Commutators, C_0−semigroups and Resolvent Estimates

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Abstract

We study the existence and the continuity properties of the boundary values $(H-\lambda\pm \mathrm{i}0)^{-1}$ of the resolvent of a selfadjoint operator H in the framework of the conjugate operator method initiated by E. Mourre. We allow the conjugate operator A to be the generator of a C_0-semigroup (finer estimates require A to be maximal symmetric) and we consider situations where the first commutator $[H, \mathrm{i}A]$ is not comparable to H. The applications include the spectral theory of zero mass quantum field models.

1 Introduction

In this paper we describe an extension of the Mourre version of the positive commutator method which can be used in situations where the commutator of the Hamiltonian H with the conjugate operator A is not comparable with H and/or A is not a selfadjoint operator. This extension is especially adapted to the study of the spectral theory of quantum field Hamiltonians.

The origin of the positive commutator method can be traced back to the following theorem, proved by C. R. Putnam in 1956: if A is bounded and $[H, \mathrm{i}A] \geq 0$, then the range of $[H, \mathrm{i}A]$ is contained in the absolute continuity subspace of H (see [P1] or [P2, page 20]). This result has been improved and used by several authors to prove absolute spectral continuity of Schrödinger operators, see [RS, vol. 4] for a review of some of these works. However, the applications were rather restricted by the boundedness condition on A and the global positivity requirement on the commutator. In 1981 E. Mourre [Mo] succeeded to treat the case when A is unbounded

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(but selfadjoint) and the positivity of the commutator holds (but in a stronger form) only on the open set J where we want to prove absolute continuity, more precisely:

$$[H, \mathrm{i}A] \geq a\mathbb{1}_J(H) - b\mathbb{1}_{\mathbb{R}\setminus J}(H)(\mathbb{1} + |H|) \tag{1.1}$$

for some numbers $a, b > 0$. It is easy to see that (1.1) is equivalent (under Mourre's conditions) to the usual form of the so called *strict Mourre estimate*. The main idea of Mourre was to control the behaviour of the resolvent $R(\lambda \pm \mathrm{i}\mu) = (H - \lambda \mp \mathrm{i}\mu)^{-1}$, where $\lambda \in J$ and $\mu \to 0^+$, with the help of A. There was a price to pay for this: the commutator $[H, \mathrm{i}A]$ and the second commutator $[[H, \mathrm{i}A], \mathrm{i}A]$ had to be dominated, in a suitable sense, by H. It turned out, however, that many physically interesting Hamiltonians can be easily studied in this framework. Moreover, the domination conditions were weakened by various authors, which increased the power and elegance of the theory (see [ABG] and references therein).

Among the various extensions of the Mourre theorem which exist in the literature, two are especially interesting for us: the first is due to M. Hübner and H. Spohn [HuS] and the second to E. Skibsted [Sk2] (this is further developed in [MS]). In [HuS] it is shown that Mourre's results remain true if A is only maximal symmetric, the main technical result being the extension of the virial theorem to this context (the resolvent estimates extend easily because domination conditions similar to those of Mourre are imposed). Moreover, the authors show the usefulness of this generalization in the study of the spin-boson model with a particle number cut-off. But the results of [HuS] cannot be used in the case of a massless quantum field because then the commutator $[H, \mathrm{i}A]$ is not dominated by any power of H. This difficulty was overcome in [Sk2] by assuming that there is a sequence of selfadjoint operators A_n which converge strongly to A and such that $[H, \mathrm{i}A_n]$ is H-bounded, so Mourre's computations makes sense for each fixed n. Suitable uniform in n bounds on the approximating operators allow one to take the limit $n \to \infty$ in the final estimate.

1.1 Presentation of the main results

In this paper we shall prove resolvent estimates under the assumption that the Hamiltonian H is regular in a certain sense with respect to the operator $[H, \mathrm{i}A]$. If $[H, \mathrm{i}A]$ is H-bounded then this condition is satisfied, hence our condition on the first commutator is weaker than that from [Mo, HuS], where H-boundedness (*and* boundedness from below) of $[H, \mathrm{i}A]$ is required.

On the other hand, some further technical conditions are necessary for the development of the theory, and ours are not directly comparable with those of the quoted papers (these conditions involve boundedness properties of the second commutator $[[H, \mathrm{i}A], \mathrm{i}A]$ and the stability of certain spaces under the (semi)group generated by A; see [ABG, Sec. 7.5.1, 7.5.2] for a detailed discussion). However, in Section 5 we show that it is easy to deduce from Theorem 3.5 a result which, for Hamiltonians with a spectral gap, covers those from [Mo] and also the more general results of P. Perry, I. Sigal and B. Simon [PSS] (note that in [PSS] H is assumed bounded from below). Since the operator A will be assumed maximal symmetric, it will be clear that the abstract theory developed in [HuS] is also a particular case of ours. Later on in this introduction we shall explain why the results of [Sk2] are consequences of ours and we shall discuss the relation with [MS].

We shall now summarize our main results. For simplicity, we present here only a particular case, when A is assumed to be maximal symmetric. We refer to Section 3 for the general case, when (a multiple of) A is the generator of a C_0-semigroup. We begin with some definitions and notations.

The regularity notion we need in order to formulate our results is the following. Let S and T be closed densely defined operators on a Hilbert space $\mathcal{H}$. Assume that S has the following property: there is a sequence of complex numbers z_ν in the resolvent set of S such that $|z_\nu| \to \infty$ and $\sup_\nu |z_\nu| \|(S - z_\nu)^{-1}\| < \infty$. We say that S *is of full class* $C^1(T)$ if for each number z in the resolvent set of S the sesquilinear form $[T, (S - z)^{-1}]$ with domain $\mathcal{D}(T^*) \times \mathcal{D}(T)$ is continuous for the topology of $\mathcal{H} \times \mathcal{H}$. Or, equivalently, if for each such z one has $(S - z)^{-1}\mathcal{D}(T) \subset \mathcal{D}(T)$ and $T(S - z)^{-1} - (S - z)^{-1}T : \mathcal{D}(T) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$.

Now let A be a maximal symmetric operator on $\mathcal{H}$. Then A has deficiency indices $(N, 0)$ or $(0, N)$. In the first case there is a strongly continuous one parameter semigroup $\{W_t\}_{t\geq 0}$ of isometries on $\mathcal{H}$ such that $u \in \mathcal{D}(A)$ if and only if $\|W_t u - u\| \leq ct$, and in this case $\mathrm{i}Au = \lim_{t\to 0^+}(W_t u - u)/t$; then we write $W_t = \mathrm{e}^{\mathrm{i}tA}$. In the second case, $-A$ generates such a semigroup. In order to have uniform notations, we define in the second case $W_t \equiv \mathrm{e}^{\mathrm{i}tA} := \mathrm{e}^{\mathrm{i}|t|(-A)}$ for $t \leq 0$. Thus we have in both cases $\mathrm{i}Au = \lim_{t\to 0}(W_t u - u)/t$ for $u \in \mathcal{D}(A)$, the parameter t being restricted by the conditions $t > 0$ and $t < 0$ in the first and second case respectively. We note that $\{W_t^*\}$ will be a C_0-semigroup of contractions with generator $-A^*$.

Assume that $\mathcal{G}$ is a Hilbert space continuously and densely embedded in $\mathcal{H}$ such that $W_t^*\mathcal{G} \subset \mathcal{G}$ for all t and $\sup_{0<|t|<1} \|W_t^*\|_{B(\mathcal{G})} < \infty$. We denote by $\mathcal{H}^*$ and $\mathcal{G}^*$ the adjoint spaces and we identify $\mathcal{H}^* = \mathcal{H}$ with the help of the Riesz isomorphism, which implies $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$ continuously and densely. Denote by $\mathcal{G}_1^*$ the completion of $D(A)$ under the norm $\|u\|_{\mathcal{G}_1^*} = \|u\|_{\mathcal{G}^*} + \|Au\|_{\mathcal{G}^*}$. We will see that $\mathcal{G}_1^* \subset \mathcal{G}^*$ continuously and densely. Now we define the spaces we are interested in by real interpolation:

$$\mathcal{G}_{s,p}^* = (\mathcal{G}_1^*, \mathcal{G}^*)_{1-s,p} \quad \text{if } 0 < s < 1 \text{ and } 1 \leq p \leq \infty.$$

Then let $\mathcal{G}_{-1}$ be the completion of $\mathcal{G}$ under the norm $\inf_v(\|v\|_{\mathcal{G}} + \|u - A^*v\|_{\mathcal{G}})$ with $v \in \mathcal{G} \cap \mathcal{D}(A^*)$ such that $A^*v \in \mathcal{G}$. Then $\mathcal{G} \subset \mathcal{G}_{-1}$ continuously and densely and we define again by real interpolation

$$\mathcal{G}_{-s,p} = (\mathcal{G}, \mathcal{G}_{-1})_{s,p} \quad \text{if } 0 < s < 1 \text{ and } 1 \leq p \leq \infty.$$

Details on these spaces and more explicit definitions in terms of the semigroup $\{W_t\}$ can be found in Subsection 3.1. We mention only that $\mathcal{G}_{s,p}^* \subset \mathcal{G}_{t,q}^*$ if $s > t$ or $s = t$ but $p \leq q$. Moreover, one has a canonical identification $(\mathcal{G}_{s,p}^*)^* = \mathcal{G}_{-s,p'}$ if $p < \infty$, where $1/p + 1/p' = 1$. The space $\mathcal{G}_s^* := \mathcal{G}_{s,2}^*$ and its adjoint $\mathcal{G}_{-s} := \mathcal{G}_{-s,2}$ are more commonly used: they are Hilbert spaces and can be defined by complex interpolation or as domains of certain operators naturally associated to A and A^* in $\mathcal{G}^*$ and $\mathcal{G}$ respectively.

We can state now the hypotheses of the next theorem. Besides the operator A introduced above we consider a selfadjoint operator H and a symmetric closed densely defined operator H' (one of the conditions below says that H' is a realization of the formal commutator $[H, \mathrm{i}A]$). We denote by $\mathcal{D}$ the space $\mathcal{D}(H) \cap \mathcal{D}(H')$ (equipped with the intersection topology). We assume:

(M1) *H is of full class $C^1(H')$, $\mathcal{D}$ is a core of H', and $\mathcal{D}(H) \cap \mathcal{D}(H'^*) = \mathcal{D}$.*

(M2) *A bounded open set $J \subset \mathbb{R}$ is given and there are numbers $a, b > 0$ such that*

$$H' \geq [a\mathbb{1}_J(H) - b\mathbb{1}_{\mathbb{R}\setminus J}(H)]\langle H\rangle \quad \textit{as forms on } \mathcal{D}.$$

Then there is $c > 0$ such that $H' + c\langle H\rangle \geq \langle H\rangle$ as forms on $\mathcal{D}$. We define $\mathcal{G}$ as the completion of $\mathcal{D}$ under the norm $\|u\|_{\mathcal{G}} = (u, (H' + c\langle H\rangle)u)^{1/2}$; we have $\mathcal{G} \subset \mathcal{H}$ continuously and densely.

Clearly H and H' extend to continuous symmetric operators $\mathcal{G} \to \mathcal{G}^*$. Our last hypotheses are:

(M3′) *$W_t^*\mathcal{G} \subset \mathcal{G}$ for all t and $\sup_{0<|t|<1} \|W_t^*\|_{B(\mathcal{G})} < \infty$.*

(M4) *For all $u \in \mathcal{D}$ we have: $\lim_{t\to 0} \frac{1}{t}\left[(Hu, W_t u) - (u, W_t H u)\right] = (u, H'u)$.*

(M5) *There is $H'' \in B(\mathcal{G}, \mathcal{G}^*)$ such that $\lim_{t\to 0} \frac{1}{t}\left[(H'u, W_t u) - (u, W_t H'u)\right] = (u, H''u), u \in \mathcal{D}$.*

We shall use the notations:

$$J_\pm^\circ = \{\lambda \pm \mathrm{i}\mu \mid \lambda \in J,\ \mu > 0\}, \quad J_\pm = \{\lambda \pm \mathrm{i}\mu \mid \lambda \in J,\ \mu \geq 0\}.$$

Our main result is the next theorem.

Theorem 1.1 *If $z \in J_+^\circ \cup J_-^\circ$ then $R(z)(\mathcal{H} \cap \mathcal{G}_{1/2,1}^*) \subset \mathcal{G}_{-1/2,\infty}$ and the restriction of the map $R(z)$ to $\mathcal{H} \cap \mathcal{G}_{1/2,1}^*$ extends to a continuous operator $R(z) : \mathcal{G}_{1/2,1}^* \to \mathcal{G}_{-1/2,\infty}$. The functions $J_\pm^\circ \ni z \mapsto R(z) \in B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$ are holomorphic and extend to weak* continuous maps on $J_\pm$. In particular, the limits $R(\lambda \pm \mathrm{i}0) := \lim_{\mu\to\pm 0} R(\lambda + \mathrm{i}\mu)$ exist in the weak* topology of $B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$ locally uniformly in $\lambda \in J$ and the boundary values maps*

$$J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$$

are weak continuous. If $1/2 < s \leq 1$ then the maps*

$$J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{G}_{s,\infty}^*, \mathcal{G}_{-s,1})$$

are locally Hölder continuous of order $s - 1/2$.

Let us state explicitly the main estimates that we prove: for each compact $I \subset J$ there is a number C_I such that

$$(1.2) \qquad \|R(\lambda + \mathrm{i}\mu)f\|_{\mathcal{G}_{-1/2,\infty}} \leq C_I \|f\|_{\mathcal{G}_{1/2,1}^*} \quad \text{for } \lambda \in I, \mu \neq 0 \text{ and } f \in \mathcal{H} \cap \mathcal{G}_{1/2,1}^*$$

and for each $1/2 < s \leq 1$ there is $C_I(s)$ such that the boundary values satisfy

$$(1.3) \qquad \|R(\lambda_1 \pm \mathrm{i}0) - R(\lambda_2 \pm \mathrm{i}0)\|_{B(\mathcal{G}_{s,\infty}^*, \mathcal{G}_{-s,1})} \leq C_I(s)|\lambda_1 - \lambda_2|^{s-1/2} \quad \text{for } \lambda_1, \lambda_2 \in I.$$

We stress that $R(z)$ does not send $\mathcal{G}^*$ into $\mathcal{G}$ in general, hence (1.2) is a rather subtle estimate.

Our Hölder continuity estimate is in fact more precise than (1.3). Indeed, we prove that

$$(1.4) \qquad \|R(z_1) - R(z_2)\|_{B(\mathcal{G}_{s,\infty}^*, \mathcal{G}_{-s,1})} \leq C_I(s)|z_1 - z_2|^{s-1/2} \quad \text{for } z_1, z_2 \in I_\pm$$

where $I_\pm$ is defined similarly to $J_\pm$. In particular, we see that $R(z)$ converges in norm to its boundary values in the space $B(\mathcal{G}_{s,\infty}^*, \mathcal{G}_{-s,1})$:

$$(1.5) \qquad \|R(\lambda \pm \mathrm{i}\mu) - R(\lambda \pm \mathrm{i}0)\|_{B(\mathcal{G}_{s,\infty}^*, \mathcal{G}_{-s,1})} \leq C_I(s)|\mu|^{s-1/2} \quad \text{for } \lambda \in I_\pm \text{ and } \mu > 0.$$

We do not have any particular application in mind of these Hölder continuity results. We just mention here that they are an ingredient in the derivation of the Fermi Golden Rule in second order perturbation theory of embedded eigenvalues, as presented in [AHS]. See also [MS] and [DJ]. Higher orders of regularity have been used recently by S. Agmon and I. Herbst [AH] to make a precise study of perturbations under which an embedded eigenvalue persists. Finally we mention that higher order regularity of the resolvent is also an ingredient in the study of the smoothness of scattering matrices, cf. e.g. [Sk1], and in some methods for obtaining propagation estimates in scattering theory, see e.g. [Je].

The methods we use allow one to eliminate the condition on H involving the second commutator H'' (as in [ABG, Chapter 7]) and to determine the order of regularity of the boundary values for all allowed s. The idea is to replace the operators $H_\varepsilon = H - \mathrm{i}\varepsilon H'$ used in Subsections 3.4 and 3.5 by a more general family of operators $H_\varepsilon \in B(\mathcal{G}, \mathcal{G}^*)$ such that $H_\varepsilon = H - \mathrm{i}\varepsilon H' + o(\varepsilon)$ as $\varepsilon \to 0$. For example, if $\mathcal{G}$ is b-stable under $\{W_t\}$ and $\{W_t^*\}$ (see Definition 2.32) then the condition $H \in C^2(A; \mathcal{G}, \mathcal{G}^*)$ from Remark 3.1 can be replaced by $H \in C^{1,1}(A; \mathcal{G}, \mathcal{G}^*)$, where the last space is defined by real interpolation: $C^{1,1}(A; \mathcal{G}, \mathcal{G}^*) = (C^2(A; \mathcal{G}, \mathcal{G}^*), B(\mathcal{G}, \mathcal{G}^*))_{1/2,1}$. We do not give details because in the main application we have in mind (see [GGM]) such an extension is not really relevant.

The condition that A be the generator of a C_0-semigroup is not always easy to check, so we make an effort to go as far as possible without it. In fact we use it only in order to establish the relation (3.22) (the approximations (3.30) could be replaced by other expressions suggested by interpolation theory). In many cases a direct justification of (3.22) is easy. Incidentally, this is the case in the examples presented later on in the introduction, but this is definitely not so in the situation of [GGM]. Also, the case when H and H' commute is rather elementary and one can obtain without much effort results of a certain interest. Of course, our main concern is the case when H and H' do not commute.

Our assumption **(M2)** is a *strict* Mourre type estimate, i.e. it involves no compact remainder. Such an estimate is difficult to obtain directly and the usual way to bypass this problem is to invoke various versions of the virial theorem. In Section 4 we present some new results, adapted to the context of [GGM], concerning this topic. We improve the standard version (see [ABG, Proposition 7.2.10] and [GG]) of the virial theorem in two directions. First, we consider conjugate operators A of a general form, including generators of C_0-semigroups. Thus we cover the known results concerning selfadjoint A as well as the extension to maximal symmetric A obtained in [HuS, Proposition 9]. Then we treat the case when the Hamiltonian H is not of class $C^1(A)$ under some supplementary assumptions (H' should be approximable by operators with better properties). These results are sufficient for the situation studied in [GGM].

We shall make now some comments concerning the relation between our paper and [Sk2, MS]. The assumptions of [Sk2] involve, besides the selfadjoint operator H and the maximal symmetric operator A, an auxiliary selfadjoint operator $M \geq 1$ (in applications this is the particle number operator plus the projection onto the vacuum state) and a sequence of selfadjoint operators A_n which converge to A. Most of Skibsted's hypotheses are formulated in terms of the commutators between H and A_n and are similar to those of Mourre [Mo] with one notable exception: $[H, \mathrm{i}A_n]$ does not satisfy the Mourre positivity condition. Instead, it is required that $\lim_{n\to\infty}[H, \mathrm{i}A_n]$ exists in some sense and is of the form $M + G$, where G is an H-bounded symmetric operator. Then Mourre positivity is imposed only on this limit operator. Also, instead of working directly with second order commutators, Skibsted requires the existence of the limits $\lim_{n\to\infty}[M, \mathrm{i}A_n]$ and $\lim_{n\to\infty}[G, \mathrm{i}A_n]$ in a suitable sense.

The connection with our formalism is obtained by defining H' as the closure of the operator $M + G$ (see Lemma 2.26). One can then show that Skibsted's assumptions imply ours. We shall not explain this in detail, although the proof is not completely trivial. Note, however, that in our notations Assumption 2.1(1) from [Sk2] can be written as: $H \in C^1(M)$ and $[H, \mathrm{i}M]^\circ \mathcal{D}(H) \subset \mathcal{H}$.

In particular, $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}(M)$ is a core for H and for M. It is also important to notice that the semigroup of isometries W_t generated by A satisfies a rather strong condition: $W_t\mathcal{D} \subset \mathcal{D}$ and $W_t^*\mathcal{D} \subset \mathcal{D}$ for all $t \geq 0$. In particular, these groups also leave invariant the interpolation space $(\mathcal{D}, \mathcal{H})_{1/2,2}$ and this is, under the conditions of [Sk2], our space $\mathcal{G}$ (see Proposition 3.8).

In the papers [Sk2] and [MS] the family H_ε is taken to be $H_\varepsilon^{MS} = H - \mathrm{i}\varepsilon(M + f(H)Gf(H))$. This choice differs from ours (namely $H_\varepsilon = H - \mathrm{i}\varepsilon H'$) due to the appearance of the energy cutoffs $f(H)$. In [MS] it is observed that H and M appear symmetrically in H_ε^{MS}, up to the need for uniformity of estimates in ε. This observation makes it possible to ease the assumption on $[H, \mathrm{i}M]$ mentioned above, such as to cover the application considered there. An assumption of the following form is introduced instead: $[H, \mathrm{i}M]^0 = T_1 + T_2$, where T_1 is H-bounded and T_2 is $M^{\frac{1}{2}}$-bounded. (That $M^{\frac{1}{2}}$ and not M is used is due to the need for uniformity in ε.) This type of assumption could possibly also be considered in our context, but at the cost of some elegance.

We furthermore mention that the limiting absorption principle proved in [MS] is of the form $\|M^{\frac{1}{2}-\beta}\langle A\rangle^{-\frac{1}{2}-\alpha}(H-z)^{-1}\langle A\rangle^{-\frac{1}{2}-\alpha}M^{\frac{1}{2}-\beta}\|$ be uniformly bounded as z approaches a part of the spectrum where a Mourre estimate holds. Here $\alpha > 0$ and in particular it is required that β be strictly positive. Our limiting absorption principle holds with $\beta = 0$ and is in this direction an improvement over [MS] (when both sets of assumptions hold). Technically the need for $\beta > 0$ in [MS] is a consequence of the use of the energy cutoff $f(H)$ in H_ε^{MS}.

1.2 Elementary applications

The main application of Theorem 1.1 is presented in [GGM], where we study the spectral theory of *massless Nelson models* describing a quantum field of massless particles interacting with non-relativistic electrons.

We shall give here two simple examples which allow us to illustrate some advantages of our results in comparison with previous ones.

We consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ and, if $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, we denote by $f(Q)$ the operator of multiplication by f in $\mathcal{H}$ and we set $f(P) := \mathcal{F}^* f(Q)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation. In particular, if $k \in \mathbb{R}^n$ then $\mathrm{e}^{\mathrm{i}kQ}$ acts as follows: $(\mathrm{e}^{\mathrm{i}kQ}u)(x) = \mathrm{e}^{\mathrm{i}kx}u(x)$. Note that $\mathrm{e}^{-\mathrm{i}kQ}f(P)\mathrm{e}^{\mathrm{i}kQ} = f(P+k)$.

A real number λ is a *threshold value* of a C^1 function $\omega : \mathbb{R}^n \to \mathbb{R}$ if there is a sequence of points $k_j \in \mathbb{R}^n$ such that $\omega(k_j) \to \lambda$ and $\omega'(k_j) \to 0$, where ω' is the gradient of ω. The set of threshold values of ω is denoted by $\tau(\omega)$. Clearly, $\lambda \notin \tau(\omega)$ if and only if there is an open neighborhood J of λ and a constant $m > 0$ such that $|\omega'(k)| \geq m$ if $\omega(k) \in J$. If $|\omega(k)| + |\omega'(k)| \to \infty$ when $k \to \infty$ then $\tau(\omega)$ is the set of critical values of ω.

Now we fix $\omega : \mathbb{R}^n \to \mathbb{R}$ of class C^2 such that the Jacobian matrix ω'' is a bounded function. We set $\widetilde{\omega} = (1 + \omega^2 + \omega'^2)^{1/2}$ and for $0 \leq \sigma \leq 1$ we introduce the space

$$\mathcal{K}^\sigma = \mathcal{D}(\widetilde{\omega}^\sigma(P)) = \mathcal{D}(|\omega|^\sigma(P)) \cap \mathcal{D}(|\omega'|^\sigma(P))$$

equipped with norm $\|\widetilde{\omega}^\sigma(P)u\|$. Then $\mathcal{K}^{-\sigma} := (\mathcal{K}^\sigma)^*$ is the completion of $\mathcal{H}$ under the norm $\|\widetilde{\omega}^{-\sigma}(P)u\|$ and we have as usual $\mathcal{K}^\sigma \subset \mathcal{H} \subset \mathcal{K}^{-\sigma}$.

It is easy to show that for each $\sigma \in [-1, 1]$ one has $\mathrm{e}^{\mathrm{i}kQ}\mathcal{K}^\sigma = \mathcal{K}^\sigma$ for all $k \in \mathbb{R}^n$ and that the n-parameter group induced by $\{\mathrm{e}^{\mathrm{i}kQ}\}_{k\in\mathbb{R}^n}$ in $\mathcal{K}^\sigma$ is strongly continuous and of polynomial growth: $\|\mathrm{e}^{\mathrm{i}kQ}\|_{B(\mathcal{K}^\sigma)} \leq C\langle k\rangle^2$, where $\langle k\rangle \equiv (1 + |k|^2)^{1/2}$. Thus one can define the scale of Besov spaces $\mathcal{K}^\sigma_{s,p}$ with $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. If $\omega(k) = \langle k\rangle$ we get the usual weighted (Besov type)

Sobolev spaces $\mathcal{H}^\sigma_{s,p}$. If $\sigma = 0$ then $\mathcal{K}^0_{s,p} \equiv \mathcal{H}_{s,p}$ are usual weighted (Besov type) L^2 spaces. Details of the construction can be found in [ABG, Chapters 3 and 4].

The proof of the following proposition can be found in the Appendix A.

Proposition 1.2 *Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2 with bounded second order derivatives and let Λ be a compact real set disjoint from $\tau(\omega)$. Set $R(z) = (\omega(P) - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. If $\Re z \in \Lambda$ then $R(z)\mathcal{H}_{1/2,1} \subset \mathcal{K}^1_{-1/2,\infty}$ and for each $-1 \leq \sigma \leq 0$ the restriction $R(z)|\mathcal{H}_{1/2,1}$ extends to a continuous operator $\mathcal{K}^\sigma_{1/2,1} \subset \mathcal{K}^{\sigma+1}_{-1/2,\infty}$ satisfying*

$$\|R(z)u\|_{\mathcal{K}^{\sigma+1}_{-1/2,\infty}} \leq C\|u\|_{\mathcal{K}^\sigma_{1/2,1}} \tag{1.6}$$

for some constant C independent of z and u.

