On the existence of ground states for massless Pauli-Fierz Hamiltonians

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1 Introduction

We consider in this paper the problem of the existence of a ground state for a class of Hamiltonians used in physics to describe a confined quantum system ("matter") interacting with a massless bosonic field. These Hamiltonians were called *Pauli-Fierz Hamiltonians* in [DG]. Examples, like the spin-boson model or a simplified model of a confined atom interacting with a bosonic field are given in [DG, Sect. 3.3].

Pauli-Fierz Hamiltonians can be described as follows: Let $\mathcal{K}$ and K be respectively the Hilbert space and the Hamiltonian describing the matter. The assumption that the matter is confined is expressed mathematically by the fact that $(K+\mathrm{i})^{-1}$ is *compact* on $\mathcal{K}$.

The bosonic field is described by the Fock space $\Gamma(\mathfrak{h})$ with the one-particle space $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, where $\mathbb{R}^d$ is the momentum space, and the free Hamiltonian

$$\mathrm{d}\Gamma(\omega(k)) = \int \omega(k) a^*(k) a(k) \mathrm{d}k.$$

The positive function $\omega(k)$ is called the *dispersion relation*. The constant $m := \inf \omega$ can be called the *mass* of the bosons, and we will consider here the case of *massless* bosons , ie we assume that $m = 0$.

The interaction of the "matter" and the bosons is described by the operator

$$V = \int v(k) \otimes a^*(k) + v^*(k) \otimes a(k) \mathrm{d}k,$$

where $\mathbb{R}^d \ni k \to v(k)$ is a function with values in operators on $\mathcal{K}$. Thus, the system is described by the Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ and the Hamiltonian

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega(k)) + gV, \tag{1.1}$$

g being a coupling constant.

If $\mathcal{K} = \mathbb{C}$, the Hamiltonian H is solvable (see eg [Be, Sect. 7]) and H is defined as a selfadjoint operator if

$$\int \frac{1}{\omega(k)} |v(k)|^2 \mathrm{d}k < \infty,$$

and admits a ground state in $\mathcal{H}$ if and only if

$$\int \frac{1}{\omega(k)^2}|v(k)|^2 \mathrm{d}k < \infty.$$

In this paper we show that H admits a ground state in $\mathcal{H}$ for all values of the coupling constant under corresponding assumptions in the general case.

The existence of a ground state of H in the Hilbert space $\mathcal{H}$ is an important physical property of the system described by H. For example it has the following consequence for the scattering theory of H: assume for example that $\omega \in C^\infty(\{k|\omega(k) > 0\})$ and $\nabla\omega(k) \neq 0$ in $\{k|\omega(k) > 0\}$. Assume also that

$$\mathbb{R}^d \ni k \mapsto \|v(k)(K+1)^{-\frac{1}{2}}\|_{B(\mathcal{K})}$$

is locally in the Sobolev space H^s in $\{k|\omega(k > 0\}$ for some $s > 1$ (a short-range condition on the interaction). Then under the conditions *(H0)*, *(H1)*, *(I1)* below, it is easy to prove the existence of the limits

$$W^\pm(h) := \text{s-}\lim_{t\to\pm\infty} e^{\mathrm{i}tH} e^{\mathrm{i}\phi(h_t)} e^{-\mathrm{i}tH}$$

for $h \in \mathfrak{h}_0 := \{h \in \mathfrak{h}|\omega^{-\frac{1}{2}}h \in \mathfrak{h}\}$ and $h_t = e^{-\mathrm{i}t\omega}h$. The operators $W^\pm(h)$ are called *asymptotic Weyl operators*. They satisfy

$$W^\pm(h)W^\pm(g) = \mathrm{e}^{-\mathrm{i}\frac{1}{2}Im(h|g)}W(h+g),\ h, g \in \mathfrak{h}_0,$$

and

$$\mathrm{e}^{\mathrm{i}tH}W^\pm(h)\mathrm{e}^{-\mathrm{i}tH} = W^\pm(h_{-t}).$$

In particular they form two regular CCR representations over the preHilbert space $\mathfrak{h}_0$. It is easy to show that the space of bound states $\mathcal{H}_{\mathrm{pp}}(H)$ of H is included into the space of vacua for these representations (see for example [DG]). Hence the existence of a ground state for H implies that the CCR representations defined by the asymptotic Weyl operators admit Fock subrepresentations. As a consequence wave operators can be defined.

When the Hamiltonian H admits no ground state in the Hilbert space $\mathcal{H}$, the ground state of H has to be interpreted as a state ω on some C^*-algebra of field observables. Similarly the scattering theory for H has to be significantly modified. These phenomena have been extensively studied by Froehlich [Fr]. In particular the arguments in the proof of Lemma 4.5 are inspired by [Fr, Sect. 2.3], where it is shown that the state ω is locally normal.

Let us end the introduction by making some comments on related works. In [AH], the existence of a ground state is shown under rather similar conditions, if the coupling constant g is sufficiently small. In [Sp], the same problem is considered in the case the small system described by $(\mathcal{K}, K)$ is a confined atom, and the coupling function $k \mapsto v(k)$ is a *real multiplication operator* in the atomic variables (ie $v^*(k) = v(-k)$ is a multiplication operator on $\mathcal{K}$). Using functional integral methods and Perron-Frobenius arguments, the existence of a ground state is shown for all values of the coupling constant.

Our result is hence a generalization of the results both of [AH] and [Sp].

If we drop the assumption that the small system is confined (mathematically this amounts to drop the hypothesis *(H0)* below), then the only result is the one of [BFS], where the existence of a ground state is shown for small coupling constant if K is an atomic Hamiltonian and assumptions similar to *(I1)*, *(I2)* are made.

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2 Result

2.1 Introduction

In this section we introduce the class of Hamiltonians that we will study in this paper. We have stated our result under rather general hypotheses, allowing for a mild UV divergency of the interaction. Clearly the behavior of the interaction for large momenta should not be important for the existence of a ground state, which essentially depends only on the low momentum behavior of the interaction. Therefore the reader wishing to avoid some technicalities can for example assume that the operator K is bounded and that the function $\mathbb{R}^d \ni k \mapsto v(k)$ is compactly supported.

