GROTHENDIECK–SERRE IN THE QUASI-SPLIT UNRAMIFIED CASE

KĘSTUTIS ČESNAVIČIUS

Abstract. The Grothendieck–Serre conjecture predicts that every generically trivial torsor under a reductive group scheme G over a regular local ring R is trivial. We settle it in the case when G is quasi-split and R is unramified. Some of the techniques that allow us to overcome obstacles that have so far kept the mixed characteristic case out of reach include Artin's construction of "good neighborhoods" carried out over discrete valuation rings, equivariant compactifications of tori over higher-dimensional bases, and the geometry of the affine Grassmannian in bad characteristics.

1. The Grothendieck–Serre conjecture

The subject of this article is the following conjecture, due to Serre [Ser58, p. 31, Rem.] and Grothendieck [Gro58, pp. 26–27, Rem. 3], [Gro68, Rem. 1.11 a)], on triviality of torsors under reductive groups.

Conjecture 1.1 (Grothendieck–Serre)**.** *For a regular local ring R and a reductive R-group scheme G, no nontrivial G-torsor trivializes over the fraction field of R, in other words,*

$$\mathrm{Ker}\left(H^1(R,G) \to H^1(\mathrm{Frac}(R),G)\right) = \{*\}.$$

CNRS, Université Paris-Saclay, Laboratoire de mathématiques d'Orsay, F-91405, Orsay, France
E-mail address: kestutis@math.u-psud.fr.
Date: April 30, 2021.
2020 *Mathematics Subject Classification.* Primary 14L15; Secondary 11E81, 14M17, 20G10.
Key words and phrases. Affine Grassmannian, Bertini, compactification, reductive group, regular ring, torsor.

The conjecture is settled when R contains a field but its remaining mixed characteristic case has so far been widely open: see the recent survey [Pan18] for a detail review of the state of the art, as well as §1.4 below for a summary. The goal of this article is to resolve the mixed characteristic case under the assumption that R is unramified and G is quasi-split. We recall that a regular local ring $(R, \mathfrak{m})$ with $p := \mathrm{char}(R/\mathfrak{m})$ is *unramified* if either R contains a field or $p \notin \mathfrak{m}^2$.

Theorem 1.2. *For an unramified regular local ring R and a quasi-split reductive R-group scheme G,*

$$\mathrm{Ker}\left(H^1(R, G) \to H^1(\mathrm{Frac}(R), G)\right) = \{*\};$$

moreover, a reductive R-group scheme H is split if and only if its generic fiber $H_{\mathrm{Frac}(R)}$ is split.

In fact, our result is stronger: we allow the regular ring R to be semilocal and geometrically regular over some Dedekind ring $\mathcal{O}$, so that the case $\mathcal{O} = \mathbb{Z}$ with R local recovers the above, see Theorems 5.3.1 and 5.3.3 and Example 5.3.2. The semilocal version is worth the extra effort because in many ways it is a more natural starting point. In equal characteristic, we strengthen the last aspect of Theorem 1.2: for an equicharacteristic semilocal regular R, we show that a reductive R-group scheme H is quasi-split if and only if its generic fiber $H_{\mathrm{Frac}(R)}$ is quasi-split, see Theorem 5.3.5.

The Grothendieck–Serre conjecture is known for its numerous concrete consequences. We illustrate them with the following product formula that seems to resist any direct attack. For a further consequence that concerns quadratic forms over semilocal regular rings, see Corollary 5.3.6.

Corollary 1.3. *For an unramified regular local ring R, an $r \in R\backslash\{0\}$, and the r-adic completion $\widehat{R}$,*

$$G(\widehat{R}[\tfrac{1}{r}]) = G(\widehat{R})G(R[\tfrac{1}{r}]) \quad \textit{for every quasi-split reductive } R\textit{-group } G.$$

Indeed, if the double coset on the right side did not exhaust the left side, then one could use patching (for instance, Lemma 4.2.1) to build a nontrivial G-torsor that trivialized over $R[\frac{1}{r}]$ (and also over $\widehat{R}$).

1.4. Known cases. Previous results on Conjecture 1.1 fall into the following categories.

(1) The case when G is a torus was settled by Colliot-Thélène and Sansuc in [CTS87].

(2) The case when R is 1-dimensional, that is, a discrete valuation ring, was settled by Nisnevich in [Nis82], [Nis84], with corrections and a generalization to semilocal Dedekind rings by Guo in [Guo20]. Subcases of the 1-dimensional case (resp., of its semilocal generalization) appeared in [Har67], [BB70], [BT87] (resp., [PS16], [BVG14], [BFF17], [BFFH19]).

(3) The case when R is Henselian was settled in [BB70] and [CTS79, 6.6.1]. For such R, one may test the triviality of a G-torsor after base change to the residue field, so one may choose a height 1 prime $\mathfrak{p} \subset R$ for which $R/\mathfrak{p}$ is regular, apply Nisnevich's result, and induct on $\dim R$.

(4) The case when R contains a field, that is, when R is of equicharacteristic, was settled by Fedorov–Panin in [FP15] when the field is infinite (with significant inputs from [PSV15], [Pan16]), and by Panin [Pan20] when the field is finite, with substantial further simplifications due to Fedorov [Fed18]. Various subcases of the equicharacteristic case appeared in [Oja80], [CTO92], [Rag94], [PS97], [Zai00], [OP01], [OPZ04], [Pan05], [Zai05], [Che10], [PSV15].

(5) Sporadic cases, in which either R or G is of specific form but with R possibly of mixed characteristic, appeared in [Gro68, Rem. 1.11 a)], [Oja82], [Nis89], [Fed16b], [Fir19], [BFFP20].

In the cases (1)–(4), the results also include the variant when the regular ring R is only semilocal.

For arguing Theorems 1.2 and 5.3.1, we use the toral case [CTS87] and the semilocal Dedekind case [Guo20] but no other known case of Conjecture 1.1. In fact, our argument simultaneously streamlines the case when R contains a field, although we do not pursue this here beyond the case of quasi-split G contained in Theorem 1.2 because we see no point in repeating the same additional reductions that Fedorov–Panin, Panin, and Fedorov used for handling general G over such R.

1.5. The point of departure. A key feature of the Grothendieck–Serre conjecture and, in fact, of problems of its flavor, for example, of the Bass–Quillen conjecture, is that one cannot easily "enlarge" the ring R, essentially, because this may trivialize torsors, one can only "shrink" it. The key to progress therefore lies in better understanding the geometry of R, and our point of departure is precisely in this for unramified R of mixed characteristic $(0,p)$: we apply Popescu approximation to assume that R is essentially smooth over $\mathbb{Z}_{(p)}$ and then use ideas from Artin's technique of "good neighborhoods" from [SGA 4_{III}, XI] to spread R out to a fibration $U \to S$ into smooth affine curves over an open $S \subset \mathbb{A}^{\dim(R)-1}_{\mathbb{Z}_{(p)}}$ in such a way that a given small closed subscheme $Y \subset \mathrm{Spec}(R)$ spread out to be finite over S, see Proposition 2.2.1 for a precise statement. This structural result may be viewed as a version of the Noether normalization in mixed characteristic and is reminiscent of presentation lemmas of Quillen and Gabber from [Qui73, Lem. 5.12] and [Gab94, Lem. 3.1].

For us, Y is such that a generically trivial G-torsor E that we wish to trivialize reduces to a B-torsor over $\mathrm{Spec}(R)\backslash Y$ for a Borel $B \subset G$. The valuative criterion of properness applied to E/B allows us to make this Y be of codimension $\geqslant 2$ in $\mathrm{Spec}(R)$, and this codimension requirement appears difficult to relax while arguing our mixed characteristic "Noether normalization." In equal characteristic, Y being of codimension $\geqslant 1$ suffices and is immediate to arrange from the generic triviality of E without using any Borel, and this distinction is one of the main sources of complications in comparison to works of Panin and Fedorov. Although in mixed characteristic virtually every step seems to require either new ideas or new techniques, the works of Panin and Fedorov in equicharacteristic have provided us with invaluable guidance for what the structure of the overall argument might be.

1.6. The stages of the proof of Theorem 1.2. In Theorem 1.2, the key assertion is the triviality of every generically trivial G-torsor E. For this, our argument proceeds as follows.

(1) In §2.1, we use the Bertini theorem and the "good neighborhoods" technique in the context of semilocal Dedekind bases $\mathcal{O}$ to build the aforementioned fibration $U \to S$ starting from a projective, flat compactification of R over $\mathcal{O}$, see Proposition 2.1.6. We do not separate into cases according to whether the residue fields of $\mathcal{O}$ are all infinite or not, but finite residue fields lead to complications that concern Bertini theorems, as do imperfect residue fields because of inseparable extensions. We resolve these complications with Gabber's approach [Gab01] to Bertini theorems in positive characteristic. *Op. cit.* is more convenient for us than the generally finer approach of Poonen [Poo04] because it can guarantee that a suitable hypersurface exists for every large enough degree divisible by the characteristic, which helps in making sure that this degree is uniform across all the residue fields of $\mathcal{O}$ at maximal ideals.

In this step, a major simplification in comparison to the strategy in equicharacteristic is that we do not seek a fibration into *projective* curves (nor even the complement of a relatively finite subscheme in a projective relative curve) but are nevertheless able to ensure that Y spreads out to a *finite* S-scheme in the notation of §1.5. Even in equicharacteristic, this allows us to dispose of much effort usually spent in analyzing the "boundary" in subsequent steps.

(2) In §2.2, we deduce the mixed characteristic "Noether normalization" mentioned in §1.5 and then use it to lift our generically trivial G-torsor E to a torsor $\mathscr{E}$ over a smooth affine R-curve C equipped with a section $s \in C(R)$ such that $\mathscr{E}$ pulls back to E via s and reduces to a torsor under the unipotent radical of a Borel over $C\backslash Z$ for some R-finite $Z \subset C$. The R-finiteness (as opposed to mere R-quasi-finiteness) of Z is critical for later steps and comes

from the finiteness of the spreading out of Y. The appearance of the unipotent radical is a new phenomenon: in equicharacteristic, $C\backslash Z$ is affine and $\mathscr{E}|_{C\backslash Z}$ is a trivial torsor.

Our (C, s, Z) is a simplification of what Panin and Fedorov keep track of with the notion of a "nice triple." The latter is a variant of a "standard triple" of Voevodsky [MVW06, Def. 11.5] used in his construction of the triangulated category of motives. In general, it is convenient to work in terms of the relative R-curve C instead of directly with the fibration $U \to S$ because this gives the flexibility of changing C. In this, we reap the benefits of our C being affine: we need to work less in subsequent reductions than "nice triples" would require.

(3) Our $\mathscr{E}$ is not a G_C-torsor but a $\mathscr{G}$-torsor for some reductive C-group scheme $\mathscr{G}$ equipped with a Borel $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, so, in order to proceed, in §3 we modify C to reduce to $\mathscr{B} \subset \mathscr{G}$ being $B_C \subset G_C$. First, in §3.1, we show that $\mathscr{G}$ and G_C become isomorphic (compatibly with the Borels) over the Henselization of C along s. This does not suffice because we need to preserve the R-finiteness of Z, so we seek to equate G and $\mathscr{G}$ over some *finite* étale $\widetilde{C} \to C$ to which s lifts. We build such a $\widetilde{C}$ in §3.3 at the cost of shrinking C by applying the Bertini theorem to compactifications of torsors under the torus $B/\mathscr{R}_u(B)$. Critically, we need torsors to be *fiberwise* dense in these compactifications, so building them is delicate. For this, in §3.2 we build equivariant compactifications of tori beyond the case when the base is a field by extending the method from [CTHS05] based on toric geometry.

The phenomenon $\mathscr{G} \not\simeq G_C$ also appears in equicharacteristic and was addressed, for instance, in [PSV15, §5]. In that setting, however, required compactifications are simpler to come by, in essence, because the closure of a subvariety inside an ambient variety is automatically dense. The mixed characteristic difficulty is the failure of *fiberwise* density of such closures when one is working, say, over a discrete valuation ring $\mathcal{O}$ (as is seen already for closures of quasi-finite $\mathcal{O}$-schemes). The presence of Borel subgroups allows us to get by with only compactifying tori, but it would be interesting to know the general answer to the following question, at least for some classes of A and G, compare with Theorem 3.2.1 below.

Question 1.7. *For a reductive group scheme G over a ring A, are there a proper A-scheme $\overline{G}$ equipped with a left G-action and a G-equivariant A-fiberwise dense open immersion $G \hookrightarrow \overline{G}$?*

(4) After simplifying $\mathscr{G}$, we turn to simplifying C in §4, namely, to replacing C by $\mathbb{A}^1_R$. In §4.1, we construct an affine open $U \subset C$ containing Z and s as well as a quasi-finite R-morphism $\pi\colon C \to \mathbb{A}^1_R$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_R$ whose preimage in U is precisely Z. The R-finiteness of Z is critical for this, and the argument is simpler than its versions in the literature because C is affine (as opposed to projective). It uses Panin's tricks with finite fields to prepare C and Z for building π: when some residue fields of R are finite, the initial Z could, for instance, have too many rational points to fit inside $\mathbb{A}^1_R$.

Since C is Cohen–Macaulay, our quasi-finite π is necessarily flat, so the idea is to descend $\mathscr{E}$ to a $G_{\mathbb{A}^1_R}$-torsor via patching. We carry this out in §4.2: the main complication is the *a priori* nontriviality of $\mathscr{E}_{C\backslash Z}$, which we overcome by bootstrapping enough excision for $H^1(-, \mathscr{R}_u(B))$ from excision for quasi-coherent cohomology, see Lemma 4.2.2. Relatedly, since Z need not be principal, the patching is slightly more delicate than usual and uses [MB96].

(5) The final step is the analysis of a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z' \subset \mathbb{A}^1_R$. This is a problem of Bass–Quillen type that is somewhat delicate when G is neither semisimple, simply connected nor split. We approach it using the ideas of Fedorov from [Fed18], with the geometry of the affine Grassmannian Gr_G playing a central

role. Namely, even though the map $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}_G$ is ill-behaved in small characteristics, it is nevertheless an isomorphism on Schubert cells, and all field-valued points of the neutral component of Gr_G lift to $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}$, see Proposition 5.1.3 and Corollary 5.1.4. The key consequence of this for $G_{\mathbb{A}^1_R}$-torsors is recorded in Corollary 5.2.6 (with the crucial more general statement contained in Proposition 5.2.4). With this in hand, we conclude in §5.3.

Fedorov's simplification [Fed18] of the equicharacteristic case uses the same idea: the aforementioned geometric input on affine Grassmannians allows him to bypass a lengthy initial reduction to the semisimple, simply connected case that was used in [FP15].

Globally, our method may be viewed as a geometric reduction of the Grothendieck–Serre conjecture for G over R to its case for the torus $T := B/\mathscr{R}_u(B)$ over R. It is tempting to expect that if G is no longer quasi-split but has a parabolic subgroup $P \subset G$ with a Levi M, then one could find a way to reduce from G to M. As it stands, the sticking point in achieving this generalization is in the proof of Proposition 2.2.3: there we extend a T-torsor across a closed subscheme of codimension $\geqslant 2$ (across Y in the notation of §1.5) and such extendibility fails beyond tori (although knowing how to resolve the Colliot-Thélène–Sansuc purity question [CTS79, 6.4] would help).

1.8. Conventions and notation. For a scheme S (resp., a ring R), we let k_s (resp., $k_{\mathfrak{p}}$) denote the residue field of a point $s \in S$ (resp., a prime ideal $\mathfrak{p} \subset R$). Intersections $Y \cap Y'$ of closed subschemes $Y, Y' \subset S$ are always scheme-theoretic, and we recall from [EGA IV_1, 0.14.1.2] that $\dim(\varnothing) = -\infty$. We denote the (always open) S-smooth locus of an S-scheme X by X^{sm}. A scheme is *Cohen–Macaulay* if it is locally Noetherian and its local rings are Cohen–Macaulay. We use the definition [EGA IV_1, 0.15.1.7, 0.15.2.2] of a regular sequence (so there is no condition on quotients being nonzero). A ring $\mathcal{O}$ is *Dedekind* if it is Noetherian, normal, and of dimension $\leqslant 1$; by [SP, 034X], any such $\mathcal{O}$ is a finite product of Dedekind domains.

We always consider *right* torsors, for instance, we want sections of G/H to give rise to H-torsors. As already seen in §1.6 (3)–(5), we use scheme-theoretic notation when talking about torsors, that is, we base change the group in order to be unambiguous about what the base is; in the rare exceptions when this would make notation too cumbersome, we make sure that no confusion is possible. For a reductive group scheme G, we let $Z(G)$, G^{ad}, and G^{der} denote its center, derived subgroup, and adjoint quotient (see [SGA $3_{\mathrm{III\ new}}$, XXII, 4.1.7, 4.3.6, and 6.2.1 (iv)]); for a semisimple group scheme G, we let G^{sc} denote its simply connected cover (see [Con14, 6.5.2 (i)]). For a parabolic group scheme P, we let $\mathscr{R}_u(P)$ denote its unipotent radical constructed in [SGA $3_{\mathrm{III\ new}}$, XXVI, 1.6 (i)] (and already in [SGA $3_{\mathrm{III\ new}}$, XXII, 5.6.9 (ii)] when P is a Borel).

Acknowledgements. On several occasions during past years, I discussed the Grothendieck–Serre conjecture with Johannes Anschütz, Alexis Bouthier, Jean-Louis Colliot-Thélène, Ning Guo, Roman Fedorov, Timo Richarz, and Peter Scholze, among others. I thank them for these conversations. I thank Jean-Louis Colliot-Thélène, Uriya First, Ivan Panin, Michael Rapoport, Timo Richarz, and Nguyen Quoc Thang for helpful remarks or correspondence. I thank Viện Toán Học for hospitality—part of this project was completed during a visit there. This project received funding from the European Research Council under the European Union's Horizon 2020 research and innovation program (grant agreement No. 851146).

2. Lifting the torsor to a smooth relative curve

We begin with the geometric part of our approach to the Grothendieck–Serre conjecture for quasi-split G: we lift a generically trivial G-torsor over R to a torsor under a reductive group scheme over a smooth affine R-curve equipped with a section, see Proposition 2.2.3. The construction of this curve uses ideas related to Artin's technique of good neighborhoods, which we adapt to our setting in §2.1.

2.1. **Fibrations into smooth relative curves**

For a discrete valuation ring $\mathcal{O}$ and a smooth, affine $\mathcal{O}$-scheme U of relative dimension $d > 0$, we wish to construct a smooth morphism $\pi\colon U \to \mathbb{A}^{d-1}_{\mathcal{O}}$ whose nonempty fibers are of dimension 1. Roughly, the idea is to cut U by $d-1$ suitably transversal hypersurfaces supplied by Bertini theorem and then let their defining equations be the images of the standard coordinates of $\mathbb{A}^{d-1}_{\mathcal{O}}$. The actual argument given in Proposition 2.1.6 is slightly more subtle because in practice our U comes with an $\mathcal{O}$-fiberwise nowhere dense closed subscheme $Y \subset U$ that we wish to be *finite* over some open neighborhood of zero in $\mathbb{A}^{d-1}_{\mathcal{O}}$. To achieve this finiteness, we start from a projective compactification of U and adapt some ideas from Artin's construction of "good neighborhoods" in [SGA 4_{III}, XI, 3.3].

Before turning to Bertini, we review the following avoidance lemma that we will use repeatedly.

Lemma 2.1.1. *For a ring R, a quasi-projective, finitely presented R-scheme X, a very R-ample line bundle $\mathscr{L}$ on X, a finitely presented closed subscheme $Z \subset X$ not containing any positive-dimensional irreducible component of any R-fiber of X, and points $y_1, \ldots, y_n \in X \backslash Z$, there is an $N_0 > 0$ such that for every $N \geqslant N_0$ there is an $h \in \Gamma(X, \mathscr{L}^{\otimes N})$ whose vanishing scheme is a hypersurface $H \subset X$ containing Z but not any y_i or any positive-dimensional irreducible component of any R-fiber of X.*

Proof. The claim is a special case of [GLL15, 5.1] (with definitions reviewed in [GLL15, p. 1207]). ☐

In the case when R is a field, the following Bertini lemma allows us to impose a smoothness requirement on $X^{\mathrm{sm}} \cap H$. Its most delicate case is when the base field is finite, in which it amounts to a mild extension of [Gab01, Cor. 1.6 and Cor. 1.7], whose argument is actually our key technique.

Lemma 2.1.2. *Let k be a field, let X be a projective k-scheme of pure dimension, let $Y_1, \ldots, Y_m, Z \subset X$ be closed subschemes with $Z = Z_1 \sqcup Z_0$ for a set of reduced closed points $Z_0 \subset X^{\mathrm{sm}}$ whose residue fields are separable extensions of k, and fix a*

$$t \leqslant \min(\dim(X), \dim(X) - \dim(Z)) \quad (\textit{recall from §1.8 that } \dim(\varnothing) = -\infty).$$

For an ample line bundle $\mathscr{O}_X(1)$ on X, there are hypersurfaces $H_1, \ldots, H_t \subset X$ such that

(i) *$H_1 \cap \ldots \cap H_t$ is of pure dimension $\dim(X) - t$ and contains Z;*

(ii) *$(X^{\mathrm{sm}} \backslash Z_1) \cap H_1 \cap \ldots \cap H_t$ is k-smooth;*

(iii) *$\dim((Y_j \backslash Z) \cap \bigcap_{\ell \in L} H_\ell) \leqslant \dim(Y_j \backslash Z) - \#L$ for $1 \leqslant j \leqslant m$ and $L \subset \{1, \ldots, t\}$;*

moreover, we may iteratively choose $H_1, \ldots, H_t$ so that, for each i, with $H_1, \ldots, H_{i-1}$ already fixed, H_i may be chosen to have any sufficiently large degree divisible by the characteristic exponent of k.

Proof. The pure dimension hypothesis means that all the irreducible components of X have the same dimension, so [EGA IV_2, 5.2.1] ensures that X is biequidimensional in the sense that the saturated chains of specializations of its points all have the same length. Similarly to [Čes20, §4.1], part (i) then ensures that each $X \cap H_1 \cap \ldots \cap H_i$ inherits biequidimensionality, so is also of pure dimension. This reduces us to $t = 1$: by applying this case iteratively and at each step adjoining to the Y_j's all their possible intersections with some of the already chosen H_i's (to ensure (iii)), we will obtain the general case. In the case $t = 1$, we fix closed points $y_1, \ldots, y_n \in X \backslash Z$ that jointly meet every irreducible component of X and of every $Y_j \backslash Z$. Both (iii) and the dimension aspect of (i) will hold as soon as H_1 contains no y_j, so at the cost of requiring this we may forget about the Y_j.