This result is interesting in the context of the remark after (1.3) because the function ω does not necessarily dominate its gradient (e.g. let $n = 2$ and $\omega(k) = k_1^2 - k_2^2$ or $\omega(k) = k_1 + k_2^2$) so we do not have $R(z)\mathcal{H} \subset \mathcal{K}^1$. Estimates like (1.6) have first been obtained by S. Agmon and L. Hörmander [Hor, Theorem 14.2.3] for simply characteristic differential operators of an arbitrary order; see [BG] for a proof involving conjugate operators and a localization argument.

The condition naturally suggested by Theorem 1.1 is $|\omega''| \leq C\widetilde{\omega}$ rather than $|\omega''| \leq C$. It is indeed easy to obtain an analogue of Proposition 1.2 under this condition but with the spaces $\mathcal{K}^{\pm 1/2}_{s,p}$ replaced by the spaces $\mathcal{G}^{(*)}_{s,p}$ defined in terms of the operator A that we introduce below (or other similar operators). In particular, one can deduce the results of [Hos] from Theorem 1.1. Note also that $|\omega''| \leq C\widetilde{\omega}$ is the only condition needed to construct the Besov scales $\mathcal{K}^\sigma_{s,p}$ (this follows easily from Proposition 2.34) but unfortunately it is not possible, in general, to pass from the spaces $\mathcal{G}^{(*)}_{s,p}$ to the more natural $\mathcal{K}^\sigma_{s,p}$.

In order to prove Proposition 1.2 we make the choice $A = \frac{1}{2}(F(P)Q + QF(P))$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field $F(k) = \omega'(k)\langle\omega'(k)\rangle^{-1}$. This forces us to take $H' = |\omega'(P)|^2\langle\omega'(P)\rangle^{-1}$ which is of order $|\omega'(P)|$, hence not comparable with H, so the known versions of the Mourre theorem cannot be used. However, A is selfadjoint. Our next purpose is to explain the usefulness of considering non selfadjoint conjugate operators. The following example is relevant in the context of [GGM]. The corresponding class of operators A is considered in detail in the Appendix A.

Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a continuous positive function such that the set $Z = \{k \in \mathbb{R}^n \mid \omega(k) = 0\}$ is of measure zero. Let $\Omega := \mathbb{R}^n \setminus Z$. We suppose that the function ω is of class C^2 on Ω and has no critical points there. We take $H = \omega(P)$ and formally define A by $A = \frac{1}{2}(F(P)Q + QF(P))$ with $F(k) = \frac{\omega'(k)}{|\omega'(k)|}$. Then, again formally, we have $[H, \mathrm{i}A] = |\omega'(P)|$. The advantage of such a choice is seen in the case when $|\omega'(k)| \geq a$ for some number $a > 0$; this happens, for example, if $\omega(k)$ is the distance from k to a given closed convex set Z of measure zero (the simplest example being $\omega(k) = |k|$). Then we have a global strict positivity estimate $H' \geq a$ which is quite useful in many circumstances (the purpose of the theory developed in [HuS] was to cover such situations).

We shall now make rigorous these facts. Let A be the closure of the operator $\frac{1}{2}(F(P)Q + QF(P))$ with domain $\mathcal{F}C_0^\infty(\Omega)$. Then A is symmetric closed and densely defined, but this is not sufficient for our purposes: we have to show that it is maximal symmetric.

Lemma 1.3 *The symmetric operator A has deficiency indices of the form $(N, 0)$.*

The proof is given in the Appendix. We have $W_t \equiv \mathrm{e}^{\mathrm{i}tA} = \mathcal{F}^* \widehat{W}_t \mathcal{F}$ where the C_0-semigroup of isometries $\{\widehat{W}_t\}_{t \geq 0}$ is constructed as follows. For each $k \in \Omega$ let $t \mapsto p_t(k)$ be the unique solution of $\frac{\mathrm{d}}{\mathrm{d}t} p_t(k) = F(p_t(k))$ satisfying $p_0(k) = k$. We show that the maximal domain of existence of this solution is of the form $]\tau(k), \infty[$ with $\tau(k) < 0$. If $t \geq 0$ then p_t is a C^1-diffeomorphism of Ω onto some open set $\Omega_{-t} \subset \Omega$ whose inverse is denoted p_{-t} (the notations are consistent). Let $f = \operatorname{div} F$ and $\alpha_{-t}(k) := \exp(\int_0^{-t} f(p_s(k)) \mathrm{d}s)$ for $k \in \Omega_{-t}$. If $t \geq 0$ we set $\widehat{W}_t u := \chi_{\Omega_{-t}} \sqrt{\alpha_{-t}} u \circ p_{-t}$. Then $\widehat{W}_t$ is an isometry in $L^2(\mathbb{R}^n) = L^2(\Omega)$ with range equal to $L^2(\Omega_{-t})$. The case when ω' is only locally Lipschitz can also be treated along these lines.

Let H' be the (selfadjoint) operator $|\omega'(P)|$. Since H and H' commute, condition **(M1)** is satisfied and $\mathcal{D} = \mathcal{K}^1$ with the same notations as before. In the present context it is convenient to define thresholds by an obvious extension of the definition given above, but also to include zero in the threshold set. In particular, if $\omega(k) \to \infty$ as $k \to \infty$ then only zero is a threshold (note, however, that zero could be a "ghost threshold", i.e. irrelevant for spectral analysis; this happens in the massless Nelson model [GGM], where $\omega(k) = |k|$). Then condition **(M2)** is satisfied if the closure of J is compact and disjoint from the threshold set. As operators defined on $\mathcal{F}C_0^\infty(\Omega)$, we have: $[H', \mathrm{i}A] = w(P)$ where the function w is defined on Ω by $w = \frac{\omega'}{|\omega'|}(\frac{\omega'}{|\omega'|}\nabla)\omega'$. We see again that the condition $|\omega''| \leq C\widetilde{\omega}$ suffices to develop the theory, cf. the proof of Proposition 1.2.

In order to explain why it is useful for applications to admit non selfadjoint conjugate operators, we shall consider a "toy version" of the model treated in [GGM]. Let us first describe the one-particle space and one-particle kinetic energy: we take $n = 1$ and $\omega(k) = |k|$. Then $\Omega = \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $F(k) = \pm 1$ if $\pm k > 0$. The generator $\widehat{A}$ of $\{\widehat{W}_t\}_{t \geq 0}$ has domain $\mathcal{D}(\widehat{A}) = \mathcal{H}_0^1(\mathbb{R}^*)$, the set of functions of Sobolev class $\mathcal{H}^1(\mathbb{R})$ which vanish at zero, and it acts as follows: $\widehat{A}u(k) = \mathrm{i}u'(k)$ if $k > 0$ and $\widehat{A}u(k) = -\mathrm{i}u'(k)$ if $k < 0$. Let $\pi_\pm$ be the orthogonal projection onto the subspace $L^2(\mathbb{R}_\pm)$ of $L^2(\mathbb{R})$ and let $U_t f(x) = f(x - t)$. Then $\widehat{W}_t = \pi_+ U_t \pi_+ + \pi_- U_{-t} \pi_-$. If the Hamiltonian is $\omega(P)$ then $[\omega(P), \mathrm{i}A] = \mathbb{1}$, so the theory trivially applies. Below we change notations: we denote by a the operator A introduced above.

Now we consider a perturbed second quantized version of this situation. Let $\mathcal{H} = \Gamma(L^2(\mathbb{R}))$ be the symmetric Fock space over $L^2(\mathbb{R})$ and $H = \mathrm{d}\Gamma(\omega(P)) + \phi(v)$, where $\phi(v)$ is the field operator associated to some $v \in \mathcal{H}_0^1(\mathbb{R}^*)$. If we take $A = \mathrm{d}\Gamma(a)$ then $H' \equiv [H, \mathrm{i}A] = N + \phi(\widetilde{v})$, where N is the particle number operator and $\widetilde{v} = -\mathrm{i}av \in L^2(\mathbb{R})$. Now we are in a situation when A is not selfadjoint and H' is not comparable with H. This choice of A is in fact the most natural in order to prove the Mourre estimate (cf. [GGM]): it replaces the strict positivity of the mass used in an essential way in [DG2]. We note, however, that other (selfadjoint) operators have been used to obtain Mourre estimates under a weak coupling condition, cf. [BFSS].

1.3 Plan of the paper

We make now some remarks concerning the organization of the paper. Section 2 is devoted to the study of a regularity property of linear operators on a Hilbert space $\mathcal{H}$ with respect to a closed operator A. The fact that the corresponding class $C^1(A)$ plays a role in our argument could seem strange: after all, the main point of this paper is to develop the commutator method beyond the usual $C^1(A)$ setting (especially emphasized in [ABG]). More precisely, we are interested in Hamiltonians H which are *not* of class $C^1(A)$. But an important point of our approach is that not the Hamiltonian but the non-selfadjoint approximations H_ε (chosen *here* equal to $H - \mathrm{i}\varepsilon H'$) have to be of class $C^1(A)$. This explains why we study arbitrary closed operators of class $C^1(A)$.

On the other hand, we consider a general closed densely defined A for two reasons. First, we develop a substantial part of the theory for conjugate operators A which are only generators of C_0-semigroups. Second, our Hamiltonian has to be of class $C^1(H')$ and H' is not more than symmetric. In this context, we emphasize the role played by Theorem 2.25 in our arguments. For completeness we have included in Subsections 2.1 and 2.2 some elementary material most of which is part of the general theory of derivations on C^*-algebras (see [BR]).

Section 3 is the heart of the paper: we present and prove there our main results. We did not try to get statements of maximal generality in order to avoid heavy formulations. However, the proofs are arranged so that extensions are easy. In Section 4 we discuss some new results on the virial theorem, part of them adapted to the context of [GGM]. In Section 5 we prove a theorem extending various classical results for operators of class $C^2(A)$ to the case when A is maximal symmetric; we assume there that H has a spectral gap, which allows us to deduce it very easily from our main result, Theorem 3.5. Finally, Appendix A contains more technical results used in other parts of the paper.

2 The $C^1(A)$ class

In this section we consider a linear operator A and define a regularity property of linear operators on a Hilbert space $\mathcal{H}$ with respect to A which is an extension of the $C^1(A)$ property (see [ABG]) when A is selfadjoint. Throughout this section A will be closed and densely defined on a Hilbert space $\mathcal{H}$. Note that since A is closed, $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$.

2.1 $C^1(A)$ class of bounded operators

If $S \in B(\mathcal{H})$ we denote by $[A, S]$ the sesquilinear form on $\mathcal{D}(A^*) \times \mathcal{D}(A)$ defined by:

$$(u, [A, S]v) := (A^*u, Sv) - (S^*u, Av),\ u \in \mathcal{D}(A^*),\ v \in \mathcal{D}(A).$$

Definition 2.1 *An operator $S \in B(\mathcal{H})$ is* of class $C^1(A)$ *if the sesquilinear form $[A, S]$ is continuous for the topology of $\mathcal{H} \times \mathcal{H}$. If this is the case, we denote by $[A, S]^\circ$ the unique bounded operator on $\mathcal{H}$ associated to the quadratic form $[A, S]$ (note that $\mathcal{D}(A^*) \times \mathcal{D}(A)$ is dense in $\mathcal{H} \times \mathcal{H}$). We denote by $C^1(A)$ the linear space*

$$C^1(A) := \{S \in B(\mathcal{H}) | S \text{ is of class } C^1(A)\}.$$

Proposition 2.2 *An operator $S \in B(\mathcal{H})$ is of class $C^1(A)$ if and only if S maps $\mathcal{D}(A)$ into itself and $AS - SA : \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$. In this case*

$$AS = SA + [A, S]^\circ \text{ as an identity on } \mathcal{D}(A).$$

Proof. If $S \in C^1(A)$ then we have:

$$(A^*u, Sv) = (u, [A, S]^\circ v + SAv),\ u \in \mathcal{D}(A^*),\ v \in \mathcal{D}(A).$$

Since $A^{**} = A$, this implies that if $v \in \mathcal{D}(A)$ then $Sv \in \mathcal{D}(A)$ and $ASv = [A, S]^\circ v + SAv$. Conversely, assume that $S\mathcal{D}(A) \subset \mathcal{D}(A)$ and that there is $T \in B(\mathcal{H})$ such that $Tv = (AS - SA)v$ on $\mathcal{D}(A)$. If $u \in \mathcal{D}(A^*), v \in \mathcal{D}(A)$ then $(A^*u, Sv) - (u, SAv) = (u, ASv - SAv) = (u, Tv)$, hence $S \in C^1(A)$ and $[A, S]^\circ = T$. □

Lemma 2.3 *Let A_1, A_2 be closed densely defined operators such that $A_1 + A_2$ (with domain $\mathcal{D}(A_1) \cap \mathcal{D}(A_2)$) is closeable and densely defined. Denote by A the closure of $A_1 + A_2$. If $S \in C^1(A_1) \cap C^1(A_2)$, then $S \in C^1(A)$ and $[A, S]^\circ = [A_1, S]^\circ + [A_2, S]^\circ$.*

Proof. If $u \in \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$ then $Su \in \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$ and, by Proposition 2.2,

$$\|ASu\| \le \|(AS - SA)u\| + \|SAu\| = \|[A_1, S]^\circ u + [A_2, S]^\circ u\| + \|SAu\| \le C(\|u\| + \|Au\|).$$

Thus $S\mathcal{D}(A) \subset \mathcal{D}(A)$ and for $u \in \mathcal{D}(A)$ we have $(AS - SA)u = [A_1, S]^\circ u + [A_2, S]^\circ u$. The second part of Proposition 2.2 gives the result. □

Proposition 2.4 *Let $S \in B(\mathcal{H})$. Then $S \in C^1(A)$ if and only if $S^* \in C^1(A^*)$ and then:*

$$[A^*, S^*]^\circ = -([A, S]^\circ)^*.$$

Proof. Since $A^{**} = A, S^{**} = S$, it suffices to prove the $\Rightarrow$ part of the proposition. Let $u \in \mathcal{D}(A^*)$, $v \in \mathcal{D}(A)$. Then

$$(A^*u, Sv) - (S^*u, Av) = (u, [A, S]^\circ v).$$

Taking the complex conjugates and using that $v \in \mathcal{D}(A^{**}) = \mathcal{D}(A)$, we obtain:

$$(A^{**}v, S^*u) - (S^{**}v, A^*u) = -(v, ([A, S]^\circ)^* u), \ u \in \mathcal{D}(A^*), \ v \in \mathcal{D}(A^{**}),$$

i.e. $S^* \in C^1(A^*)$ and $[A^*, S^*]^\circ = -([A, S]^\circ)^*$. □

2.2 Properties of the space $C^1(A)$

Lemma 2.5 *The linear map $\mathcal{A} : C^1(A) \to B(\mathcal{H})$ defined by $S \mapsto [A, S]^\circ$ is closed for the weak operator topology.*

Proof. Let $\{S_\alpha\}_{\alpha \in I}$ be a net in $B(\mathcal{H})$ such that $S_\alpha \in C^1(A)$, $\mathrm{w-}\lim_\alpha S_\alpha = S$ weakly and $\mathrm{w-}\lim_\alpha [A, S_\alpha] = T$. weakly. Then for $u \in \mathcal{D}(A^*)$, $v \in \mathcal{D}(A)$ we have:

$$(u, [A, S_\alpha]^\circ v) = (A^*u, S_\alpha v) - (u, S_\alpha Av).$$

Taking the limit over the directed set I, we obtain:

$$(u, Tv) = (A^*u, Sv) - (u, SAv) = (A^*u, Sv) - (S^*u, Av).$$

Hence $S \in C^1(A)$ and $[A, S]^\circ = T$, which proves that $\mathcal{A}$ is closed for the weak topology. □

Proposition 2.6 (i) *The space $C^1(A)$ is a sub-algebra of $B(\mathcal{H})$ and $\mathcal{A}$ is a derivation on it, i.e.*

$$[A, ST]^\circ = [A, S]^\circ T + S[A, T]^\circ \ \text{ if } S, T \in C^1(A).$$

(ii) *If $S \in C^1(A)$ and z is a complex number in the connected component of infinity of $\mathbb{C} \setminus \sigma(S)$, then $R(z) := (S - z)^{-1} \in C^1(A)$ and*

$$[A, R(z)]^\circ = -R(z)[A, S]^\circ R(z).$$

Proof. Let us first prove (i). Let $u \in \mathcal{D}(A^*)$, $v \in \mathcal{D}(A)$. Then

$$\begin{aligned}(u,[A,ST]v) &= (A^*u, STv) - (u, STAv)\\ &= (A^*u, STv) - (u, SATv) + (u, (SAT - STA)v),\end{aligned}$$

using Proposition 2.2 for T. Hence

$$(u,[A,ST]v) = (u,[A,S]^\circ Tv) + (A^*S^*u, Tv) - (S^*u, TAv),$$

using Proposition 2.2 and Proposition 2.4 for S^*. Since $T \in C^1(A)$, we get:

$$\begin{aligned}(u,[A,ST]v) &= (u,[A,S]^\circ Tv) + (S^*u,[A,T]^\circ v)\\ &= (u,[A,S]^\circ Tv) + (u, S[A,T]^\circ v),\end{aligned}$$

which proves (i).

To prove (ii) we first consider the particular case $z = 1$ and $\|S\| < 1$, and we follow the method of proof of [BR, Lemma 3.2.29]. From (i) we see that if $S \in C^1(A)$ then $S^n \in C^1(A)$ and

$$[A, S^n]^\circ = \sum_{k=0}^{n-1} S^k [A,S]^\circ S^{n-1-k}.$$

Since $\|S\| < 1$ the Neumann series

$$(1-S)^{-1} = \sum_{k=0}^{\infty} S^k$$

is norm convergent. By Lemma 2.5, $(1-S)^{-1} \in C^1(A)$ and:

$$[A,(1-S)^{-1}]^\circ = \text{w-}\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} S^k [A,S]^\circ S^{n-1-k}.$$

This series is norm convergent to $(1-S)^{-1}[A,S]^\circ(1-S)^{-1}$, which completes the proof of (ii) in the particular case considered above. The general case is treated in two steps. If $|z| > \|S\|$ then we can use $(S-z)^{-1} = -z^{-1}(1-z^{-1}S)^{-1}$ and what was proved above. Then observe that

$$R(z) = R(z_0)(\mathbb{1} - (z - z_0)R(z_0))^{-1} \text{ for } z, z_0 \in \mathbb{C}\backslash\sigma(S). \tag{2.7}$$

If $R(z_0) \in C^1(A)$ and $|z - z_0|\|R(z_0)\| < 1$, then $R(z) \in C^1(A)$. By analytic continuation, we obtain that $R(z) \in C^1(A)$ for each z in the connected component of $\mathbb{C}\backslash\sigma(S)$ containing z_0. □

Remark 2.7 We do not know if the stronger version "$T \in C^1(A)$ invertible $\Rightarrow T^{-1} \in C^1(A)$ and $[A,T^{-1}]^\circ = -T^{-1}[A,T]^\circ T^{-1}$" of part (ii) of the preceding proposition is true. A positive answer would significantly simplify and improve some of our later arguments.

It is possible to avoid this problem for the following class of operators.

Definition 2.8 *A closed and densely defined operator A on $\mathcal{H}$ is called* regular *if there is a sequence (α_n) in $\mathbb{C}\backslash\sigma(A)$ and a constant C such that $|\alpha_n| \to \infty$ and $\|(A-\alpha_n)^{-1}\| \le C|\alpha_n|^{-1}$.*

The notion of regular operators is suggested by that of *positive* operator in a Banach space, which plays an important role in interpolation theory (see [ABG, Proposition 2.7.2] and [Tr]). We note that selfadjoint operators, maximal symmetric operators and, more generally, generators of C_0-semigroups are regular operators (see Section 2.5). Symmetric but not maximal symmetric operators are not regular. On the other hand, the condition of regularity is more convenient and less restrictive than that of being the generator of a C_0-semigroup. For example, let A be the operator of multiplication by x^3 in the Sobolev space $\mathcal{H} = H^1(\mathbb{R})$. It is clear that A is regular, but e^{itA} is not a bounded operator in $\mathcal{H}$ for $t \in \mathbb{R}\setminus\{0\}$.

If A is a regular operator we set $I_n = \alpha_n(\alpha_n - A)^{-1}$ and $A_n = AI_n$, with α_n as in Definition 2.8 (the operators A_n are the usual bounded regularizations of A when A is the generator of a C_0-semigroup, see [HP]; they also appear in [Mo] for selfadjoint A). Then I_n and A_n are bounded operators such that $\text{s-}\lim_{n\to\infty} I_n = \mathbb{1}$ in $\mathcal{H}$ and $\mathcal{D}(A)$, in particular $Au = \lim_{n\to\infty} A_n u$ if $u \in \mathcal{D}(A)$. Moreover, for each $u \in \mathcal{H}$ one has $u \in \mathcal{D}(A) \Leftrightarrow \sup_n \|A_n u\| < \infty$. Note that A^* is also regular (consider the sequence $(\bar{\alpha}_n)$).

Proposition 2.9 *Let A be a regular operator and let A_n be as above. A bounded operator S is of class $C^1(A)$ if and only if $\|[A_n, S]\| \le C$ for some constant C. If this is the case, then* $[A, S]^\circ = \text{s-}\lim_{n\to\infty}[A_n, S]$.

Proof. If $u \in \mathcal{D}(A^*)$ and $v \in \mathcal{D}(A)$ then $(u, [A_n, S]v) = (A_n^* u, Sv) - (u, SA_n v)$, so we have $\lim(u, [A_n, S]v) = (A^* u, Sv) - (u, SAv)$. If $\|[A_n, S]\| \le C$ then we clearly get $S \in C^1(A)$ and then $[A, S]v = \lim(A_n S - SA_n)v$ by the remarks made before the statement of the proposition and the relation $S\mathcal{D}(A) \subset \mathcal{D}(A)$ (see Proposition 2.2). Reciprocally, we have $A_n = \alpha_n I_n - \alpha_n$, hence for an arbitrary S we have $[A_n, S] = \alpha_n [I_n, S]$. Assume that $S \in C^1(A)$. From Proposition 2.2 it follows now easily that $[A_n, S] = I_n [A, S]^\circ I_n$, hence $\|[A_n, S]\| \le C^2 \|[A, S]^\circ\|$, where C is as in Definition 2.8. □

Corollary 2.10 *If A is regular and $S \in C^1(A)$ is invertible then $S^{-1} \in C^1(A)$ and*

$$[A, S^{-1}]^\circ = -S^{-1}[A, S]^\circ S^{-1}.$$

2.3 $C^1(A)$ class of unbounded operators

In this subsection we extend the $C^1(A)$ property to unbounded operators, as in the case when A is selfadjoint (see [ABG]).

Definition 2.11 *If S is a closed and densely defined operator on $\mathcal{H}$, then $\rho(S, A)$ is the set of $z \in \mathbb{C}\setminus\sigma(S)$ such that $R(z) := (S - z)^{-1}$ is of class $C^1(A)$.*

Remarks 2.12 (1) *$\rho(S, A)$ is a union of connected components of* $\mathbb{C}\setminus\sigma(S)$ (because (2.7) is valid for an arbitrary closed S, so we can use the last argument of the proof of Proposition 2.6).
(2) *If A is regular then either $\rho(S, A) = \emptyset$ or $\rho(S, A) = \mathbb{C}\setminus\sigma(S)$* (use (2.7) and Corollary 2.10).
(3) $\rho(S^*, A^*) = \rho(S, A)^*$ (see Proposition 2.4).

Proposition 2.13 *For all $z, z_0 \in \rho(S, A)$ one has:*

$$\begin{aligned} [A, R(z)]^\circ &= (\mathbb{1} + (z - z_0)R(z))[A, R(z_0)]^\circ(\mathbb{1} + (z - z_0)R(z)) \\ &= (S - z_0)R(z)[A, R(z_0)]^\circ R(z)(S - z_0). \end{aligned}$$

Proof. We write $R(z) - R(z_0) = (z - z_0)R(z)R(z_0)$ which by Proposition 2.6 yields:

$$[A, R(z)]^\circ - [A, R(z_0)]^\circ = (z - z_0)\Big([A, R(z)]^\circ R(z_0) + R(z)[A, R(z_0)]^\circ\Big),$$

or equivalently:

$$[A, R(z)]^\circ(\mathbb{1} - (z - z_0)R(z_0)) = (\mathbb{1} + (z - z_0)R(z))[A, R(z_0)]^\circ.$$

This gives the required relation, since

$$(\mathbb{1} - (z - z_0)R(z_0)) = R(z_0)R(z)^{-1} = (\mathbb{1} + (z - z_0)R(z))^{-1}.\square$$

Definition 2.14 *Let S be a closed and densely defined operator. We say that S is* of class $C^1(A)$ *if there is a sequence of complex numbers $z_\nu \in \rho(S, A)$ with $|z_\nu| \to \infty$ and such that the operators $J_\nu := z_\nu(z_\nu - S)^{-1}$ satisfy $\|J_\nu\| \le C$ for some constant C. If S is of class $C^1(A)$ and $\rho(S, A) = \mathbb{C}\backslash\sigma(S)$ then we say that S is* of full class $C^1(A)$.

Remarks 2.15 (1) An operator S of class $C^1(A)$ is regular.
(2) For bounded operators the two definitions of the class $C^1(A)$ coincide (see Proposition 2.6).
(3) $S \in C^1(A)$ if and only if $S^* \in C^1(A^*)$ (use Proposition 2.4).
(4) If S is of class $C^1(A)$ and A is regular, then S is of full class $C^1(A)$.

Lemma 2.16 *Let S be an operator of class $C^1(A)$ and let J_ν be as in Definition 2.14. Equip $\mathcal{D}(A) \cap \mathcal{D}(S)$ with the intersection topology, defined by the norm $\|u\| + \|Au\| + \|Su\|$. Then:*
(i) *The space $\mathcal{D} := R(z)\mathcal{D}(A)$ is independent of $z \in \rho(S, A)$ and is a core for S.*
(ii) *If $u \in \mathcal{D}(A) \cap \mathcal{D}(S)$ then $J_\nu u \in \mathcal{D}(A) \cap \mathcal{D}(S)$ and $J_\nu u \to u$ in $\mathcal{D}(A) \cap \mathcal{D}(S)$.*
In particular, $\mathcal{D}$ is a dense subset of $\mathcal{D}(A) \cap \mathcal{D}(S)$.

