The Mourre Theory for Analytically Fibered Operators

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Abstract

We develop in this paper the Mourre theory for an abstract class of fibered self-adjoint operators which we call analytically fibered operators. We construct a conjugate operator for which we prove that a Mourre estimate holds. Examples of analytically fibered operators are given and finally perturbations of such operators are considered.

1 Introduction

This paper is devoted to the Mourre theory for an abstract class of self-adjoint operators, called *analytically fibered operators*. The Mourre theory for a self-adjoint operator H_0 acting on some Hilbert space $\mathcal{H}$ is based on the construction of another self-adjoint operator A, called a *conjugate operator* so that the following estimate holds:

$$1_{\Delta}(H_0)[H_0, iA]1_{\Delta}(H_0) \geq c_0 1_{\Delta}(H_0) + K, \tag{1.1}$$

where $1_{\Delta}(H_0)$ denotes the spectral projection on the interval $\Delta \subset \mathbb{R}$ for the operator H_0, c_0 is a positive constant and K is a compact operator. The estimate (1.1) is called a *Mourre estimate*. If one can take $K = 0$ in (1.1), then it is called a *strict Mourre estimate*.

The Mourre estimate has several important consequences for the spectral and scattering theory of H_0. The first ones are the discreteness of the point spectrum $\sigma_{\rm pp}(H_0)$ in Δ, and under some additional assumptions, the existence of a limiting absorption principle, i.e. the existence of the limits $\lim_{\epsilon\to 0}(H_0 - \lambda \pm i\epsilon)^{-1}$, for $\lambda \in \Delta\backslash\sigma_{\rm pp}(H_0)$ as a bounded operator between suitable weighted spaces. The estimates leading to the limiting absorption principle are called *resolvent estimates*. In turn the limiting absorption principle implies that the singular continuous spectrum of H_0 $\sigma_{\rm sc}(H_0)$ is empty in Δ. Moreover there exists a natural class of perturbations V for which one can deduce from (1.1) a Mourre estimate for $H = H_0 + V$ with the same conjugate operator A.

However the most intuitive consequences of the Mourre estimate (1.1) are probably properties of the unitary group e^{-itH_0} for large times t, which go under the name of *propagation estimates*.They are based on the fact that $[H_0, iA]$ is the time derivative of $t \mapsto e^{itH_0}Ae^{-itH_0}$ at $t = 0$. An example of such a propagation estimate is

$$\|F(\frac{A}{t} < c_0)e^{-itH_0}1^{\rm c}_{\Delta}(H_0)\| \to 0 \text{ when } t \to \pm\infty,$$

where $1^{\rm c}_{\Delta}(H_0)$ is the spectral projection on the continuous spectral subspace of H_0 in Δ. Such propagation estimates allow to develop in a very natural way the scattering theory for perturbations $H = H_0 + V$ of H_0. For example there exists a natural class of perturbations V (that one can call short-range perturbations) for which the local wave operators

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0} 1_{\Delta}(H_0) =: \Omega^{\pm}$$

can be shown to exist and to be *asymptotically complete* i.e.

$$1^{\rm c}_{\Delta}(H)\mathcal{H} = \Omega^{\pm}\mathcal{H}.$$

Finally using extensions of the Mourre method one can prove more detailed resolvent estimates for H and H_0 which lead to results on the scattering matrix $S(\lambda)$ for the pair (H_0, H).

Let us end this very brief overview of the Mourre method by some brief historical comments and some bibliographical references, which do not intend to be complete. The Mourre method was invented by Eric Mourre in [11] and subsequently developed and applied in [6], [10], [12], [13]. An essentially optimal version of the Mourre method was developed in [1], [3]. In particular the book [1] contains a detailed exposition of the subject. Time-dependent propagation estimates originate in the papers of Sigal and Soffer [16], [18], [17].

In this paper we will construct a conjugate operator and prove a strict Mourre estimate for an abstract class of self-adjoint operators which we call *analytically fibered operators.* Let us recall that an operator H_0 on a Hilbert space $\mathcal{H}$ is called *fibered* (or also a direct integral) if the following conditions hold (see as example [14]):

i) $\mathcal{H}$ can be written as $L^2(M, \mu; \mathcal{H}')$ where $\mathcal{H}'$ is a separable Hilbert space and (M, μ) a σ-finite measure space.

ii) the operator H can be written as the direct integral

$$H_0 = \int_M^{\oplus} H_0(k) \mathrm{d}\mu(k), \tag{1.2}$$

where the function $M \ni k \to H_0(k)$ is measurable with values in the self-adjoint operators on $\mathcal{H}'$.

The set

$$\Sigma := \{(\lambda, k) \in \mathbb{R} \times M,\ \lambda \in \sigma(H_0(k))\},$$

which will be defined further without ambiguity about null measure sets, plays an important role. We shall call it the *set of energy-momentum* of H_0.

Hamiltonians describing 'free ' systems , which admit generally a rich set of constants of motion are often fibered.

We introduce in this paper a particular class of fibered operators, which is characterized by three additional properties which can be summarized as:

i) the space M is a real-analytic manifold,

ii) the resolvent $(H_0(k) + i)^{-1}$ is analytic with respect to k and $H_0(k)$ has only discrete spectrum for $k \in M$,

iii) the projection $p_{\mathbb{R}} : \Sigma \ni (\lambda, k) \mapsto \lambda \in \mathbb{R}$ is a proper map.

Examples of analytically fibered operators will be given at the end of Section 2. The main example is Schrödinger operators with a periodic potential. Application of the results in this paper to the scattering theory of perturbed periodic Schrödinger operators will be treated in a subsequent publication.

Let us now give the plan of our paper.

In Section 2 we define analytically fibered operators and give some examples. In Section 3 a conjugate operator A is constructed for an analytically fibered operator H_0. We also recall standard results about the perturbations $H = H_0 + V$ of analytically fibered operators which follows from the strict Mourre estimate for H_0.