For the rest of the argument, we begin with the case when $\operatorname{char} k = 0$, in which we will use the "classical" Bertini theorem. For this, we first claim that for every large $N > 0$ there are global sections h_i of $\mathscr{O}_X(N)$ whose common zero locus contains Z and set-theoretically equals to it. Indeed, by repeatedly applying [EGA III$_1$, 2.2.4] to shrink the base locus, we first build such $h'_{i'}$ (resp., $h''_{i''}$) for some N' (resp., N'') that is a power of 2 (resp., of 3), then express every large N as $aN' + bN''$ with $a, b > 0$, and, finally, let h_i be the collection of all the $h'^{a}_{i'} h''^{b}_{i''}$. By [EGA III$_1$, 2.2.4] and [EGA IV$_4$, 17.15.9] (which uses the separable residue field assumption), granted that N is sufficiently large, we may build another global section h_0 of $\mathscr{O}_X(N)$ whose associated hypersurface contains Z and is smooth at every point of Z_0. We adjoin h_0 to the h_i to ensure that the common zero locus of the global sections h_i is k-smooth at the points in Z_0. We also discard linear dependencies to assume that the h_i are k-linearly independent. By [EGA II, 4.2.3], the h_i determine a morphism

$$X\backslash Z \to \mathbb{P}^{N'}_k$$

such that the pullback of $\mathscr{O}_{\mathbb{P}^{N'}_k}(1)$ is our $\mathscr{O}_{X\backslash Z}(N)$. The hyperplanes in $\mathbb{P}^{N'}_k$ and, compatibly, the nonzero k-linear combinations of the h_i up to scaling are parametrized by the dual projective space. Due to the existence of a k-linear combination of the h_i whose associated hypersurface does not contain a fixed y_j, a generic such hypersurface contains no y_j. Likewise, due to the openness of the smooth locus, the existence of a k-linear combination of the h_i whose associated hypersurface is smooth at all the points in Z_0, and [EGA IV$_3$, 11.3.8 b$'$)$\Leftrightarrow$c)], a generic such hypersurface is smooth at all the points in Z_0. Finally, by the Bertini theorem [Jou83, 6.11 2)], the hypersurface H associated to a generic k-linear combination of the h_i is such that $(X^{\mathrm{sm}}\backslash Z) \cap H$ is k-smooth. In conclusion, since k is infinite, we may choose our desired H_1 to be a generic such H.

The remaining case when $\operatorname{char} k = p > 0$ is a very minor sharpening of [Gab01, Cor. 1.6] that is proved as there. Namely, we use the pure dimension hypothesis to apply [Gab01, Thm. 1.1][1] with

- U there being our $X^{\mathrm{sm}}\backslash(Z \cup \{y_1, \dots, y_n\})$ and $\mathcal{E}$ there being Ω^1_U;
- Z there being our $Z_1 \cup \{y_1, \dots, y_n\} \cup \bigcup_{z\in Z_0} \underline{\operatorname{Spec}}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}_z^2)$;
- Σ there being our $Z_0 \cup \{y_1, \dots, y_n\}$;
- m_0 there being 0; and
- σ_0 there being 0 on our Z_1, a unit on each of our $y_1, \dots, y_n$, and a nonzero cotangent vector at $z \in Z_0$ on each of our $\underline{\operatorname{Spec}}_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}_z^2)$;

to find a finite set of closed points $F \subset X^{\mathrm{sm}}\backslash(Z \cup \{y_1, \dots, y_n\})$ and, for every large N divisible by p, a global section h of $\mathscr{O}_X(N)$ whose associated hypersurface contains Z, has a k-smooth intersection with $X^{\mathrm{sm}}\backslash(F \cup Z \cup \{y_1, \dots, y_n\})$, passes through every $z \in Z_0$ and is k-smooth there (for this we use [EGA IV$_4$, 17.15.9] and the separable residue field assumption as in the characteristic 0 case), and does not pass through any y_j. By [EGA III$_1$, 2.2.4], if this N is sufficiently large, then there is a global section h' of $\mathscr{O}_X(N/p)$ that vanishes on $Z \cup \{y_1, \dots, y_n\}$ and is such that $h + h'^p$ does not vanish at any point of F. We may then let H_1 be the hypersurface associated to $h + h'^p$. □

Remark 2.1.3. In (iii), if $(Y_j\backslash Z) \cap \bigcap_{\ell\in L} H_\ell \neq \varnothing$, then the inequality is actually an equality because, unless the intersection is empty, cutting by $\#L$ hypersurfaces decreases dimension by at most $\#L$.

We do not know how to ensure that the hypersurfaces H_i in Lemma 2.1.2 all have the same degree. This complication will force us to use weighted blowups and weighted projective spaces, so we review some basic properties of these notions.

[1] *Loc. cit.* is stated in the case when the base field k is finite but its proof continues to work whenever k is any field of positive characteristic p.

2.1.4. Weighted projective spaces. For $w_0, \dots, w_d \in \mathbb{Z}_{>0}$, we consider the polynomial algebra $\mathbb{Z}[t_0, \dots, t_d]$ that is $\mathbb{Z}_{\geqslant 0}$-graded by declaring each t_i to be of weight w_i (and the constants $\mathbb{Z}$ to be of weight 0), and we let the resulting *weighted projective space* be

$$\mathbb{P}_{\mathbb{Z}}(w_0, \dots, w_d) := \mathrm{Proj}(\mathbb{Z}[t_0, \dots, t_d]).$$

We repeat the same construction over any scheme S to build $\mathbb{P}_S(w_0, \dots, w_d)$, although the latter is just $\mathbb{P}_{\mathbb{Z}}(w_0, \dots, w_d) \times_{\mathbb{Z}} S$ because the formation of Proj commutes with base change [EGA II, 3.5.3]. We will only use weighted projective spaces when $w_0 = 1$, in which case the open subscheme of $\mathbb{P}_S(1, w_1, \dots, w_d)$ given by $\{t_0 \neq 0\}$ is the affine space $\mathbb{A}^d_S$ with coordinates $t_1/t_0^{w_1}, \dots, t_d/t_0^{w_d}$.

2.1.5. Weighted blowups. For a scheme X, a line bundle $\mathscr{L}$ on X, and sections

$$h_0 \in \Gamma(X, \mathscr{L}^{\otimes w_0}), \quad \dots, \quad h_d \in \Gamma(X, \mathscr{L}^{\otimes w_d}) \quad \text{with} \quad w_0, \dots, w_d > 0,$$

we define the *weighted blowup* of X with respect to $h_0, \dots, h_d$ as

$$\mathrm{Bl}_X(h_0, \dots, h_d) := \underline{\mathrm{Proj}}_{\mathscr{O}_X}(\mathscr{O}_X[h_0, \dots, h_d]), \quad \text{where} \quad \mathscr{O}_X[h_0, \dots, h_d] \subset \textstyle\bigoplus_{n \geqslant 0} \mathscr{L}^{\otimes n}$$

is the quasi-coherent, graded $\mathscr{O}_X$-subalgebra generated by the h_i. The *center* of this weighted blowup is the closed subscheme of X cut out by the h_i. By [EGA II, 3.1.8 (i)], the map

$$\mathrm{Bl}_X(h_0, \dots, h_d) \to X \quad \text{is an isomorphism away from the center.}$$

By §2.1.4 and the functoriality of Proj, the weighted blowup $\mathrm{Bl}_X(h_0, \dots, h_d)$ admits a morphism

$$\mathrm{Bl}_X(h_0, \dots, h_d) \to \mathbb{P}_{\mathbb{Z}}(w_0, \dots, w_d) \quad \text{determined by} \quad t_i \mapsto h_i.$$

In the case when $w_0 = \dots = w_d$, our $\mathrm{Bl}_X(h_0, \dots, h_d)$ is identified with the usual blowup of X along the closed subscheme cut out by the h_i: this is evident when also $\mathscr{L} = \mathscr{O}_X$, and the general case reduces to this one because Proj is insensitive to twisting by line bundles [EGA II, 3.1.8 (iii)].

We are ready for the promised construction of a fibration into relative curves. In the following statement, the reader could assume that the points $x_1, \dots, x_n$ are closed in $\overline{X}$: this case will suffice for us and is slightly simpler. We decided to include the general case because this does not dramatically complicate the argument and may be useful for generalizing Theorem 5.3.1.

Proposition 2.1.6. *For*

- *a semilocal Dedekind ring* $\mathcal{O}$;
- *a projective, flat morphism* $f\colon \overline{X} \to \mathrm{Spec}(\mathcal{O})$ *with fibers of pure dimension* $d > 0$;
- *a closed subscheme* $\overline{Y} \subset \overline{X}$ *that is* $\mathcal{O}$*-fiberwise of codimension* $\geqslant 1$ *in* $\overline{X}$;
- *an* $\mathcal{O}$*-smooth open* $X \subset \overline{X}$ *such that* $Y' := \overline{Y} \backslash X$ *is* $\mathcal{O}$*-fiberwise of codimension* $\geqslant 2$ *in* $\overline{X}$;
- *points* $x_1, \dots, x_n \in X$ *with* $k_{x_i}/k_{f(x_i)}$ *separable such that*

$$\overline{\{x_i\}} \cap Y' = \varnothing \quad \text{in } \overline{X} \text{ for all} \quad i \tag{2.1.6.1}$$

(*if the points* x_i *are closed in* $\overline{X}$*, then* (2.1.6.1) *simply says that* $x_1, \dots, x_n \notin Y'$);

there are

(i) *an affine open* $U \subset X$ *containing* $x_1, \dots, x_n$;

(ii) *an affine open* $S \subset \mathbb{A}^{d-1}_{\mathcal{O}}$ *containing the zero section* $z := \mathrm{Spec}(\mathcal{O}) \subset \mathbb{A}^{d-1}_{\mathcal{O}}$;

(iii) *a flat* $\mathcal{O}$*-morphism* $\pi\colon U \to S$ *whose fibers are Cohen–Macaulay of pure dimension* 1 *such that*

(1) π *maps those* x_i *that are closed in their* $\mathcal{O}$*-fibers of* $\overline{X}$ *into the zero section* z;

(2) *π is smooth at each x_i and at those $x \in U \setminus \bigcup_{i=1}^{n} \overline{\{x_i\}}$ that map to a closed point of z;*

(3) *$\overline{Y} \cap U$ is S-finite and if its (set-theoretic) intersection with the closed $\mathcal{O}$-fibers of $\bigcup_{i=1}^{n} \overline{\{x_i\}}$ is a subset of $\{x_1, \ldots, x_n\}$, then $\overline{Y} \cap U$ lies in the smooth locus of π.*

Of course, the assumption in (3) is automatic in the case when the points x_i are closed in $\overline{X}$.

Proof. In principle, we imitate Artin's argument from [SGA 4_{III}, XI, 3.3], but we have to overcome complications caused by the base $\mathcal{O}$ not being a field. On the other hand, our goal is slightly simpler than there in that we do not seek to simultaneously compactify U to a projective relative curve over S.[2] The subtleties caused by some residue fields of $\mathcal{O}$ possibly being finite are already subsumed into Lemma 2.1.2. In comparison, in *op. cit.* one worked over an algebraically closed field and used a suitable variant of Lemma 2.1.2 in which one could choose the hypersurfaces there to be hyperplanes.

To begin, we pass to connected components of $\mathrm{Spec}(\mathcal{O})$ to assume that $\mathcal{O}$ is a domain. By [EGA IV_4, 17.15.9], the $k_{f(x_i)}$-fiber of $\overline{\{x_i\}}$ is generically smooth, so, by Lemma 2.1.2, each x_i has a specialization in X that is a closed point in its $\mathcal{O}$-fiber of $\overline{X}$ and whose residue field is separable over $k_{f(x_i)}$. We replace the x_i by these specializations to assume that $x_1, \ldots, x_n$ are closed in their $\mathcal{O}$-fibers. Granted these reductions, our construction of U, S, and π is based on the following claim.

Claim 2.1.6.2. There are a closed immersion $\overline{X} \subset \mathbb{P}^N_{\mathcal{O}}$, its associated very ample line bundle $\mathscr{O}_{\overline{X}}(1)$, and hypersurfaces $H_0, \ldots, H_{d-1} \subset \mathbb{P}^N_{\mathcal{O}}$ with H_0 a hyperplane such that

(a) H_0 does not contain any of the $x_1, \ldots, x_n$;

(b) $X \cap H_1 \cap \ldots \cap H_{d-1}$ is Cohen–Macaulay, $\mathcal{O}$-flat of relative dimension 1, contains $x_1, \ldots, x_n$;

(c) $X \cap H_1 \cap \ldots \cap H_{d-1}$ is $\mathcal{O}$-smooth at each x_i;

(d) $(X \cap H_1 \cap \ldots \cap H_{d-1}) \setminus \bigcup_{i=1}^{n} \overline{\{x_i\}}$ has smooth closed $\mathcal{O}$-fibers;

(e) $\overline{Y} \cap H_1 \cap \ldots \cap H_{d-1}$ is $\mathcal{O}$-finite and $\overline{Y} \cap H_0 \cap \ldots \cap H_{d-1} = Y' \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

Proof. We first fix some closed immersion $\overline{X} \subset \mathbb{P}^{N'}_{\mathcal{O}}$ and use Lemma 2.1.1 to find a hypersurface $H_0 \subset \mathbb{P}^{N'}_{\mathcal{O}}$ that does not contain any $x_1, \ldots, x_n$, does not contain any generic point of an $\mathcal{O}$-fiber of $\overline{X}$, does not contain any generic point η of an $\mathcal{O}$-fiber $\overline{Y}$ such that η is of height 1 in the corresponding $\mathcal{O}$-fiber of $\overline{X}$, and, using (2.1.6.1), does not contain any point of any $\overline{\{x_i\}} \cap \overline{Y}$ (since x_i is closed in its $\mathcal{O}$-fiber, $\overline{\{x_i\}}$ has only finitely many points). In particular, H_0 satisfies (a). We then postcompose with some Veronese embedding $\mathbb{P}^{N'}_{\mathcal{O}} \subset \mathbb{P}^N_{\mathcal{O}}$ to find a closed immersion $\overline{X} \subset \mathbb{P}^N_{\mathcal{O}}$ in which our H_0 becomes a hyperplane. By construction, $\overline{Y} \cap H_0$ is $\mathcal{O}$-fiberwise of dimension $\leqslant d-2$. With this H_0 fixed, the requirement (e) becomes a requirement on $H_1, \ldots, H_{d-1}$.

The rest of the claim is simpler when our Dedekind domain $\mathcal{O}$ is a field. Namely, then Lemma 2.1.2 (applied with $Z := \{x_1, \ldots, x_n\}$ and $Z_0 = Z$) supplies hypersurfaces $H_1, \ldots, H_{d-1}$ such that

- $\overline{X} \cap H_1 \cap \ldots \cap H_{d-1}$ is of pure dimension 1 and contains $x_1, \ldots, x_n$;
- $(\overline{X}^{\mathrm{sm}} \setminus \{x_1, \ldots, x_n\}) \cap H_1 \cap \ldots \cap H_{d-1}$ is $\mathcal{O}$-smooth;
- $\overline{X} \cap H_1 \cap \ldots \cap H_{d-1}$ is $\mathcal{O}$-smooth at each x_i;
- $\overline{Y} \cap H_1 \cap \ldots \cap H_{d-1}$ is finite and $\overline{Y} \cap H_0 \cap \ldots \cap H_{d-1} = Y' \cap H_1 \cap \ldots \cap H_{d-1} = \varnothing$.

[2] See the first arXiv version of this article for a variant of Proposition 2.1.6 that also builds such a compactification.

Since X is Cohen–Macaulay and we are iteratively cutting by transversal hypersurfaces, the equations of the $H_0, \dots, H_{d-1}$ form a regular sequence locally on X and (a)–(e) hold.

The remaining case of a semilocal Dedekind domain $\mathcal{O}$ that is not a field is more subtle because some of the x_i may lie in the generic $\mathcal{O}$-fiber of $\overline{X}$. Thus, we set $K := \mathrm{Frac}(\mathcal{O})$, let $C \subset \mathrm{Spec}(\mathcal{O})$ be the union of the closed points of $\mathrm{Spec}(\mathcal{O})$, and order $x_1, \dots, x_n$ so that $x_1, \dots, x_{n'}$ with $n' \leqslant n$ lie over C and $x_{n'+1}, \dots, x_n$ lie over the generic point. We consider the reduced closed subscheme

$$\varepsilon := \textstyle\bigsqcup_{n'+1 \leqslant i \leqslant n} x_i \subset \overline{X}_K$$

and let $Z \subset \overline{X}$ be its schematic closure in $\overline{X}$. Since $x_{n'+1}, \dots, x_n$ are closed in the generic fiber and the local rings of $\overline{X}$ at the closed points are of dimension $\leqslant d+1$, the scheme Z is semilocal and either empty or of dimension 1, with closed points that lie over C, with generic points $x_{n'+1}, \dots, x_n$, and with $Z_K \cong \varepsilon$. The assumption (2.1.6.1) gives $Z \cap Y' = \varnothing$. We consider the schematic closure $\widetilde{\overline{X}} \subset \overline{X}$ of $\overline{X}_C \sqcup \varepsilon$: concretely, its ideal sheaf is $\mathscr{I}_{\widetilde{\overline{X}}} = \mathscr{I}_Z \cap \mathscr{I}_{\overline{X}_C}$, so

$$\mathscr{O}_{\widetilde{\overline{X}}} \xrightarrow{\sim} \mathscr{O}_{\overline{X}_C} \times_{\mathscr{O}_{Z_C}} \mathscr{O}_Z,$$

to the effect that giving a function on $\widetilde{\overline{X}}$ amounts to giving compatible functions on $\overline{X}_C$ and on Z.

As in the case when $\mathcal{O}$ was a field, by Lemma 2.1.2 applied to the C-fibers of $\overline{X}$, there are hypersurfaces

$$H'_1, \dots, H'_{d-1} \subset \mathbb{P}^N_C$$

of arbitrarily large and, by the last aspect of Lemma 2.1.2, constant on C degrees that contain Z_C, contain $x_1, \dots, x_{n'}$, are such that $\overline{X}_C \cap H'_1 \cap \ldots \cap H'_{d-1}$ is of pure dimension 1, and satisfy the analogues of (a)–(e) with C, $\overline{X}_C$, $(H_0)_C$, X_C, $\overline{Y}_C$, $x_1, \dots, x_{n'}$, Y'_C in place of $\mathcal{O}$, $\overline{X}$, H_0, X, $\overline{Y}$, $x_1, \dots, x_n$, Y', granted that in (d) we also exclude points in Z_C: for (e), we use that $Z \cap \overline{Y} \cap H_0 = \varnothing$ by the construction of H_0 and that $Z \cap Y' = \varnothing$ by (2.1.6.1). With the H'_i chosen of large enough degrees, we then combine the last conclusion of the previous paragraph with [EGA III_1, 2.2.4] to lift the $H'_1, \dots, H'_{d-1}$ to hypersurfaces $\widetilde{H}_1, \dots, \widetilde{H}_{d-1} \subset \mathbb{P}^N_{\mathcal{O}}$ that vanish on Z, and hence also on each $x_{n'+1}, \dots, x_n$. We let $\widetilde{h}_j$ be a homogeneous polynomial defining $\widetilde{H}_j$.

By prime avoidance [SP, 00DS], for every maximal ideal $\mathfrak{m} \subset \mathcal{O}$ there is a $\varpi_{\mathfrak{m}} \in \mathcal{O}$ that maps to a uniformizer of $\mathcal{O}_{\mathfrak{m}}$ and to a unit in every other local ring of $\mathcal{O}$. Thus, with the degrees of the H'_j chosen large enough, we now apply [EGA III_1, 2.2.4] to the generic fiber $\overline{X}_K$ and afterwards multiply by suitable powers of the $\varpi_{\mathfrak{m}}$ if needed to iteratively build hypersurfaces $H_{1,\,\mathrm{aux}}, \dots, H_{d-1,\,\mathrm{aux}} \subset \mathbb{P}^N_K$ of the same degrees as $\widetilde{H}_1, \dots, \widetilde{H}_{d-1}$ that pass through $x_{n'+1}, \dots, x_n$ and whose defining homogeneous polynomials $h_{j,\,\mathrm{aux}}$ have coefficients in $\mathcal{O}$, vanish identically modulo every maximal ideal $\mathfrak{m} \subset \mathcal{O}$, and are such that the $h_j := \widetilde{h}_j + h_{j,\,\mathrm{aux}}$ form a part of a regular system of parameters of $\mathscr{O}_{\overline{X},\, x_i}$. We define the hypersurface $H_j \subset \mathbb{P}^N_{\mathcal{O}}$ as the vanishing locus of the homogeneous polynomial h_j. By [EGA IV_4, 17.15.9] and our construction, $\overline{X}_K \cap H_1 \cap \ldots \cap H_{d-1}$ contains $x_{n'+1}, \dots, x_n$ and is smooth of dimension 1 at every such x_i.

By construction, $(H_j)_C \cong H'_j$, so, by [SP, 0D4I], the $\mathcal{O}$-fibers of $\overline{X} \cap H_1 \cap \ldots \cap H_{d-1}$ are of pure dimension 1. Moreover, we use [EGA IV_3, 11.3.8] to check over C that locally on X the h_j form a regular sequence and that the $\mathcal{O}$-scheme $X \cap H_1 \cap \ldots \cap H_{d-1}$ is $\mathcal{O}$-flat and Cohen–Macaulay. Thus, (b) holds. Similarly, (c) and (d) follow from the construction and from their counterparts over C. We use the openness of the quasi-finite locus [SP, 01TI] and the finiteness of proper, quasi-finite morphisms [SP, 02OG] to check (e) over C. □

We choose $\mathscr{O}_{\overline{X}}(1)$, $H_0, \dots, H_{d-1}$ as in the claim and let h_i be a homogeneous defining polynomial of H_i viewed as a global section of $\mathscr{O}_{\overline{X}}(w_i)$ with $w_0 = 1$. Since h_0 trivializes $\mathscr{O}_{\overline{X}}(1)$ away from H_0, the complement $X' := X \backslash H_0$ is affine and the $h_1, \dots, h_{d-1}$ determine an $\mathcal{O}$-morphism

$$\pi \colon X' \to \mathbb{A}^{d-1}_{\mathcal{O}} \quad \text{that maps} \quad x_1, \dots, x_n \quad \text{inside the zero section} \quad z \in \mathbb{A}^{d-1}_{\mathcal{O}}(\mathcal{O}).$$

The z-fiber of π is $X' \cap H_1 \cap \dots \cap H_{d-1}$. Our $S \subset \mathbb{A}^{d-1}_{\mathcal{O}}$ will be a small affine open neighborhood of z (since $\mathcal{O}$ is semilocal, by [SP, 00DS], affine opens are cofinal among the neighborhoods of z) and our U will be a suitably chosen affine open of $\pi^{-1}(S)$. For such S and U, (i), (ii), and (1) will hold.

As in §2.1.5, we consider the weighted blowup

$$\widetilde{X} := \mathrm{Bl}_{\overline{X}}(h_0, \dots, h_{d-1}), \quad \text{which contains} \quad \overline{X} \backslash (\overline{X} \cap H_0 \cap \dots \cap H_{d-1}) \quad \text{as an open.}$$

As there, granted that we identify $\mathbb{A}^{d-1}_{\mathcal{O}}$ with the affine open of the weighted projective space $\mathbb{P}_{\mathcal{O}}(1, w_1, \dots, w_{d-1})$ given by the locus on which the 0-th standard projective coordinate does not vanish, π is the restriction of an $\mathcal{O}$-morphism

$$\overline{\pi} \colon \widetilde{X} \to \mathbb{P}_{\mathcal{O}}(1, w_1, \dots, w_{d-1}).$$

By [EGA II, 5.5.5], this $\overline{\pi}$ inherits projectivity from f. By (e), the closed subscheme $\overline{Y} \subset \overline{X}$ is simultaneously a closed subscheme of $\widetilde{X}$ and the π-image of Y' does not meet z. This image is closed, so we shrink S to arrange that $\overline{\pi}^{-1}(S) \cap Y' = \varnothing$. Likewise, we use (e) again and shrink S further to arrange that it does not meet the $\overline{\pi}$-image of $\overline{Y} \cap H_0$. It then follows that $\overline{\pi}^{-1}(S) \cap \overline{Y} = \pi^{-1}(S) \cap \overline{Y}$, so that, in particular, $\pi^{-1}(S) \cap \overline{Y}$ is projective over S. By (e), the z-fiber of $\overline{Y}$ is finite, so the same references as at the end of the proof of Claim 2.1.6.2 allow us to shrink S more to arrange that $\pi^{-1}(S) \cap \overline{Y}$ is even finite over S.