2.2 Hamiltonian

Let $\mathcal{K}$ be a separable Hilbert space representing the degrees of freedom of the atomic system. The Hamiltonian describing the atomic system is denoted by K. We assume that K is selfadjoint on $\mathcal{D}(K) \subset \mathcal{K}$ and bounded below. Without loss of generality we can assume that K is positive. We assume

$$(H0) \qquad (K + \mathrm{i})^{-1} \text{ is compact.}$$

The physical interpretation is that the atomic system is confined.

Let $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$ be the $1-$particle Hilbert space in the momentum representation and let $\Gamma(\mathfrak{h})$ be the bosonic Fock space over $\mathfrak{h}$, representing the field degrees of freedom. We will denote by k the momentum operator of multiplication by k on $L^2(\mathbb{R}^d, \mathrm{d}k)$, and by $x = \mathrm{i}\nabla_k$ the position operator on $L^2(\mathbb{R}^d, \mathrm{d}k)$. Let $\omega \in C(\mathbb{R}^d, \mathbb{R})$ be the boson dispersion relation. We assume

$$(H1) \qquad \begin{cases} \nabla\omega \in L^\infty(\mathbb{R}^d), \\ \lim_{|k|\to\infty} \omega(k) = +\infty, \\ \inf \omega(k) = 0. \end{cases}$$

To stay close to the usual physical situation, we will also assume that $\omega(0) = 0, \omega(k) \neq 0$ for $k \neq 0$, although the results below hold also in the general case. The typical example is of course the massless relativistic dispersion relation $\omega(k) = |k|$. The Hamiltonian describing the field is equal to $\mathrm{d}\Gamma(\omega)$. The Hilbert space of the interacting system is

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h}).$$

The Hamiltonian $H_0 := K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega)$ of the non-interacting system is associated with the quadratic form

$$Q_0(u, u) := (K^{\frac{1}{2}} \otimes \mathbb{1}u, K^{\frac{1}{2}} \otimes \mathbb{1}u) + \int \omega(k)(\mathbb{1} \otimes a(k)u, \mathbb{1} \otimes a(k)u)\mathrm{d}k,$$

with $D(Q_0) = D((K + \mathrm{d}\Gamma(\omega))^{\frac{1}{2}})$.

The interaction between the atom and the boson field is described with a coupling function v

$$\mathbb{R}^d \ni k \mapsto v(k),$$

such that for a.e. $k \in \mathbb{R}^d$, $v(k)$ is a bounded operator from $D(K^{\frac{1}{2}})$ into $\mathcal{K}$ and from $\mathcal{K}$ into $D(K^{\frac{1}{2}})^*$. We associate to the coupling function v the quadratic form

$$V(u,u) = \int (\mathbb{1} \otimes a(k)u, v(k) \otimes \mathbb{1}u) + (v(k) \otimes \mathbb{1}u, \mathbb{1} \otimes a(k)u)\mathrm{d}k, \tag{2.1}$$

A rather minimal assumption under which the quadratic form $Q = Q_0 + V$ gives rise to a selfadjoint operator is

$$(I1) \quad \begin{array}{l} \text{for a.e. } k \in \mathbb{R}^d \; v(k)(K+1)^{-\frac{1}{2}},\; (K+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K}), \\ \forall u_1, u_2 \in \mathcal{K},\; k \mapsto (u_2, v(k)(K+1)^{-\frac{1}{2}}u_1),\; k \mapsto (u_2, (K+1)^{-\frac{1}{2}}v(k)u_1) \text{ are measurable,} \\ C(R) := \int \frac{1}{\omega(k)}(\|v(k)(K+R)^{-\frac{1}{2}}\|^2 + \|(K+R)^{-\frac{1}{2}}v(k)\|^2)\mathrm{d}k < \infty, \\ \lim_{R\to+\infty} C(R) = 0. \end{array}$$

Note that it follows from the results quoted in the Appendix that the functions $k \mapsto \|v(k)(K+R)^{-\frac{1}{2}}\|$, $k \mapsto \|(K+R)^{-\frac{1}{2}}v(k)\|$ are measurable, and hence the last condition in $(I1)$ has a meaning.

Proposition 2.1 *Assume hypothesis (I1). Then the quadratic form V is Q_0−form bounded with relative bound* 0. *Consequently one can associate with the quadratic form $Q = Q_0 + V$ a unique bounded below selfadjoint operator H with $D(H^{\frac{1}{2}}) = D(H_0^{\frac{1}{2}})$.*

The Hamiltonian H is called a *Pauli-Fierz Hamiltonian.*

Proof. We apply the estimate (A.1) in Lemma A.1 with $B = K$, $m = \omega$. □

2.3 Results

Under assumption *(I1)*, one can associate a bounded below, selfadjoint Hamiltonian H to the quadratic form Q. Let us introduce the following assumption on the behavior of $v(k)$ near $\{k|\omega(k) = 0\}$:

$$(I2) \quad \int \frac{1}{\omega(k)^2}\|v(k)(K+1)^{-\frac{1}{2}}\|^2\mathrm{d}k < \infty.$$

Theorem 1 *Assume hypotheses (H0), (H1), (I1), (I2). Then* $\inf\operatorname{spec}(H)$ *is an eigenvalue of H. In other words H admits a ground state in $\mathcal{H}$.*

3 The cut-off Hamiltonians

3.1 Operator bounds

Let us introduce the following assumption:

$$(I1') \quad \begin{array}{l} C'(R) := \int (1 + \frac{1}{\omega(k)})(\|v(k)(K+R)^{-\frac{1}{2}}\|^2 + \|(K+1)^{-\frac{1}{2}}v(k)\|^2)\mathrm{d}k < \infty, \\ \lim_{R\to+\infty} C'(R) = 0. \end{array}$$

Proposition 3.1 *Assume (I1), (I1'). Then the operator*

$$V = a^*(v) + a(v) = \int v(k) \otimes a^*(k) + v^*(k) \otimes a(k) d\ k$$

is H_0-bounded with relative bound 0. Consequently $H = H_0 + V$ is a bounded below selfadjoint operator with $D(H) = D(H_0)$.

