1. Filling some gaps in the lectures

Let (X,σ) be a real pre-symplectic space. For $\eta \in L_s(X)$ we have the conditions

$$(C)\ \eta \geq 0,\ |x_1 \cdot \sigma x_2| \leq 2(x_1 \cdot \eta x_1)^{\frac{1}{2}}(x_2 \cdot \eta x_2)^{\frac{1}{2}},\ x_1, x_2 \in X.$$

1.1. How to obtain a quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$ from a quasi-free state on $\mathrm{CCR}^{Weyl}(X,\sigma)$.

Let (X,σ) a real pre-symplectic space and ω be a quasi-free state on $\mathrm{CCR}^{Weyl}(X,\sigma)$, with covariance η. Let $(\mathcal{H},\pi,\Omega)$ its GNS triple. We denote by $\mathcal{D}\subset\mathcal{H}$ the dense subspace $\mathcal{D} = \{\pi(A)\Omega\ :\ A\in \mathrm{CCR}^{Weyl}(X,\sigma)\}$.

Lemma 1.1. *set $W_\pi(x) := \pi(W(x)) \in U(\mathcal{H})$ (unitary operators on $\mathcal{H}$). Then for $x\in X$ the one-parameter group $\mathbb{R}\ni t\mapsto W_\pi(tx)$ is strongly continuous.*

Proof. By standard arguments it suffices to prove the strong continuity at $t=0$. By a density argument it suffices to show that for $u\in\mathcal{D}$ one has $W(tx)u - u \to 0$ in $\mathcal{H}$ when $t\to 0$. We can assume by linearity that $u = W_\pi(y)\Omega$, $y\in X$. Then

$$\|u - W_\pi(tx)u\|^2 = (\Omega|W_\pi(-y)(\mathbb{1} - W_\pi(-tx))(\mathbb{1} - W_\pi(tx))W_\pi(y)\Omega),$$

and using the CCR :

$$\begin{aligned} & W_\pi(-y)(\mathbb{1} - W_\pi(-tx))(\mathbb{1} - W_\pi(tx))W_\pi(y) \\ = \ & 2\mathbb{1} - W(-tx)\mathrm{e}^{-\mathrm{i}y\cdot\sigma x} - W(tx)\mathrm{e}^{\mathrm{i}y\cdot\sigma x}. \end{aligned}$$

Therefore

$$\begin{aligned} \|u - W_\pi(tx)u\|^2 = \ & \omega(2\mathbb{1} - W(-tx)\mathrm{e}^{-\mathrm{i}y\cdot\sigma x} - W(tx)\mathrm{e}^{\mathrm{i}y\cdot\sigma x}) \\ = \ & 2 - \mathrm{e}^{-\frac{1}{2}t^2 x\cdot\eta x - \mathrm{i}y\cdot\sigma x} - \mathrm{e}^{-\frac{1}{2}t^2 x\cdot\eta x + \mathrm{i}y\cdot\sigma x}, \end{aligned}$$

which tends to 0 when $t\to 0$. $\Box$

From Lemma 1.1 we can define the *field operator* $\phi_\pi(x)$ as the generator of the strongly continuous unitary group $\mathbb{R}\ni t\mapsto W_\pi(tx)$. The operator $\phi_\pi(x)$ will be selfadjoint and actually unbounded. The definition is

$$\phi_\pi(x)u := \mathrm{i}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}W_\pi(tx)u_{|t=0},\ u\in \mathrm{Dom}\phi_\pi(x),$$

where by definition the domain $\mathrm{Dom}\phi_\pi(x)$ is the set of u such that the derivative exists (in the norm topology of $\mathcal{H}$).

Lemma 1.2. *$\mathcal{D}\subset\mathrm{Dom}\phi_\pi(x)$, actually $\phi_\pi(x)$ is essentially selfadjoint on $\mathcal{D}$.*

Proof. the first part of the claim is easy : it suffices to check that for $u = W_\pi(y)\Omega$, the map $t\mapsto W_\pi(tx)u$ is strongly differentiable at $t=0$. This is done by the same computation as in Step 1. The essential selfadjointness can be shown using the following theorem of Nelson :

if $U(t) = \mathrm{e}^{\mathrm{i}tH}$ is a strongly continuous unitary group, and $\mathcal{D}\subset\mathcal{H}$ is a dense subspace included in the domain of H which is invariant under $U(t)$, then H is essentially selfadjoint on $\mathcal{D}$.