Since X' is Cohen–Macaulay, $\mathcal{O}$-flat, with $\mathcal{O}$-fibers of pure dimension d, the flatness criteria [EGA IV$_2$, 6.1.5], [EGA IV$_3$, 11.3.11] and (b) ensure that π is flat at every point that lies over z. In other words, the S-flat locus of X', which, by [EGA IV$_3$, 11.3.1], is open, contains every point of X' above z. Thus, we may use [SP, 00DS] to choose an S-flat affine open $U \subset \pi^{-1}(S)$ that contains $x_1, \dots, x_n$ and the z-fiber of $\overline{Y} \cap \pi^{-1}(S)$. Thus, the π-image of $(\overline{Y} \cap \pi^{-1}(S)) \backslash U$ is closed in S and does not meet z, so we shrink S further to arrange that $\overline{Y} \cap U = \overline{Y} \cap \pi^{-1}(S)$. This ensures that the first assertion of (3) holds. Moreover, since U is S-flat, (2) follows from (c) and (d). We may then shrink S further to ensure that $\overline{Y} \cap U$ lies in the S-smooth locus of U, so that (3) holds in full.

Being an open of X', our U is Cohen–Macaulay, so its S-flatness ensures that the π-fibers of U are also Cohen–Macaulay. Thus, by [SP, 02NM], our U is a disjoint union of clopens each of whose π-fibers are of fixed pure dimension. Thus, since, by (a), the π-fibers of U above points in z are of pure dimension 1, we may shrink S one final time to ensure that (iii) holds. □

2.2. Generically trivial torsors under quasi-split groups lift to relative curves

In practice, we start from a smooth affine open X, not from its projective compactification $\overline{X}$. The following proposition recasts the results of the previous section from this vantage point. We thank Panin for extracting its formulation from the initial version of this article.

Proposition 2.2.1. *For*

- *a semilocal Dedekind ring* $\mathcal{O}$;
- *a smooth* $\mathcal{O}$*-algebra* A *that is everywhere of positive relative dimension over* $\mathcal{O}$;
- $x_1, \dots, x_n \in \mathrm{Spec}(A)$ *such that* x_i *maps to a maximal ideal* $\mathfrak{m}_i \subset \mathcal{O}$ *with* $k_{x_i}/k_{\mathfrak{m}_i}$ *separable*;

- *a closed subscheme $Y \subset \mathrm{Spec}(A)$ that is of codimension $\geqslant 2$;*

there are

(i) *an affine open $U \subset \mathrm{Spec}(A)$ containing all the x_i;*

(ii) *an affine open $S \subset \bigsqcup_{d \geqslant 0} \mathbb{A}^d_{\mathcal{O}}$;*

(iii) *a smooth $\mathcal{O}$-morphism $\pi \colon U \to S$ of pure relative dimension 1 such that $Y \cap U$ is S-finite.*

Proof. We decompose $\mathrm{Spec}(\mathcal{O})$ and $X := \mathrm{Spec}(A)$ into connected components to assume that $\mathcal{O}$ and A are domains, so that A is of pure relative dimension $d > 0$ over $\mathcal{O}$. For each x_i, as in beginning of the proof of Proposition 2.1.6, we use Lemma 2.1.2 to choose an $x'_i \in \overline{\{x_i\}} \subset X$ that is closed in its $\mathcal{O}$-fiber such that $k_{x'_i}$ is still a separable extension of $k_{\mathfrak{m}_i}$. We replace the x_i by the x'_i to ensure without losing generality that the $x_1, \dots, x_n$ are closed in their $\mathcal{O}$-fibers.

We embed X into some affine space over $\mathcal{O}$ and then take the schematic image in the corresponding projective space to build an open immersion $X \hookrightarrow \overline{X}$ into a projective, flat $\mathcal{O}$-scheme $\overline{X}$. On the generic $\mathcal{O}$-fiber this immersion has dense image, so $\overline{X}_K$ is of pure dimension d. It then follows from [SP, 0D4J, 02FZ] that $\overline{X}$ is of pure relative dimension d over $\mathcal{O}$. We will build our U and π from $\overline{X}$ via Proposition 2.1.6. To apply the latter, we note that our $x_1, \dots, x_n$ are all closed in X and we now define the relevant X, $\overline{Y}$, and Y'.

We let $\overline{Y} \subset \overline{X}$ be the schematic image of Y and let Y' be the reduced complement $\overline{Y} \backslash Y$, so that $\overline{Y} \backslash Y' = Y \subset X$. By [SP, 01R8], set-theoretically we have $\overline{Y} = \bigcup_y \overline{\{y\}}$ where y ranges over the generic points of Y and the closures are in $\overline{X}$. Each y is of height $\geqslant 2$ in $\overline{X}$, so each $\overline{\{y\}}$ intersects the $\mathcal{O}$-fiber of $\overline{X}$ containing y in a closed subscheme of dimension $\leqslant d-1$ (even $\leqslant d-2$ if the fiber is generic). Thus, since $\overline{\{y\}}$ has a nonempty open $\overline{\{y\}} \cap X$, the contribution of y to its $\mathcal{O}$-fiber of Y' is of dimension $\leqslant d-2$. The only situation in which $\overline{\{y\}}$ may contribute to other $\mathcal{O}$-fibers of Y' is when y lies in the generic $\mathcal{O}$-fiber of $\overline{X}$ and $\mathcal{O}$ is not a field. However, since the local rings of $\overline{X}$ are of dimension $\leqslant d+1$, then the intersection of $\overline{\{y\}}$ with any closed $\mathcal{O}$-fiber of $\overline{X}$ is of dimension $\leqslant d-2$. In conclusion, Y' is $\mathcal{O}$-fiberwise of dimension $\leqslant d-2$, that is, $\mathcal{O}$-fiberwise of codimension $\geqslant 2$ in $\overline{X}$ and, likewise, $\overline{Y}$ is $\mathcal{O}$-fiberwise of codimension $\geqslant 1$ in $\overline{X}$. By construction, $x_1, \dots, x_n \notin Y'$.

Proposition 2.1.6 applies to $\overline{X}$, X, $x_1, \dots, x_n$, $\overline{Y}$, and Y' and gives us the desired U, S, and π, except that π is only smooth at $x_1, \dots, x_n$ and at the points of $Y \cap U$. However, the locus where π is smooth is open in U, so after semilocalizing S at the images of the x_i we may use [SP, 00DS] to shrink U to ensure that the base change of π to this semilocalization is smooth (with (i) and (iii) intact over the semilocalization). Spreading out then allows us to shrink S and U to ensure that π is smooth even without the base change to the semilocalization (with (i) and (iii) still intact). □

Remark 2.2.2. It is desirable to remove the assumption that the x_i all lie over maximal ideals of $\mathcal{O}$: this would pave the way for an analogous improvement of Theorem 5.3.1. Without it, however, we do not know how to ensure that $\overline{\{x_i\}} \cap Y' = \varnothing$, as is needed for applying Proposition 2.1.6.

The following consequence of Proposition 2.2.1 starts a string of reductions that will eventually lead us to Theorem 1.2. In comparison to versions in the literature, for instance, to [Fed16b, Prop. 3.5], the main new phenomena are that the group $\mathscr{G}$ is only defined over a small affine C and that $C \backslash Z$ need not be affine. This will, in particular, cause additional subtleties in §4.2.

Proposition 2.2.3. *For a semilocal Dedekind ring $\mathcal{O}$, the localization R of a smooth $\mathcal{O}$-algebra A at finitely many primes $\mathfrak{p}$ that all lie over maximal ideals $\mathfrak{p}' \subset \mathcal{O}$ with $k_{\mathfrak{p}}/k_{\mathfrak{p}'}$ separable, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

(i) *a smooth, affine R-scheme C of pure relative dimension* 1;

(ii) *a section $s \in C(R)$;*

(iii) *an R-finite closed subscheme $Z \subset C$;*

(iv) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$;*

(v) *a $\mathscr{G}$-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(\mathscr{B})$-torsor over $C \backslash Z$.*

Proof. We decompose $\mathrm{Spec}(\mathcal{O})$ and $\mathrm{Spec}(R)$ into connected components to assume that $\mathcal{O}$ and R are domains, and then likewise assume that A is a domain. If A is of relative dimension 0, then R is a Dedekind domain, so, by [Guo20, Thm. 1.2], our torsor E is trivial and we may choose $C = \mathbb{A}^1_R$ and $\mathscr{E} := E_{\mathbb{A}^1_R}$, the closed subscheme Z being empty and s being the zero section. Thus, we may assume that A is $\mathcal{O}$-fiberwise of pure dimension $d > 0$. Moreover, we localize A and spread out to assume (abusively, from a notational standpoint) that G, B, and E begin life over A.

By [SGA $3_{\text{III new}}$, XXVI, 3.6, 3.20], the quotient E/B is representable by a projective A-scheme. Thus, due to the generic triviality of E and the valuative criterion of properness, there is a closed subscheme $Y \subset \mathrm{Spec}(A)$ of codimension $\geqslant 2$ such that $(E/B)_{\mathrm{Spec}(A)\backslash Y}$ has a section that generically lifts to E, in other words, such that $E_{\mathrm{Spec}(A)\backslash Y}$ reduces to a generically trivial $B_{\mathrm{Spec}(A)\backslash Y}$-torsor E^B. Consider the torus

$$T := B/\mathscr{R}_u(B) \quad \text{and the induced } T_{\mathrm{Spec}(A)\backslash Y}\text{-torsor} \quad E^T := E^B/\mathscr{R}_u(B).$$

Since Y is of codimension $\geqslant 2$ in the regular scheme $\mathrm{Spec}(A)$, by [CTS79, Cor. 6.9], there is a unique T-torsor $\widetilde{E^T}$ that extends E^T to all of $\mathrm{Spec}(A)$. Since the Grothendieck–Serre conjecture is known for tori [CTS87, Thm. 4.1 (i)] and $\widetilde{E^T}$ is generically trivial, the base change of $\widetilde{E^T}$ to $\mathrm{Spec}(R)$ is trivial. Thus, we may localize A further around the maximal ideals of R to assume that $\widetilde{E^T}$ is trivial, so that E^T is also trivial and $E_{\mathrm{Spec}(A)\backslash Y}$ reduces to an $\mathscr{R}_u(B)$-torsor.

We now apply Proposition 2.2.1 to obtain

- an affine open $U \subset \mathrm{Spec}(A)$ containing $\mathrm{Spec}(R)$;
- an affine open $S \subset \mathbb{A}^{d-1}_{\mathcal{O}}$; and
- a smooth $\mathcal{O}$-morphism $U \to S$ of pure relative dimension 1 such that $Y \cap U$ is S-finite.

Since R is a localization of the coordinate ring of U, we then set

$$C := U \times_S \mathrm{Spec}(R) \quad \text{and} \quad Z := (Y \cap U) \times_S \mathrm{Spec}(R).$$

The R-scheme C comes equipped with an R-point $s \in C(R)$ induced by the diagonal of $\mathrm{Spec}(R)$ over S, and, by base change, (i)–(iii) hold. Finally, we let $\mathscr{G}$, $\mathscr{B}$, and $\mathscr{E}$ be the base changes G_W, B_W, and E_W, so that, by construction, their s-pullbacks are G, B, and E, respectively. Since $U \subset \mathrm{Spec}(A)$ and, by construction, $E_{\mathrm{Spec}(A)\backslash Y}$ reduces to an $\mathscr{R}_u(B)$-torsor, the restriction of $\mathscr{E}$ to $C\backslash Z$ reduces to an $\mathscr{R}_u(\mathscr{B})$-torsor. In particular, (iv) and (v) hold. □

Remark 2.2.4. Proposition 2.2.3 is significantly simpler in the case when $\mathcal{O}$ is a field, in which the assumption that G be quasi-split could be removed with $\mathscr{E}$ in (v) being trivial over $C\backslash Z$. The main point is that in this case, even if Y is only of codimension $\geqslant 1$, as is immediate to arrange from the generic triviality of E, one nevertheless gets that Y' in the proof of Proposition 2.2.1 of codimension $\geqslant 2$ and the argument goes through with no need to worry about B and with $\mathscr{E}|_{C\setminus Z}$ even being a trivial torsor.

3. Changing the relative curve to equate $\mathscr{G}$ and G_C

To reach our main result on the Grothendieck–Serre conjecture we will gradually simplify the structure exhibited in Proposition 2.2.3 and eventually reduce to studying G-torsors over $\mathbb{A}^1_R$. As the first step towards this, in §3.3 we reduce to the case when the C-group scheme $\mathscr{G}$ appearing there is constant, that is, when $\mathscr{G}$ is simply G_C. The basic idea is captured by the invariance under Henselian pairs discussed in §3.1, although retaining the R-finiteness of Z requires a finer technique and uses suitable compactifications of torsors under tori built in §3.2.

3.1. Invariance under Henselian pairs for isomorphism classes of reductive groups

The goal of this section is to show in Corollary 3.1.3 that reductive group schemes lift uniquely across Henselian pairs. This result is not used elsewhere in the paper, but it helps motivate the constructions of §3.3 and it also generalizes [SGA $3_{\text{III new}}$, XXIV, 1.21], which treated the case of Henselian local rings. Its argument is based on the following mild extension of results from [BČ20].

Proposition 3.1.1. *Let (A, I) be a Henselian pair and let G be an A-group scheme.*

(a) *For a smooth A-scheme X,*

$$X(A) \twoheadrightarrow X(A/I), \quad \text{and, if } X \text{ is constant, then even} \quad X(A) \xrightarrow{\sim} X(A/I).$$

(b) *If G is smooth and quasi-separated, then*

$$H^1(A, G) \hookrightarrow H^1(A/I, G).$$

(c) *If G is smooth and quasi-affine, then*

$$H^1(A, G) \xrightarrow{\sim} H^1(A/I, G).$$

(d) *If $G \simeq H \rtimes \overline{G}$ is a semidirect product of an A-group $\overline{G}$ that becomes constant over a finite étale cover of A and a quasi-affine, smooth, normal A-subgroup H, the ring A/I is Noetherian, and its localizations at prime ideals are geometrically unibranch, then*

$$H^1(A, G) \xrightarrow{\sim} H^1(A/I, G).$$

Proof. In (a), every A-point and every A/I-point of X factor through a quasi-compact open subscheme, so we lose no generality by assuming that X is quasi-compact. In the constant case, X is then a finite union of copies of $\operatorname{Spec}(A)$. In effect, in this case, by [SP, 09XI], the clopen subsets of X are identified with those of $X_{A/I}$ via base change and, by [SP, 09ZL], this identification respects the property of mapping isomorphically to $\operatorname{Spec}(A)$ (resp., to $\operatorname{Spec}(A/I)$). Since sections of X (resp., of $X_{A/I}$) are precisely such clopens, the claimed bijectivity in the constant case of (a) follows.

For a general X in (a), we only seek the surjectivity, and we will reduce it to when X is not only quasi-compact but also quasi-separated, which is a case contained in [BČ20, Ex. 2.1.5] (and in which X was even allowed to be an algebraic space). For this reduction, we use a technique of Gabber that appeared in [Bha16, Rem. 4.6]. Namely, by [SP, 03K0] and its proof, there is a filtered direct system of étale X-schemes X_i that are quasi-compact and quasi-separated, are such that Zariski locally on X_i the structure map $X_i \to X$ is an open immersion, and are such that

$$\varinjlim_i X_i(R) \xrightarrow{\sim} X(R) \quad \text{for every } A\text{-algebra } R.$$

In particular, a fixed A/I-point of X lifts to an A/I-point of some X_i. Since X_i inherits A-smoothness from X, the known quasi-compact and quasi-separated case of (a) applied to X_i then implies the desired surjectivity $X(A) \twoheadrightarrow X(A/I)$.

Parts (b) and (c) are special cases of [BČ20, Thm. 2.1.6] (whose key input is Tannaka duality for algebraic stacks [HR19b, 1.5 (ii)] or [BHL17, 1.5], as already alluded to above).

For the rest of proof, we focus on the remaining part (d), in which, by the effectivity of fpqc descent for ind-quasi-affine schemes [SP, 0APK], our smooth A-group G is ind-quasi-affine.[3] In particular, (b) reduces us to only arguing the surjectivity in (d). For the latter, we first reduce to the case $G = \overline{G}$ with the help of the morphism of short exact sequences of pointed sets

$$\begin{array}{ccccc} H^1(A,H) & \longrightarrow & H^1(A,G) & \twoheadrightarrow & H^1(A,\overline{G}) \\ \downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow \\ H^1(A/I,H) & \longrightarrow & H^1(A/I,G) & \longrightarrow & H^1(A/I,\overline{G}) \end{array}$$

and twisting, as follows. The semidirect product decomposition ensures the displayed surjectivity of a top horizontal arrow and, by (c), the analogue of left vertical map stays bijective for every smooth, quasi-affine A-group, for instance, for every form of H for the fpqc topology. Moreover, any inner form of G is an extension of an inner form of $\overline{G}$ by a form of H. Thus, a diagram chase and the twisting bijections [Gir71, III, 2.6.1 (i)] show that every $G_{A/I}$-torsor lifts to a G-torsor granted that the same holds for $\overline{G}$, which achieves the promised reduction to $G = \overline{G}$.

In the remaining case in which G becomes constant over a finite étale cover A' of A, we fix a $G_{A/I}$-torsor E that is to be lifted to a G-torsor. By [SP, 0CB4], the localizations of A'/IA' at prime ideals are geometrically unibranch, so [SGA 3_{II}, X, 5.14] applied to the base changes of E to the connected components of $\mathrm{Spec}(A'/IA')$ shows that E trivializes over some finite étale cover $A'/IA' \to B$. That is, E is described by a section $g \in G(B \otimes_{A/I} B)$ that satisfies the cocycle condition. We use [SP, 09ZL] to lift B to a finite étale cover $A' \to \widetilde{B}$ and we apply (a) over $\widetilde{B} \otimes_A \widetilde{B}$ and over $\widetilde{B} \otimes_A \widetilde{B} \otimes_A \widetilde{B}$ to lift g to a section $\widetilde{g} \in G(\widetilde{B} \otimes_A \widetilde{B})$ that satisfies the cocycle condition with respect to $A \to \widetilde{B}$. This $\widetilde{g}$ gives rise to the desired G-torsor $\widetilde{E}$ that lifts E (by the effectivity of fpqc descent for ind-quasi-affine schemes [SP, 0APK], this $\widetilde{E}$ is even an ind-quasi-affine A-scheme). ☐

Remark 3.1.2. Without geometric unibranchedness, (d) fails even for $G = \mathbb{Z}$, see [BČ20, 2.1.7].

Corollary 3.1.3. *Let (A, I) be a Henselian pair and let G and G' be reductive A-groups.*

(a) *G has a Borel subgroup (resp., is quasi-split[4]) if and only if the same holds for $G_{A/I}$.*

(b) *Every A/I-group isomorphism $\iota\colon G_{A/I} \xrightarrow{\sim} G'_{A/I}$ lifts to an A-group isomorphism $\widetilde{\iota}\colon G \xrightarrow{\sim} G'$, and $\widetilde{\iota}$ may be chosen to respect fixed Borel subgroups G and G' granted that ι respects them.*

(c) *A reductive A/I-group H lifts (uniquely, by (b)) to a reductive A-group $\widetilde{H}$ if either*

 (i) *A/I is Noetherian and its localizations at prime ideals are geometrically unibranch; or*

[3] We recall from [SP, 0AP6] that an A-scheme is *ind-quasi-affine* if each of its quasi-compact opens is quasi-affine. By [SP, 0AP8], this property is fpqc local on A.

[4] Beyond semilocal bases, quasi-splitness is more than a Borel subgroup: recall from [SGA $3_{\mathrm{III\ new}}$, XXIV, 3.9] that G is quasi-split if it has a Borel subgroup $B \subset G$ containing a maximal torus $T \subset B$ such that on the scheme $\underline{\mathrm{Dyn}}(G)$ of Dynkin diagrams the line bundle given by the universal root space that is simple with respect to B is trivial.

(ii) *the center of H is A/I-fiberwise of rank $\leqslant 1$, for instance, H is semisimple.*

Proof.

(a) By [SGA $3_{\text{III new}}$, XXII, 5.8.3], the functor that parametrizes the Borel subgroups of G is representable by a smooth, projective A-scheme $\mathscr{B}$, so [BČ20, 2.1.4] shows that every A/I-point of $\mathscr{B}$ lifts to an A-point. Moreover, since the base is affine, any Borel subgroup of G has a maximal torus. In addition, by [SGA $3_{\text{III new}}$, §3.2ff], the scheme of Dynkin diagrams $\underline{\text{Dyn}}(G)$ is A-finite, so that, by [SP, 09XK], its coordinate ring is Henselian with respect to the ideal induced by I. To conclude that G is quasi-split if so is $G_{A/I}$, it then suffices to apply Proposition 3.1.1 (c) to the coordinate ring of $\underline{\text{Dyn}}(G)$ to conclude that

$$\text{Pic}(\underline{\text{Dyn}}(G)) \xrightarrow{\sim} \text{Pic}(\underline{\text{Dyn}}(G)_{A/I}).$$

(b) By [SGA $3_{\text{III new}}$, XXIV, 1.9], the functor $\underline{\text{Isom}}(G, G')$ that parametrizes group isomorphisms is a torsor under the automorphism functor $\underline{\text{Aut}}(G)$. Thus, [SGA $3_{\text{III new}}$, XXIV, 1.3] and [SP, 0AP8] ensure that $\underline{\text{Isom}}(G, G')$ is representable by an ind-quasi-affine, smooth A-scheme. In particular, by [BČ20, 2.1.4], every A/I-point ι of $\underline{\text{Isom}}(G, G')$ lifts to a desired A-point $\widetilde{\iota}$. With fixed Borel subgroups, the argument is the same but uses [SGA $3_{\text{III new}}$, XXIV, 2.2 (ii) and 2.1] to ensure analogous properties for the functor that parametrizes those group isomorphisms that preserve fixed Borel subgroups.