Proof. We apply the estimates (A.2), (A.3) in Lemma A.1 with $B = K$, $m = \omega$.$\Box$

3.2 Cut-off Hamiltonians

In the sequel we will need to introduce various cut-off Hamiltonians. For $0 < \sigma \ll 1$ an infrared cutoff parameter and $\tau \gg 1$ an ultraviolet cutoff parameter, we denote by V_σ, $V_{\sigma,\tau}$ the quadratic forms defined as V in (2.1) with the coupling function v replaced respectively by v_σ, $v_{\sigma,\tau}$ for

$$v_\sigma = 1\!\!1_{\{\sigma\leq\omega\}}(k)v,\ v_{\sigma,\tau} = 1\!\!1_{\{\sigma\leq\omega\leq\tau\}}(k)v.$$

We denote by $H_\sigma, H_{\sigma,\tau}$ the selfadjoint operators associated with the quadratic forms $Q_0 + V_\sigma, Q_0 + V_{\sigma,\tau}$. Note that since $v_{\sigma,\tau}$ satisfies *(I1')*, we have $D(H_{\sigma,\tau}) = D(H_0)$.

Applying Lemma A.2 in the Appendix and the fact that $D(H^{\frac{1}{2}}) = D(H_0^{\frac{1}{2}})$ we obtain

$$\begin{aligned} &\lim_{\tau\to+\infty}(H_{\sigma,\tau} - \lambda)^{-1} = (H_\sigma - \lambda)^{-1}, \\ &\lim_{\sigma\to 0}(H_\sigma - \lambda)^{-1} = (H - \lambda)^{-1}, \end{aligned} \tag{3.1}$$

for $\lambda \in \mathbb{R}, \lambda \ll -1$, and

$$\begin{aligned} &\|((H_{\sigma,\tau} - z)^{-1} - (H_\sigma - z)^{-1})(H_0 + 1)^{\frac{1}{2}}\| \in o(1)|Imz|^{-1}\ \tau \to +\infty, \\ &\|((H_\sigma - z)^{-1} - (H - z)^{-1})(H_0 + 1)^{\frac{1}{2}}\| \in o(1)|Imz|^{-1}\ \sigma \to 0, \end{aligned} \tag{3.2}$$

for $z \in \mathbb{C}\backslash\mathbb{R}$.

3.3 Existence of ground states for the cut-off Hamiltonians

Let $\tilde{\omega}_\sigma : \mathbb{R}^d \to \mathbb{R}$ be a dispersion relation satisfying

$$\begin{cases} \nabla\tilde{\omega}_\sigma \in L^\infty(\mathbb{R}^d), \\ \tilde{\omega}_\sigma(k) = \omega(k) \text{ if } \omega(k) \geq \sigma, \\ \tilde{\omega}_\sigma(k) \geq \sigma/2. \end{cases} \tag{3.3}$$

Let $\tilde{H}_\sigma$ be the operator associated to the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int \tilde{\omega}_\sigma(k)\|a(k)u\|^2 \mathrm{d}k + V_\sigma(u,u)$.

Lemma 3.2 *H_σ admits a ground state in $\mathcal{H}$ if and only if $\tilde{H}_\sigma$ admits a ground state in $\mathcal{H}$.*

Proof. Let $\mathfrak{h}_\sigma := L^2(\{k|\omega(k) < \sigma\}, \mathrm{d}k)$, $\mathfrak{h}_\sigma^\perp = L^2(\{k|\omega(k) \geq \sigma\}, \mathrm{d}k)$. Let U be the canonical unitary map

$$U : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}_\sigma^\perp) \otimes \Gamma(\mathfrak{h}_\sigma)$$

(see for example [DG, Sect. 2.7]). Let us still denote by U the unitary map $\mathbb{1}_{\mathcal{K}} \otimes U$ from $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$ into $\mathcal{K} \otimes \Gamma(\mathfrak{h}_\sigma^\perp) \otimes \Gamma(\mathfrak{h}_\sigma)$. By [DG, Sect. 2.7], the operator $UH_\sigma U^*$ is equal to

$$\mathbb{1}_{\mathcal{K}\otimes\Gamma(\mathfrak{h}_\sigma^\perp)} \otimes \mathrm{d}\Gamma(\omega_{\sigma,1}) + H_\sigma^2 \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\sigma)},$$

where $\omega_{\sigma,1} = \omega_{|\mathfrak{h}_\sigma}$ and H_σ^2 is the operator associated with the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int_{\{\omega(k)\geq\sigma\}} \omega_\sigma(k)\|a(k)u\|^2\mathrm{d}k + V_\sigma(u,u)$. Similarly $U\tilde{H}_\sigma U^*$ is equal to

$$\mathbb{1}_{\mathcal{K}\otimes\Gamma(\mathfrak{h}_\sigma^\perp)} \otimes \mathrm{d}\Gamma(\tilde{\omega}_{\sigma,1}) + H_\sigma^2 \otimes \mathbb{1}_{\Gamma(\mathfrak{h}_\sigma)},$$

where $\tilde{\omega}_{\sigma,1} = \tilde{\omega}_{\sigma|\mathfrak{h}_\sigma}$. Now H_σ^2 has a ground state ψ if and only if $U\tilde{H}_\sigma U^*$ or $UH_\sigma U^*$ have a ground state (equal to $\psi \otimes \Omega$, where $\Omega \in \Gamma(\mathfrak{h}_\sigma)$ is the vacuum vector). This proves the lemma. $\Box$

The following result is essentially well known (see [AH], [BFS]) and rather easy to show.

Proposition 3.3 *Assume hypotheses (H0), (H1), (I1). Then for any $\sigma > 0$ H_σ admits a ground state.*

Proof. By Lemma 3.2 it suffices to show that $\tilde{H}_\sigma$ admits a ground state. Let for $\tau \in \mathbb{N}$ $\tilde{H}_{\sigma,\tau}$ be the Hamiltonian associated with the quadratic form $\|K^{\frac{1}{2}}u\|^2 + \int \tilde{\omega}_\sigma(k)\|a(k)u\|^2\mathrm{d}k + V_{\sigma,\tau}(u,u)$. Let

$$\tilde{E}_{\sigma,\tau} = \inf \mathrm{spec}(\tilde{H}_{\sigma,\tau}),\ \tilde{E}_\sigma = \inf \mathrm{spec}(\tilde{H}_\sigma).$$

Applying Lemma A.2, we have for $z \in \mathbb{C}\backslash\mathbb{R}$

$$(z - \tilde{H}_\sigma)^{-1} = \lim_{n\to+\infty} (z - \tilde{H}_{\sigma,n})^{-1}. \tag{3.4}$$