2 Analytically fibered operators

In this section we define analytically fibered operators and give some examples.

2.1 Definition

Let $\mathcal{H}'$ be a separable Hilbert space and (M, μ) a σ-finite measure space. We denote by $\mathcal{H}$ the Hilbert space

$$\int_M^{\oplus} \mathcal{H}' \mathrm{d}\mu = L^2(M, \mu; \mathcal{H}').$$

We recall that a function

$$M \ni k \to H_0(k)$$

with values in the self-adjoint operators (not necessarily bounded) on $\mathcal{H}'$ is *measurable* if the functions

$$M \ni k \to (\psi, (H_0(k) + i)^{-1}\psi)$$

are measurable for all $\psi \in \mathcal{H}'$. We define then the operator

$$H_0 = \int_M^{\oplus} H_0(k) \mathrm{d}\mu(k) \tag{2.1}$$

acting on $\mathcal{H}$ by

$$\begin{aligned} D(H_0) &:= \left\{ \psi \in \mathcal{H},\ \psi(k) \in D(H_0(k)) \text{ a.e.}, \int_M \|H_0(k)\psi(k)\|^2 \mathrm{d}\mu(k) < \infty \right\}, \\ (H_0\psi)(k) &:= H_0(k)\psi(k), \quad \text{for } \psi \in D(H_0). \end{aligned}$$

Operators of the form (2.1) are called *fibered operators.* The operators $H_0(k)$ are the *fibers* of H_0. In this paper the space M will be called the *momentum space.* Since we want to use some analyticity property of $H_0(k)$, we next specify the analytic structure in which it makes sense.

Definition 2.1. *Let M be a real analytic manifold and $\mathcal{H}'$ a separable Hilbert space.*

a) *If M is a real analytic manifold, the real analytic vector bundle $p_M : \mathcal{F} \to M$ with fiber $\mathcal{H}'$ is called a real analytic Hilbert bundle on M if its structure group is the group of unitary operators on $\mathcal{H}'$.*

b) *With the previous bundle is naturally associated the real analytic vector bundle with fiber $\mathcal{L}(\mathcal{H}')$ and structure group the group of conjugations by unitary operators on $\mathcal{H}'$. This bundle will be denoted by $\mathcal{L}(\mathcal{H}')$.*

c) *The spaces of real analytic sections of these bundles will be respectively denoted by $\mathcal{C}^\omega(M;\mathcal{F})$ and $\mathcal{C}^\omega(M;\mathcal{L}(\mathcal{F}))$. The spaces of $\mathcal{C}^\infty$ and compactly supported $\mathcal{C}^\infty$ sections will denoted similarly with the symbols $\mathcal{C}^\infty$ and $\mathcal{C}^\infty_{\mathrm{comp}}$ instead of $\mathcal{C}^\omega$.*

When μ is a measure on M, the space $L^2(M,\mu;\mathcal{F})$ of μ-square-integrable sections of $\mathcal{F}$ is naturally identified with $\mathcal{H}$. We can now introduce the exact definition of analytically fibered operators . for which we will be able to construct conjugate operators.

Definition 2.2. *Assume that M is a real analytic manifold and that the measure μ is given by a positive (>0) C^∞ $1-$density. The operator (2.1) will be said* analytically fibered *if there exists a real analytic Hilbert bundle $p_M:\mathcal{F}\to M$ with fiber $\mathcal{H}'$ so that*

i) *the resolvent $(H_0(k)+i)^{-1}$ defines an element of $\mathcal{C}^\omega(M;\mathcal{L}(\mathcal{F}))$;*

ii) *for all $k\in M$, the self-adjoint operator $H_0(k)$ has purely discrete spectrum;*

iii) *the projection $p_{\mathbb{R}}:\Sigma\ni(\lambda,k)\to\lambda$ is proper.*

The energy-momentum set,

$$\Sigma := \{(\lambda,k)\in\mathbb{R}\times M,\ \lambda\in\sigma(H_0(k))\},$$

is here well defined because $H_0(k)$ depends continuously on $k\in M$ in the resolvent sense.

Let us now give some examples of analytically fibered operators where the fiber bundle $\mathcal{F}$ is the trivial bundle $M\times\mathcal{H}'$.

2.2 Matrix valued differential operators with constant coefficients

Let H_0 be the differential operator $P(D)$ on $\mathbb{R}^n$ where $P(\xi)=(a_{ij}(\xi))\in\mathcal{M}_p(\mathbb{C})$ is a self-adjoint matrix with polynomial coefficients. We have the following lemma:

Lemma 2.3. *Assume that there exists a function $\mathbb{R}^n\ni\xi\to f(\xi)\in\mathbb{R}$ so that $\lim\limits_{|\xi|\to+\infty} f(\xi)=+\infty$, and*

$$|P(\xi)|\geq f(\xi)1_p. \tag{2.2}$$

Then the operator $H=P(D)$ is analytically fibered.

The elementary proof is left to the reader.

2.3 Neutral two-particle systems in a magnetic field

Our second example concerns the Hamiltonian describing a neutral system of two interacting particles in a constant magnetic field, in two space dimensions.

This Hamiltonian is of the form

$$H_0 = \sum_{i=1}^{2} \frac{1}{2m_i}(D_{y_i} - q_i J y_i)^2 + V(y_1 - y_2), \text{ acting on } L^2(\mathbb{R}^4),$$

where $m_i, q_i, i = 1, 2$ are the masses and charges of the particles, V the interaction potential and

$$J := \frac{1}{2}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

where b is the intensity of the magnetic field. We assume that the potential V satisfies:

(V1) V is a multiplication operator on $L^2(\mathbb{R}^2)$ $\Delta-$bounded with relative bound 0,

(V2) $\|F(\frac{|x|}{R} \geq 1)V(-\Delta + 1)^{-1}\| = o(1)$, when $R \to \infty$.

