $\Gamma_0(\frac{N}{p})$ -structure satisfies the claimed compatibility with respect to any other $\tilde{E} \rightarrow \tilde{S}$ due to [Čes17, 5.7].
- (b) To make sense of the characterizing property, one notes that $(G[p]_{(m)})_{\tilde{S}}$ identifies with $\tilde{G}[p]$ inside $E_{\tilde{S}}$ and that $(E/G_{(m)}[p])_{\tilde{S}}$ identifies with a contraction of $\tilde{E}/\tilde{G}[p]$, as is ensured by the uniqueness aspect of [DR73, IV.1.2] (see also the review in [Čes17, 3.2.1]). Granted this, the proof is the same as that of (a) with the role of $\tilde{G}[\frac{N}{p}]$ replaced by $\tilde{G}/\tilde{G}[p]$. \square

3.4. The maps π_{forg} and π_{quot} on coarse spaces. Returning to the setting of §3.1, we let

$$\pi_{\text{forg}}, \pi_{\text{quot}}: X \rightrightarrows X' \tag{3.4.1}$$

denote the maps induced on the coarse moduli schemes by the maps $\pi_{\text{forg}}, \pi_{\text{quot}}: \mathcal{X} \rightrightarrows \mathcal{X}'$ constructed in §§3.2–3.3 (we take $N = n$ and $p = 2$). The base changes of the maps (3.4.1) to \mathbb{C} identify with degeneracy maps that appear in a discussion of the theory of newforms, so pullbacks of Kähler differentials along $(\pi_{\text{forg}})_{\mathbb{Q}}$ or $(\pi_{\text{quot}})_{\mathbb{Q}}$ are (associated to) oldforms. The map π_{forg} induces

$$\text{an isomorphism } X_{\mathbb{F}_2}^{\mu} \cong X'_{\mathbb{F}_2} \quad \text{and a purely inseparable degree 2 morphism } X_{\mathbb{F}_2}^{\text{ét}} \rightarrow X'_{\mathbb{F}_2},$$

as is seen on the elliptic curve locus using [DR73, diagram on p. 287]. Analogously, π_{quot} induces

$$\text{a purely inseparable degree 2 morphism } X_{\mathbb{F}_2}^{\mu} \rightarrow X'_{\mathbb{F}_2} \quad \text{and an isomorphism } X_{\mathbb{F}_2}^{\text{ét}} \cong X'_{\mathbb{F}_2}.$$

With the maps π_{forg} and π_{quot} at our disposal, we turn to producing oldforms that sweep out the 2-torsion subgroup of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$. This is accomplished in Proposition 3.6, with the key step carried out in the following lemma.

Lemma 3.5. *Let $S \subset X_{\mathbb{F}_2}^{\text{ét}}$ be the reduced divisor of supersingular points, and let $\Omega|_{X_{\mathbb{F}_2}^{\text{ét}}}$ be the pullback of the relative dualizing sheaf of $X_{\mathbb{Z}(2)}$. Every $g \in H^0(X_{\mathbb{F}_2}^{\text{ét}}, \Omega|_{X_{\mathbb{F}_2}^{\text{ét}}}(-S))$ lifts to an oldform $\tilde{g} \in H^0(X_{\mathbb{Z}(2)}, \Omega)$ whose restriction to $X_{\mathbb{F}_2}^\mu$ vanishes.*

Proof. Since $X_{\mathbb{F}_2}$ is semistable (see §2.6), we have the $\mathcal{O}_{X_{\mathbb{F}_2}^{\text{ét}}}$ -module identification

$$\Omega|_{X_{\mathbb{F}_2}^{\text{ét}}} = \Omega_{X_{\mathbb{F}_2}^{\text{ét}}/\mathbb{F}_2}^1(S) \quad \text{inside the generic stalk of } \Omega_{X_{\mathbb{F}_2}^{\text{ét}}/\mathbb{F}_2}^1, \quad (3.5.1)$$

as may be checked over $\overline{\mathbb{F}_2}$ by using the theory of regular differentials exposed in [Con00, §5.2]. Therefore, g lies in $H^0(X_{\mathbb{F}_2}^{\text{ét}}, \Omega^1)$, and hence, via the isomorphism $X_{\mathbb{F}_2}^{\text{ét}} \cong X'_{\mathbb{F}_2}$ induced by π_{quot} (see §3.4), corresponds to a unique $g' \in H^0(X'_{\mathbb{F}_2}, \Omega^1)$. By Grothendieck–Serre duality and cohomology and base change (see [Con00, Thm. 5.1.2]), we may lift g' to a $G' \in H^0(X'_{\mathbb{Z}(2)}, \Omega^1)$. We set

$$\tilde{g} := (\pi_{\text{quot}}|_{X_{\mathbb{Z}(2)}} - S)^*(G'|_{X'_{\mathbb{Z}(2)}} - \pi_{\text{quot}}(S)) \in H^0(X_{\mathbb{Z}(2)} - S, \Omega^1) = H^0(X_{\mathbb{Z}(2)}, \Omega)$$

(the normality of $X_{\mathbb{Z}(2)}$ ensures the equality because Ω is a line bundle whose restriction to $X_{\mathbb{Z}(2)} - S$ is Ω^1). By construction, \tilde{g} is an oldform in $H^0(X_{\mathbb{Z}(2)}, \Omega)$ that agrees with g on $X_{\mathbb{F}_2}^{\text{ét}}$ and that vanishes on $X_{\mathbb{F}_2}^\mu$ because the map $X_{\mathbb{F}_2}^\mu \rightarrow X'_{\mathbb{F}_2}$ induced by π_{quot} is purely inseparable of degree 2. \square

Proposition 3.6. *Every element of $(H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega))[2]$ lifts to an oldform in $H^0(U^\mu, \Omega^1)$.*

Proof. The stalks of $\mathcal{O}_{X_{\mathbb{Z}(2)}}$ at the generic points of $X_{\mathbb{F}_2}^\mu$ and $X_{\mathbb{F}_2}^{\text{ét}}$ are discrete valuation rings with 2 as a uniformizer. Thus, as explained in [BLR90, p. 104], there are “order functions” v^μ and $v^{\text{ét}}$ that measure the valuations of sections of Ω at these respective points. In this notation, $H^0(U^\mu, \Omega^1)$ (resp., $H^0(X_{\mathbb{Z}(2)}, \Omega)$) identifies with the set of $f \in H^0(X_{\mathbb{Q}}, \Omega^1)$ for which $v^\mu(f) \geq 0$ (resp., for which $v^\mu(f) \geq 0$ and $v^{\text{ét}}(f) \geq 0$), similarly to the proof of Lemma 2.7.

If $f \in H^0(U^\mu, \Omega^1)$ represents an element of $(H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega))[2]$, then $2f$ is a global section of Ω that vanishes on $X_{\mathbb{F}_2}^\mu$. The restriction of $2f$ to $X_{\mathbb{F}_2}^{\text{ét}}$ therefore lies in $H^0(X_{\mathbb{F}_2}^{\text{ét}}, \Omega|_{X_{\mathbb{F}_2}^{\text{ét}}}(-S))$, as is required for Lemma 3.5 to apply. Lemma 3.5 supplies an oldform $\tilde{g} \in H^0(X_{\mathbb{Z}(2)}, \Omega)$ that agrees with $2f$ on $X_{\mathbb{F}_2}^{\text{ét}}$ and vanishes on $X_{\mathbb{F}_2}^\mu$. It remains to note that, by the discussion of the previous paragraph, $\frac{\tilde{g}}{2}$ is an oldform in $H^0(U^\mu, \Omega^1)$ for which

$$\tilde{g} - 2f \in 2 \cdot H^0(X_{\mathbb{Z}(2)}, \Omega), \quad \text{i.e.,} \quad \frac{\tilde{g}}{2} - f \in H^0(X_{\mathbb{Z}(2)}, \Omega). \quad \square$$

Remark 3.7. In the case $\Gamma = \Gamma_0(n)$, the discussion of §3.4 and the proofs of Lemma 3.5 and Proposition 3.6 are not specific to the prime $p = 2$ (see also Remark 2.8). In particular, if $n \in \mathbb{Z}_{\geq 1}$ and p is a prime with $\text{ord}_p(n) \leq 1$, then, adopting the analogous notation $U^\mu \subset X_0(n)_{\mathbb{Z}(p)}$ also for odd p , they show that every p -torsion element of $H^0(U^\mu, \Omega^1)/H^0(X_0(n)_{\mathbb{Z}(p)}, \Omega)$ lifts to an oldform in $H^0(U^\mu, \Omega^1)$. Since for odd p every element of this quotient is p -torsion by [Edi06, 2.5 and 2.7] (equivalently, by the proof of Proposition 3.14 below), this combines with Theorem 2.10 and Remark 2.11 to reprove that the Manin constant of a new elliptic optimal quotient of $J_0(n)$ is not divisible by any odd prime p with $\text{ord}_p(n) \leq 1$. The key distinction of this proof is that it does not use exactness results for semiabelian Néron models over bases of low ramification (compare with the well-known argument recalled in the proof of Proposition 4.3).