(c) By decomposing into clopens and lifting idempotents via [SP, 09XI], we may assume that the type of the geometric fibers of H is constant (see [SGA $3_{\text{III new}}$, XXII, 2.8]). We let $\mathbf{H}$ be a split reductive group over A of the same type as H, so that H is a form of $\mathbf{H}_{A/I}$ (see [SGA $3_{\text{III new}}$, XXII, 2.3]), and hence, by [SGA $3_{\text{III new}}$, XXIV, 1.17 (i)], corresponds to an element $x \in H^1(A/I, \underline{\text{Aut}}(\mathbf{H}))$. By Proposition 3.1.1 (c) and (d) and the structure of $\underline{\text{Aut}}(\mathbf{H})$ described in [SGA $3_{\text{III new}}$, XXIV, 1.3, 1.4, 1.6], the latter lifts to an element $\widetilde{x} \in H^1(A, \underline{\text{Aut}}(\mathbf{H}))$. This $\widetilde{x}$ corresponds to a desired reductive A-group $\widetilde{H}$ that lifts H. ☐

Remark 3.1.4. In Corollary 3.1.3 (c), some condition on A or H is necessary: it is not true that for every Henselian pair (A, I), every reductive A/I-group lifts to a reductive A-group. Indeed, such liftability already fails for tori: if it held, then, by considering those pairs in which A is normal (or even in which A is a Henselization of some affine space), we could conclude from [SGA 3_{II}, X, 1.3] that every torus over an affine base splits over a finite étale cover, contradicting [SGA 3_{II}, X, 1.6].

3.2. Compactifications of torsors under tori in a relative setting

The most delicate aspect of equating $\mathscr{G}$ and the base change of G lies in retaining a *projective* relative curve compactifying the affine open over which $\mathscr{G}$ is defined. Our technique for achieving this hinges on the Bertini theorem applied to suitable compactifications of torsors. Building the required compactifications is straight-forward when the base is a field: one may simply form a closure in a projective space. In our mixed characteristic setting, however, we need a finer technique because our torsor needs to be *fiberwise* dense in the compactification. The following result, which is a mild extension of [CTHS05, Cor. 1] (so also of previous work of Brylinski and Künnemann) will suffice for us because we will use the quasi-splitness of G to reduce to only compactifying torsors under tori.

Theorem 3.2.1. *For a Noetherian scheme S, an S-torus T that splits over a finite étale cover of S, and a T-torsor E that trivializes over a finite étale cover of S, there are a projective, smooth S-scheme $\overline{E}$, a right T-action on $\overline{E}$, and a T-equivariant S-fiberwise dense open immersion*

$$E \hookrightarrow \overline{E}.$$

Example 3.2.2. To illustrate the assumption on T, we recall from [SGA 3_{II}, X, 5.16] that T splits over a finite étale cover if the local rings of S are geometrically unibranch, for instance, if S is normal. In addition, if S is semilocal, then any finite étale cover that splits T also trivializes E.

Proof of Theorem 3.2.1. We begin by reducing to the case $E = T$ as follows. The contracted product $\overline{E} := \overline{T} \times_T E$ is an algebraic space (see [SP, 06PH]) that, since T is commutative, inherits a right T-action from E. Moreover, it comes equipped with an open immersion $E \hookrightarrow \overline{E}$ that is étale locally on S isomorphic to $T \hookrightarrow \overline{T}$. Thus, for the promised reduction, all we need to check is that this $\overline{E}$ is a projective S-scheme. For this, we will only use that there is a finite étale cover S' of S such that $\overline{E}_{S'}$ is a projective S'-scheme, for instance, S' could be a finite étale cover of S trivializing E. Indeed, consider the restriction of scalars $\overline{E}' := \mathrm{Res}_{S'/S}(\overline{E}_{S'})$. Its base change to a larger finite étale cover of S decomposes as a product of copies of $\overline{E}$, so, by [CGP15, A.5.8 and its proof] and [EGA II, 5.5.4 (i)], this $\overline{E}'$ is a projective S-scheme. By checking étale locally on S, the adjunction morphism $\overline{E} \hookrightarrow \overline{E}'$ is a closed immersion (compare with [CGP15, A.5.7]), so $\overline{E}$ is a projective S-scheme, as promised.

For the remainder of the proof we assume that $E = T$ and we show how to construct the desired $\iota\colon T \hookrightarrow \overline{T}$ by using the results of [CTHS05], where ι was constructed when the base is a field by using the theory of toric varieties. We decompose S into connected components to assume that it is connected and let S' be a finite étale cover of S splitting T. We may assume that S' is connected and then enlarge it to ensure that it is Galois over S with group G. We claim that it suffices to construct an analogous equivariant compactification $\iota'\colon T_{S'} \hookrightarrow \overline{T}'$ over S' granted that $\overline{T}'$ is equipped with a G-action (compatibly with the G-action on S', so that the action will be free on $\overline{T}'$ because it so already on S') and ι' is G-equivariant. Indeed, by [SP, 07S7], the projectivity of $\overline{T}'$ will ensure that the quotient $\overline{T} := \overline{T}'/G$ is an S-scheme. Moreover, by [SP, 0BD0 (with 0BD2, 0AH6, and 05B5)], this $\overline{T}$ will automatically be projective and smooth over S. Thus, we will be able to choose ι to be

$$T \cong T_{S'}/G \hookrightarrow \overline{T}'/G = \overline{T}.$$

To build $\overline{T}'$, we will use [CTHS05, Thm. 1] and the theory of toric varieties, and we begin by noting that, by functoriality, G acts on the cocharacter lattice $L := X_*(T_{S'})$, as well as on $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathcal{F}$ be a (rational, polyhedral) fan in $L_{\mathbb{R}}$ whose associated toric variety is $\mathbb{P}^{\mathrm{rk}(L)}$ (see, for instance, [Dan78, 5.3]). This fan need not be G-invariant but, by [CTHS05, Thm. 1], there is a (rational, polyhedral) fan $\mathcal{F}'$ in $L_{\mathbb{R}}$ that is G-invariant, is a subdivision of $\mathcal{F}$, and is projective and smooth in the sense that its associated toric variety is projective and smooth (projectivity and smoothness can also be expressed combinatorially in terms of $\mathcal{F}'$, see [Dan78, 3.3] and [CTHS05, Prop. 1]). The construction [Dan78, 5.2] that builds the toric variety associated to $\mathcal{F}'$ adapts to any base, so we obtain a flat, finitely presented S'-scheme $\overline{T}'$ equipped with a $T_{S'}$-action, a compatible G-action, and an S'-fiberwise dense, $T_{S'}$-equivariant and G-equivariant open immersion $\iota'\colon T_{S'} \hookrightarrow \overline{T}'$ over S'. By [Dan78, 3.3] applied S'-fiberwise, $\overline{T}'$ is S'-smooth, so it remains to argue that it is projective over S'.

Due to its combinatorial definition, the S'-scheme $\overline{T}'$ descends to a scheme over $\mathrm{Spec}(\mathbb{Z})$, so [Dan78, 5.5.6] and its proof, which is based on the finer than usual form [EGA II, 7.3.10 (ii)] of the valuative criterion of properness, imply that $\overline{T}'$ is proper over S'. In combinatorial terms, the fact that $\mathcal{F}'$ is projective means that there exists a function $h\colon L_{\mathbb{R}} \to \mathbb{R}$ that is *strictly upper convex* in the sense that, letting $\mathcal{F}'^{\,\mathrm{top}} \subset \mathcal{F}'$ denote the subset of top-dimensional cones, there are linear forms

$$\{\ell_\sigma\}_{\sigma \in \mathcal{F}'^{\,\mathrm{top}}} \subset \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = X^*(T)$$

satisfying $\ell_\sigma(x) \geqslant h(x)$ for all $x \in L_{\mathbb{R}}$ with equality if and only if $x \in \sigma$ (see [CTHS05, Prop. 1] and [Oda88, 2.12]). This last requirement uniquely determines the characters ℓ_σ because each σ is top-dimensional. Thus, as in [Oda88, Prop. 2.1 (i) and its proof], the function h, more precisely, the ℓ_σ, define a line bundle $\mathscr{L}_h$ on $\overline{T}'$. By [EGA IV$_3$, 9.6.4], checking that $\mathscr{L}_h$ is ample over S' can be done S'-fiberwise. Consequently, [Oda88, Prop. 2.1 (vi), Cor. 2.14, and their proofs] imply the S'-ampleness of $\mathscr{L}_h$, and hence the S'-projectivity of $\overline{T}'$. □

We now use the compactifications supplied by Theorem 3.2.1 and the Bertini theorem of Lemma 2.1.2 to build *finite* étale covers that over a semilocal, geometrically unibranch bases trivialize torsors under tori compatibly with predetermined trivializations over closed subschemes. This is based on the following general lemma that we learned from the argument of [OP01, Lem. 7.2] and that was also pointed out to us by Uriya First.

Lemma 3.2.3. *For a semilocal ring A, an ideal $I \subset A$, a projective, finitely presented A-scheme X, an A-smooth open $U \subset X$ of pure relative dimension d over A that is dense in the closed A-fibers of X, and an A/I-point $u \in U(A/I)$, there are a finite étale A-algebra $\widetilde{A}$ with an A/I-point $a \colon \widetilde{A} \twoheadrightarrow A/I$ and a $\widetilde{u} \in U(\widetilde{A})$ whose a-pullback is u.*

Proof. By replacing U by a finite union of some of its open affines if needed, we may assume that U is quasi-compact. Then a limit argument allows us to assume (mostly for comfort) that A is Noetherian. By decomposing into connected components, we may also assume that $\mathrm{Spec}(A)$ is connected. Finally, we fix a projective embedding $X \hookrightarrow \mathbb{P}^n_A$ and postcompose it with a linear change of projective coordinates if necessary to arrange that u is the origin A/I-point $[0 : \ldots : 0 : 1] \in \mathbb{P}^n_A(A/I)$.

Let $C \subset \mathrm{Spec}(A)$ be the union of the closed points. Since $(X \backslash U)_C$ is of dimension less than d, we may apply Lemma 2.1.2, with $Z = Z_0$ there being the image of our u_C, to find hypersurfaces $H_1, \ldots, H_d \subset X_C$ of large enough and constant on C degrees such that $H_1 \cap \ldots \cap H_d$ lies in U_C, is finite étale over C, and contains u. Granted that these degrees are sufficiently large, [EGA III$_1$, 2.2.4] allows us to lift the H_i to hypersurfaces $H'_1, \ldots, H'_d \subset X_{C'}$ where $C' \subset \mathrm{Spec}(A)$ is the closed subscheme that is the union of $\mathrm{Spec}(A/I)$ and C. Moreover, we may choose these lifts in such a way that they contain u: indeed, we arranged $u_{A/I}$ to be $[0 : \ldots : 0 : 1]$, so ensuring $u \in H'_i(A/I)$ amounts to lifting a defining equation of H_i in such a way that the coefficient of the monomial that is a power of the last variable stays zero. Once such H'_i of large degrees are fixed, we apply [EGA III$_1$, 2.2.4] again to lift them to hypersurfaces $\widetilde{H}_1, \ldots, \widetilde{H}_d \subset X$, which, by construction, contain u.

By construction, the scheme-theoretic intersection $\widetilde{Z} := \widetilde{H}_1 \cap \ldots \cap \widetilde{H}_d$ lies in U and contains u. By the openness of the quasi-finite locus [SP, 01TI] and the finiteness or proper, quasi-finite morphisms [SP, 02OG], the A-scheme $\widetilde{Z}$ is finite. By [EGA IV$_3$, 11.3.8], it is A-flat at its closed points, so the openness of the flat locus [EGA IV$_3$, 11.3.1] ensures that it is A-flat. Thus, by checking over C, we find that $\widetilde{Z}$ is étale over A. In conclusion, $\widetilde{Z} = \mathrm{Spec}(\widetilde{A})$ for a finite étale A-algebra $\widetilde{A}$ that is equipped with an A/I-point $a \colon \widetilde{A} \twoheadrightarrow A/I$ that corresponds to u. The inclusion $\widetilde{Z} \subset U$ corresponds to the desired $\widetilde{A}$-point $\widetilde{u} \in U(\widetilde{A})$ whose a-pullback is u. □

Corollary 3.2.4. *For a semilocal Noetherian ring A whose localizations at prime ideals are geometrically unibranch, an ideal $I \subset A$, an A-torus T, a T-torsor E, and an $e \in E(A/I)$, there are a finite étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a \colon \widetilde{A} \twoheadrightarrow A/I$ and an $\widetilde{e} \in E(\widetilde{A})$ whose a-pullback is e.*

If we do not require $\widetilde{A}$ to be *finite* over A, then the claim follows from a general invariance of sets of torsors under Henselian pairs [BČ20, 2.1.6] applied to the Henselization of A along I.

Proof. By Theorem 3.2.1 and Example 3.2.2, there are a projective, smooth A-scheme $\overline{E} \subset \mathbb{P}^n_A$ of relative dimension $d := \mathrm{rk}(T)$ and an A-fiberwise dense open immersion $E \hookrightarrow \overline{E}$. To obtain the desired conclusion, it then remains to apply Lemma 3.2.3 with $X = \overline{E}$ and $U = E$. ☐

Remark 3.2.5. For context, we recall from [SGA $3_{\mathrm{III\ new}}$, XXIV, 4.1.6] that any torsor under a reductive group scheme over a semilocal Noetherian ring whose localizations at prime ideals are geometrically unibranch trivializes over a finite étale cover. We do not know, however, if the finer property recorded in Corollary 3.2.4 also holds beyond the case of torsors under tori settled above.

3.3. Changing the Cohen–Macaulay relative curve to make $\mathscr{G}$ constant

We now reduce to the situation in which the group $\mathscr{G}$ is simply the base change G_D in Proposition 3.3.2. The intuition for this is supplied by Corollary 3.1.3: the group $\mathscr{G}$ becomes isomorphic to G_D over an étale neighborhood of the section s in D, so the task is to shrink D around s. The main subtlety lies in retaining the R-finiteness of the closed subscheme $Z \subset D$, which means that we need to equate $\mathscr{G}$ and G_D over a *finite* étale neighborhood of s in D. The eventual source of such a neighborhood is the following refinement of Corollary 3.1.3 that is similar in spirit to [OP01, Prop. 7.1], [PSV15, Prop. 5.1], or [Pan16, Prop. 6.0.18]. In its proof, to overcome mixed characteristic phenomena not witnessed in these references we rely on compactifications of torsors under tori constructed in §3.2.

Proposition 3.3.1. *For a semilocal Noetherian ring A whose localizations at prime ideals are geometrically unibranch, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibers have the same type, Borel subgroups $B \subset G$ and $B' \subset G'$, and an A/I-group isomorphism*

$$\iota\colon G_{A/I} \xrightarrow{\sim} G'_{A/I} \quad \textit{with} \quad \iota(B_{A/I}) = B'_{A/I},$$

there are

(i) *a finite étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a\colon \widetilde{A} \twoheadrightarrow A/I$; and*

(ii) *an $\widetilde{A}$-group isomorphism $\widetilde{\iota}\colon G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ with $\widetilde{\iota}(B_{\widetilde{A}}) = B'_{\widetilde{A}}$ whose a-pullback is ι.*

It is key that $A \to \widetilde{A}$ be finite: without this, the assertion would be a special case of Corollary 3.1.3.

Proof. Similarly to the proof of Corollary 3.1.3, we consider the smooth, ind-quasi-affine scheme

$$X := \underline{\mathrm{Isom}}((G, B), (G', B')).$$

We need to show that X has the following property, which we call *property* $(\star)$: for every $\iota \in X(A/I)$, there are a finite étale A-algebra $\widetilde{A}$, an A/I-point $a\colon \widetilde{A} \twoheadrightarrow A/I$, and a $\widetilde{\iota} \in X(\widetilde{A})$ with a-pullback ι.

By [SGA $3_{\mathrm{III\ new}}$, XXIV, 2.2 (ii)], the Borel subgroup $\overline{B} \subset G^{\mathrm{ad}}$ that corresponds to $B \subset G$ acts freely on X and the quotient $\overline{X} := X/\overline{B}$ is the functor that parametrizes exterior isomorphisms between G and G'. By [SGA $3_{\mathrm{III\ new}}$, XXIV, 1.10 and 1.3 (iii)], the condition on the geometric fibers of G and G' ensures that $\overline{X}$ is representable by a fiberwise nonempty A-scheme that becomes constant étale locally on A. Thus, by [SGA 3_{II}, X, 5.14] (with [EGA I, 6.1.9]), the geometrically unibranch assumption ensures that the connected components of $\overline{X}$ are open subschemes that are finite étale over A. The A/I-point $\overline{\iota}$ of $\overline{X}$ induced by ι meets finitely many such components, whose union is then the spectrum of a finite étale A-algebra A'. In effect, we may replace A by A' (whose localizations at prime ideals are still geometrically unibranch by [SP, 06DM]), the ideal I by $\mathrm{Ker}(A' \twoheadrightarrow A/I)$, and X by the $\overline{B}_{A'}$-torsor $X \times_{\overline{X}} \mathrm{Spec}(A')$ to reduce to showing that every B-torsor Y has property $(\star)$.

By [SGA $3_{\mathrm{III\ new}}$, XXVI, 2.1], the unipotent radical $U := \mathscr{R}_u(\overline{B})$ is an iterated extension of powers of $\mathbb{G}_a$. Thus, for any $\widetilde{A}$ and a that appear in property $(\star)$, we have $H^1(\widetilde{A}, U) = \{*\}$ and the map

$U(a)\colon U(\widetilde{A}) \twoheadrightarrow U(A/I)$ is surjective. Consequently, letting $\overline{T} := \overline{B}/U$ be the indicated A-torus, we are reduced to showing property ($\star$) for the $\overline{T}$-torsor $\overline{Y} := Y/U$. This, however, is Corollary 3.2.4. ☐

We are ready to simplify Proposition 2.2.3 as follows.

Proposition 3.3.2. *For a semilocal Dedekind ring $\mathcal{O}$, the localization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$ that all lie over maximal ideals $\mathfrak{p}' \subset \mathcal{O}$ with $k_{\mathfrak{p}}/k_{\mathfrak{p}'}$ separable, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

(i) *a smooth, affine R-scheme C of pure relative dimension* 1;

(ii) *a section $s \in C(R)$;*

(iii) *an R-finite closed subscheme $Z \subset C$;*

(iv) *a G_C-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(B)$-torsor over $C \backslash Z$.*

Proof. We decompose $\mathrm{Spec}(R)$ into connected components to assume that R is a domain. By Proposition 2.2.3, there are such C, s, Z, and $\mathscr{E}$, except that $\mathscr{E}$ there is a torsor under a quasi-split reductive C-group scheme $\mathscr{G}$ that may not be G_C but that comes equipped with a Borel C-subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$. We replace C by its connected component containing the image of s to arrange that C be connected. Thus, the geometric C-fibers of $\mathscr{G}$ and G_C are of constant types, so that, by the condition on the s-pullback, these types are the same.

To replace $\mathscr{B} \subset \mathscr{G}$ by $B_C \subset G_C$, we first use prime avoidance [SP, 00DS] to construct the semilocalization $\mathrm{Spec}(A)$ of C at the union of the closed points of Z and of those of the image of s. Since $\mathrm{Spec}(A)$ lies in the R-smooth locus of C, the ring A is regular. The image of s gives rise to a closed subscheme $\mathrm{Spec}(R) \subset \mathrm{Spec}(A)$ cut out by an ideal $I \subset A$ and, by assumption, $\mathscr{B}_{A/I} \subset \mathscr{G}_{A/I}$ agrees with $B \subset G$. Thus, by Proposition 3.3.1, there is a finite étale $\mathrm{Spec}(A)$-scheme $\mathrm{Spec}(\widetilde{A})$ equipped with an R-point $\widetilde{s}$ lifting s such that $\mathscr{B}_{\widetilde{A}} \subset \mathscr{G}_{\widetilde{A}}$ is isomorphic to $B_{\widetilde{A}} \subset G_{\widetilde{A}}$ compatibly with the fixed identification of $\widetilde{s}$-pullbacks. We may spread out $\mathrm{Spec}(\widetilde{A}) \to \mathrm{Spec}(A)$ to a finite étale morphism $\widetilde{C} \to C'$ for some affine open $C' \subset C$ that contains Z and the image of s, while preserving an $\widetilde{s} \in \widetilde{C}(R)$ and an isomorphism between $\mathscr{B}_{\widetilde{C}} \subset \mathscr{G}_{\widetilde{C}}$ and $B_{\widetilde{C}} \subset G_{\widetilde{C}}$. To arrive at the desired conclusion, it then remains to replace C, s, Z, and $\mathscr{E}$ by $\widetilde{C}$, $\widetilde{s}$, $Z \times_C \widetilde{C}$, and $\mathscr{E} \times_C \widetilde{C}$, respectively. ☐

4. Changing the relative curve to $\mathbb{A}^1_R$

Having corrected $\mathscr{G}$, our next goal is to reduce to the case when the affine relative curve C is $\mathbb{A}^1_R$. After building a required map $C \to \mathbb{A}^1_R$ in §4.1, we achieve this reduction by patching in §4.2.

4.1. A Lindel trick in the setting of Cohen–Macaulay relative curves

In §4.2, we will use patching to replace the relative curve C in Proposition 3.3.2 by $\mathbb{A}^1_R$. For this, we need a suitable flat morphism $C \to \mathbb{A}^1_R$, whose construction is the goal of this section. We summarize the resulting relevant for us refinement of Proposition 3.3.2 in Proposition 4.1.5.

For patching to apply, it is key to arrange that on some open subscheme $C' \subset C$ containing Z our desired flat map $C \to \mathbb{A}^1_R$ does not change Z in the sense that the latter is precisely the scheme-theoretic preimage in C' of some closed subscheme $Z' \subset \mathbb{A}^1_R$ to which Z maps isomorphically. This is reminiscent of Lindel's insight [Lin81, p. 321, Lemma] that led to the resolution of the Bass–Quillen conjecture in the "geometric" case and says that an étale map $B \to A$ of local rings with the same residue field is an isomorphism modulo powers of a well-chosen element in the maximal ideal of B (compare also with [CTO92, Lem. 1.2] or [CT95, §3.7 and proof of Thm. 5.1.1]). In our situation, however, there is a basic obstruction to the existence of Z': if some residue fields of R are finite, then Z could have too many rational points to fit into $\mathbb{A}^1_R$. The purpose of the following minor adjustment essentially taken from the literature is to circumvent this obstacle.

Lemma 4.1.1. *For a semilocal ring R, a quasi-projective, finitely presented R-scheme C, its R-finite closed subscheme Z, and an $s \in Z(R)$, there is a finite morphism $\widetilde{C} \to C$ that is étale at the points in $\widetilde{Z} := Z \times_C \widetilde{C}$ such that s lifts to $\widetilde{s} \in \widetilde{C}(R)$ and, for every maximal ideal $\mathfrak{m} \subset R$, we have*

$$\#\{z \in \widetilde{Z}_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} < \#\{z \in \mathbb{A}^1_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} \quad \text{for every} \quad d \geqslant 1$$

(*a vacuous condition if the residue field $k_\mathfrak{m}$ is infinite*).

Proof. The lemma is a variant of, for instance, [Pan17, Lem. 6.1] or [Fed16b, Lem. 5.1], and we will prove it by using similar arguments as there due to Panin. Since R is semilocal, the finite R-scheme Z has finitely many closed points, which all lie over maximal ideals of R. Thus, we begin by using Lemma 2.1.1 to construct the semilocalization S of C at the closed points of Z, so that Z is also a closed subscheme of S and $s \in S(R)$. It then suffices to construct a finite étale S-scheme $\widetilde{S}$ such that s lifts to an R-point $\widetilde{s} \in \widetilde{S}(R)$ and the preimage $\widetilde{Z} \subset \widetilde{S}$ of Z satisfies the displayed inequalities: indeed, once this is done, we may first spread $\widetilde{S}$ out to a finite étale scheme over an open neighborhood of S in C and then form its schematic image [SP, 01R8] in the factorization supplied by Zariski's main theorem [EGA IV$_4$, 18.12.13] to further extend to a desired finite $\widetilde{C} \to C$.