Proof. Since $R(z) : \mathcal{H} \to \mathcal{D}(S)$ is a homeomorphism and $\mathcal{D}(A)$ is dense in $\mathcal{H}$, we see that $\mathcal{D}$ is dense in $\mathcal{D}(S)$, i.e. $\mathcal{D}$ is a core for S. Next, for $z_1, z_2 \in \rho(S, A)$ we have:

$$R(z_1)\mathcal{D}(A) = R(z_2)(\mathbb{1} + (z_1 - z_2)R(z_1))\mathcal{D}(A) \subset R(z_2)\mathcal{D}(A),$$

by Proposition 2.2 applied to $R(z_1)$. This shows that $\mathcal{D}$ is independent of z and completes the proof of (i). Note that $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(S)$ by Proposition 2.2. Let us now prove (ii). Let now $u \in \mathcal{D}(A) \cap \mathcal{D}(S)$ and set $u_\nu = J_\nu u \in \mathcal{D}$. Since by Remark 2.15(1) S is regular, we have $u_\nu \to u$ in $\mathcal{D}(S)$ and $u_\nu \in \mathcal{D}(A)$. Next $A(u_\nu - u) = (J_\nu - \mathbb{1})Au + [A, J_\nu]^\circ u$. The first term tends to 0 when $\nu \to \infty$ because $\text{s-}\lim J_\nu = \mathbb{1}$. For the second term we use Proposition 2.13 and obtain:

$$[A, J_\nu]^\circ u = (S - z_0)R(z_\nu)[A, R(z_0)]^\circ J_\nu(S - z_0)u.$$

Since $\text{s-}\lim J_\nu = \mathbb{1}$ and $\text{s-}\lim\,(S - z_0)R(z_\nu) = 0$, we get $\lim\,[A, J_\nu]^\circ u = 0$, so $\lim Au_\nu = Au$. $\square$

Remark 2.17 We stress that $\mathcal{D}(A) \cap \mathcal{D}(S)$ is not, in general, a core for A (see, however, Remark 2.35). For example, let $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ and let A be the usual selfadjoint realization of $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. Observe that if S is the operator of multiplication by a real rational function (arbitrarily defined at the poles of the function) then S is of full class $C^1(A)$. Let S be the operator of multiplication by $1/x$, so that S is of full class $C^1(A)$. Then $\mathcal{D}(A) \cap \mathcal{D}(S)$ is the set of functions f in the first order Sobolev space $\mathcal{H}^1(\mathbb{R})$ such that $\int_{\mathbb{R}} x^{-2}|f(x)|^2\mathrm{d}x < \infty$. By Hardy's inequality this is just the set of $f \in \mathcal{H}^1(\mathbb{R})$ such that $f(0) = 0$, which is not dense in $\mathcal{D}(A) = \mathcal{H}^1(\mathbb{R})$.

We now characterize the $C^1(A)$ property in terms of the commutator $[S, A]$.

Definition 2.18 *Let A and S be two closed and densely defined linear operators on $\mathcal{H}$. We define $[A, S]$ as the sesquilinear form with domain $[\mathcal{D}(A^*) \cap \mathcal{D}(S^*)] \times [\mathcal{D}(A) \cap \mathcal{D}(S)]$ given by:*

$$(u, [A, S]v) := (A^*u, Sv) - (S^*u, Av).$$

Proposition 2.19 *Let S be an operator of class $C^1(A)$. Then $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ are cores for S and S^* respectively and the form $[A, S]$ has a unique extension to a continuous sesquilinear form $[A, S]^\circ$ on $\mathcal{D}(S^*) \times \mathcal{D}(S)$. One has:*

$$[A, R(z)]^\circ = -R(z)[A, S]^\circ R(z),\ z \in \rho(S, A) \tag{2.8}$$

where on the right hand side of (2.8) we consider $[A, S]^\circ$ as a bounded operator $\mathcal{D}(S) \to \mathcal{D}(S^*)^*$.

Proof. Let $\mathcal{D} = R(z)\mathcal{D}(A)$ and $\mathcal{D}^* = R(z)^*\mathcal{D}(A^*)$. By Remark 2.15(3) and Lemma 2.16, $\mathcal{D}$ and $\mathcal{D}^*$ are cores for S and S^*, $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(S)$, $\mathcal{D}^* \subset \mathcal{D}(A^*) \cap \mathcal{D}(S^*)$, so $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ are cores for S and S^*. Let now $u \in \mathcal{D}^*, v \in \mathcal{D},\ z \in \rho(S, A)$. Note that $\bar{z} \in \rho(S^*, A^*)$ by Remark 2.12(3). We have:

$$u = R(z)^*u_0,\ v = R(z)v_0,\ \text{for some } u_0 \in \mathcal{D}(A^*),\ v_0 \in \mathcal{D}(A).$$

Then:

$$\begin{aligned} (u, [A, S]v) = &\ (A^*u, Sv) - (S^*u, Av) \\ = &\ (A^*u, (S - z)v) - ((S^* - \overline{z})u, Av) \\ = &\ (A^*u, v_0) - (u_0, Av) \\ = &\ (R(z)^*u_0, Av_0) - (A^*u_0, R(z)v_0) \\ = &\ -(u_0, [A, R(z)]^\circ v_0). \end{aligned} \tag{2.9}$$

Since $R(z) \in C^1(A)$ this yields:

$$|(u, [A, S]v)| \leq \|[A, R(z)]^\circ\| \|(S - z)^*u\| \|(S - z)v\|. \tag{2.10}$$

Since, by Lemma 2.16, $\mathcal{D}$ and $\mathcal{D}^*$ are dense in $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ for the intersection topology, (2.10) extends to $u \in \mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ and $v \in \mathcal{D}(A) \cap \mathcal{D}(S)$, i.e. $[A, S]$ is bounded for the topology of $\mathcal{D}(S^*) \times \mathcal{D}(S)$. Since $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ are dense in $\mathcal{D}(S)$ and $\mathcal{D}(S^*)$, $[A, S]$ admits a unique extension to a bounded sesquilinear form $[A, S]^\circ$ on $\mathcal{D}(S^*) \times \mathcal{D}(S)$.

We can now rewrite (2.9) as:

$$(R(z)^*u_0, [A, S]^\circ R(z)v_0) = -(u_0, [A, R(z)]^\circ v_0),\ u_0 \in \mathcal{D}(A^*),\ v_0 \in \mathcal{D}(A).$$

This identity extends to $u_0, v_0 \in \mathcal{H}$ and gives (2.8). □

Remark 2.20 The last assertion of Proposition 2.19 must be interpreted in the following sense. Since S is closed and densely defined, if we equip $\mathcal{D}(S^*)$ with the graph topology then we get a dense continuous embedding $\mathcal{D}(S^*) \subset \mathcal{H}$. Then, identifying the adjoint space $\mathcal{H}^*$ with $\mathcal{H}$ with the help of the Riesz lemma, we get a dense continuous embedding $\mathcal{H} \subset \mathcal{D}(S^*)^*$. Then the operator $S : \mathcal{D}(S) \to \mathcal{H}$ has a unique extension to a continuous operator $S : \mathcal{H} \to \mathcal{D}(S^*)^*$, namely the adjoint of $S^* : \mathcal{D}(S^*) \to \mathcal{H}$. We similarly get a continuous extension $R(z) : \mathcal{D}(S^*)^* \to \mathcal{H}$, which is the first operator on the right hand side of (2.8).

The next result allows one to deduce that S is of class $C^1(A)$ starting from properties of the formal commutator $[A, S]$ and some (necessary) supplementary conditions. We define $\|u\|_S = (\|u\|^2 + \|Su\|^2)^{1/2}$ and $\|u\|_{S^*}$ similarly.

Proposition 2.21 *Let A and S be closed densely defined operators and let $z \in \mathbb{C}\backslash\sigma(S)$. Assume that:*
(i) *there exists $c \geq 0$ such that $|(A^*u, Sv) - (S^*u, Av)| \leq c\|u\|_{S^*}\|v\|_S$ for all $u \in \mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ and $v \in \mathcal{D}(A) \cap \mathcal{D}(S)$,*
(ii) *$\{u \in \mathcal{D}(A^*) | R(z)^*u \in \mathcal{D}(A^*)\}$ is a core for A^*,*
(iii) *$\{u \in \mathcal{D}(A) | R(z)u \in \mathcal{D}(A)\}$ is a core for A.*
Then $R(z) \in C^1(A)$.

Proof. It is clear that we can assume $z = 0$. Set $R = S^{-1}$ and let $u \in \mathcal{D}(A^*)$ with $R^*u \in \mathcal{D}(A^*)$, $v \in \mathcal{D}(A)$ with $Rv \in \mathcal{D}(A)$. Then:

$$\begin{aligned}(u, [A, R]v) &= (A^*u, Rv) - (R^*u, Av) = (u, ARv) - (A^*R^*u, v) \\ &= (S^*R^*u, ARv) - (A^*R^*u, SRv) = -(R^*u, [A, S]Rv),\end{aligned}$$

because $R^*u \in \mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ and $Rv \in \mathcal{D}(A) \cap \mathcal{D}(S)$. By (i) we have:

$$|(u, [A, R]v)| \leq c\|R^*u\|_{S^*}\|Rv\|_S \leq C\|u\|\|v\|.$$

Since by (ii) and (iii) the space of (u, v) for which this estimate holds is dense in $\mathcal{D}(A^*) \times \mathcal{D}(A)$, we obtain that $R \in C^1(A)$. □

2.4 $C^1(A)$ class of selfadjoint operators

If S is a selfadjoint operator then S is of class $C^1(A)$ if and only if $\rho(S, A)$ contains one of the half-planes $\{\Im z > 0\}$ or $\{\Im z < 0\}$, and S is of full class $C^1(A)$ if and only if $\rho(S, A) = \mathbb{C}\backslash\sigma(S)$. If S has a spectral gap or if A is regular then the two conditions are equivalent. From Proposition 2.19 and Proposition 2.21 we get:

Proposition 2.22 *Let S be a selfadjoint operator on $\mathcal{H}$. Then S is of class $C^1(A)$ if and only if the following two conditions are satisfied:*
(i) *$|(u, [A, S]v)| \leq c\|u\|_S\|v\|_S$ for some $c \geq 0$ and all $u \in \mathcal{D}(A^*) \cap \mathcal{D}(S)$, $v \in \mathcal{D}(A) \cap \mathcal{D}(S)$,*
(ii) *there exists $z \in \mathbb{C}\backslash\sigma(S)$ such that $\{u \in \mathcal{D}(A) \mid R(z)u \in \mathcal{D}(A)\}$ is a core for A and $\{u \in \mathcal{D}(A^*) \mid R(\bar{z})u \in \mathcal{D}(A^*)\}$ is a core for A^*.*

Note that $[A, S]$ is a quadratic form on $[\mathcal{D}(A^*) \cap \mathcal{D}(S)] \times [\mathcal{D}(A) \cap \mathcal{D}(S)]$. If S is of class $C^1(A)$ then $\mathcal{D}(A) \cap \mathcal{D}(S)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(S)$ are cores for S and $[A, S]$ has a unique extension to a continuous sesquilinear form $[A, S]^\circ$ on $\mathcal{D}(S) \times \mathcal{D}(S)$. One has:

$$[A, R(z)]^\circ = -R(z)[A, S]^\circ R(z), \ z \in \rho(S, A).$$

Observe also that $\rho(S, A^*) = \rho(S, A)^*$ (see Remark 2.12(3)). In particular S is of class $C^1(A)$ if and only if it is of class $C^1(A^*)$ and in this case a simple computation gives $[A, S]^{\circ*} = -[A^*, S]^\circ$ as forms on $\mathcal{D}(S)$ or as continuous operators $\mathcal{D}(S) \to \mathcal{D}(S)^*$.

Proposition 2.23 *Let S be selfadjoint operator of full class $C^1(A)$. If $\varphi \in C_0^\infty(\mathbb{R})$ then $\varphi(S) \in C^1(A)$.*

Proof. The proof is based on a representation of $[A, \varphi(S)]$ in terms of $[A, R(z)]^\circ$ with the help of the Helffer-Sjöstrand formula (see [HeS]) and it is quite easy and standard, so we do not give details. We mention, however, the main estimate one needs, namely

$$\|[A, R(z)]^\circ\| = \|(S - \mathrm{i})R(z)[A, R(\mathrm{i})]^\circ(S - \mathrm{i})R(z)\| \leq \|[A, R(\mathrm{i})]^\circ\|(1 + |z - \mathrm{i}||\Im z|^{-1})^2$$

which follows from Proposition 2.13. □

Remark 2.24 The class of functions for φ for which the assertion of the lemma remains true can be considerably extended. For example, it suffices that φ be of class C^3 and such that $|\varphi^{(k)}(s)| \leq c(1 + |s|)^{-k-1-\varepsilon}$ for $0 \leq k \leq 3$ (this requires only a small modification of the proof given on page 43 of [BGS] for the case of selfadjoint A). But we stress that $\varphi(S)$ could be not of class $C^1(A)$ if φ is a symbol of class $S^0(\mathbb{R})$. For example, let S be the operator of multiplication by $1/x$ (arbitrarily defined at $x = 0$) in $L^2(\mathbb{R}, \mathrm{d}x)$ and let $A = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. Then S is of class $C^1(A)$ but $\varphi(S)$ is not of class $C^1(A)$ if φ is a C^∞ function equal to 0 for $x \leq 0$ and to 1 for $x \geq 1$.

Now assume that A is symmetric (closed and densely defined) and $S \in C^1(A)$. Then $[S, \mathrm{i}A]^\circ$ is a symmetric continuous sesquilinear form on $\mathcal{D}(S)$ equal to $[S, \mathrm{i}A^*]^\circ$. Indeed, these forms are continuous and equal on $[\mathcal{D}(A) \cap \mathcal{D}(S)] \times [\mathcal{D}(A) \cap \mathcal{D}(S)]$, which is dense in $\mathcal{D}(S) \times \mathcal{D}(S)$. In particular, there exists $C \geq 0$ such that: $\pm[S, \mathrm{i}A]^\circ \leq C(S^2 + 1)$ as forms on $\mathcal{D}(S)$. We set:

$$\alpha(S, A) := \inf\{\alpha \in \mathbb{R}^+ | \exists \beta \in \mathbb{R}^+ \text{ such that } \pm [S, \mathrm{i}A]^\circ \leq \alpha S^2 + \beta\}.$$

The following theorem is an extension of [Sk2, Lemma 2.6], cf. also [MS, Lemma 2.6]. The main idea of the proof is due to Skibsted but the technical details are rather different, so we give a complete proof.

Theorem 2.25 *Let S be selfadjoint and A symmetric, closed and densely defined. Assume that S is of full class $C^1(A)$, $\mathcal{D}(S) \cap \mathcal{D}(A)$ is a core for A, $\mathcal{D}(S) \cap \mathcal{D}(A) = \mathcal{D}(S) \cap \mathcal{D}(A^*)$ and $\alpha(S, A) < 2/3$. Then $(S \pm \mathrm{i}A)^* = S \mp \mathrm{i}A$, where $\mathcal{D}(S \pm \mathrm{i}A) = \mathcal{D}(S) \cap \mathcal{D}(A)$.*

Proof. The operators $S \pm \mathrm{i}A$ are obviously densely defined and $S \mp \mathrm{i}A \subset (S \pm \mathrm{i}A)^*$, so it suffices to show the opposite inclusion. We shall prove $\mathcal{D}(X^*) = \mathcal{D}(S) \cap \mathcal{D}(A) \equiv \mathcal{D}$, where $X = S + \mathrm{i}A$.

For ν real with $0 < |\nu| \leq 1$ we set $J_\nu = (\mathbb{1} + \mathrm{i}\nu S)^{-1}$. Note that these are the operators introduced in Definition 2.14 for the choice $z_\nu = \mathrm{i}/\nu$, in particular Remark 2.15(1) and Lemma 2.16(ii) are valid. Also $J_\nu^* = J_{-\nu}$ and $\|J_\nu\| \leq 1$.

We shall first prove that $J_\nu \mathcal{D}(X^*) \subset \mathcal{D}$. If $v \in \mathcal{D}(X^*)$ then for $u \in \mathcal{D}$ we have $J_\nu u \in \mathcal{D}$, so:

$$\begin{aligned}
|(Au, J_\nu v)| &= |(J_{-\nu}Au, v)| = |([J_{-\nu}, A]^\circ u + AJ_{-\nu}u, v)| \\
&\leq \|[J_{-\nu}, A]^\circ\|\|u\|\|v\| + |((X - S)J_{-\nu}u, v)| \\
&\leq (\|[J_{-\nu}, A]^\circ\| + \|SJ_{-\nu}\|)\|u\|\|v\| + |(XJ_{-\nu}u, v)| \\
&\leq \Big(\|[J_{-\nu}, A]^\circ\|\|v\| + \|SJ_{-\nu}\|\|v\| + \|X^*v\|\Big)\|u\|.
\end{aligned}$$

Thus we have $|(Au, J_\nu v)| \le C\|u\|$ for a constant C and all $u \in \mathcal{D}$. Since $\mathcal{D}$ is a core for A, we get $J_\nu v \in \mathcal{D}(A^*)$. But $J_\nu v \in \mathcal{D}(S)$ and $\mathcal{D}(S) \cap \mathcal{D}(A^*) = \mathcal{D}(S) \cap \mathcal{D}(A) = \mathcal{D}$. This finishes the proof of the relation $J_\nu \mathcal{D}(X^*) \subset \mathcal{D}$.

Observe that $\mathrm{i}\nu S J_\nu^* J_\nu = J_\nu^* J_\nu - J_\nu$, hence $S J_\nu^* J_\nu \mathcal{D}(X^*) \subset \mathcal{D}$. Thus if $v \in \mathcal{D}(X^*)$ we get

$$|(XSJ_\nu^* J_\nu v, v)| \le \|SJ_\nu^* J_\nu v\| \|X^* v\| \le \|SJ_\nu v\| \|X^* v\|. \tag{2.11}$$

Let $L = SJ_\nu^* J_\nu$, then

$$\Re(XSJ_\nu^* J_\nu v, v) = \|SJ_\nu v\|^2 + \Re(\mathrm{i}ALv, v). \tag{2.12}$$

We shall compute the last term as follows. We know that $\text{s-}\lim_{\varepsilon\to 0} J_\varepsilon = 1$ in $B(\mathcal{D})$ (see Lemma 2.16(ii)). Since $Lv \in \mathcal{D}$ we get $Lv = \lim_{\varepsilon\to0} J_\varepsilon Lv = \lim_{\varepsilon\to0} Lv_\varepsilon$ strongly in $\mathcal{D}(A)$, where $v_\varepsilon = J_\varepsilon v \in \mathcal{D}$. Now clearly

$$2\Re(\mathrm{i}ALv_\varepsilon, v_\varepsilon) = (\mathrm{i}ALv_\varepsilon, v_\varepsilon) + (v_\varepsilon, \mathrm{i}ALv_\varepsilon) = \mathrm{i}(Av_\varepsilon, Lv_\varepsilon) - \mathrm{i}(Lv_\varepsilon, Av_\varepsilon) = \mathrm{i}(v_\varepsilon, [A, L]^\circ v_\varepsilon).$$

Taking the limit $\varepsilon \to 0$ here we obtain $2\Re(\mathrm{i}ALv, v) = \mathrm{i}(v, [A, L]^\circ v)$ because $[A, L]^\circ$ is a bounded operator (L being a linear combination of resolvents of S). Then (2.12) can be written as

$$\Re(XSJ_\nu^* J_\nu v, v) = \|SJ_\nu v\|^2 + \frac{1}{2}(v, [\mathrm{i}A, L]^\circ v).$$

Now from (2.11) we get

$$\|SJ_\nu v\|^2 - \frac{1}{2}|(v, [\mathrm{i}A, L]^\circ v)| \le \|SJ_\nu v\| \|X^* v\|. \tag{2.13}$$

A straightforward computation based on the relation $\mathrm{i}\nu L = J_\nu^* J_\nu - J_\nu$ and involving Propositions 2.6 and 2.19 gives

$$[\mathrm{i}A, L]^\circ = J_\nu^* [\mathrm{i}A, S]^\circ J_\nu + J_\nu^* [\mathrm{i}A, \mathrm{i}\nu S]^\circ J_\nu^* S J_\nu - J_\nu^* S J_\nu [\mathrm{i}A, \mathrm{i}\nu S]^\circ J_\nu.$$

Set $K_\nu = \mathrm{i}\nu S J_\nu^*$. We then have

$$(v, [\mathrm{i}A, L]^\circ v) = (v_\nu, [\mathrm{i}A, S]^\circ v_\nu) + (v_\nu, [\mathrm{i}A, S]^\circ K_\nu v_\nu) + (K_\nu v_\nu, [\mathrm{i}A, S]^\circ v_\nu).$$

But $[\mathrm{i}A, S]^\circ$ is a symmetric form on $\mathcal{D}(S)$, hence

$$(v, [\mathrm{i}A, L]^\circ v) = (v_\nu, [\mathrm{i}A, S]^\circ v_\nu) + 2\Re(K_\nu v_\nu, [\mathrm{i}A, S]^\circ v_\nu). \tag{2.14}$$

The assumption $\alpha(S, A) < 2/3$ implies the existence of $\alpha \in]0, 2/3[$ and $\beta \ge 0$ such that $\pm[\mathrm{i}A, S]^\circ \le \alpha(S^2 + \beta^2)$. Then for all $f, g \in \mathcal{D}(S)$ we shall have

$$|(f, [\mathrm{i}A, S]^\circ g)| \le \alpha \|(S^2 + \beta^2)^{1/2} f\| \|(S^2 + \beta^2)^{1/2} g\|.$$

Taking into account that $\|K_\nu\| \le 1$, the relation (2.14) gives

$$|(v, [\mathrm{i}A, L]^\circ v)| \le 3\alpha \|(S^2 + \beta^2)^{1/2} v_\nu\|^2.$$

We insert this estimate into (2.13) and get

$$\|Sv_\nu\|^2 - \frac{3\alpha}{2} \|(S^2 + \beta^2)^{1/2} v_\nu\|^2 \le \|Sv_\nu\| \|X^* v\|,$$

which can be written as

$$(1-\frac{3\alpha}{2})\|Sv_\nu\|^2 \leq \|Sv_\nu\|\|X^*v\| + \frac{3\alpha\beta^2}{2}\|v_\nu\|^2.$$

Since $3\alpha/2 < 1$ and $\|v_\nu\| \leq \|v\|$ this implies that $\|Sv_\nu\| \leq C$ for a constant C. Letting $\nu \to 0$ we obtain $v \in \mathcal{D}(S)$ for each $v \in \mathcal{D}(X^*)$.

Finally, since we have $|(Xu, v)| \leq \|u\|\|X^*v\|$, we get $|(Au, v)| \leq \|u\|(\|X^*v\| + \|Sv\|)$ for all $u \in \mathcal{D}$. But $\mathcal{D}$ is a core for A, hence $v \in \mathcal{D}(A^*)$. So $v \in \mathcal{D}(S) \cap \mathcal{D}(A^*) = \mathcal{D}$. □

The following lemma will be used in [GGM] to check the hypotheses of Theorem 2.25 in a concrete situation.

Lemma 2.26 *Let S and M be selfadjoint operators such that $S \in C^1(M)$ and $\mathcal{D}(S) \cap \mathcal{D}(M)$ is a core for M. Let R be a symmetric operator with $\mathcal{D}(R) \supset \mathcal{D}(S)$ and let us denote by A the closure of the operator $M + R$ defined on $\mathcal{D}(S) \cap \mathcal{D}(M)$. Then A is symmetric closed and densely defined, S is of full class $C^1(A)$, and:*

$$\mathcal{D}(S) \cap \mathcal{D}(A) = \mathcal{D}(S) \cap \mathcal{D}(A^*) = \mathcal{D}(S) \cap \mathcal{D}(M). \tag{2.15}$$

Proof. Clearly A is symmetric, closed and densely defined and $\mathcal{D}(S) \cap \mathcal{D}(M)$ is a core for A. Let us now check that $\mathcal{D}(S) \cap \mathcal{D}(A) = \mathcal{D}(S) \cap \mathcal{D}(A^*) = \mathcal{D}(S) \cap \mathcal{D}(M)$. Clearly $\mathcal{D}(S) \cap \mathcal{D}(M) \subset \mathcal{D}(S) \cap \mathcal{D}(A) \subset \mathcal{D}(S) \cap \mathcal{D}(A^*)$ since A is symmetric, so it suffices to check that $\mathcal{D}(S) \cap \mathcal{D}(A^*) \subset \mathcal{D}(M)$. If $u \in \mathcal{D}(S) \cap \mathcal{D}(A^*)$, we have $|(Av, u)| \leq C\|v\|$ for $v \in \mathcal{D}(A)$. In particular, if $v \in \mathcal{D}(S) \cap \mathcal{D}(M)$ this yields:

$$|(Mv, u)| \leq C\|v\| + \|Ru\|\|v\|. \tag{2.16}$$

But $\mathcal{D}(S) \cap \mathcal{D}(M)$ is a core for M by hypothesis, hence (2.16) extends to all $v \in \mathcal{D}(M)$, which implies that $u \in \mathcal{D}(M)$.

It remains to prove that S is of full class $C^1(A)$, i.e. that for each $z \in \mathbb{C} \backslash \sigma(S)$ the operator $T \equiv (S - z)^{-1}$ is of class $C^1(A)$. We could use Lemma 2.3, but a direct check of the conditions of Definition 2.1 is easy. We consider the quadratic form $[A, T]$ on $\mathcal{D}(A^*) \times \mathcal{D}(A)$. Using that $T : \mathcal{D}(S) \cap \mathcal{D}(M) \to \mathcal{D}(S) \cap \mathcal{D}(M)$, we have:

$$(v, [A, T]u) = (v, ATu) - (v, TAu) = (v, (M + R)Tu) - (v, T(M + R)u),$$

for $v \in \mathcal{D}(A^*)$, $u \in \mathcal{D}(S) \cap \mathcal{D}(M)$. Now we note that since $S \in C^1(M)$, we have $T : \mathcal{D}(M) \to \mathcal{D}(M)$ and $[M, T]$, a well defined operator on $\mathcal{D}(M)$, extends to a bounded operator on $\mathcal{H}$. Using also the fact that $\mathcal{D}(S) \subset \mathcal{D}(R)$, we obtain that

$$|(v, [A, T]u)| \leq C\|v\|\|u\|, \; v \in \mathcal{D}(A^*), \; u \in \mathcal{D}(S) \cap \mathcal{D}(M).$$

Since $\mathcal{D}(S) \cap \mathcal{D}(M)$ is a core for A, this proves that $T \in C^1(A)$. □

2.5 Regularity with respect to C_0-semigroups

In this subsection we study the $C^1(A)$ class when A is the generator of a C_0-semigroup. We first recall the definition of such a semigroup in a version convenient in our context.

Definition 2.27 *A map $\mathbb{R}^+ \ni t \mapsto W_t \in B(\mathcal{H})$ is a* C_0-semigroup *if:*
(i) $W_0 = \mathbb{1}$, $W_t W_s = W_{t+s}$, $t, s \geq 0$;
(ii) $\text{w-}\lim_{t \to 0^+} W_t = \mathbb{1}$.