For extending Proposition 3.6 beyond the 2-torsion subgroup of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$, it will be handy to work with the Deligne–Mumford stack $\mathcal{X}_{\mathbb{Z}(2)}$ instead of its coarse moduli scheme $X_{\mathbb{Z}(2)}$. To facilitate for this, we supplement the discussion of §2.6 with a similar discussion of $\mathcal{X}_{\mathbb{Z}(2)}$. The principal advantage of $\mathcal{X}_{\mathbb{Z}(2)}$ over $X_{\mathbb{Z}(2)}$ is its regularity (see §3.1), which will permit an effective use of techniques from intersection theory. The principal disadvantage is the loss of Grothendieck–Serre duality, which was an important component of the proof of Lemma 3.5.

3.8. The structure of $\mathcal{X}_{\mathbb{Z}(2)}$. Due to the moduli interpretation of $\mathcal{X}_{\mathbb{Z}(2)}$, the map

$$\mathcal{X}_{\mathbb{Z}(2)} \rightarrow X_{\mathbb{Z}(2)}$$

towards the coarse moduli scheme is étale over the locus of $X_{\mathbb{Z}(2)}$ on which the j -invariant satisfies $j \neq 0$ and $j \neq 1728$, see [Čes17, 4.1.4 and proof of Thm. 6.7]. Thus, since \mathcal{X} is normal, $\mathcal{X}_{\mathbb{Z}(2)} \rightarrow \text{Spec } \mathbb{Z}(2)$ is smooth away from the supersingular points of its \mathbb{F}_2 -fiber $\mathcal{X}_{\mathbb{F}_2}$ (see §2.6). In particular, by the $(R_0)+(S_1)$ criterion, $\mathcal{X}_{\mathbb{F}_2}$ is reduced and, by [DR73, VI.6.10], $X_{\mathbb{F}_2}$ is the coarse moduli space of $\mathcal{X}_{\mathbb{F}_2}$.

In contrast, the smooth locus of $\mathcal{X}'_{\mathbb{Z}(2)} \rightarrow \text{Spec } \mathbb{Z}(2)$ is the entire $\mathcal{X}'_{\mathbb{Z}(2)}$ by [DR73, IV.6.7].

The decomposition of $X_{\mathbb{F}_2}$ into irreducible components $X_{\mathbb{F}_2}^\mu$ and $X_{\mathbb{F}_2}^{\text{ét}}$ corresponds to the decomposition of $\mathcal{X}_{\mathbb{F}_2}$ into irreducible components

$$\mathcal{X}_{\mathbb{F}_2}^\mu \quad \text{and} \quad \mathcal{X}_{\mathbb{F}_2}^{\text{ét}}.$$

We let $\mathcal{S} \subset \mathcal{X}_{\mathbb{F}_2}$ denote the reduced closed substack consisting of the supersingular points, so that $\mathcal{X}_{\mathbb{F}_2}^\mu$ and $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ meet precisely at \mathcal{S} . Due to [DR73, V.1.16 (ii)], the intersections of $\mathcal{X}_{\mathbb{F}_2}^\mu$ and $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ in $\mathcal{X}_{\mathbb{F}_2}$ at the points of \mathcal{S} are transversal. Thus, due to the regularity of $\mathcal{X}_{\mathbb{Z}(2)}$, the intersections of $\mathcal{X}_{\mathbb{F}_2}^\mu$ and $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ in $\mathcal{X}_{\mathbb{Z}(2)}$ are also transversal.

By [DR73, V.1.16 (i)], if we let the replacement of \mathcal{X} by \mathcal{Y} indicate the elliptic curve locus, then we have the isomorphism

$$\mathcal{Y}'_{\mathbb{F}_2} \xrightarrow{\sim} \mathcal{Y}_{\mathbb{F}_2}^\mu \quad (\text{resp., } \mathcal{Y}'_{\mathbb{F}_2} \xrightarrow{\sim} \mathcal{Y}_{\mathbb{F}_2}^{\text{ét}}) \quad (3.8.1)$$

obtained by supplementing the universal elliptic curve of $\mathcal{Y}'_{\mathbb{F}_2}$ (resp., the Frobenius pullback of the universal elliptic curve of $\mathcal{Y}'_{\mathbb{F}_2}$) with the subgroup of order 2 given by the kernel of Frobenius (resp., the kernel of Verschiebung).

3.9. The line bundle ω . The cotangent space at the identity section of the universal generalized elliptic curve gives a line bundle ω on \mathcal{X} (resp., on \mathcal{X}') of weight 1 modular forms (we sometimes write $\omega_{\mathcal{X}}$, etc. to emphasize the space on which ω lives). We will primarily be concerned with cusp forms, so we let ‘cusps’ denote the reduced relative effective Cartier divisor on \mathcal{X} (resp., on \mathcal{X}') over \mathbb{Z} cut out by the degeneracy locus of the universal generalized elliptic curve (see [Čes17, 4.4.2 (b) and 5.13 (b)]); *a posteriori*, ‘cusps’ is also the reduced complement of the elliptic curve locus of \mathcal{X} or \mathcal{X}' . Depending on the context, we also write ‘cusps’ for base changes or restrictions of this divisor. The line bundle whose global sections are weight 2 cusp forms is therefore $\omega^{\otimes 2}(-\text{cusps})$.

The modular definitions of the maps π_{forg} and π_{quot} given in §§3.2–3.3 also produce underlying \mathcal{X} -morphisms from the universal generalized elliptic curve of \mathcal{X} to the pullback of the universal generalized elliptic curve of \mathcal{X}' . The effect of these \mathcal{X} -morphisms on the cotangent spaces at the identity sections gives rise to $\mathcal{O}_{\mathcal{X}}$ -module morphisms

$$\pi_{\text{forg}}^*(\omega_{\mathcal{X}'}) \rightarrow \omega_{\mathcal{X}} \quad \text{and} \quad \pi_{\text{quot}}^*(\omega_{\mathcal{X}'}) \rightarrow \omega_{\mathcal{X}}. \quad (3.9.1)$$

Thus, since π_{forg} and π_{quot} are finite locally free (see §3.2) and when restricted to ‘cusps’ on \mathcal{X} factor through ‘cusps’ on \mathcal{X}' , the morphisms (3.9.1) lead to $\mathcal{O}_{\mathcal{X}}$ -module morphisms

$$\pi_{\text{forg}}^*(\omega_{\mathcal{X}'}^{\otimes 2}(-\text{cusps})) \rightarrow \omega_{\mathcal{X}}^{\otimes 2}(-\text{cusps}) \quad \text{and} \quad \pi_{\text{quot}}^*(\omega_{\mathcal{X}'}^{\otimes 2}(-\text{cusps})) \rightarrow \omega_{\mathcal{X}}^{\otimes 2}(-\text{cusps}).$$

We analyze the restrictions of these morphisms to $\mathcal{X}_{\mathbb{F}_2}^{\mu}$ and $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ in the following lemma.

Lemma 3.10.

- (a) The map π_{forg} restricts to an isomorphism $\pi_{\text{forg}}: \mathcal{X}_{\mathbb{F}_2}^{\mu} \xrightarrow{\sim} \mathcal{X}'_{\mathbb{F}_2}$ for which the pullback

$$\pi_{\text{forg}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}(-\text{cusps})) \rightarrow \omega_{\mathcal{X}_{\mathbb{F}_2}^{\mu}}^{\otimes 2}(-\text{cusps})$$

is also an isomorphism.

- (b) The map π_{quot} restricts to an isomorphism $\pi_{\text{quot}}: \mathcal{X}_{\mathbb{F}_2}^{\text{ét}} \xrightarrow{\sim} \mathcal{X}'_{\mathbb{F}_2}$ for which the pullback

$$\pi_{\text{quot}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}(-\text{cusps})) \rightarrow \omega_{\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}}^{\otimes 2}(-\text{cusps})$$

induces an identification $\pi_{\text{quot}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}(-\text{cusps})) \cong \omega_{\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}}^{\otimes 2}(-\text{cusps} - 2\mathcal{S})$.

- (c) The map π_{quot} restricts to a morphism $\pi_{\text{quot}}: \mathcal{X}_{\mathbb{F}_2}^{\mu} \rightarrow \mathcal{X}'_{\mathbb{F}_2}$ for which the pullback

$$\pi_{\text{quot}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}(-\text{cusps})) \rightarrow \omega_{\mathcal{X}_{\mathbb{F}_2}^{\mu}}^{\otimes 2}(-\text{cusps})$$

vanishes.

Proof. The divisor ‘cusps’ on $\mathcal{X}'_{\mathbb{Z}(2)}$ is étale over $\mathbb{Z}(2)$, and hence is also $\mathbb{Z}(2)$ -fiberwise reduced—this follows from [Čes17, 4.4.2 (b) and 5.13 (b)].