We view s as a closed subscheme $\mathrm{Spec}(R) \subset Z$ and we list the closed points of Z (that is, of S):

- the closed points $y_1, \dots, y_m$ of Z not in s with an infinite residue field;
- the closed points $z_1, \dots, z_n$ of Z not in s with a finite residue field;
- the closed points $y'_1, \dots, y'_{m'}$ of s with an infinite residue field;
- the closed points $z'_1, \dots, z'_{n'}$ of s with a finite residue field.

For any $N > 1$, we may choose monic polynomials

- $f_{y_i} \in k_{y_i}[t]$ that are products of N distinct linear factors; and
- $f_{z_j} \in k_{z_j}[t]$ that are irreducible of degree N.

Likewise, we we may choose a monic polynomial $f_s \in tR[t]$ of degree N such that

- the image of f_s in each $k_{y'_i}[t]$ is a product of N distinct linear factors; and
- the image of f_s in each $k_{z'_j}[t]$ is a product of t and an irreducible polynomial not equal to t.

Finally, since $s \sqcup \bigsqcup_{i=1}^m y_i \sqcup \bigsqcup_{j=1}^n z_j$ is a closed subscheme of S, by lifting coefficients we may choose a monic polynomial $f \in \Gamma(S, \mathscr{O}_S)[t]$ of degree N that restricts to f_{y_i} on each y_i, to f_{z_j} on each z_j, and to f_s on s. This f defines a finite étale S-scheme $\widetilde{S}$, which, by construction, is equipped with an R-point $\widetilde{s} \in \widetilde{S}(R)$ lifting s (cut out by the factor t of f_s) and is such that the number of closed points with finite residue fields in the preimage $\widetilde{Z} \subset \widetilde{S}$ of Z stays bounded as N grows but, except for the points in $\widetilde{s}$, the cardinalities of these residue fields grow uniformly. Thus, since, for a finite field $\mathbb{F}$, the number of closed points of $\mathbb{A}^1_\mathbb{F}$ with a given residue field grows unboundedly together with the degree of that residue field over $\mathbb{F}$, for large N our $\widetilde{S}$ meets the requirements. □

We turn to the Lindel trick in our setting, namely, to building the desired flat map $C \to \mathbb{A}^1_R$ in Lemma 4.1.3. Its numerous variants appeared in works of Panin, for instance, in [OP99, §5], [PSV15, Thm. 3.4], or [Pan17, Thm. 3.8], but with the more stringent smoothness assumption on C, and preparation lemmas of similar flavor can be traced back at least to [Gab94, Lem. 3.1] (compare also with [CTHK97, Thm. 3.1.1]). As we show, Cohen–Macaulayness of C suffices. The argument uses the following simple lemma that characterizes residue fields of closed points on smooth curves.

Lemma 4.1.2. *For a field k, a smooth connected k-curve C, and a closed point $c \in C$, the extension k_c/k is generated by a single element, that is, k_c is the residue field of a closed point of $\mathbb{A}^1_k$.*

Proof. By [EGA IV_4, 17.11.4], an open neighborhood $U \subset C$ of c has an étale k-morphism $U \to \mathbb{A}^1_k$. Thus, there is a subextension ℓ/k of k_c/k generated by a single element with k_c/ℓ separable. By the primitive element theorem, we need to check that this forces k_c/k to only have finitely many subextensions k'/k. Since there are finitely many possibilities for $k' \cap \ell$, we replace k by $k' \cap \ell$ to reduce to considering those k' for which $k' \cap \ell = k$. Like any finite separable extension, the separable closure of k in k_c has only finitely many subextensions. Thus, there are finitely many possibilities for the maximal separable subextension k''/k of k'/k. By replacing k by k'' and ℓ by $k''\ell$, we therefore reduce to the case when k'/k is purely inseparable. Then the subextension $k'\ell/\ell$ of the separable extension k_c/ℓ is also purely inseparable, to the effect that $k' \subset \ell$. However, ℓ/k is generated by a single element, so, by the primitive element theorem, it has only finitely many subextensions. □

Lemma 4.1.3. *For*

- *a semilocal ring R;*
- *a flat, affine R-scheme C with Cohen–Macaulay fibers of pure dimension* 1;
- *R-finite closed subschemes $Y \subset C$ and $Z \subset C^{\mathrm{sm}}$ such that, for every maximal ideal $\mathfrak{m} \subset R$,*
$$\#\{z \in Z_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} < \#\{z \in \mathbb{A}^1_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} \quad \text{for every} \quad d \geqslant 1$$
(*a vacuous condition if the residue field $k_\mathfrak{m}$ is infinite*);

there are

(i) *an affine open $C' \subset C$ containing Y and Z;*

(ii) *a quasi-finite, flat R-map $C' \to \mathbb{A}^1_R$ that maps Z isomorphically onto a closed subscheme*
$$Z' \subset \mathbb{A}^1_R \quad \text{such that} \quad Z \cong Z' \times_{\mathbb{A}^1_R} C';$$

so that, in particular, $C' \to \mathbb{A}^1_R$ is étale along Z and, for every $n \geqslant 0$, maps the n-th infinitesimal neighborhood of Z in C' isomorphically to the n-th infinitesimal neighborhood of Z' in $\mathbb{A}^1_R$.

Proof.[5] The étaleness follows from the flatness and the isomorphy over Z' of the map $C' \to \mathbb{A}^1_R$, and it implies the infinitesimal neighborhood aspect. For the rest, every closed point $z \in Z$ lies over some maximal ideal $\mathfrak{m} \subset R$ and, since $z \in C^{\mathrm{sm}}_{k_\mathfrak{m}}$, the ideal sheaf $\mathscr{I}_z \subset \mathscr{O}_{C_{k_\mathfrak{m}}}$ is generated at z by a uniformizer $u_z \in \mathscr{O}_{C_{k_\mathfrak{m}}, z}$. Consequently, by [BouAC, IX, §3, no. 3, Thm. 1], the thickening
$$\varepsilon_z := \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{k_\mathfrak{m}}}}(\mathscr{O}_{C_{k_\mathfrak{m}}}/\mathscr{I}_z^2) \quad \text{is isomorphic to} \quad \mathrm{Spec}(k_z[u_z]/(u_z^2)).$$
Letting y range over the closed points of Y not in Z and z range over the closed points of Z, we set
$$\varepsilon_Y := \textstyle\bigsqcup_y y \subset C, \quad \varepsilon_Z := \bigsqcup_z \varepsilon_z \subset C, \quad \text{and} \quad \varepsilon := \varepsilon_Y \sqcup \varepsilon_Z = \bigsqcup_y y \sqcup \bigsqcup_z \varepsilon_z \subset C.$$

[5] We loosely follow [Pan17, proof of Thm. 3.8], with several improvements and simplifications whose purpose is to avoid assuming that C be R-smooth or that R be the semilocal ring at finitely many closed points of a smooth variety over a field. Notably, in Remark 4.1.4 we give a more direct and more general argument for the final portion of *loc. cit.*

By Lemma 4.1.2 and the assumption on the numbers of points of $Z_{k_\mathfrak{m}}$, we may find an R-morphism

$$j\colon \varepsilon \to \bigsqcup_{\mathfrak{m}} \mathbb{A}^1_{k_\mathfrak{m}} \subset \mathbb{A}^1_R \quad \text{that restricts to a closed immersion} \quad \varepsilon_Z \hookrightarrow \mathbb{A}^1_R$$

and, for each $\mathfrak{m}$, maps the points of ε_Y above $\mathfrak{m}$ to an $k_\mathfrak{m}$-point of $\mathbb{A}^1_{k_\mathfrak{m}} \backslash \varepsilon_Z$. We fix two disjoint sets of closed points

$$d_1, \ldots, d_n \in C\backslash(Y \cup Z) \quad \text{and} \quad d'_1, \ldots, d'_{n'} \in C\backslash(Y \cup Z)$$

lying over maximal ideals of R such that each set jointly meets every irreducible component of every closed R-fiber of C. Since $\varepsilon \cup \bigsqcup_{i=1}^{n} d_i \cup \bigsqcup_{i=1}^{n'} d'_i$ is a closed subscheme of C, we may find an $s \in \Gamma(C, \mathscr{O}_C)$ that

- vanishes on every d_i but does not vanish on any d'_i;
- on ε equals the j-pullback of the coordinate of $\mathbb{A}^1_R$.

By mapping the coordinate of $\mathbb{A}^1_R$ to s, we obtain an R-morphism

$$\pi\colon C \to \mathbb{A}^1_R.$$

The behavior of s at d_i and d'_i ensures that the locus where π is quasi-finite, which, by [SP, 01TI], is an open of C, contains every closed R-fiber of C. In particular, we may use prime avoidance [SP, 00DS] to replace C by some affine open subset containing Y and Z (equivalently, containing the closed points of Y and Z) to arrange that π is quasi-finite.

Since C is R-flat with Cohen–Macaulay fibers of pure dimension 1, the flatness criteria [EGA IV_2, 6.1.5], [EGA IV_3, 11.3.11] ensure that π is flat. By construction $\pi|_\varepsilon = j$, so, by checking on the closed R-fibers, [EGA IV_4, 17.11.1] shows that π is étale around Z. Since $Z_{k_\mathfrak{m}}$ and ε_Z have the same underlying reduced subscheme $\bigsqcup_z z$, the agreement with j also shows that $\pi|_{Z_{k_\mathfrak{m}}}$ is a closed immersion. Since Z is R-finite, Nakayama lemma [SP, 00DV] then ensures that $\pi|_Z$ is also a closed immersion, so that π maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_R$.

A section of a separated, étale morphism is an isomorphism onto a clopen subscheme, so the étaleness of π around Z gives a decomposition

$$\pi^{-1}(Z') = Z \sqcup Z''$$

for some R-quasi-finite closed subscheme $Z'' \subset C$. By the agreement with j, the image under π of every closed point of Y not in Z does not lie in Z', to the effect that $Y \cap Z'' = \varnothing$. Thus, prime avoidance [SP, 00DS] supplies a global section of C that vanishes on Z'' but does not vanish at any closed point of Y or Z. By inverting this section, we obtain the desired affine open $C' \subset C$. □

Remark 4.1.4. If $\mathrm{Spec}(R)$ is connected, then any R-(finite locally free) closed subscheme $Z' \subset \mathbb{A}^1_R$ is cut out by a monic polynomial. This holds for any ring R with a connected spectrum: the coordinate t of $\mathbb{A}^1_R$ acts by multiplication on the projective R-module $\Gamma(Z', \mathscr{O}_{Z'})$, the characteristic polynomial of this action is monic and cuts out an R-(finite locally free) closed subscheme $H \subset \mathbb{A}^1_R$, and Cayley–Hamilton implies that $Z' \subset H$ inside $\mathbb{A}^1_R$, so, by comparing ranks over R, even $Z' = H$.

We now refine Proposition 3.3.2 to the following statement adapted to passing to $\mathbb{A}^1_R$ via patching.

Proposition 4.1.5. *For a semilocal Dedekind ring $\mathcal{O}$, the localization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$ that all lie over maximal ideals $\mathfrak{p}' \subset \mathcal{O}$ with $k_\mathfrak{p}/k_{\mathfrak{p}'}$ separable, a quasi-split reductive R-group G, a Borel R-subgroup $B \subset G$, and a generically trivial G-torsor E, there are*

(i) *a smooth, affine R-scheme C of pure relative dimension* 1;

(ii) *a section $s \in C(R)$;*

(iii) *an R-finite closed subscheme $Z \subset C$;*

(iv) *a G_C-torsor $\mathscr{E}$ whose s-pullback is E such that $\mathscr{E}$ reduces to a $\mathscr{R}_u(B)$-torsor over $C \backslash Z$;*

(v) *a flat R-map $C \to \mathbb{A}^1_R$ that maps Z isomorphically onto a closed subscheme $Z' \subset \mathbb{A}^1_R$ with*

$$Z \cong Z' \times_{\mathbb{A}^1_R} C.$$

Proof. Proposition 3.3.2 supplies C, s, Z, and $\mathscr{E}$ that satisfy the present (i)–(iv). We view s as a closed subscheme of C and we apply Lemma 4.1.1 to the R-finite closed subscheme $(Z \cup s)^{\mathrm{red}}$ of C to see that we may change C to assume that, in addition, for every maximal ideal $\mathfrak{m} \subset R$,

$$\#\{z \in Z_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} < \#\{z \in \mathbb{A}^1_{k_\mathfrak{m}} \,|\, [k_z : k_\mathfrak{m}] = d\} \quad \text{for every} \quad d \geqslant 1.$$

This allows us to apply Lemma 4.1.3 with $Y = s$ to shrink C and to arrange (v). □

4.2. Descending to $\mathbb{A}^1_R$ via patching

With the suitable flat map $C \to \mathbb{A}^1_R$ already built in Proposition 4.1.5, descending the G_C-torsor $\mathscr{E}_C$ to $\mathbb{A}^1_R$ concerns patching along the closed subscheme Z. Since our Z need not be cut out by a single equation (relatedly, $C \backslash Z$ need not be affine), this patching is slightly more delicate than its most frequently encountered instances. Its precise statement is captured by the following lemma, which follows from more general results of Moret-Bailly [MB96] (for our purposes, we could also get by with the more basic patching of Ferrand–Raynaud [FR70, Prop. 4.2]).

Lemma 4.2.1. *Let $S' \to S$ be an affine, flat scheme map whose base change to a closed subscheme $Z \subset S$ cut out by a quasi-coherent ideal sheaf of finite type is an* isomorphism *and let $U' \to U$ be the base change to $U := S \backslash Z$. For a quasi-affine, flat, finitely presented S-group scheme G, base change induces an equivalence from the category of G-torsors to the category of triples consisting of a $G_{S'}$-torsor, a G_U-torsor, and a $G_{U'}$-torsor isomorphism between the two base changes to U'.*

Of course, the isomorphism condition $Z \times_S S' \xrightarrow{\sim} Z$ ensures that S' and U jointly cover S.

Proof. By [SP, 06FI], the classifying S-stack $\mathbf{B}G$ is algebraic and, by descent, its diagonal inherits quasi-affineness from G. Thus, the assertion is a special case of [MB96, Cor. 6.5.1 (a)]. □

To be able to apply Lemma 4.2.1 in our setting, we need to descend $\mathscr{E}_{U \backslash Z}$ to a G-torsor over $\mathbb{A}^1_R \backslash Z'$. To achieve this, we will use the following excision result that is similar (but simpler) than its counterparts that recently appeared in [ČS19, Thm. 5.4.4] and in [BČ20, §2.3].

Lemma 4.2.2. *Let $S' \to S$ be a flat morphism of affine, Noetherian schemes whose base change to a closed subscheme $Z \subset S$ is an isomorphism, and let $U' \to U$ be the base change to $U := S \backslash Z$.*

(a) *For a quasi-coherent $\mathscr{O}_S$-module $\mathscr{F}$ (or even a complex of such $\mathscr{O}_S$-modules), we have*

$$R\Gamma_Z(S, \mathscr{F}) \xrightarrow{\sim} R\Gamma_Z(S', \mathscr{F}_{S'}).$$

(b) *For an affine, smooth S-group (resp., U-group) F with a filtration $F = F_0 \supset F_1 \supset \ldots \supset F_n = 0$ by normal, affine, smooth S-subgroups (resp., U-subgroups) such that, for all $i \geqslant 0$, the quotient F_i/F_{i+1} is a vector group (resp., that is also central in F/F_{i+1}), the map*

$$H^1(U, F) \to H^1(U', F) \quad \textit{has trivial kernel (resp., is surjective).}$$

Proof.

(a) We let A and A' be the coordinate rings of S and S', respectively. By [SP, 0ALZ, 0955],

$$R\Gamma_Z(S, \mathscr{F}) \otimes^{\mathbb{L}}_A A' \xrightarrow{\sim} R\Gamma_Z(S', \mathscr{F}_{S'}).$$

Thus, since A' is A-flat, to obtain (a) it remains to note that, by [SP, 05E9], we have

$$H^i_Z(S, \mathscr{F}) \xrightarrow{\sim} H^i_Z(S, \mathscr{F}) \otimes_A A' \quad \text{for all} \quad i \in \mathbb{Z}.$$

(b) In the case when F is an S-group, the vanishing of quasi-coherent cohomology of affine schemes and the assumed filtration show that both $H^1(S, F)$ and $H^1(S', F)$ vanish. Thus, the assertion about the kernel simply amounts to the claim that every F_U-torsor that trivializes over U' extends to an F-torsor. This, however, is immediate from Lemma 4.2.1.

For the surjectivity assertion, we will induct on n. We begin with the case $n = 1$, in which F itself is the vector group associated to some vector bundle $\mathscr{F}$ on U. By applying (a) to $j_*(\mathscr{F})$, where $j\colon U \hookrightarrow S$ is the indicated open immersion, and again using the vanishing of quasi-coherent cohomology of affine schemes, we find that, for all $i \geqslant 1$, even

$$H^i(U, F) \cong H^i(U, \mathscr{F}) \cong H^{i+1}_Z(S, j_*(\mathscr{F})) \xrightarrow[\sim]{\text{(a)}} H^{i+1}_Z(S', j_*(\mathscr{F})) \cong H^i(U', \mathscr{F}_{U'}) \cong H^i(U', F).$$

For the inductive step, we assume that $n > 1$ and combine the inductive hypothesis, the preceding display for F_{n-1}, and the nonabelian cohomology sequences [Gir71, IV, 4.2.10] of a central extension to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(U, F_{n-1}) & \longrightarrow & H^1(U, F) & \longrightarrow & H^1(U, F/F_{n-1}) & \longrightarrow & H^2(U, F_{n-1}) \\ \downarrow\wr & & \downarrow & & \downarrow\!\!\downarrow & & \downarrow\wr \\ H^1(U', F_{n-1}) & \longrightarrow & H^1(U', F) & \longrightarrow & H^1(U', F/F_{n-1}) & \longrightarrow & H^2(U', F_{n-1}). \end{array}$$

We fix an $\alpha' \in H^1(U', F)$ that we wish to lift to $H^1(U, F)$ and note that, by a diagram chase, there at least is an $\alpha \in H^1(U, F)$ whose image in $H^1(U', F/F_{n-1})$ agrees with that of α'. Every inner fpqc form of F inherits an analogous filtration, even with the same subquotients F_i/F_{i+1}, so the change of origin bijections [Gir71, III, 2.6.1 (i)] allow us to twist F and reduce to the case when the common image of α and α' in $H^1(U', F/F_{n-1})$ vanishes. In this case, however, the surjectivity of the left vertical arrow suffices. □

Example 4.2.3. For example, F in Lemma 4.2.2 (b) could be the unipotent radical $\mathscr{R}_u(P)$ of a parabolic S-subgroup (resp., U-subgroup) P of a reductive S-group (resp., U-group) G: in this case, [SGA $3_{\text{III new}}$, XXVI, 2.1] supplies the desired filtration.

We can now reduce to the case when the relative curve C in Proposition 4.1.5 is $\mathbb{A}^1_R$.

Proposition 4.2.4. *For a semilocal Dedekind ring $\mathcal{O}$, the localization R of a smooth $\mathcal{O}$-algebra at finitely many primes $\mathfrak{p}$ that all lie over maximal ideals $\mathfrak{p}' \subset \mathcal{O}$ with $k_{\mathfrak{p}}/k_{\mathfrak{p}'}$ separable, a quasi-split reductive R-group G, and a generically trivial G-torsor E, there are*

(i) *a closed subscheme $Z \subset \mathbb{A}^1_R$ that is finite over R;*

(ii) *a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ whose pullback along the zero section is E such that $\mathscr{E}$ is trivial over $\mathbb{A}^1_R \backslash Z$.*

Proof. Let $B \subset G$ be a Borel R-subgroup. Proposition 4.1.5 supplies a quasi-finite, affine, flat R-morphism $\pi\colon C \to \mathbb{A}^1_R$ whose base change to an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ (called Z' there) is an isomorphism, as well as an $s \in C(R)$ and a G_U-torsor $\widetilde{\mathscr{E}}$ (called $\mathscr{E}$ there) with s-pullback E such that $\widetilde{\mathscr{E}}$ reduces to a $\mathscr{R}_u(B)$-torsor over $C \backslash \pi^{-1}(Z)$. By Lemma 4.2.2 (b) and Example 4.2.3, this $\mathscr{R}_u(B)$-torsor descends to a $\mathscr{R}_u(B)$-torsor over $\mathbb{A}^1_R \backslash Z$, so $\widetilde{\mathscr{E}}_{C\backslash \pi^{-1}(Z)}$ descends to a $G_{\mathbb{A}^1_R\backslash Z}$-torsor. The patching lemma 4.2.1 then ensures that $\widetilde{\mathscr{E}}$ itself descends to a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that reduces to a $\mathscr{R}_u(B)$-torsor over $\mathbb{A}^1_R \backslash Z$. By postcomposing with a change of coordinate automorphism of $\mathbb{A}^1_R$ to ensure that s map to the zero section of $\mathbb{A}^1_R$, we make the pullback of $\mathscr{E}$ along the zero section be E. Finally, we apply Lemma 2.1.1 to $\mathbb{P}^1_R$ to enlarge $Z \subset \mathbb{A}^1_R$ to be a hypersurface in $\mathbb{P}^1_R$. This ensures that $\mathbb{A}^1_R \backslash Z$ is affine, so that, due to the filtration of $\mathscr{R}_u(B)$ by vector groups as in Example 4.2.3 and the vanishing of quasi-coherent cohomology of affine schemes, our $\mathscr{E}_{\mathbb{A}^1_R\backslash Z}$ is even trivial. □

5. Torsors over $\mathbb{A}^1_R$ via the geometry of the affine Grassmannian

Our final task is to study generically trivial torsors over $\mathbb{A}^1_R$, which may be viewed as a problem of Bass–Quillen type beyond the case of vector bundles (that is, beyond GL_n-torsors). For this, we follow arguments of Fedorov from [Fed18], with some minor simplifications. The cases when the group G is either semisimple simply connected or split are simpler, see Proposition 5.2.2. To approach more general G, one employs geometric properties of affine Grassmannians described in §5.1, the main conclusion about torsors now being Proposition 5.2.4 (see also Corollary 5.2.6 for a simpler statement when G is quasi-split). Once we carry out this analysis in §5.2, we conclude our proof of the unramified case of the Grothendieck–Serre conjecture for quasi-split G in §5.3.

5.1. Lifting field-valued points of the neutral component of the affine Grassmannian

The geometric input about affine Grassmannians that we need is the surjectivity of $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}^0_G$ on field-valued points. This map is even an isomorphism if G is semisimple and the degree of the isogeny $G^{\mathrm{sc}} \to G$ is invertible on the base (see [Zhu17, 1.3.11 (3)]). Without the assumption on this degree, however, the geometry of the affine Grassmannian Gr_G becomes more delicate (see [HLR20]), and to reach the desired lifting in Corollary 5.1.4 we use an argument suggested by Timo Richarz.