Then by [HP, Theorem 10.6.5], the map $\mathbb{R}^+ \ni t \mapsto W_t \in B(\mathcal{H})$ is strongly continuous, hence we get the more usual version of the definition. For an elementary introduction to the theory of C_0-semigroups we refer to [RS], here we recall only some basic facts.

We define the generator A of $\{W_t\}$ by the rule

$$\mathcal{D}(A) := \{u \in \mathcal{H} \mid \lim_{t \to 0^+} (\mathrm{i}t)^{-1}(W_t u - u) =: Au \text{ exists}\}.$$

Thus we formally have $W_t = \mathrm{e}^{\mathrm{i}tA}$, which is not the usual convention but is natural in our context. Note that $\mathcal{D}(A)$ can also be characterized as:

$$\mathcal{D}(A) = \{u \in \mathcal{H} | \text{ there is a number } C \text{ such that } \|W_t u - u\| \leq Ct \text{ if } 0 \leq t \leq 1\}.$$

The generator A is closed and densely defined. It is easy to see that there are real numbers M, ω such that $\|W_t\| \leq M\mathrm{e}^{\omega t}$. In particular, if z is a complex number such that $\Re z > \omega$, then z belongs to the resolvent set of A and $(A-z)^{-1} = \mathrm{i}\int_0^\infty W_t \mathrm{e}^{-\mathrm{i}tz}\mathrm{d}z$, so $\|(A-z)^{-1}\| \leq M(\Re z - \omega)^{-1}$. This clearly implies that A is regular, so by Remark 2.15(4) *the full $C^1(A)$ class coincides with the $C^1(A)$ class.*

The map $\mathbb{R}^+ \ni t \mapsto W_t^* \in B(\mathcal{H})$ is weakly continuous, hence defines a C_0-semigroup. It is easy to see that the generator of W_t^* is $-A^*$.

Before going on into more technical aspects of the theory let us point out a formal relation, reminiscent to Duhamel's formula, which will play an important role below: if $S \in B(\mathcal{H})$ then

$$[S, W_t] = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} W_{t-s} S W_s \mathrm{d}s = \int_0^t W_{t-s}[S, \mathrm{i}A] W_s \mathrm{d}s \tag{2.17}$$

for all $t \geq 0$. This formal computation, and natural extensions, will be rigorously justified when we shall use it.

Definition 2.28 *Let $\{W_{1,t}\}, \{W_{2,t}\}$ be two C_0-semigroups on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with generators A_1, A_2. We say that $S \in B(\mathcal{H}_1, \mathcal{H}_2)$ is* of class $C^1(A_1, A_2)$ *if:*

$$\|W_{2,t} S - S W_{1,t}\|_{B(\mathcal{H}_1, \mathcal{H}_2)} \leq Ct, \ 0 \leq t \leq 1.$$

Proposition 2.29 *S is of class $C^1(A_1, A_2)$ if and only if the sesquilinear form ${}_2[S, A]_1$ on $\mathcal{D}(A_2^*) \times \mathcal{D}(A_1)$ defined by $(u_2, {}_2[S, A]_1 u_1) = (S^* u_2, A_1 u_1) - (A_2^* u_2, S u_1)$ is bounded for the topology of $\mathcal{H}_2 \times \mathcal{H}_1$. If we denote by ${}_2[S, A]_1^0 \in B(\mathcal{H}_1, \mathcal{H}_2)$ the associated operator we have:*

$${}_2[S, A]_1^0 = \text{s-}\lim_{t \to 0^+} (\mathrm{i}t)^{-1}(S W_{1,t} - W_{2,t} S). \tag{2.18}$$

Proof. Assume first that the sesquilinear form ${}_2[S, A]_1$ is bounded for the topology of $\mathcal{H}_2 \times \mathcal{H}_1$ and let $u_1 \in \mathcal{D}(A_1)$, $u_2 \in \mathcal{D}(A_2^*)$. We have:

$$\begin{aligned}
(u_2, (S W_{1,t} - W_{2,t} S) u_1) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(W_{2,t-s}^* u_2, S W_{1,s} u_1)\mathrm{d}s \\
&= \int_0^t (\mathrm{i}A_2^* W_{2,t-s}^* u_2, S W_{1,s} u_1) + (W_{2,t-s}^* u_2, S\mathrm{i}A_1 W_{1,s} u_1)\mathrm{d}s \\
&= \int_0^t (W_{2,t-s}^* u_2, {}_2[S, \mathrm{i}A]_1 W_{1,s} u_1)\mathrm{d}s \\
&= \int_0^t (W_{2,t-s}^* u_2, {}_2[S, \mathrm{i}A]_1^\circ W_{1,s} u_1)\mathrm{d}s.
\end{aligned}$$

This gives:

$$SW_{1,t} - W_{2,t}S = \int_0^t W_{2,t-s\,2}[S, \mathrm{i}A]_1^\circ W_{1,s}\mathrm{d}s, \tag{2.19}$$

as a strong integral, and hence:

$$\begin{aligned} \|SW_{1,t} - W_{2,t}S\| &\leq \|{}_2[S,A]_1^0\| \int_0^t M_1 M_2 \mathrm{e}^{\omega_2(t-s)}\mathrm{e}^{\omega_1 s}\mathrm{d}s \\ &\leq C\|{}_2[S,A]_1^0\| t \text{ for } 0 \leq t \leq 1. \end{aligned}$$

This shows that S is of class $C^1(A_1, A_2)$. It follows also from (2.19) and the fact that $\{W_{1,t}\}$ and $\{W_{2,t}\}$ are C_0-semigroups that (2.18) holds.

It remains to prove the converse implication. Assume that $\|SW_{1,t} - W_{2,t}S\| \leq Ct$ for $0 \leq t \leq 1$. For $u_1 \in \mathcal{D}(A_1)$ and $u_2 \in \mathcal{D}(A_2^*)$ we have:

$$(u_{2,2}[S, \mathrm{i}A]_1 u_1) = (u_2, \mathrm{i}SA_1u_1) + (\mathrm{i}A_2^* u_2, Su_1) = \lim_{t\to 0^+} t^{-1}(u_2, SW_{1,t} - W_{2,t}Su_1).$$

Since $\|SW_{1,t} - W_{2,t}S\| \leq Ct$ for $0 \leq t \leq 1$ we obtain that $|(u_2, {}_2[S,A]_1u_1)| \leq C\|u_2\|\|u_1\|$. □

Remark 2.30 In particular, if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and $W_{1,t} = W_{2,t} = W_t$ then $C^1(A_1, A_2) = C^1(A)$.

We give now a simple and natural characterization of the $C^1(A)$ property when A is the generator of a C_0-semigroup. Note that this allows one to extend the notion to arbitrary closed and densely defined operators.

Proposition 2.31 *Let S be a closed densely defined regular operator. Then S is of class $C^1(A)$ if and only if for each $u \in \mathcal{D}(S^*), v \in \mathcal{D}(S)$ there is $c < \infty$ such that $|(S^*u, W_tv) - (u, W_tSv)| \leq ct$ if $0 \leq t \leq 1$. If this is the case, then $\lim_{t\to 0^+} t^{-1}[(S^*u, W_tv) - (u, W_tSv)] = (u, [S, \mathrm{i}A]^\circ v)$.*

Proof. Suppose first that S is of class $C^1(A)$. Let $z \in \mathbb{C} \setminus \sigma(S)$ and $R = (S - z)^{-1}$. If $u', v' \in \mathcal{H}$ then

$$\begin{aligned} (u', [R, \frac{1}{t}W_t]v') &= (R^*u', \frac{1}{t}W_t(S-z)Rv') - ((S^* - \bar{z})R^*u', \frac{1}{t}W_tRu') \\ &= (R^*u', \frac{1}{t}W_tSRv') - (S^*R^*u', \frac{1}{t}W_tRu'). \end{aligned}$$

But $u = R^*u' \in \mathcal{D}(S^*)$ and $v = Rv' \in \mathcal{D}(S)$, so the right hand side is bounded by a constant $c(u', v')$ for $0 < t \leq 1$. The uniform boundedness principle (for sesquilinear forms) implies $\sup_{0<t\leq 1} \|[R, \frac{1}{t}W_t]\| < \infty$. From Proposition 2.29 we get $R \in C^1(A)$ and

$$(u', [R, \mathrm{i}A]^\circ v') = \lim_{t\to 0} \frac{1}{t}[(R^*u', W_tSRv') - (S^*R^*u', W_tRu')].$$

The last formula from the statement of the proposition follows now from Proposition 2.19. The reciprocal assertion can be proved in the same way. □

Let $\mathcal{G}, \mathcal{H}$ be two Hilbert spaces with $\mathcal{G} \subset \mathcal{H}$ continuously and densely. We identify the adjoint space $\mathcal{H}^*$ with $\mathcal{H}$ by using the Riesz isomorphism. Then by taking adjoints we get a scale of Hilbert spaces with dense and continuous embeddings $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$.

Definition 2.32 *Let $\mathcal{G}, \mathcal{H}$ be as above and let $\{W_t\}$ be a C_0-semigroup in $\mathcal{H}$. We say that $\mathcal{G}$ is* b-stable (boundedly stable) *under $\{W_t\}$, or that $\{W_t\}$* b-preserves $\mathcal{G}$, *if $W_t\mathcal{G} \subset \mathcal{G}$ for all $t > 0$ and $\sup_{0<t<1} \|W_t u\|_{\mathcal{G}} < \infty$ for each $u \in \mathcal{G}$.*

Observe that from the closed graph theorem it follows that $W_t^\circ := W_{t|\mathcal{G}} \in B(\mathcal{G})$.

Lemma 2.33 *If $\mathcal{G}$ is* b-*stable under $\{W_t\}$ then $\{W_t^\circ\}$ is a C_0-semigroup in $\mathcal{G}$.*

Proof. We must show $\lim_{t\to 0}(W_t u, v) = (u, v)$ if $u \in \mathcal{G}, v \in \mathcal{G}^*$. Since $\sup_{0<t<1} \|W_t u\|_{\mathcal{G}} < \infty$ it suffices to prove this for v in the dense subspace $\mathcal{H}$ of $\mathcal{G}^*$, and then it is obvious. □

Note that if we only require that $W_t\mathcal{G} \subset \mathcal{G}$ for all t, then it is not difficult to show that $t \mapsto W_t^\circ \in B(\mathcal{G})$ is strongly continuous on $]0, \infty[$ (use [HP, Theorem 10.2.3]). However, it is not clear whether this map is continuous at $t = 0$.

In general we shall not distinguish $\{W_t^\circ\}$ from $\{W_t\}$ since it will be clear from the context which of the semigroups is involved. If needed, we denote by $A_{\mathcal{G}}$ the generator of $\{W_t^\circ\}$. It is not difficult to show that

$$(2.20) \quad \mathcal{D}(A_{\mathcal{G}}) = \mathcal{D}(A;\mathcal{G}) := \{u \in \mathcal{G} \cap \mathcal{D}(A) \mid Au \in \mathcal{G}\} \text{ and } A_{\mathcal{G}}u = Au \text{ if } u \in \mathcal{D}(A_{\mathcal{G}}).$$

The space $C^1(A_{\mathcal{G}}, A)$ will be denoted by $C^1(A; \mathcal{G}, \mathcal{H})$.

It is easy to show that $\{W_t\}$ extends to a C_0-semigroup in $\mathcal{G}^*$ (i.e. each W_t extends to a continuous operator in $\mathcal{G}^*$ and these extensions form a C_0-semigroup in $\mathcal{G}^*$) if and only if $\mathcal{G}$ is b-stable under $\{W_t^*\}$. The extended semigroup is also denoted by $\{W_t\}$ but its generator is sometimes denoted $A_{\mathcal{G}^*}$. It is easy to verify that A is the closure of $A_{\mathcal{G}}$ considered as acting in $\mathcal{H}$ and that $A_{\mathcal{G}^*}$ is the closure of A considered as acting in $\mathcal{G}^*$. In particular $\mathcal{D}(A_{\mathcal{G}})$ is dense in $\mathcal{D}(A)$ and $\mathcal{D}(A)$ is dense in $\mathcal{D}(A_{\mathcal{G}^*})$. The space $C^1(A_{\mathcal{G}}, A_{\mathcal{G}^*})$ will be denoted by $C^1(A; \mathcal{G}, \mathcal{G}^*)$.

The next proposition is analogous to [GG, Lemma 2].

Proposition 2.34 *Let A be the generator of the C_0-semigroup W_t and S a selfadjoint operator on $\mathcal{H}$. Then the following properties are equivalent:*
(i) *S is of class $C^1(A)$ and $[S, A]^\circ$ is bounded for the topology of $\mathcal{H} \times \mathcal{D}(S)$,*
(ii) $\{W_t\}$ b-*preserves $\mathcal{D}(S)$ and S is of class $C^1(A; \mathcal{D}(S), \mathcal{H})$.*

Proof. Let us first prove that (i) implies (ii). Replacing W_t by $W_t e^{-\omega t}$, which amounts to adding a constant to A and does not change $[A, S]$, we may assume that $\|W_t\| \le M$ for $t \ge 0$. Let for $\nu > 0$:

$$J_\nu := (\mathbb{1} + i\nu S)^{-1},\ S_\nu := SJ_\nu = -i\nu^{-1}\mathbb{1} + i\nu^{-1}J_\nu.$$

Then $S_\nu \in C^1(A)$ and by (2.19) for $W_{1t} = W_{2t} = W_t$, we obtain:

$$(2.21) \qquad S_\nu W_t u - W_t S_\nu u = \int_0^t W_{t-s}[S_\nu, iA]^\circ W_s u \mathrm{d}s.$$

By Proposition 2.19, we have:

$$(2.22) \qquad [S_\nu, iA]^\circ = J_\nu[S, iA]^\circ J_\nu = J_\nu[S, iA]^\circ (S + i)^{-1}(S + i)J_\nu = J_\nu T S_\nu + iJ_\nu T J_\nu,$$

for $T = [S, iA]^\circ (S + i)^{-1} \in B(\mathcal{H})$. This yields since $\|J_\nu\| \le 1$:

$$\begin{aligned} \|S_\nu W_t u\| \le & \ \|W_t S_\nu u\| + \textstyle\int_0^t M\|T\|(\|S_\nu W_s u\| + \|W_s u\|)\mathrm{d}s \\ \le & \ M\|S_\nu u\| + \textstyle\int_0^t M\|T\|(\|S_\nu W_s u\| + M\|u\|)\mathrm{d}s, \end{aligned}$$

uniformly in $0 < \nu \leq 1$. By Gronwall's lemma, we obtain:

$$\|S_\nu W_t u\| \leq C\|S_\nu u\| + Ct\|u\|, \ \ 0 \leq t \leq 1, \tag{2.23}$$

uniformly in $0 < \nu \leq 1$. By Fatou's lemma we deduce from (2.23) that $W_t u \in \mathcal{D}(S)$ if $u \in \mathcal{D}(S)$ and that $\sup_{0\leq t\leq 1} \|SW_t u\| < \infty$. It remains to prove that $S \in C^1(A; \mathcal{D}(S), \mathcal{H})$, i.e.

$$\|(W_t S - SW_t)(S + \mathrm{i})^{-1}\|_{B(\mathcal{H})} \leq Ct, \ \ 0 \leq t \leq 1. \tag{2.24}$$

We first note that it follows from (2.22) that $[S_\nu, \mathrm{i}A](S + \mathrm{i})^{-1}$ is uniformly bounded in ν. Now for $u \in \mathcal{D}(S)$, we have since $W_t u \in \mathcal{D}(S)$:

$$\begin{aligned}
\|(SW_t - W_t S)u\| = \ & \lim_{\nu\to 0} \|(S_\nu W_t - W_t S_\nu)u\| \\
\leq \ & \sup_{0<\nu\leq 1} \textstyle\int_0^t \|W_{t-s}[S_\nu, \mathrm{i}A]^\circ W_s u\| \mathrm{d}s \\
\leq \ & Mt \sup_{0<\nu\leq 1} \|[S_\nu, \mathrm{i}A](S + \mathrm{i})^{-1}\| \sup_{0\leq s\leq 1} \|W_s\|_{B(\mathcal{D}(S))} \|u\|_{\mathcal{D}(S)},
\end{aligned}$$

which proves (2.24).

Let us now prove that (ii) implies (i). Assume that $S \in C^1(A; \mathcal{D}(S), \mathcal{H})$. By Definition 2.28 we have for $z \in \mathbb{C}\backslash\sigma(S)$:

$$\|(W_t S - SW_t)(S - z)^{-1}\| \leq Ct, \ 0 \leq t \leq 1. \tag{2.25}$$

Now

$$\begin{aligned}
& (S - z)^{-1} W_t S(S - z)^{-1} - (S - z)^{-1} SW_t (S - z)^{-1} \\
= \ & (S - z)^{-1} W_t (\mathbb{1} + z(S - z)^{-1}) - (\mathbb{1} + z(S - z)^{-1}) W_t (S - z)^{-1} \\
= \ & (S - z)^{-1} W_t - W_t (S - z)^{-1},
\end{aligned}$$

hence by (2.25) we have:

$$\|(S - z)^{-1} W_t - W_t (S - z)^{-1}\| \leq Ct, \ 0 \leq t \leq 1,$$

i.e. $(S - z)^{-1} \in C^1(A; \mathcal{H}, \mathcal{H})$ for all $z \in \mathbb{C}\backslash\sigma(S)$ and hence $S \in C^1(A)$ by Remark 2.30.

It remains to prove that $[S, A]^\circ (S + \mathrm{i})^{-1} \in B(\mathcal{H})$. Let A_S be the generator of $\{W_t\}$ as a C_0-semigroup on $\mathcal{D}(S)$. We recall that (see (2.20)):

$$\mathcal{D}(A_S) = \{u \in \mathcal{D}(A) \cap \mathcal{D}(S) | Au \in \mathcal{D}(S)\}, \ A_S u = Au \text{ for } u \in \mathcal{D}(A_S).$$

We first claim that $\mathcal{D}(A_S)$ is dense in $\mathcal{D}(A) \cap \mathcal{D}(S)$ for the intersection topology. In fact let $u \in \mathcal{D}(A) \cap \mathcal{D}(S)$ and set

$$T_\nu u := \nu^{-1} \int_0^\nu W_t u \mathrm{d}t, \ \nu > 0.$$

Then $T_\nu u \in \mathcal{D}(A_S)$ and $T_\nu u \to u$ in $\mathcal{D}(S)$ when $\nu \to 0$ since $\{W_t\}$ is a C_0-semigroup on $\mathcal{D}(S)$. Moreover, since $[A, T_\nu] = 0$, we have $T_\nu u \to u$ in $\mathcal{D}(A)$ when $\nu \to 0$.

By Proposition 2.29 we have:

$$|(A^* v, Su) - (v, SAu)| \leq C\|(S + \mathrm{i})u\|\|v\|, \ v \in \mathcal{D}(A^*), \ u \in \mathcal{D}(A_S).$$

If $v \in \mathcal{D}(A^*) \cap \mathcal{D}(S)$ and $u \in \mathcal{D}(A_S)$ we get:

$$|(v, [A, S]u)| \leq C\|(S + \mathrm{i})u\|\|v\|. \tag{2.26}$$

Since we have seen that $\mathcal{D}(A_S)$ is dense in $\mathcal{D}(A) \cap \mathcal{D}(S)$ for the intersection topology, we see that (2.26) extends to $v \in \mathcal{D}(A^*) \cap \mathcal{D}(S)$, $u \in \mathcal{D}(A) \cap \mathcal{D}(S)$, i.e. that $[S, A]^\circ (S + \mathrm{i})^{-1} \in B(\mathcal{H})$. This completes the proof of the lemma. □

Remark 2.35 This remark is relevant in the context of Theorem 2.25 and of Hypothesis **(M1)** of Subsection 3.1. We saw that if S is of class $C^1(A)$ then $\mathcal{D}(A) \cap \mathcal{D}(S)$ is a core of S but not of A in general (see Remark 2.17). However, *if A is the generator of a C_0-semigroup W_t and if $W_t\mathcal{D}(S) \subset \mathcal{D}(S)$ for all $t > 0$, then $\mathcal{D}(A) \cap \mathcal{D}(S)$ is a core of A.* This is an immediate consequence of the following lemma due to E. Nelson: if $\mathcal{K} \subset \mathcal{D}(A)$ is dense in $\mathcal{H}$ and $W_t\mathcal{K} \subset \mathcal{K}$ for all $t > 0$, then $\mathcal{K}$ is a core of A (see [BR, Corollary 3.1.7]).

We end this section with some comments concerning the case when the operator A is symmetric, closed and densely defined. We recall that such an operator is a generator of a C_0-semigroup if and only if it has deficiency $(N, 0)$ for some cardinal N and that $\{W_t\}$ is then a C_0-semigroup of isometries. Assuming that this is the case, let $S \in B(\mathcal{H})$ and let us consider the following conditions: there is $C > 0$ such that

$$(1)\ \|W_t^* S W_t - S\| \leq Ct;\ \ (2)\ \|[S, W_t]\| \leq Ct;\ \ (3)\ \|[S, W_t^*]\| \leq Ct;\ \ (4)\ \|W_t S W_t^* - S\| \leq Ct$$

for $0 \leq t \leq 1$. Our definition of the class $C^1(A)$ amounts to ask (2) to hold and (3) is equivalent to $S \in C^1(A^*)$. On the other hand, since $W_t^* W_t = \mathbb{1}$ we have

$$\|W_t^* S W_t - S\| = \|W_t^*(SW_t - W_t S)\| \leq \|SW_t - W_t S\| = \|(S - W_t S W_t^*)W_t\| \leq \|W_t S W_t^* - S\|$$

hence (4)⇒(2)⇒(1). Taking adjoints we also get (4)⇒(3)⇒(1). It is easy to find examples which show that these implications are strict if A is not selfadjoint.

It is possible to describe (1) and (4) directly in terms of A and to obtain characterizations similar to Definition 2.1 and Proposition 2.2 in the case of condition (2) (the corresponding facts in the case (3) follow from $S \in C^1(A^*)$). For this we use the rigorous quadratic form version of the formal relation

$$W_t^* S W_t - S = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} W_s^* S W_s \mathrm{d}s = -\mathrm{i} \int_0^t W_s^*(A^* S - SA) W_s \mathrm{d}s$$

and arguments similar to those of the proof of Proposition 2.29. Thus we see that (1) is equivalent to the fact that the sesquilinear form $(Au, Sv) - (S^* u, Av)$ with domain $\mathcal{D}(A) \times \mathcal{D}(A)$ is continuous for the topology of $\mathcal{H} \times \mathcal{H}$. And this is easily seen to be equivalent to: $S\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $A^* S - SA^* : \mathcal{D}(A) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$. Similarly we see that (4) is equivalent to the fact that the sesquilinear form $(A^* u, Sv) - (S^* u, A^* v)$ with domain $\mathcal{D}(A^*) \times \mathcal{D}(A^*)$ is continuous for the topology of $\mathcal{H} \times \mathcal{H}$, which in turn is equivalent to: $S\mathcal{D}(A^*) \subset \mathcal{D}(A)$ and $A^* S - SA^* : \mathcal{D}(A^*) \to \mathcal{H}$ extends to a bounded operator on $\mathcal{H}$.

A condition like (1) seems too weak for our purposes because an operator S satisfying it does not leave the domain of A invariant. For example, if S is the orthogonal projection on the subspace generated by a vector $f \in \mathcal{H}$, then (1) is equivalent to $f \in \mathcal{D}(A^*)$ and (2) is equivalent to $f \in \mathcal{D}(A)$; so if $f \in \mathcal{D}(A^*) \setminus \mathcal{D}(A)$ we have (1) but $S\mathcal{D}(A) \not\subset \mathcal{D}(A)$.

On the other hand, the most restrictive condition (4) has the interesting feature that it is expressed in terms of the map $S \mapsto W_t S W_t^*$ which is a $*$-morphism (not unital if A is not selfadjoint) of the C^*-algebra $B(\mathcal{H})$, and this could be useful in an algebraic setting.

3 Boundary values of resolvent families

3.1 Hypotheses

Let us first introduce the abstract set of hypotheses under which we will study the boundary values of the resolvent $R(z) := (H-z)^{-1}$, $z \in \mathbb{C}\backslash\sigma(H)$, of a selfadjoint operator H. We consider three operators H, H' and A such that *H is selfadjoint, H' is symmetric closed and densely defined, and A is closed and densely defined.* Note that one of the conditions below says that H' is a realization of the formal commutator $[H, \mathrm{i}A]$. We set $\mathcal{D} := \mathcal{D}(H) \cap \mathcal{D}(H')$ (equipped with the intersection topology).

Our first two assumptions are:

(M1) *H is of full class $C^1(H')$, $\mathcal{D}$ is a core of H', and $\mathcal{D}(H) \cap \mathcal{D}(H'^*) = \mathcal{D}$.*

(M2) *A bounded open set $J \subset \mathbb{R}$ is given and there are numbers $a, b > 0$ such that the inequality $H' \geq [a\mathbb{1}_J(H) - b\mathbb{1}_{\mathbb{R}\backslash J}(H)]\langle H\rangle$ holds in the sense of forms on $\mathcal{D}$.*

These are the most important hypotheses and the only ones used in the proof of our main estimates (established in Subsection 3.4). Note that they do not explicitly involve the operator A.

There is a rather large freedom in the choice of the other assumptions, as we shall see later on. We consider here a set of conditions convenient for later applications (see [GGM]). We first introduce some notations. Observe that the next condition (which will be of independent interest later on) is a consequence of hypothesis **(M2)** (e.g. let $c = b + 1$).

(M2′) *There exists a number $c > 0$ such that $H' + c\langle H\rangle \geq \langle H\rangle$ as forms on $\mathcal{D}$.*

We consider a new norm on $\mathcal{D}$, namely

(3.1) $$\|u\|_{\mathcal{G}} = \sqrt{(u, (H' + c\langle H\rangle)u)}.$$

and we introduce the new space

(3.2) $$\mathcal{G} := \text{completion of } (\mathcal{D}, \|\cdot\|_{\mathcal{G}}).$$

Observe that the topology on $\mathcal{D}$ associated to the norm (3.1) is independent of the choice of c (two different c's produce equivalent norms). Let G be the Friedrichs extension of the positive operator $H' + c\langle H\rangle$ on $\mathcal{D}$. Then G is a selfadjoint operator satisfying $G \geq \langle H\rangle$ and $\|u\|_{\mathcal{G}} = \|\sqrt{G}u\|$. It follows that (3.1) is a closeable norm, in particular $\mathcal{G} = \mathcal{D}(G^{1/2})$ is embedded in $\mathcal{H}$.