- (a) On the elliptic curve locus the claim follows from the description of the first map of (3.8.1). Thus, $\pi_{\text{forg}}: \mathcal{X}_{\mathbb{F}_2}^{\mu} \rightarrow \mathcal{X}'_{\mathbb{F}_2}$ is finite locally free of rank 1 (see §3.2), and hence is an isomorphism. Moreover, since its restriction to ‘cusps’ of the source factors through ‘cusps’ of the target, the reducedness of the latter ensures that $\pi_{\text{forg}}: \mathcal{X}_{\mathbb{F}_2}^{\mu} \xrightarrow{\sim} \mathcal{X}'_{\mathbb{F}_2}$ identifies ‘cusps’ of its source and target. It remains to note that $\pi_{\text{forg}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}) \xrightarrow{\sim} \omega_{\mathcal{X}_{\mathbb{F}_2}^{\mu}}^{\otimes 2}$ because π_{forg} does not change the relative identity component of the smooth locus of the universal generalized elliptic curve.
- (b) As in the proof of (a), the map $\pi_{\text{quot}}: \mathcal{X}_{\mathbb{F}_2}^{\text{ét}} \rightarrow \mathcal{X}'_{\mathbb{F}_2}$ is an isomorphism that identifies ‘cusps’ of its source and target. It remains to show that the pullback map

$$\pi_{\text{quot}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}) \rightarrow \omega_{\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}} \tag{3.10.1}$$

induces an identification

$$\pi_{\text{quot}}^*(\omega_{\mathcal{X}'_{\mathbb{F}_2}}) \cong \omega_{\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}}(-\mathcal{S}).$$

The subgroup of order 2 coming from the 2-primary part of the Γ -structure on the universal generalized elliptic curve of $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}} - \mathcal{S}$ is étale because the locus where this subgroup is of multiplicative type is *a priori* open and does not meet the elliptic curve locus (and hence is empty). Therefore, (3.10.1), being induced by pullback along the quotient by this 2-primary part, is an isomorphism away from the supersingular points.

For the remaining claim that the divisor cut out by (3.10.1) is precisely \mathcal{S} , we may work on the elliptic curve locus and after restriction along the isomorphism $\mathcal{Y}'_{\mathbb{F}_2} \xrightarrow{\sim} \mathcal{Y}_{\mathbb{F}_2}^{\text{ét}}$ of (3.8.1). After this restriction, (3.10.1) identifies with the map induced by pullback along the Verschiebung

isogeny of the universal elliptic curve of $\mathcal{Y}'_{\mathbb{F}_2}$, i.e., with the Hasse invariant. It remains to recall from [KM85, 12.4.4] that the Hasse invariant has simple zeroes at the supersingular points.

- (c) For the sought vanishing we may work on the elliptic curve locus, so, due to (3.8.1), it suffices to note that for an elliptic curve over a base scheme of characteristic $p > 0$ the pullback of any differential form along the relative Frobenius morphism vanishes. \square

We are ready for the following key proposition, which will replace Lemma 3.5 when attempting to lift arbitrary elements of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$ to oldforms. The role of its surjectivity assumption is to serve as a replacement of the Grothendieck–Serre duality for the Deligne–Mumford stack $\mathcal{X}'_{\mathbb{Z}(2)}$ (such duality for the scheme $X'_{\mathbb{Z}(2)}$ was important in the proof of Lemma 3.5; see also Lemma 3.12 (a)).

Proposition 3.11. *If the pullback map*

$$H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps})) \rightarrow H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps}))$$

is surjective, then every $g \in H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps} - \mathcal{S}))$ lifts to an oldform $\tilde{g} \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$ that vanishes on $\mathcal{X}'_{\mathbb{F}_2}$.

Proof. By Lemma 3.10 (a), g is the π_{forg} -pullback of a unique $g' \in H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps} - \mathcal{S}'))$, where $\mathcal{S}' \subset \mathcal{X}'_{\mathbb{F}_2}$ is the reduced closed substack supported at the supersingular points. Due to the surjectivity assumption, g' lifts to a $G' \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$. We set

$$\tilde{g}_0 := \pi_{\text{forg}}^*(G') \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps})),$$

so that \tilde{g}_0 is an oldform that lifts g . We claim that the restriction

$$h := \tilde{g}_0|_{\mathcal{X}'_{\mathbb{F}_2}} \in H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps})) \quad \text{lies in} \quad H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps} - 2\mathcal{S})). \quad (3.11.1)$$

For this, we may work on the elliptic curve locus and after restricting along the isomorphism $\mathcal{Y}'_{\mathbb{F}_2} \xrightarrow{\sim} \mathcal{Y}'_{\mathbb{F}_2}$ of (3.8.1). Under this isomorphism, $\omega_{\mathcal{Y}'_{\mathbb{F}_2}}^{\otimes 2}$ identifies with $\omega_{\mathcal{Y}'_{\mathbb{F}_2}}^{\otimes 4}$ and $\pi_{\text{forg}}|_{\mathcal{Y}'_{\mathbb{F}_2}}$ identifies with the Frobenius morphism of $\mathcal{Y}'_{\mathbb{F}_2}$, so h identifies with the Frobenius pullback of g' . Since g' , when viewed as a global section of $\omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}$, vanishes on \mathcal{S}' , (3.11.1) follows from the fact that the Frobenius pullback of $\mathcal{O}_{\mathcal{Y}'_{\mathbb{F}_2}}(-\mathcal{S}')$ is $\mathcal{O}_{\mathcal{Y}'_{\mathbb{F}_2}}(-2\mathcal{S}')$.

Due to (3.11.1) and Lemma 3.10 (b), h is the π_{quot} -pullback of a unique $h' \in H^0(\mathcal{X}'_{\mathbb{F}_2}, \omega^{\otimes 2}(-\text{cusps}))$. The surjectivity assumption lifts h' to an $H' \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$ and we set

$$\tilde{h} := \pi_{\text{quot}}^*(H') \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps})).$$

By construction and Lemma 3.10 (c), \tilde{h} is an oldform, agrees with h on $\mathcal{X}'_{\mathbb{F}_2}$, and vanishes on $\mathcal{X}'_{\mathbb{F}_2}$. In conclusion, the oldform $\tilde{g} := \tilde{g}_0 - \tilde{h} \in H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$ lifts g and vanishes on $\mathcal{X}'_{\mathbb{F}_2}$. \square

In order to take advantage of Proposition 3.11, we seek to relate $\omega^{\otimes 2}(-\text{cusps})$ to the “relative dualizing sheaf” Ω of §A.1 via Kodaira–Spencer type isomorphisms supplied by the following lemma.

Lemma 3.12.

- (a) *On $\mathcal{X}'_{\mathbb{Z}(2)}$, there is an $\mathcal{O}_{\mathcal{X}'_{\mathbb{Z}(2)}}$ -module isomorphism $\Omega^1 \cong \omega^{\otimes 2}(-\text{cusps})$.*
- (b) *On $\mathcal{X}'_{\mathbb{Z}(2)}$, there is an $\mathcal{O}_{\mathcal{X}'_{\mathbb{Z}(2)}}$ -module isomorphism $\Omega \cong \omega^{\otimes 2}(-\text{cusps} + \mathcal{X}'_{\mathbb{F}_2})$.*

Proof. We will bootstrap the claims from their analogue for $\mathcal{X}(1)$ supplied by [Kat73, A1.3.17]:

$$\Omega_{\mathcal{X}(1)/\mathbb{Z}}^1 \cong \omega_{\mathcal{X}(1)}^{\otimes 2}(-\text{cusps}). \quad (3.12.1)$$

For any congruence subgroup $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$, the structure map $\pi: \mathcal{X}_H \rightarrow \mathcal{X}(1)$ is finite locally free, so, due to the base change compatibility of the formation of the relative dualizing sheaf (see [Con00, Thm. 3.6.1]), there is a “relative dualizing” $\mathcal{O}_{\mathcal{X}_H}$ -module $\Omega_{\mathcal{X}_H/\mathcal{X}(1)}$ constructed étale locally on $\mathcal{X}(1)$. Explicitly, due to [Con00, bottom half of p. 31 and pp. 137–139, esp. (VAR6) on p. 139, supplemented by Cor. 3.6.4], $\Omega_{\mathcal{X}_H/\mathcal{X}(1)}$ identifies with $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}(1)}}(\pi_*(\mathcal{O}_{\mathcal{X}_H}), \mathcal{O}_{\mathcal{X}(1)})$ regarded as an $\mathcal{O}_{\mathcal{X}_H}$ -module.

By working étale locally on $\mathcal{X}(1)$, [Con00, Thm. 4.3.3 and (4.3.7); see also bottom of p. 206] supply an $\mathcal{O}_{\mathcal{X}_H}$ -module isomorphism

$$\Omega_{\mathcal{X}_H/\mathcal{X}(1)} \otimes_{\mathcal{O}_{\mathcal{X}_H}} \pi^* \Omega_{\mathcal{X}(1)/\mathbb{Z}} \cong \Omega_{\mathcal{X}_H/\mathbb{Z}}. \quad (3.12.2)$$