Lemma 1.3. *on $\mathcal{D}$ one has :*

1. *$X\ni x\mapsto\phi_\pi(x)$ is $\mathbb{R}-$linear,*
2. *$[\phi_\pi(x),\phi_\pi(y)] = \mathrm{i}x\cdot\sigma y\mathbb{1}$, for $x,y\in X$.*

It follows that

$$\pi : \mathrm{CCR}^{\mathrm{pol}}(X,\sigma)\in\phi(x)\mapsto\phi_\pi(x)\in L(\mathcal{D})$$

generates a representation of the $*-$algebra $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$.

Moreover we can define a state ω^{pol} on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$ by :

$$\omega^{\mathrm{pol}}(A) := (\Omega|\pi(A)\Omega),\ A\in\mathrm{CCR}^{\mathrm{pol}}(X,\sigma).$$

Proof. the first part are routine computations. The second part is obvious. □

Lemma 1.4. *one has*

$$\omega^{\mathrm{pol}}(\phi(x_1)\phi(x_2)) = x_1 \cdot \eta x_2 + \frac{i}{2} x_1 \cdot \sigma x_2.$$

Proof. We have :

$$\begin{aligned}
\omega^{\mathrm{pol}}(\phi(x_1)\phi(x_2)) &= (\Omega|\phi_\pi(x_1)\phi_\pi(x_2)\Omega) \\
&= (\mathrm{i})^{-2}\tfrac{\mathrm{d}}{\mathrm{d}t_1}\tfrac{\mathrm{d}}{\mathrm{d}t_2}(\Omega|W_\pi(t_1x_1)W_\pi(t_2x_2)\Omega)_{|t_1=t_2=0} \\
&= (\mathrm{i})^{-2}\tfrac{\mathrm{d}}{\mathrm{d}t_1}\tfrac{\mathrm{d}}{\mathrm{d}t_2}(\Omega|W_\pi(t_1x_1+t_2x_2)\mathrm{e}^{-\frac{\mathrm{i}}{2}t_1t_2x_1\cdot\sigma x_2}\Omega)_{|t_1=t_2=0} \\
&= (\mathrm{i})^{-2}\tfrac{\mathrm{d}}{\mathrm{d}t_1}\left(\mathrm{e}^{-\frac{1}{2}(t_1x_1+t_2x_2)\cdot\eta(t_1x_1+t_2x_2)})\mathrm{e}^{-\frac{\mathrm{i}}{2}t_1t_2x_1\cdot\sigma x_2}\right)_{|t_1=t_2=0}.
\end{aligned}$$

Computing the last derivative proves the claim. □

Lemma 1.5. *one has :*

$$(H^{\mathrm{pol}})\begin{cases} \omega^{\mathrm{pol}}\big(\phi(x_1)\cdots\phi(x_{2m-1})\big) = 0, \\ \omega^{\mathrm{pol}}\big(\phi(x_1)\cdots\phi(x_{2m})\big) = \sum_{\sigma\in\mathrm{Pair}_{2m}}\prod_{j=1}^{m}\omega\big(\phi(x_{\sigma(2j-1)})\phi(x_{\sigma(2j)})\big). \end{cases}$$

Proof. same proof as before, writing :

$$\begin{aligned}
&\omega\big(\phi(x_1)\cdots\phi(x_n)\big) \\
=\ &(\mathrm{i})^{-n}\tfrac{\mathrm{d}}{\mathrm{d}t_1}\tfrac{\mathrm{d}}{\mathrm{d}t_2}\cdots\tfrac{\mathrm{d}}{\mathrm{d}t_n}\omega(\textstyle\prod_1^n W(t_ix_i))_{|t_1=\cdots=t_n=0},
\end{aligned}$$

then using the CCR and clever computations. □

2. How to complete the missing points in my lectures

Assume first that one is given a quasi-free state ω on $\mathrm{CCR}^{Weyl}(X,\sigma)$ with covariance η, ie by definition

$$(H^{\mathrm{Weyl}})\ \omega(W(x)) = \mathrm{e}^{-\frac{1}{2}x\cdot\eta x},\ x\in X.$$

One constructs by Lemma 1.3 the associated state ω^{pol} on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$. Using Lemma 1.3 one obtains the condition (C). The fact that if η satisfies (C) then ω given by $(H)^{\mathrm{Weyl}}$ is a state on $\mathrm{CCR}^{\mathrm{Weyl}}(X,\sigma)$ was completely proved in the lectures.

Assume next that one is given a quasi-free state ω_1 on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$, ie a state given (H^{pol}), and the covariance η is given by Lemma 1.4.

By Lemma 1.3 this implies condition (C).

Conversely let $\eta \in L_s(X,\sigma)$ satisfying condition (C).

We can define a state ω on $\mathrm{CCR}^{Weyl}(X,\sigma)$. Consider the state ω^{pol} on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$, which is defined by condition (H^{pol}). This is the quasi-free state on $\mathrm{CCR}^{\mathrm{pol}}(X,\sigma)$ that we want.