On the other hand applying the bounds in Lemma A.1 we have $D(\tilde{H}_{\sigma,\tau}) = D(K + \mathrm{d}\Gamma(\tilde{\omega}_\sigma))$. The Hamiltonian $\tilde{H}_{\sigma,\tau}$ is very similar to the class of massive Pauli-Fierz Hamiltonians studied in [DG]. It is easy to see that the arguments of [DG] extend to $\tilde{H}_{\sigma,\tau}$. In particular, following the proofs of [DG, Lemma 3.4], [DG, Thm. 4.1], we obtain that $\chi(\tilde{H}_{\sigma,\tau})$ is compact if $\chi \in C_0^\infty(]-\infty, \tilde{E}_{\sigma,\tau} + \sigma/2[)$. Using (3.4) and the fact that $\tilde{E}_\sigma = \lim_{n\to+\infty} \tilde{E}_{\sigma,\tau}$, we obtain that $\chi(\tilde{H}_\sigma)$ is compact if $\chi \in C_0^\infty(]-\infty, \tilde{E}_\sigma + \sigma/2[)$. This implies that $\tilde{H}_\sigma$ and hence H_σ admit a ground state. $\Box$

3.4 The pullthrough formula

As in [BFS], we shall use the pullthrough formula to get control on the ground states of H_σ. Since the domain H_σ is not explicitely known under assumption *(I1)*, some care is needed to prove the pullthrough formula in our situation.

Proposition 3.4 *As an identity on $L^2_{\mathrm{loc}}(\mathbb{R}^d\backslash\{0\}, \mathrm{d}k; \mathcal{H})$, we have:*

$$(H_\sigma + \omega(k) - z)^{-1}a(k)\psi = a(k)(H_\sigma - z)^{-1}\psi + (H_\sigma + \omega(k) - z)^{-1}v_\sigma(k)(H_\sigma - z)^{-1}\psi,\ \psi \in \mathcal{H}.$$

Proof. For $u_1, u_2 \in D(H_0)$, the following identity makes sense as an identity on $L^2_{\rm loc}(\mathbb{R}^d\backslash\{0\}, {\rm d}k)$:

$$(a^*(k)u_1, (H_{\sigma,\tau} - z)u_2) = ((H_{\sigma,\tau} + \omega(k) - \overline{z})u_1, a(k)u_2) + (u_1, v_{\sigma,\tau}(k)u_2).$$

Setting $u_2 = (H_{\sigma,\tau} - z)^{-1}v_2$, we obtain that for $v_2 \in \mathcal{H}$, $a(k)v_2 \in L^2_{\rm loc}(\mathbb{R}^d\backslash\{0\}, {\rm d}k; D(H_0)^*)$ and

$$a(k)v_2 = (H_{\sigma,\tau} + \omega(k) - z)a(k)(H_{\sigma,\tau} - z)^{-1}v_2 + v_{\sigma,\tau}(k)(H_{\sigma,\tau} - z)^{-1}v_2.$$

Hence for $\psi \in \mathcal{H}$, $(H_\sigma + \omega(k) - z)^{-1}a(k)\psi \in L^2_{\rm loc}(\mathbb{R}^d\backslash\{0\}, {\rm d}k; \mathcal{H})$ and

$$\begin{aligned} &(H_{\sigma,\tau} + \omega(k) - z)^{-1}a(k)\psi \\ &= a(k)(H_{\sigma,\tau} - z)^{-1}\psi + (H_{\sigma,\tau} + \omega(k) - z)^{-1}v_{\sigma,\tau}(k)(H_{\sigma,\tau} - z)^{-1}\psi, \end{aligned} \tag{3.5}$$

holds as an identity in $L^2_{\rm loc}(\mathbb{R}^d\backslash\{0\}, {\rm d}k; \mathcal{H})$.

By *(I1)*, $(v_{\sigma,\tau}(k) - v_\sigma(k))(H_0 + 1)^{-\frac{1}{2}}$ tends to 0 in $L^2(\mathbb{R}^d\backslash\{0\}, {\rm d}k; B(\mathcal{K}))$ when $\tau \to +\infty$. Using also (3.2) and letting $\tau \to +\infty$ we obtain

$$(H_\sigma + \omega(k) - z)^{-1}a(k)\psi = a(k)(H_\sigma - z)^{-1}\psi + (H_\sigma + \omega(k) - z)^{-1}v_\sigma(k)(H_\sigma - z)^{-1}\psi,$$

as claimed. $\square$

4 Proof of Thm. 1

Let

$$E_\sigma := \inf {\rm spec}(H_\sigma),\ E := \inf {\rm spec}(H).$$

We denote by ψ_σ, $\sigma > 0$ a normalized ground state of H_σ. Applying the pullthrough formula to ψ_σ, we obtain easily the following identity on $L^2(\mathbb{R}^d, {\rm d}k; \mathcal{H})$:

$$a(k)\psi_\sigma = (E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma. \tag{4.1}$$

The first rather obvious bound on the family of ground states ψ_σ is the following.

Lemma 4.1 *Assume hypotheses (H0), (H1), (I1). Then*

$$(\psi_\sigma, H_0\psi_\sigma) \le C, \ \textit{uniformly in } \sigma > 0. \tag{4.2}$$

The bound (4.2) follows immediately from the fact that the quadratic forms Q_σ are equivalent to Q_0, uniformly in σ. The following lemma is also well-known (see eg [BFS, Thm. II.5], [AH, Lemma 4.3]). We denote by N the number operator on $\Gamma(\mathfrak{h})$.

Lemma 4.2 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$(\psi_\sigma, N\psi_\sigma) \le C, \ \textit{uniformly in } \sigma > 0. \tag{4.3}$$

Proof. We have using (4.1)

$$\begin{array}{rl}(\psi_\sigma, N\psi_\sigma) & = \int \|a(k)\psi_\sigma\|^2 \mathrm{d}k \\ & = \int \|(E_\sigma - H_\sigma(k) - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|^2 \mathrm{d}k \\ & \leq \|(H_0+1)^{\frac{1}{2}}\psi_\sigma\|^2 \int \frac{1}{\omega(k)^2}\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2 \mathrm{d}k \\ & \leq C,\end{array}$$

uniformly in $\sigma > 0$ using *(I2)* and (4.2). $\Box$

Lemma 4.3 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$E - E_\sigma \in o(\sigma). \tag{4.4}$$