To proceed further, we assume that \mathcal{X}_H is regular, so that π is a local complete intersection (see [Liu02, 6.3.18]), and hence has Gorenstein fibers, to the effect that $\Omega_{\mathcal{X}_H/\mathcal{X}(1)}$ is a line bundle (see [Con00, Thm. 3.5.1]). Then, since π is étale over a dense open of $\mathcal{X}(1)$, the element

$$\text{trace} \in \text{Hom}_{\mathcal{O}_{\mathcal{X}(1)}}(\pi_*(\mathcal{O}_{\mathcal{X}_H}), \mathcal{O}_{\mathcal{X}(1)}) \cong \Gamma(\mathcal{X}_H, \Omega_{\mathcal{X}_H/\mathcal{X}(1)})$$

gives rise to the identification

$$\Omega_{\mathcal{X}_H/\mathcal{X}(1)} \cong \mathcal{O}_{\mathcal{X}_H} \left(\sum_{x \in |\mathcal{X}_H|^{(1)}} d_x \cdot \overline{\{x\}} \right), \quad (3.12.3)$$

where the sum runs over the height 1 points x of \mathcal{X}_H , the corresponding to x irreducible Weil divisor on \mathcal{X}_H is denoted by $\overline{\{x\}}$, and d_x denotes the valuation of the different ideal of the extension $\mathcal{O}_{\mathcal{X}_H, x}^{\text{sh}}/\mathcal{O}_{\mathcal{X}(1), \pi(x)}^{\text{sh}}$ of discrete valuation rings. Since $d_x = 0$ whenever this extension is étale, each x that contributes to the sum either is the generic point of an irreducible component of a closed fiber of $\mathcal{X}_H \rightarrow \text{Spec } \mathbb{Z}$ or lies on the cusps of $(\mathcal{X}_H)_{\mathbb{Q}}$. Moreover, at the latter x the ramification is tame and $d_x = e_x - 1$, where e_x is the ramification index of $\mathcal{O}_{\mathcal{X}_H, x}^{\text{sh}}/\mathcal{O}_{\mathcal{X}(1), \pi(x)}^{\text{sh}}$. By combining this with (3.12.1)–(3.12.3), we arrive at the identification

$$\Omega_{\mathcal{X}_H/\mathbb{Z}} \cong (\pi^*(\omega_{\mathcal{X}(1)}))^{\otimes 2}(-\text{cusps} + \sum_y d_y \cdot \overline{\{y\}}), \quad (3.12.4)$$

where y runs over the generic points of the irreducible components of the closed \mathbb{Z} -fibers of \mathcal{X}_H and ‘cusps’ denotes the reduced complement of the elliptic curve locus of \mathcal{X}_H .

In the case when \mathcal{X}_H is \mathcal{X} or \mathcal{X}' , the map π is the forgetful contraction and does not change the relative identity component of the smooth locus of the universal generalized elliptic curve, so $\pi^*(\omega_{\mathcal{X}(1)})$ identifies with $\omega_{\mathcal{X}}$ or $\omega_{\mathcal{X}'}$, respectively. Therefore, since $\Omega_{\mathcal{X}'_{\mathbb{Z}(2)}/\mathbb{Z}(2)} \cong \Omega_{\mathcal{X}'_{\mathbb{Z}(2)}/\mathbb{Z}(2)}^1$ due to the $\mathbb{Z}(2)$ -smoothness of $\mathcal{X}'_{\mathbb{Z}(2)}$, the sought conclusion will follow from (3.12.4) once we identify the d_y for y of residue characteristic 2 in the case when $\mathcal{X}_H = \mathcal{X}$ or $\mathcal{X}_H = \mathcal{X}'$.

- (a) Since the “level” $\frac{n}{2}$ of \mathcal{X}' is odd, the map $\mathcal{X}'_{\mathbb{Z}(2)} \rightarrow \mathcal{X}(1)_{\mathbb{Z}(2)}$ is étale over a fiberwise dense open of $\mathcal{X}(1)_{\mathbb{Z}(2)}$. Therefore, $d_y = 0$ whenever y has residue characteristic 2.
- (b) By Lemma 3.10 (a) and the proof of (a), π is generically étale on $\mathcal{X}'_{\mathbb{F}_2}$. In contrast, (3.8.1) identifies the map $\mathcal{Y}_{\mathbb{F}_2}^{\text{ét}} \rightarrow \mathcal{Y}'_{\mathbb{F}_2}$ induced by π_{forg} with the Frobenius of $\mathcal{Y}'_{\mathbb{F}_2}$, which is not generically étale. We therefore conclude from (3.12.4) and [AS02, A.3] that

$$\Omega_{\mathcal{X}'_{\mathbb{Z}(2)}/\mathbb{Z}(2)} \cong \omega^{\otimes 2}(-\text{cusps} + d \cdot \mathcal{X}'_{\mathbb{F}_2}^{\text{ét}}) \quad \text{for some } d \geq 1. \quad (3.12.5)$$

To see that also $d \leq 1$, we note that on strict Henselizations of \mathcal{X} and \mathcal{X}' at the generic points of $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ and $\mathcal{X}'_{\mathbb{F}_2}$ the map π_{forg} induces a degree 2 extension of absolutely unramified discrete valuation rings (in particular, the trace of 1 in this extension is a uniformizer). \square

Remark 3.13. The isomorphism $H^0(\mathcal{X}_{\mathbb{Q}}, \Omega^1) \cong H^0(\mathcal{X}_{\mathbb{Q}}, \omega^{\otimes 2}(-\text{cusps}))$ supplied by Lemma 3.12 (b) or even by any $\mathcal{O}_{\mathcal{X}_{\mathbb{Q}}}$ -module isomorphism $\Omega^1 \simeq \omega^{\otimes 2}(-\text{cusps})$, any two of which differ by \mathbb{Q}^\times -scaling, identifies the spaces of oldforms on both sides, as may be checked over \mathbb{C} (see also [Gro90, 3.15]).

To proceed beyond the 2-torsion subgroup treated in Proposition 3.6, we begin by recording the following basic fact about the structure of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$. Its proof below is modeled on that of [Edi06, 2.7] given there in the $\Gamma_0(n)$ context (see also [DR73, VII.3.19–20] and [BDP16, B.3.2] for similar results). The use of the Atkin–Lehner involution in the proof is for convenience and compensates for the fact that Lemma 3.10 (a) is specific to the irreducible component $\mathcal{X}_{\mathbb{F}_2}^\mu$ of $\mathcal{X}_{\mathbb{F}_2}$.

Proposition 3.14. *The quotient $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ is killed by 4.*

Proof. As noted towards the end of §2.6, both $H^0(U^\mu, \Omega^1)$ and $H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ are $\mathbb{Z}_{(2)}$ -lattices inside $H^0(X_{\mathbb{Q}}, \Omega^1)$. Therefore, every element of their quotient is killed by a power of 2.

Due to the moduli interpretation of \mathcal{X} (combined with [KM85, 6.1.1 (1)] in the $\Gamma_1(n)$ -case), the Atkin–Lehner involution w_2 of $X_{\mathbb{Q}}$ extends (uniquely) to an involution of the elliptic curve locus of $X_{\mathbb{Z}_{(2)}}$, and this extension interchanges the generic points of the irreducible components of $X_{\mathbb{F}_2}$. Therefore, the automorphism of $H^0(X_{\mathbb{Q}}, \Omega^1)$ induced by w_2 respects the $\mathbb{Z}_{(2)}$ -lattice $H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ and interchanges $H^0(U^\mu, \Omega^1)$ and $H^0(U^{\text{ét}}, \Omega^1)$ (see the first paragraph of the proof of Proposition 3.6). Our task becomes showing that $H^0(U^{\text{ét}}, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ is killed by 4.

By Theorem A.4 (b) and Remark A.6 (see also §3.8), we have compatible identifications

$$H^0(U^{\text{ét}}, \Omega^1) \cong H^0(\mathcal{U}^{\text{ét}}, \Omega^1) \quad \text{and} \quad H^0(X_{\mathbb{Z}_{(2)}}, \Omega) \cong H^0(\mathcal{X}_{\mathbb{Z}_{(2)}}, \Omega), \quad (3.14.1)$$

so we may switch to working with stacks. The principal advantage in this is that due to its regularity, $\mathcal{X}_{\mathbb{Z}_{(2)}}$ admits a robust intersection theory formalism (see, for instance, [BDP16, §B.2.2]) analogous to the case of a proper regular arithmetic surface.

We fix an $f \in H^0(\mathcal{U}^{\text{ét}}, \Omega^1) \setminus H^0(\mathcal{X}_{\mathbb{Z}_{(2)}}, \Omega)$, let $m > 0$ be minimal such that $2^m f \in H^0(\mathcal{X}_{\mathbb{Z}_{(2)}}, \Omega)$, and seek to show that $m \leq 2$ by using the fact that $2^m f$ does not vanish on $\mathcal{X}_{\mathbb{F}_2}^\mu$ but vanishes to order at least m along $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$. Since the intersections of $\mathcal{X}_{\mathbb{F}_2}^\mu$ and $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ in $\mathcal{X}_{\mathbb{Z}_{(2)}}$ are transversal, the restriction of $2^m f$ to $\mathcal{X}_{\mathbb{F}_2}^\mu$ identifies with a nonzero global section of the line bundle

$$\Omega|_{\mathcal{X}_{\mathbb{F}_2}^\mu}(-m\mathcal{S}) \stackrel{3.12 \text{ (b)}}{\cong} \omega_{\mathcal{X}_{\mathbb{F}_2}^\mu}^{\otimes 2}(-\text{cusps} + (1-m)\mathcal{S}) \stackrel{3.10 \text{ (a)}}{\cong} \omega_{\mathcal{X}'_{\mathbb{F}_2}}^{\otimes 2}(-\text{cusps} + (1-m)\mathcal{S}'),$$

whose degree must therefore be nonnegative (we let \mathcal{S}' be the image of \mathcal{S} under $\pi_{\text{forg}}: \mathcal{X}_{\mathbb{F}_2}^\mu \xrightarrow{\sim} \mathcal{X}'_{\mathbb{F}_2}$). The sought $m \leq 2$ follows by taking into account the isomorphism $\omega_{\mathcal{X}'_{\mathbb{F}_2}} \cong \mathcal{O}_{\mathcal{X}'_{\mathbb{F}_2}}(\mathcal{S}')$ supplied by the Hasse invariant (see the proof of Lemma 3.10 (b)) and by recalling that ‘cusps’ $\neq \emptyset$. \square

Remark 3.15. By Proposition 3.6, every 2-torsion element of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}_{(2)}}, \Omega)$ lifts to an oldform in $H^0(U^\mu, \Omega^1)$, so Proposition 3.14 shows that the finite cyclic group

$$\frac{H^0(U^\mu, \Omega^1)}{H^0(X_{\mathbb{Z}_{(2)}}, \Omega) + H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp}$$

that appears in Theorem 2.10 is killed by 2. Therefore, Theorems 2.5 and 2.10 reprove a result of Mazur–Raynaud, [AU96, Prop. 3.1]: in the setting of §2.1, if $\text{ord}_2(n) \leq 1$, then the Manin–Stevens

constant c_π satisfies $\text{ord}_2(c_\pi) \leq 1$. Similarly to Remark 3.7, the distinction of this reproof is that it does not use exactness results for semiabelian Néron models.

