INTEGRAL RELATIONS BETWEEN p-ADIC COHOMOLOGY THEORIES

A PLAN FOR THE UC BERKELEY NUMBER THEORY LEARNING SEMINAR, SPRING 2017

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The goal of the seminar is to discuss the relations that result from the techniques of [BMS16] between different integral cohomology theories (p-adic étale, de Rham, crystalline...) of varieties over p-adic fields. Such relations may be viewed as refinements of the comparison theorems of "rational" p-adic Hodge theory—these comparisons generally do not preserve integral structures.

0. Introduction. (References: [BMS15].)

After giving an overview, I will distribute the lectures to volunteers.

1. Algebraic de Rham cohomology. (References: [Ked08, §1], [Hai14], [DI87].)

Define the de Rham complex of an algebraic variety over an arbitrary field. Review necessary background on derived categories and introduce algebraic de Rham cohomology. Discuss the Hodge–de Rham spectral sequence and the Hodge filtration on H^i_{dR}. Point out the relative versions of the formal aspects of the theory. Discuss selected material from [DI87], especially, the degeneration of the Hodge–de Rham spectral sequence in characteristic 0.

2. Crystalline cohomology. (References: [CL98, I.§2], [Ill94, §§1–2], [BO83, esp., (2.5)], [BC09, §§7.2–7.3].)

Introduce divided power structures. Review the definition and the basic properties of crystalline cohomology of schemes in characteristic p. Discuss the crystalline–de Rham comparison isomorphism. Illustrate the theory with the case of abelian varieties. Mention connections with Dieudonné modules.

3. Rational comparisons between p-adic cohomology theories. (References: [Bha16, §2.3], [BMS16, §1.1, §3.1, §3.3] (possibly also [Čes16, §3.1]), [Ill94, §3], [Tsu02].)

Define Fontaine's ring A_{inf} and discuss its basic properties. Discuss the étale, de Rham, and crystalline "specializations" of A_{inf}. Introduce the rings B_{dR} and B_{cris}. Discuss the cohomological versions of the de Rham–étale and crystalline–étale comparison isomorphisms of p-adic Hodge theory.

4. The functor $L\eta$. (References: [Bha16, §5], [BMS16, §6], [Mor16, §2], [SP, 00X9, 01D2, 0940], [BS15, §§3.4–3.5].)

Discuss the definition of a ringed topos $(\mathcal{T}, R)$ and of morphisms between such. Briefly mention the notion of a replete topos. Discuss the derived category of R-modules, derived tensor products, derived p-adic completions. Introduce the décalage functor $L\eta$ and discuss its properties. Work in the setting of ringed topoi but stress the basic case of usual rings.

5. The étale specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$. (References: [Sch13, §3, Lem. 4.10 (v), §6], [Sch13e], [BMS16, §5.1, Def. 9.1, Thm. 14.3 (iv)], [Mor16, §4].)

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Introduce the proétale site of a locally Noetherian adic space. After specializing to the p-adic case, introduce various sheaves on this site, especially, $\widehat{\mathcal{O}}_X^+$ and $\mathbb{A}_{\mathrm{inf},X}$. Mention the almost purity theorem. Define $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ and, in the proper case, identify its étale specialization.

6. **The cotangent complex and its derived p-adic completion.** (References: [Ill05, §8.5.G], [Ill71, Ch. II], [LMB00, §17.1], [BMS16, Lem. 3.14], [Bha16, §6.2], [Česn16, §§2.18–2.20].)

   Review the definition and the basic properties of the cotangent complex (focus on the case of a ring morphism but mention that everything works for ringed topoi). Sketch the proof of the vanishing of $\widehat{\mathbb{L}}_{R'/R}$ in the perfectoid case. Review Fontaine's computation of $T_p(\Omega_{\mathcal{O}_C/\mathbb{Z}_p})$ and introduce $\mathcal{O}_C\{1\}$. Construct the comparison map
$$\widehat{\mathbb{L}}_{\mathfrak{X}/\mathbb{Z}_p}\{-1\}[-1] \to R\nu_*\widehat{\mathcal{O}}_X^+.$$

7. **The de Rham specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$.** (References: [BMS16, §§7–8, Thm. 14.1 (ii), Thm. 14.3 (ii)], possibly also [Čes16, §2 and Thm. 4.4] (specialize to the smooth case).)

   For a smooth $\mathcal{O}_C$-scheme $\mathcal{X}$, introduce the object $\widetilde{\Omega}_{\mathcal{X}} \in D^{\geqslant 0}(\mathcal{O}_{\widehat{\mathcal{X}}})$. Review the computation of continuous group cohomology via Koszul complexes. Sketch the proof of the identification $H^i(\widetilde{\Omega}_{\mathcal{X}}) \cong \widehat{\Omega}^i_{\mathcal{X}/\mathcal{O}_C}\{-i\}$. Identify the de Rham specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$.

8. **The relative de Rham–Witt complex.** (References: [CL98, I.§§3–4], [BMS16, §10], [Mor16, §6], [LZ04].)

   Review the theory of the de Rham–Witt complex, highlighting connections to crystalline cohomology. Introduce the relative de Rham–Witt complex of Langer–Zink and discuss its properties.

9. **The crystalline specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$.** (References: [BMS16, §9, §11, Thm. 14.1 (i), Thm. 14.3 (i)], [Mor16, §5 and §7].)

   Overview the proof of the identification of the crystalline specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$.

10. **Integral relations between p-adic cohomology theories.** (References: [BMS16, §2, §4, Rem. 14.4, Thm. 14.5 (ii)], [Mor16, §1.1], [Bha16, §2.4], possibly also [Čes16, §4] (specialize to the smooth case).)

    Use the theory discussed in the previous talks to deduce integral relations between torsion in different p-adic cohomology theories. Discuss examples that illustrate sharpness of the relations. Prove that H^i_{dR} is torsion free if and only if H^i_{cris} is torsion free.

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