Using Kato's inequality (see [8, lemma 3.1]), we deduce easily from *(V1)* that H_0 is $H_{00}-$bounded with relative bound 0, for

$$H_{00} := \sum_{i=1}^{2} \frac{1}{2m_i}(D_{y_i} - q_i J y_i)^2.$$

The operator H_0 is hence self-adjoint with domain the magnetic Sobolev space

$$D(H_0) = D(H_{00}) = \{u \in L^2(\mathbb{R}^4) | H_{00}u \in L^2(\mathbb{R}^4)\}.$$

The operator H_0 commute with the *pseudomomentum of the center of mass*, defined by:

$$K := (D_{y_1} + q_1 J y_1) + (D_{y_2} + q_2 J y_2),$$

and if $q_1 + q_2 = 0$, the two components of K commute. Using the arguments of [8], we construct a unitary transformation :

$$U : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^2_k \times \mathbb{R}^2),$$

so that

$$UKU^* = (k_1, k_2), \qquad UH_0U^* = \int_{\mathbb{R}^2_k}^{\oplus} H_0(k)dk,$$

The operator $H_0(k)$ acting on $L^2(\mathbb{R}^2)$ is given by

$$H_0(k) := \frac{1}{2M}(k - 2qJy)^2 + \frac{1}{2m}\left(D_y - q\frac{m_1 - m_2}{M} Jy\right)^2 + V(y),$$

with $M = m_1 + m_2, m = m_1 m_2 M^{-1}$ and $q = q_1 = -q_2$. We denote

$$H_{00}(k) := \frac{1}{2M}(k - 2qJy)^2 + \frac{1}{2m}(D_y - q\frac{m_1 - m_2}{M} Jy)^2.$$

We have the following lemma.

Lemma 2.4. *Let Σ be the energy-momentum set of H_0 and we denote by τ_0 the set*

$$\tau_0 := \{\sum_1^2 \frac{|q|}{m_i} b(n_i + \frac{1}{2}), n_i \in \mathbb{N}\}.$$

Under the hypotheses (V1) and (V2) we have the following properties:

i) *The function*

$$\mathbb{R}^2 \ni k \to (H_0(k) + i)^{-1} \in \mathcal{L}(L^2(\mathbb{R}^2))$$

is analytic with values in the compact operators on $L^2(\mathbb{R}^2)$.

ii) *The projection $p_{\mathbb{R}} : \Sigma \backslash p_{\mathbb{R}}^{-1}(\tau_0) \to \mathbb{R} \backslash \tau_0$ is proper.*

The set τ_0 represents additional thresholds associated with the free channel where the two particles are far from each other. (The construction of a conjugate operator allows to prove the asymptotic completeness for three particle systems in two space dimensions in the case where there exists a neutral cluster of two particles [7].)

Proof : Hypothesis *(V1)* ensures that V is $H_{00}(k)-$bounded with relative bound 0, and hence that

$$D(H_0(k)) = D(H_{00}(k)) = \{u \in L^2(\mathbb{R}^2) | \|D^2 u\|^2 + \|y^2 u\|^2 < \infty\}.$$

The resolvent $(H_0(k)+i)^{-1}$ is hence compact and clearly analytic with respect to k. It remains to check that the projection $p_{\mathbb{R}} : \Sigma \to \mathbb{R} \backslash \tau_0$ is proper. Denote by $T(k)$ the unitary transformation, which quantizes the symplectic transformation:

$$y \to y + (2q)^{-1} J^{-1} k,$$

$$\eta \to \eta - \tfrac{m_1 - m_2}{2M} k.$$

We have

$$\begin{aligned} T(k) H_{00}(k) T^*(k) &= H_{00}(0), \\ T(k) H_0(k) T(k)^* &= H_{00}(0) + V(y + (2q)^{-1} J^{-1} k). \end{aligned} \tag{2.3}$$

We deduce first from (2.3) that $\sigma(H_{00}(k))$ is independent of k. On the other hand the direct integral

$$\int_{\mathbb{R}^2}^{\oplus} H_{00}(k) dk$$

is unitarily equivalent to H_{00}. So we have

$$\sigma(H_{00}(k)) = \tau_0. \tag{2.4}$$

On the other hand hypotheses *(V1)*, *(V2)* imply:

$$\lim_{k \to \infty} V(\cdot + (2q)^{-1} J^{-1} k)(H_{00}(0) + i)^{-1} = 0,$$

and hence

$$\lim_{k \to \infty} \|(H_0(k) + i)^{-1} - (H_{00}(k) + i)^{-1}\| = 0. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain *ii)*. □

2.4 Periodic Schrödinger operators

Our third example concerns the periodic Schrödinger operators. We consider the Hamiltonian

$$H_0 = \frac{1}{2}D^2 + V_\Gamma(x), \text{ we } L^2(\mathbb{R}^n),$$

where V_Γ is a real potential, $\Gamma-$periodic for a lattice Γ in $\mathbb{R}^n$:

$$V_\Gamma(x+\gamma) = V_\Gamma(x),\ \gamma \in \Gamma.$$

We assume that V_Γ satisfies the following hypothesis:

(V) V_Γ is $\Delta-$bounded with relative bound strictly smaller than 1.