5.1.1. The affine Grassmannian. For a reductive group G over a field k, the *affine Grassmannian* Gr_G is the functor that to a k-algebra R associates the set of isomorphism classes of pairs $(\mathscr{E}, \tau)$ consisting of a $G_{R[\![t]\!]}$-torsor $\mathscr{E}$ and a trivialization $\tau\colon \mathscr{E}_{R(\!(t)\!)} \xrightarrow{\sim} G_{R(\!(t)\!)}$ of the induced torsor over $R(\!(t)\!)$. By, for instance, [Zhu17, 1.2.2], the functor Gr_G is representable by an ind-projective ind-scheme.

Concretely, consider the loop and the positive loop groups of G defined as the respective functors

$$LG\colon R \mapsto G(R(\!(t)\!)) \quad \text{and} \quad L^+G\colon R \mapsto G(R[\![t]\!]),$$

which are representable by a group ind-affine ind-scheme (resp., by an affine group scheme) over k. The subfunctor of Gr_G that parametrizes those pairs in which $\mathscr{E}$ is trivial is the presheaf quotient

$$LG/L^+G \subset \mathrm{Gr}_G\,. \tag{5.1.1.1}$$

A general $\mathscr{E}$ trivializes over $R'[\![t]\!]$ for a faithfully flat, étale R-algebra R' (see Proposition 3.1.1 (c)), so this inclusion exhibits Gr_G as the étale sheafification of LG/L^+G. Moreover, whenever no nontrivial $G_{R[\![t]\!]}$-torsor trivializes over $R(\!(t)\!)$, as happens, for instance, when R is a field (see §1.4 (2)) or for any R when G is either a torus or a strongly inner form of GL_n (see Proposition 3.1.1 (c) and [BČ20, 2.1.24, 3.1.7]), the inclusion (5.1.1.1) induces an equality on R-points:

$$\mathrm{Gr}_G(R) \cong G(R(\!(t)\!))/G(R[\![t]\!]).$$

In addition, in general the group L^+G acts on Gr_G by left multiplication, and one may write Gr_G as the increasing union of L^+G-invariant projective subschemes (for this one fixes a closed immersion $G \hookrightarrow \mathrm{GL}_n$ and uses the resulting closed immersion $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$, see the proof of [Zhu17, 1.2.2]).

The scheme L^+G is connected because the fibers of the "reduction modulo t" map $L^+G \to G$ are infinite-dimensional affine spaces. In turn, by [PR08, Thm. 5.1], the map $LG \to \mathrm{Gr}_G$ induces a bijection on sets of geometric connected components, these components are all clopen, and, if G is semisimple and simply connected, then both LG and Gr_G are geometrically connected. In general, the *neutral components*, that is, the connected components $LG^0 \subset LG$ and $\mathrm{Gr}_G^0 \subset \mathrm{Gr}_G$ containing the class of the identity, are geometrically connected (as is any connected k-scheme X with $X(k) \neq \varnothing$, see [EGA IV$_2$, 4.5.13]). Since L^+G is geometrically connected, its left multiplication action on LG and Gr_G respects connected components. The map

$$\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}_G^0 \tag{5.1.1.2}$$

is surjective on topological spaces, in fact, it is even surjective on K-points for every algebraically closed field extension K of k.[6] By [Zhu17, 1.3.11 (3)], if G is semisimple with $G^{\mathrm{sc}} \to G$ of degree prime to $\mathrm{char}\, k$, then the map (5.1.1.2) is even an isomorphism.

5.1.2. Schubert cells. With G over k as in §5.1.1, let $T \subset G$ be a maximal k-torus with its cocharacter group $X_*(T) := \underline{\mathrm{Hom}}_{\mathrm{gp}}(\mathbb{G}_m, T)$. By [SGA $3_{\mathrm{I\ new}}$, $\mathrm{VI_A}$, 3.3.2], the L^+G-orbit of any $x \in \mathrm{Gr}_G(k)$ is a smooth k-subscheme of Gr_G. When x is the image of t under the base change to $k(\!(t)\!)$ of the k-morphism given by a $\lambda \in X_*(T)(k)$, the resulting subscheme is the *Schubert cell*

$$\mathrm{Gr}_G^{\lambda} \subset \mathrm{Gr}_G\,.$$

Its closure (schematic image) in Gr_G is the *Schubert variety*

$$\mathrm{Gr}_G^{\leqslant\lambda} \subset \mathrm{Gr}_G,$$

which is a reduced, projective k-scheme containing Gr_G^{λ} as a dense open. In the case when T is split, the Gr_G^{λ} topologically exhaust Gr_G: then, by [PR08, Appendix, Prop. 8], every field-valued (equivalently, (algebraically closed field)-valued) point of Gr_G factors through some Gr_G^{λ}. In general, the same holds for the k-subschemes

$$\mathrm{Gr}_G^{[\lambda]} := \textstyle\bigcup_{\lambda' \in \mathrm{Gal}(k^{\mathrm{sep}}/k)\cdot\lambda} \mathrm{Gr}_G^{\lambda'} \subset \mathrm{Gr}_G \quad \text{with} \quad \lambda \in X_*(T)(k^{\mathrm{sep}}).$$

Thus, letting $T^{\mathrm{sc}} \subset (G^{\mathrm{der}})^{\mathrm{sc}}$ be the maximal torus corresponding to $T \subset G$, we see from (5.1.1.2) that the $\mathrm{Gr}_G^{[\lambda]}$ with $\lambda \in X_*(T^{\mathrm{sc}})(k^{\mathrm{sep}}) \subset X_*(T)(k^{\mathrm{sep}})$ topologically exhaust the neutral component Gr_G^0.

We now argue that these latter $\mathrm{Gr}_G^{[\lambda]}$ are insensitive to replacing G by $(G^{\mathrm{der}})^{\mathrm{sc}}$.

Proposition 5.1.3. *For a reductive group G over a field k, a maximal k-torus $T \subset G$, the corresponding maximal torus $T^{\mathrm{sc}} \subset (G^{\mathrm{der}})^{\mathrm{sc}}$, and a $\lambda \in X_*(T^{\mathrm{sc}})(k^{\mathrm{sep}})$, the k-morphism*

$$\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}^{[\lambda]} \xrightarrow{\sim} \mathrm{Gr}_G^{[\lambda]} \quad \textit{induced by} \quad \mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}_G \quad \textit{is an isomorphism.}$$

Proof. The argument is similar to that of [Fed18, Prop. 2.8] and was suggested to us by Timo Richarz. The claim is insensitive to enlarging k, so we reduce to k being algebraically closed and then, by passing to individual Schubert cells, to showing that $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}^{\lambda} \xrightarrow{\sim} \mathrm{Gr}_G^{\lambda}$. This last isomorphism, however, is a special case of [HR19a, Lem. 3.8]. □

We are ready for the conclusion about the behavior of $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}_G$ on field-valued points.

[6] To justify the assertion about K-points, one may argue as follows. Since $LG \to \mathrm{Gr}_G$ is surjective on K-points and a bijection on sets of connected components, by [PR08, Thm. 5.1 and the end of the proof of Lemma 17 on p. 198 (with the definition of $G(L)_1$ after Rem. 2 on p. 189)] (their $G(L)_1$ is our $(LG)^0(K)$), we may replace G by a z-extension to reduce to G^{der} being simply connected. For such G, however, the surjectivity of $\mathrm{Gr}_{G^{\mathrm{der}}}(K) \to \mathrm{Gr}_G^0(K)$ follows from [PR08, last line on p. 197 and proof of Lemma 5 on p. 191] (by the latter, $T(L)_1$ there is $T(K[\![t]\!])$ for us).

Corollary 5.1.4. *For a reductive group G over a field k, the map $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \xrightarrow{(5.1.1.2)} \mathrm{Gr}_G^0$ is surjective on k-points, and the image of the following map is stable under left multiplication by $G(k[\![t]\!])$:*

$$\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}(k) \overset{\S 5.1.1}{\cong} (G^{\mathrm{der}})^{\mathrm{sc}}(k(\!(t)\!))/(G^{\mathrm{der}})^{\mathrm{sc}}(k[\![t]\!]) \to G(k(\!(t)\!))/G(k[\![t]\!]) \overset{\S 5.1.1}{\cong} \mathrm{Gr}_G(k).$$

Proof. By §5.1.1, the ind-scheme $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}$ is connected, so the map $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}} \to \mathrm{Gr}_G$ factors through the clopen $\mathrm{Gr}_G^0 \subset \mathrm{Gr}_G$. Moreover, by §5.1.2, a k-point of Gr_G^0 factors through some $\mathrm{Gr}_G^{[\lambda]}$ for a $\lambda \in X_*(T^{\mathrm{sc}})(k^{\mathrm{sep}})$, where $T \subset G$ is a maximal torus and $T^{\mathrm{sc}} \subset (G^{\mathrm{der}})^{\mathrm{sc}}$ is the corresponding maximal torus of $(G^{\mathrm{der}})^{\mathrm{sc}}$. Thus, by Proposition 5.1.3, every such point lifts to $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}$, as claimed.

By §5.1.1, the source of the left multiplication map $L^+G \times_k \mathrm{Gr}_G^0 \to \mathrm{Gr}_G$ is connected, so this map factors through $\mathrm{Gr}_G^0 \subset \mathrm{Gr}_G$. Thus, the image of this map on k-points is $\mathrm{Gr}_G^0(k)$, that is, by the above, it agrees with the image of $\mathrm{Gr}_{(G^{\mathrm{der}})^{\mathrm{sc}}}(k) \to \mathrm{Gr}_G(k)$. Thus, the latter is $G(k[\![t]\!])$-stable. ☐

5.2. The analysis of torsors over $\mathbb{A}^1_R$

To analyze torsors over $\mathbb{A}^1_R$, we strengthen several results of Fedorov from [Fed16b] and [Fed18]. By the following lemma, which generalizes the main result of [Tsy19], the key point is to extend to a torsor over $\mathbb{P}^1_R$ in such a way that the latter be trivial over the closed R-fibers.

Lemma 5.2.1. *For a semilocal ring R and a reductive R-group G that is a closed subgroup of some $\mathrm{GL}_{n,R}$ (a vacuous condition if R is normal or if G is split or semisimple), every $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ whose base change to $\mathbb{P}^1_{k_{\mathfrak{m}}}$ is trivial for every maximal ideal $\mathfrak{m} \subset R$ is the base change of a G-torsor.*

Proof. The parenthetical assertion is a special case of [Tho87, 3.2 (3)]. For the rest, we first use a limit argument to reduce to Noetherian R and then pass to connected components to also assume that $\mathrm{Spec}(R)$ is connected. Moreover, we begin with the case $G = \mathrm{GL}_{n,R}$, in which we may regard $\mathscr{E}$ as a vector bundle of rank n.

In this vector bundle case, $\mathscr{V} := \mathscr{H}om_{\mathscr{O}_{\mathbb{P}^1_R}}(\mathscr{O}_{\mathbb{P}^1_R}^{\oplus n}, \mathscr{E}) \cong \mathscr{E}^{\oplus n}$ is also a vector bundle on $\mathbb{P}^1_R$. By [EGA III_1, 3.2.1], the R-module $V := \Gamma(\mathbb{P}^1_R, \mathscr{V})$ is finite. By assumption, $\mathscr{E}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ is trivial for every maximal ideal $\mathfrak{m} \subset R$, so for such $\mathfrak{m}$ we choose an isomorphism

$$\mathscr{O}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}^{\oplus n} \xrightarrow{\sim} \mathscr{E}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}, \quad \text{which corresponds to some} \quad v_{\mathfrak{m}} \in \Gamma(\mathbb{P}^1_{k_{\mathfrak{m}}}, \mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}).$$

Likewise, each $\mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ is trivial, so $H^1(\mathbb{P}^1_{k_{\mathfrak{m}}}, \mathscr{V}|_{\mathbb{P}^1_{k_{\mathfrak{m}}}}) \cong 0$. Thus, by cohomology and base change [EGA III_1, 4.6.1], there is a $\widetilde{v}_{\mathfrak{m}} \in V/\mathfrak{m}V$ that maps to $v_{\mathfrak{m}}$. Since R is semilocal and $\mathfrak{m}$ ranges over its maximal ideals, we may then find a $v \in V$ that maps to all the $\widetilde{v}_{\mathfrak{m}}$, so also to all the $v_{\mathfrak{m}}$. By construction and the Nakayama lemma [SP, 00DV], the $\mathscr{O}_{\mathbb{P}^1_R}$-module homomorphism $\mathscr{O}_{\mathbb{P}^1_R}^{\oplus n} \to \mathscr{E}$ corresponding to v is surjective at every closed point, so it is surjective. Cayley–Hamilton [SP, 05G8] then ensures that this homomorphism is an isomorphism, so that $\mathscr{E}$ is trivial, as desired.

To deduce the general case, we use our closed embedding $G \hookrightarrow \mathrm{GL}_{n,R}$. Namely, the settled case of $\mathrm{GL}_{n,R}$ and the nonabelian cohomology sequence [Gir71, III, 3.2.2] show that our $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ comes from a some $\mathbb{P}^1_R$-point of $\mathrm{GL}_{n,R}/G$. However, G is reductive, so, by [Alp14, 9.4.1 and 9.7.5], this quotient is affine. By [MFK94, Prop. 6.1] (to reduce to an R-fiber), this means that the only R-morphisms from $\mathbb{P}^1_R$ to $\mathrm{GL}_{n,R}/G$ are constant, in particular, that our $\mathbb{P}^1_R$-point comes from an R-point. This then implies that our $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ is the base change of a G-torsor, as desired. ☐

The preceding lemma leads to the triviality of generically trivial reductive group torsors over $\mathbb{A}^1_R$ under the assumptions of the following proposition, whose case (i) was relevant in [FP15]. Examples from [Fed16a] show that without some isotropicity condition on G such triviality does not hold.

Proposition 5.2.2. *For a semilocal ring R and a reductive R-group G such that either*

(i) *G is semisimple, simply connected, absolutely almost simple, and isotropic; or*

(ii) *G is split, semisimple, simply connected; or*

(iii) *G is split and R is local;*

every $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$ is trivial.

In (i), 'absolutely almost simple' means that the Dynkin diagrams of the geometric R-fibers of G are connected and 'isotropic' means that G contains $\mathbb{G}_{m,R}$ as a subgroup.

Proof. By [SGA $3_{\text{III new}}$, XXIV, 5.3, 5.10 (i)], a split, semisimple, simply connected group is a direct product of split, semisimple, simply connected, absolutely almost simple groups, so (ii) reduces to (i).

We let t be the inverse of the coordinate on $\mathbb{A}^1_R$ and consider $R[[t]]$ as the completion of $\mathbb{P}^1_R$ along infinity. Due to its R-finiteness, Z is closed in $\mathbb{P}^1_R$, so its pullback to $\mathrm{Spec}(R[[t]])$ is also closed and hence is even empty because it does not meet the locus $\{t = 0\}$. Thus, we may use formal glueing supplied by, for instance, [BČ20, 2.2.11 (b)] (or Lemma 4.2.1 when R is Noetherian) to extend $\mathscr{E}$ to a $G_{\mathbb{P}^1_R}$-torsor $\overline{\mathscr{E}}$ by glueing $\mathscr{E}$ with the trivial $G_{R[[t]]}$-torsor. It suffices to argue that we can glue like this so that $\overline{\mathscr{E}}_{\mathbb{P}^1_{k_\mathfrak{m}}}$ be trivial for every maximal ideal $\mathfrak{m} \subset R$: Lemma 5.2.1 will then imply that $\overline{\mathscr{E}}$ is the base change of its pullback by the section at infinity, and hence that $\overline{\mathscr{E}}$ and $\mathscr{E}$ are trivial.

Explicitly, the glueings of $\mathscr{E}$ and the trivial $G_{R[[t]]}$-torsor to a $G_{\mathbb{P}^1_R}$-torsor are parametrized by elements of $G(R((t)))/G(R[[t]])$, and likewise over the residue fields $k_\mathfrak{m}$. We will first build trivial $G_{\mathbb{P}^1_{k_\mathfrak{m}}}$-bundles $\overline{\mathscr{E}}_{\mathbb{P}^1_{k_\mathfrak{m}}}$ from $\mathscr{E}_{\mathbb{P}^1_{k_\mathfrak{m}}}$ by such a glueing and then argue that these glueings come from a glueing over R. These two steps reduce, respectively, to the following claims.

(1) For every maximal ideal $\mathfrak{m} \subset R$, the $G_{\mathbb{A}^1_{k_\mathfrak{m}}}$-torsor $\mathscr{E}_{\mathbb{A}^1_{k_\mathfrak{m}}}$ is trivial.

(2) The following map is surjective, where $\mathfrak{m}$ ranges over the maximal ideals of R:

$$G(R((t)))/G(R[[t]]) \twoheadrightarrow \textstyle\prod_\mathfrak{m} G(k_\mathfrak{m}((t)))/G(k_\mathfrak{m}[[t]]).$$

For (1), since $\mathscr{E}_{\mathbb{A}^1_{k_\mathfrak{m}}}$ is trivial away from $Z_{k_\mathfrak{m}}$, we may glue it arbitrarily with the trivial $G_{k_\mathfrak{m}[[t]]}$-torsor to obtain a $G_{\mathbb{P}^1_{k_\mathfrak{m}}}$-torsor whose pullback along the infinity section is trivial. By [Gil02, 3.12] (see also [Gil05]), such torsors are trivial over $\mathbb{A}^1_{k_\mathfrak{m}}$, so $\mathscr{E}_{\mathbb{A}^1_{k_\mathfrak{m}}}$ is trivial, that is, (1) holds.

The claim (2) is where we will use the assumptions (i) or (iii), and we pass to connected components to reduce to the case when $\mathrm{Spec}(R)$ is connected. We begin with (i), whose isotropy assumption, by [SGA $3_{\text{III new}}$, XXVI, 6.12], implies that G has a proper parabolic subgroup $P \subset G$. Moreover, the assumptions of (i) are such that the Whitehead group of the base changes of G is of unramified nature, more precisely and more concretely, by [Gil09, Fait 4.3, Lem. 4.5], under (i) we have

$$\textstyle\prod_\mathfrak{m} G(k_\mathfrak{m}((t))) = \prod_\mathfrak{m} G(k_\mathfrak{m}((t)))^+ G(k_\mathfrak{m}[[t]]), \tag{5.2.2.1}$$

where $G(k_\mathfrak{m}((t)))^+ \subset G(k_\mathfrak{m}((t)))$ is the subgroup generated by $(\mathscr{R}_u(P))(k_\mathfrak{m}((t)))$ and $(\mathscr{R}_u(P^-))(k_\mathfrak{m}((t)))$ with $P^- \subset G$ being a parabolic opposite to P in the sense of [SGA $3_{\text{III new}}$, XXVI, 4.3.3, 4.3.5 (i)].

To conclude (2) in the case (i), it suffices to show that the following pullback maps are surjective:

$$(\mathscr{R}_u(P))(R((t))) \twoheadrightarrow \prod_{\mathfrak{m}} (\mathscr{R}_u(P))(k_{\mathfrak{m}}((t))) \quad \text{and} \quad (\mathscr{R}_u(P^-))(R((t))) \twoheadrightarrow \prod_{\mathfrak{m}} (\mathscr{R}_u(P^-))(k_{\mathfrak{m}}((t))).$$

For this, we combine the surjectivity of the map $R_{\mathfrak{m}}((t)) \twoheadrightarrow \prod_{\mathfrak{m}} k_{\mathfrak{m}}((t))$ with [SGA $3_{\text{III new}}$, XXVI, 2.5], according to which both $\mathscr{R}_u(P)$ and $\mathscr{R}_u(P^-)$ are isomorphic to affine spaces $\mathbb{A}^d_R$.

In the case (iii), by assumption, there are a split maximal R-torus and a Borel R-subgroup $T \subset B \subset G$. The Iwasawa decomposition, so, in essence, the valuative criterion of properness, gives the equality

$$G(k_{\mathfrak{m}}((t))) = B(k_{\mathfrak{m}}((t)))G(k_{\mathfrak{m}}[[t]]) = (\mathscr{R}_u(B))(k_{\mathfrak{m}}((t)))T(k_{\mathfrak{m}}((t)))G(k_{\mathfrak{m}}[[t]])$$

for every maximal ideal $\mathfrak{m} \subset R$. Thus, the concluding portion of the argument for (i) applied to $\mathscr{R}_u(B)$ now reduces (2) to the case when G is $\mathbb{G}_m$. For the latter, it suffices to note that, since R is local, the map $R((t))^\times \to k_{\mathfrak{m}}((t))^\times \cong t^{\mathbb{Z}} \times k_{\mathfrak{m}}[[t]]^\times$ is surjective. □

Remark 5.2.3. One difference between Proposition 5.2.2 and some of its versions in the literature is that we work directly with the $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ instead of first glueing it arbitrarily to a $G_{\mathbb{P}^1_R}$-torsor and then modifying this extension. Ultimately, this is an expository point, but it highlights that in (2) there is no need to pursue the analogous surjectivity before taking the quotients.

Conditions (i)–(iii) of Proposition 5.2.2 are too restrictive for our purposes, but its conclusion is also stronger than we actually need. The following sharpening, which is a variant of the core result of [Fed18], applies without restrictions on G but only gives triviality away from a finite étale subscheme.

Proposition 5.2.4. *Let R be a semilocal ring with connected spectrum, let G be a reductive R-group, let $\widetilde{R}_i$ (resp., $\widetilde{G}_i$) be the finite étale R-algebras with connected spectra (resp., the simple adjoint $\widetilde{R}_i$-groups $\widetilde{G}_i$) such that the canonical decomposition of G^{ad} from* [SGA $3_{\text{III new}}$, XXIV, 5.10 (i)] *is*

$$G^{\mathrm{ad}} \cong \prod_i G_i \quad \textit{with} \quad G_i := \mathrm{Res}_{\widetilde{R}_i/R}(\widetilde{G}_i),$$

and let $Y_i \subset Y \subset \mathbb{A}^1_R$ be nonempty R-(finite étale) closed subschemes such that

(i) *$\mathscr{O}(1)$ is trivial on $\mathbb{P}^1_R - Y$;*

(ii) *each $(G_i)_{Y_i}$ is isotropic; and*

(iii) *$\mathscr{O}(1)$ is trivial on $\mathbb{P}^1_{k_{\mathfrak{m}}} - (Y_i)_{k_{\mathfrak{m}}}$ for every i and every maximal ideal $\mathfrak{m} \subset R$ with $(G_i)_{k_{\mathfrak{m}}}$ isotropic.*

For a $G_{\mathbb{P}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R \backslash Y$, if for every maximal ideal $\mathfrak{m} \subset R$ the $G^{\mathrm{ad}}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$-torsor induced by $\mathscr{E}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$ lifts to a Zariski locally trivial *$(G^{\mathrm{ad}})^{\mathrm{sc}}_{\mathbb{P}^1_{k_{\mathfrak{m}}}}$-torsor, then $\mathscr{E}$ is trivial away from Y, that is, then $\mathscr{E}_{\mathbb{P}^1_R \backslash Y}$ is a trivial $G_{\mathbb{P}^1_R \backslash Y}$-torsor.*

The assumptions of this proposition become simpler when G is quasi-split, see Corollary 5.2.6.

Proof. The statement is mild generalization of [Fed18, Thm. 6] and the proof is similar, even if presented slightly differently. It combines the techniques and references we used in the proof Proposition 5.2.2 with the analysis of affine Grassmannians that we carried out in §5.1.