We shall denote by $\|\cdot\|_{\mathcal{G}^*}$ the norm dual to $\|\cdot\|_{\mathcal{G}}$. Thus for $v \in \mathcal{H}$

$$\|v\|_{\mathcal{G}^*} = \sup\{|(u, v)| \mid u \in \mathcal{D}, \|u\|_{\mathcal{G}} \leq 1\} = \|G^{-1/2}v\|.$$

The completion of $(\mathcal{H}, \|\cdot\|_{\mathcal{G}^*})$ is canonically identified with the adjoint space $\mathcal{G}^*$. Thus we get a scale of spaces

(3.3) $$\mathcal{D} \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \subset \mathcal{D}^*$$

with dense and continuous embeddings.

We note that *H and H' extend to continuous symmetric operators $\mathcal{G} \to \mathcal{G}^*$* (the extensions will be denoted by the same symbols). Indeed, this is an immediate consequence of the inequalities $\pm H \leq \langle H\rangle \leq G$ and $-bG \leq -b\langle H\rangle \leq H' \leq G$.

We can now state our last hypotheses.

(M3) *A is the generator of a C_0-semigroup $\{W_t\}_{t\geq 0}$ in $\mathcal{H}$.*

(M4) *For all $u\in\mathcal{D}$ we have: $\lim_{t\to 0^+}\frac{1}{t}\left[(Hu,W_tu)-(u,W_tHu)\right]=(u,H'u)$.*

(M5) *There is $H''\in B(\mathcal{G},\mathcal{G}^*)$ such that $\lim_{t\to 0^+}\frac{1}{t}\left[(H'u,W_tu)-(u,W_tH'u)\right]=(u,H''u), u\in\mathcal{D}$.*

Remark 3.1 If $\mathcal{G}$ is b-stable under $\{W_t\}$ and $\{W_t^*\}$ then the conditions **(M4)** and **(M5)** follow from: *$H\in C^1(A;\mathcal{G},\mathcal{G}^*)$ with $[H,\mathrm{i}A]^\circ=H'$ and $H'\in C^1(A;\mathcal{G},\mathcal{G}^*)$ with $[H',\mathrm{i}A]^\circ=H''$.* So a stronger but more natural version of **(M4)**–**(M5)** is: *$H\in C^2(A;\mathcal{G},\mathcal{G}^*)$ and $[H,\mathrm{i}A]^\circ=H'$.* The notation $C^2(A;\mathcal{G},\mathcal{G}^*)$ has an obvious meaning (in fact, all the classes of operators introduced in [ABG] for selfadjoint A have analogues in our context).

Remark 3.2 Our proofs extend trivially to the case when a multiple αA of A is the generator of a C_0-semigroup, where $\alpha\in\mathbb{C},\ \alpha\neq 0$ (then the operators H' and H'' in the second members of the relations from **(M4)** and **(M5)** should be replaced by $\alpha H'$ and $\alpha H''$). In particular, the operator A from Theorem 3.5 can be maximal symmetric with deficiency indices $(N,0)$ or $(0,N)$.

3.2 A general result

We can now state our first version of the so-called "limiting absorption principle":

Theorem 3.3 *Assume* **(M1)**–**(M5)**. *For each compact $I\subset J$ there is a constant C_I such that*

$$|(u,R(z)u)|\leq C_I\left(\|u\|^2_{\mathcal{G}^*}+\|Au\|^2_{\mathcal{G}^*}+\|A^*u\|^2_{\mathcal{G}^*}\right). \tag{3.4}$$

for all $u\in\mathcal{D}(A)\cap\mathcal{D}(A^*)$ and $z=\lambda+\mathrm{i}\mu,\ \lambda\in I,\ \mu\neq 0$ real. Moreover, if $z_1=\lambda_1+\mathrm{i}\mu_1$ and $z_2=\lambda_2+\mathrm{i}\mu_2$ are two such numbers, and if μ_1 and μ_2 have the same sign, then

$$|(u,(R(z_1)-R(z_2))u)|\leq C_I|z_1-z_2|^{1/2}\left(\|u\|^2_{\mathcal{G}^*}+\|Au\|^2_{\mathcal{G}^*}+\|A^*u\|^2_{\mathcal{G}^*}\right). \tag{3.5}$$

In particular, if $u\in\mathcal{D}(A)\cap\mathcal{D}(A^*)$ then the limits $\lim_{\mu\to\pm 0}(u,R(\lambda+\mathrm{i}\mu)u)=:(u,R(\lambda\pm \mathrm{i}0)u)$ exist uniformly in $\lambda\in I$ and for all $\lambda_1,\lambda_2\in I$ we have:

$$|(u,(R(\lambda_1\pm\mathrm{i}0)-R(\lambda_2\pm\mathrm{i}0))u)|\leq C_I|\lambda_1-\lambda_2|^{1/2}\left(\|u\|^2_{\mathcal{G}^*}+\|Au\|^2_{\mathcal{G}^*}+\|A^*u\|^2_{\mathcal{G}^*}\right). \tag{3.6}$$

3.3 Improved results in the symmetric case

The case when A is symmetric is especially interesting in applications. Then the result of Theorem 3.3 can be substantially improved. To formulate these improvements, we first recall some terminology and results about interpolation spaces associated to C_0-semigroups, whose proofs can be found in [ABG, Chaps. 2 and 3].

Let F be a Hilbert space and $\{U_t\}_{t\geq 0}$ a C_0-semigroup on F with generator A. We set $F_0:=F$ and denote by $\|f\|_0$ the norm on F, $F_1:=\mathcal{D}(A)$ equipped with the graph norm $\|f\|_1=(\|f\|^2+\|Af\|^2)^{1/2}$, and let F_{-1} be the completion of F for the norm

$$\|f\|_{-1}:=\inf\{(\|f_0\|^2+\|f_1\|^2)^{\frac{1}{2}}\mid f_i\in F_i,\ f=f_0+Af_1\}.$$

Then $F_1 \subset F_0 \subset F_{-1}$ and one defines the scale of Besov spaces $F_{s,p}$ for $-1 < s < 1$, $1 \le p \le +\infty$ by real interpolation: $F_{s,p} = (F_1, F_{-1})_{\frac{1-s}{2},p}$. If $s < t$ then $F_{t,q} \subset F_{s,p}$ continuously for all p, q, densely if $p < \infty$. If $1 \le p \le q < \infty$ then $F_{s,p} \subset F_{s,q}$ continuously and densely and $F_{s,p} \subset F_{s,\infty}$ continuously. We recall that $\{U_t^*\}$ is a C_0-semigroup on F^* and we associate to it the spaces $F^*_{\pm 1}$ and the Besov scale $F^*_{s,p}$ for $-1 < s < 1$, $1 \le p \le +\infty$. Then $(F_{\pm 1})^* = F^*_{\mp 1}$ and $(F_{s,p})^* = F^*_{-s,p'}$ if $1 \le p < \infty$ and $(p')^{-1} + p^{-1} = 1$ (see [ABG, Thms. 3.3.28, 2.4.2]).

We shall now give a description of the spaces $F_{s,p}$ for $s \neq 0$ in terms of the semigroup $\{U_t\}$ (we use [ABG, Prop. 2.7.3 and Th. 3.3.23]). Note that $\{U_t\}$ restricts to a C_0-semigroup on F_1 and extends to a C_0-semigroup on F_{-1}, both still denoted by $\{U_t\}$ (see [ABG, Prop. 3.3.8]). For $0 < s < 1$, $1 \le p \le +\infty$ we set:

$$\|f\|_{s,p} := \begin{cases} \|f\|_0 + (\int_0^1 \|t^{-s}(U_t - 1\!\!1)f\|_0^p \frac{\mathrm{d}t}{t})^{\frac{1}{p}}, \ p < \infty, \\ \|f\|_0 + \sup_{[0,1]} \|t^{-s}(U_t - 1\!\!1)f\|_0, \ p = +\infty. \end{cases}$$

Then $F_{s,p}$ is the space of $f \in F$ such that $\|f\|_{s,p} < \infty$ equipped with the norm $\|f\|_{s,p}$. Similarly we set:

$$\|f\|_{-s,p} := \begin{cases} \|f\|_{-1} + (\int_0^1 \|t^{-s}(U_t - 1\!\!1)f\|_{-1}^p \frac{\mathrm{d}t}{t})^{\frac{1}{p}}, \ p < \infty, \\ \|f\|_{-1} + \sup_{[0,1]} \|t^{-s}(U_t - 1\!\!1)f\|_{-1}, \ p = +\infty. \end{cases}$$

Then $F_{-s,p}$ is the space of $f \in F_{-1}$ such that$\|f\|_{-s,p} < \infty$. Finally if we denote by $F^{*0}_{s,\infty}$ the closure of F_1^* in $F^*_{s,\infty}$, then $F_{-s,1}$ is canonically identified with the dual of $F^{*0}_{s,\infty}$ (see [ABG, Theorem 2.4.2]).

We will assume in this subsection that A is symmetric (hence maximal symmetric, because it generates a C_0-semigroup) and that $\mathcal{G}$ is b-stable under $\{W_t^*\}$. Then $\{W_t^*\}$ induces a C_0-semigroup on $\mathcal{G}$ and $\{W_t\}$ induces a C_0-semigroup on $\mathcal{G}^*$. We denote by $\mathcal{D}(A; \mathcal{G}^*) \equiv \mathcal{D}(A_{\mathcal{G}^*})$ the domain of the generator of the semigroup induced by $\{W_t\}$ in $\mathcal{G}^*$. Then the spaces $\mathcal{G}_{\pm 1}$ and $\mathcal{G}^*_{\pm 1}$ and the Besov scales $\mathcal{G}_{s,p}$ and $\mathcal{G}^*_{s,p}$ $(-1 < s < 1, 1 \le p \le +\infty)$ are defined as explained above, taking $F = \mathcal{G}$ and the semigroup $\{U_t\} := \{W_t^*\}$ acting on $\mathcal{G}$. With the notations introduced after Definition 2.32 we have $\mathcal{G}_1 = \mathcal{D}(A^*; \mathcal{G})$ and $\mathcal{G}_1^* = \mathcal{D}(A; \mathcal{G}^*)$.

We will denote by $\mathcal{E}$ the space:

$$\mathcal{E} := \mathcal{G}^*_{1/2,1},$$

and as recalled above $\mathcal{E}^* = \mathcal{G}_{-1/2,\infty}$.

For later use we prove the following lemma.

Lemma 3.4 *Assume that A is symmetric and that $\mathcal{G}$ is* b*-stable under $\{W_t^*\}$. Then $\mathcal{D}(A)$ is dense in $\mathcal{D}(A; \mathcal{G}^*)$, in $\mathcal{E}$ and in $\mathcal{E} \cap \mathcal{H}$ for the intersection topology.*

Proof. For $\lambda \gg 1$ the operator $(A + \mathrm{i}\lambda)^{-1}$ is equal to the restriction to $\mathcal{H}$ of $(A_{\mathcal{G}^*} + \mathrm{i}\lambda)^{-1}$. Since $\mathcal{H}$ is dense in $\mathcal{G}^*$, $\mathcal{D}(A) = (A + \mathrm{i}\lambda)^{-1}\mathcal{H} = (A_{\mathcal{G}^*} + \mathrm{i}\lambda)^{-1}\mathcal{H}$ is dense in $(A_{\mathcal{G}^*} + \mathrm{i}\lambda)^{-1}\mathcal{G}^* = \mathcal{D}(A; \mathcal{G}^*)$. Since $\mathcal{D}(A; \mathcal{G}^*) = \mathcal{G}_1^*$ is dense in $\mathcal{E} = \mathcal{G}^*_{1/2,1}$, $\mathcal{D}(A)$ is dense in $\mathcal{E}$.

Finally $\{W_t\}$ is a C_0-semigroup on $\mathcal{G}^*$ and on $\mathcal{D}(A, \mathcal{G}^*)$ hence on $\mathcal{E}$ by real interpolation. Let $R_\epsilon = \epsilon^{-1} \int_0^\epsilon W_t \mathrm{d}t$. Then $R_\epsilon : \mathcal{H} \to \mathcal{D}(A)$, and by semigroup theory $\text{s-}\lim_{\epsilon \to 0} R_\epsilon = 1\!\!1$ in $\mathcal{H}$ and in $\mathcal{E}$, so that $\mathcal{D}(A)$ is also dense in $\mathcal{E} \cap \mathcal{H}$ for the intersection topology. □

Before stating our next results, we introduce two more notations:

$$J^\circ_\pm = \{\lambda \pm \mathrm{i}\mu \mid \lambda \in J, \ \mu > 0\}, \quad J_\pm = \{\lambda \pm \mathrm{i}\mu \mid \lambda \in J, \ \mu \ge 0\}. \tag{3.7}$$

Theorem 3.5 *Assume, besides* **(M1)**–**(M5)***, that* A *is symmetric and that* $\mathcal{G}$ *is* b*-stable under* $\{W_t^*\}$*. Then:*
(i) *if* $I \subset J$ *is compact, there is a constant* C_I *such that*

$$|(u, R(z)u)| \le C_I \|u\|_{\mathcal{E}}^2 \tag{3.8}$$

for all $u \in \mathcal{E} \cap \mathcal{H}$ *and* $z = \lambda + \mathrm{i}\mu,\ \lambda \in I,\ \mu \neq 0$ *real.*
(ii) *for each* $z = \lambda + \mathrm{i}\mu$ *with* $\lambda \in J,\ \mu \neq 0$*, the restriction of the sesquilinear form* $(u, v) \mapsto (u, R(z)v)$ *to* $\mathcal{E} \cap \mathcal{H}$ *extends to a continuous sesquilinear form on* $\mathcal{E}$ *and this extension has the following property: for each* $f, g \in \mathcal{E}$ *the maps* $J_\pm^\circ \ni z \mapsto (f, R(z)g)$ *are holomorphic and extend to continuous maps on* $J_\pm$*. In particular, the limits* $\lim_{\mu\to\pm 0}(f, R(\lambda + \mathrm{i}\mu)g)$ *exist locally uniformly in* $\lambda \in J$*.*

The polarization identity applied to estimates like (3.4), (3.5), or (3.8) allows one to express the limiting absorption principle in more standard terms. More precisely, let Q be a sesquilinear form on a complex vector space V and let us set $q(u) = Q(u, u)$. Then $4Q(u, v) = \sum \varepsilon q(\varepsilon u + v)$, where the sum is over $\varepsilon \in \mathbb{C}$ with $\varepsilon^4 = 1$ and Q is antilinear in the variable u. Now assume that $\|\cdot\|_V$ is a norm on V such that $|q(u)| \le \|u\|_V^2$ for all $u \in V$. Writing $Q(u, v) = Q(tu, t^{-1}v)$ with $t^2 = \|v\|_V / \|u\|_V$ and then using the polarization identity we get $|Q(u, v)| \le 4\|u\|_V \|v\|_V$ (one can replace 4 by 2 if $\|\cdot\|_V$ is a quadratic norm). For example, (3.8) gives

$$|(u, R(z)v)| \le 4C_I \|u\|_{\mathcal{E}} \|v\|_{\mathcal{E}}$$

for $u, v \in \mathcal{E} \cap \mathcal{H}$, which implies that $R(z)\mathcal{E} \cap \mathcal{H} \subset \mathcal{E}^*$ and that the map $R(z) : \mathcal{E} \cap \mathcal{H} \to \mathcal{E}^*$ extends to a bounded operator $\mathcal{E} \to \mathcal{E}^*$ satisfying

$$\|R(z)\|_{B(\mathcal{E},\mathcal{E}^*)} \le 4C_I. \tag{3.9}$$

Corollary 3.6 *Assume that the hypotheses of Theorem 3.5 hold. Then, if* $z = \lambda + \mathrm{i}\mu$ *with* $\lambda \in J$ *and* $\mu \neq 0$*,* $R(z)$ *induces a continuous operator* $R(z) : \mathcal{G}_{1/2,1}^* \to \mathcal{G}_{-1/2,\infty}$*. The maps* $J_\pm^\circ \ni z \mapsto R(z) \in B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$ *are holomorphic and extend to weak* continuous maps on* $J_\pm$*. In particular, the limits* $R(\lambda \pm \mathrm{i}0) := \lim_{\mu\to\pm 0} R(\lambda + \mathrm{i}\mu)$ *exist in the weak* topology of* $B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$ *locally uniformly in* $\lambda \in J$ *and the boundary values* $J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{G}_{1/2,1}^*, \mathcal{G}_{-1/2,\infty})$ *are weak* continuous maps.*

We mention that the space $\mathcal{E}^* \equiv \mathcal{G}_{-1/2,\infty}$ is not reflexive and one can not replace in the Corollary 3.6 the weak* topology by the weak topology even in the simplest situations. More precisely, even if u is a quite nice vector, e.g. $u \in \mathcal{D}(A)$, the vector $R(\lambda \pm \mathrm{i}0)u$ does not belong to the closure of $\mathcal{G}$ in $\mathcal{G}_{-1/2,\infty}$ and the map $\lambda \mapsto R(\lambda \pm \mathrm{i}0)u \in \mathcal{G}_{-1/2,\infty}$ is not weakly continuous, cf. [BGS]. But the situation improves if the "small" space $\mathcal{G}_{-1/2,\infty}$ is replaced by larger ones. As an example, we give below an optimal Hölder continuity result which can be proved without much effort. We note that the result remains true if $s < 3/2$, but is not so elementary. The methods of [BGS] allow one to cover the case $s = 3/2$, but then a new type of regularity is involved (the boundary values are not locally Lipschitz, as one could expect, but only of Zygmund class). One cannot take $s > 3/2$ because the order of regularity of H with respect to A is too small; this restriction can, however, be removed.

For $s > 1/2$ we have continuous embeddings $\mathcal{G}_{s,\infty}^* \subset \mathcal{G}_{1/2,1}^*$ and $\mathcal{G}_{-1/2,\infty} \subset \mathcal{G}_{-s,1}$, hence the operators $R(z)$ and $R(\lambda \pm \mathrm{i}0)$ induce continuous maps $\mathcal{G}_{s,\infty}^* \to \mathcal{G}_{-s,1}$.

Theorem 3.7 *If $1/2 < s \leq 1$ and if $I \subset J$ is a compact subset, then there is a number $C_I(s)$ such that*

$$\|R(z_1) - R(z_2)\|_{B(\mathcal{G}^*_{s,\infty},\mathcal{G}_{-s,1})} \leq C_I(s)|z_1 - z_2|^{s-1/2} \tag{3.10}$$

for all complex numbers z_1, z_2 with real parts in I and $\Im z_1 \cdot \Im z_2 > 0$.

In particular

$$\|R(\lambda_1 \pm \mathrm{i}0) - R(\lambda_2 \pm \mathrm{i}0)\|_{B(\mathcal{G}^*_{s,\infty},\mathcal{G}_{-s,1})} \leq C_I(s)|\lambda_1 - \lambda_2|^{s-1/2} \quad \text{for } \lambda_1, \lambda_2 \in I, \tag{3.11}$$

so the maps $J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{G}^*_{s,\infty}, \mathcal{G}_{-s,1})$ are locally Hölder continuous of order $s - 1/2$. Moreover:

$$\|R(\lambda \pm \mathrm{i}\mu) - R(\lambda \pm \mathrm{i}0)\|_{B(\mathcal{G}^*_{s,\infty},\mathcal{G}_{-s,1})} \leq C_I(s)|\mu|^{s-1/2} \quad \text{for } \lambda \in I \text{ and } \mu > 0. \tag{3.12}$$

These results remain true for the more usual space $\mathcal{G}^*_s \equiv \mathcal{G}^*_{s,2}$ and its adjoint $\mathcal{G}_{-s} \equiv \mathcal{G}_{-s,2}$ (these spaces can also be obtained by complex interpolation or as domains of suitably defined fractional powers of A). For example, $\mathcal{G}^*_s \subset \mathcal{G}^*_{s,\infty}$ and $\mathcal{G}_{-s,1} \subset \mathcal{G}_{-s}$ continuously, hence the maps $J \ni \lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{G}^*_s, \mathcal{G}_{-s})$ are also locally Hölder continuous of order $s - 1/2$.

We give an explicit description of the space $\mathcal{G}$ in an important particular case, cf. [GGM].

Proposition 3.8 *Let H, M be selfadjoint operators and R a symmetric operator on $\mathcal{H}$. Assume that: (1) $H \in C^1(M)$ and $[H, \mathrm{i}M]^\circ \mathcal{D}(H) \subset \mathcal{H}$; (2) $M \geq 0$; (3) $\mathcal{D}(H) \subset \mathcal{D}(R)$. Let H' be the closure of the operator $M + R$ defined on $\mathcal{D}(M) \cap \mathcal{D}(R)$. Then H' is symmetric closed and densely defined and the conditions* **(M1)***,* **(M2′)** *are satisfied. Moreover, we have*

$$\mathcal{D} = \mathcal{D}(H) \cap \mathcal{D}(M) \quad \text{and} \quad \mathcal{G} = (\mathcal{D}, \mathcal{H})_{1/2,2} = \mathcal{D}(|H|^{1/2}) \cap \mathcal{D}(M^{1/2}). \tag{3.13}$$

Proof. From Proposition 2.34 and Remark 2.35 it follows that $\mathcal{D}(H) \cap \mathcal{D}(M)$ is a core for M. Thus we can check **(M1)** with the help of Lemma 2.26. This also gives the first relation in (3.13). The condition **(M2′)** is satisfied because $M + R \geq R \geq -C\langle H\rangle$ for some number C. Now observe that we can choose $\|\cdot\|_\mathcal{G}$ such that

$$\|u\|^2_\mathcal{G} = (u, (M + \langle H\rangle)u) = \|M^{1/2}u\|^2 + \|\langle H\rangle^{1/2}u\|^2.$$

This implies that $\mathcal{G}$ is a closed subspace of $\mathcal{K} := \mathcal{D}(|H|^{1/2}) \cap \mathcal{D}(M^{1/2})$. If $(\mathcal{D}, \mathcal{H})_{1/2,2} = \mathcal{K}$ then $\mathcal{D}$, hence $\mathcal{G}$, is a dense subspace of $\mathcal{K}$, so we get $\mathcal{G} = \mathcal{K}$. So it remains to show that

$$(\mathcal{D}(H) \cap \mathcal{D}(M), \mathcal{H})_{1/2,2} = (\mathcal{D}(H), \mathcal{H})_{1/2,2} \cap (\mathcal{D}(M), \mathcal{H})_{1/2,2} \tag{3.14}$$

because the right hand side is clearly equal to $\mathcal{K}$. To get (3.14) we apply a non-commutative interpolation theorem due to Grisvard, see [ABG, Proposition 2.7.4]. More precisely, we take there $\mathbf{E} = \mathcal{D}(H)$, $\mathbf{G} = \mathcal{D}(M)$ and $V_\tau = (\mathbb{1} + \mathrm{i}\tau H)^{-1}$ for $0 \leq \tau \leq 1$. From Proposition 2.2 we get $V_\tau \mathbf{G} \subset \mathbf{G}$. Then Proposition 2.13 implies

$$[M, V_\tau]^\circ = V_\tau[\mathrm{i}\tau H, M]^\circ V_\tau = V_\tau[\mathrm{i}H, M]^\circ V_1(\tau + \mathrm{i}\tau H)V_\tau$$

hence $\|[M, V_\tau]^\circ\| \leq \|[\mathrm{i}H, M]^\circ V_1\|$ and for $u \in \mathcal{D}(M)$ we have:

$$\|MV_\tau u\| \leq \|Mu\| + \|[M, V_\tau]^\circ u\| \leq \|Mu\| + \|[\mathrm{i}H, M]^\circ V_1\|\|u\|.$$

Thus we see that $\|V_\tau\|_{\mathbf{G}\to\mathbf{G}}$ is bounded by a constant independent of τ. Hence we can apply [ABG, Proposition 2.7.4] to finish the proof of the proposition. □

3.4 Main estimates

In this subsection we collect the more technical estimates which will be needed to prove the limiting absorption principle. We use only the pair of operators H, H' satisfying **(M1)**, **(M2)**.

We introduce some new notations. For real $\varepsilon \neq 0$ we set

$$H_\varepsilon = H - \mathrm{i}\varepsilon H', \text{ with } \mathcal{D}(H_\varepsilon) = \mathcal{D}.$$

From Theorem 2.25 it follows that there is $\varepsilon_0 > 0$ such that $H_\varepsilon^* = H_{-\varepsilon}$ if $0 < |\varepsilon| \le \varepsilon_0$. From now on ε will always satisfy these inequalities (later on we shall require that ε_0 satisfy stronger conditions). In particular, note that H_ε is closed and densely defined.

From now on, if S is an operator on $\mathcal{H}$, we denote $S^\perp := \mathbb{1} - S$.

Lemma 3.9 *There is a number $C > 0$ such that*

$$|\varepsilon|\|u\|_{\mathcal{G}}^2 + |\mu|\|u\|^2 \le C|\Im((H_\varepsilon - z)u, u)| + C|\varepsilon|\|\mathbb{1}_J(H)^\perp\langle H\rangle^{1/2}u\|^2$$

for all $u \in \mathcal{D}$, $z = \lambda + \mathrm{i}\mu$, $\lambda \in \mathbb{R}$ and μ, ε real and having the same sign.

Proof. We shall assume that μ, ε are positive. Observe first that $H' + (a+b)\mathbb{1}_J(H)^\perp\langle H\rangle$ is greater than $a\langle H\rangle$ and H', hence

$$(u, (H' + (a+b)\mathbb{1}_J(H)^\perp\langle H\rangle)u) \ge \frac{a}{a+c}\|u\|_{\mathcal{G}}^2.$$

Then:

$$\begin{aligned}\Im((H_\varepsilon - z)u, u) &= ((\varepsilon H' + \mu)u, u)\\ &= \varepsilon(u, (H' + (a+b)\mathbb{1}_J(H)^\perp\langle H\rangle)u) - \varepsilon(a+b)\|\mathbb{1}_J(H)^\perp\langle H\rangle^{1/2}u\|^2 + \mu\|u\|^2\\ &\ge \frac{\varepsilon a}{a+c}\|u\|_{\mathcal{G}}^2 + \mu\|u\|^2 - \varepsilon(a+b)\|\mathbb{1}_J(H)^\perp\langle H\rangle^{1/2}u\|^2\end{aligned}$$

which is the required estimate. □

From now on we fix a compact set $I \subset J$ and a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi(s) = 1$ on a neighborhood of I and $\operatorname{supp}\varphi \subset J$. We set $\phi = \varphi(H)$ and $\phi^\perp = 1 - \phi$. The complex number z will always be of the form $z = \lambda + \mathrm{i}\mu$ with $\lambda \in I$ and $\mu \in \mathbb{R}$. We assume $0 < |\varepsilon| \le \varepsilon_0$ with ε_0 small enough and $\varepsilon\mu \ge 0$, so ε and μ have the same sign. We denote by C a generic constant independent of the numbers $\varepsilon, \lambda, \mu$ and of the vector $u \in \mathcal{D}$.