Proof. Let $0 < \sigma' < \sigma$. We have

$$\begin{array}{l} E_{\sigma'} - E_\sigma \leq (Q_{\sigma'} - Q_\sigma)(\psi_\sigma, \psi_\sigma) = (V_{\sigma'} - V_\sigma)(\psi_\sigma, \psi_\sigma), \\ E_\sigma - E_{\sigma'} \leq (Q_\sigma - Q_{\sigma'})(\psi_{\sigma'}, \psi_{\sigma'}) = (V_\sigma - V_{\sigma'})(\psi_{\sigma'}, \psi_{\sigma'}), \end{array} \tag{4.5}$$

Applying (A.1) with $m(k) = 1$, we obtain

$$|(V_{\sigma'} - V_\sigma)(u,u)| \leq C(\sigma', \sigma)(u, Nu)^{\frac{1}{2}}(u, (K+1)u)^{\frac{1}{2}}, \tag{4.6}$$

for

$$C(\sigma', \sigma) = (\int_{\{\sigma' < \omega(k) \leq \sigma\}} \|v(k)(K+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}}$$

Using (4.6) for $u = \psi_\sigma$ or $\psi_{\sigma'}$, the right hand side of (4.5) is bounded by $C_0 C(\sigma', \sigma)$, uniformly in σ, σ', using (4.2) and (4.3). We note that by (3.1) $E = \lim_{\sigma' \to 0} E_{\sigma'}$. Hence letting σ' tend to 0 we get $|E - E_\sigma| \leq C_0 C(0, \sigma) \in o(\sigma)$, using hypothesis *(I2)*. $\Box$

Proposition 4.4 *Assume hypotheses (H0), (H1), (I1), (I2). Then*

$$a(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \to 0$$

when $\sigma \to 0$ *in* $L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$.

Proof. We have, using (4.1)

$$\begin{array}{rl} & a(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ = & (E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma - (E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ = & -1\!\!1_{\{\omega(k) \leq \sigma\}}(k)(E - H - \omega(k))^{-1} v(k)\psi_\sigma \\ & +(E - H - \omega(k))^{-1}(H - H_\sigma)(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ & +(E_\sigma - E)(E - H - \omega(k))^{-1}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma \\ =: & R_{\sigma,1}(k) + R_{\sigma,2}(k) + R_{\sigma,3}(k). \end{array}$$

We will estimate separately $R_{\sigma,i}$, $1 \leq i \leq 3$. First

$$\|R_{\sigma,1}(k)\|_{\mathcal{H}} \leq \mathbb{1}_{\{\omega(k)\leq\sigma\}}(k)\frac{1}{\omega(k)}\|v(k)(K+1)^{-\frac{1}{2}}\|_{B(\mathcal{K})}\|(K+1)^{\frac{1}{2}}\psi_\sigma\|_{\mathcal{H}},$$

which shows using hypothesis *(I2)* and (4.2) that

$$R_{\sigma,1} \in o(\sigma^0) \text{ in } L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H}). \tag{4.7}$$

Let us next estimate $R_{\sigma,2}$. Using the fact that $(v - v_\sigma)(k)(K+1)^{-\frac{1}{2}}$ belongs to $L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$, it is easy to verify that

$$\begin{aligned} & (E - H - \omega(k))^{-1}(H - H_\sigma)(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma \\ = \; & (E - H - \omega(k))^{-1}(a^*(v - v_\sigma) + a(v - v_\sigma))(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma. \end{aligned}$$

Note that it follows from functional calculus that

$$\|(E - H - \omega(k))^{-1}(H + b)^{\frac{1}{2}}\| \leq C\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}). \tag{4.8}$$

Using also the fact that $(K+1)^{\frac{1}{2}}(H+b)^{-\frac{1}{2}}$ is bounded, we have:

$$\begin{aligned} & \|(E - H - \omega(k))^{-1}(a^*(v - v_\sigma) + a(v - v_\sigma))(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\| \\ \leq \; & C\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}})\|(K+1)^{-\frac{1}{2}}(a^*(v - v_\sigma) + a(v - v_\sigma))(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\| \\ \leq \; & C\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}})\Big(\int_{\{\omega(k)\leq\sigma\}}\|v(k)(K+1)^{-\frac{1}{2}}\|^2 + \|(K+1)^{-\frac{1}{2}}v(k)\|^2\mathrm{d}k\Big)^{\frac{1}{2}} \times \\ & \|(N+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|, \end{aligned}$$

applying the estimates (A.2), (A.3) in Lemma A.1 to $B = \mathbb{1}$, $m = 1$, $v(k) = (K+1)^{-\frac{1}{2}}(v - v_\sigma)(k)$.

To bound $\|(N+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma\|$, we write using again the pullthrough formula (4.1):

$$\begin{aligned} & a(k')(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma \\ = \; & (E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1}a(k')v_\sigma(k)\psi_\sigma \\ & +(E_\sigma - H_\sigma - \omega(k'))^{-1}v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma \\ = \; & (E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1}v_\sigma(k)(E_\sigma - H_\sigma - \omega(k'))^{-1}v_\sigma(k')\psi_\sigma \\ & +(E_\sigma - H_\sigma - \omega(k'))^{-1}v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1}v_\sigma(k)\psi_\sigma. \end{aligned}$$

This gives

$$\begin{aligned}
& \|N^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|^2 \\
= \ & \int \|a(k')(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|^2 \mathrm{d}k' \\
\leq \ & 2\int \|(E_\sigma - H_\sigma - \omega(k) - \omega(k'))^{-1} v_\sigma(k)(E_\sigma - H_\sigma - \omega(k'))^{-1} v_\sigma(k')\psi_\sigma\|^2 \mathrm{d}k' \\
& +2\int \|(E_\sigma - H_\sigma - \omega(k'))^{-1} v_\sigma(k')(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|^2 \mathrm{d}k' \\
\leq \ & C\int \tfrac{1}{\omega(k)^2}\|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2 \|(K+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k'))^{-1}\|^2 \times \\
& \|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2 \|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \mathrm{d}k' \\
& +C\int \tfrac{1}{\omega(k')^2}\|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2 \|(K+1)^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1}\|^2 \times \\
& \|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2 \|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \mathrm{d}k'.
\end{aligned}$$