We are ready to investigate the liftability to oldforms in $H^0(U^\mu, \Omega^1)$ of arbitrary elements of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$ (Proposition 3.6 only addressed elements killed by 2).

Theorem 3.16. *If the pullback map*

$$H^0(\mathcal{X}'_{\mathbb{Z}(2)}, \Omega^1) \rightarrow H^0(\mathcal{X}'_{\mathbb{F}_2}, \Omega^1)$$

is surjective, then every element of $H^0(U^\mu, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$ lifts to an oldform in $H^0(U^\mu, \Omega^1)$.

Proof. Similarly to the proof of Proposition 3.14, the Atkin–Lehner involution w_2 reduces us to showing that every element of

$$H^0(U^{\text{ét}}, \Omega^1)/H^0(X_{\mathbb{Z}(2)}, \Omega)$$

lifts to an oldform in $H^0(U^{\text{ét}}, \Omega^1)$, and we already know such liftability for 2-torsion elements due to Proposition 3.6. Moreover, the identifications (3.14.1) permit us to switch to working with stacks. In conclusion, we seek to show that for every

$$f_0 \in H^0(\mathcal{U}^{\text{ét}}, \Omega^1) \quad \text{such that} \quad 2f_0 \notin H^0(\mathcal{X}_{\mathbb{Z}(2)}, \Omega) \quad (3.16.1)$$

there exists some oldform $\tilde{f}_0 \in H^0(\mathcal{U}^{\text{ét}}, \Omega^1)$ for which $2(f_0 - \tilde{f}_0) \in H^0(\mathcal{X}_{\mathbb{Z}(2)}, \Omega)$.

We set $f := 4f_0$, so that, by Proposition 3.14 and (3.16.1), f is a global section of Ω on $\mathcal{X}_{\mathbb{Z}(2)}$ that vanishes to order 2 along $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$ but does not vanish on $\mathcal{X}_{\mathbb{F}_2}^\mu$. In particular, under the isomorphism

$$\Omega \stackrel{3.12 \text{ (b)}}{\cong} \omega^{\otimes 2}(-\text{cusps} + \mathcal{X}_{\mathbb{F}_2}^{\text{ét}}), \quad (3.16.2)$$

f lies in $\omega^{\otimes 2}(-\text{cusps})$ and vanishes on $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$, so its pullback to $H^0(\mathcal{X}_{\mathbb{F}_2}^\mu, \omega^{\otimes 2}(-\text{cusps}))$ lies in $H^0(\mathcal{X}_{\mathbb{F}_2}^\mu, \omega^{\otimes 2}(-\text{cusps} - \mathcal{S}))$. Therefore, Proposition 3.11 (with Lemma 3.12 (a)) supplies an oldform

$$\tilde{f} \in H^0(\mathcal{X}_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$$

that agrees with f on $\mathcal{X}_{\mathbb{F}_2}^\mu$ and vanishes on $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$. This \tilde{f} satisfies $f - \tilde{f} \in 2 \cdot H^0(\mathcal{X}_{\mathbb{Z}(2)}, \omega^{\otimes 2}(-\text{cusps}))$ and, when viewed as a global section of Ω via (3.16.2), is an oldform (see Remark 3.13) that vanishes to order at least 2 along $\mathcal{X}_{\mathbb{F}_2}^{\text{ét}}$. The oldform $\frac{\tilde{f}}{4}$ is then a sought \tilde{f}_0 . \square

The following lemma helps us recognize situations in which Theorem 3.16 applies, i.e., in which the surjectivity assumption holds.

Lemma 3.17. *Fix an odd $m \in \mathbb{Z}_{\geq 1}$.*

(a) *The pullback map*

$$H^0(\mathcal{X}_1(m)_{\mathbb{Z}(2)}, \Omega^1) \rightarrow H^0(\mathcal{X}_1(m)_{\mathbb{F}_2}, \Omega^1)$$

is surjective whenever $\mathcal{X}_1(m)_{\mathbb{Z}(2)}$ is a scheme, for instance, whenever $m > 3$.

(b) *The pullback map*

$$H^0(\mathcal{X}_0(m)_{\mathbb{Z}(2)}, \Omega^1) \rightarrow H^0(\mathcal{X}_0(m)_{\mathbb{F}_2}, \Omega^1)$$

is surjective if and only if so is the pullback map

$$H^0(X_0(m)_{\mathbb{F}_2}, \Omega^1) \rightarrow H^0(\mathcal{X}_0(m)_{\mathbb{F}_2}, \Omega^1), \quad (3.17.1)$$

and this is the case if m is a prime or if m has a prime factor q with $q \equiv 3 \pmod{4}$.

Proof.

- (a) Since m is odd, $\mathcal{X}_1(m)_{\mathbb{Z}(2)} \rightarrow \text{Spec } \mathbb{Z}(2)$ is proper and smooth of relative dimension 1 (see [DR73, IV.6.7]). Therefore, if $\mathcal{X}_1(m)_{\mathbb{Z}(2)}$ is a scheme, then the surjectivity in question follows from the formalism of Grothendieck–Serre duality and cohomology and base change (see [Con00, Thm. 5.1.2]). If $m > 3$, then $\mathcal{X}_1(m)_{\mathbb{Z}(2)}$ is a scheme by [KM85, 2.7.4] and [Čes17, 4.1.4].
- (b) On the level of coarse moduli schemes, the pullback map

$$H^0(X_0(m)_{\mathbb{Z}(2)}, \Omega^1) \rightarrow H^0(X_0(m)_{\mathbb{F}_2}, \Omega^1)$$

is surjective as in the proof of (a) (see §2.4 for basic properties of $X_0(m)_{\mathbb{Z}(2)}$). Therefore, the ‘if and only if’ claim follows from Remark A.5, which supplies the identification

$$H^0(X_0(m)_{\mathbb{Z}(2)}, \Omega^1) \cong H^0(\mathcal{X}_0(m)_{\mathbb{Z}(2)}, \Omega^1).$$

Granted that we address the case when there exists a suitable q , if m is a prime, then we may assume that $m \geq 5$, so that the surjectivity of (3.17.1) results from [Maz77, II.4.4 (1)].

For the rest of the proof we set $\mathcal{Z} := \mathcal{X}_0(m)_{\mathbb{F}_2}$ and $Z := X_0(m)_{\mathbb{F}_2}$ for brevity and recall that Z is the coarse moduli space of \mathcal{Z} because $2 \nmid m$ (see [Čes17, 6.4 (b)]). It suffices to show that

$$\Omega_{Z/\mathbb{F}_2}^1 \rightarrow \pi_* \Omega_{\mathcal{Z}/\mathbb{F}_2}^1 \quad (3.17.2)$$

induced by pullback along the coarse moduli scheme morphism $\pi: \mathcal{Z} \rightarrow Z$ is an isomorphism under the assumption of the existence of q . The proof of this is similar to the proof of Theorem A.4 (a), and the role of q is to ensure that the ramification of π is tame.

For every odd m , (3.17.2) is an isomorphism over the open $V \subset Z$ on which the j -invariant satisfies $j \neq 0$ because $\pi|_{\pi^{-1}(V)}$ is étale (see [Čes17, proof of Thm. 6.7]) so that (3.17.2) over V identifies with the $\Omega_{V/\mathbb{F}_2}^1$ -twist of the isomorphism $\mathcal{O}_V \xrightarrow{\sim} (\pi|_{\pi^{-1}(V)})_*(\mathcal{O}_{\pi^{-1}(V)})$. It remains to analyze (3.17.2) after base change to the completion $\hat{\mathcal{O}}_{Z,z}^{\text{sh}}$ of the strict Henselization of Z at a variable $z \in Z(\overline{\mathbb{F}}_2)$ with $j(z) = 0$. The \mathbb{F}_2 -smoothness of Z and \mathcal{Z} gives an isomorphism

$$\hat{\mathcal{O}}_{Z,z}^{\text{sh}} \simeq \overline{\mathbb{F}}_2[[t]] \quad \text{under which} \quad (\Omega_{Z/\mathbb{F}_2}^1)_{\hat{\mathcal{O}}_{Z,z}^{\text{sh}}} \simeq \overline{\mathbb{F}}_2[[t]] \cdot dt$$

and also, using the identification $Z(\overline{\mathbb{F}}_2) \cong \mathcal{Z}(\overline{\mathbb{F}}_2)$ to view z inside $\mathcal{Z}(\overline{\mathbb{F}}_2)$, an isomorphism

$$\hat{\mathcal{O}}_{\mathcal{Z},z}^{\text{sh}} \simeq \overline{\mathbb{F}}_2[[\tau]] \quad \text{under which} \quad (\Omega_{\mathcal{Z}/\mathbb{F}_2}^1)_{\hat{\mathcal{O}}_{\mathcal{Z},z}^{\text{sh}}} \simeq \overline{\mathbb{F}}_2[[\tau]] \cdot d\tau.$$

