INTRODUCTION TO QUANTUM FIELD THEORY ON CURVED SPACE-TIMES

Christian Gérard, M2 course, second semester

I. Presentation

Quantum field theory is usually formulated on the flat Minkowski space-time.

If one wants to take into account the effect of strong gravitational fields, or take some steps towards quantum gravity, it is essential to also consider quantum fields on *curved* space-times (Lorentzian manifolds). One then discovers new difficulties and new phenomena :

for example the symmetries of Minkowski space-time (Poincaré group) allow to give a clear meaning to the notion of a *vacuum state*, which is the basis of the perturbative approach to quantum fields. This notion disappears on a curved space-time without any particular symmetry.

Similarly the standard presentation of quantum fields on Minkowski space uses the Fourier transform, which also disappears on curved spaces, and has to be replaced by microlocal analysis.

Among the new phenomena, the most famous is the *Hawking effect*, which predicts that a black-hole can emit quantum particle, in contrast to the classical situation, where nothing can escape from the black-hole horizon.

The aim of this course is to give an introduction to some aspects of quantum field theory on curved space-time which is sufficiently rigorous and accessible to mathematicians. We will in particular describe the important progresses of this field due to the use of microlocal analysis, following the seminal works of M. Radzikowski.

II. Plan of the course

I. **Free quantum field theory on Minkowski space**

- Klein-Gordon (resp. Dirac) equation as a symplectic (resp. unitary) evolution.
- bosonic (resp. fermionic) quantization of Klein-Gordon (resp ; Dirac) fields.
- vacuum state in Minkowski space, space-time fields, two-point function.

II. **Algebraic framework.**

- canonical commutation and ant-commutation relations.
- CCR and CAR algebras.
- quasi-free states, pure quasi-free states, Fock spaces.

III. **Quantum field theory on curved space-times.**

- Lorentzian manifolds, causality, Cauchy surfaces.
- globally hyperbolic space-times.
- Klein-Gordon and Dirac equations on globally hyperbolic space-times
- conserved currents, advanced and retarded Green's functions, Pauli-Jordan function.
- the space of Klein-Gordon (resp. Dirac) solutions as a symplectic (resp. hermitian) space.
-space-time fields, two point functions.

IV. **Hadamard states.**

-background on microlocal analysis, wavefront set of a distribution, Hörmander's propagation of singularities theorem.
- Hadamard states as substitutes for vacuum states on curved space-times, existence of the renormalized stress-energy tensor.
-construction of Hadamard states and some examples.

III. References

[1] J. Derezinski, C. Gérard : *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2013).

[2] C. Bär, N. Ginoux, F. Pfäffle : *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, Springer (2007).

[3] *Quantum Field Theory on Curved Spacetimes*, C. Bär, K. Fredenhagen editors, Springer Lecture Notes in Physics, Springer (2009).

[4] S. Fulling : *Aspects of Quantum Field Theory in Curved Space-Time*, London Mathematical Society Student Texts, Cambridge University Press (1989).