The operator H_0 is hence self-adjoint with domain $H^2(\mathbb{R}^n)$. We recall now the Floquet-Bloch reduction for periodic operators. We refer to [19] for proofs. We associate with Γ the torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$, the fundamental domain

$$F := \{x = \sum_{j=1}^{n} x_j\gamma_j, 0 \le x_j < 1\},$$

where $\{\gamma_j\}_1^n$ is a basis of Γ, and the dual lattice

$$\Gamma^* := \{\gamma^* \in \mathbb{R}^n | \langle \gamma, \gamma^* \rangle \in 2\pi\mathbb{Z},\ \forall \gamma \in \Gamma\}.$$

Similarly we define the fundamental domain F^* and the torus $\mathbb{T}^{n*}$. Finally we denote by μ_Γ (resp. μ_{Γ^*}) the volume of the fundamental domain F (resp.F^*). With this notations, the Floquet-Bloch transform associated with Γ is defined by:

$$Uu(k,x) := \mu_{\Gamma^*}^{-\frac{1}{2}} \sum_{\gamma\in\Gamma} e^{-i\langle k,\gamma\rangle} u(x+\gamma), \tag{2.6}$$

for $u \in S(\mathbb{R}^n)$ as example. We have:

$$(Uu)(k+\gamma^*, x) = (Uu)(k,x), \text{ for } \gamma^* \in \Gamma^*,$$

and U extends as a unitary operator

$$U : L^2(\mathbb{R}^n, dx) \to L^2\left(\mathbb{T}^{n*}, dk; \mathcal{H}'\right),$$

for $\mathcal{H}' = L^2(F, dx)$, whose inverse is given by

$$U^{-1}v(x+\gamma) = \mu_\Gamma^{-\frac{1}{2}} \int_{\mathbb{T}^{n*}} e^{i\langle k,\gamma\rangle} v(k,x)dk,\ x \in F.$$

The operator UH_0U^{-1} is equal to

$$UH_0U^{-1} = \int_{\mathbb{T}^{n*}}^{\oplus} H_0(k)dk, \tag{2.7}$$

with

$$H_0(k) = \tfrac{1}{2}D^2 + V_\Gamma(x),$$
$$D(H_0(k)) = \{u = v\Big|_F,\ v \in H^2_{\mathrm{loc}}(\mathbb{R}^n) | v(x+\gamma) = e^{i\langle k,\gamma\rangle}v(x), \forall\gamma \in \Gamma\}.$$

The energy-momentum set Σ is traditionally called the *Bloch variety* and the fibers $\Sigma_\lambda = \Sigma \cap p_{\mathbb{R}}^{-1}(\{\lambda\})$ *Fermi surfaces.*

Lemma 2.5. *Under hypothesis (V), the operator* UH_0U^{-1} *is analytically fibered.*

Proof : The bijection $F^* \ni \xi \to \xi \pmod{\Gamma^*} \in \mathbb{T}^{n*}$ restricted to the interior of the fundamental domain $\dot{F}^*$ is a real analytic diffeomorphism. Let us identify $\dot{F}^*$ and its image. For $k \in \dot{F}^*$, we consider the unitary operator $T(k)$ on $L^2(F)$ defined by $T(k)v(x) = e^{-i\langle k,x\rangle}v(x)$. We have then

$$T(k)H_0(k)T(k)^{-1} =: \tilde{H}_0(k), \tag{2.8}$$

with

$$\tilde{H}_0(k) = \frac{1}{2}(D+k)^2 + V_\Gamma(x),$$
$$D(\tilde{H}_0(k)) = \{u = v\Big|_F,\ v \in H^2_{\mathrm{loc}}(\mathbb{R}^n) | v(x+\gamma) = v(x), \forall \gamma \in \Gamma\}.$$

If we identify the opposite faces of F we can consider $\tilde{H}_0(k)$ as $\frac{1}{2}(D+k)^2 + V_\Gamma(x)$ naturally defined on $L^2(\mathbb{T}^n)$. The resolvent $(\tilde{H}_0(k)+i)^{-1}$ is compact and clearly analytic with respect to $k \in \dot{F}^*$. The same property holds for $(H_0(k)+i)^{-1}$. this property extends to all $k \in \mathbb{T}^{n*}$ by taking several charts of the form $\xi_0 + \dot{F}^*$. □

3 Mourre estimate for analytically fibered operators

In this section, we construct a family of conjugate operators associated with analytically fibered operator H_0.

Theorem 3.1. *There exists a discrete set* τ *determined by* H_0 *so that for any interval* $I \subset\subset \mathbb{R}\backslash\tau$ *there exists an operator* A_I *essentially self-adjoint on* $D(A_I) = \mathcal{C}^\infty_{\mathrm{comp}}(M;\mathcal{F})$ *satisfying the following properties:*

i) *For all* $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(I)$, *there exists a constant* $c_\chi > 0$ *so that*

$$\chi(H_0)\,[H_0, iA_I]\,\chi(H_0) \geq c_\chi \chi(H_0)^2.$$

ii) *The multi-commutators* $\mathrm{ad}^k_{A_I}(H_0)$ *are bounded for all* $k \in \mathbb{N}$.

iii) *The operator* A_I *is a first order differential operator in* k *of which the coefficients belong to* $\mathcal{C}^\infty(M;\mathcal{L}(\mathcal{H}'))$ *and there exists* $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}\setminus\tau)$ *so that* $A = \chi(H_0)A = A\chi(H_0)$.

Let us first recall the following well-known consequences of Theorem 3.1 (see [11], [13]).

Corollary 3.2. a) $\sigma_{\mathrm{pp}}(H_0) \subset \tau$.

b) *For all* $\Delta \subset I \subset\subset \mathbb{R}\backslash\tau$, *one has*

$$\sup_{\lambda\in\Delta,\varepsilon>0} \left\| (1+|A_I|)^{-s}\,(H_0-\lambda \pm i\varepsilon)^{-1}\,(1+|A_I|)^{-s} \right\| < \infty, \quad s > \frac{1}{2}.$$

and the singular continuous spectrum of H_0, $\sigma_{\mathrm{sc}}(H_0)$, *is empty.*

Before giving the proof of Theorem 3.1, we state the results for some natural class of perturbed Hamiltonians $H = H_0 + V$. We will simply recall some well known results in the Mourre theory (see [11], [13]) and refer the reader to the book [1] for a complete exposition of the Mourre method. In particular a sharper version of Theorem 3.3 is given in [1, Prop. 7.5.6].