We use the bound (4.8) and we obtain

$$\begin{aligned}
& \|N^{\frac{1}{2}}(E_\sigma - H_\sigma - \omega(k))^{-1} v_\sigma(k)\psi_\sigma\|^2 \\
\leq \ & C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2 \|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2 \times \\
& \int (\sup(\omega(k')^{-1}, \omega(k')^{-\frac{1}{2}}))^2 \|v_\sigma(k')(K+1)^{-\frac{1}{2}}\|^2 \mathrm{d}k' \times \\
& \|(K+1)^{\frac{1}{2}}\psi_\sigma\|^2 \\
\leq \ & C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2 \|v_\sigma(k)(K+1)^{-\frac{1}{2}}\|^2,
\end{aligned}$$

using (4.2) and hypothesis *(I2)*. Hence

$$\|R_{\sigma,2}(k)\|_{\mathcal{H}} \leq C(\sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}))^2 \|v_\sigma(k)(K+1)^{-\frac{1}{2}}\| (\int_{\{\omega(k)\leq\sigma\}} \|(K+1)^{-\frac{1}{2}} v(k)\|^2 \mathrm{d}k)^{\frac{1}{2}}.$$

By *(I2)*,

$$(\int_{\{\omega(k)\leq\sigma\}} \|(K+1)^{-\frac{1}{2}} v(k)\|^2 \mathrm{d}k)^{\frac{1}{2}} \in o(\sigma),$$

and since $\mathrm{supp} v_\sigma \subset \{\omega(k) \geq \sigma\}$, we obtain

$$\|R_{\sigma,2}(k)\| \leq o(\sigma^0) \sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}) \|v(k)(K+1)^{-\frac{1}{2}}\|. \tag{4.9}$$

Finally using Lemma 4.3, (4.2) and the fact that $\mathrm{supp} v_\sigma \subset \{\omega(k) \geq \sigma\}$, we obtain

$$\|R_{3,\sigma}(k)\| \leq o(\sigma^0) \sup(\omega(k)^{-1}, \omega(k)^{-\frac{1}{2}}) \|v(k)(K+1)^{-\frac{1}{2}}\|. \tag{4.10}$$

Combining (4.7), (4.9), (4.10) and using *(I2)* we obtain the proposition. □

As a consequence of Prop. 4.4, we have the following lemma, which is the main part of the proof of Thm. 1. We recall that $x := \mathrm{i}\nabla_k$ is the position operator on $L^2(\mathbb{R}^d, \mathrm{d}k)$.

Lemma 4.5 *Let $F \in C_0^\infty(\mathbb{R})$ be a cutoff function with $0 \le F \le 1$, $F(s) = 1$ for $|s| \le \frac{1}{2}$, $F(s) = 0$ for $|s| \ge 1$. Let $F_R(x) = F(\frac{|x|}{R})$. Then*

$$\lim_{\sigma\to 0, R\to+\infty} (\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = 0. \tag{4.11}$$

Proof. Recall that if B is a bounded operator on $\mathfrak{h}$ with distribution kernel $b(k, k')$, we have

$$(u, \mathrm{d}\Gamma(B)u) = \int\int b(k, k')(a(k)u, a(k')u)\mathrm{d}k\mathrm{d}k',\ u \in D(N^{\frac{1}{2}}).$$

Using this identity, we obtain

$$(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = (a(\cdot)\psi_\sigma, (1 - F(\frac{|D_k|}{R}))a(\cdot)\psi_\sigma)_{L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})}.$$

By Prop. 4.4, we have

$$(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) = ((E - H - \omega(\cdot))^{-1}v(\cdot)\psi_\sigma, (1 - F(\tfrac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\psi_\sigma) + o(\sigma^0),$$

uniformly in R. This yields

$$\begin{aligned}(\psi_\sigma, \mathrm{d}\Gamma(1 - F_R)\psi_\sigma) \le\ & \|(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d, \mathrm{d}k, B(\mathcal{H}))} \times \\ & \|(1 - F(\tfrac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d, \mathrm{d}k, B(\mathcal{H}))} + o(\sigma^0).\end{aligned}$$

Now it follows from hypothesis *(I2)* and (4.8) that $(E - H - \omega(\cdot))^{-1}v(\cdot)$ belongs to $L^2(\mathbb{R}^d, \mathrm{d}k, B(\mathcal{H}))$, and hence

$$\|(1 - F(\frac{|D_k|}{R}))(E - H - \omega(\cdot))^{-1}v(\cdot)\|_{L^2(\mathbb{R}^d, \mathrm{d}k, B(\mathcal{H}))} \in o(R^0).$$

This proves (4.11). $\square$

We can now prove Thm. 1.

Proof of Thm. 1

Let us first recall the a priori bounds on the family of ground states $\{\psi_\sigma\}$. From (4.2), (4.3), we have

$$\|N^{\frac{1}{2}}\psi_\sigma\| \le C,\ \|H_0^{\frac{1}{2}}\psi_\sigma\| \le C,\ \text{uniformly in } \sigma. \tag{4.12}$$

Let also F be a cutoff function as in Lemma 4.5. Then it is easy to verify, using the fact that $0 \le F \le 1$, that

$$(1 - \Gamma(F_R))^2 \le (1 - \Gamma(F_R)) \le \mathrm{d}\Gamma(1 - F_R).$$

Using Lemma 4.5, we obtain

$$\lim_{\sigma\to 0, R\to\infty} \|1 - \Gamma(F_R)\psi_\sigma\| = 0. \tag{4.13}$$

Let us denote by $\chi(s \le s_0)$ a cutoff function supported in $\{|s| \le s_0\}$, equal to 1 in $\{|s| \le s_0/2\}$.

Since the unit ball in $\mathcal{H}$ is compact for the weak topology, there exist a sequence $\sigma_n \to 0$ and a vector $\psi \in \mathcal{H}$ such that ψ_{σ_n} tends weakly to ψ. By Lemma A.3 in the Appendix, it suffices to show that $\psi \ne 0$ to prove the theorem.

Assume that $\psi = 0$. Note using hypotheses *(H0)*, *(H1)*, that for any λ, R the operator $\chi(N \leq \lambda)\chi(H_0 \leq \lambda)\Gamma(F_R)$ is compact on $\mathcal{H}$. Then

$$\lim_{n\to\infty} \chi(N \leq \lambda)\chi(H_0 \leq \lambda)\Gamma(F_R)\psi_{\sigma_n} = 0, \tag{4.14}$$

for any λ, R. By (4.13), we can pick R large enough such that for $n \geq n_0$

$$\|(1 - \Gamma(F_R))\psi_{\sigma_n}\| \leq 10^{-1}. \tag{4.15}$$

Since $(1 - \chi(s \leq s_0)) \leq s_0^{-\frac{1}{2}} s^{\frac{1}{2}}$, we can using (4.12) pick λ large enough such that

$$\|(1 - \chi(N \leq \lambda))\psi_{\sigma_n}\| \leq 10^{-1},\ \|(1 - \chi(H_0 \leq \lambda))\psi_{\sigma_n}\| \leq 10^{-1}. \tag{4.16}$$

But (4.15), (4.16) and (4.14) imply that for n large enough $\|\psi_{\sigma_n}\| \leq 10^{-1}$ which is a contradiction. Hence $\psi \neq 0$ and the theorem is proved.