Moreover, with $G := \text{Aut}(z)/\{\pm 1\}$ we have compatible identifications

$$\overline{\mathbb{F}}_2[[t]] \cong (\overline{\mathbb{F}}_2[[\tau]])^G \quad \text{and} \quad (\pi_* \Omega_{\mathcal{Z}/\mathbb{F}_2}^1)_{\hat{\mathcal{O}}_{Z,z}^{\text{sh}}} \cong (\overline{\mathbb{F}}_2[[\tau]] \cdot d\tau)^G$$

with G acting faithfully on $\overline{\mathbb{F}}_2[[\tau]]$ (see [DR73, I.8.2.1] or [Ols06, 2.12]). If π is tamely ramified at z (i.e., if $2 \nmid \#G$), then $G \simeq \mu_{\#G}(\overline{\mathbb{F}}_2)$ and we may choose τ in such a way that $t = \tau^{\#G}$ and any $\zeta \in \mu_{\#G}(\overline{\mathbb{F}}_2)$ acts by $\tau \mapsto \zeta \cdot \tau$. Therefore, in the tamely ramified case the map $\overline{\mathbb{F}}_2[[t]] \cdot dt \rightarrow (\overline{\mathbb{F}}_2[[\tau]] \cdot d\tau)^G$ that identifies with the $\hat{\mathcal{O}}_{Z,z}^{\text{sh}}$ -pullback of (3.17.2) is an isomorphism.

To complete the proof we show that π is tamely ramified at z if some prime q with $q \equiv 3 \pmod{4}$ divides m . Let $E \rightarrow \text{Spec } \mathbb{F}_2$ be an elliptic curve that underlies $z \in \mathcal{Z}(\overline{\mathbb{F}}_2)$. The action of $\text{Aut}(z)$ on $E[q](\overline{\mathbb{F}}_2)$ is faithful (because $q \geq 3$) and preserves the Weil pairing and a cyclic subgroup C of order q . Thus, since a 2-Sylow subgroup of $\text{Aut}(z)$ acts semisimply, its action on C embeds it into $\text{Aut}(C) \simeq (\mathbb{Z}/q\mathbb{Z})^\times$. To conclude that the inclusion of $\{\pm 1\}$ into this

2-Sylow subgroup is an equality, as desired, it remains to note that $\#((\mathbb{Z}/q\mathbb{Z})^\times[2^\infty]) = 2$ because $q \equiv 3 \pmod{4}$. \square

Remarks.

3.18. The equivalent conditions of Lemma 3.17 (b) are also equivalent to the inequality

$$\dim_{\mathbb{F}_2} H^0(\mathcal{X}_0(m)_{\mathbb{F}_2}, \Omega^1) \leq g, \quad \text{where } g \text{ is the genus of } X_0(m)_{\mathbb{C}}.$$

To see this it suffices to note that $\dim_{\mathbb{F}_2} H^0(X_0(m)_{\mathbb{F}_2}, \Omega^1) = g$ and that the generic isomorphy of the map (3.17.2) ensures the injectivity of (3.17.1).

3.19. We are unaware of examples of odd m for which the equivalent conditions of Lemma 3.17 (b) fail to hold. The proof of Theorem 1.5 (i)–(ii) given in §§4.1–4.2 shows that if these conditions hold for $m = \frac{n}{2}$, then the Manin constant of any new elliptic optimal quotient of $J_0(n)$ is odd.

4. PROOFS OF THE MAIN RESULTS

With the results of §§2–3 at our possession, we are ready to present the proofs of Theorems 1.2 and 1.5. Most of the $p = 2$ cases of these theorems are proved in §§4.1–4.2, whereas the remaining cases are postponed until §§4.5–4.6 because they rely on a direct relationship between the conjectures of Manin and Stevens, a relationship captured by Lemma 4.4 and encapsulated by the formula (4.4.2).

4.1. Proof of Theorem 1.2 in the case $p = 2$. For a new elliptic optimal quotient

$$\pi: J_1(n) \twoheadrightarrow E \quad \text{with} \quad \text{ord}_2(n) \leq 1,$$

we seek to show that $\text{ord}_2(c_\pi) = 0$. Theorem 2.5 settles the case of an odd n , so we assume that $\text{ord}_2(n) = 1$. In this case, since $c_\pi \in \mathbb{Z}$ (see §2.1), Theorem 2.10 reduces us to showing that

$$\frac{H^0(U^\mu, \Omega^1)}{H^0(X_1(n)_{\mathbb{Z}(2)}, \Omega) + H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp} = 0, \tag{4.1.1}$$

where f is the normalized new eigenform that corresponds to π and $U^\mu \subset X_1(n)_{\mathbb{Z}(2)}$ is the complement of an irreducible component of $X_1(n)_{\mathbb{F}_2}$. Since the pullback map

$$H^0(\mathcal{X}_1(\frac{n}{2})_{\mathbb{Z}(2)}, \Omega^1) \rightarrow H^0(\mathcal{X}_1(\frac{n}{2})_{\mathbb{F}_2}, \Omega^1)$$

is surjective by Lemma 3.17 (a), Theorem 3.16 shows that $H^0(U^\mu, \Omega^1)/H^0(X_1(n)_{\mathbb{Z}(2)}, \Omega)$ consists of images of oldforms in $H^0(U^\mu, \Omega^1)$. To obtain (4.1.1), it therefore remains to note that every such oldform lies in $H^0(U^\mu, \Omega^1) \cap (\mathbb{Q} \cdot f)^\perp$. \square

4.2. Proof of Theorem 1.5 (i)–(ii). For a new elliptic optimal quotient

$$\pi: J_0(n) \twoheadrightarrow E \quad \text{with} \quad \text{ord}_2(n) \leq 1,$$

we seek to show that $\text{ord}_2(c_\pi) = 0$ whenever n has a prime factor q with $q \equiv 3 \pmod{4}$ and whenever $n = 2p$ for some prime p . The argument is the same as that of §4.1, except that we use Lemma 3.17 (b) in place of Lemma 3.17 (a). \square

The proof of Theorem 1.2 for an odd p is given in §4.5 and proceeds by reduction to $J_0(n)$, more precisely, to Proposition 4.3. Even though we have already proved this proposition in Remark 3.7, we also include its more standard proof based on exactness properties of semiabelian Néron models.

Proposition 4.3. *For a new elliptic optimal quotient $\pi: J_0(n) \twoheadrightarrow E$ and an odd prime p with $\text{ord}_p(n) \leq 1$, the Manin constant $c_\pi \in \mathbb{Z}$ satisfies $\text{ord}_p(c_\pi) = 0$.*

Proof. Since $J_0(n)$ has semiabelian reduction at p and $p-1 > 1$, [BLR90, 7.5/4 and its proof] ensure that π induces a smooth map $\pi_{\mathbb{Z}(p)} : \mathcal{J}_{\mathbb{Z}(p)} \rightarrow \mathcal{E}_{\mathbb{Z}(p)}$ on the Néron models over $\mathbb{Z}(p)$. Due to the resulting surjectivity of $\text{Lie}(\pi_{\mathbb{Z}(p)}) : \text{Lie } \mathcal{J}_{\mathbb{Z}(p)} \rightarrow \text{Lie } \mathcal{E}_{\mathbb{Z}(p)}$, the dual map $\iota : H^0(\mathcal{E}_{\mathbb{Z}(p)}, \Omega^1) \hookrightarrow H^0(\mathcal{J}_{\mathbb{Z}(p)}, \Omega^1)$ has a torsion free cokernel. Since $\text{Im}(\iota) = \mathbb{Z}(p) \cdot c_\pi f$, where f is the normalized newform that corresponds to π , to conclude that $\text{ord}_p(c_\pi) = 0$ it remains to recall that $f \in H^0(\mathcal{J}_{\mathbb{Z}(p)}, \Omega^1)$ (see Remark 2.8). \square

The following lemma supplies a direct relationship between Conjectures 1.1 and 1.3.

Lemma 4.4. *Let $\pi_0 : J_0(n) \rightarrow E_0$ and $\pi_1 : J_1(n) \rightarrow E_1$ be new elliptic optimal quotients that correspond to the same normalized new eigenform f .*

(a) *There is a unique isogeny e that fits into the commutative diagram*

$$\begin{array}{ccc} J_1(n) & \xrightarrow{\pi_1} & E_1 \\ \downarrow j^\vee & & \downarrow e \\ J_0(n) & \xrightarrow{\pi_0} & E_0 \end{array} \quad (4.4.1)$$

in which j^\vee is the dual of the pullback map $j : J_0(n) \rightarrow J_1(n)$. Moreover, the \mathbb{Q} -group scheme $\text{Ker } e$ is constant and is a quotient of the Cartier dual $\Sigma(n)^\vee$ of the Shimura subgroup

$$\Sigma(n) := \text{Ker} \left(J_0(n) \xrightarrow{j} J_1(n) \right).$$

(b) *The Manin constant c_{π_0} of π_0 and the Manin–Stevens constant c_{π_1} of π_1 are related by*

$$c_{\pi_0} = c_{\pi_1} \cdot \# \text{Coker} \left(H^0(\mathcal{E}_0, \Omega^1) \xrightarrow{e^*} H^0(\mathcal{E}_1, \Omega^1) \right), \quad (4.4.2)$$

where \mathcal{E}_0 and \mathcal{E}_1 are the Néron models of E_0 and E_1 over \mathbb{Z} . In particular, Conjecture 1.3 for π_0 implies Conjecture 1.1 for π_1 .