Theorem 3.3. *Let A_I be a conjugate operator for H_0 associated with an arbitrary compact interval $I \subset \mathbb{R} \setminus \tau$. Let V be a symmetric operator on $\mathcal{H}$ so that:*

i) $V(H_0 + i)^{-1}$ is compact,

ii) $(H_0 + i)^{-1}[V, iA_I](H_0 + i)^{-1}$ is compact,

iii) $\int_0^1 \|(H_0 + i)^{-1} \left(e^{itA_I}[V, iA_I]e^{-itA_I} - [V, iA_I]\right) (H_0 + i)^{-1}\| \frac{dt}{t} < \infty$.

Then the following results hold:

i) *There exists a constant $c > 0$ and a compact operator K so that if $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(I)$*

$$\chi(H)[H, iA_I]\chi(H) \geq c\chi^2(H) + K.$$

Consequently $\sigma_{\mathrm{pp}}(H)$ is of finite multiplicity in $\mathbb{R}\backslash\tau$ and has no accumulation points in $\mathbb{R}\backslash\tau$.

ii) *For each $\lambda \in I\backslash\sigma_{\mathrm{pp}}(H)$, there exists $\epsilon > 0$ and $c > 0$ so that*

$$1_{[\lambda-\epsilon,\lambda+\epsilon]}(H)[H, iA_I]1_{[\lambda-\epsilon,\lambda+\epsilon]}(H) \geq c1_{[\lambda-\epsilon,\lambda+\epsilon]}(H).$$

iii) *The limiting absorption principle holds on $I\backslash\sigma_{\mathrm{pp}}(H)$:*

$$\lim_{\epsilon\to\pm 0}(1 + |A_I|)^{-s}(H - \lambda + i\epsilon)^{-1}(1 + |A_I|)^{-s} \text{ exists and is bounded for all } s > \frac{1}{2}.$$

Consequently the singular continuous spectrum of H is empty.

iv) *If the operator $(1+|A_I|)^s V(1+|A_I|)^s$ is bounded for some $s > \frac{1}{2}$, then for any open interval $\Delta \subset I$, the wave operators*

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}1_\Delta(H_0) =: \Omega^\pm_\Delta$$

exist and are asymptotically complete,

$$1^{\mathrm{c}}_\Delta(H)\mathcal{H} = \Omega^\pm_\Delta\mathcal{H}.$$

The proof of Theorem 3.1 relies on two elementary lemmas and a classical result from the theory of analytic singularities.

Notation : For I a Borel set in $\mathbb{R}$, we denote by $\pi_I(k)$ the spectral projection $1_I(H_0(k))$ on I.

Lemma 3.4. a) *For all $(\lambda_0, k_0) \in \mathbb{R} \times M$, there exists neighborhoods $I_0 \subset \mathbb{R}$ of λ and $V_0 \subset M$ of k so that $\pi_{I_0}(k_0) = \pi_{\{\lambda_0\}}(k_0)$ and $\pi_{I_0}(k) \in \mathcal{L}(\mathcal{H}')$ is real analytic with respect to $k \in V_0$.*

b) *The function* $\mathbf{mul} : \mathbb{R} \times M \to \mathbb{N}$ *defined by* $\mathbf{mul}(\lambda, k) := \dim \pi_{\{\lambda\}}(k)\mathcal{H}'$*, is upper semicontinuous. It reaches its maximum* N_K *when we restrict* λ *to a compact set* $K \subset \mathbb{R}$*.*

Proof : a) The only non trivial case is when $\lambda_0 \in \sigma(H_0(k_0))$. Let I_0 be an open interval so that $\overline{I_0} \cap \sigma(H_0(k_0)) = I_0 \cap \sigma(H_0(k_0)) = \{\lambda_0\}$. For k close to k_0, we have

$$\pi_{I_0}(k) = \oint_{\Gamma_0} (z - H_0(k))^{-1}\, dz$$

where the contour Γ_0 is chosen sufficiently close to I_0. The projection $\pi_{I_0}(k)$ is clearly analytic with respect to k for k close to k_0.
b) If $\mathbf{mul}(\lambda_0, k_0) < n \in \mathbb{N}$, we choose a neighborhood $I_0 \times V_0$ as in the proof of a) and by continuity we obtain

$$\mathbf{mul}(\lambda, k) \leq \dim \pi_{I_0}(k)\mathcal{H}' = \dim \pi_{I_0}(k_0)\mathcal{H}' = \mathbf{mul}(\lambda_0, k_0) < n,$$

for all $(\lambda, k) \in I_0 \times V_0$. So the function **mul** is upper semicontinuous. Finally for all compact set $K \subset \mathbb{R}$ we have

$$\sup_{\substack{\lambda \in K \\ k \in M}} \mathbf{mul}(\lambda, k) = \sup_{(\lambda,k) \in \Sigma \cap p_{\mathbb{R}}^{-1}(K)} \mathbf{mul}(\lambda, k).$$

Since the projection $p_{\mathbb{R}}\Big|_{\Sigma}$ is proper, the set $\Sigma \cap p_{\mathbb{R}}^{-1}(K)$ is compact. **mul** is upper semicontinuous and bounded above and hence reaches its maximum on $\Sigma \cap p_{\mathbb{R}}^{-1}(K)$. $\square$

Lemma 3.5. *The sets* $\Sigma_i = \{(\lambda, k) \in \mathbb{R} \times M,\ \mathbf{mul}(\lambda, k) = i\}$*,* $i \in \mathbb{N}$*, are semi-analytic sets. of* $\mathbb{R} \times M$*.*