A Appendix

We use the notations of Sect. 2. The following lemma is well known if the coupling function $v(k)$ is of the form $v\lambda(k)$ for v a fixed linear operator on $\mathcal{K}$ and $k \mapsto \lambda(k)$ a scalar function. In our general setting it seems not to be in the literature.

Let us first recall some terminology and results about measurability of vector and operator-valued functions. Let $\mathcal{K}$ be a Hilbert space. A map $k \mapsto \psi(k) \in \mathcal{K}$ is said measurable if it is measurable if we equip $\mathcal{K}$ with the norm topology. Let now $\mathbb{R}^d \ni k \mapsto T(k) \in B(\mathcal{K})$ be defined for a.e. k. The map $k \mapsto T(k)$ is said weakly measurable if for all $\psi_1, \psi_2 \in \mathcal{K}$ the map $k \mapsto (\psi_2, T(k)\psi_1)$ is measurable. If $\mathcal{K}$ is *separable* the following facts are true (see eg [Di, Chap. II §2]):

i) the function $k \mapsto \|T(k)\|$ is measurable,

ii) for any $k \mapsto \psi(k) \in \mathcal{K}$ measurable, the function $k \mapsto T(k)\psi(k)$ is measurable.

In particular for $\psi \in \mathcal{K}$ the function $k \mapsto T(k)\psi$ is measurable. These facts will be used in the proof of Lemma A.1 below.

Lemma A.1 *Let $B \geq 0$ be a selfadjoint operator on the separable Hilbert space $\mathcal{K}$, $v : \mathbb{R}^d \ni k \mapsto v(k)$ a function such that for a.e. $k \in \mathbb{R}^d$, $v(k)(B+1)^{-\frac{1}{2}} \in B(\mathcal{K})$, $\mathbb{R}^d \ni k \mapsto v(k)(B+1)^{-\frac{1}{2}} \in B(\mathcal{K})$ is weakly measurable and $m : \mathbb{R}^d \ni k \mapsto m(k) \in \mathbb{R}^+$ be a measurable function. Then*

$$|\int (v(k)u, a(k)u)\mathrm{d}k| \leq C(R)(u, \mathrm{d}\Gamma(m)u)^{\frac{1}{2}}(u, (B+R)u)^{\frac{1}{2}}, \tag{A.1}$$

for

$$C(R) = (\int \frac{1}{m(k)} \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}}.$$

If moreover for a.e. $k \in \mathbb{R}^d$, $(B+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K})$ and $\mathbb{R}^d \ni k \mapsto (B+1)^{-\frac{1}{2}}v(k) \in B(\mathcal{K})$ is weakly measurable

$$\|\int v^*(k) \otimes a(k)u\, \mathrm{d}k\| \leq C_1(R)\|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}}u\|, \tag{A.2}$$

for

$$C_1(R) = (\int \frac{1}{m(k)} \|(B+R)^{-\frac{1}{2}} v(k)\|^2 \mathrm{d}k)^{\frac{1}{2}},$$

and

$$\| \int v(k) \otimes a^*(k) u \, \mathrm{d}k \| \leq C_2(R) \|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}} u\| + C_3(R) \|u\|, \tag{A.3}$$

for

$$C_2(R) = (\int \frac{1}{m(k)} \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}},$$

$$C_3(R) = (\int \|v(k)(B+R)^{-\frac{1}{2}}\|^2 \mathrm{d}k)^{\frac{1}{2}}.$$

Proof. The estimate (A.1) follows directly from Cauchy-Schwarz inequality. (We use the fact that for $u \in \mathcal{K} \otimes D(N^{\frac{1}{2}}) \cap D(\mathrm{d}\Gamma(m)^{\frac{1}{2}})$ the map $k \mapsto a(k)u \in \mathcal{H}$ is measurable). To prove (A.2), we consider the operator

$$w_R : \mathcal{K} \ni u \mapsto w_R(k) u := m(k)^{-\frac{1}{2}} (B+R)^{-\frac{1}{2}} v(k) u \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K}) = \mathcal{K} \otimes \mathfrak{h}.$$

Clearly $\|w_R\|_{B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} \leq C_1(R)$ and hence $\|w_R w_R^*\|_{B(\mathcal{K}\otimes\mathfrak{h})} \leq C_1(R)^2$. This gives

$$| \int \int (w_R^*(k)\psi(k), w_R^*(k')\psi(k'))_{\mathcal{K}} \mathrm{d}k \mathrm{d}k' | \leq C_1(R)^2 \int \|\psi(k)\|_{\mathcal{K}}^2 \mathrm{d}k, \tag{A.4}$$

for $\psi \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K})$. The bound (A.4) still holds for $\psi \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{H})$ if we replace the scalar product $(.,.)_{\mathcal{K}}$ by the scalar product $(.,.)_{\mathcal{H}}$. We have:

$$\begin{aligned} \|a(v)u\|^2 = & \quad \| \int v^*(k) a(k) u \, \mathrm{d}k \|^2 \\ = & \quad \int \int (v^*(k) a(k) u, v^*(k') a(k') u)_{\mathcal{H}} \mathrm{d}k \mathrm{d}k' \\ = & \quad \int \int (w_R^*(k)\psi(k), w_R^*(k')\psi(k'))_{\mathcal{H}} \mathrm{d}k \mathrm{d}k,' \end{aligned}$$

for $\psi(k) = m(k)^{\frac{1}{2}} a(k) (B+R)^{\frac{1}{2}} u$. Using (A.4) we obtain

$$\begin{aligned} & \|a(v)u\|^2 \leq C_1(R)^2 \int \omega(k) \|a(k)(B+R)^{\frac{1}{2}} u\|^2 \mathrm{d}k \\ & = C_1(R)^2 \|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}} u\|^2. \end{aligned}$$

This proves (A.2).

