Invariance under Henselian pairs for flat cohomology

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The goal of this talk is to present results from [BČ20] and [ČS20] that established invariance under Henselian pairs for several cohomological functors, see Theorem 2.1. For context, we first explain a common utility of this invariance.

1. The role of Henselian pairs in algebraization and approximation

Definition 1.1. A pair (A, I) consisting of a commutative ring A and an ideal $I \subset A$ is *Henselian* if for every affine, étale A-scheme X, we have

$$X(A) \twoheadrightarrow X(A/I).$$

Example 1.2. If A is I-adically complete in the sense that $A \xrightarrow{\sim} \varprojlim_{n>0}(A/I^n)$, then (A, I) is Henselian. If A is merely derived I-adically complete in the sense that $A \xrightarrow{\sim} R\lim_{n>0}(A/^{\mathbb{L}}a^n)$ for $a \in I$, then (A, I) is still Henselian [ČS20, Lem. 5.6.2].

The properties of (A, I) discussed in Example 1.2 all depend only on the nonunital ring I. For example, by [Gab92, Prop. 1] or [SP, 09XI], the pair (A, I) is Henselian if and only if elements of $1 + I$ have multiplicative inverses and every polynomial

$$T^n(T-1) + a_nT^n + \ldots + a_1T + a_0 \quad \text{with} \quad a_n, \ldots, a_0 \in I$$

has a (necessarily unique) root in $1 + I$.

We seek to exhibit *invariant under Henselian pairs* functors F in the sense that $F(A) \xrightarrow{\sim} F(A/I)$ for Henselian pairs (A, I) such that F is defined on A and A/I.

Example 1.3 ([Gab94, Thm. 1])**.** For an abelian, torsion sheaf $\mathscr{F}$ on the étale site of some commutative ring A_0, on the category of A_0-algebras A, the functor

$$A \mapsto R\Gamma_{\text{ét}}(A, \mathscr{F}) \quad \text{is invariant under Henselian pairs.}$$

In Theorem 2.1 (b), we will establish a similar property of fppf cohomology.

The following result of Gabber shows that the functors that are invariant under Henselian pairs behave well with respect to algebraization and approximation.

Theorem 1.4 (Gabber (unpublished), [BČ20, Thm. 2.1.16])**.** *Let B be a topological ring that has a Henselian open nonunital subring $B' \subset B$ whose induced topology has a neighborhood base of 0 consisting of ideals of B'. If a functor*

$$F\colon B\text{-}algebras \to Sets$$

commutes with filtered direct limits and is invariant under Henselian pairs, then

$$F(B) \xrightarrow{\sim} F(\widehat{B}). \tag{1.4.1}$$

Here $\widehat{B}$ is the completion of the topological ring B. For example, we could have

(1) $B := R\{t\}[\frac{1}{t}]$ with $B' := tR\{t\}$, where R is a commutative ring and $R\{t\}$ is the Henselization of $R[t]$ along $\{t = 0\}$, so that $\widehat{B} \cong R((t))$; or
(2) B is a Henselian Huber ring as defined in [Hub96, Def. 3.1.2] with B' an ideal of definition in a ring of definition, so that $\widehat{B}$ is a complete Huber ring; or
(3) $B := A$ with $B' := I$ for some Henselian pair (A, I) such that B' with its coarse topology is open in B, so that $\widehat{B} \cong A/I$.

The idea of the proof of Theorem 1.4. We let S be a neighborhood base of 0 in B considered as a poset with the order $U \leq U'$ iff $U' \subset U$, and we consider the ring

$$\mathrm{Cauchy}_S(B) := \{\text{germs of Cauchy nets } f\colon S \to B\}$$

and its ideal

$$\mathrm{Null}_S(B) := \{\text{germs of null nets } f\colon S \to B\}.$$

The nonunital ring $\mathrm{Null}_S(B)$ is Henselian because it agrees with $\mathrm{Null}_S(B')$ which, in turn, is an ideal in $\varinjlim_{U\in S} \left(\prod_{S_{\geq U}} B'\right)$. The identification

$$\widehat{B} \cong \mathrm{Cauchy}_S(B)/\,\mathrm{Null}_S(B)$$

then serves as a basic link between (1.4.1) and invariance under Henselian pairs.

This technique based on rings of Cauchy nets also leads to a reproof and a non-Noetherian generalization of the Elkik approximation theorem, see [BČ20, §2.2].

2. Flat cohomology and invariance under Henselian pairs

The following is the promised invariance under Henselian pairs for flat cohomology.