By Remark 4.1.4, the R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$ is cut out by a global section. Thus, the coordinate ring of the formal completion of $\mathbb{A}^1_R$ along Y is $R'[[t]]$ for a finite étale R-algebra R' that is the coordinate ring of Y (where t is a monic polynomial in the coordinate of $\mathbb{A}^1_R$). Likewise, for each i, the formal completion of $\mathbb{A}^1_R$ along Y_i is, compatibly, $R_i[[t]]$ for a finite étale R-algebra R_i that is the coordinate ring of Y_i and hence is a direct factor of R', so that $R' \cong R_i \times R'_i$.

We fix a trivialization $\tau \in \mathscr{E}(\mathbb{P}^1_R \backslash Z)$ of $\mathscr{E}|_{\mathbb{P}^1_R \backslash Z}$. Since $Y \subset \mathbb{P}^1_R \backslash Z$, this τ trivializes the restriction of $\mathscr{E}$ to $R'[\![t]\!]$ and, similarly to the proof of Proposition 5.2.2, we use τ to regard $\mathscr{E}$ as the glueing corresponding to $1 \in G(R'((t)))/G(R'[\![t]\!])$ of $\mathscr{E}|_{\mathbb{P}^1_R \backslash Y}$ and the trivial $G_{R'[\![t]\!]}$-torsor.

We let $\mathfrak{m}$ range over the maximal ideals of R, set $k := \prod_{\mathfrak{m}} k_{\mathfrak{m}}$, let $\mathscr{E}_i$ be the G_i-torsor over $\mathbb{P}^1_k$ induced by $\mathscr{E}$, and let $\mathscr{E}_i^{\mathrm{sc}}$ be a Zariski locally trivial G_i^{sc}-torsor over $\mathbb{P}^1_k$ that lifts $\mathscr{E}_i$ (the existence of $\mathscr{E}_i^{\mathrm{sc}}$ is a part of the assumptions). By [Gil02, 3.10 (b)], the Zariski local triviality implies that $\mathscr{E}_i^{\mathrm{sc}}$ is trivial on a Zariski neighborhood of $(Y_i)_k$ in $\mathbb{P}^1_k$. We fix a trivialization τ_i over such a neighborhood and, similarly to the proof of Proposition 5.2.2, use it to regard $\mathscr{E}_i^{\mathrm{sc}}$ as the glueing corresponding to

$$1 \in G_i^{\mathrm{sc}}((R_i \otimes_R k)((t)))/G_i^{\mathrm{sc}}((R_i \otimes_R k)[\![t]\!])$$

of $\mathscr{E}_i^{\mathrm{sc}}|_{\mathbb{P}^1_k \backslash (Y_i)_k}$ and the trivial $(G_i^{\mathrm{sc}})_{(R_i \otimes_R k)[\![t]\!]}$-torsor.

Of course, the trivializations τ and τ_i need not be compatible, that is, using τ as the reference, the image of τ_i in $G_i((R_i \otimes_R k)((t)))$ need not be the identity. Nevertheless, this image of τ_i as well as that of τ both describe the same $(G_i)_{\mathbb{P}^1_k}$-torsor (the one induced by $\mathscr{E}$) as the glueing of the same $G^{\mathrm{ad}}_{(\mathbb{P}^1 \backslash Y_i)_k}$-torsor $\mathscr{E}_i|_{(\mathbb{P}^1 \backslash Y_i)_k}$ and the trivial $(G_i)_{(R_i \otimes_R k)[\![t]\!]}$-torsor. Concretely, this identification of the glueings means that the image of τ_i lies in

$$G_i((R_i \otimes_R k)[\![t]\!]) \subset G_i((R_i \otimes_R k)((t))),$$

in other words, that the images of τ and τ_i are $G_i((R_i \otimes_R k)[\![t]\!])$-translates of each other. Thus, Corollary 5.1.4 implies—and this is a crucial point—that, at the cost of $\mathscr{E}_i^{\mathrm{sc}}$ only lifting $\mathscr{E}_i$ over $\mathbb{P}^1_k \backslash (Y_i)_k$, we may change the glueings $\mathscr{E}_i^{\mathrm{sc}}$ and the trivializations τ_i to arrange that they be compatible with τ: namely, still with τ as the reference, that the image of τ_i in $G_i((R_i \otimes_R k)((t)))/G_i((R_i \otimes_R k)[\![t]\!])$ would be the class of the identity 1.

By [Gil02, 3.8 (b)], the Zariski local triviality of $\mathscr{E}_i^{\mathrm{sc}}$ means that this torsor comes from a torsor under a split subtorus, and hence, thanks to (iii), that $\mathscr{E}_i^{\mathrm{sc}}|_{\mathbb{P}^1_k \backslash (Y_i)_k}$ is a trivial torsor. In particular, the trivial G_i^{sc}-torsor over $\mathbb{P}^1_k$ is a glueing of $\mathscr{E}_i^{\mathrm{sc}}|_{(\mathbb{P}^1 \backslash Y_i)_k}$ and the trivial $(G_i^{\mathrm{sc}})_{(R_i \otimes_R k)[\![t]\!]}$-torsor, and, continuing to use τ_i as reference, this glueing is given by some

$$\alpha_i \in G_i^{\mathrm{sc}}((R_i \otimes_R k)((t))).$$

As in the proof of Proposition 5.2.2, the isotropy condition (ii) allows us to fix a proper parabolic subgroup $P_i \subset (G_i)_{R_i}$, which induces a proper parabolic subgroup of $(G_i^{\mathrm{sc}})_{R_i}$. Since

$$G_i^{\mathrm{sc}} \cong \mathrm{Res}_{\widetilde{R}_i/R}(\widetilde{G}_i^{\mathrm{sc}}),$$

this latter parabolic is the restriction of scalars of a unique proper parabolic of $(\widetilde{G}_i^{\mathrm{sc}})_{\widetilde{R}_i \otimes_R R_i}$ (as one sees after base change to a further finite étale cover splitting $\widetilde{R}_i$). We then use this parabolic together with the analogue of (5.2.2.1) again supplied by [Gil09, Fait 4.3, Lem. 4.5] (which applies because $\widetilde{G}_i^{\mathrm{sc}}$ is semisimple, simply connected, absolutely almost simple, and isotropic) to arrange that

$$\alpha_i \in G_i^{\mathrm{sc}}((R_i \otimes_R k)((t)))^+,$$

where the Whitehead group $G_i^{\mathrm{sc}}(-)^+$ is defined as in the proof of Proposition 5.2.2 using the chosen parabolic. As there, this reduction allows us to lift α_i to an $\widetilde{\alpha}_i \in G_i^{\mathrm{sc}}(R_i((t)))$. We consider $\widetilde{\alpha}_i$ as an element of $G_i^{\mathrm{sc}}(R((t)))$ by letting it be the identity on the complementary factor $G_i^{\mathrm{sc}}(R_i'((t)))$.

Jointly, the $\widetilde{\alpha}_i$ assemble to an element $\widetilde{\alpha} \in (G^{\mathrm{ad}})^{\mathrm{sc}}(R((t)))$. The map $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$ factors through the isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$, where $G^{\mathrm{der}} \subset G$ is the derived subgroup, so $\widetilde{\alpha}$ maps to an element of $G(R((t)))$. With τ as the reference trivialization, this image of $\widetilde{\alpha}$ in $G(R((t)))$ gives rise to a $G_{\mathbb{P}^1_R}$-torsor $\widetilde{\mathscr{E}}$ that is the glueing of $\mathscr{E}_{\mathbb{P}^1_R \backslash Y}$ and the trivial $G_{R'[\![t]\!]}$-torsor. The $G^{\mathrm{ad}}_{\mathbb{P}^1_k}$-torsor $\overline{\mathscr{E}}$ induced by $\widetilde{\mathscr{E}}$ is the analogous glueing over $\mathbb{P}^1_k$ that arises from the image of $\prod_i \widetilde{\alpha}_i$ in $\prod_i G_i((R_i \otimes_R k)((t)))$. Thus, by construction and by the prearranged compatibility between τ and τ_i, this $\overline{\mathscr{E}}$ is a trivial torsor.

Lemma 5.2.1 now implies that $\widetilde{\mathscr{E}}$ induces a $G^{\mathrm{ad}}_{\mathbb{P}^1_R}$-torsor that is the pullback of a G^{ad}-torsor. Thus, since $\widetilde{\mathscr{E}}|_{\mathbb{P}^1_R\backslash(Y\cup Z)}$ is trivial and since the infinity section factors through $\mathbb{P}^1_R\backslash(Y \cup Z)$, we conclude that $\widetilde{\mathscr{E}}$ induces a trivial $G^{\mathrm{ad}}_{\mathbb{P}^1_R}$-torsor, to the effect that $\widetilde{\mathscr{E}}$ comes from a $Z(G)_{\mathbb{P}^1_R}$-torsor $\mathscr{F}$. It now suffices to argue that $\mathscr{F}|_{\mathbb{P}^1_R\backslash Y}$ is the pullback of a $Z(G)$-torsor: then $\widetilde{\mathscr{E}}|_{\mathbb{P}^1_R\backslash Y} \cong \mathscr{E}|_{\mathbb{P}^1_R\backslash Y}$ will be the pullback of a G-torsor, so, by again considering pullbacks at ∞, it will be trivial.

For showing that $\mathscr{F}|_{\mathbb{P}^1_R\backslash Y}$ descends to a $Z(G)$-torsor, we twist to assume that the pullback of $\mathscr{F}$ along the infinity section is trivial, and we then fix a trivialization of this pullback. With this rigidification in place, [MFK94, Prop. 6.1] ensures that $\mathscr{F}$ has no nontrivial automorphisms. We now consider the line bundle $\mathscr{O}(1)$ on $\mathbb{P}^1_R$, rigidify it by trivializing its pullback along the infinity section, and use (i) to reduce to showing that there is a unique cocharacter $\mu\colon \mathbb{G}_{m,R} \to Z(G)$ such that $\mathscr{F}$ is isomorphic to the extension along μ of $\mathscr{O}(1)$ regarded as a $\mathbb{G}_{m,R}$-torsor. By what we already observed, such an isomorphism is unique granted that we require it to be compatible with rigidifications at infinity, so the claim is étale local on R. Thus, we may assume that the multiplicative R-group scheme $Z(G)$ is split and reduce to when $Z(G)$ is either $\mathbb{G}_{m,R}$ or $\mu_{n,R}$. In the first case, the uniqueness of μ follows from the classification of line bundles on $\mathbb{P}^1_R$ that results from Lemma 5.2.1 and [BLR90, 9.1/2]. In the second case, since $\mathrm{Pic}(R)$ is torsion-free and $R^\times \xrightarrow{\sim} \Gamma(R, \mathscr{O}^\times_{\mathbb{P}^1_R})$, our $\mathscr{F}$ descends to a $\mu_{n,R}$-torsor that, by checking at infinity, is necessarily trivial, and the unique choice $\mu = 0$ works. ☐

In practice, we will ensure the condition about lifting torsors to $(G^{\mathrm{ad}})^{\mathrm{sc}}$ via the following minor generalization of [Fed18, Prop. 2.2] that we settle with the same argument as there.

Lemma 5.2.5. *For a field k, a semisimple k-group G, opens $U, U' \subset \mathbb{P}^1_k$, and a generically trivial G_U-torsor $\mathscr{E}$, the pullback of $\mathscr{E}$ along any finite k-morphism $U' \to U$ whose degree is divisible by the degree of the isogeny $G^{\mathrm{sc}} \to G$ (or merely by the exponent of the quotient $X_*(T)/X_*(T^{\mathrm{sc}})$ for a maximal split k-torus $T^{\mathrm{sc}} \subset G^{\mathrm{sc}}$ and its image $T \subset G$) lifts to a Zariski locally trivial $G^{\mathrm{sc}}_{U'}$-torsor.*

Proof. The kernel of the isogeny $T^{\mathrm{sc}} \to T$ is a subgroup of the kernel of $G^{\mathrm{sc}} \to G$, so the degree d_T of the former divides that of the latter. Since d_T equals the order of $X_*(T)/X_*(T^{\mathrm{sc}})$, it is divisible by the exponent e_T of this quotient. Thus, the parenthetical assertion is indeed more general and we seek to show the claim granted that e_T divides the degree d of the finite k-morphism $U' \to U$. For this, we first note that, by §1.4 (2), our G_U-torsor $\mathscr{E}$ is Zariski locally trivial.

The key input to the proof is [Gil02, 3.10 (a)], according to which $\mathscr{E}$ is the extension of $\mathscr{O}(1)|_U$ (viewed as a $\mathbb{G}_m$-torsor) along some cocharacter $\mu\colon \mathbb{G}_m \to T$. The pullback of $\mathscr{O}(1)|_U$ to U' is $\mathscr{O}(d)$, so the pullback of $\mathscr{E}$ to U' is the extension of $\mathscr{O}(1)_{U'}$ along the cocharacter $d\mu\colon \mathbb{G}_m \to T$. However, the assumption $e_T \mid d$ ensures that d kills $X_*(T)/X_*(T^{\mathrm{sc}})$, so $d\mu$ factors through a cocharacter $\mathbb{G}_m \to T^{\mathrm{sc}}$. Consequently, the pullback of $\mathscr{E}$ to U' lifts to a $G^{\mathrm{sc}}_{U'}$-torsor that comes from a $(\mathbb{G}_m)_{U'}$-torsor, and hence is Zariski locally trivial as desired. ☐

Corollary 5.2.6. *For a semilocal ring R, a quasi-split reductive R-group G, a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ that is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$, and a $d \in \mathbb{Z}_{>0}$ divisible by the R-fibral degrees of the isogeny $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$, the pullback of $\mathscr{E}$ along the map $f_d\colon \mathbb{A}^1_R \to \mathbb{A}^1_R$ given by $t \mapsto t^d$ is trivial over $\mathbb{A}^1_R\backslash Y$ for some R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$ disjoint from $f_d^{-1}(Z)$.*

Proof. Since $\mathscr{E}$ is trivial away from Z, we may extend it to a $G_{\mathbb{P}^1_R}$-torsor $\widetilde{\mathscr{E}}$ that is trivial over $\mathbb{P}^1_R \backslash Z$. Likewise, we extend f_d to the morphism $\widetilde{f}_d \colon \mathbb{P}^1_R \to \mathbb{P}^1_R$ given by $t \mapsto t^d$. We then replace $\widetilde{\mathscr{E}}$ and Z by their $\widetilde{f}_d$-pullbacks and apply Lemma 5.2.5 to reduce to the case when $d = 1$ and for every maximal ideal $\mathfrak{m} \subset R$ our $\widetilde{\mathscr{E}}$ induces a $G^{\mathrm{ad}}_{\mathbb{P}^1_{k_\mathfrak{m}}}$-torsor that lifts to a Zariski locally trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}_{\mathbb{P}^1_{k_\mathfrak{m}}}$-torsor. Moreover, we pass to connected components if needed and reduce to $\operatorname{Spec} R$ being connected.

Since G is quasi-split, by [SGA $3_{\mathrm{III\ new}}$, XXVI, 6.12], its base change to any finite R-scheme contains a noncentral $\mathbb{G}_m$. Thus, to deduce the desired conclusion from Proposition 5.2.4, it suffices to exhibit an R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$ such that $\mathscr{O}(1)$ is trivial on $\mathbb{P}^1_R \backslash Y$ and $\mathbb{P}^1_{k_\mathfrak{m}} \backslash Y_{k_\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$. For this, for every large $n > 0$ it suffices to find an R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$ of degree n: then the same triviality will hold for $\mathscr{O}(n)$ in place of $\mathscr{O}(1)$ and, by subsequently adjoining to Y its disjoint analogue with $n + 1$ in place of n, also for $\mathscr{O}(1)$.

For every large n and every maximal ideal $\mathfrak{m} \subset R$, there is a $k_\mathfrak{m}$-(finite étale) closed subscheme $Y_\mathfrak{m} \subset (\mathbb{A}^1 \backslash Z)_{k_\mathfrak{m}}$ of degree n that is either a single closed point (the case when $k_\mathfrak{m}$ is finite) or n distinct $k_\mathfrak{m}$-rational points (the case when $k_\mathfrak{m}$ is infinite). Such a Y_m is cut out by a monic polynomial in $k_\mathfrak{m}[t]$ of degree n, and a common lift of all these monic polynomials to a monic polynomial in $R[t]$ of degree n cuts out the desired R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R \backslash Z$ of degree n. ☐

5.3. **The quasi-split unramified case of the Grothendieck–Serre conjecture**

We are ready to settle the following mild strengthening of the Grothendieck–Serre conjecture in the case of quasi-split reductive groups over unramified regular local rings. By choosing $\mathcal{O}$ to be either $\mathbb{Z}$, or $\mathbb{Q}$, or $\mathbb{F}_p$ for some prime p and R to be local, this version recovers the first assertion in Theorem 1.2, see Example 5.3.2.

Theorem 5.3.1. *For a Dedekind ring $\mathcal{O}$, a semilocal regular $\mathcal{O}$-algebra R whose $\mathcal{O}$-fibers are geometrically regular*[7] *and whose maximal ideals $\mathfrak{m}$ all lie over maximal ideals $\mathfrak{m}' \subset \mathcal{O}$ with $k_\mathfrak{m}/k_{\mathfrak{m}'}$ separable (for example, with $k_{\mathfrak{m}'}$ perfect), and a quasi-split reductive R-group G, no nontrivial G-torsor trivializes over the total fraction ring* $\operatorname{Frac}(R)$ *of R, that is,*

$$\operatorname{Ker}(H^1(R, G) \to H^1(\operatorname{Frac}(R), G)) = \{*\}.$$

Proof. We pass to connected components to assume that $\operatorname{Spec}(R)$ is connected, so that R is a domain and, in particular, $R \neq 0$. Let E be a G-torsor that trivializes over $\operatorname{Frac}(R)$, so also over $R[\frac{1}{r}]$ for some $r \in R \backslash \{0\}$. By Popescu's theorem [SP, 07GC], the ring R is a filtered direct limit of smooth $\mathcal{O}$-algebras. Thus, a limit argument allows us to assume that R is the localization of a smooth R-algebra at finitely many primes $\mathfrak{p}$ that all lie over maximal ideals $\mathfrak{p}' \subset \mathcal{O}$ with $k_\mathfrak{p}/k_{\mathfrak{p}'}$ separable. In this case, Proposition 4.2.4 gives a $G_{\mathbb{A}^1_R}$-torsor $\mathscr{E}$ whose pullback along the zero section is E such that $\mathscr{E}$ is trivial away from an R-finite closed subscheme $Z \subset \mathbb{A}^1_R$, which we may enlarge to contain the zero section. By Corollary 5.2.6, the pullback of $\mathscr{E}$ under the map $\mathbb{A}^1_R \to \mathbb{A}^1_R$ given by $t \mapsto t^d$ is trivial away from some R-(finite étale) closed subscheme $Y \subset \mathbb{A}^1_R$ disjoint from the zero section. In particular, the pullback of $\mathscr{E}$ by the zero section is trivial, as desired. ☐

Example 5.3.2. In the case when $\mathcal{O}$ is a perfect field, such as $\mathbb{Q}$ or $\mathbb{F}_p$, any regular $\mathcal{O}$-algebra is geometrically regular, so, for quasi-split G, Theorem 5.3.1 simultaneously reproves the equicharacteristic case of the Grothendieck–Serre conjecture settled in [FP15] and [Pan20]. Similarly, in the case when $\mathcal{O} = \mathbb{Z}$, the $\mathcal{O}$-fibers of R are geometrically regular if and only if for every prime p and

[7] We recall from [SP, 0382] that a Noetherian algebra over a field k is *geometrically regular* if its base change to every finite purely inseparable (equivalently, to every finitely generated) field extension of k is regular.

every maximal ideal $\mathfrak{m} \subset R$ of residue characteristic p, we have $p \notin \mathfrak{m}^2$, equivalently, p is a regular parameter for the regular local ring $R_{\mathfrak{m}}$. In particular, Theorem 5.3.1 recovers Theorem 1.2.

With our main result in hand, we are ready to settle the second assertion of Theorem 1.2.

Theorem 5.3.3. *For a Dedekind ring $\mathcal{O}$ and a semilocal regular $\mathcal{O}$-algebra R whose $\mathcal{O}$-fibers are geometrically regular and whose maximal ideals $\mathfrak{m}$ all lie over maximal ideals $\mathfrak{m}' \subset \mathcal{O}$ with $k_{\mathfrak{m}}/k_{\mathfrak{m}'}$ separable, a reductive R-group G is split if and only if its generic fiber $G_{\mathrm{Frac}(R)}$ is split.*

Proof. We pass to connected components to assume that $\mathrm{Spec}(R)$ is connected, so that R is a domain, and we set $K := \mathrm{Frac}(R)$. Only the 'if' part requires an argument, so we assume that G_K is split. The geometric fibers of G have a constant type (see [SGA $3_{\text{III new}}$, XXII, 1.13]), and we let $\mathbf{G}$ be a split reductive R-group of this type, so that G is a form of $\mathbf{G}$ that corresponds to some $x \in H^1(R, \underline{\mathrm{Aut}}(\mathbf{G}))$ whose pullback to $H^1(K, \underline{\mathrm{Aut}}(\mathbf{G}))$ is trivial. We wish to show that x is trivial.

By [SGA $3_{\text{III new}}$, XXIV, 1.3], we have a short exact sequence of group schemes

$$1 \to \mathbf{G}^{\mathrm{ad}} \to \underline{\mathrm{Aut}}(\mathbf{G}) \to \underline{\mathrm{Autext}}(\mathbf{G}) \to 1$$

that, via a fixed pinning of $\mathbf{G}$, is split by a homomorphism $\underline{\mathrm{Autext}}(\mathbf{G}) \hookrightarrow \underline{\mathrm{Aut}}(\mathbf{G})$, whose source is a constant R-group. Any $\underline{\mathrm{Autext}}(\mathbf{G})$-torsor E is constant étale locally on R, so, by [SGA 3_{II}, X, 5.14], its connected components are finite étale over R. Thus, by, for instance, [Čes17, 3.1.9], every K-point of E extends to an R-point, to the effect that no nontrivial $\underline{\mathrm{Autext}}(\mathbf{G})$-torsor trivializes over K.

The nonabelian cohomology exact sequence now lifts x to an $\widetilde{x} \in H^1(R, \mathbf{G}^{\mathrm{ad}})$ and, since the map $\underline{\mathrm{Aut}}(\mathbf{G})(K) \twoheadrightarrow \underline{\mathrm{Autext}}(\mathbf{G})(K)$ is surjective due to the splitting, it also shows that the pullback of $\widetilde{x}$ to $H^1(K, \mathbf{G}^{\mathrm{ad}})$ is trivial. Theorem 5.3.1 then implies that $\widetilde{x}$ itself is trivial, and then so is x. ☐

The ideas of the preceding proof also give a version for quasi-split groups in Theorem 5.3.5. To put it into context, we recall the following conjecture, which may be traced to results of [CT79] or [Pan09]. Even though not formulated there explicitly, it is sometimes attributed to Colliot-Thélène or Panin.

Conjecture 5.3.4. *For a regular local ring R, if the generic fiber of a reductive R-group scheme G has a parabolic subgroup, then G itself has a parabolic subgroup of the same type.*

This conjecture "of Grothendieck–Serre type" seems to lie deeper than the Grothendieck–Serre conjecture: even in equicharacteristic, it is only known in few cases, see [CT79], [Pan09], [PP10], [PP15], [Scu18] for precise results. We use the ideas of this article to settle its equicharacteristic case for minimal parabolics, that is, for Borel subgroups, as follows.