We recall that a subset S of a real analytic manifold M is *semi-analytic* if for every $x_0 \in M$ there exists a neighborhood U in M of x_0 so that $S \cap U = \{x \in U | f_i(x) > 0, i = 1, \cdots, N\}$, where the functions f_i are analytic on U.
Proof : Let $i \in \mathbb{N}$ and $(\lambda_0, k_0) \in \mathbb{R} \times M$. Let us check that Σ_i is given in a neighborhood of (λ_0, k_0) by a system of analytic inequalities. Let $i_0 = \mathbf{mul}(\lambda_0, k_0)$. Since the function **mul** is u.s.c. it suffices to consider the case $i_0 \geq i$. We introduce again the neighborhood $I_0 \times V_0$ of Lemma 3.4 a), with V_0 small enough so that

$$\|\pi_{I_0}(k) - \pi_{I_0}(k_0)\|_{\mathcal{L}(\mathcal{H}')} \leq \frac{1}{2}, \quad \forall k \in V_0.$$

The operator $1 + \pi_{I_0}(k)\,(\pi_{I_0}(k) - \pi_{I_0}(k_0))$ is invertible with a real analytic inverse for $k \in V_0$. We denote by Π_k the image of $\pi_{I_0}(k)$ and $\Theta(k) = \pi_{I_0}(k)\Big|_{\Pi_{k_0}}$. We have

$$[1 + \pi_{I_0}(k)\,(\pi_{I_0} - \pi_{I_0}(k_0))]^{-1}\,\pi_{I_0}(k_0)\Theta(k)u = u - [\ldots]^{-1}\left(1 - \pi_{I_0}(k_0)^2\right)u = u, \quad \forall u \in \Pi_{k_0},$$

with $\dim \Pi_k = \dim \Pi_{k_0}$. So $\Theta(k)$ defines an isomorphism from Π_{k_0} to Π_k for all $k \in V_0$, so that $\Theta(k)$ and $\Theta(k)^{-1}$ are real analytic with respect to $k \in V_0$. For $(\lambda, k) \in I_0 \times V_0$ belongs to Σ_i

if and only if λ is an eigenvalue of multiplicity i of $H(k)\Big|_{\Pi_k}$ or equivalently an eigenvalue of multiplicity i of

$$\Theta(k)^{-1}H(k)\Theta(k) : \Pi_{k_0} \to \Pi_{k_0}.$$

The determinant

$$\delta(\lambda, k) = \det\left[\lambda - \Theta(k)^{-1}H(k)\Theta(k)\right]$$

is real analytic in $(\lambda, k) \in I_0 \times V_0$ and $\Sigma_i \cap (I_0 \times V_0)$ is given by the equations

$$\delta(\lambda, k) = \ldots = \partial_\lambda^{i-2}\delta(\lambda, k) = 0, \tag{3.1}$$

$$\partial_\lambda^{i-1}\delta(\lambda, k) = 0 \text{ and } \partial_\lambda^{i}\delta(\lambda, k) \neq 0. \tag{3.2}$$

□

We now recall some definitions of the theory of analytic singularities. We refer to [2], [4],[9] for more details.

Definition 3.6. *A **stratification** of an analytic manifold N is a partition $\mathcal{S} = (\mathcal{S}_i)_{i\in I}$ of N satisfying the following conditions:*
i) for all $i \in I$, $\mathcal{S}_i$ is a connected analytic submanifold of N,
ii) the family $(\mathcal{S}_i)_{i\in I}$ is locally finite,
iii) if $\overline{\mathcal{S}}_i \cap \mathcal{S}_j \neq \emptyset$ then $\mathcal{S}_j \subset \overline{\mathcal{S}}_i$.

We say that a stratification $\mathcal{S}$ of N is *compatible* with a locally finite family $(C_j)_{j\in J}$ of semi-analytic subsets of N if for all $i \in I$, $j \in J$ either $\mathcal{S}_i \cap C_j = \emptyset$, or $\mathcal{S}_i \subset C_j$.

Definition 3.7. *Let N, M be two real analytic manifolds and $f : M \to N$ a real analytic map. A stratification of f is a pair $(\mathcal{S}, \mathcal{T})$ where $\mathcal{S}$ is a stratification of M and $\mathcal{T}$ a stratification of N such that for all strata $\mathcal{S}_i$ of $\mathcal{S}$, $f(\mathcal{S}_i) \in \mathcal{T}$ and* $\mathrm{rang}(f_{|\mathcal{S}_i}) = \mathrm{dim} f(\mathcal{S}_i)$.

The following result is classical (see as example [4]). A complete proof which requires the introduction of the class of subanalytic sets (stable by proper projection) may be found in [9].

Theorem 3.8. *Let M' and N' be two real analytic manifolds and $f : M' \to N'$ a real analytic map. Assume that there exists an open set Ω of M' so that $f\Big|_\Omega$ is proper. If $\mathcal{C}$ and $\mathcal{D}$ are two locally finite families of semi-analytic subsets of M' and N', then there exists a stratification $(\mathcal{S}, \mathcal{T})$ of $f\Big|_\Omega$ with $\mathcal{S}$ and $\mathcal{T}$ compatible with $\mathcal{C}$ and $\mathcal{D}$.*

We will apply this general result with $M' = \mathbb{R} \times M$, $N' = \mathbb{R}$, $f = p_\mathbb{R}$, $\mathcal{C} = (\Sigma_i)_{i\in\mathbb{N}}$ and $\mathcal{D} = (\mathbb{R})$. We easily construct Ω as assumed in the theorem by covering $p_\mathbb{R}^{-1}(K)\cap\Sigma$, for K compact interval of $\mathbb{R}$, by a finite number of open balls. The stratification of $p_\mathbb{R}\Big|_\Omega$ is then given by two locally finite families of strata $\mathcal{S} = (S_\alpha)_\alpha$ and $\mathcal{T} = (T_\beta)_\beta$ satisfying :

$$\forall\alpha, \exists\beta,\ p_\mathbb{R}(S_\alpha) = T_\beta \text{ and } \mathrm{rank}(f\Big|_{S_\alpha}) = \dim T_\beta.$$

We define then the set of *thresholds* of Theorem 3.1 as follows:

Definition 3.9. *The set of* thresholds τ *is defined by*

$$\tau = \underset{\dim T_\beta = 0}{\cup} T_\beta.$$

The local finiteness of the stratification implies that τ is a discrete subset of $\mathbb{R}$.