Proof.

(a) The existence of the unique e follows from the Hecke equivariance of $\pi_0 \circ j^\vee$. By [LO91, Thm. 2], the finite \mathbb{Q} -group $\Sigma(n)$ is of multiplicative type, so $\Sigma(n)^\vee$ is constant. Thus, it suffices to argue that $\text{Ker } e$ is a quotient of $\Sigma(n)^\vee$ or, since $\text{Ker } \pi_0$ is connected, that the component group of $\text{Ker}(j^\vee)$ is $\Sigma(n)^\vee$. The latter follows from the exact sequences

$$0 \rightarrow (\text{Coker } j)^\vee \rightarrow J_1(n) \xrightarrow{a} (\text{Im } j)^\vee \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \Sigma(n)^\vee \rightarrow (\text{Im } j)^\vee \xrightarrow{b} J_0(n) \rightarrow 0$$

in which $\text{Coker } j$ and $\text{Im } j$ are abelian varieties and $b \circ a = j^\vee$.

(b) The formula (4.4.2) follows from (4.4.1) once we note that the alternative description of (1.9.1) reviewed in §1.9 ensures that the j^\vee -pullback of f is f (see Remark 2.8). The last sentence follows from (4.4.2) because $c_{\pi_1} \in \mathbb{Z}$ (see §2.1). \square

4.5. Proof of Theorem 1.2 in the case of an odd p . For an odd prime p and a new elliptic optimal quotient

$$\pi_1 : J_1(n) \rightarrow E_1 \quad \text{with} \quad \text{ord}_p(n) \leq 1,$$

we seek to show that $\text{ord}_p(c_{\pi_1}) = 0$. For this, due to (4.4.2), it suffices to recall that $c_{\pi_1} \in \mathbb{Z}$ and to note that, by Proposition 4.3, the elliptic new optimal quotient $\pi_0 : J_0(n) \rightarrow E_0$ that corresponds to the same normalized new eigenform as π_1 has $\text{ord}_p(c_{\pi_0}) = 0$. \square

4.6. Proof of Theorem 1.5 (iii). For a new elliptic optimal quotient

$$\pi_0: J_0(n) \rightarrow E_0 \quad \text{with} \quad \text{ord}_2(n) \leq 1,$$

we seek to show that $\text{ord}_2(c_{\pi_0}) = 0$ whenever $E_0(\mathbb{Q})[2] = 0$. For this, in the notation of Lemma 4.4 and thanks to (4.4.2), it suffices to prove $2 \nmid \#\text{Ker } e$ because $\text{ord}_2(c_{\pi_1}) = 0$ by Theorem 1.2. However, $\text{Ker } e$ is constant, so it remains to note that E_1 , being isogenous to E_0 , satisfies $E_1(\mathbb{Q})[2] = 0$. \square

Remark 4.7. By [LO91, Cor. 2 on p. 173], if $n = 2 \cdot q^r$ for a prime q with either $q \equiv 3 \pmod{4}$ or $q \equiv 5 \pmod{8}$, then the order $\#\Sigma(n)$ of the Shimura subgroup is odd. Therefore, since Lemma 4.4 (a) ensures that $\text{Ker } e$ is a quotient of $\Sigma(n)^\vee$, for such n the argument of §4.6 shows that $\text{ord}_2(c_{\pi_0}) = 0$ for every new elliptic optimal quotient $\pi_0: J_0(n) \rightarrow E_0$.

APPENDIX A. THE “RELATIVE DUALIZING SHEAF” OF \mathcal{X}_H AND OF ITS COARSE SPACE

The main goal of this appendix is to prove a certain comparison result between the relative dualizing sheaf on the modular curve X_H and an analogous sheaf on \mathcal{X}_H . This is accomplished in Theorem A.4 after introducing the relevant sheaf on \mathcal{X}_H in §A.1 and detailing some of its properties in §A.3. As the proof of Proposition 3.14 illustrates, the practical role of Theorem A.4 is to facilitate passage between X_H and \mathcal{X}_H in the study of integral structures on spaces of weight 2 cusp forms (a link between the latter and the relative dualizing sheaf is supplied by Kodaira–Spencer, see Lemma 3.12).

A.1. “Relative dualizing sheaves” of Deligne–Mumford stacks. For a scheme S and a Cohen–Macaulay (and hence flat) morphism $X \rightarrow S$ that has a pure relative dimension, the theory of Grothendieck duality associates a quasi-coherent, locally finitely presented, S -flat relative dualizing \mathcal{O}_X -module $\Omega_{X/S}$ (see [Con00, bottom halves of p. 157 and p. 214]), which identifies with the determinant of $\Omega_{X/S}^1$ if $X \rightarrow S$ is in addition smooth. The formation of $\Omega_{X/S}$ is compatible with étale localization on X : for every étale S -morphism $f: X' \rightarrow X$ one has a canonical isomorphism

$$\iota_f: f^*(\Omega_{X/S}) \xrightarrow{\sim} \Omega_{X'/S} \tag{A.1.1}$$

supplied by [Con00, Thm. 4.3.3 and bottom half of p. 214]. Moreover, if $f': X'' \rightarrow X'$ is a further étale S -morphism, then [Con00, (4.3.7) and bottom half of p. 214] supply the following compatibility between the isomorphisms of (A.1.1):

$$\iota_{f \circ f'} = \iota_{f'} \circ ((f')^*(\iota_f)): (f')^*(f^*(\Omega_{X/S})) \xrightarrow{\sim} \Omega_{X''/S}. \tag{A.1.2}$$

Therefore, if \mathcal{X} is a Deligne–Mumford stack over S such that $\mathcal{X} \rightarrow S$ Cohen–Macaulay and has a pure relative dimension, then the compatibilities (A.1.2) ensure that the \mathcal{O}_X -modules $\Omega_{X/S}$ for étale morphisms $X \rightarrow \mathcal{X}$ from a scheme X glue to form a quasi-coherent, locally of finite presentation, S -flat $\mathcal{O}_{\mathcal{X}}$ -module

$$\Omega_{\mathcal{X}/S}, \quad \text{the “relative dualizing sheaf” of} \quad \mathcal{X} \rightarrow S$$

(see [LMB00, 12.2.1] for a discussion of analogous compatibilities). If $\mathcal{X} \rightarrow S$ is in addition smooth, then $\Omega_{\mathcal{X}/S}$ identifies with the determinant of $\Omega_{\mathcal{X}/S}^1$. Due to [Con00, Thm. 4.4.4 and bottom half of p. 214], the formation of $\Omega_{\mathcal{X}/S}$ commutes with base change in S .

Remark A.2. In the case when $\mathcal{X} \rightarrow S$ is proper (and \mathcal{X} is not a scheme), we *do not* claim any dualizing properties of the $\mathcal{O}_{\mathcal{X}}$ -module $\Omega_{\mathcal{X}/S}$ constructed in §A.1. Nevertheless, if a sufficiently robust Grothendieck–Serre duality formalism for $\mathcal{X}_0(m)_{\mathbb{Z}_{(2)}} \rightarrow \text{Spec } \mathbb{Z}_{(2)}$ with $2 \nmid m$ existed, then it would prove the surjectivity in Lemma 3.17 (b) without assuming the primality of m or the existence of q (see the proof of Lemma 3.17 (a)), which would settle the semistable case of the Manin conjecture (see Remark 3.19).

A.3. The case of modular curves. For us, the key case in §A.1 is when $S = \text{Spec } \mathbb{Z}$ and \mathcal{X} is either a modular stack \mathcal{X}_H or its coarse moduli scheme X_H for some open subgroup $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$ (see §1.8), as we now assume. The resulting $\mathcal{X} \rightarrow S$ is flat, of finite presentation, purely of relative dimension 1, and Cohen–Macaulay (the latter due to the normality of \mathcal{X} and [EGA IV₂, 6.3.5 (i)]), so the discussion of §A.1 applies. Moreover, [EGA IV₂, 6.12.6 (i)] and the normality of \mathcal{X} ensure that after removing finitely many closed points \mathcal{X} becomes regular and hence also a local complete intersection over \mathbb{Z} (see [Liu02, 6.3.18]). In particular, each of the finitely many nonsmooth \mathbb{Z} -fibers of \mathcal{X} has a dense open Gorenstein locus. The resulting coherent \mathbb{Z} -flat $\mathcal{O}_{\mathcal{X}}$ -module $\Omega_{\mathcal{X}/\mathbb{Z}}$ is therefore a line bundle on a \mathbb{Z} -fiberwise dense open of \mathcal{X} . It then follows from [EGA IV₂, 6.4.1 (ii)] and from the proof of [Con00, Lem. 5.2.1] (carried out for the compactification of an étale scheme cover of a \mathbb{Z} -fiber of \mathcal{X}) that $\Omega_{\mathcal{X}/\mathbb{Z}}$ is Cohen–Macaulay⁵ on the entire \mathcal{X} .