Lemma 3.10 *One can choose ε_0 such that*

$$\|\phi^\perp\langle H\rangle u\| \le C\|(H_\varepsilon - z)u\| + C|\varepsilon|\|u\|$$

if $0 < |\varepsilon| \le \varepsilon_0$ and $u \in \mathcal{D}$.

Proof. A simple computation gives

$$\|(H_\varepsilon - z)u\|^2 + \varepsilon(u, [H, \mathrm{i}H']^\circ u) = \|(H-\lambda)u\|^2 + \|(\varepsilon H' + \mu)u\|^2.$$

But $\pm[H, \mathrm{i}H']^\circ \le C\langle H\rangle^2$ by **(M1)**, hence $\|(H-\lambda)u\|^2 \le \|(H_\varepsilon - z)u\|^2 + C|\varepsilon|\|\langle H\rangle u\|^2$, or

$$(u, [(H-\lambda)^2 - C|\varepsilon|(1+H^2)]u) \le \|(H_\varepsilon - z)u\|^2.$$

From Proposition 2.23 we get $\phi \in C^1(H')$. In particular $\phi^\perp \mathcal{D} \subset \mathcal{D}$, cf. Proposition 2.2, so that we can replace above u by $\phi^\perp u$. Then

$$(u, \left((H-\lambda)^2 - C|\varepsilon|(1+H^2)\right)\phi^{\perp 2} u) \leq \|(H_\varepsilon - z)\phi^\perp u\|^2.$$

We estimate the left hand side from below by observing that one can choose ε_0 such that

$$\left((s-\lambda)^2 - C|\varepsilon|(1+s^2)]\right)(1-\varphi(s))^2 \geq C_0(1+s^2)(1-\varphi(s))^2$$

for some constant $C_0 > 0$ and all real s, if $0 < |\varepsilon| \leq \varepsilon_0$. For the right hand side we use

$$\|(H_\varepsilon - z)\phi^\perp u\| \leq \|\phi^\perp(H_\varepsilon - z)u\| + |\varepsilon|\|[H', \phi]^\circ u\| \leq \|(H_\varepsilon - z)u\| + C|\varepsilon|\|u\|.\square$$

The next proposition is the main technical result of this section.

Proposition 3.11 *There are $C, \varepsilon_0 > 0$ such that for $0 < |\varepsilon| \leq \varepsilon_0,\ \lambda \in I$ and μ real with $\mu\varepsilon \geq 0$, the operator $H_\varepsilon - z : \mathcal{D} \to \mathcal{H}$ is bijective and its inverse $R_\varepsilon \equiv R_\varepsilon(z)$ extends to a continuous operator $R_\varepsilon : \mathcal{G}^* \to \mathcal{G}$ satisfying for each $f \in \mathcal{G}^*$:*

$$|\varepsilon|^{1/2}\|R_\varepsilon f\|_{\mathcal{G}} \leq C|\Im(f, R_\varepsilon f)|^{1/2} + C\|f\|_{\mathcal{G}^*}. \tag{3.15}$$

Proof. By using $\mathbb{1}_J(H)^\perp \leq \phi^{\perp 2}$ in Lemma 3.10 we get

$$\|\mathbb{1}_J(H)^\perp \langle H\rangle^{1/2} u\|^2 \leq \|u\|\|\mathbb{1}_J(H)^\perp \langle H\rangle u\| \leq C\|u\|\|(H_\varepsilon - z)u\| + C|\varepsilon|\|u\|^2.$$

Inserting this into the estimate of Lemma 3.9 and taking into account that $\|u\| \leq \|u\|_{\mathcal{G}}$, we get for ε_0 small enough:

$$|\varepsilon|\|u\|^2_{\mathcal{G}} + |\mu|\|u\|^2 \leq C|\Im((H_\varepsilon - z)u, u)| + C|\varepsilon|\|u\|\|(H_\varepsilon - z)u\| \leq C(1+|\varepsilon|)\|u\|\|(H_\varepsilon - z)u\|.$$

In particular, since $|\varepsilon| + |\mu| = |\varepsilon + \mu|$, we get $|\varepsilon + \mu|\|u\| \leq C(1+|\varepsilon|)\|(H_\varepsilon - z)u\|$. This implies that $H_\varepsilon - z : \mathcal{D} \to \mathcal{H}$ is injective with closed range. But $(H_\varepsilon - z)^* = H_{-\varepsilon} - \bar{z}$ by Theorem 2.25, so the adjoint operator $(H_\varepsilon - z)^*$ is also injective, hence $H_\varepsilon - z : \mathcal{D} \to \mathcal{H}$ is surjective. Thus the bijectivity assertion is proved and we also have shown that there is C such that

$$\|R_\varepsilon\| \leq C|\varepsilon + \mu|^{-1}. \tag{3.16}$$

From Lemma 3.10 we now obtain $\|\phi^\perp \langle H\rangle R_\varepsilon\| \leq C$. Taking adjoints we get $\|R_\varepsilon \phi^\perp \langle H\rangle\| \leq C$, which is equivalent to:

$$\|R_\varepsilon \phi^\perp v\| \leq C\|\langle H\rangle^{-1} v\|,\ v \in \mathcal{H}. \tag{3.17}$$

We recall that G is the Friedrichs extension of $H' + c\langle H\rangle$ on $\mathcal{D}$. Then if $v \in \mathcal{H}$, we know that $R_\varepsilon \phi^\perp v \in \mathcal{D} \subset \mathcal{D}(G)$ and

$$\|G R_\varepsilon \phi^\perp v\| \leq \|H' R_\varepsilon \phi^\perp v\| + c\|\langle H\rangle R_\varepsilon \phi^\perp v\|.$$

We estimate the last term as follows:

$$\|\langle H\rangle R_\varepsilon \phi^\perp v\| \leq \|\phi \langle H\rangle R_\varepsilon \phi^\perp v\| + \|\phi^\perp \langle H\rangle R_\varepsilon \phi^\perp v\| \leq C\|v\|. \tag{3.18}$$

To estimate the first term, we write

$$\|H' R_\varepsilon \phi^\perp v\| = |\varepsilon|^{-1}\|((H_\varepsilon - z) - (H-z))R_\varepsilon \phi^\perp v\| \le |\varepsilon|^{-1}(\|\phi^\perp v\| + \|(H-z)R_\varepsilon \phi^\perp v\|) \le C|\varepsilon|^{-1}\|v\|$$

where (3.16) and (3.18) have been used. Hence we have:

$$\|G R_\varepsilon \phi^\perp v\| \le C|\varepsilon|^{-1}\|v\|,\ v \in \mathcal{H}. \tag{3.19}$$

A quadratic interpolation between (3.17) and (3.19) gives

$$\|G^{1/2} R_\varepsilon \phi^\perp v\| \le C|\varepsilon|^{-1/2}\|\langle H\rangle^{-1/2} v\|,\ v \in \mathcal{H},$$

and this is equivalent to $\|G^{1/2} R_\varepsilon \phi^\perp \langle H\rangle^{1/2}\| \le C|\varepsilon|^{-1/2}$. In other words, $R_\varepsilon \phi^\perp \langle H\rangle^{1/2} \in B(\mathcal{H}, \mathcal{G})$ with norm less than $C|\varepsilon|^{-1/2}$. Taking adjoints we see that $\phi^\perp \langle H\rangle^{1/2} R_\varepsilon \in B(\mathcal{G}^*, \mathcal{H})$ with norm less than $C|\varepsilon|^{-1/2}$. In particular

$$\|\phi^\perp \langle H\rangle^{1/2} R_\varepsilon f\| \le C|\varepsilon|^{-1/2}\|f\|_{\mathcal{G}^*},\ f \in \mathcal{H}.$$

Finally, we use this estimate in Lemma 3.9 with $u = R_\varepsilon f,\ f \in \mathcal{H}$, and obtain:

$$\begin{aligned} |\varepsilon|\|R_\varepsilon f\|_{\mathcal{G}}^2 + |\mu|\|R_\varepsilon f\|^2 &\le C|\Im(f, R_\varepsilon f)| + C|\varepsilon|\|\mathbb{1}_J(H)^\perp \langle H\rangle^{1/2} R_\varepsilon f\|^2 \\ &\le C|\Im(f, R_\varepsilon f)| + C\|f\|_{\mathcal{G}^*}^2 \end{aligned} \tag{3.20}$$

which implies (3.15). This estimate implies also that $R_\varepsilon \in B(\mathcal{G}^*, \mathcal{G})$ because if follows from (3.20) that

$$|\varepsilon|\|R_\varepsilon f\|_{\mathcal{G}}^2 \le \frac{|\varepsilon|}{2}\|R_\varepsilon f\|_{\mathcal{G}}^2 + \frac{1}{2|\varepsilon|}C^2\|f\|_{\mathcal{G}^*}^2 + C\|f\|_{\mathcal{G}^*}^2$$

hence $|\varepsilon|\|R_\varepsilon f\|_{\mathcal{G}}^2 \le C_0(1 + |\varepsilon|^{-1})\|f\|_{\mathcal{G}^*}^2$, or

$$\|R_\varepsilon\|_{B(\mathcal{G}^*, \mathcal{G})} \le C|\varepsilon|^{-1}. \tag{3.21}$$

□

We shall discuss now some consequences of Proposition 3.11 which will be useful in the last step of the proof of Theorem 3.5.

Lemma 3.12 *Under the conditions of Proposition 3.11, we have:*
(i) *the operator* $R_\epsilon(z) : \mathcal{G}^* \to \mathcal{G}$ *is the inverse of* $(H_\epsilon - z) : \mathcal{G} \to \mathcal{G}^*$,
(ii) *for* $0 < |\epsilon| < \epsilon_0$, *the map* $z \mapsto R_\epsilon(z) \in B(\mathcal{G}^*, \mathcal{G})$ *is holomorphic in a neighborhood of* I^+,
(iii) $\text{s-}\lim_{\epsilon \to 0^\pm} R_\epsilon(z) = (H-z)^{-1}$ *in* $B(\mathcal{H})$ *for* $\pm\Im z > 0$.

Proof. We have $(H_\epsilon - z)(R_\epsilon)(z)u = u,\ u \in \mathcal{H}$ and $R_\epsilon(z)(H_\epsilon - z)u = u,\ u \in \mathcal{D}$. Recall $\mathcal{D}$ is dense in $\mathcal{G}$ and $\mathcal{H}$ is dense in $\mathcal{G}^*$. Since $(H_\epsilon - z) \in B(\mathcal{G}, \mathcal{G}^*)$ and $R_\epsilon(z) \in B(\mathcal{G}^*, \mathcal{G})$ by Proposition 3.11, these identities extend to $u \in \mathcal{G}$ and $u \in \mathcal{G}^*$ respectively, which proves (i).

Let us now prove (ii). We recall (see (3.21)) that $\|R_\epsilon(z)\|_{B(\mathcal{G}^*, \mathcal{G})} \le C|\epsilon|^{-1}$. Since $R_\epsilon(z_1) - R_\epsilon(z_2) = (z_1 - z_2)R_\epsilon(z_1)R_\epsilon(z_2)$, for $z_1, z_2 \in I_+^\circ$, we see that $I_+^\circ \ni z \to R_\epsilon(z)$ is holomorphic with $\frac{\mathrm{d}^n}{\mathrm{d}z^n}R_\varepsilon(z) = n! R_\varepsilon(z)^{n+1}$. Hence the Taylor expansion of $R_\epsilon(z)$ at $z = z_0 \in I_+^\circ$ converges in $|z - z_0| < |\epsilon| C^{-1}$, which shows that $z \to R_\epsilon(z)$ extends as an holomorphic function in a neighborhood of I_+.

Finally it follows from (3.16) that for $z \in I_+^\circ$, $R_\epsilon(z)$ is uniformly bounded in $B(\mathcal{H})$ for $0 < \epsilon < \epsilon_0$. Next if $v \in \mathcal{D}$, $R(z)v \in \mathcal{D}$ because $R(z) \in C^1(H')$ by hypothesis **(M1)**, so

$$(R_\epsilon(z) - R(z))v = \mathrm{i}\epsilon R_\epsilon(z) H' R(z) v \to 0 \text{ when } \epsilon \to 0.$$

This proves (iii) since $\mathcal{D}$ is dense in $\mathcal{H}$. □

3.5 Differential inequalities

The estimate (3.15) of Proposition 3.11 is only one of the ingredients needed for the proof of the limiting absorption principle. The second one is the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_\varepsilon(z) = [R_\varepsilon(z), A]^\circ - \varepsilon R_\varepsilon(z) H'' R_\varepsilon(z). \tag{3.22}$$

which we shall establish below in a general context. This will then be used for the derivation of the fundamental differential inequality from which the limiting absorption principle follows (this is (3.26) in the case of Theorem 3.3 and (3.35) in that of Theorem 3.5). One can use this scheme in order to treat situations when A is not a generator of a semigroup or when the semigroup generated by it (or its adjoint) does not leave $\mathcal{G}$ invariant. Moreover, one can also consider situations when H'' is not a map $\mathcal{G} \to \mathcal{G}^*$. We shall not describe these possible extensions of the theory because the main ideas will be clear from the concrete situation treated below, which suffices for the applications we have in mind.

As before we assume that **(M1)** and **(M2)** hold and keep the convention used in the previous subsection concerning the parameters ε, λ and μ. In order to fix the ideas we shall, however, take $\varepsilon > 0$ and $\mu > 0$. The constants C and ε_0 are as in Proposition 3.11. Observe that by (3.16) the operator H_ε is regular.

The crucial property on which the rest of the proof depends is isolated in the next lemma. Let A be a closed densely defined operator on $\mathcal{H}$ such that H_ε is of full class $C^1(A)$. Then, according to Proposition 2.19, $\mathcal{D}(A) \cap \mathcal{D}$ and $\mathcal{D}(A^*) \cap \mathcal{D}$ are dense subspaces of $\mathcal{D}$ and the quadratic form $[H_\varepsilon, \mathrm{i}A]$ with domain $[\mathcal{D}(A^*) \cap \mathcal{D}] \times [\mathcal{D}(A) \cap \mathcal{D}]$ extends to a continuous form $[H_\varepsilon, \mathrm{i}A]^\circ$ on $\mathcal{D}$. We identify this form with a continuous operator $[H_\varepsilon, \mathrm{i}A]^\circ : \mathcal{D} \to \mathcal{D}^*$ and obtain

$$[R_\varepsilon(z), \mathrm{i}A]^\circ = -R_\varepsilon(z)[H_\varepsilon, \mathrm{i}A]^\circ R_\varepsilon(z). \tag{3.23}$$

Lemma 3.13 *Assume that H_ε is of full class $C^1(A)$ and that there exists $H'' \in B(\mathcal{D}, \mathcal{D}^*)$ such that*

$$[H_\varepsilon, \mathrm{i}A]^\circ u = H'u - \mathrm{i}\varepsilon H'' u \quad \text{if } u \in \mathcal{D}.$$

Then $[R_\varepsilon(z), A]^\circ = R_\varepsilon(z)(\mathrm{i}H' + \varepsilon H'')R_\varepsilon(z)$, the map $\varepsilon \mapsto R_\varepsilon(z) \in B(\mathcal{H})$ is C^1 in norm on $]0, \varepsilon_0]$, and (3.22) is satisfied.

Proof. We write $R_{\varepsilon'}(z) - R_\varepsilon(z) = (\varepsilon' - \varepsilon)\mathrm{i}R_{\varepsilon'}(z)H'R_\varepsilon(z)$. We know that $R_\varepsilon(z)$ is bounded from $\mathcal{H}$ to $\mathcal{D}$ which implies that $]0, \varepsilon_0[\ni \varepsilon \to R_\varepsilon(z)$ is norm continuous and then norm differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} R_\varepsilon(z) = \mathrm{i}R_\varepsilon(z) H' R_\varepsilon(z). \tag{3.24}$$

Now we use (3.23) and the hypothesis of the lemma. □

Lemma 3.14 *Assume that the conditions* **(M3)** *and* **(M4)** *are satisfied and that the limit $\lim_{t \to 0} t^{-1}[(H'u, W_t u) - (u, W_t H'u)]$ exists for all $u \in \mathcal{D}$. Then the conditions of Lemma 3.13 are satisfied.*

Proof. By the uniform boundedness principle, there is $H'' \in B(\mathcal{D}, \mathcal{D}^*)$ such that for each $u \in \mathcal{D}$ one has $\lim_{t \to 0} t^{-1}[(H'u, W_t u) - (u, W_t H'u)] = (u, H''u)$. Let $u \in \mathcal{D}(A^*), v \in \mathcal{D}(A)$.

Then $-\mathrm{i}A^*u = \lim_{t\to 0} t^{-1}(W_t^* - 1)u$ and $\mathrm{i}Av = \lim_{t\to 0} t^{-1}(W_t - 1)v$, hence

$$\begin{aligned}(u, [R_\varepsilon(z), \mathrm{i}A]v) &= \lim_{t\to 0}\Big((R_\varepsilon(z)^*u, t^{-1}W_t v) - (t^{-1}W_t^* u, R_\varepsilon(z)v)\Big)\\ &= \lim_{t\to 0}\Big((R_\varepsilon(z)^*u, t^{-1}W_t H_\varepsilon R_\varepsilon(z)v) - (H_\varepsilon^* R_\varepsilon^* u, t^{-1}W_t R_\varepsilon v)\Big).\end{aligned}$$

Since by Theorem 2.25 $R_\epsilon(z)^* = R_{-\epsilon}(\overline{z})$, we have $R_\varepsilon(z)^*u$, $R_\varepsilon(z)v \in \mathcal{D}$ and the last limit equals

$$-(R_\varepsilon(z)^*u, H' R_\varepsilon(z)v) + \mathrm{i}\varepsilon(R_\varepsilon(z)^*u, H'' R_\varepsilon(z)v) = (u, R_\varepsilon(z)(-H' + \mathrm{i}\varepsilon H'')R_\varepsilon(z)v).$$

This shows that $R_\varepsilon(z) \in C^1(A)$ for all $z \in \mathbb{C}\setminus\sigma(H_\varepsilon)$ and hence H_ε is of full class $C^1(A)$. □

Proof of Theorem 3.3. We omit the z-dependence to simplify notations. Let u be an element of $\mathcal{D}(A)\cap\mathcal{D}(A^*)$ and let us set $F_\varepsilon = (u, R_\varepsilon u)$. From (3.22) we get

$$F'_\varepsilon = (u, [R_\varepsilon, A]^\circ u) - \varepsilon(R_\varepsilon^* u, H'' R_\varepsilon u) = (R_\varepsilon^* u, Au) - (A^*u, R_\varepsilon u) - \varepsilon(R_\varepsilon^* u, H'' R_\varepsilon u). \tag{3.25}$$

Then:

$$|F'_\varepsilon| \le \|R_\varepsilon^* u\|_{\mathcal{G}}\|Au\|_{\mathcal{G}^*} + \|A^*u\|_{\mathcal{G}^*}\|R_\varepsilon u\|_{\mathcal{G}} + \varepsilon\|H''\|_{B(\mathcal{G},\mathcal{G}^*)}\|R_\varepsilon^* u\|_{\mathcal{G}}\|R_\varepsilon u\|_{\mathcal{G}}.$$

In the sequel, when we write an estimate containing the symbol $L^{(*)}$ where L denotes some linear operator, we mean that the estimate holds both for L and L^*. From (3.15) (used for ε and $-\varepsilon$) we get $\|R_\varepsilon^{(*)}u\|_{\mathcal{G}} \le C\varepsilon^{-1/2}(|F_\varepsilon|^{1/2} + \|u\|_{\mathcal{G}^*})$. Now we set $[u] = \|u\|_{\mathcal{G}^*} + \|Au\|_{\mathcal{G}^*} + \|A^*u\|_{\mathcal{G}^*}$ and obtain

$$|F'_\varepsilon| \le C|F_\varepsilon| + C\varepsilon^{-1/2}[u]|F_\varepsilon|^{1/2} + C\varepsilon^{-1/2}[u]^2 \le C'\varepsilon^{-1/2}([u]^2 + |F_\varepsilon|). \tag{3.26}$$

By Gronwall's lemma we get for $0 < \varepsilon \le \varepsilon_0$:

$$|(u, R_\varepsilon u)| \le C''|(u, R_{\varepsilon_0}u)| + C''[u]^2 \le C[u]^2, \tag{3.27}$$

because $R_{\varepsilon_0} \in B(\mathcal{G}^*, \mathcal{G})$. But $\text{s-}\lim_{\varepsilon\to 0} R_\varepsilon(z) = R(z)$ in $B(\mathcal{H})$ by Lemma 3.12. Combined with (3.27), this gives the estimate (3.4).

It is easy to deduce now from (3.26) that the limit $(u, R(\lambda + \mathrm{i}0)u) := \lim_{\mu\downarrow 0}(u, R(\lambda + \mathrm{i}\mu)u)$ exists uniformly in $\lambda \in I$ (see (3.39) below). We shall now prove that $z \mapsto (u, R(z)u)$ is Hölder continuous of order 1/2, cf. (3.5). Let I_0 be an open real interval whose closure is included in the interior of I and $U = \{z \mid \Re z \in I_0, \Im z > 0\}$. Let us define $\phi(z,\epsilon) = (u, R_\varepsilon(z)u)$ for $z \in U$ and $0 < \varepsilon < \varepsilon_0$. Our purpose is to show that ϕ satisfies the conditions of Lemma A.2. From (3.26) and (3.27) we get

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(z,\varepsilon)\Big| = |F'_\varepsilon| \le C\varepsilon^{-1/2}[u]^2. \tag{3.28}$$

On the other hand, from (3.25) we get:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon) = (R_\varepsilon^{*2}u, Au) - (A^*u, R_\varepsilon^2 u) - \varepsilon(R_\varepsilon^{*2}u, H'' R_\varepsilon u) - \varepsilon(R_\varepsilon^* u, H'' R_\varepsilon^2 u)$$

hence

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)\Big| &\le \|Au\|_{\mathcal{G}^*}\|R_\varepsilon^{*2}u\|_{\mathcal{G}} + \|A^*u\|_{\mathcal{G}^*}\|R_\varepsilon^2 u\|_{\mathcal{G}}\\ &+ \varepsilon\|H''\|_{B(\mathcal{G},\mathcal{G}^*)}\Big(\|R_\varepsilon u\|_{\mathcal{G}}\|R_\varepsilon^{*2}u\|_{\mathcal{G}} + \|R_\varepsilon^* u\|_{\mathcal{G}}\|R_\varepsilon^2 u\|_{\mathcal{G}}\Big).\end{aligned}$$

Now we use the estimate (3.21) and get

$$\|R_\varepsilon^{(*)2}u\|_{\mathcal{G}} \leq \|R_\varepsilon^{(*)}\|_{B(\mathcal{G}^*,\mathcal{G})}\|R_\varepsilon^{(*)}u\|_{\mathcal{G}^*} \leq C\varepsilon^{-1}\|R_\varepsilon^{(*)}u\|_{\mathcal{G}}.$$

Thus

$$|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)| \leq C\varepsilon^{-1}[u]\left(\|R_\varepsilon u\|_{\mathcal{G}} + \|R_\varepsilon^* u\|_{\mathcal{G}}\right) + C\|R_\varepsilon u\|_{\mathcal{G}}\|R_\varepsilon^* u\|_{\mathcal{G}}.$$

From (3.15) and (3.27) we have $\varepsilon^{1/2}\|R_\varepsilon^{(*)}u\|_{\mathcal{G}} \leq C[u]$ which inserted above implies

$$|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)| \leq C\varepsilon^{-3/2}[u]^2. \tag{3.29}$$

Finally, we also have the easy estimate $|\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon_0)| = |(u, R_{\varepsilon_0}(z)^2u)| \leq C[u]^2$. Thus, from (3.28) and (3.29) we see that the conditions of Lemma A.2 are satisfied with $\sigma = 1/2$ and $M = C[u]^2$. As a consequence, since $\phi(z,+0) = (u, R(z)u)$, we see that

$$|(u,R(z_1)u) - (u,R(z_2)u)| \leq C[u]^2|z_1 - z_2|^{1/2}$$

for some number C independent of u and $z_1, z_2 \in U$. Since I_0 and I are arbitrary, this completes the proof of the theorem. □

Proof of Theorem 3.5. Let $f \in \mathcal{E}$ and let us consider the vectors

$$f_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon W_t f\,\mathrm{d}t \in \mathcal{D}(A;\mathcal{G}^*). \tag{3.30}$$

Then $Af_\varepsilon = (\mathrm{i}\varepsilon)^{-1}(W_\varepsilon - 1)f$ and the map $\varepsilon \mapsto f_\varepsilon \in \mathcal{G}^*$ is C^1 on $]0,\infty[$ with

$$f'_\varepsilon = \frac{1}{\varepsilon}W_\varepsilon f - \frac{1}{\varepsilon}f_\varepsilon = \frac{1}{\varepsilon}(W_\varepsilon - 1)f + \frac{1}{\varepsilon}\int_0^1 (1 - W_{\varepsilon s})f\,\mathrm{d}s.$$

Let us abbreviate $l(\varepsilon) = \|Af_\varepsilon\|_{\mathcal{G}^*} + \|f'_\varepsilon\|_{\mathcal{G}^*}$. Then

$$l(\varepsilon) \leq \frac{2}{\varepsilon}\|(W_\varepsilon - 1)f\|_{\mathcal{G}^*} + \frac{1}{\varepsilon}\int_0^1 \|(W_{\varepsilon s} - 1)f\|_{\mathcal{G}^*}\mathrm{d}s. \tag{3.31}$$

It follows by a change of variables and [ABG, Proposition 2.7.3] that

$$\int_0^\infty l(\varepsilon)\frac{\mathrm{d}\varepsilon}{\sqrt{\varepsilon}} \leq \frac{8}{3}\int_0^\infty \|(W_\varepsilon - 1)f\|_{\mathcal{G}^*}\frac{\mathrm{d}\varepsilon}{\varepsilon^{3/2}} \leq C\|f\|_{\mathcal{E}} \tag{3.32}$$

where C is a constant independent of f.