Proof of Theorem 3.1 : The proof will be divided in two steps. In the first step we construct the operator A_I by a local procedure. Note that, for small enough open sets $V \subset M$, the space $\mathcal{C}^\infty_{\text{comp}}(V;\mathcal{F})$ can be identified with $\mathcal{C}^\infty_{\text{comp}}(V;\mathcal{H}')$. In the second step we check that A_I is essentially self-adjoint on $D(A_I) = \mathcal{C}^\infty_{\text{comp}}(M;\mathcal{F})$. Let us first recall the identity

$$\chi(H_0) = \int_M^\oplus \chi(H_0(k))\mathrm{d}\mu(k),$$

where due to the fact that $H_0(k)$ has purely discrete spectrum:

$$\chi(H_0(k)) = \sum_{\lambda,\ (\lambda,k)\in\Sigma} \chi(H_0(k))\pi_{\{\lambda\}}(k).$$

a) Let us pick $\chi_I \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}\backslash\tau)$, $\chi_I \equiv 1$ on I. Let $V = p_M(\Sigma \cap p_{\mathbb{R}}^{-1}(I))$. Note that since the projection $p_{\mathbb{R}}$ is proper, V is relatively compact.

For $k_0 \in V, \lambda_0 \in \sigma(H_0(k_0)) \cap I$, there exists $j \in \{1, \dots N_I\}$ and a stratum $S_\alpha \in S$ so that $(\lambda_0, k_0) \in S_\alpha \subset \Sigma_j$. We put ourselves in a neighborhood $I_0 \times V_0$ of (λ_0, k_0) as in Lemma 3.4. By the implicit function theorem applied to (3.2) $\Sigma_j \cap (I_0 \times V_0)$ is included in an analytic submanifold of $\mathbb{R} \times M$ of the form $W_0 = \left\{\lambda = \tilde\lambda(k),\ k \in V_0\right\}$. Decreasing V_0, we can choose coordinates $k = (k', k'')$ so that

$$S_\alpha \cap (I_0 \times V_0) = \left\{\left(\tilde\lambda(k',0), k', 0\right),\ (k',0) \in V_0\right\}.$$

Note that by the definition of Σ_j and of S_α, $\tilde\lambda(k',0)$ is an eigenvalue of $H_0(k',0)$ but this property is not necessarily true for $k'' \neq 0$. Consider the symmetric operator A_{λ_0,k_0}, defined on $L^2(V_0,\mu;\mathcal{H}')$ with domain $\mathcal{C}^\infty_{\text{comp}}(V_0;\mathcal{H}')$ by

$$A_{\lambda_0,k_0} = \pi_{I_0}(k) \circ \left[\frac{1}{2}\partial_{k'}\tilde\lambda(k)D_{k'} + \text{h. c.}\right] \circ \pi_{I_0}(k). \tag{3.3}$$

The commutator $[H_0(k), iA_{\lambda_0,k_0}]$ (defined as a form on $\mathcal{C}^\infty_{\text{comp}}(V_0;\mathcal{H}') \cap D(H_0)$) is equal to

$$\pi_{I_0}(k)\partial_{k'}\tilde\lambda(k).\partial_{k'}\left[H_0(k)\pi_{I_0(k)}\right]\pi_{I_0}(k) =: B(k). \tag{3.4}$$

$B(k)$ extends to a bounded self-adjoint operator on $L^2(V_0,\mu;\mathcal{H}')$. We note first the identity $H_0(k',0)\pi_{I_0}(k',0) = \tilde\lambda(k',0)\pi_{I_0}(k',0)$. Moreover, differentiating with respect to k the identity $\pi_{I_0}^2(k) = \pi_{I_0}(k)$, we obtain that

$$\pi_{I_0}(k)\partial_k\pi_{I_0}(k)\pi_{I_0}(k) = 0,$$

which gives

$$B(k_0) = \pi_{I_0}(k_0)|\partial_{k'}\tilde{\lambda}(k_0)|^2\pi_{I_0}(k_0).$$

By the Definition 3.9 of the set τ there exists a constant $c_0 > 0$ so that $|\partial_{k'}\tilde{\lambda}(k_0)|^2 > c_0$. This gives

$$B(k_0) \geq c_0\pi_{I_0}(k_0). \tag{3.5}$$

Since the spectrum of $H_0(k)$ is discrete, there exists a finite number of $\lambda_i \in I$ so that $\lambda_i \in \sigma(H_0(k_0))$. Let

$$A_{k_0} := \sum_{\lambda_i\in\sigma(H_0(k_0))} A_{\lambda_i,k_0},$$

where A_{λ_i,k_0} is defined as above. We deduce then from (3.5) that for $\chi_1 \in \mathcal{C}^\infty_{\mathrm{comp}}(I), \chi_1 \equiv 1$ on $\operatorname{supp}\chi$, we have:

$$\chi_1(H_0(k_0))[H_0(k), iA_k]_{|k=k_0}\chi_1(H_0(k_0)) \geq c_1\chi_1^2(H_0(k_0)). \tag{3.6}$$

The maps $k \to [H_0(k), iA_{k_0}]$ and $k \to \chi_1(H_0(k))$ are continuous in the norm topology. So for V_0 small enough, we have:

$$\chi_1(H_0(k))[H_0(k), iA_k]\chi_1(H_0(k)) \geq c_1\chi_1^2(H_0(k)) - c_1/2,\ k \in V_0. \tag{3.7}$$