Similarly, introducing the operator

$$\tilde{w}_R : \mathcal{K} \ni u \mapsto \tilde{w}_R(k) u = m(k)^{-\frac{1}{2}} v(k) (B+R)^{-\frac{1}{2}} \in L^2(\mathbb{R}^d, \mathrm{d}k; \mathcal{K}) = \mathcal{K} \otimes \mathfrak{h},$$

we have $\|\tilde{w}_R\|_{B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})} \leq C_2(R)$ and hence $\|\tilde{w}_R^* \tilde{w}_R\|_{B(\mathcal{K})} \leq C_2(R)^2$. This yields

$$\| \int \tilde{w}_R^*(k) \tilde{w}_R(k) \mathrm{d}k \|_{B(\mathcal{K})} \leq C_2(R)^2. \tag{A.5}$$

(The integral in (A.5) should be considered in the weak sense on $B(\mathcal{K})$, ie as a quadratic form on $\mathcal{K}$). We have

$$\begin{aligned}\|a^*(v)u\|^2 = & \int\int(v(k)a^*(k)u, v(k')a^*(k')u)_{\mathcal{H}}\mathrm{d}k\mathrm{d}k' \\ = & \int\int(v(k)a(k')u, v(k')a(k)u)_{\mathcal{H}}\mathrm{d}k\mathrm{d}k' \\ & + \int(v(k)u, v(k)u)\mathrm{d}k.\end{aligned}$$

The second term in the r.h.s. is bounded by

$$\begin{aligned}& \int \|v(k)(B+R)^{-\frac{1}{2}}\|^2\|(B+R)^{\frac{1}{2}}u\|^2\mathrm{d}k \\ & \leq C_3^2(R)\|(B+R)^{\frac{1}{2}}u\|^2.\end{aligned}$$

We write then the first term as

$$\begin{aligned}& \int\int(\tilde{w}_R(k)\psi(k'), \tilde{w}_R(k')\psi(k))_{\mathcal{H}}\mathrm{d}k\mathrm{d}k' \\ & \leq \int\int \|\tilde{w}_R(k)\psi(k')\|^2_{\mathcal{H}}\mathrm{d}k\mathrm{d}k' \\ & \leq \|\int \tilde{w}_R^*(k)\tilde{w}_R(k)\mathrm{d}k\| \int \|\psi(k')\|^2_{\mathcal{H}}\mathrm{d}k' \\ & \leq C_2(R)^2\|(B+R)^{\frac{1}{2}} \otimes \mathrm{d}\Gamma(m)^{\frac{1}{2}}u\|^2,\end{aligned}$$

which proves (A.3). $\square$

Lemma A.2 *Let Q be a closed, positive quadratic form, Q_n be closed quadratic forms on $D(Q)$ such that Q_n converges to Q when $n \to +\infty$ in the topology of $D(Q)$. Let H, H_n be the associated selfadjoint operators. Then for z in a bounded set $U \subset \mathbb{C}\backslash\mathbb{R}$, we have:*

$$\|((H-z)^{-1} - (H_n - z)^{-1})(H+1)^{-\frac{1}{2}}\| \in o(1)|Imz|^{-1}, \text{ when } n \to +\infty.$$

and for $\lambda \in \mathbb{R}, \lambda \ll -1$

$$\|((H-\lambda)^{-1} - (H_n - \lambda)^{-1})(H+1)^{-\frac{1}{2}}\| \in o(1) \text{ when } n \to +\infty.$$

Proof. Let for $z \in \mathbb{C}, u \in \mathcal{H}$, $R_n(z) = (H_n - z)^{-1}, R(z) = (H-z)^{-1}$, $r = R_n(z)u - R(z)u$. We have for $v \in D(Q)$:

$$\begin{aligned}(v, u) & = Q(v, R(z)u) - z(v, R(z)u) \\ & = Q_n(v, R_n(z)u) - z(v, R_n(z)u).\end{aligned}$$

Hence for $v = r$ we obtain

$$Q(r, R(z)u) - Q_n(r, R_n(z)u) + z\|r\|^2 = 0,$$

or

$$Q(r,r) - z\|r\|^2 = (Q - Q_n)(r, R(z)u). \tag{A.6}$$

If $\lambda \in \mathbb{R}, \lambda \ll -1$, we deduce from (A.6) that

$$(Q+1)(r,r) \in o(1)(Q+1)(r,r)^{\frac{1}{2}}(Q+1)(R(\lambda)u, R(\lambda)u)^{\frac{1}{2}}.$$

This implies that $(Q+1)(r,r)^{\frac{1}{2}}$ is $o(1)\|u\|$, as claimed.

Let now $z \in U \subset \mathbb{C}\backslash\mathbb{R}$. Taking the imaginary part of (A.6) we obtain

$$\begin{aligned}\|r\|^2 &\in o(1)|Imz|^{-1}(Q+1)(r,r)^{\frac{1}{2}}(Q+1)(R(z)u,R(z)u)^{\frac{1}{2}} \\ &\in o(1)|Imz|^{-2}(Q+1)(r,r)^{\frac{1}{2}}\|u\|^2,\end{aligned}$$

since $(Q_0+1)(R(z)u,R(z)u)$ is bounded by $|Imz|^{-2}\|u\|^2$ for $z \in U$. Taking then the real part of (A.6) we obtain

$$\begin{aligned}|Q(r,r)| &\in o(1)(Q_0+1)(r,r)^{\frac{1}{2}}(Q+1)(R(z)u,R(z)u)^{\frac{1}{2}} + o(1)|Imz|^{-2}(Q+1)(r,r)^{\frac{1}{2}}\|u\|^2 \\ &\in o(1)|Imz|^{-2}(Q+1)^{\frac{1}{2}}(r,r)\|u\|^2.\end{aligned}$$

This implies that $(Q+1)(r,r)^{\frac{1}{2}} \in o(1)|Imz|^{-1}\|u\|$ as claimed. $\Box$

The following result is shown in [AH, Lemma 4.9]

Lemma A.3 *Let H, H_n for $n \in \mathbb{N}$ be selfadjoint operators on a Hilbert space $\mathcal{H}$. Let ψ_n be a normalized eigenvector of H_n with eigenvalue E_n. Assume that*

i) $H_n \to H$ *when* $n \to \infty$ *in strong resolvent sense,*

ii) $\lim_{n\to\infty} E_n = E$,

iii) $\mathrm{w}-\lim_{n\to\infty}\psi_n = \psi \neq 0$.

Then ψ is an eigenvector of H with eigenvalue E.

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