With the discussion of §A.3, we are ready for the promised comparison result.

Theorem A.4. *Fix an open subgroup $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$ and let $\pi: \mathcal{X}_H \rightarrow X_H$ be the coarse moduli space morphism.*

(a) *Pullback of Kähler differentials along $\pi_{\mathbb{Q}}$ induces an $\mathcal{O}_{(X_H)_{\mathbb{Q}}}$ -module isomorphism*

$$\Omega_{(X_H)_{\mathbb{Q}}/\mathbb{Q}}^1 \xrightarrow{\sim} (\pi_{\mathbb{Q}})_*(\Omega_{\mathcal{X}_H/\mathbb{Q}}^1). \quad (\text{A.4.1})$$

(b) *For every open subscheme $U \subset X_H$ with $\mathcal{U} := \pi^{-1}(U)$ such that $\pi|_{\mathcal{U}}: \mathcal{U} \rightarrow U$ is étale over a \mathbb{Z} -fiberwise dense open of U , the isomorphism $H^0(U_{\mathbb{Q}}, \Omega^1) \cong H^0(\mathcal{U}_{\mathbb{Q}}, \Omega^1)$ of (a) identifies*

$$H^0(U, \Omega) \subset H^0(U_{\mathbb{Q}}, \Omega^1) \quad \text{with} \quad H^0(\mathcal{U}, \Omega) \subset H^0(\mathcal{U}_{\mathbb{Q}}, \Omega^1).$$

Proof.

(a) Let $V \subset (X_H)_{\mathbb{Q}}$ be a dense open over which $\pi_{\mathbb{Q}}$ is étale, and set $\mathcal{V} := (\pi_{\mathbb{Q}})^{-1}(V)$. The restriction of (A.4.1) to V identifies with the $\Omega_{V/\mathbb{Q}}^1$ -twist of the isomorphism $\mathcal{O}_V \xrightarrow{\sim} (\pi_{\mathbb{Q}}|_{\mathcal{V}})_*(\mathcal{O}_{\mathcal{V}})$, so is an isomorphism. It remains to prove that the base change of (A.4.1) to the completion $\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}}$ of the strict Henselization of $(X_H)_{\mathbb{Q}}$ at a variable $x \in X_H(\overline{\mathbb{Q}})$ is an isomorphism.

We have an isomorphism

$$\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}} \simeq \overline{\mathbb{Q}}[[t]] \quad \text{under which} \quad (\Omega_{(X_H)_{\mathbb{Q}}/\mathbb{Q}}^1)_{\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}}} \simeq \overline{\mathbb{Q}}[[t]] \cdot dt,$$

and also (using the identification $X_H(\overline{\mathbb{Q}}) \cong \mathcal{X}_H(\overline{\mathbb{Q}})$ to view x inside $\mathcal{X}_H(\overline{\mathbb{Q}})$)

$$\widehat{\mathcal{O}}_{(\mathcal{X}_H)_{\mathbb{Q}}, x}^{\text{sh}} \simeq \overline{\mathbb{Q}}[[\tau]] \quad \text{under which} \quad (\Omega_{(\mathcal{X}_H)_{\mathbb{Q}}/\mathbb{Q}}^1)_{\widehat{\mathcal{O}}_{(\mathcal{X}_H)_{\mathbb{Q}}, x}^{\text{sh}}} \simeq \overline{\mathbb{Q}}[[\tau]] \cdot d\tau.$$

Taking into account the action of the automorphism group of $x \in \mathcal{X}_H(\overline{\mathbb{Q}})$, we have, compatibly,

$$\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}} \cong (\widehat{\mathcal{O}}_{(\mathcal{X}_H)_{\mathbb{Q}}, x}^{\text{sh}})^G \quad \text{and} \quad ((\pi_{\mathbb{Q}})_*(\Omega_{(\mathcal{X}_H)_{\mathbb{Q}}/\mathbb{Q}}^1))_{\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}}} \cong ((\Omega_{(\mathcal{X}_H)_{\mathbb{Q}}/\mathbb{Q}}^1)_{\widehat{\mathcal{O}}_{(\mathcal{X}_H)_{\mathbb{Q}}, x}^{\text{sh}}})^G$$

for a certain group G acting faithfully on $\widehat{\mathcal{O}}_{(\mathcal{X}_H)_{\mathbb{Q}}, x}^{\text{sh}}$ (see [DR73, I.8.2.1] or [Ols06, 2.12]).

Moreover, the ramification of $\pi_{\mathbb{Q}}$ is tame, so we may assume that $G \simeq \mu_{\#G}(\overline{\mathbb{Q}})$ and that $\zeta \in \mu_{\#G}(\overline{\mathbb{Q}})$ acts by $\tau \mapsto \zeta \cdot \tau$ with $t = \tau^{\#G}$. It then follows that $\overline{\mathbb{Q}}[[t]] \cdot dt \xrightarrow{\sim} (\overline{\mathbb{Q}}[[\tau]] \cdot d\tau)^G$, i.e., that the base change of (A.4.1) to $\widehat{\mathcal{O}}_{(X_H)_{\mathbb{Q}}, x}^{\text{sh}}$ is an isomorphism.

⁵The S -fibral Cohen–Macaulayness of $\Omega_{\mathcal{X}/S}$ is actually a general fact.

- (b) Let $U' \subset U$ be a \mathbb{Z} -fiberwise dense open over which π is étale and let $\mathcal{U}' \subset \mathcal{U}$ be its preimage. By §A.3, the \mathcal{O}_{X_H} -module $\Omega_{X_H/\mathbb{Z}}$ is (S₂), and likewise for $\Omega_{\mathcal{X}_H/\mathbb{Z}}$, so, due to [EGA IV₂, 5.10.5],

$$\begin{aligned} H^0(U, \Omega) &= H^0(U', \Omega) \cap H^0(U_{\mathbb{Q}}, \Omega^1) && \text{inside } H^0(U'_{\mathbb{Q}}, \Omega^1), && \text{and} \\ H^0(\mathcal{U}, \Omega) &= H^0(\mathcal{U}', \Omega) \cap H^0(\mathcal{U}_{\mathbb{Q}}, \Omega^1) && \text{inside } H^0(\mathcal{U}'_{\mathbb{Q}}, \Omega^1). \end{aligned}$$

Therefore, (a) reduces us to the case when $U = U'$. Moreover, the (S₂) property ensures that neither $H^0(U, \Omega)$ nor $H^0(\mathcal{U}, \Omega)$ changes if we remove finitely many closed points from U , so, thanks to §A.3, we assume further that U and \mathcal{U} are regular and that $\Omega_{U/\mathbb{Z}}$ and $\Omega_{\mathcal{U}/\mathbb{Z}}$ are line bundles. Then, due to the étaleness of $\pi|_{\mathcal{U}}$, we have $(\pi|_{\mathcal{U}})^*(\Omega_{U/\mathbb{Z}}) \cong \Omega_{\mathcal{U}/\mathbb{Z}}$ (see (A.1.1)), so that, since $\Omega_{U/\mathbb{Z}}$ is a line bundle, the resulting pullback map

$$\Omega_{U/\mathbb{Z}} \rightarrow (\pi|_{\mathcal{U}})_*(\Omega_{\mathcal{U}/\mathbb{Z}}) \tag{A.4.2}$$

identifies with the $\Omega_{U/\mathbb{Z}}$ -twist of the isomorphism $\mathcal{O}_U \xrightarrow{\sim} (\pi|_{\mathcal{U}})_*(\mathcal{O}_{\mathcal{U}})$ and hence is an isomorphism. The sought claim then follows by taking global sections in (A.4.2). \square

Remarks.

- A.5.** If H contains $\text{Ker}(\text{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \text{GL}_2(\mathbb{Z}/n\mathbb{Z}))$, then the \mathbb{Z} -fibrational generic étaleness assumption of Theorem A.4 (b) holds for every U on which n is invertible (see [Čes17, last paragraph of the proof of Prop. 6.4 (b)]). In particular, since \mathcal{X}_H and X_H are $\mathbb{Z}[\frac{1}{n}]$ -smooth (see [DR73, IV.6.7 and VI.6.7] or also [Čes17, 6.4 (a)]), Theorem A.4 proves that pullback induces an isomorphism

$$\Omega_{(X_H)_{\mathbb{Z}[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]}^1 \xrightarrow{\sim} (\pi_{\mathbb{Z}[\frac{1}{n}]})_*(\Omega_{(\mathcal{X}_H)_{\mathbb{Z}[\frac{1}{n}]/\mathbb{Z}[\frac{1}{n}]}^1).$$

- A.6.** An important case in which the \mathbb{Z} -fibrational generic étaleness assumption of Theorem A.4 (b) holds for every U is when $H = \Gamma_0(n)$ (see [Čes17, proof of Thm. 6.7]). In this case, Theorem A.4 provides an $\mathcal{O}_{X_0(n)}$ -module isomorphism

$$\Omega_{X_0(n)/\mathbb{Z}} \xrightarrow{\sim} \pi_*(\Omega_{\mathcal{X}_0(n)/\mathbb{Z}})$$

which on the \mathbb{Q} -fiber is induced by pullback of Kähler differentials.

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