Theorem 5.3.5. *Let R a semilocal regular ring, set $K := \mathrm{Frac}(R)$, and let G be a reductive R-group scheme such that every form $\mathcal{G}$ of G^{ad} satisfies $H^1(R, \mathcal{G}) \hookrightarrow H^1(K, \mathcal{G})$ (this condition holds for every G if R contains a field). Then G is quasi-split if and only if G_K is quasi-split.*

Proof. The injectivity assumption is a special case of the Grothendieck–Serre conjecture and of the "change of origin" twisting bijections in nonabelian cohomology [Gir71, III, 2.6.1 (i)], so the parenthetical assertion follows from the known equicharacteristic case of the Grothendieck–Serre conjecture, see §1.4. By [Guo20, 6.1] (whose proof is similar to that of Theorem 5.3.3 above), this assumption implies that G is the unique reductive model of its generic fiber, so all we need to do is to assume that G_K is quasi-split and to produce a quasi-split reductive R-model of G_K. By the properness of the scheme of Borel subgroups, there is an open subscheme $U \subset \mathrm{Spec}(R)$ whose complement is of codimension $\geqslant 2$ such that even G_U has a Borel subgroup.

Analogously to the proof of Theorem 5.3.3, we reduce to the setting when $\mathrm{Spec}(R)$ is connected, we have a split reductive R-group $\mathbf{G}$, and G corresponds to an element $x \in H^1(R, \underline{\mathrm{Aut}}(\mathbf{G}))$. We fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ that arises from a pinning of $\mathbf{G}$, and we consider the subfunctor

$$\underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B}) \subset \underline{\mathrm{Aut}}(\mathbf{G})$$

that parametrizes those automorphisms that preserve $\mathbf{B}$. In a reductive group, any two Borels are Zariski locally conjugate, so we are reduced to showing that for our $x \in H^1(R, \underline{\mathrm{Aut}}(\mathbf{G}))$ such that $x|_U$ lifts to $H^1(U, \underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B}))$, the restriction $x|_K \in H^1(K, \underline{\mathrm{Aut}}(\mathbf{G}))$ lifts to $H^1(R, \underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B}))$.

By [SGA $3_{\mathrm{III\ new}}$, XXIV, 1.3, 2.1], letting $\mathbf{B}^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}$ be the Borel subgroup of $\mathbf{G}^{\mathrm{ad}}$ corresponding to $\mathbf{B}$, we have a morphism of short exact sequences of group schemes

$$\begin{array}{ccccccccc} 1 & \to & \mathbf{B}^{\mathrm{ad}} & \to & \underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B}) & \to & \underline{\mathrm{Autext}}(\mathbf{G}) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathbf{G}^{\mathrm{ad}} & \to & \underline{\mathrm{Aut}}(\mathbf{G}) & \to & \underline{\mathrm{Autext}}(\mathbf{G}) & \to & 1 \end{array}$$

that, due to our fixed pinning, are compatibly split by some homomorphism $\underline{\mathrm{Autext}}(\mathbf{G}) \hookrightarrow \underline{\mathrm{Aut}}(\mathbf{G}, \mathbf{B})$. We may first map x to an $\overline{x} \in H^1(R, \underline{\mathrm{Autext}}(\mathbf{G}))$ and then map $\overline{x}$ via the splitting to obtain a $y \in H^1(R, \underline{\mathrm{Autext}}(\mathbf{G}, \mathbf{B}))$ whose image in $H^1(R, \underline{\mathrm{Autext}}(\mathbf{G}))$ is also $\overline{x}$. Twisting by (the images of) y gives us the morphism of short exact sequences of R-groups of corresponding forms:

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{B} & \to & \mathcal{A}_0 & \to & \mathcal{E} & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & \mathcal{G} & \to & \mathcal{A} & \to & \mathcal{E} & \to & 1 \end{array}$$

and, via the "change of origin" bijections [Gir71, III, 2.6.1 (i)], we obtain an $x' \in H^1(R, \mathcal{A})$ such that $x'|_U$ lifts to $H^1(U, \mathcal{A}_0)$ for which we need to lift $x'|_K \in H^1(K, \mathcal{A})$ to $H^1(R, \mathcal{A}_0)$ or even to $H^1(R, \mathcal{B})$.

By the nonabelian cohomology sequence, $x'|_U$ even lifts to some $b \in H^1(U, \mathcal{B})$. By descent, $\mathcal{B} \subset \mathcal{G}$ is the inclusion of a Borel R-subgroup, and we let $\mathcal{T} := \mathcal{B}/\mathscr{R}_u(\mathcal{B})$ be the indicated torus. The image of b is a $t \in H^1(U, \mathcal{T})$, which, by purity for torsors under tori [CTS79, Cor. 6.9], extends uniquely to a $\widetilde{t} \in H^1(R, \mathcal{T})$. Any Levi R-subgroup of $\mathcal{B}$ splits the surjection $\mathcal{B} \twoheadrightarrow \mathcal{T}$, and then $\widetilde{t}$ gives a $\widetilde{b} \in H^1(R, \mathcal{B})$ whose image in $H^1(K, \mathcal{B})$, thanks to [SGA $3_{\mathrm{III\ new}}$, XXVI, 2.3], is nothing else but $b|_K$. In particular, the image of $\widetilde{b}$ in $H^1(K, \mathcal{A})$ is $x'|_K$, to the effect that $\widetilde{b}$ is the desired lift. ☐

We thank Uriya First for pointing out the following further consequence about quadratic forms.

Corollary 5.3.6. *For a semilocal regular ring R as in Theorem 5.3.1 with $2 \in R^{\times}$, we have*

$$H^1(R, \mathrm{SO}_n) \hookrightarrow H^1(\mathrm{Frac}(R), \mathrm{SO}_n) \quad \textit{and} \quad H^1(R, \mathrm{O}_n) \hookrightarrow H^1(\mathrm{Frac}(R), \mathrm{O}_n) \quad \textit{for all} \quad n \geqslant 1;$$

moreover, no two nonisomorphic quadratic forms over R that are nondegenerate (in the sense that their associated symmetric bilinear forms are perfect) become isomorphic over $\mathrm{Frac}(R)$.

Proof. Every inner form of SO_n is an $\mathrm{SO}(E)$ for a nondegenerate quadratic space E over R of rank n. Thus, by twisting [Gir71, III, 2.6.1 (i)], the injectivity assertion for SO_n reduces to showing that

$$\mathrm{Ker}(H^1(R, \mathrm{SO}(E)) \to H^1(\mathrm{Frac}(R), \mathrm{SO}(E))) = \{*\}.$$

By the analysis of the long exact cohomology sequence [CT79, p. 17, proof of (D)⇔(E)], this triviality of the kernel is, in turn, equivalent to its analogue for $\mathrm{O}(E)$. Thus, by twisting again, we are reduced to the injectivity assertion for O_n, which itself, for varying n, is a reformulation of the assertion about quadratic forms. For the latter, however, due to the cancellation theorem for quadratic forms, specifically, due to [CT79, Prop. 1.2 (D)⇔(F)], we may assume that one of the forms is a sum of copies of the hyperbolic plane. In terms of O_n-torsors, this means that it suffices to show that

$$\mathrm{Ker}(H^1(R, \mathrm{O}_n) \to H^1(\mathrm{Frac}(R), \mathrm{O}_n)) = \{*\} \quad \text{for all even} \quad n \geqslant 1.$$

We then use [CT79, p. 17, proof of (D)⇔(E)] again to replace O_n by SO_n in this display. With this replacement, however, the desired triviality of the kernel is a special case of Theorem 5.3.1. □

References

[Alp14] Jarod Alper, *Adequate moduli spaces and geometrically reductive group schemes*, Algebr. Geom. **1** (2014), no. 4, 489–531, DOI 10.14231/AG-2014-022. See also https://arxiv.org/abs/1005.2398v2 for an updated post-publication version. MR3272912

[BB70] A. Białynicki-Birula, *Rationally trivial homogeneous principal fibrations of schemes*, Invent. Math. **11** (1970), 259–262, DOI 10.1007/BF01404652. MR276250

[BČ20] Alexis Bouthier and Kęstutis Česnavičius, *Torsors on loop groups and the Hitchin fibration*, Ann. Sci. École Norm. Sup., to appear (2020). Available at http://arxiv.org/abs/1908.07480v2.

[BFF17] Eva Bayer-Fluckiger and Uriya A. First, *Rationally isomorphic hermitian forms and torsors of some non-reductive groups*, Adv. Math. **312** (2017), 150–184, DOI 10.1016/j.aim.2017.03.012. MR3635808

[BFFH19] E. Bayer-Fluckiger, U. A. First, and M. Huruguen, *Orders that are étale-locally isomorphic*, Algebra i Analiz **31** (2019), no. 4, 1–15. MR3985253

[BFFP20] Eva Bayer-Fluckiger, Uriya A. First, and Raman Parimala, *On the Grothendieck–Serre conjecture for classical groups*, preprint (2020). Available at http://arxiv.org/abs/1911.07666v2.

[Bha16] Bhargav Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461, DOI 10.4310/CJM.2016.v4.n4.a1. MR3572635

[BHL17] Bhargav Bhatt and Daniel Halpern-Leistner, *Tannaka duality revisited*, Adv. Math. **316** (2017), 576–612, DOI 10.1016/j.aim.2016.08.040. MR3672914

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034)

[BouAC] Nicolas Bourbaki, *Éléments de mathématique. Algèbre commutative*, chap. I-VII, Hermann (1961, 1964, 1965); chap. VIII-X, Springer (2006, 2007) (French).

[BT87] F. Bruhat and J. Tits, *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 671–698 (French). MR927605

[BVG14] Sofie Beke and Jan Van Geel, *An isomorphism problem for Azumaya algebras with involution over semilocal Bézout domains*, Algebr. Represent. Theory **17** (2014), no. 6, 1635–1655, DOI 10.1007/s10468-013-9463-6. MR3284324

[Čes17] Kęstutis Česnavičius, *A modular description of $\mathscr{X}_0(n)$*, Algebra Number Theory **11** (2017), no. 9, 2001–2089. MR3735461

[Čes20] ———, *Macaulayfication of Noetherian schemes*, Duke Math. J., to appear (2020). Available at http://arxiv.org/abs/1810.04493.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, 2nd ed., New Mathematical Monographs, vol. 26, Cambridge University Press, Cambridge, 2015. MR3362817

[Che10] V. Chernousov, *Variations on a theme of groups splitting by a quadratic extension and Grothendieck-Serre conjecture for group schemes F_4 with trivial g_3 invariant*, Doc. Math. **Extra vol.: Andrei A. Suslin sixtieth birthday** (2010), 147–169. MR2804252

[Con14] Brian Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444 (English, with English and French summaries). MR3362641

[ČS19] Kęstutis Česnavičius and Peter Scholze, *Purity for flat cohomology*, preprint (2019). Available at http://arxiv.org/abs/1912.10932.

[CT79] Jean-Louis Colliot-Thélène, *Formes quadratiques sur les anneaux semi-locaux réguliers*, Bull. Soc. Math. France Mém. **59** (1979), 13–31 (French). Colloque sur les Formes Quadratiques, 2 (Montpellier, 1977). MR532002

[CT95] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64. MR1327280

[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn, *The Bloch-Ogus-Gabber theorem*, Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 31–94. MR1466971

[CTHS05] J.-L. Colliot-Thélène, D. Harari, and A. N. Skorobogatov, *Compactification équivariante d'un tore (d'après Brylinski et Künnemann)*, Expo. Math. **23** (2005), no. 2, 161–170, DOI 10.1016/j.exmath.2005.01.016 (French, with English summary). MR2155008

[CTO92] Jean-Louis Colliot-Thélène and Manuel Ojanguren, *Espaces principaux homogènes localement triviaux*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 97–122 (French). MR1179077

[CTS79] J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134, DOI 10.1007/BF01420486 (French). MR550842

[CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra **106** (1987), no. 1, 148–205, DOI 10.1016/0021-8693(87)90026-3. MR878473

[Dan78] V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247 (Russian). MR495499

[EGA I] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960), 228. MR0217083 (36 #177a)

[EGA II] ———, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 222 (French). MR0163909 (29 #1208)

[EGA III$_1$] ———, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. MR0217085 (36 #177c)

[EGA IV$_1$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. **20** (1964), 259 (French). MR0173675 (30 #3885)

[EGA IV$_2$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330)

[EGA IV$_3$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966), 255. MR0217086 (36 #178)

[EGA IV$_4$] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361 (French). MR0238860 (39 #220)

[Fed16a] Roman Fedorov, *Affine Grassmannians of group schemes and exotic principal bundles over $\mathbb{A}^1$*, Amer. J. Math. **138** (2016), no. 4, 879–906, DOI 10.1353/ajm.2016.0036. MR3538146

[Fed16b] ———, *On the Grothendieck–Serre conjecture on principal bundles in mixed characteristic*, preprint (2016). Available at http://arxiv.org/abs/1501.04224v2.

[Fed18] ———, *On the Grothendieck–Serre Conjecture about principal bundles and its generalizations*, preprint (2018). Available at http://arxiv.org/abs/1810.11844.

[Fir19] Uriya A. First, *An 8-Periodic Exact Sequence of Witt Groups of Azumaya Algebras with Involution*, preprint (2019). Available at http://arxiv.org/abs/1910.03232.

[FP15] Roman Fedorov and Ivan Panin, *A proof of the Grothendieck-Serre conjecture on principal bundles over regular local rings containing infinite fields*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 169–193, DOI 10.1007/s10240-015-0075-z. MR3415067

[FR70] Daniel Ferrand and Michel Raynaud, *Fibres formelles d'un anneau local noethérien*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 295–311 (French). MR0272779

[Gab94] Ofer Gabber, *Gersten's conjecture for some complexes of vanishing cycles*, Manuscripta Math. **85** (1994), no. 3-4, 323–343, DOI 10.1007/BF02568202. MR1305746

[Gab01] O. Gabber, *On space filling curves and Albanese varieties*, Geom. Funct. Anal. **11** (2001), no. 6, 1192–1200, DOI 10.1007/s00039-001-8228-2. MR1878318

[Gil02] P. Gille, *Torseurs sur la droite affine*, Transform. Groups **7** (2002), no. 3, 231–245, DOI 10.1007/s00031-002-0012-3 (French, with English summary). MR1923972

[Gil05] ———, *Errata: "Torsors on the affine line" (French) [Transform. Groups* **7** *(2002), no. 3, 231–245; MR1923972]*, Transform. Groups **10** (2005), no. 2, 267–269, DOI 10.1007/s00031-005-1010-z (French, with English summary). MR2195603

[Gil09] Philippe Gille, *Le problème de Kneser-Tits*, Astérisque **326** (2009), Exp. No. 983, vii, 39–81 (2010) (French, with French summary). Séminaire Bourbaki. Vol. 2007/2008. MR2605318

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971 (French). Die Grundlehren der mathematischen Wissenschaften, Band 179. MR0344253 (49 #8992)

[GLL15] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270, DOI 10.1215/00127094-2877293. MR3347315

[Gro58] Alexandre Grothendieck, *Torsion homologique et sections rationnelles*, Seminaire Claude Chevalley **3** (1958), no. 5, 1–29 (French).

[Gro68] Alexander Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 67–87 (French). MR0244270 (39 #5586b)

[Guo20] Ning Guo, *The Grothendieck–Serre conjecture over semilocal Dedekind rings*, Transform. Groups, to appear (2020). Available at https://arxiv.org/abs/1902.02315v3.

[Har67] Günter Harder, *Halbeinfache Gruppenschemata über Dedekindringen*, Invent. Math. **4** (1967), 165–191, DOI 10.1007/BF01425754 (German). MR225785

[HLR20] Thomas J. Haines, João Lourenço, and Timo Richarz, *On the normality of Schubert varieties: remaining cases in positive characteristic*, preprint (2020). Available at https://arxiv.org/abs/1806.11001v4.

[HR19a] Thomas J. Haines and Timo Richarz, *Normality and Cohen–Macaulayness of parahoric local models*, preprint (2019). Available at https://arxiv.org/abs/1903.10585v4.

[HR19b] Jack Hall and David Rydh, *Coherent Tannaka duality and algebraicity of Hom-stacks*, Algebra Number Theory **13** (2019), no. 7, 1633–1675, DOI 10.2140/ant.2019.13.1633. MR4009673

[Jou83] Jean-Pierre Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983 (French). MR725671

[Lin81] Hartmut Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981/82), no. 2, 319–323, DOI 10.1007/BF01389017. MR641133

[MB96] Laurent Moret-Bailly, *Un problème de descente*, Bull. Soc. Math. France **124** (1996), no. 4, 559–585 (French, with English and French summaries). MR1432058

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR2242284

[Nis82] Yevsey A. Nisnevich, *Etale cohomology and arithmetic of semisimple groups*, ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–Harvard University. MR2632405

[Nis84] ———, *Espaces homogènes principaux rationnellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 1, 5–8 (French, with English summary). MR756297

[Nis89] Yevsey Nisnevich, *Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional regular local rings*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 10, 651–655 (English, with French summary). MR1054270

[Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties; Translated from the Japanese. MR922894

[Oja80] Manuel Ojanguren, *Quadratic forms over regular rings*, J. Indian Math. Soc. (N.S.) **44** (1980), no. 1-4, 109–116 (1982). MR752647

[Oja82] ———, *Unités représentées par des formes quadratiques ou par des normes réduites*, Algebraic K-theory, Part II (Oberwolfach, 1980), Lecture Notes in Math., vol. 967, Springer, Berlin-New York, 1982, pp. 291–299 (French). MR689397

[OP99] Manuel Ojanguren and Ivan Panin, *A purity theorem for the Witt group*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 1, 71–86, DOI 10.1016/S0012-9593(99)80009-3 (English, with English and French summaries). MR1670591

[OP01] ———, *Rationally trivial Hermitian spaces are locally trivial*, Math. Z. **237** (2001), no. 1, 181–198, DOI 10.1007/PL00004859. MR1836777

[OPZ04] M. Ojanguren, I. Panin, and K. Zainoulline, *On the norm principle for quadratic forms*, J. Ramanujan Math. Soc. **19** (2004), no. 4, 289–300. MR2125505

[Pan05] Ivan Panin, *Purity for multipliers*, Algebra and number theory, Hindustan Book Agency, Delhi, 2005, pp. 66–89. MR2193345

[Pan09] ———, *Rationally isotropic quadratic spaces are locally isotropic*, Invent. Math. **176** (2009), no. 2, 397–403, DOI 10.1007/s00222-008-0168-0. MR2495767

[Pan16] I. A. Panin, *On Grothendieck-Serre's conjecture concerning principal G-bundles over reductive group schemes: II*, Izv. Ross. Akad. Nauk Ser. Mat. **80** (2016), no. 4, 131–162, DOI 10.4213/im8452; English transl., Izv. Math. **80** (2016), no. 4, 759–790. MR3535361

[Pan17] Ivan Panin, *Nice triples and a moving lemma for motivic spaces*, preprint (2017). Available at https://arxiv.org/abs/1707.01755.

[Pan18] ———, *On Grothendieck-Serre conjecture concerning principal bundles*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 201–221. MR3966763

[Pan20] I. A. Panin, *Proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing a field*, Izv. Ross. Akad. Nauk Ser. Mat. **84** (2020), no. 4, 169–186, DOI 10.4213/im8982 (Russian). MR4133391

[Poo04] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127, DOI 10.4007/annals.2004.160.1099. MR2144974

[PP10] Ivan Panin and Konstantin Pimenov, *Rationally isotropic quadratic spaces are locally isotropic: II*, Doc. Math. **Extra vol.: Andrei A. Suslin sixtieth birthday** (2010), 515–523. MR2804263

[PP15] I. Panin and K. Pimenov, *Rationally isotropic quadratic spaces are locally isotropic. III*, Algebra i Analiz **27** (2015), no. 6, 234–241, DOI 10.1090/spmj/1433; English transl., St. Petersburg Math. J. **27** (2016), no. 6, 1029–1034. MR3589229

[PR08] G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties*, Adv. Math. **219** (2008), no. 1, 118–198, DOI 10.1016/j.aim.2008.04.006. With an appendix by T. Haines and Rapoport. MR2435422

[PS97] I. A. Panin and A. A. Suslin, *On a conjecture of Grothendieck concerning Azumaya algebras*, Algebra i Analiz **9** (1997), no. 4, 215–223 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **9** (1998), no. 4, 851–858. MR1604322

[PS16] I. A. Panin and A. K. Stavrova, *On the Grothendieck-Serre conjecture concerning principal G-bundles over semi-local Dedekind domains*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **443** (2016), no. Voprosy Teorii Predstavlenii Algebr i Grupp. 29, 133–146, DOI 10.1007/s10958-017-3316-5; English transl., J. Math. Sci. (N.Y.) **222** (2017), no. 4, 453–462. MR3507770

[PSV15] I. Panin, A. Stavrova, and N. Vavilov, *On Grothendieck-Serre's conjecture concerning principal G-bundles over reductive group schemes: I*, Compos. Math. **151** (2015), no. 3, 535–567, DOI 10.1112/S0010437X14007635. MR3320571

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129

[Rag94] M. S. Raghunathan, *Principal bundles admitting a rational section*, Invent. Math. **116** (1994), no. 1-3, 409–423, DOI 10.1007/BF01231567. MR1253199

[Scu18] Stephen Scully, *The Artin-Springer theorem for quadratic forms over semi-local rings with finite residue fields*, Proc. Amer. Math. Soc. **146** (2018), no. 1, 1–13, DOI 10.1090/proc/13744. MR3723116

[Ser58] J.-P. Serre, *Espaces fibrés algébriques*, Seminaire Claude Chevalley **3** (1958), no. 1, 1–37 (French).

[SGA $3_{\mathrm{I\ new}}$] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome I. Propriétés générales des schémas en groupes*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 7, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre; Revised and annotated edition of the 1970 French original. MR2867621

[SGA 3_{II}] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970 (French). MR0274459 (43 #223b)

[SGA $3_{\mathrm{III\ new}}$] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 8, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre; Revised and annotated edition of the 1970 French original. MR2867622

[SGA 4_{III}] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin, 1973 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4); Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. MR0354654 (50 #7132)

[SP] A. J. de Jong et al., *The Stacks Project.* Available at `http://stacks.math.columbia.edu`.

[Tho87] R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*, Adv. in Math. **65** (1987), no. 1, 16–34. MR893468

[Tsy19] A. E. Tsybyshev, *A step toward the non-equicharacteristic Grothendieck-Serre conjecture*, Algebra i Analiz **31** (2019), no. 1, 246–254, DOI 10.1090/spmj/1591 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **31** (2020), no. 1, 181–187. MR3932824

[Zai00] K. V. Zainullin, *On Grothendieck's conjecture on principal homogeneous spaces for some classical algebraic groups*, Algebra i Analiz **12** (2000), no. 1, 150–184 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **12** (2001), no. 1, 117–143. MR1758566

[Zai05] K. Zainoulline, *On Knebusch's norm principle for quadratic forms over semi-local rings*, Math. Z. **251** (2005), no. 2, 415–425, DOI 10.1007/s00209-005-0809-6. MR2191034

[Zhu17] Xinwen Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence*, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 59–154. MR3752460