Observe now that for each $u \in \mathcal{D}(A;\mathcal{G}^*)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(u, R_\varepsilon u) = (R_\varepsilon^* u, Au) - (Au, R_\varepsilon u) - \varepsilon(R_\varepsilon^* u, H'' R_\varepsilon u). \tag{3.33}$$

Indeed, if $u \in \mathcal{D}(A)$ then this follows from (3.25) and the symmetry of A. Now arguing as in the proof of Lemma 3.13 we see that $\varepsilon \mapsto R_\varepsilon \in B(\mathcal{G}^*,\mathcal{G})$ is of class C^1 on $]0,\varepsilon_0[$ and that (3.24) holds in $B(\mathcal{G}^*,\mathcal{G})$. Thus the left hand side in (3.33) is a continuous function of $u \in \mathcal{D}(A)$ for the topology induced by $\mathcal{G}^*$. The same is true for the last term in (3.33), thanks to hypothesis

(M5), while the two remaining terms are continuous for the topology induced by $\mathcal{D}(A;\mathcal{G}^*)$, since $R_\epsilon : \mathcal{G}* \to \mathcal{G}$. Since by Lemma 3.4, $\mathcal{D}(A)$ is dense in $\mathcal{D}(A;\mathcal{G}^*)$, we get (3.33) for all $u \in \mathcal{D}(A;\mathcal{G}^*)$.

Now let us set $F_\varepsilon = (f_\varepsilon, R_\varepsilon f_\varepsilon)$. Then from (3.33) we get

$$F'_\varepsilon = (R^*_\varepsilon f_\varepsilon, f'_\varepsilon + Af_\varepsilon) + (f'_\varepsilon - Af_\varepsilon, R_\varepsilon f_\varepsilon) - \varepsilon(R^*_\varepsilon f_\varepsilon, H'' R_\varepsilon f_\varepsilon), \tag{3.34}$$

hence

$$|F'_\varepsilon| \le l(\varepsilon)(\|R_\varepsilon f_\varepsilon\|_\mathcal{G} + \|R^*_\varepsilon f_\varepsilon\|_\mathcal{G}) + \varepsilon \|H''\|_{B(\mathcal{G},\mathcal{G}^*)} \|R^*_\varepsilon f_\varepsilon\|_\mathcal{G} \|R_\varepsilon f_\varepsilon\|_\mathcal{G}.$$

Our main estimate (3.15) (used for ε and $-\varepsilon$) gives

$$\|R^{(*)}_\varepsilon f_\varepsilon\|_\mathcal{G} \le C\varepsilon^{-1/2}(|F_\varepsilon|^{1/2} + \|f_\varepsilon\|_{\mathcal{G}^*}).$$

We obtain

$$|F'_\varepsilon| \le C|F_\varepsilon| + Cl(\varepsilon)\varepsilon^{-1/2}|F_\varepsilon|^{1/2} + C\|f_\varepsilon\|_{\mathcal{G}^*}\left(l(\varepsilon)\varepsilon^{-1/2} + \|f_\varepsilon\|_{\mathcal{G}^*}\right).$$

But clearly $\|f_\varepsilon\|_{\mathcal{G}^*} \le C\|f\|_{\mathcal{G}^*} \le C\|f\|_\mathcal{E}$. Thus

$$|F'_\varepsilon| \le C|F_\varepsilon| + Cl(\varepsilon)\varepsilon^{-1/2}|F_\varepsilon|^{1/2} + C\|f\|_\mathcal{E}\left(l(\varepsilon)\varepsilon^{-1/2} + \|f\|_\mathcal{E}\right). \tag{3.35}$$

We apply now the improved version of Gronwall's lemma stated as Lemma A.1 with $c(\varepsilon) = C$, $b(\varepsilon) = Cl(\varepsilon)\varepsilon^{-1/2}$ and $a(\varepsilon) = C\|f\|_\mathcal{E}(l(\varepsilon)\varepsilon^{-1/2} + \|f\|_\mathcal{E})$. Taking into account the estimate (3.32) we see that there is a constant C independent of ε, z and $f \in \mathcal{E}$ such that

$$|(f_\varepsilon, R_\varepsilon(z) f_\varepsilon)| \le C|(f_{\varepsilon_0}, R_{\varepsilon_0}(z) f_{\varepsilon_0})| + C\|f\|^2_\mathcal{E}. \tag{3.36}$$

By (3.21) the right hand side is less than $C\|f\|^2_\mathcal{E}$. Thus

$$|(f_\varepsilon, R_\varepsilon(z) f_\varepsilon)| \le C\|f\|^2_\mathcal{E}. \tag{3.37}$$

Assume for a moment that $f \in \mathcal{E} \cap \mathcal{H}$. Recall that by Lemma 3.12 $\text{s-}\lim_{\epsilon\to 0} R_\epsilon(z) = R(z)$ in $B(\mathcal{H})$. Clearly $f_\varepsilon \to f$ in $\mathcal{H}$. Since R_ϵ is uniformly bounded in $B(\mathcal{H})$, we get $|(f, R(z)f)| \le C\|f\|^2_\mathcal{E}$ for $f \in \mathcal{E} \cap \mathcal{H},\ \lambda \in I$ and $\mu > 0$. Using the polarization identity (see the comment after Theorem 3.5) it follows that the sesquilinear form $(f, R(z)g)$ on $\mathcal{E} \cap \mathcal{H}$ uniquely extends to a continuous sesquilinear form on $\mathcal{E}$, for which we shall keep the same notation (recall that by Lemma 3.4 $\mathcal{E} \cap \mathcal{H}$ is dense in $\mathcal{E}$).This completes the proof of (i).

Let us now give another description of $(f, R(z)g)$ for $f, g \in \mathcal{E}$. From (3.35) and (3.37) we see that there is a constant C such that

$$|F'_\varepsilon| \le C\|f\|_\mathcal{E}\left(l(\varepsilon)\varepsilon^{-1/2} + \|f\|_\mathcal{E}\right) \tag{3.38}$$

and, because of (3.32), the right hand side is an integrable function of ε on $]0, \infty[$. Thus $\lim_{\varepsilon\to 0} F_\varepsilon =: q(f)$ exists and $q(f)$ defines via the polarization identity and (3.36) a continuous sesquilinear form on $\mathcal{E}$. Since this form coincides with $(f, R(z)g)$ on $\mathcal{E} \cap \mathcal{H}$ they are identical. Hence $(f, R(z)f) = \lim_{\varepsilon\to 0} F_\varepsilon$ for all $f \in \mathcal{E}$.

It remains to prove (ii). We write for $f \in \mathcal{E}$

$$(f, R(z)f) = (f_{\varepsilon_0}, R_{\varepsilon_0}(z) f_{\varepsilon_0}) - \int_0^{\varepsilon_0} F'_\varepsilon(z) \mathrm{d}\varepsilon \tag{3.39}$$

This holds for all $z = \lambda + i\mu$ with $\lambda \in I$ and $\mu > 0$, i.e. for $z \in I_+^\circ$. We saw in Lemma 3.12 that for $0 < \varepsilon \le \varepsilon_0$ the map $z \mapsto R_\varepsilon(z) \in B(\mathcal{G}^*, \mathcal{G})$ extends to a holomorphic function on a neighborhood of I_+. Thus the first term on the right-hand side above extends to a holomorphic function of z on a neighborhood of I_+. From (3.34) and since $H'' \in B(\mathcal{G}, \mathcal{G}^*)$ it follows that for each $\varepsilon > 0$ the map $z \mapsto F'_\varepsilon(z)$ has the same property; note that its domain of holomorphy depends on ε but contains I_+°. Moreover, we have the bound (3.38). A standard application of the dominated convergence theorem (or use [Di, Theorem 13.8.6(iii)]) shows that the last term in (3.39) is a holomorphic function of z on I_+°. So the map $z \mapsto (f, R(z)f)$ is holomorphic on I_+°.

Finally, we show that for $f \in \mathcal{E}$, $\lim_{\mu\to 0^+}(f, R(\lambda + i\mu)f)$ exists uniformly in $\lambda \in I$. It suffices to treat the integral term in (3.39). For each $\varepsilon > 0$ the limit $\lim_{\mu\to 0^+} F'_\varepsilon(\lambda + i\mu) =: F'_\varepsilon(\lambda + i0)$ exists uniformly in $\lambda \in I$, the function F'_ε being holomorphic on a neighborhood of I_+. Let us set $\phi_\varepsilon(\mu) = \sup_{\lambda\in I} |F'_\varepsilon(\lambda + i\mu) - F'_\varepsilon(\lambda + i0)|$. Thus $\phi_\varepsilon(\mu) \to 0$ when $\mu \to 0^+$ and, because of (3.38), we have $\phi_\varepsilon(\mu) \le \theta(\epsilon)$ for some integrable function θ. Hence $\int_0^{\varepsilon_0} \phi_\varepsilon d\varepsilon \to 0$ if $\mu \to 0^+$, which is more than required. □

Proof of Theorem 3.7. We shall proceed as in the proof of the corresponding assertion of Theorem 3.3. Let U be as in that proof and let $\phi(z, \epsilon) = F_\varepsilon(z) = (f_\varepsilon, R_\varepsilon(z)f_\varepsilon)$ for $z \in U$ and $0 < \varepsilon < \varepsilon_0$. By (3.38), there is constant C such that

$$|\frac{d}{d\varepsilon}\phi(z, \epsilon)| \le C\|f\|_{\mathcal{E}} \left(l(\varepsilon)\varepsilon^{-1/2} + \|f\|_{\mathcal{E}} \right). \tag{3.40}$$

Then from (3.34) we get

$$\frac{d}{d\varepsilon}\frac{d}{dz}\phi(z, \epsilon) = (R_\varepsilon^{*2} f_\varepsilon, f'_\varepsilon + Af_\varepsilon) + (f'_\varepsilon - Af_\varepsilon, R_\varepsilon^2 f_\varepsilon) - \varepsilon(R_\varepsilon^{*2} f_\varepsilon, H'' R_\varepsilon f_\varepsilon) - \varepsilon(R_\varepsilon^* f_\varepsilon, H'' R_\varepsilon^2 f_\varepsilon)$$

hence

$$\begin{aligned} |\frac{d}{d\varepsilon}\frac{d}{dz}\phi(z, \epsilon)| &\le l(\varepsilon) \left(\|R_\varepsilon^2 f_\varepsilon\|_{\mathcal{G}} + \|R_\varepsilon^{*2} f_\varepsilon\|_{\mathcal{G}} \right) \\ &+ \varepsilon\|H''\|_{B(\mathcal{G},\mathcal{G}^*)} \left(\|R_\varepsilon f_\varepsilon\|_{\mathcal{G}} \|R_\varepsilon^{*2} f_\varepsilon\|_{\mathcal{G}} + \|R_\varepsilon^* f_\varepsilon\|_{\mathcal{G}} \|R_\varepsilon^2 f_\varepsilon\|_{\mathcal{G}} \right). \end{aligned}$$

But, according to (3.21), we have $\|R_\varepsilon^{(*)2} f_\varepsilon\|_{\mathcal{G}} \le C\varepsilon^{-1}\|R_\varepsilon^{(*)} f_\varepsilon\|_{\mathcal{G}}$, so we have

$$|\frac{d}{d\varepsilon}\frac{d}{dz}\phi(z, \epsilon)| \le Cl(\varepsilon)\varepsilon^{-1} \left(\|R_\varepsilon f_\varepsilon\|_{\mathcal{G}} + \|R_\varepsilon^* f_\varepsilon\|_{\mathcal{G}} \right) + C\|R_\varepsilon f_\varepsilon\|_{\mathcal{G}} \|R_\varepsilon^* f_\varepsilon\|_{\mathcal{G}}$$

for some new constant C. On the other hand, the estimate (3.15) gives us

$$\varepsilon^{1/2}\|R_\varepsilon^{(*)} f_\varepsilon\|_{\mathcal{G}} \le C|F_\varepsilon|^{1/2} + C\|f_\varepsilon\|_{\mathcal{G}^*} \le C'\|f\|_{\mathcal{E}}.$$

Inserting this into the preceding inequality we finally see that there is C such that

$$|\frac{d}{d\varepsilon}\frac{d}{dz}\phi(z, \epsilon)| \le C\varepsilon^{-1}\|f\|_{\mathcal{E}} \left(l(\varepsilon)\varepsilon^{-1/2} + \|f\|_{\mathcal{E}} \right). \tag{3.41}$$

Until now there was no assumption on f besides $f \in \mathcal{E}$. Now we choose $1/2 < s \le 1$ and assume $f \in \mathcal{G}_{s,\infty}^*$. As explained in Subsection 3.1, this is equivalent to $f \in \mathcal{G}^*$ and

$$\|f\|_{\mathcal{G}_{s,\infty}^*} := \|f\|_{\mathcal{G}^*} + \sup_{0<t<1} \|t^{-s}(W_t - 1)f\|_{\mathcal{G}^*} < \infty.$$

Then from (3.31) we obtain

$$l(\varepsilon) \leq 3\varepsilon^{-1} \sup_{0<t<\varepsilon} \|(W_t - 1)f\|_{\mathcal{G}^*} \leq 3\varepsilon^{s-1}\|f\|_{\mathcal{G}^*_{s,\infty}}.$$

We use this estimate in (3.40) and obtain

$$|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(z,\epsilon)| \leq C\|f\|_{\mathcal{E}} \left(\varepsilon^{s-3/2}\|f\|_{\mathcal{G}^*_{s,\infty}} + \|f\|_{\mathcal{E}}\right) \leq C'\varepsilon^{s-3/2}\|f\|^2_{\mathcal{G}^*_{s,\infty}}.$$

Similarly, from (3.41) we get

$$|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(z,\epsilon)| \leq C'\varepsilon^{s-5/2}\|f\|^2_{\mathcal{G}^*_{s,\infty}}.$$

Note also the trivial estimate $|\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon_0)| = |(f_{\varepsilon_0}, R_{\varepsilon_0}(z)^2 f_{\varepsilon_0})| \leq C\|f\|^2_{\mathcal{G}^*}$. Thus we can apply Lemma A.2 with $\sigma = s - 1/2$ and M of the form $C\|f\|^2_{\mathcal{G}^*_{s,\infty}}$. We obtain

$$|(f, R(z_1)f) - (f, R(z_2)f)| \leq C\|f\|^2_{\mathcal{G}^*_{s,\infty}}|z_1 - z_2|^{s-1/2}$$

for some number C independent of f and $z_1, z_2 \in U$. From the polarization identity (see the comment after Theorem 3.5) it follows now that

$$|(g, R(z_1)f) - (g, R(z_2)f)| \leq 4C\|g\|_{\mathcal{G}^*_{s,\infty}}\|f\|_{\mathcal{G}^*_{s,\infty}}|z_1 - z_2|^{s-1/2} \tag{3.42}$$

for all $f, g \in \mathcal{G}^*_{s,\infty}$. We know that $R(z)f \in \mathcal{G}_{-1/2,\infty} \subset \mathcal{G}_{-s,1}$ for $z = z_1, z_2$.

We recalled in the beginning of Subsectionimproved that if $\mathcal{G}^{*\circ}_{s,\infty}$ is the closure of $\mathcal{G}^*_1 = \mathcal{D}(A;\mathcal{G}^*)$ in $\mathcal{G}^*_{s,\infty}$, then the adjoint space of $\mathcal{G}^{*\circ}_{s,\infty}$ is canonically identified with $\mathcal{G}_{-s,1}$. Taking into account that the anti-duality between $\mathcal{G}$ and $\mathcal{G}^*$ is defined with the help of $\mathcal{H}$, we obtain after taking in (3.42) the supremum with respect to $g \in \mathcal{G}^{*\circ}_{s,\infty}$ with norm equal to one that

$$\|R(z_1)f - R(z_2)f\|_{\mathcal{G}_{-s,1}} \leq 4C\|f\|_{\mathcal{G}^*_{s,\infty}}|z_1 - z_2|^{s-1/2}.$$

This finishes the proof of the theorem. □

4 The virial theorem

In this section we improve the standard version of the virial theorem [ABG, Proposition 7.2.10] in two directions. First, we allow a general class of conjugate operators A, thus extending [HuS, Proposition 9]. Then we consider the case when the Hamiltonian H is not of class $C^1(A)$: we have in mind the framework of Section 3, but we are forced to require that the commutator H' can be approximated by operators with better properties.

This version of the virial theorem will be needed in [GGM] for application to massless Nelson models. Let us also mention the recent paper [FM], where another version of the virial theorem has been shown. In our case the virial theorem for a pair H, A and an eigenvector ψ of H is shown by approximating A by a sequence A_n of operators such that $H \in C^1(A_n)$. The method used in [FM] is different and consists in approximating ψ by a sequence ψ_n of vectors in $\mathcal{D}(H) \cap \mathcal{D}(A)$.

Proposition 4.1 *Let H be a selfadjoint operator and A a regular operator (e.g. a generator of a C_0-semigroup) such that H is of class $C^1(A)$. If u is an eigenvector of H, then* $(u, [H, \mathrm{i}A]^\circ u) = 0$.

Proof. Assume $Hu = \lambda u$, let $z \in \rho(H, A) = \mathbb{C} \backslash \sigma(H)$ (because A is regular), and let us set $\mu = (z - \lambda)^{-1}$. Then $R(z)u = \mu u$ and $R(z)^*u = \bar{\mu}u$, hence

$$(u, [A, H]^\circ u) = \mu^{-2}(R(z)^*u, [A, H]^\circ R(z)u) = -(u, [A, R(z)]^\circ u),$$

because of (2.8). Now we use Proposition 2.9:

$$\begin{aligned}(u, [A, R(z)]^\circ u) &= \lim_n (u, [A_n, R(z)]u) = \lim_n [(u, A_n R(z)u) - (R(z)^*u, A_n u)] \\ &= \lim_n [(u, A_n \mu u) - (\bar{\mu}u, A_n u)] = 0. \square\end{aligned}$$

We mention one more result of a similar nature. If $\mathcal{D}(H) \subset \mathcal{G}$ then the next proposition, although its proof is quite trivial, is an extension of the virial theorems from [Mo] and [HuS]. The general case requires a supplementary condition on the eigenvector u (which is fulfilled in our applications). The notations are chosen to fit those of Section 3, see Remark 3.1.

Proposition 4.2 *Let H be a selfadjoint operator and A the generator of a C_0-semigroup $\{W_t\}$ in $\mathcal{H}$. Let $\mathcal{G}$ be a Hilbert space with $\mathcal{G} \subset \mathcal{H}$ continuously and densely, identify $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$, and assume that $\mathcal{G}$ is* b*-stable under $\{W_t\}$ and $\{W_t^*\}$. Finally, assume that $\mathcal{D}(H) \cap \mathcal{G}$ is dense in $\mathcal{G}$ and that the restriction of H to $\mathcal{D}(H) \cap \mathcal{G}$ extends to a continuous operator $\widetilde{H} \in B(\mathcal{G}, \mathcal{G}^*)$ which is of class $C^1(A; \mathcal{G}, \mathcal{G}^*)$. If u is an eigenvector of H such that $u \in \mathcal{G}$, then $(u, [\widetilde{H}, \mathrm{i}A]^\circ u) = 0$.*

Proof. Note first that $\widetilde{H} : \mathcal{G} \to \mathcal{G}^*$ will be a symmetric operator. Then the result is an immediate consequence of $[\widetilde{H}, \mathrm{i}A]^\circ = \lim_{t \to 0} \frac{1}{t}(\widetilde{H}W_t - W_t\widetilde{H})$ with the usual interpretations of the two symbols W_t (the first one acts in $\mathcal{G}$ the second one in $\mathcal{G}^*$). $\square$

Proposition 4.1 covers the case when the commutator $[H, \mathrm{i}A]$ is dominated by H^2. The next result can be used in the context of Subsection 3.1, where H and $[H, \mathrm{i}A]$ are not comparable, as it happens in the main application considered in [GGM].

In the sequel we adopt the following standard convention: if Q is a symmetric bounded below quadratic form on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(Q)$, then we extend Q to $\mathcal{H}$ by setting $Q(u) := +\infty$ if $u \notin \mathcal{D}(Q)$. We recall the following easy fact, which can be checked using the concept of *gauges* on topological vector spaces (see e.g. [ABG, Prop. 2.1.1]):
Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with $\mathcal{H}_2 \subset \mathcal{H}_1$ continuously. Then if Q is a symmetric bounded below quadratic form on $\mathcal{H}_1$, Q is closed (resp. closeable) on $\mathcal{H}_2$ if Q is closed (resp. closeable) on $\mathcal{H}_1$. Moreover if Q is closeable on $\mathcal{H}_1$, then the domain of the closure of Q on $\mathcal{H}_2$ is $\mathcal{D}(\overline{Q}) \cap \mathcal{H}_2$.

Let H, H' and $\mathcal{G}$ be as in Subsection 3.1. We assume that condition **(M1)** and the weakened version **(M2′)** of **(M2)** hold. Let $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = \mathcal{D}(H)$ and $Q(u) = (u, H'u) + c(u, \langle H \rangle u)$, with domain $\mathcal{D}$. We saw in Subsection 3.1 that Q is closeable on $\mathcal{D}$ with closure (u, Gu) with domain $\mathcal{G}$. By the above remark, the quadratic form given by $(u, H'u)$ with domain $\mathcal{D} \cap \mathcal{D}(H)$ is closeable on $\mathcal{D}(H)$. We denote its closure by $(u, \dot{H}u)$, which has domain $\mathcal{G} \cap \mathcal{D}(H)$. The following result is an immediate consequence of Proposition 4.1.

Proposition 4.3 *Assume that there is a sequence of regular operators A_n such that for each n the operator H is of class $C^1(A_n)$ and $[H, \mathrm{i}A_n]^\circ$ is a symmetric form on $\mathcal{D}(H)$ and such that*

$$\lim_{n \to \infty} (v, [H, \mathrm{i}A_n]^\circ v) = (v, \dot{H}v),$$

for all $v \in \mathcal{D}(H)$, where in the l.h.s. we mean the limit in $\mathbb{R} \cup +\infty$. Then, if u is an eigenvector of H one has $u \in \mathcal{G}$ and $(u, \dot{H}u) = 0$.

5 Hamiltonians of class $C^2(A)$

In this section we fix a *maximal symmetric operator A on $\mathcal{H}$*. The semigroup of isometries $W_t = \mathrm{e}^{\mathrm{i}tA}$ associated to it is defined as in Subsection 1.1, so $t \geq 0$ if A has deficiency indices $(N, 0)$ and $t \leq 0$ if A has deficiency indices $(0, N)$. For Hamiltonians H with a spectral gap, we shall deduce from Theorem 3.5 a result which covers those from [Mo, HuS, PSS], as well as the results from [ABG] under the $C^2(A)$ assumption.

Let us say that a bounded operator S is of class $C^2(A)$ if it is of class $C^1(A)$ and $S' := [S, \mathrm{i}A]^\circ$ is also of class $C^1(A)$; then we set $S'' := [S', \mathrm{i}A]^\circ$. A selfadjoint operator H is called of class $C^2(A)$ if there is $z \in \mathbb{C} \setminus \sigma(H)$ such that $R(z) \in C^2(A)$. Note that this property is independent of z. Indeed, assume that it holds for some z_0 and let us set $R_0 = R(z_0), R = R(z)$. Then $R \in C^1(A)$ by Remark 2.15(4). From Proposition 2.13 and with the notation $\alpha = z - z_0$ we get $R' = (\mathbb{1} + \alpha R)R_0'(\mathbb{1} + \alpha R)$, hence $R' \in C^1(A)$ and

$$R'' = 2\alpha(\mathbb{1} + \alpha R)R_0'(\mathbb{1} + \alpha R)R_0'(\mathbb{1} + \alpha R) + (\mathbb{1} + \alpha R)R_0''(\mathbb{1} + \alpha R). \tag{5.1}$$

Proposition 2.19 and the comment before Theorem 2.25 give us a continuous symmetric operator $H' := [H, \mathrm{i}A]^\circ : \mathcal{D}(H) \to \mathcal{D}(H)^*$ such that $R' = -RH'R$. In particular, the Mourre estimate makes sense in the usual form.

We denote by $\mathcal{H}_{s,p}$ the Besov spaces associated to the operator A. We stress that the adjoint spaces $\mathcal{H}^*_{s,p}$ are similarly constructed, but involve the operator A^* (see Subsection 3.3). The main result of this section is the next theorem. We shall not explicitly mention the Hölder continuity properties of the boundary values, but it should be clear from the proof how to deduce them from Theorem 3.7.

Theorem 5.1 *Let H be a selfadjoint operator of class $C^2(A)$ and having a spectral gap. Assume that $J \subset \mathbb{R}$ is a bounded open set and that there are a number $a > 0$ and a compact operator K such that $\mathbb{1}_J(H)H'\mathbb{1}_J(H) \geq a\mathbb{1}_J(H) + K$. Then J contains at most a finite number of eigenvalues of H and these eigenvalues are of finite multiplicity. The limits $\lim_{\mu\to\pm 0} R(\lambda+\mathrm{i}\mu)$ exist in the weak* topology of $B(\mathcal{H}_{1/2,1}, \mathcal{H}^*_{1/2,1})$ locally uniformly in $\lambda \in J \setminus \sigma_p(H)$.*

Proof. The assertion concerning the eigenvalues follows by an easy and standard argument from the virial theorem (Proposition 4.1). For the rest of the proof we may assume that $0 \notin J \cap \sigma(H)$ and we denote $S = -H^{-1} \in B(\mathcal{H})$. Let I be a compact subset of J which does not contain eigenvalues of H. Then $z \mapsto \zeta = -z^{-1}$ is a holomorphic map of the open upper half plane $\mathbb{C}_+$ onto itself which extends to a homeomorphism of $\mathbb{C}_+ \cup I$ onto $\mathbb{C}_+ \cup L$, where the compact real set L is the image of I. For $z \in \mathbb{C}_+$ we have $R(z) = -\zeta(S - \zeta)^{-1}S$ and $S \in B(\mathcal{H}_{1/2,1})$ (this follows by real interpolation from $S \in C^1(A)$ and Proposition 2.2). Hence it suffices to prove that for each $u \in \mathcal{H}_{1/2,1}$ the map $\mathbb{C}_+ \ni \zeta \mapsto (u, (S - \zeta)^{-1}u)$ extends to a weak* continuous function on $\mathbb{C}_+ \cup L$. Since $S \in C^2(A)$, from Theorem 3.5 it follows that it suffices to prove that S satisfies a strict Mourre estimate on small subsets of L. As we explained before relation (5.1), we have $S' = SH'S$ and clearly $\mathbb{1}_L(S) = \mathbb{1}_I(H)$, hence

$$\mathbb{1}_L(S)S'\mathbb{1}_L(S) \geq a\mathbb{1}_L(S)S^2 + SKS \geq ab\mathbb{1}_L(S) + SKS$$

where $b = \min_{x\in L} x^2 > 0$. Let $c < ab$. Since S has no eigenvalues in L and SKS is compact, for $M \subset L$ small we clearly get $\mathbb{1}_M(S)S'\mathbb{1}_M(S) \geq c\mathbb{1}_M(S)$. □

One point remains to be discussed: how should one check the C^2 property of H? The problem arises because in general the resolvent of an operator is not a simple object. Proposition 2.31 allows one to check rather easily the C^1 property: indeed, it suffices to show that for each $u \in \mathcal{D}(H)$ there is a number C_u such that

$$|(Hu, W_t u) - (u, W_t Hu)| \leq C_u |t| \quad \text{if } 0 \leq |t| \leq 1. \tag{5.2}$$

We do not have such a simple criterion for the C^2 property. One can show that a bounded operator S is of class $C^2(A)$ if and only if there is a number C such that

$$\|[[S, W_t], W_t]\| \equiv \|SW_{2t} - 2W_t S W_t + W_{2t} S\| \leq Ct^2 \quad \text{if } 0 \leq |t| \leq 1. \tag{5.3}$$

However, if $S = R(z)$ it is not possible to eliminate the resolvent completely from this relation. Instead, we have the following criterion. The space $\mathcal{G}$ that we introduce below could be the form domain $\mathcal{G} = \mathcal{D}(|H|^{1/2})$ of H, but this choice is not always convenient (cf. [ABG, p. 316]).