Theorem 2.1. *Let (A, I) be a Henselian pair.*

(a) ([BČ20, Prop. 2.1.4, Thm. 2.1.7]). *For a smooth, quasi-compact, algebraic A-stack $\mathscr{X}$ whose diagonal is quasi-affine,*

$$\mathscr{X}(A) \twoheadrightarrow \mathscr{X}(A/I).$$

In particular, for a smooth, quasi-affine A-group G,

$$H^1_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(A/I, G),$$
$$\mathrm{Ker}(H^2_{\mathrm{fppf}}(A, G) \to H^2_{\mathrm{fppf}}(A/I, G)) = \{*\},$$

where the H^1 and H^2 are interpreted in terms of torsors and gerbes.

(b) ([ČS20, Cor. 5.6.8]). *For a commutative, finite, locally free A-group G,*

$$H^i_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} H^i_{\mathrm{fppf}}(A/I, G) \quad \textit{for} \quad i \geq 2.$$

(c) ([ČS20, Thm. 5.6.5]). *If A is derived I-adically complete, $I = (a_1, \ldots, a_r)$ is finitely generated, and G is as in (b), then*

$$R\Gamma_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} R\lim_{n>0}(R\Gamma_{\mathrm{fppf}}(A/^{\mathbb{L}}(a_1^n, \ldots, a_r^n), G)). \qquad (2.1.1)$$

In particular, if A is I-adically complete, I is finitely generated, and G is as in (b), then we have a short exact sequence

$$0 \to \varprojlim^1_{n>0}(G(A/I^n)) \to H^1(A, G) \to \varprojlim_{n>0}(H^1(A/I^n, G)) \to 0.$$

The continuity formula (2.1.1) continues to hold when A is merely an animated ring in the sense of [ČS20, §5.1], and this added generality is crucial for the proof.

The map in (b) for $i = 1$ is still surjective but no longer injective: for instance,

$$H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mu_p) \ncong 0 \quad \text{but} \quad H^1_{\mathrm{fppf}}(\mathbb{F}_p, \mu_p) \cong 0.$$

3. An overview of the proof of Theorem 2.1

In (a), one begins with an affine $\mathscr{X}$, for which one uses the local structure of smooth morphisms to eventually reduce to affine étale $\mathscr{X}$ (see [Gru72, I.8]) covered by Definition 1.1. One combines the affine case, limit arguments, and Popescu's theorem [SP, 07GC] to reduce to Noetherian, I-adically complete A. One concludes by combining the infinitesimal smoothness criterion with the formula

$$\mathscr{X}(A) \xrightarrow{\sim} \varprojlim_{n>0} \mathscr{X}(A/I^n)$$

that follows from Tannaka duality for algebraic stacks settled in [BHL17] or [HR19].

An analogous passage to the Noetherian, complete case reduces (b) to (c), except that (b) for $i = 2$ is actually an input to (c). One deduces this low degree case from (a) by combining the Bégueri sequence $0 \to G \to \mathrm{Res}_{G^*/A}(\mathbb{G}_m) \to Q \to 0$, the identification $\mathrm{Br}(R) \xrightarrow{\sim} H^2(R, \mathbb{G}_m)_{\mathrm{tors}}$ due to Gabber [Gab81, Ch. II, Thm. 1], and the definition of the Brauer group $\mathrm{Br}(-)$ in terms of PGL_n-torsors.

In (c), one assumes at the outset that A is animated and reduces to $r = 1$ with $a := a_1$ and G of p-power order for a prime p. One then uses the $i = 2$ case of (b) to establish (2.1.1) "by hand" when the appearing $H^{\geq 3}_{\mathrm{fppf}}$ all vanish. Bounds on the p-cohomological dimension of $\mathbb{F}_p$-algebras (essentially, the Artin–Schreier sequence) ensure that this includes the case when A is an $\mathbb{F}_p$-algebra. One then deduces (2.1.1) for p-Henselian A by combining deformation theory with the p-adic continuity formula [ČS20, Thm. 5.3.5], which, essentially, is the $a = p$ case of (2.1.1) and is simpler because $R\Gamma_{\text{ét}}((-)^h_{(p)}[\frac{1}{p}], G)$ satisfies p-complete arc hyperdescent [BM20, Cor. 6.17]. With this in hand, the idea is to show a-completely faithfully flat hyperdescent for the functor $A \mapsto R\Gamma_{\mathrm{fppf}}(A, G)$ on a-adically complete inputs by replacing A by its p-Henselization and using excision both for flat cohomology [ČS20, Thm. 5.4.4] and for étale cohomology [BM20, Thm. 5.4] to reduce the study of the flat cohomology of G to that of the étale cohomology of $j_!(G)$ with $j\colon \mathrm{Spec}(A[\frac{1}{p}]) \hookrightarrow \mathrm{Spec}(A)$, for which Example 1.3 applies. The acquired a-completely faithfully flat hyperdescent allows one to replace A by the terms A^i of a large a-completely faithfully flat hypercover, constructed so that each A^i has no nonsplit étale covers. This last property ensures that all the appearing $H^{\geq 2}_{\mathrm{fppf}}$ all vanish, to the effect that one is in the case that was already settled "by hand."

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