Composing (3.7) to the left and right by $\chi(H_0(k))$, we obtain

$$\chi(H_0(k))[H_0(k), iA_k]\chi(H_0(k)) \geq c_1/2\chi^2(H_0(k)),\ k \in V_0. \tag{3.8}$$

We now cover V by a finite number of open sets $V_i \ni k_i$, similar to $V_0 \ni k_0$ and take a partition of unity $\sum_{0\leq i<N}\chi_i^2(k) \equiv 1$ on V, with $\chi_i \in \mathcal{C}^\infty_{\mathrm{comp}}(V_i)$ and $0 \leq \chi_i \leq 1$. Let

$$A_I = A_I(k, D_k) := \sum_{0\leq i<N}\chi_i(k)A_{k_i}\chi_i(k). \tag{3.9}$$

We deduce from (3.8) that for some $c_2 > 0$:

$$\chi(H_0)[H_0, iA_I]\chi(H_0) \geq c_2\chi^2(H_0),$$

which proves i). Property ii) is easy to check.

It remains to check that A_I is essentially self-adjoint on $D(A_I) = \mathcal{C}^\infty_{\mathrm{comp}}(M;\mathcal{F})$. We may assume that the covering $\cup_{0\leq i'<N}V_{i'}$ is contained in an open set with $\mathcal{C}^\infty$ boundary $\Omega_I \subset\subset M$. Since the density μ is positive, we can find a metric $g > 0$ on $\overline{\Omega_I}$ such that $\mathrm{d}\mu(k) = |\det g|^{\frac{1}{2}}(k)|dk|$, in some coordinate system. We note Δ_g the Laplace-Beltrami operator on Ω_I and we choose $\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(\Omega_I)$ so that $\chi \equiv 1$ on $\cup_{0\leq i'<N}\operatorname{supp}\chi_{i'}$. The Lemma 3.10 below, states that the operator $\mathcal{N} = (1 - \chi\Delta_g\chi)$, with domain $D(\mathcal{N}) = \mathcal{C}^\infty_{\mathrm{comp}}(M;\mathcal{F})$, is essentially self-adjoint. Moreover, it follows from (3.3) that A_I is a differential operator of order 1 in $k \in M$. We have then for all $\varphi \in D(A_I) = D(\mathcal{N})$

$$\|A_I\varphi\|_{\mathcal{H}} \leq C\,\|\mathcal{N}\varphi\|_{\mathcal{H}} \tag{3.10}$$

$$\text{and}\quad |(A_I\varphi, \mathcal{N}\varphi)_{\mathcal{H}} - (\mathcal{N}\varphi, A_I\varphi)_{\mathcal{H}}| \leq C'\left\|\overline{\mathcal{N}}^{1/2}\varphi\right\|^2_{\mathcal{H}}. \tag{3.11}$$

By Nelson's commutator theorem [14], A_I is essentially self-adjoint on $\mathcal{C}^\infty_{\mathrm{comp}}(M;\mathcal{F})$. $\square$

Lemma 3.10. *The operator $\mathcal{N} = 1 - \chi\Delta_g\chi$ defined on $L^2(M, \mu; \mathcal{F})$ with $D(\mathcal{N}) = \mathcal{C}^\infty_{\text{comp}}(M; \mathcal{F})$ is essentially self-adjoint.*

Proof : a) The operator $\mathcal{N}$ is symmetric and strictly positive. It suffices to check that $\ker(\mathcal{N}^*) = 0$. The domain of $\mathcal{N}^*$ is the set $u \in L^2(M, \mu; \mathcal{F})$ so that the distribution $(1 - \chi\Delta_g\chi)u$ belongs to $L^2(M, \mu; \mathcal{F})$. The $\mathcal{F}$ valued distribution are 0-densities which are defined locally as elements of $\mathcal{S}'(\mathbb{R}^n; \mathcal{H}')$ for which the Fourier transform is well defined (see [21] for details about vector valued distributions). The function χ vanishes to infinite order on $\partial\Omega_I$. We can find a sequence of functions $\Phi_n \in \mathcal{C}^\infty_{\text{comp}}(\Omega_I)$ so that $0 \leq \Phi_n \leq 1$, $\Phi_n \to 1$ a.e. and $\|(\nabla_g \Phi_n)\chi\|_{L^\infty} \to 0$ when $n \to \infty$. We follow now the method in [20], [22]. If $u \in \ker(\mathcal{N}^*)$, then u satisfies in distribution sense

$$\chi(-\Delta_g)\chi u = -u \tag{3.12}$$

and the elliptic regularity theorem implies that $\Phi_n u \in \mathcal{C}^\infty_{\text{comp}}(\Omega_I)$ for all $n \in \mathbb{N}$. We deduce that

$$\begin{aligned} -\left(\Phi_n^2 u, u\right) &= \left(\Phi_n^2 u, -\chi(-\Delta_g)\chi u\right) \\ &= \left(\Phi_n^2 \chi u, (-\Delta_g)\chi u\right) \\ &= \left(\Phi_n^2 \nabla_g(\chi u), \nabla_g(\chi u)\right) + 2\left(\nabla_g \Phi_n \chi u, \Phi_n \nabla_g(\chi u)\right). \end{aligned}$$

Hence for all $n \in \mathbb{N}$ we have:

$$\|\Phi_n \nabla_g(\chi u)\|^2_{L^2(M,\mu;\mathcal{F})} \leq 2\,\|(\nabla_g \Phi_n)\chi\|_{L^\infty}\,\|u\|_{L^2(M,\mu;\mathcal{F})}\,\|\Phi_n \nabla_g(\chi u)\|_{L^2(M,\mu;\mathcal{F})}.$$

Letting n tend to ∞, we obtain that $\|\nabla_g(\chi u)\|_{L^2,\mu;\mathcal{F})} = 0$, i.e. $\chi u = 0$ and using (3.12) $u = 0$.
$\square$

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