Proposition 5.2 *Let H be a selfadjoint operator of class $C^1(A)$. Assume that $\mathcal{G}$ is a Hilbert space with $\mathcal{D}(H) \subset \mathcal{G} \subset \mathcal{H}$ continuously and densely and such that $\mathcal{G}$ is* b*-stable under $\{W_t\}$ and $\{W_t^*\}$. We identify $\mathcal{D}(H) \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \subset \mathcal{D}(H)^*$ and assume $(H - \mathrm{i})^{-1}\mathcal{G}^* \subset \mathcal{G}$ and $H'\mathcal{D}(H) \subset \mathcal{G}^*$. Then H is of class $C^2(A)$ if and only if for each $u \in \mathcal{D}(H)$ there is a number C_u such that*

$$|(H'u, W_t u) - (u, W_t H'u)| \leq C_u |t| \quad \textit{if } 0 \leq |t| \leq 1. \tag{5.4}$$

Proof. Below we abbreviate $\mathcal{D}(H) = \mathcal{D}$. Note first that, by the closed graph theorem and our hypotheses, the operator $H' \in B(\mathcal{D}, \mathcal{D}^*)$ belongs in fact to $B(\mathcal{D}, \mathcal{G}^*)$. By symmetry we also get $H' \in B(\mathcal{G}, \mathcal{D}^*)$. Then observe that $\{W_t\}$ extends to a C_0-semigroup in $\mathcal{G}^*$ (cf. the comments after Lemma 2.33). Thus for $u \in \mathcal{D}$ the term $(H'u, W_t u) = (u, H'W_t u)$ is well defined because $H'u \in \mathcal{G}^*$ and $W_t u \in \mathcal{G}$, and $(u, W_t H'u)$ is well defined because $W_t H'u \in \mathcal{G}^*$. Moreover, by the uniform boundedness principle, the relation (5.4) is equivalent to $\|H'W_t - W_t H'\|_{B(\mathcal{D},\mathcal{D}^*)} \leq C|t|$.

Let $R = (\mathrm{i} - H)^{-1}$. Then R is of class $C^1(A)$ and $R' = RH'R$. We have to show that R' is of class $C^1(A)$. By Remark 2.30 it suffices to prove $\|[R', W_t]\| \leq C|t|$ for some constant C. Taking into account the preceding explanations one can check that the following formally obvious relation is indeed true (the two operators $[R, W_t]$ in the right hand side act in different spaces):

$$[R', W_t] = [R, W_t]H'R + R[H', W_t]R + RH'[R, W_t]. \tag{5.5}$$

Thus there is a number C such that

$$\|[R', W_t]\| \leq C\|[R, W_t]\|_{B(\mathcal{G}^*,\mathcal{H})} + C\|[H', W_t]\|_{B(\mathcal{D},\mathcal{D}^*)} + C\|[R, W_t]\|_{B(\mathcal{H},\mathcal{G})}.$$

We shall prove that $\|[R, W_t]\|_{B(\mathcal{H},\mathcal{G})} \leq C|t|$; the first term in the right hand side above is similarly estimated and this finishes the proof. Since $R \in C^1(A)$ we have $[R, W_t] = \int_0^t W_{t-s} R' W_s \mathrm{d}s$ as a strong integral in $B(\mathcal{H})$, cf. (2.17) and (2.19). But clearly $R' = RH'R \in B(\mathcal{H}, \mathcal{G})$ so the integrand is a strongly continuous $B(\mathcal{H}, \mathcal{G})$ valued function. This gives the required estimate. □

To see the relation with the results from [Mo, HuS, PSS] we use their notations $\mathcal{H}^2 = \mathcal{D}(H)$, $\mathcal{H}^1 = \mathcal{D}(|H|^{1/2})$ and $\mathcal{H}^{-2} = (\mathcal{H}^2)^*$, $\mathcal{H}^{-1} = (\mathcal{H}^1)^*$. This gives us the scale of Hilbert spaces:

$$\mathcal{H}^2 \subset \mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1} \subset \mathcal{H}^{-2}. \tag{5.6}$$

We choose $\mathcal{G} = \mathcal{H}^1$ in Proposition 5.2 and we assume that $\mathcal{H}^2$ is b-stable under $\{W_t\}$ and $\{W_t^*\}$. By interpolation it follows that $\mathcal{H}^1$ has the same property. With notations introduced in Subsection 2.5 we see that if $H \in C^2(A; \mathcal{H}^2, \mathcal{H}^{-2})$ and $H'\mathcal{H}^2 \subset \mathcal{H}^{-1}$ then $H \in C^2(A)$ and we can apply Theorem 5.1. Thus we obtain an extension of the results of [PSS] (here A is assumed maximal symmetric, not necessarily selfadjoint, and there are some supplementary hypotheses in [PSS]). In particular, we also cover those from [Mo, HuS] when H has a spectral gap.

The preceding result is quite efficient when the domain $\mathcal{H}^2$ of H is known, e.g. if $H = H_0 + V$ and one can use the Kato-Rellich theorem (the operator H_0 being easy to control). A second possibility one may consider is that when the preceding sum exists in the sense of forms, so only the form domain $\mathcal{H}^1$ is explicitly known. In this case it suffices to require b-stability of $\mathcal{H}^1$ under $\{W_t\}$ and $\{W_t^*\}$, which is weaker than the stability of $\mathcal{H}^2$, but then one needs $H \in C^2(A; \mathcal{H}^1, \mathcal{H}^{-1})$, an assumption stronger than $H \in C^2(A; \mathcal{H}^2, \mathcal{H}^{-2})$. From Proposition 5.2 it follows that Theorem 5.1 covers this situation too, in fact it is obvious that it implies [ABG, Theorem 7.5.4] (for H with spectral gap and with C^2 type conditions). One can also replace in Theorem 5.1 the space $\mathcal{H}_{1/2,1}$ by $\mathcal{H}_{1/2,1}^{-1}$. This follows easily from [ABG, Proposition 7.4.4].

We stress, however, that in Theorem 5.1 there is no assumption concerning the stability of $\mathcal{H}^1$ or $\mathcal{H}^2$ under the semigroups and this is useful when there is not enough information concerning these spaces (see, e.g. [Am, DG2]). Proposition 5.2 describes just one method of checking the $C^2(A)$ property, in some concrete situations other methods could be more efficient.

One final comment on the regularity condition we imposed on H in Theorem 5.1. One can replace the assumption $H \in C^2(A)$ by $H \in C^{1,1}(A)$, which means $\int_0^1 \|[[R(z), W_t], W_t]\| \frac{\mathrm{d}t}{t^2} < \infty$ for some (hence for all) $z \in \mathbb{C} \setminus \sigma(H)$ (compare with (5.3)). The resulting theorem extends all the results from [ABG, Sec. 7.4] to the case when A is only maximal symmetric. We do not give details because this extension is of no interest for [GGM].

A Appendix

A.1 We state here two results needed in Subsection 3.5. The first one is an improved version of the Gronwall's lemma; the proof can be found in [ABG, Appendix 7.A]:

Lemma A.1 *Let* $(0, \varepsilon_0] \ni \varepsilon \mapsto F_\varepsilon \in \mathbb{C}$ *be a* C^1 *function such that*

$$|F_\varepsilon'| \leq a(\varepsilon) + b(\varepsilon)|F_\varepsilon|^{1/2} + c(\varepsilon)|F_\varepsilon|$$

for some locally integrable functions a, b *and* c*. Then for all* $\varepsilon \leq \varepsilon_0$ *one has:*

$$|F_\varepsilon| \leq \left[\left(|F_{\varepsilon_0}| + \int_\varepsilon^{\varepsilon_0} a(\tau)\mathrm{d}\tau \right)^{1/2} + \frac{1}{2}\int_\varepsilon^{\varepsilon_0} b(\tau)\exp\left(-\frac{1}{2}\int_\tau^{\varepsilon_0} c(\sigma)\mathrm{d}\sigma \right)\mathrm{d}\tau \right]^2 \exp\left(\int_\varepsilon^{\varepsilon_0} c(\sigma)\mathrm{d}\sigma \right)$$

We also need the following elementary fact.

Lemma A.2 *Let* $U \subset \mathbb{C}$ *be an open convex set,* ε_0 *a number in* $]0,1]$*, and* $\phi : U \times]0, \varepsilon_0[\to \mathbb{C}$ *a function such that for each* ε *the map* $z \mapsto \phi(z, \varepsilon)$ *is holomorphic and for each* z *the maps* $\varepsilon \mapsto \phi(z, \varepsilon)$ *and* $\varepsilon \mapsto \frac{\mathrm{d}}{\mathrm{d}z}\phi(z, \varepsilon)$ *are of class* C^1*. Assume that there are numbers* $0 < \sigma < 1$ *and* $M > 0$ *such that for all* $(z, \varepsilon) \in U \times]0, \varepsilon_0[$ *the following inequalities hold:*

$$|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(z,\varepsilon)| \leq M\varepsilon^{\sigma-1}, \quad |\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)| \leq M\varepsilon^{\sigma-2}, \quad \liminf_{\varepsilon \to \varepsilon_0} |\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)| \leq M. \tag{A.1}$$

The the limit $\lim_{\varepsilon\to 0}\phi(z,\varepsilon) =: \phi(z,+0)$ *exists uniformly in* $z\in U$ *and satisfies*

$$\text{(A.2)}\qquad |\phi(z_1,+0)-\phi(z_2,+0)| \le \frac{2M}{\sigma(1-\sigma)}|z_1-z_2|^\sigma \quad \text{if } z_1,z_2\in U \text{ and } |z_1-z_2|\le\varepsilon_0.$$

Proof. Writing for $\varepsilon<\varepsilon_1<\varepsilon_0$

$$\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon_1) - \int_\varepsilon^{\varepsilon_1}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\tau)\mathrm{d}\tau$$

and taking $\varepsilon_1\to\varepsilon_0$ along a convenient sequence after the obvious estimate, we get

$$\text{(A.3)}\qquad |\frac{\mathrm{d}}{\mathrm{d}z}\phi(z,\varepsilon)| \le M + M\int_\varepsilon^{\varepsilon_0}\tau^{\sigma-2}\mathrm{d}\tau \le \frac{2M}{1-\sigma}\varepsilon^{\sigma-1}.$$

Now we have $\phi(z,+0)=\phi(z,\varepsilon)-\int_0^\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}\phi(z,\tau)\mathrm{d}\tau$ hence

$$\text{(A.4)}\qquad |\phi(z_1,+0)-\phi(z_2,+0)| \le |\phi(z_1,\varepsilon)-\phi(z_2,\varepsilon)| + 2\sup_{k=1,2}\int_0^\varepsilon |\frac{\mathrm{d}}{\mathrm{d}\tau}\phi(z_k,\tau)|\mathrm{d}\tau.$$

The first term on the right hand side is less than $|z_1-z_2|\varepsilon^{\sigma-1}2M/(1-\sigma)$ by (A.3). The last term is less than $2M\varepsilon^\sigma/\sigma$ because of (A.1). It suffices now to take $\varepsilon=|z_1-z_2|$.□

A.2 We prove now Proposition 1.2. In all this proof we keep the notations and refer to the relations from the Introduction. We take $H=\omega(P)$ and $H'=|\omega'(P)|^2\langle\omega'(P)\rangle^{-1}$ in Theorem 1.1. Thus H and H' are commuting selfadjoint operators and $H'\ge 0$. Clearly $\mathcal{D}=\mathcal{K}^1$ and $\mathcal{G}=\mathcal{K}^{1/2}$. Then J will be a bounded open set containing Λ with closure disjoint from $\tau(\omega)$. By the comments after the definition of the threshold set we see that there is $m>0$ such that $|\omega'(k)|\ge m$ if $\omega(k)\in J$. Thus $H'\ge H'\mathbb{1}_J(H)\ge m^2\langle m\rangle^{-1}\mathbb{1}_J(H)$ and conditions **(M1)** and **(M2)** are satisfied.

We define A such that formally $A=\frac{1}{2}(F(P)Q+QF(P))$, where $F:\mathbb{R}^n\to\mathbb{R}^n$ is the vector field $F(k)=\omega'(k)\langle\omega'(k)\rangle^{-1}$. In order to have a rigorous definition and to show that A is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ it suffices to note that F is a Lipschitz vector field and to use [ABG, Proposition 7.6.3(a)]. However, we stress that the proof in [ABG] of the quoted proposition is wrong (it works if $\operatorname{div}F$ is of class C^1 for example). A correct proof when $F\in C^1$ is given in the next subsection in a more general context; observe that the proof, modulo some measure theoretic technicalities, extends to locally Lipschitz F. This also proves that $W_t=\mathrm{e}^{\mathrm{i}tA}$ leaves invariant the set $\mathcal{F}C_0(\mathbb{R}^n)$ of Fourier transforms of continuous functions with compact support. Let us show that W_t leaves invariant $\mathcal{K}^\sigma$ for each σ and t (in particular condition **(M3′)** is fulfilled). By interpolation and duality it suffices to consider the case $\sigma=1$. We have, with natural notations:

$$[\widetilde{\omega}(P),\mathrm{i}A] = F(P)\widetilde{\omega}'(P) = \frac{\omega'(P)\omega''(P)F(P)+\omega(P)F(P)\omega'(P)}{\widetilde{\omega}(P)}$$

as operators on $\mathcal{F}C_0(\mathbb{R}^n)$. Thus $[\widetilde{\omega}(P),\mathrm{i}A]$ is bounded with respect to $\widetilde{\omega}(P)$ and we can apply Proposition 2.34.

Now it is easy to check that condition **(M4)** is fulfilled and that H'' as an operator on $\mathcal{K}^1$ is given by the relation: $H''=(2+|\omega'(P)|^2)(1+|\omega'(P)|^2)^{-1/2}(F\omega''F)(P)$. Hence H'' is a bounded operator and thus condition **(M5)** is satisfied. We see that all the hypotheses of Theorem 1.1

are fulfilled so $R(z) \in B(\mathcal{G}^*_{1/2,1}, \mathcal{G}_{-1/2,\infty})$ with norm bounded by a constant independent of z. We recall that $\mathcal{G}^*_{1/2,1}$ is obtained by interpolating between $\mathcal{G}^* = \mathcal{K}^{-1/2}$ and $\mathcal{G}^*_1$, which is the completion of $\mathcal{D}(A)$ under the norm $\|u\|_{\mathcal{G}^*} + \|Au\|_{\mathcal{G}^*}$. But $A = F(P)Q - \frac{\mathrm{i}}{2}(\operatorname{div} F)(P)$ and F and $\operatorname{div} F$ are bounded, hence the preceding norm is dominated by $\sum_j \|Q_j u\|_{\mathcal{G}^*} + \|u\|_{\mathcal{G}^*}$ on $\mathcal{F}C_0(\mathbb{R}^n)$. Thus we get $\mathcal{K}^{-1/2}_1 \subset \mathcal{G}^*_1$ and then, by interpolation, $\mathcal{K}^{-1/2}_{1/2,1} \subset \mathcal{G}^*_{1/2,1}$. Then taking adjoints we have $\mathcal{G}_{-1/2,\infty} \subset \mathcal{K}^{+1/2}_{-1/2,\infty}$. Thus the estimate (1.6) holds if $\sigma = -1/2$. Now we prove it for an arbitrary σ. For this we set $\tau = 1/2 + \sigma$ and we observe that $\widetilde{\omega}^\tau(P)$ is a unitary map $\mathcal{K}^\sigma \to \mathcal{K}^{-1/2}$. Since $|\nabla \widetilde{\omega}^\tau| \le C\widetilde{\omega}^\tau$ we then see that $\widetilde{\omega}^\tau(P)$ is an isomorphism of $\mathcal{K}^\sigma_1$ onto $\mathcal{K}^{-1/2}_1$. Duality and interpolation give us that $\widetilde{\omega}^\tau(P) : \mathcal{K}^\sigma_{1/2,1} \to \mathcal{K}^{-1/2}_{1/2,1}$. Exactly in the same way we see that $\widetilde{\omega}^\tau(P) : \mathcal{K}^{\sigma+1}_{-1/2,\infty} \to \mathcal{K}^{1/2}_{-1/2,\infty}$ is an isomorphism. To finish the proof of Proposition 1.2 it suffices to replace u by $\widetilde{\omega}^\tau(P)u$ in (1.6).

A.3 We prove here Lemma 1.3 and more general facts. Recall that $F(k) = \frac{\omega'(k)}{|\omega'(k)|}$ defines a vector field $F : \Omega \to \mathbb{R}^n$ of class C^1. For each $k \in \Omega$ consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = F(p(t)) \text{ with } p(0) = k. \tag{A.5}$$

Since F is of class C^1 on Ω, a unique solution exists for t in a neighborhood of 0. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(p(t)) = \omega'(p(t))\frac{\mathrm{d}}{\mathrm{d}t}p(t) = |\omega'(p(t))| > 0$$

on the interval of existence of the solution. Thus the function $t \mapsto \omega(p(t))$ is strictly increasing, in particular for $t > 0$ we have $\omega(p(t)) > \omega(p(0)) = \omega(k) > 0$. On the other hand, $|F(k)| = 1$ so the local solution satisfies $|p(t)| \le |k| + |\int_0^t F(p(s))\mathrm{d}s| \le |k| + |t|$. By [Ha, Theorem 3.1] the maximal interval of existence of the local solution is of the form $]\tau(k), \infty[$ with $\tau(k) < 0$.

In the rest of the proof we shall not use the explicit form of F. The only fact which matters is that the solution of (A.5) is defined on such an interval. This is useful, for example, in the proof of Proposition 1.2, where we are in the situation $\Omega = \mathbb{R}^n$ and F of class C^1 and bounded; then the solution of (A.5) exists on $\mathbb{R}$.

Lemma A.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set with complement of measure zero and let $F : \Omega \to \mathbb{R}^n$ be of class C^1 and such that for each $k \in \Omega$ the equation (A.5) has a solution $t \mapsto p(t) \in \Omega$ defined for all $t \ge 0$. Then the closure in $L^2(\mathbb{R}^n)$ of the operator $-\frac{1}{2}(F(Q)P + PF(Q))|C_0^\infty(\Omega)$ is a maximal symmetric densely defined operator with deficiency indices of the form $(N, 0)$.*

Proof. Let $t \mapsto p_t(k)$ be the solution of (A.5) with maximal domain $]\tau(k), \infty[$ with $\tau(k) < 0$. For each real t we denote Ω_t the open set of $k \in \Omega$ such that $\tau(k) < t$. In other words, Ω_t is the domain of the map p_t. Then $\Omega_s \subset \Omega_t$ if $s \le t$ and $\Omega_t = \Omega$ if $t \ge 0$. For $t \ge 0$ we have $\Omega_{-t} = p_t(\Omega)$ and p_t is a C^1-diffeomorphism of Ω onto Ω_{-t} with inverse p_{-t}.

Let us denote $f = \operatorname{div} F$. Then for each real t and $k \in \Omega_t$ we have

$$\alpha_t(k) := \det \nabla p_t(k) = \exp\left(\int_0^t f(p_s(k))\mathrm{d}s\right) > 0. \tag{A.6}$$

Note that α_t is a continuous function and, by the inverse function theorem, $\alpha_t(p_{-t}(k))\alpha_{-t}(k) = 1$ if $k \in \Omega_{-t}$. We define for $t \ge 0$

$$\widehat{W}_t u := \chi_{\Omega_{-t}}\sqrt{\alpha_{-t}}\, u \circ p_{-t} = \chi_{\Omega_{-t}}(\alpha_t \circ p_{-t})^{-1/2} u \circ p_{-t}. \tag{A.7}$$

A change of variables shows that $\widehat{W}_t$ is an isometry in $L^2(\Omega) = L^2(\mathbb{R}^n)$ with range equal to $L^2(\Omega_{-t})$, and $\widehat{W}_t^* u = \sqrt{\alpha_t} u \circ p_t$. Clearly $\{\widehat{W}_t\}_{t\geq 0}$ is C_0-semigroup of isometries in $L^2(\mathbb{R}^n)$.

Let us compute the generator of this semigroup. Let $u \in C_0^1(\Omega)$ and $t \geq 0$. Then $\widehat{W}_t u \in C_0(\Omega_{-t})$ and if $k \in \Omega_{-t}$ we have $\widehat{W}_t u(k) = \alpha_{-t}(k)^{1/2} u(p_{-t}(k))$. From (A.6) and (A.5) we get

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}_t u(k) &= \frac{1}{2} f(p_{-t}(k))\alpha_{-t}(k)^{1/2} u(p_{-t}(k)) + \alpha_{-t}(k)^{1/2} F(p_{-t}(k))(\nabla u)(p_{-t}(k)) \\ &= [\widehat{W}_t(F\nabla + \frac{1}{2}f)u](k). \end{aligned}$$

Denote $\widehat{A}$ the closure of the operator acting on $C_0^1(\Omega)$ as follows:

$$\widehat{A}u = \frac{\mathrm{i}}{2} fu + \mathrm{i}F\nabla u = \frac{\mathrm{i}}{2} fu - FPu = -\frac{1}{2}(F(Q)P + PF(Q))u.$$

Thus for $u \in C_0^1(\Omega)$ we have $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{W}_t u = \widehat{W}_t \widehat{A} u$.

If $f \in C^1(\Omega)$, e.g. if F is of class $C^2(\Omega)$, then clearly $\widehat{W}_t C_0^1(\Omega) \subset C_0^1(\Omega_{-t}) \subset C_0^1(\Omega)$ and we can apply Nelson's lemma (see Remark 2.35) to obtain that $\widehat{A}$ is the generator of $\{\widehat{W}_t\}$ (note that $C_0^1(\Omega)$ is a dense subspace of $L^2(\mathbb{R}^n)$). Thus, the closure $\widehat{A}$ of $-\frac{1}{2}(F(Q)P + PF(Q))|C_0^1(\Omega)$ is a symmetric densely defined operator with deficiency indices $(N, 0)$ and $\widehat{W}_t = \mathrm{e}^{\mathrm{i}t\widehat{A}}$.

In general, f is only continuous and the argument has to be modified as follows. Let us denote, for a moment, $\widetilde{A}$ the generator of $\{\widehat{W}_t\}$. By what we proved above, we have $\widehat{A} \subset \widetilde{A}$. Since $\widetilde{A}$ is symmetric, we have $\widehat{A} \subset \widetilde{A} \subset \widetilde{A}^* \subset \widehat{A}^*$. It is easy to see $\widehat{A}^*$ is the operator $-\frac{1}{2}(F(Q)P + PF(Q))$ acting in the sense of distributions on the domain

$$\mathcal{D}(\widehat{A}^*) = \{u \in L^2(\mathbb{R}^n) \mid (F\nabla + \frac{1}{2}f)u \in L^2(\Omega)\}.$$

In particular, both $\widehat{A}$ and $\widetilde{A}$ are restrictions of the operator $-\frac{1}{2}(F(Q)P + PF(Q))$ acting in the sense of distributions.

Now let $C_0^F(\Omega)$ be the set of $u \in C_0(\Omega)$ such that $F\nabla u \in C_0(\Omega)$ (distributional derivatives). We shall prove later on that $C_0^F(\Omega) \subset \mathcal{D}(\widehat{A})$. More precisely, for each $u \in C_0^F(\Omega)$ we shall construct a sequence of functions $u_\varepsilon \in C_0^1(\Omega)$, with support in a fixed compact subset of Ω, such that $u_\varepsilon \to u$ and $\widehat{A}u_\varepsilon \to \widehat{A}u$ uniformly. We make $\varepsilon \to 0$ in $\widehat{W}_t\widehat{A}u_\varepsilon = \widehat{W}_t\widetilde{A}u_\varepsilon = \widetilde{A}\widehat{W}_t u_\varepsilon$ and take into account that $\widetilde{A}$ is a closed operator. We obtain that $\widehat{W}_t u \in \mathcal{D}(\widetilde{A})$ for all t if $u \in C_0^F(\Omega)$ and $\widehat{W}_t\widehat{A}u = \widetilde{A}\widehat{W}_t u_\varepsilon$. Taking into account the way $\widetilde{A}$ acts we thus obtain $\widehat{W}_t C_0^F(\Omega) \subset C_0^F(\Omega)$ for all $t > 0$. Now we can apply Nelson's lemma and get that $C_0^F(\Omega)$ is a core for $\widetilde{A}$. Hence $\widetilde{A} = \widehat{A}$.

It remains to construct the functions u_ε. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball and such that $\int \theta \mathrm{d}x = 1$ and let $\theta_\varepsilon(x) = \varepsilon^{-n}(x/\varepsilon)$. If $u \in C_0^F(\Omega)$ we set $u_\varepsilon = \theta_\varepsilon * u$ and from now on $\varepsilon > 0$ is small enough. We have $u_\varepsilon \to u$ and $\theta_\varepsilon * (F\nabla u) \to F\nabla u$ uniformly because $F\nabla u \in C_0$. Thus it suffices to show that $F\nabla u_\varepsilon - \theta_\varepsilon * (F\nabla u) \to 0$ uniformly. A straightforward computation gives:

$$F\nabla\theta_\varepsilon * u(x) - \theta_\varepsilon * (F\nabla u)(x) = \int \left(\frac{F(x) - F(x - \varepsilon y)}{\varepsilon} \nabla\theta(y) + f(x - \varepsilon y)\theta(y) \right) u(x - \varepsilon y)\mathrm{d}y.$$

But this clearly converges uniformly to $\int (y\nabla)F(x)\nabla\theta(y)\mathrm{d}y + f(x)u(x) = 0.\square$

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